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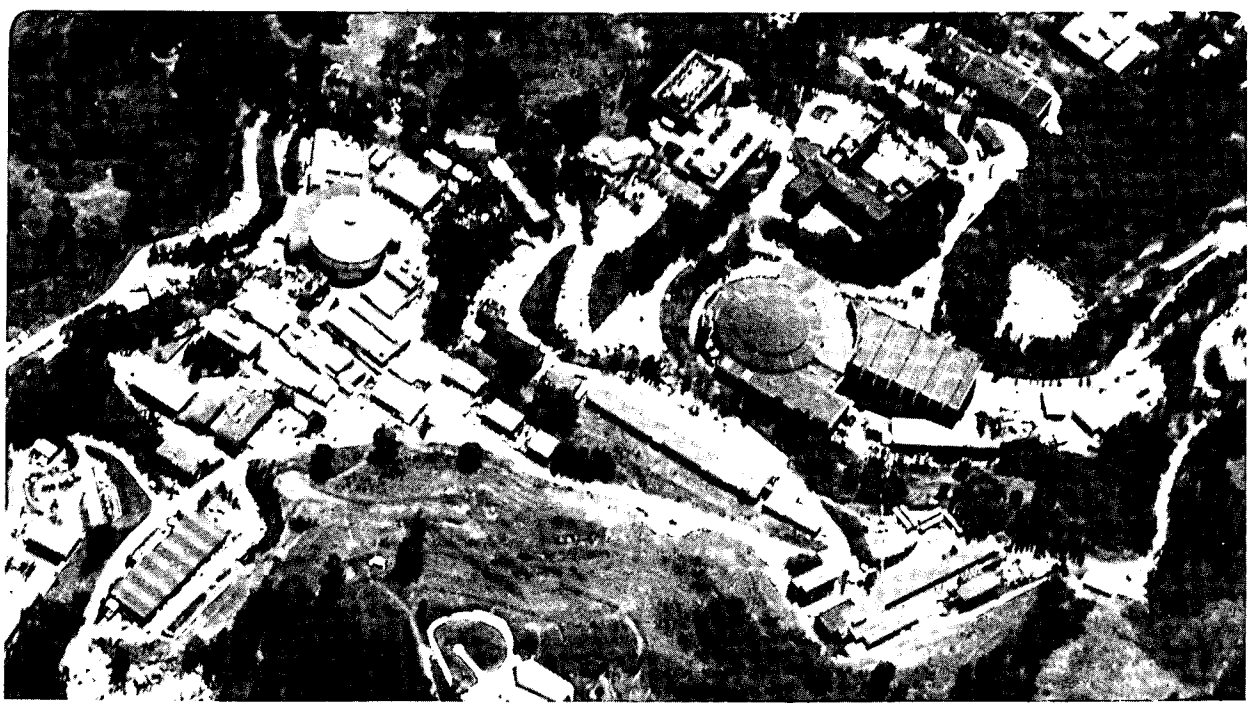

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RANDOM VORTEX METHODS FOR THE NAVIER-STOKES EQUATIONS[1]

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Random Vortex Methods for the Navier-Stokes Equations

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ABSTRACT

Two random vortex methods of Runge-Kutta type are presented for solving the two-dimensional Navier-Stokes equations. We investigate the accuracy of these methods by considering the model problem of a rotating flow with initial vorticity concentrated uniformly on a disk of finite radius. Functionals of the numerical solution are computed by Monte-Carlo estimates with efficient variance reduction, and the results are compared to those obtained from Euler's method. The numerical results show that both of the methods produce errors smaller by one power of the time step size than Euler's method, one seemingly even better than the other. These Runge-Kutta methods are derivations of similar schemes proposed by us in an earlier time for solving stochastic differential equations with constant diffusion coefficients.

0. **Introduction.** Some time ago, we proposed two numerical methods of Runge-Kutta type for solving stochastic differential equations [9],[10]. Here we explore the possibility of obtaining high accuracy random vortex methods by applying these methods to the two-dimensional Navier-Stokes equations. In a stochastic differential equation, there is a drift corresponding to the convection term of the Navier-Stokes equations, and a diffusion--the Wiener process--corresponding to the viscous term. This connection provides a basis for the study of the Navier-Stokes equations through stochastic differential equations, particularly the random vortex method which we are interested in here.

In this article, we present three random vortex methods and compare their accuracy in numerical examples. The methods that we consider are Euler's method and two other methods--A and B, say--based on the mid-point rule. For the case of stochastic differential equations, the equivalent of Euler's method has order 1 in the L_2 sense, that of Method B has order $1\frac{1}{2}$, while the equivalent of Method A we conjecture to have order 2 in a weak sense.

Our model problem is a two-dimensional rotating flow with initial vorticity distributed uniformly on a disk of finite radius a $(= .5)$. The viscosity ν $(= .002)$ is chosen within the range of a typical slightly viscous flow. Following [37],[41], we compare these methods by estimating two functionals of the flow field which can be evaluated exactly. Indeed, as we shall see in section 3, the velocity field is actually a functional of the Wiener process. The total time interval T $(= 4)$ of the test is about the order of the period of rotation of the corresponding inviscid flow.

There are two main sources of error in the stochastic schemes for estimating the functionals: the time discretization and the random sampling The former because the discretization in time of the stochastic differential equation produces local truncation error at each time step. This error depends only on the specific scheme used, and can be reduced to some extent by decreasing the time step size Δt. The latter comes from Monte-Carlo estimation of the functional, and can vary considerably for various Monte-Carlo estimates. The error variance

of a typical Monte-Carlo estimate is usually proportional to $1/\sqrt{N}$, so that the random sampling error often dominates the time discretization error, and therefore should be minimized as much as possible. A variance reduction technique is derived here; we make time differences of the functionals to be estimated and employ the non-anticipating property of the solutions of stochastic differential equations.

Other factors affecting the calculations include the cutoff δ, the cutoff function f_δ and the number N $(= 856)$ of the vortex blobs used. In order to exhibit the accuracy of the schemes considered, the cutoffs are chosen so that their effects are minimal, and, instead of being randomly sampled, the initial vortex blobs are placed on a uniform grid, as suggested by Roberts [41].

Our numerical results show that the sampling errors for the usual Monte-Carlo estimate, in Euler's method, are roughly of the same order as the time discretization errors, but dominate in the other two. As expected, these estimates cannot show the high accuracy of Methods A and B for such relatively small samples $(N = 856)$ without efficient variance reduction. The variance reduction technique mentioned above produces, for Euler's method, errors of the same order as those by a usual Monte-Carlo estimate, and exhibits clearly the first order accuracy of the method. However, the variance reduction technique produces, for Method A, errors less by a factor of between 2.5 and 10 than those of the usual estimate; and for Method B, less by a factor of between 20 and 100. Therefore, it is only interesting to discuss the reduced errors. In all cases $(\Delta t = .2, .1, .05)$, we find that the errors for Method A are smaller by a factor of from 20 to 500 than those of Euler's method, while for Method B the factor is in the range of between 33 and 500, somewhat more than those for Method A. Nevertheless, due to the interactions between the time discretization errors and the reduced random sampling errors, definite relations between the reduced random sampling errors and the time step sizes for Methods A and B are not observed. Further observations on the accuracy of these schemes would require more elaborate variance reduction techniques.

These results provide numerical evidence that either Method A or B is a practical second order method for solving the Navier-Stokes equations, as long as an efficient variance reduction technique is available. The seeming superiority of Method A to Method B is due to the non-anticipating property of solutions of stochastic differential equations, and will be explained further in a later section.

Historically, the first random vortex method was conceived by Chorin, and used to study of a slightly viscous flow with boundary conditions [12]. This method consists of solving Euler's equations by a vortex method, and sampling Gaussian random variables to model the diffusion equation. It is therefore a fractional step type method. The legitimacy of methods of this kind has been studied by several authors, e.g., [11],[18],[30] and [46]. The vortex method for solving Euler's equations can be briefly described as follows. In a vortex method the initial vorticity field is partitioned to a sum of vortex blobs--called vortices; and Euler's equations are replaced by a finite set of ordinary differential equations. These vortices then evolve according to this set of equations. Chorin's random vortex method is simply to let, at each time step, the vortices take random jumps to model the diffusion equation. Therefore, both the vortex method and the random vortex method are grid free--no spatial discretization is needed to evolve the vortex blobs.

The main difference between Chorin's methods and our Methods A and B lies that, at each time step, the velocity field in the latter case is set to be interlaced with the purely random field (the Wiener process), that is, intermediate random interactions are introduced. These intermediate random effects complicate the numerical diffusion process, requiring our schemes to use more information about the Wiener process. This might be crucial to the success of designing high accuracy random vortex methods. A theoretical study along this direction is still lacking at the present.

