Title
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Journal
Journal of Economic Theory, 152(1)

ISSN
0022-0531

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Publication Date
2014-07-01

DOI
10.1016/j.jet.2014.05.001

Peer reviewed

Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 152 (2014) 237–248

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Transitive regret over statistically independent lotteries ☆

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Received 14 May 2012; final version received 16 December 2013; accepted 27 February 2014

Available online 6 May 2014

Abstract

Preferences may arise from regret, i.e., from comparisons with alternatives forgone by the decision maker. When each outcome in a random variable is compared with the parallel outcome in an alternative random variable, regret preferences are transitive iff they are expected utility. In this paper we show that when the choice set consists of pairwise statistically independent lotteries and the regret associated with each outcome is with respect to the entire alternative distribution, then transitive regret-based behavior is consistent with betweenness preferences and with a family of preferences that is characterized by a consistency property. Examples of consistent preferences include CARA, CRRA, and anticipated utility.

JEL classification: D81

Keywords: Regret; Transitivity; Non-expected utility

1. Introduction

One of the most convincing psychological alternatives to expected utility theory is regret theory, which was independently developed by Bell [1] and by Loomes and Sugden [8]. The basic form of the theory applies to choices made between pairs of random variables. While in Savage's [13] model the decision maker evaluates a random variable by weighting its outcomes,

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http://dx.doi.org/10.1016/j.jet.2014.05.001

regret theory suggests that the decision maker should also take into consideration the alternative outcome from the other random variable. This comparison may cause rejoicing — if the actual outcome is better than the alternative — or regret.

Both Bell and Loomes and Sugden assumed that the evaluation of the regret should be additive. That is, for two random variables $X = (x_1, s_1; \dots; x_n, s_n)$ and $Y = (y_1, s_1; \dots; y_n, s_n)$,

$$X \succeq Y \iff \sum_i p(s_i)\psi(x_i, y_i) \geqslant 0$$

where $p(s_i)$ is the probability of event s_i and ψ is a regret function. If $\psi(x, y) = u(x) - u(y)$ then this theory reduces to expected utility theory and it is easy to verify that unless this is the case, such preferences cannot be transitive. One may suspect the restrictive additive form to be the source of intransitive cycles, but as is proved in Bikhchandani and Segal [2], intransitivity is built into the regret model: even when one adopts the more general form

$$X \succeq Y \iff V\big(\psi(x_1, y_1), p(s_1); \dots; \psi(x_n, y_n), p(s_n)\big) \geqslant 0, \tag{1}$$

where V is any increasing (with respect to first-order stochastic dominance) functional, transitivity still implies expected utility.[1]

Regret theory can be used to compare statistically independent lotteries (see Loomes and Sugden [8]), where the regret felt upon winning x_i and not y_j is weighted by $p_i q_j$. But consider a gambler who chooses to play the Roulette instead of Craps. While betting on Black in Roulette when the outcome turns out Red, it seems unnatural that he will compare this outcome to each specific roll of the dice in a Craps game he did not play (and probably did not even observe). Rather, it may drive him to regret the fact that he did not play Craps, and the regret is with respect to the whole alternative distribution. Such feelings are the topic of the present paper.

We discuss a choice problem between two statistically independent lotteries $X = (x_1, p_1; \dots; x_n, p_n)$ and $Y = (y_1, q_1; \dots; y_m, q_m)$. When evaluating the lottery X, the decision maker forecasts his ex-post feelings, and considers his regret or rejoice when he will know that he won x_i but did not play the lottery Y.[2] Formally, we analyze binary relations that are defined by $X \succeq Y$ iff

$$V\big(\psi(x_1, Y), p_1; \dots; \psi(x_n, Y), p_n\big) \geqslant 0 \geqslant V\big(\psi(y_1, X), q_1; \dots; \psi(y_m, X), q_m\big) \tag{2}$$

where $\psi(x, Y)$ is the rejoice or regret felt by the decision maker upon learning that he won x in lottery X which he chose to play out of the set $\{X, Y\}$.[3] We call this property *distribution regret*. The question we ask is this: Under what conditions are distribution-regret relations transitive?