The random vortex method has been successful in the study of several physical phenomena, for example, turbulent combustion and [22],[44]. However, it should be noted

that the very physical vortex blobs in the vortex method, except initially, do not serve in a similar way in the random vortex method. Unlike solving Euler's equations, one cannot, for the Navier-Stokes equations, keep track of the paths of physical vortex blobs by solving a system of ordinary differential equations, due to the existence of a viscous term. In a random vortex method, each vortex blob carries a certain weight determined by the initial vorticity field, and their motion generates at each time step a probability distribution; the velocity field of the flow in turn is determined via the distribution and the weights through the Biot-Savart law (see Section 3). In short, the random vortex method is a method that provides an approximation to the velocity field through the distribution of (random) vortex blobs, each of them being specified by a quadruple: its position, its weight, the cutoff and the cutoff function.

We summarize some fundamental aspects of the vortex method and some of its development. The success of the vortex method consists in the use of vortex blobs, suggested by Chorin [12]. The early study, by Chorin and Bernard [17], of a vortex method without using vortex blobs--called point vortex method, showed that the method was unstable in predicting roll up of nonuniform vortex sheets. In a vortex method, the velocity field is determined by integrating the vorticity field against a kernel with singularity at the origin. The above instability is thus due to when two point vortices coming very close to each other. The idea of using vortex blobs is therefore to cut off this singularity. For this purpose, a class of so-called cutoff functions was introduced by Hald [25],[26], who, under mild conditions on initial data, gave the first convergence proof of the vortex method. Subsequently, Beale and Majda [6] designed vortex methods of arbitrary accuracy by careful choice of cutoff functions. All these convergence proofs have been done for paths of vortex blobs and the velocity field--which justifies the idea of using vortex methods, and also for the vorticity field in three dimensions. For other aspects of the vortex method, especially in three dimensions, we refer to Beale and Majda [5], Leonard [33], Greengard [24], and Anderson and Greengard [2]. Of special interest are Chorin [15],[16] where the random vortex sheet method is introduced, and Anderson [1] who treats flows of slightly variable density.

This paper is organized as follows. In the next section, some aspects of, and numerical methods for stochastic differential equations are briefly reviewed. Derivation of the random vortex methods follows: the random vortex equations are introduced, the use of cutoff functions is explained, and then discretized random vortex algorithms are presented. Then a section is devoted to describing the model problem and the functionals to be computed. Numerical results are then presented, and conclusions are drawn.

1. **The Stochastic Schemes.** We begin with a quick review of stochastic differential equations and two numerical methods for solving them. Let $\{\Omega, \mathbf{M}, P\}$ denote a probability space. We the consider d-dimensional stochastic differential equation

$$d\underline{Y} = \underline{f}(t, \underline{Y})\, dt + \mu\, d\underline{W}_t \ , \quad 0 \leq t \leq T \ , \tag{1.1}$$

$$\underline{Y}(0) = \underline{Y}_0 \tag{1.2}$$

with drift vector $\underline{f} = \underline{f}(t, \underline{Y})$ and diffusion coefficient μ, where $\underline{W}_t = \underline{W}(t, \cdot)$ is a normalized d-dimensional Wiener process (Brownian motion). Since the Brownian path is nowhere differentiable, a solution of equations (1.1), (1.2) is defined to be the stochastic process $\underline{Y} = \underline{Y}(t, \cdot)$ such that the integral form

$$\underline{Y}(t) = \underline{Y}_0 + \int_0^t \underline{f}(s, \underline{Y}(s))\, ds + \mu\, \underline{W}_t \tag{1.3}$$

holds with probability one. In the form (1.3), equations (1.1) and (1.2) make sense, and can be interpreted either in Ito's or in Stratonovich's sense. A unique solution exists if $\underline{f} = \underline{f}(t, \underline{y})$ is Lipschitz continuous in $\underline{y}$ and satisfies a certain growth condition [3].

There are two natural ways of measuring error in numerical solution of equations (1.1), (1.2). Let E denote expectation. We call a numerical method of order p in the L_2 sense, if there exists a constant $C = C(T)$ such that

$$[E[\underline{Y}^{(n)} - \phi(\underline{Y}(t_n))]^2]^{\frac{1}{2}} = [\int_\Omega |\, \underline{Y}^{(n)} - \underline{Y}(t_n)\,|^2 dP\,]^{\frac{1}{2}} \leq C(\Delta t)^p \ , \tag{1.4}$$

for all $t_n \leq T$, where Δt is the time step size, and $\underline{Y}^{(n)}$ is the numerical solution obtained at

time t_n. Furthermore, a numerical method is said to be of order p in the weak sense, if there exists a constant $C' = C'(T)$ such that

$$| E[\phi(\underline{Y}^{(n)}) - \phi(\underline{Y}(t_n))] | = | \int_\Omega [\phi(\underline{Y}^{(n)}) - \phi(\underline{Y}(t_n)] \, dP | \leq C' (\Delta t)^p . \qquad (1.5)$$

for the class $C_0^\infty(R^d)$ of smooth functions $\phi = \phi(\underline{x})$ with compact supports in R^d.

Let $t_n^* = t_n + \frac{1}{2}\Delta t$; we consider two schemes for the equation (1.1):

Method A_0:

$$\underline{P}^{(n)} = \underline{Y}^{(n)} + \frac{1}{2}\Delta t \, \underline{f}(t_n, \underline{Y}^{(n)}) ,$$

$$\underline{Q}^{(n)} = \underline{Y}^{(n)} + \frac{1}{2}\Delta t \, \underline{f}(t_n, \underline{Y}^{(n)}) + \frac{1}{2}\sqrt{\Delta t} \, \mu \, \underline{\Gamma}^{(n)} ,$$

$$\underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \sqrt{\Delta t} \, \mu \, \underline{\Gamma}^{(n)} + \frac{1}{2}\Delta t \, [\, \underline{f}(t_n^*, \underline{P}^{(n)}) + \underline{f}(t_n^*, \underline{Q}^{(n)}) \,] ;$$

Method B_0:

$$\underline{P}^{(n)} = \underline{Y}^{(n)} + \frac{1}{2}\Delta t \, \underline{f}(t_n, \underline{Y}^{(n)}) ,$$

$$\underline{Q}^{(n)} = \underline{Y}^{(n)} + \frac{1}{2}\Delta t \, \underline{f}(t_n, \underline{Y}^{(n)}) + \frac{3}{2}\sqrt{\Delta t} \, \mu \, \underline{\Lambda}^{(n)} ,$$

$$\underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \sqrt{\Delta t} \, \mu \, \underline{\Gamma}^{(n)} + \frac{1}{3}\Delta t \, [\, \underline{f}(t_n^*, \underline{P}^{(n)}) + 2 \, \underline{f}(t_n^*, \underline{Q}^{(n)}) \,] .$$

Here, $\{\underline{\Gamma}^{(n)}\}$ are independent normalized Gaussian random variables; and $\{\underline{\Lambda}^{(n)}\}$ are independent Gaussian random variables of distribution $N(\underline{0}, 1/3 I_d)$, where I_d denotes the $d \times d$-identity matrix. For each n, the correlated matrix for $\underline{\Gamma}^{(n)}$ and $\underline{\Lambda}^{(n)}$ is $1/3 I_d$. In fact, each of these methods belongs to one of the following one-parameter family of schemes of the same accuracy [9],[10]. The choices made here are so that all the coefficients appearing are rational and are not essential.

Family A_0:

$$\underline{P}^{(n)} = \underline{Y}^{(n)} + \frac{1}{2}\Delta t\, \underline{f}\,(t_n\,,\, \underline{Y}^{(n)}) + p\,\sqrt{\Delta t}\,\mu\,\underline{\Gamma}^{(n)}\,,$$

$$\underline{Q}^{(n)} = \underline{Y}^{(n)} + \frac{1}{2}\Delta t\, \underline{f}\,(t_n\,,\, \underline{Y}^{(n)}) + q\,\sqrt{\Delta t}\,\mu\,\underline{\Gamma}^{(n)}\,,$$

$$\underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \sqrt{\Delta t}\;\underline{\Gamma}^{(n)} + \Delta t\,[\; a\,\underline{f}\,(t_n^*,\, \underline{P}^{(n)}) + b\,\underline{f}\,(t_n^*,\, \underline{Q}^{(n)})\;]\;;$$

here the parameters a , b , p and q satisfy

$$a\, +\, b\, =1\,,\quad a\cdot p\, +\, b\cdot q\, =\frac{1}{2}\,,\quad a\cdot p^{\,2}+\, b\cdot q^{\,2}=\frac{1}{2}\,.$$

Family B_0:

$$\underline{P}^{(n)} = \underline{Y}_{(n)} + \frac{1}{2}\Delta t\, \underline{f}\,(t_n\,,\, \underline{Y}^{(n)}) + p\,\sqrt{\Delta t}\,\mu\,\underline{\Lambda}^{(n)}\,,$$

$$\underline{Q}^{(n)} = \underline{Y}^{(n)} + \frac{1}{2}\Delta t\, \underline{f}\,(t_n\,,\, \underline{Y}^{(n)}) + q\,\sqrt{\Delta t}\,\mu\,\underline{\Lambda}^{(n)}\,,$$

$$\underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \sqrt{\Delta t}\,\mu\,\underline{\Gamma}^{(n)} + \Delta t\,[\; a\,\underline{f}\,(t_n^*,\, \underline{P}^{(n)}) + b\,\underline{f}\,(t_n^*,\, \underline{Q}^{(n)})\;]\;;$$

here the parameters a , b , p and q satisfy the condition:

$$a\, +\, b\, =1\,,\quad a\cdot p\, +\, b\cdot q\, =1\,,\quad a\cdot p^{\,2}+\, b\cdot q^{\,2}=\frac{3}{2}\,.$$

Note that as μ goes to 0, the above schemes become the Runge-Kutta method based on mid-point rule for ordinary differential equations. Though, for our applications, we do not discuss methods of Taylor series type, we refer to [36],[39],[40] for their interesting, mainly theoretical development. Most methods of this type have been developed for general (Ito) stochastic differential equations.

The results in [9],[10] show that, if initial errors are sufficiently small, then there exists a Wiener process such that Method B_0 is of order 1.5 in the sense of (1.4), and, Method A_0 is of order , i.e., the local truncation error is of order 3 in the sense of of (1.5). The goal of this paper is to test the accuracy of these methods when applied to the Navier-Stokes equations, though though their applications are not necessarily so.

Equation (1.1) has aroused interest in other branches of science, for example, control theory, biology, chemical kinetics, quantum theory, and statistical mechanics. For a recent review of some of these applications, we recommend Jazwinski [29], Lavenda [32], Schuss [43] and Schulman [45]; also of interest is the classic by Wax [47]. In particular, Chorin's lecture notes [14] contain the first ideas of the stochastic differential equation approach to the study of turbulence.

2. **Random Vortex Methods.** We consider the two-dimensional Navier-Stokes equations for incompressible fluids with constant density ρ:

$$\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\underline{u} , \tag{2.1}$$

$$\nabla\cdot\underline{u} = 0 , \tag{2.2}$$

where $\underline{u} = \underline{u}(t, \underline{x})$ is the velocity, $p = p(t, \underline{x})$ is the pressure, and the constant ν is the kinematic viscosity. By introducing the vorticity

$$\omega(t, \underline{x}) = (\partial_1 u_2 - \partial_2 u_1)(t, \underline{x}) ,$$

where $\underline{u} = (u_1, u_2)$, we can rewrite equations (2.1) and (2.2) into a single equation:

$$\omega_t + (\underline{u}\cdot\nabla)\omega = \nu\Delta\omega . \tag{2.3}$$

This equation connects the Navier-Stokes equations with stochastic differential equations. Suppose first that the initial vorticity $\omega_0(\underline{x}) = \omega(0, \underline{x})$ is a probability density on R^2. Then it follows from (2.2) that (2.3) is the Kolmogorov's forward equation of the stochastic differential equation ([3],[23])

$$d\underline{Y} = \underline{u}(t, \underline{Y}(t))\, dt + \sqrt{2\nu}\, d\underline{W}_t , \quad t > 0 \tag{2.4}$$

where, again, $\underline{W}_t$ is a normalized Wiener process. The random variable $\underline{Y}(t)$ has the distribution

$$P(\underline{Y}(t) \in A) = \int_A \omega(t, \underline{y})\, d\underline{y} , \quad t \geq 0$$

for any Borel set A in R^2. For an arbitrary initial vorticity $\omega_0(\underline{x})$, we write it in a product

$$\omega_0(\underline{x}) = \kappa(\underline{x})\, \pi_0(\underline{x}) ,$$

so that $\pi_0(\omega) = \pi(0, \underline{\sigma})$ is a probability density over R^2. Here $\kappa = \kappa(\underline{x})$ is called the weight function. From this definition, we see that the choice for π_0, and thus for κ, is not necessarily unique. We further write in Lagrangian form

$$\underline{x}(0, \underline{\sigma}) = \underline{\sigma},$$

$$\omega(t, \underline{\sigma}) \equiv \omega(t, \underline{x}(t, \underline{\sigma})) \equiv \kappa(\underline{\sigma})\, \pi(t, \underline{x}(t, \underline{\sigma})) \equiv \kappa(\underline{\sigma})\, \pi(t, \underline{\sigma}) \tag{2.5}$$

with $\pi_0(\underline{\sigma}) \equiv \pi(0, \underline{\sigma})$. Then $\pi(t, \underline{\sigma})$ satisfies the Kolmogorov's forward equation. This can be seen by writing (2.3) in Lagrangian form

$$\partial_t \omega(t, \underline{\sigma}) = \nu \Delta \omega(t, \underline{\sigma}),$$

and substituting (2.5) in the above equation, which gives

$$\kappa(\underline{\sigma}) \cdot \partial_t \pi(t, \underline{\sigma}) = \kappa(\underline{\sigma}) \cdot \nu \Delta \pi(t, \underline{\sigma}),$$

that is, in Eulerian terms,

$$\pi_t(t, \underline{x}) + (\underline{u} \cdot \nabla)\pi(t, \underline{x}) = \nu \Delta \pi(t, \underline{x}). \tag{2.6}$$

Next we look for a probabilistic representation of the velocity field. Since the flow is incompressible, we can introduce a stream function $\psi = \psi(t, \underline{x})$ such that

$$-\partial_2 \psi(t, \underline{x}) = u_1(t, \underline{x}), \quad \partial_1 \psi(t, \underline{x}) = u_2(t, \underline{x}). \tag{2.7}$$

Then,

$$\omega = \partial_1 u_2 - \partial_2 u_1 = \partial_1^2 \psi + \partial_2^2 \psi = \Delta \psi, \tag{2.8}$$

which is Poisson's equation for ψ. Assume that $\omega(t, \cdot)$ decays rapidly at infinity; the solution of equation (2.8) is then given by

$$\psi(t, \underline{x}) = (G * \omega)(t, \underline{x}) = \frac{1}{2\pi} \int \log(|x - y|)\, \omega(t, \underline{y})\, d\underline{y} \tag{2.9}$$

where $*$ denotes convolution, and

$$G(\underline{x}) = \frac{1}{2\pi} \log(|\underline{x}|) \tag{2.10}$$

is the fundamental solution of the Laplace operator. Setting

$$K_1(\underline{x}) = (-\partial_2 G)(\underline{x}), \quad K_2(\underline{x}) = (\partial_1 G)(\underline{x}), \tag{2.11}$$

we have, from (2.7) and (2.9), that

$$\underline{u}(t, \underline{x}) = (K * \omega)(t, \underline{x}) = \int \underline{K}(\underline{x} - \underline{y})\, \omega(t, \underline{y})\, d\underline{y} , \tag{2.12}$$

$\underline{K} = \underline{K}(\underline{x})$ being the velocity kernel:

$$\underline{K}(\underline{x}) = (K_1, K_2)(\underline{x}) = \frac{1}{2\pi} \frac{1}{|\underline{x}|^2} (-x_2, x_1) . \tag{2.13}$$

Equation (2.12) is the Biot-Savart law for the velocity field $\underline{u}$. We can further write this equation in Lagrangian form to discover an explicit probabilistic representation for $\underline{u}$. Actually, for any flow property, an expression of this kind is crucial for the random vortex method. Recall the definition (2.5). We have

$$\begin{aligned} \underline{u}(t, \underline{\sigma}) &\equiv \underline{u}(t, \underline{x}(t, \underline{\sigma})) , \\ \underline{u}(t, \underline{\sigma}) &= \int \underline{K}(\underline{x}(t, \underline{\sigma}) - \underline{y})\, \omega(t, \underline{y})\, d\underline{y} \\ &= \int \underline{K}(\underline{x}(t, \underline{\sigma}) - \underline{y}(t, \underline{\sigma}'))\, \omega(t, \underline{\sigma}')\, d\underline{\sigma}' \\ &= \int \underline{K}(\underline{x}(t, \underline{\sigma}) - \underline{y}(t, \underline{\sigma}'))\, \cdot\, \kappa(\underline{\sigma}')\, \pi(t, \underline{\sigma}')\, d\underline{\sigma}' \\ &= \int \underline{K}(\underline{x}(t, \underline{\sigma}) - \underline{y}(t, \underline{\sigma}'))\, \kappa(\underline{\sigma}')\, dP_{\underline{Y}}(\underline{\sigma}') , \end{aligned} \tag{2.14}$$

or, in Eulerian terms,

$$\underline{u}(t, \underline{x}) = \int \underline{K}(\underline{x} - \underline{y}(t, \underline{\sigma}'))\, \kappa(\underline{\sigma}')\, dP_{\underline{Y}}(\underline{\sigma}') . \tag{2.15}$$