Unlike [2], where it was shown that only expected utility preferences are consistent with both Eq. (1) and transitivity, here we identify two families of preferences that satisfy distribution regret, i.e. Eq. (2), and transitivity. The first are betweenness preferences, according to which $X \succeq Y$ iff for all $\alpha \in [0, 1]$, $X \succeq \alpha X + (1 - \alpha)Y \succeq Y$ (see Chew [3,4], Fishburn [7], and

[1] If $\succeq$ is defined on the set of independent lotteries then transitivity and (1) do not imply expected utility. See Machina [10, footnote 20].

[2] These feelings do not have to agree with his initial preferences over lotteries. That is, at this stage we do not rule out the possibility of preference for the outcome x_i over Y together with anticipated regret if X is chosen and x_i is drawn. But this will be ruled out by transitivity.

[3] Note that if a preference relation over independent lotteries satisfies Eq. (1) in a linear form then it also satisfies Eq. (2) in a linear form. Such a preference relation was suggested without a further discussion by Machina [10] and by Starmer [14]. We provide additional results in Section 3 below.

Dekel [6]). The other family is new and is characterized by a consistency property which includes, as a special case, constant risk-aversion preferences. We also offer conditions over the regret preferences under which these two families are the only preferences to satisfy distribution regret and transitivity.

The paper is organized as follows. The model and a simplification of the regret function ψ that is due to transitivity are described in Section 2. In Section 3 we show that betweenness preferences satisfy distribution regret with a linear regret functional V; moreover, if Eq. (2) holds with a linear V then preferences must be betweenness. Consistent preferences are defined in Section 4 and shown to satisfy distribution regret. We conclude in Section 5.

2. The model and preliminary results

The choice set, denoted by $\mathcal{L}$, is the set of finite-valued lotteries with outcomes in an interval $\mathcal{D} \subseteq \Re$. When comparing a pair of lotteries, the decision maker evaluates each possible outcome of one lottery against the entire probability distribution of the other. Thus, in evaluating $X = (x_1, p_1; \ldots; x_n, p_n)$ against $Y = (y_1, q_1; \ldots; y_m, q_m)$, the decision maker considers his feelings of regret or rejoicing if he were to obtain outcome x_i after choosing X over the alternative lottery Y. This evaluation is conducted for each outcome x_i of X. Implicit in this formulation is the assumption that X and Y are independent lotteries: the probability distribution of Y is unchanged after the outcome of X becomes known. With this background, we have the following definitions.

Definition 1. The continuous function $\psi : \mathcal{D} \times \mathcal{L} \to \Re$ is a *regret function* if for all x and Y, $\psi(x, Y)$ is strictly increasing in x and strictly decreasing as Y increases in the sense of first-order stochastic dominance.

If the lottery X yields x then $\psi(x, Y)$ is a measure of the decision maker's *ex post* feelings of regret or rejoicing about the choice of X over Y. This leads to the next definition:

Definition 2. Let $X, Y \in \mathcal{L}$. The *regret lottery* evaluating the choice of X over Y is

$$\Psi(X, Y) = \big(\psi(x_1, Y), p_1; \ldots; \psi(x_n, Y), p_n\big)$$

Denote the set of regret lotteries by $\mathcal{R} = \{\Psi(X, Y) : X, Y \in \mathcal{L}\}$.

Thus, $\Psi(X, Y)$ is the ex ante regret lottery the decision maker uses in evaluating the choice of X over Y.[4]

Definition 3. A preference relation (that is, a complete and transitive relation) $\succeq$ over $\mathcal{L}$ is *distribution-regret based* if there is a regret lottery Ψ and a strictly increasing[5] and continuous functional $V : \mathcal{R} \to \Re$ such that

$$X \succeq Y \quad \textit{iff} \quad V\big(\Psi(X, Y)\big) \geqslant 0 \quad \textit{iff} \quad 0 \geqslant V\big(\Psi(Y, X)\big). \tag{3}$$

[4] For brevity we refer to ψ and Ψ as regret functions and regret lotteries even though they encompass both regret and rejoicing.

[5] In the sense of first-order stochastic dominance over regret lotteries.

The aim of this paper is to find conditions over a preference relation such that it will satisfy distribution regret. Formally, we ask under what circumstances will a preference relation satisfy Eq. (3) above.

Our first observation is that transitivity of a distribution-regret relation leads to an enormous simplification of the regret function. Instead of evaluating the regret of receiving the outcome x out of X when the alternative lottery was Y, one can evaluate regret with respect to the certainty equivalent of Y. In other words, if Y and Y' are equally attractive, then the regret of x with respect to both is the same.

Lemma 1. *Let $\succeq$ be a distribution-regret preference relation. Then $\succeq$ admits a two-dimensional regret function $\psi^*: \mathcal{D} \times \mathcal{D} \to \Re$ and a regret functional V^* such that*

$$\begin{aligned} X \succeq Y \quad &\iff \quad V^*\big(\psi^*(x_1, c_Y), p_1; \ldots; \psi^*(x_n, c_Y), p_n\big) \geqslant 0 \\ &\iff \quad V^*\big(\psi^*(y_1, c_X), q_1; \ldots; \psi^*(y_m, c_X), q_m\big) \leqslant 0 \end{aligned}$$

where c_X and c_Y are the certainty equivalents of X and Y respectively.

Proof. Let ψ be the regret function and V the regret functional that represent $\succeq$. For $y \in \mathcal{D}$, define $\psi^*(x, y) = \psi(x, \delta_y)$, where δ_y denotes, with a little abuse of notation, the lottery that yields y with probability 1, and define

$$V^*\big(\psi^*(x_1, y), p_1; \ldots; \psi^*(x_n, y), p_n\big) = V\big(\psi(x_1, \delta_y), p_1; \ldots; \psi(x_n, \delta_y), p_n\big)$$

Then, by transitivity,

$$\begin{aligned} X \succeq Y \quad &\iff \quad X \succeq \delta_{c_Y} \\ &\iff \quad V\big(\psi(x_1, \delta_{c_Y}), p_1; \ldots; \psi(x_n, \delta_{c_Y}), p_n\big) \geqslant 0 \\ &\iff \quad V^*\big(\psi^*(x_1, c_Y), p_1; \ldots; \psi^*(x_n, c_Y), p_n\big) \geqslant 0 \qquad \square \end{aligned}$$

The requirement in Lemma 1 that the relation is a preference relation is restrictive, as not all distribution-regret relations are transitive. The following is an example of such a relation.

Example 1. Let $\mathcal{D} = [0, 1]$. Define

$$\psi(x, Y) = x^3 + x^2 + x + x^2\mathrm{E}[Y] - \mathrm{E}\big[Y^3\big] - \mathrm{E}\big[Y^2\big] - \mathrm{E}[Y] - x\mathrm{E}\big[Y^2\big]$$

and let

$$V\big(\psi(x_1, Y), p_1; \ldots; \psi(x_n, Y), p_n\big) = \sum_i p_i \psi(x_i, Y)$$

Therefore, by Eq. (3), $X \succeq Y$ iff

$$\mathrm{E}\big[X^3\big] + \mathrm{E}\big[X^2\big] + \mathrm{E}[X] + \mathrm{E}\big[X^2\big]\mathrm{E}[Y] - \mathrm{E}\big[Y^3\big] - \mathrm{E}\big[Y^2\big] - \mathrm{E}[Y] - \mathrm{E}[X]\mathrm{E}\big[Y^2\big] \geqslant 0$$

Let

- $X = (0.9, \frac{315}{1050}; 0.5, \frac{331}{1050}; 0.2, \frac{404}{1050})$
- $Y = (0.8, \frac{1}{2}; 0.2, \frac{1}{2})$
- $Z = (0.6, \frac{185}{198}; 0.3, \frac{13}{198})$

and obtain that $X \sim Y$, $Y \sim Z$, but $Z \succ X$. $\square$

As a consequence of Lemma 1, we use the following definition of distribution regret without loss of generality:

Definition 4. The preference relation $\succeq$ is *distribution-regret based* if there exists

a. A continuous function $\psi : \mathcal{D} \times \mathcal{D} \to \Re$, strictly increasing in the first argument and strictly decreasing in the second argument.
b. A strictly increasing and continuous functional $V : \mathcal{R} \to \Re$ such that

$$X \succeq Y \quad \textit{iff} \quad V\big(\Psi(X, c_Y)\big) \geqslant 0 \quad \textit{iff} \quad 0 \geqslant V\big(\Psi(Y, c_X)\big),$$

where $\Psi(X, c_Y) = (\psi(x_1, c_Y), p_1; \ldots; \psi(x_n, c_Y), p_n)$ is the regret lottery evaluating the choice of X over δ_{c_Y} (hence over Y), $\Psi(Y, c_X)$ is the regret lottery evaluating the choice of Y over δ_{c_X} (hence over X), and $\mathcal{R}$ is the set of such regret lotteries.