The second equality in (2.14) follows from the incompressibility of the fluid, because the flow map $\tau_t : \underline{\sigma} \to \underline{x}(t, \underline{\sigma})$ then has Jacobian identically one. Here $P_{\underline{Y}}$ is the probability measure induced by the random variable $\underline{Y} = \underline{Y}(t, \cdot)$--the solution of equation (2.4): for a Borel subset A of R^2,

$$P_{\underline{Y}}(A) = P(\underline{Y}(t) \in A) = \int_A \pi(t, \underline{y})\, d\underline{y} , \quad t \geq 0 . \tag{2.16}$$

Let $E_{\underline{Y}}$ denote the corresponding expectation, then (2.15) states simply that the velocity field $\underline{u}(t, \cdot)$ is the expectation of $\underline{K}(\cdot - \underline{Y}(t))$ with the weight κ. We have explicitly that

$$\underline{u}(t, \underline{x}) = E_{\underline{Y}}[\underline{K}(\underline{x} - \underline{Y}(t))\, \kappa(\tau_t^{-1}(\underline{Y}(t)))] , \tag{2.17}$$

where τ_t^{-1} denotes the inverse flow map. Thus the velocity field $\underline{u}(t, \cdot)$ is completely determined by the distribution of $\underline{Y}(t)$ and the weight assigned initially. This connection enables us to solve the Navier-Stokes equations (2.1)-(2.2) by solving the stochastic differential

equation (2.4). Equations (2.4)-(2.6), (2.16) and (2.17) constitute the basis for our study of random vortex methods.

To approximate the velocity field, great care must be taken to avoid the singularity of the kernel $\underline{K} = \underline{K}(\underline{x})$ at the origin. For this purpose, we need to introduce a class of so-called cutoff functions. Let $\Phi = \Phi(\underline{x})$ be a rapidly decreasing C^∞ with integral one: $\int_{R^2} \Phi(\underline{x})\, d\underline{x} = 1$, and put

$$\Phi_\delta(\underline{x}) = \delta^{-2}\, \Phi(\underline{x}/\delta)\ .$$

Then for every function $\underline{h} \in L^p(R^2)$ $(1 \leq p \leq \infty)$, we have $\underline{h} * \Phi \in C^\infty$, and

$$h_\delta(\underline{x}) \equiv (\underline{h} * \Phi_\delta)(\underline{x}) \to \underline{h}(\underline{x})\ ,\ \text{ as }\ \delta \to 0\ ,\ \text{ in }\quad \mathrm{L}^p(R^2)$$

(see [20]). We call $\underline{u}_\delta \equiv \underline{u} * \Phi_\delta$ the regularization (or mollification) of $\underline{u}$, δ the cutoff for the function Φ_δ, and the family $\{\Phi_\delta\}$ the set of cutoff functions.

We apply this mollification to the velocity kernel $\underline{K}$, beginning with stream function ψ. Since $\omega(t\,,\cdot)$ is assumed to be decay rapidly at infinity, the stream function $\psi(t\,,\cdot) = (G * \omega)(t\,,\cdot)$ is in $L^1 \cap L^\infty$. Suppose that $G_\delta(\underline{x}) = (G * \Phi_\delta)(\underline{x})$ exists for every sufficiently small δ, then

$$(G_\delta * \omega)(t\,,\underline{x}) = (G * \Phi_\delta) * \omega = (G * \omega) * \Phi_\delta \to \psi(t\,,\underline{x}) \tag{2.18}$$

in the $L^1 \cap L^\infty$ sense. Furthermore, we have

$$\Delta G_\delta(\underline{x}) = \Phi_\delta(\underline{x})\ , \tag{2.19}$$

since $G = G(\underline{x})$ (see (2.10)) is the fundamental solution of the Laplace operator; and therefore we have according to (2.18)

$$\omega_\delta(t\,,\underline{x}) = \Delta(G_\delta * \omega) = (\Phi_\delta * \omega)(t\,,\underline{x})\ . \tag{2.20}$$

Now, as in (2.11), we define $\underline{K}_\delta = (K_{\delta 1}, K_{\delta 2})$ by

$$K_{\delta 1} = -\partial_2 G_\delta\ ,\quad K_{\delta 2} = \partial_1 G_\delta\ .$$

Then it follows from (2.7) that an approximation of the velocity field is given by

$$u_{\delta 1}(t\,,\,\underline{x}) = -\partial_2(G_\delta * \omega) = (K_{\delta 1} * \omega)(t\,,\,\underline{x})\,,$$
$$u_{\delta 2}(t\,,\,\underline{x}) = \partial_1(G_\delta * \omega) = (K_{\delta 2} * \omega)(t\,,\,\underline{x})\,.$$

Therefore, we have obtained a complete set of cutoff equations. In summary, let $\underline{u}_\delta$ denote $(u_{\delta 1},\, u_{\delta 2})$, then we are solving equations (2.4)-(2.6), (2.15) and (2.16) with $\underline{u}$ replaced by $\underline{u}_\delta$ and $\underline{K}$ replaced by $\underline{K}_\delta$. That is, we are considering

$$d\underline{Y} = \underline{u}_\delta(t\,,\,\underline{Y}(t))\,dt + \sqrt{2\nu}\,d\underline{W}_t\,, \quad t \geq 0\,, \tag{2.21}$$

while the probability density π of $\underline{Y}(t)$ satisfies

$$\pi_t(t\,,\,\underline{x}) + (\underline{u}_\delta \cdot \nabla)\pi(t\,,\,\underline{x}) = \nu\Delta\pi(t\,,\,\underline{x})\,, \tag{2.22}$$

and

$$\underline{u}_\delta(t\,,\,\underline{\sigma}) = \int \underline{K}_\delta(\underline{x}(t\,,\,\underline{\sigma}) - \underline{y}(t\,,\,\underline{\sigma}'))\,\kappa(\underline{\sigma}')\,dP_{\underline{Y}}(\underline{\sigma}')\,, \tag{2.23}$$

or, in analogy with (2.17),

$$\underline{u}_\delta(t\,,\,\underline{x}) = E_{\underline{Y}}[\underline{K}_\delta(\underline{x} - \underline{Y}(t))\,\kappa(\tau_t^{-1}(\underline{Y}(t)))]\,, \tag{2.24}$$

with (2.5) and (2.16) unchanged. Note that the cutoff velocity field $\underline{u}_\delta(t\,,\,\underline{x})$ satisfies the incompressibility condition (2.2) automatically by its definition; and (2.23) can be derived in exactly the same way as for (2.14). Similarly, (2.20) yields a probabilistic expression for the vorticity field:

$$\omega_\delta(t\,,\,\underline{\sigma}) = \int \Phi_\delta(\underline{x}(t\,,\,\underline{\sigma}) - \underline{y}(t\,,\,\underline{\sigma}'))\,\kappa(\underline{\sigma}')\,dP_{\underline{Y}}(\underline{\sigma}')\,, \tag{2.25}$$

or,

$$\omega_\delta(t\,,\,\underline{x}) = E_{\underline{Y}}[\Phi_\delta(\underline{x} - \underline{Y}(t))\,\kappa(\tau_t^{-1}(\underline{Y}(t)))]\,. \tag{2.26}$$

Next we will derive random vortex methods based on equations (2.21) and (2.23). Above all, we need an explicit kernel $\underline{K}_\delta$. If we choose Φ to be radially symmetric, it follows from (2.19) that G_δ is also radially symmetric. Then, remembering the definition of $\underline{K}_\delta$, we have, by integrating (2.19)

$$\underline{K}_\delta(\underline{x}) = (-\partial_2,\, \partial_1)G_\delta(\underline{x}) = f_\delta(\,|\underline{x}|\,)\,\underline{K}(\underline{x})\,. \tag{2.27}$$

The similar idea was used by Beale and Majda [7] to design high order accuracy cutoff functions for the vortex method. In Table 1, we list several commonly used function pairs f_δ ,Φ

derived by Beale and Majda, together with some other cutoff functions, including those used by Chorin [12], Hald [25], Kuwahara and Takami [31] and Milinazzo and Safman [37].

We can now derive discretized random vortex methods which are used in actual computation. Recall that the equations that we are solving are (2.21) and (2.22) with definition (2.5) and consequences (2.16) and (2.22).

First we sample N independent random variables $\underline{Y}_j^{(0)}$, $1 \leq j \leq N$, each with weight $\kappa_j \equiv \kappa(\underline{Y}_j^{(0)})$, according to the initial distribution $\pi_0(\cdot) = \pi(0, \cdot)$. Each pair $(\underline{Y}_j\,,\ \kappa_j\,)$ moving with fixed cutoffs $(\delta,\ \Phi_\delta)$ is called a vortex blob, or simply vortex. Next we define an approximation to the $\underline{Y}^{(n)}$-velocity field:

$$\widetilde{\underline{u}}_{\underline{Y}}^{(n)}(\underline{x}) = \frac{1}{N}\sum_{j=1}^{N} \underline{K}_\delta(\underline{x} - \underline{Y}_j^{(n)})\,\kappa_j \ ; \tag{2.28}$$

this can be regarded as a Monte-Carlo estimate for the integral (2.23). Other estimates are possible, but the one in (2.28) is the simplest. Although the samples $\{\underline{Y}_j^{(n)}\}$ are not independent due to their mutual interaction, the law of large numbers justifies this estimate because they are identically distributed [8]. From (2.25) or (2.26), an approximation for the vorticity is given by

$$\widetilde{\omega}_{\underline{Y}}^{(n)}(\underline{x}) = \frac{1}{N}\sum_{j=1}^{N} \Phi_\delta(\underline{x} - \underline{Y}_j^{(n)})\,\kappa_j \ . \tag{2.29}$$

As we will shall see, the expressions (2.28), (2.29) are approximations to the integrals (2.22) and (2.25) following the vortex blobs, provided that uniform grids are used initially. In the notation used in the previous section, we have immediately

Euler's Method:

$$\underline{Y}_j^{(n+1)} = \underline{Y}_j^{(n)} + \sqrt{2\Delta t\,\nu}\,\underline{\Gamma}_j^{(n)} + \Delta t\ \widetilde{\underline{u}}_{\underline{Y}}^{(n)}(\underline{Y}_j^{(n)}) \ .$$

Because of its simplicity, this scheme has been used by Chorin in the study of slightly viscous flows [12], and of boundary layer approximations [15]. It is also one of the simplest schemes of fractional step type, and of Runge-Kutta type. For more complicated schemes, we define the approximation to the $\underline{P}^{(n)}$-velocity field

$$\widetilde{\underline{u}_P}^{(n)}(\underline{x}) = \frac{1}{N}\sum_{j=1}^{N} \underline{K}_\delta(\underline{x} - \underline{P}_j^{(n)})\,\kappa_j \ , \tag{2.30}$$

and the approximation to the $\underline{Q}^{(n)}$-velocity field

$$\widetilde{\underline{u}_Q}^{(n)}(\underline{x}) = \frac{1}{N}\sum_{j=1}^{N} \underline{K}_\delta(\underline{x} - \underline{Q}_j^{(n)})\,\kappa_j \ . \tag{2.31}$$

The latter approximation will serve as an intermediate interaction between the velocity field and the random field $\underline{W}_t$ with an increment of the Wiener process entering $\underline{Q}_j^{(n)}$. Looking back at Methods A_0 and B_0 for the equation (1.1), we define the following analogous schemes for the equations (2.21) and (2.23):

Method A:

$$\underline{P}_j^{(n)} = \underline{Y}_j^{(n)} + \frac{1}{2}\Delta t\ \widetilde{\underline{u}_Y}^{(n)}(\underline{Y}_j^{(n)}) \ ,$$

$$\underline{Q}_j^{(n)} = \underline{Y}_j^{(n)} + \frac{1}{2}\Delta t\ \widetilde{\underline{u}_Y}^{(n)}(\underline{Y}_j^{(n)}) \ + \frac{1}{2}\sqrt{2\Delta t\,\nu}\,\underline{\Gamma}_j^{(n)} \ ,$$

$$\underline{Y}_j^{(n+1)} = \underline{Y}_j^{(n)} + \sqrt{2\Delta t\,\nu}\,\underline{\Gamma}_j^{(n)} + \frac{1}{2}\Delta t\,[\ \widetilde{\underline{u}_P}^{(n)}(\underline{P}_j^{(n)}) + \widetilde{\underline{u}_Q}^{(n)}(\underline{Q}_j^{(n)})\ ] \ ;$$

Method B:

$$\underline{P}_j^{(n)} = \underline{Y}_j^{(n)} + \frac{1}{2}\Delta t\ \widetilde{\underline{u}_Y}^{(n)}(\underline{Y}_j^{(n)}) \ ,$$

$$\underline{Q}_j^{(n)} = \underline{Y}_j^{(n)} + \frac{1}{2}\Delta t\ \widetilde{\underline{u}_Y}^{(n)}(\underline{Y}_j^{(n)}) + \frac{3}{2}\sqrt{2\Delta t\,\nu}\,\underline{\Lambda}_j^{(n)} \ ,$$

$$\underline{Y}_j^{(n+1)} = \underline{Y}_j^{(n)} + \sqrt{2\Delta t\,\nu}\,\underline{\Gamma}_j^{(n)} + \frac{1}{3}\Delta t\,[\ \widetilde{\underline{u}_P}^{(n)}(\underline{P}_j^{(n)}) + 2\,\widetilde{\underline{u}_Q}^{(n)}(\underline{Q}_j^{(n)})\ ] \ .$$

Note that μ in the Methods A_0 and B_0 corresponds to $\sqrt{2\,\nu}$ in the above schemes since we are solving equation (2.21). The velocity field $\underline{u}_\delta(t_n\ ,\ \cdot)$, given by (2.23), is determined from the flow at time t_n ; therefore the time dependence is raised to the superscript in the approximations of the velocity fields. Note also that as ν tend to 0, these schemes become the vortex method based on mid-point rule:

$$\underline{R}_j^{(n)} = \underline{X}_j^{(n)} + \frac{1}{2}\Delta t\ \widetilde{\underline{u}}_{\underline{X}}(\underline{X}_j^{(n)}) ,$$

$$\underline{X}_j^{(n+1)} = \underline{X}_j^{(n)} + \Delta t\ \widetilde{\underline{u}}_{\underline{R}}(\underline{R}_j^{(n)}) ,$$

which has been discussed by Anderson and Greengard [2], and proved to have a second order accuracy by these authors. Convergence for the random vortex method has recently been studied by Goodman [23], and Marchioro and Pulvirenti [35], mainly for Euler's method. Also of interest are Hald [27] who proved, at an earlier time, convergence of a simplified model problem with vorticity creation, and Roberts [42] who proved independently the convergence of a random vortex method for Burgers' equation.

3. The Model Problem.

Consider the vorticity equation

$$\omega_t + (\underline{u}\cdot\nabla)\omega = \nu\Delta\omega , \tag{3.1}$$

subject to the initial condition

$$\omega_0(\underline{x}) = \Omega_0 , \ |\underline{x}| \leq a \ ; \ = 0 , \text{ elsewhere} ; \tag{3.2}$$

where $\Omega_0 = 1/(\pi a^2)$ is so that $\omega_0(\underline{x})$, and therefore $\omega(t , \underline{x})$, is a probability density. Due to the symmetry of the initial $\omega(0, \underline{x})$, the convective term vanishes [4]; and therefore equation (3.1) reduces to the diffusion equation:

$$\omega_t(t , \underline{x}) = \nu\Delta\omega(t , \underline{x}) . \tag{3.3}$$

The initial value problem (3.3) and (3.2) allows for the explicit solution

$$\omega(t , \underline{x}) = \int_{R^2} D_t(\underline{x} - \underline{y})\,\omega_0(\underline{y})\ d\underline{y} , \tag{3.4}$$

where

$$D_t(\underline{x}) = \frac{1}{\sqrt{4\pi\nu t}}\exp(-\frac{|\underline{x}|^2}{4\nu t})$$

is the diffusion kernel. For later use, we compute the initial velocity field $\underline{u}_0$ of the flow. By (2.8), the radial symmetry of the initial vorticity field implies that the initial stream function $\psi_0(\underline{x}) = \psi_0(r)$, $r = |\underline{x}|$, and in polar coordinates,

$$\frac{1}{r}\frac{d}{dr}(r^2\frac{d\psi_0}{dr}) = \Omega_0 \ , \quad |\underline{x}| \leq a \ ; \quad = 0 \ , \ \text{elsewhere} \ .$$