For any x and x', let $X = Y = \delta_x$ and $X' = Y' = \delta_{x'}$. Then $X \sim Y$ and $X' \sim Y'$ imply that $V((\psi(x, x), 1)) = 0$ and $V((\psi(x', x'), 1)) = 0$. Thus, $\psi(x, x) = \psi(x', x')$ by strict monotonicity of V.

3. Betweenness preferences

The preferences $\succeq$ satisfy betweenness [3,4,6,7] if all indifference curves are linear; that is, if $X \succeq Y$ implies $X \succeq \alpha X + (1-\alpha)Y \succeq Y$ for all $\alpha \in [0, 1]$. In this section we show that all betweenness preferences satisfy distribution regret.

Proposition 1. *Betweenness preferences satisfy distribution regret.*

Proof. For $z \in \mathcal{D}$, let u_z be a vNM utility function that determines the indifference curve through δ_z. Obviously, $X \succeq Y$ iff X is above the indifference curve through Y, which is also the indifference curve through δ_{c_Y}. That is, $X \succeq Y$ iff $\mathrm{E}[u_{c_Y}(X)] \geqslant u_{c_Y}(c_Y)$. Define $\psi(x, c_Y) = u_{c_Y}(x) - u_{c_Y}(c_Y)$ and $V(R) = \mathrm{E}[R]$ to obtain

$$\begin{aligned} X \succeq Y \quad &\Longleftrightarrow \quad \mathrm{E}\big[u_{c_Y}(X)\big] \geqslant u_{c_Y}(c_Y) \\ &\Longleftrightarrow \quad \mathrm{E}\big[\Psi(X, c_Y)\big] \geqslant 0 \\ &\Longleftrightarrow \quad V\big(\Psi(X, c_Y)\big) \geqslant 0 \qquad \square \end{aligned}$$

The proof of Proposition 1 also shows that betweenness preferences admit a linear distribution-regret functional V. In fact, betweenness preferences are unique in this respect.

Proposition 2. *The following two conditions are equivalent:*

a. *The preference relation $\succeq$ is betweenness.*
b. *The preference relation $\succeq$ satisfies distribution regret with a linear functional V of the form $V(\Psi(X, c_Y)) = \sum_i p_i \psi(x_i, c_Y)$.*

Proof. By Proposition 1 and its proof, (a) $\Longrightarrow$ (b). To show that (b) $\Longrightarrow$ (a), suppose that $V(\Psi(X, Y)) = \sum_i p_i \psi(x_i, c_Y)$. Then

$$X \sim \delta_{c_Y} \quad \Longrightarrow \quad V\big(\Psi(X, c_Y)\big) = \sum_{i=1}^{n} p_i \psi(x_i, c_Y) = 0$$

$$Y \sim \delta_{c_Y} \quad \Longrightarrow \quad V\big(\Psi(Y, c_Y)\big) = \sum_{i=1}^{m} q_i \psi(y_i, c_Y) = 0$$

Hence

$$V\left(\Psi\left(\frac{1}{2}X + \frac{1}{2}Y, c_Y\right)\right) = \sum_{i=1}^{n} \frac{p_i}{2}\psi(x_i, c_Y) + \sum_{i=1}^{m} \frac{q_i}{2}\psi(y_i, c_Y) = 0$$

Therefore $\frac{1}{2}X + \frac{1}{2}Y \sim \delta_{c_Y} \sim Y \sim X$, hence the claim. □

In particular, transitivity together with an additive regret functional of the form in Eq. (2) does not imply expected utility when the two lotteries are statistically independent.[6]

There is a relationship between risk-attitude and the regret function we have defined for betweenness preferences. Observe that we can implicitly define the certainty equivalent of a lottery X from the equation $\sum_i p_i \psi(x_i, c_X) = 0$. Then risk averse preferences imply that for each y, $\psi(x, y)$ is concave in x (from the definition of ψ in the proof of Proposition 1).

4. Consistency

We offer another set of transitive distribution-regret preferences. Unlike betweenness preferences where there is little connection between the shape of different indifference curves, the preferences we discuss in this section exhibit a very high degree of interdependency among indifference curves. We call this property consistency. To illustrate this concept, consider the set of CARA (constant absolute risk aversion) preferences: $X = (x_1, p_1; \ldots; x_n, p_n) \succeq Y = (y_1, q_1; \ldots; y_m, q_m)$ iff for every suitable λ,[7]

$$X + \lambda = (x_1 + \lambda, p_1; \ldots; x_n + \lambda_n) \succeq Y + \lambda = (y_1 + \lambda, q_1; \ldots; y_m + \lambda, q_m).$$

Once we know the shape of one indifference curve, (segments of) all other indifference curves are determined by it.

Definition 5. The preference relation $\succeq$ over $\mathcal{L}$ is *consistent* if there is a continuous function $f : \mathcal{D} \times \Lambda \to \Re$, where $\Lambda \subseteq \Re$, such that

a. $f(x, \lambda)$ is strictly increasing in both x and λ.
b. For every x, y in the interior of $\mathcal{D}$ there is $\lambda \in \Lambda$ such that $f(x, \lambda) = y$.
c. For all x, y in the interior of $\mathcal{D}$ and λ, $f(x, \lambda) > x$ if and only if $f(y, \lambda) > y$.
d. If for some $x, y \in \mathcal{D}$ and $\lambda \in \Lambda$, $f(x, \lambda) = y$ then for any $z \in \mathcal{D}$, $f(z, \lambda) \in \mathcal{D}$.
e. For all $X, Y \in \mathcal{L}$ and for every $\lambda \in \Lambda$,

$$X \succeq Y \quad \text{iff} \quad f(X, \lambda) \succeq f(Y, \lambda)$$

where $f(Z, \lambda) := (f(z_1, \lambda), r_1; \ldots; f(z_\ell, \lambda), r_\ell)$ for any $Z = (z_1, r_1; \ldots; z_\ell, r_\ell)$.

[6] See also Machina [10, footnote 20].
[7] See Safra and Segal [12] for a discussion of constant risk aversion in general non-expected utility theory.

The intuition behind consistent preferences is simple and compelling. A preference relation can be described by its indifference curves. Assuming continuity, it is enough to know the shape of a countable set of such indifference curves to know the whole relation (for example, the set of indifference curves along which V is rational). This is, of course, less information than knowing a continuum of indifference curves, but it is still a lot. At the other extreme stand preferences like CARA or CRRA (constant relative risk aversion), where from one indifference curve all other indifference curves are obtained either by a parallel shift (CARA) or by projection to zero (CRRA). In both cases, one indifference curve implies all others using one parameter to switch from one indifference curve to another. Consistent preferences extend this intuition: One indifference curve is enough to obtain all and the shift from one indifference curve to another depends on a single parameter.

The following are examples of consistent preferences.

Example 2.

a. If $\succeq$ is CARA on $\mathcal{D} = \Re$, let $f(x, \lambda) = x + \lambda$ and if $\succeq$ is CRRA on $\mathcal{D} = \Re_{++}$, let $f(x, \lambda) = \lambda x$ and obtain that both types of preferences are consistent.[8]

b. This example is related to a quadratic utility example in Machina [9]. We use one indifference curve of the preference relation defined by Machina in Eq. (6) in his paper to construct a consistent preference relation. Note, however, that the preferences we suggest are *not* quadratic. Let $\mathcal{L}$ be the set of lotteries with outcomes in $\mathcal{D} = \Re_+$, and for $\lambda > 0$, let $f(x, \lambda) = (x + 1)^\lambda - 1$. Define the indifference curve through δ_1 to be

$$\left\{X : \left(\mathrm{E}[X]\right)^2 + \mathrm{E}[\sqrt{X}] = 2\right\}$$

Define λ_X implicitly by

$$\left(\mathrm{E}\left[f(X, \lambda_X)\right]\right)^2 + \mathrm{E}\left[\sqrt{f(X, \lambda_X)}\,\right] = 2$$

The existence of λ_X follows by continuity. The preferences that are represented by $U(X) = 1/\lambda_X$ are consistent.

c. Let $\succeq$ be rank dependent[9] with the utility function u. Assume that $\mathcal{D} = \Re$. Let $f(x, \lambda) = u^{-1}(u(x) + \lambda)$ and let $X = (x_1, p_1; \ldots; x_n, p_n)$ with $x_1 \leqslant \ldots \leqslant x_n$. We obtain for the rank-dependent functional

$$U\big(f(X, \lambda)\big) := U\big(f(x_1, \lambda), p_1; \ldots; f(x_n, \lambda), p_n\big) = U(X) + \lambda$$

and $\succeq$ is consistent. □

The following is the main result of this section.