Direct integration of this equation twice, followed by the use of the formulas in (2.7) yields the initial velocity field:

$$\underline{u}_0(\underline{x}) = \begin{cases} \frac{1}{2}\Omega_0(-x_2, x_1) \ , & \text{if } |\underline{x}| \leq a \ , \\ \frac{1}{2}\frac{a^2}{|\underline{x}|^2}\Omega_0(-x_2, x_1) \ , & \text{if } |\underline{x}| > a \ . \end{cases} \tag{3.5}$$

Now we consider the problem of investigating the accuracy of the methods derived in the previous section. It seems that the simplest and least expensive way to compare the accuracy of stochastic numerical schemes is to use them to estimate functionals which can be evaluated explicitly. This indicates the accuracy both of the computed velocity field and of the vortex distribution. Following [37],[41], we consider integrals of the form:

$$\int_{R^2} f(|\underline{y}|)\ \omega(t, \underline{y})\ d\underline{y} \ , \tag{3.6}$$

which have, according to what we have done in (2.14), the probabilistic expressions:

$$\int f(\underline{y}(t, \underline{\sigma}))\ \kappa(\underline{\sigma})\ dP_{\underline{Y}}(\underline{\sigma}) \ , \tag{3.7}$$

or,

$$E_{\underline{Y}}[f(\underline{Y}(t))\ \kappa(\tau_t^{-1}(\underline{Y}(t)))] \tag{3.8}$$

where $\underline{Y}(t)$ is the solution of equation (2.4) and $E_{\underline{Y}}$ denotes the corresponding expectation. Integral (3.6), like the velocity field is a functional of the Wiener process $\underline{W}_t$. We take first $f(\underline{x}) = |\underline{x}|^2$. Substituting (3.4) in (3.6) and performing the integration, (3.6) becomes

$$U(t) = \frac{1}{2}a^2 + 4\nu t \ . \tag{3.9}$$

Then putting $f(|\underline{x}|) = \exp(-|\underline{x}|^2)$, (3.6) becomes

$$V(t) = \frac{1}{a^2}\exp(-\frac{a^2}{4\nu t + 1}) \ . \tag{3.10}$$

Since we are numerically solving a stochastic differential equation (i.e., (2.21)), all the information available at the time t_n is the set of samples $(\underline{Y}_j^{(n)}, \kappa_j)$. Therefore, we would Monte-Carlo estimate the functionals considered. The simplest estimate for (3.7) or (3.8) is the usual

estimate

$$\frac{1}{N}\sum_{j=1}^{N} f\left(\underline{Y}_j^{(n)}\right)\kappa_j \ . \tag{3.11}$$

An estimate of this kind can be justified, as mentioned before, by the law of large numbers, despite the lack of independence among the samples $\underline{Y}_j^{(n)}$ due to their mutual interaction through the velocity field.

A Monte-Carlo estimate creates statistical error--called error variance due to imperfect sampling. Since the error variance for the estimate in (3.11) is usually proportional to $1/N$ (see [38]), substantial error of order $1/\sqrt{N}$ is due to the Monte-Carlo calculation. It is then clear that, in order to exhibit the accuracy of a numerical scheme, the error variance produced by Monte-Carlo estimation should be made as small as possible. The trick is to utilize the non-anticipating property of the solutions of stochastic differential equations. Denoting by $g\left(\underline{Y}(t)\right)$ the integrand in (3.8) and forming the sum

$$g\left(\underline{Y}(t_n)\right) = g\left(\underline{Y}(0)\right) + \sum_{k=1}^{n}\left[g\left(\underline{Y}(t_k)\right) - g\left(\underline{Y}(t_{k-1})\right)\right], \tag{3.12}$$

we approximate each summand by $g^{(k)} - g^{(k-1)}$, where $g^{(k)}$ is obtained by replacing $\underline{Y}(t)$ in (3.8) by $\underline{Y}^{(k)}$--the numerical solution of the stochastic differential equation (2.21). We further observe that all the schemes considered (Euler, Methods A, B) are of the form:

$$\underline{Y}^{(k+1)} = \underline{Y}^{(k)} + \sqrt{2\Delta t\,\nu}\,\underline{\Gamma}^{(k)} + \Delta t\ \underline{\Psi}^{(k)} \ ,$$

The non-anticipating property implies that the numerical solution $\underline{Y}^{(k)}$ is independent of the newly input increment $\underline{\Gamma}^{(k)}$. This suggests the following Taylor expansion:

$$g_j^{(k+1)} = g_j^{(k)} + g_l^{(k)}\,|_j\,[\ \sqrt{2\Delta t\,\nu}\,\Gamma^{(k),l} + \Delta t\ \Psi^{(k),l}\ ] + \Delta t\,\nu\, g_{lp}^{(k)}\,|_j\ \Gamma^{(k),l}\,\Gamma^{(k),p} \ . \tag{3.13}$$

Here the summation convention is adopted, and the subscript j means that the sample pair $\{Y_j\ ,\ k_j\}$ is used, and

$$g_l^{(k)}\,|_j = f_{,l}^{(k)}\kappa_j \quad , \quad g_{lp}^{(k)}\,|_j = f_{,lp}^{(k)}\kappa_j$$

where only subscripts with a comma denote differentiation. Hence we need to assume smoothness on f but not on g. For convenience, we omit the index j temporarily. Move

the first two terms on the right hand side of (3.13) to the left and denote by $G^{(k+1)}$ the resultant expression, then

$$G^{(k+1)} = g^{(k+1)} - g^{(k)} - \sqrt{2\nu\Delta t}\, g_{,l}^{(k)}\, \Gamma^{(k),l} \ ,$$
$$= \Delta t\, g_{,l}^{(k)}\, \Psi^{(k),l} + \Delta t\, \nu\, g_{,lp}^{(k)} \Gamma^{(k),l}\, \Gamma^{(k),p} \ . \tag{3.14}$$

Recall the definition of $g^{(k)}$, and note the independence between $g_{,l}^{(k)}$ and $\underline{\Gamma}^{(k),l}$, the latter being of the distribution $N(\underline{0},\, I_d)$. Taking expectations on both sides of the first equality and summing the results over k from 0 to $n-1$, we have immediately

$$E\,[g\,(\underline{Y}^{(n)})] = E\,[g\,(\underline{Y}^{(0)})] + \sum_{k=1}^{n} E\,[G^{(k)}] \ . \tag{3.15}$$

which is equivalent to

$$E\,[g\,(\underline{Y}^{(k)})] = E\,[g\,(\underline{Y}^{(k-1)})] + E\,[G^{(k)}] \tag{3.16}$$

--a recursive relation between $E\,[g\,(\underline{Y}^{(k)})]$ and $E\,[g\,(\underline{Y}^{(k-1)})]$. In this way the functional values to be computed are successively linked, and an estimate for each $E\,[G^{(k)}]$ can be designed, according to the first identity of (3.14), as

$$\frac{1}{N}\sum_{j=1}^{N} [G_j^{(k)}] \equiv \frac{1}{N}\sum_{j=1}^{N} [\ g_j^{(k)} - g_j^{(k-1)} - \sqrt{2\Delta t\, \nu}\, g_{,l}^{(k-1)}\, \Gamma^{(k-1),l}\ ] \ , \tag{3.17}$$

where, again, $g_j^{(k)}$ means that the sampling solution $\{\underline{Y}_j^{(k)}\}$ is used. The reason for using this form is then clear: the usual estimate for $g\,(\underline{Y}^{(n)})$ is equivalent to summing up estimates of the same kind for $g\,(\underline{Y}^{(k+1)}) - g\,(\underline{Y}^{(k)})$ over k from 0 to $n-1$, while the estimate for $g\,(\underline{Y}^{(n)})$ constructed through $G^{(k)}$ eliminates the dominating sampling error due to the existence of the term $\sqrt{2\Delta t\, \nu}\, g_{,l}^{(k-1)}\, \Gamma^{(k-1),l}$. More complicated estimates can be constructed through the use of Hermite polynomials [13],[34].

4. **Numerical Implementation.** In this section we will detail the process of performing Monte-Carlo estimation of the considered functionals via the numerical solution of the stochastic differential equation. For the sake of clarity, we repeat some formulas; recall that we are solving the Navier-Stokes equations by approximating the equations

$$d\underline{Y} = \underline{u}_\delta(t\,,\,\underline{Y}(t))\,dt + \sqrt{2\nu}\,d\underline{W}_t\,, \tag{4.1}$$

$$\underline{u}_\delta(t\,,\,\underline{x}) = \int \underline{K}_\delta(\underline{x} - \underline{y}(t\,,\,\underline{\sigma}'))\,\kappa(\underline{\sigma}')\,dP_{\underline{Y}}(\underline{\sigma}')\,, \tag{4.2}$$

or,

$$\underline{u}_\delta(t\,,\,\underline{x}) = E_{\underline{Y}}[\underline{K}_\delta(\underline{x} - \underline{Y}(t))\,\kappa(\tau_t^{-1}(\underline{Y}(t)))]\,, \tag{4.3}$$

by Euler's method and Methods A and B (see section 3). In each of these methods, let $\{\underline{Y}_j^{(n)}\}$ denote the sample solution at time t_n, then the cutoff velocity $\underline{u}_\delta(t\,,\,\underline{Y}^{(n)})$ is approximated by

$$\widetilde{\underline{u}_{\underline{Y}}}^{(n)}(\underline{Y}_j^{(n)}) = \frac{1}{N}\sum_{k=1}^{N}\underline{K}_\delta(\underline{Y}_j^{(n)} - \underline{Y}_k^{(n)})\,\kappa_k\,. \tag{4.4}$$

and similarly the approximations $\widetilde{\underline{u}_{\underline{P}}}(\underline{P}_j^{(n)})$ and $\widetilde{\underline{u}_{\underline{Q}}}(\underline{Q}_j^{(n)})$ (see (2.28), (2.30),(2.31)). We further recall that our model problem is a rotating flow with initial vorticity concentrated uniformly on a disk of radius a. The two functionals to be computed are given preceding (3.9) and (3.10) where their exact values are given. We use two Monte-Carlo estimates for evaluating these functionals: the usual estimate (3.11) and the modified one, given respectively by

$$\frac{1}{N}\sum_{j=1}^{N} f(\underline{Y}_j^{(n)})\,\kappa_j\,, \quad \text{and} \quad \sum_{k=1}^{n}\left\{\frac{1}{N}\sum_{j=1}^{N} G_j^{(k)}\right\} \tag{4.5}$$

to which the exact initial value $E[g(Y(0))]$ is added. The second estimate is obtained by summing (3.17) up to t_n; the function $f = f(\underline{y})$ denotes either $|\underline{y}|^2$ or $\exp(-|\underline{y}|^2)$. For convenience, we call the first estimate in (4.5) the usual Monte-Carlo estimate, and the second one the modified Monte-Carlo estimate.

In studying the (theoretical) accuracy of these schemes, we would like to concern ourselves only with the time discretization error but we must consider the random sampling error as well. Therefore, the main aim of this section is to weigh the relative significance of these errors and estimate their interaction. However, before we can discuss the numerical results meaningfully, we need to further specify the following parameters and conditions:

(i): the time step size (Δt),

(ii): the discretization of the initial vorticity field,

(iii): the cutoff δ and the cutoff function f_δ, and

(iv): the sampling algorithm for the variables: $\{\underline{\Gamma}_j^{(n)}\}$ and $\{\underline{\Delta}_j^{(n)}\}$.

To exhibit the accuracy of the numerical schemes, we use three time step sizes $\Delta t = .2, .1$ and .05 for Monte-Carlo computation of each functional under consideration. As we have noted in Section 4, the time step size may also affect the Monte-Carlo computation, and this point will be discussed further later.

The number of vortex blobs used is actually determined by the spatial discretization of the support--a disk of radius a --of the initial vorticity field. Since we are mainly concerned with the two errors mentioned above, the initial error due to discretization will be minimized as follows. The radius a of the disk is chosen to be .5, and therefore the initial vorticity is given by

$$\omega_0(\underline{x}) = \frac{4}{\pi} , \quad |\underline{x}| \leq a \ ; \quad = 0 , \ \text{elsewhere} ,$$

so that the total vorticity is 1. Following Roberts [41], we will uniformly distribute the initial vortex blobs; thus we set

$$\omega_0(\underline{x}) = \kappa(\underline{x}) \ \pi_0(\underline{x}) .$$

where $\pi_0 = \pi_0(\underline{x})$ is uniform over the unit square with sides centered at the origin;

$$\pi_0(\underline{x}) = 1 , \ \underline{x} \in [-.5, .5]^2 ; \quad = 0 , \ \text{elsewhere} .$$

The corresponding weight function $\kappa = \kappa(\underline{x})$ is therefore identical to the initial vorticity field: $\kappa(\underline{x}) = \omega_0(\underline{x})$. Next we partition the unit square into a lattice of sidelength $1/32 = .003125$, which is comparable to that used by Perlman [38] in a vortex method. The center of each square L_j is the position of a vortex blob which carries the average weight of the square;

$$\kappa_j = \int_{L_j} \kappa(\underline{y})\, d\underline{y} \;/\; area\,(L_j)\,. \tag{4.6}$$

The weights $\{\kappa_j\}$ used here are somewhat different from those defined before; the previous ones were obtained by evaluating the weight function $\kappa = \kappa(\underline{x})$ at the initial positions of the vortex blobs. Consequently the total weight are exact;

$$\sum_j \kappa_j \cdot area\,(L_j) = \sum_j \int_{L_j} k\,(\underline{y})\, d\underline{y} = \int k\,(\underline{y})\, d\underline{y}\,.$$

It should be noted, however, that only those vortex blobs with nonzero average weights k_j enter the actual computation, because the others contribute nothing to the computed velocity fields (by (4.3)). This explains why the number N of vortex blobs that we actually use is 856 instead of the number 1024 of squares. Therefore, it suffices to determine the initial velocity field for those vortex blobs with nonzero weights, from (3.5) with $\Omega_0 = 4/\pi$,

$$\underline{u}_0(\underline{x}) = \frac{1}{2}\Omega_0(-x_2,\, x_1) = \frac{2}{\pi}(-x_2,\, x_1)\,.$$

Furthermore, we specify the cutoff δ and the cutoff function f_δ. The cutoffs serve as measures of the size of vortex blobs and of the interactions among them. These factors play a major role in determining the accuracy of a vortex method (see, e.g., [5],[6]). In the present paper, we choose δ to be the sidelength of an initial square and f_δ to be the fourth order kernel of Beale and Majda ([7]). We use a fourth order kernel to minimize the cutoff-induced error.

To simulate the Gaussian random variables $\underline{\Gamma}_j^{(n)}$ and $\underline{\Lambda}_j^{(n)}$, we recall that their correlation matrix is $N(\underline{0},\, 1/2I_d)$, and write

$$\underline{\Gamma}_j^{(n)} = \underline{N}_{1j}^{(n)}\,, \quad \underline{\Lambda}_j^{(n)} = \frac{1}{2}\underline{N}_{1j}^{(n)} + \frac{\sqrt{3}}{6}\underline{N}_{2j}^{(n)}$$

where $\underline{N}_{1j}^{(n)}$ and $\underline{N}_{2j}^{(n)}$ are two independent normalized random variables, and for different j we sample different ones independently. We sample the normalized random variables according to the following formula [28]:

$$N_1^l = \cos(2\pi U_1)\,[-2\log(U_2)]\,,$$

$$N_2^l = \sin(2\pi U_1)\,[-2\log(U_2)]\,,$$

where U_1 and U_2 are two independent (scalar) uniform random variables over the unit interval [0, 1]. The above sampling procedure is done for each vortex blob at each time step independently. Finally we choose the viscosity $\nu = .002$, within the range of a typical slightly viscous flow. We can now analyze our numerical results.

For each functional, we list the computed results in four tables at the times $t = 1, 2, 3$ and 4. The total time interval ($T = 4$) is comparable to the period of the corresponding nonviscous flow [38]. For each scheme, there are two subcolumns, containing results obtained by the usual estimate and the modified estimate respectively. We note that for Euler's method, the errors produced by these two estimates are about of the same order, and they exhibit clearly the first order accuracy of this method. At each time, the error is reduced to half of its original amount by halving the time step size. This implies that for this number $N = 856$ of vortex blobs, it suffices to employ the usual estimate for Euler's method.

The situation for Methods A and B is quite different from that for Euler's method. First we observe that the errors are drastically reduced if the modified estimate is used. Let us focus on the ratios of the errors: the errors for the usual estimate/the errors for the modified estimate. Then we find that the ratio of errors for Method A is from .1 to .4, while for Method B the ratio is from .001 to .05. However, we also observe that there is no clear dependence of the reduced errors on the time step sizes; this indicates the existence of strong interactions between the time discretization errors and the random sampling errors. As we have noted, the fact that the time step size enters into both the local truncation errors and the Monte-Carlo modified estimate, complicates the mutual interactions.

Nevertheless, in any case, the numerical results reveal that both Method A and Method B are more accurate than Euler's method by one power of the time step size. Method A reduces error by a factor of between 20 and 500, Method B by a factor of between 33 and 5000, over Euler's method. The seeming superiority of Method A to Method B can be understood as follows. Since the accuracy of Method B_0 has been proved to be the order one and

one half in the L_2 sense (see (1.4)), it is very likely that Method B would exhibit a second order accuracy in the weak sense (see (1.5)) due to the non-anticipating property of the solutions of stochastic differential equations. On the other hand, Method B accumulates local truncation errors in an invisible way, and possibly does not produce these errors with small uniform bounds, though the order of the local truncation error, at each time step, is three in the weak sense. We used the word 'invisible' because of the lack of (accurate) estimation of the accumulated error. We do not observe, in the present analysis, the dependence of errors on the viscosity. However, we expect that for small viscosity, the random effects are small as well, and our schemes would produce even better results. Finally, we would like to mention in this connection that the viscosity ($\nu = .002$) used in this study is slightly larger than those used elsewhere (e.g., [37],[42]).

5. **Concluding Remarks.** The vorticity in two-dimensional incompressible flow evolves under a convection term and a diffusion term, exactly corresponding to the drift vector and Brownian part of a diffusion process. Therefore, it is reasonable that one can derive accurate numerical methods for solving the Navier-Stokes equations from those known for stochastic differential equations. This has been verified in the present article (Methods A and B). However, as we have noted, further study in this direction should include both the accuracy of numerical schemes and variance reduction techniques.

Of equal interest is the generalization of the results to three dimensions, where an immediate hindrance is encountered: the emergence of vortex stretching. This important physics is not attached to a diffusion process, and therefore to a stochastic differential equation, and should be treated separately. Moreover, since the vorticity in three dimensions is no longer scalar, the analogy with what we have done is not clear, there may be a stochastic calculus which can model the physics of of vortex stretching. However, Esposito and Pulvirenti [19] have more or less extended the three-dimensional convergence results of Beale and Majda [5] to the stochastic case, using a splitting algorithm.

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Chorin [12]:

$$f_\delta(r) = 1\,, \quad \text{if } r > \delta\,,$$
$$= r/\delta\,, \quad \text{if } r \leq \delta\,;$$

Milinazzo and Saffman [37]:

$$f_\delta(r) = 1\,, \quad \text{if } r > \delta\,,$$
$$= r^2/\delta^2\,, \quad \text{if } r \leq \delta\,;$$

Hald [26]:

$$\Phi(r) = \frac{1}{2\pi}\left[140(1-r)^3 - 420(1-r)^4 + 252(1-r)^5\right],$$

$$f_\delta(r) = \frac{r^2}{\delta^2}\left[14 - 105\frac{r^2}{\delta^2} + 196\frac{r^3}{\delta^3} - 140\frac{r^4}{\delta^4} + 36\frac{r^5}{\delta^5}\right] \text{ if } r \leq \delta\,, \quad = 1 \text{ if } r > \delta\,;$$

$$\Phi(r) = \frac{1}{2\pi}\left[280(1-r)^4 - 780(1-r)^5 + \frac{1400}{3}(1-r)^6\right],$$

$$f_\delta(r) = \frac{r^2}{3\delta^2}\left[56 - 630\frac{r^2}{\delta^2} + 1568\frac{r^3}{\delta^3} - 1680\frac{r^4}{\delta^4} + 864\frac{r^5}{\delta^5} - 175\frac{r^6}{\delta^6}\right] \text{ if } r \leq \delta\,, \quad = 1 \text{ if } r > \delta\,;$$

Kuhawara and Takami [31]:

$$\Phi(r) = \frac{1}{\pi}e^{-r^2}\,, \quad f_\delta(r) = 1 - e^{-r^2/\delta^2}\,;$$

Beale and Majda [7]:

$$\Phi(r) = \frac{1}{\pi}\left(2e^{-r^2} - \frac{1}{2}e^{-r^2/2}\right), \quad f_\delta(r) = 1 - 2e^{-r^2/\delta^2} + e^{-r^2/2\delta^2}\,;$$

$$\Phi(r) = \frac{1}{\pi}(2 - r^2)e^{-r^2}\,, \quad f_\delta(r) = 1 - \left(\frac{r^2}{\delta^2} - 1\right)e^{-r^2/\delta^2}\,.$$

Table 1. Commonly used cutoff functions.

Numerical results of computing $\int |\underline{y}|^2 \omega(t, \underline{y})\, d\underline{y}$: for each scheme, errors for the usual and the modified Monte-Carlo estimates are listed in order in exponential form:

Example 1: t = 1.0 N = 856 Exact Value = 1.330-1						
Δ t	Euler's		Method A		Method B	
0.2000	8.267-3	9.430-3	-1.424-3	-2.684-4	-1.475-4	2.196-5
0.1000	4.789-3	3.580-3	-6.035-4	-2.405-4	-1.573-3	-9.270-5
0.0500	2.574-3	2.311-3	1.223-4	-1.409-4	-1.326-3	-6.156-5

Table 2.1

Example 1: t = 2.0 N = 856 Exact Value = 1.410-1						
Δ t	Euler's		Method A		Method B	
0.2000	1.689-2	1.764-2	1.459-3	-5.797-4	-1.227-3	-6.603-5
0.1000	9.938-3	9.060-3	5.673-4	-3.057-4	-5.726-4	-6.798-5
0.0500	1.556-3	4.521-3	-3.135-3	-2.330-4	-4.231-3	-4.898-5

Table 2.2

Example 1: t = 3.0 N = 856 Exact Value = 1.490-1						
Δ t	Euler's		Method A		Method B	
0.2000	2.670-2	2.564-2	-3.256-4	-5.701-4	1.083-4	8.836-6
0.1000	1.290-2	1.316-2	-4.989-4	-3.660-4	-3.021-3	-1.395-4
0.0500	-1.840-3	6.698-3	-5.043-3	-2.573-4	-5.457-3	-7.560-6

Table 2.3

Example 1: t = 4.0 N = 856 Exact Value = 1.570-1						
Δ t	Euler's		Method A		Method B	
0.2000	3.457-2	3.254-2	8.940-4	-8.191-4	2.073-3	-1.719-4
0.1000	1.506-2	1.712-2	-1.969-3	-4.138-4	-3.721-3	7.749-5
0.0500	6.231-3	8.872-3	-2.789-3	-1.924-4	-5.931-3	-5.898-5

Table 2.4

Numerical results of computing $\int e^{-|\underline{x}|^2} \omega(t, \underline{y})\, d\underline{y}$: for each scheme, errors for the usual and the modified Monte-Carlo estimates are listed in order in exponential form:

Example 2: t = 1.0 N = 856 Exact Value = 8.7861-1						
Δ t	Euler's		Method A		Method B	
0.2000	-5.382-3	-7.909-3	2.722-3	2.008-4	1.564-3	-4.661-5
0.1000	-2.496-3	-3.842-3	1.573-3	2.035-4	2.752-3	4.542-5
0.0500	-7.046-4	-1.960-3	1.355-3	1.006-4	2.526-3	9.954-6

Table 3.1

Example 2: t = 2.0 N = 856 Exact Value = 8.7251-1						
Δ t	Euler's		Method A		Method B	
0.2000	-1.246-2	-1.456-2	2.632-3	4.309-4	-2.500-3	-4.172-6
0.1000	-6.807-3	-7.533-3	9.334-4	2.111-4	1.814-3	-2.331-5
0.0500	1.023-4	-3.774-3	4.010-3	1.722-4	4.760-3	-3.219-5

Table 3.2

Example 2: t = 3.0 N = 856 Exact Value = 8.6649-1						
Δ t	Euler's		Method A		Method B	
0.2000	-2.025-2	-2.081-2	1.047-3	3.826-4	1.235-3	-9.602-5
0.1000	-8.937-3	-1.075-2	1.983-3	2.765-4	3.720-3	1.597-5
0.0500	-4.585-4	-5.542-3	5.223-3	1.639-4	-5.645-3	-1.494-4

Table 3.3

Example 2: t = 4.0 N = 856 Exact Value = 8.6055-1						
Δ t	Euler's		Method A		Method B	
0.2000	-2.620-2	-2.600-2	5.181-4	5.488-4	-3.717-4	9.692-5
0.1000	-1.078-2	-1.382-2	2.967-3	2.881-4	4.055-3	-2.121-4
0.0500	-4.171-3	-7.295-3	3.167-3	4.804-5	6.000-3	-1.868-4

Table 3.4

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