[8] Because the choice set $\mathcal{L}$ for CRRA preferences consists of lotteries over non-negative numbers, i.e. $\mathcal{D} = \Re_+$, Definition 5c is satisfied.

[9] According to the rank-dependent theory (Quiggin [11]), for $X = (x_1, p_1; \ldots; x_n, p_n)$ with $x_1 \leqslant \ldots \leqslant x_n$, the preferences $\succeq$ can be represented by

$$U(X) = u(x_1)g(p_1) + \sum_{i=2}^{n} u(x_i)\left[g\left(\sum_{j=1}^{i} p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right)\right]$$

Proposition 3. *If the preference relation $\succeq$ is consistent then it satisfies distribution regret.*

Proof. For every $Z \in \mathcal{L}$ (except for $Z = \delta_x$ for x on the boundary of $\mathcal{D}$), define $\lambda(Z)$ to be the number λ such that $f(c_Z, \lambda) = 0$.[10] Definition 5, together with continuity, implies the existence of $\lambda(Z)$ for every $Z \in \mathcal{L}$, including $Z = \delta_x$ where x is on the boundary of $\mathcal{D}$. Let U be the representation of $\succeq$ satisfying $U(\delta_x) = x$ for all x. Define $\psi(x, c_Y) = f(x, \lambda(Y))$.[11] That is, $\psi(x, c_Y)$ is the number into which x is transformed via f by applying $\lambda(Y)$ to it, where $\lambda(Y)$ is the number that transforms the certainty equivalent of Y to 0. For example, if $\succeq$ is CARA, then $\lambda(Z) = -c_Z$ and $\psi(x, y) = x - y$. From Definition 5d, we know that for any $x \in \mathcal{D}$ we have $f(x, \lambda(Y)) \in \mathcal{D}$. Let

$$\begin{aligned} f\big(X, \lambda(Y)\big) &= \big(f\big(x_1, \lambda(Y)\big), p_1; \ldots; f\big(x_n, \lambda(Y)\big), p_n\big) \\ &= \big(\psi(x_1, c_Y), p_1; \ldots; \psi(x_n, c_Y), p_n\big) \\ &= \Psi(X, c_Y) \end{aligned} \tag{4}$$

By consistency

$$\begin{aligned} X \succeq Y \sim \delta_{c_Y} \quad &\Longleftrightarrow \quad f\big(X, \lambda(Y)\big) \succeq f\big(Y, \lambda(Y)\big) \sim \delta_{f(c_Y, \lambda(Y))} = \delta_0 \\ &\Longleftrightarrow \quad U\big(f\big(X, \lambda(Y)\big)\big) \geqslant U\big(f\big(Y, \lambda(Y)\big)\big) = U(\delta_0) = 0 \end{aligned}$$

Let

$$V\big(\Psi(X, c_Y)\big) = U\big(f\big(X, \lambda(Y)\big)\big) = U\big(\Psi(X, c_Y)\big)$$

(the second equation follows by Eq. (4)). Therefore

$$\begin{aligned} X \succeq Y \quad &\Longleftrightarrow \quad U\big(\Psi(X, c_Y)\big) \geqslant 0 \\ &\Longleftrightarrow \quad V\big(\Psi(X, c_Y)\big) \geqslant 0 \end{aligned}$$

Hence $\succeq$ satisfies distribution regret. □

To illustrate, we show the explicit form of distribution regret associated with the rank-dependent model (see Example 2c above). Assume, without loss of generality, $u(0) = 0$. For $Z = (z_1, r_1; \ldots; z_n, r_n)$ with $z_1 \leqslant \ldots \leqslant z_n$, the certainty equivalent c_Z of Z satisfies

$$c_Z = u^{-1}\left(u(z_1)g(r_1) + \sum_{i=2}^{n} u(z_i)\left[g\left(\sum_{j=1}^{i} p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right)\right]\right)$$

For the rank-dependent model, we defined $f(x, \lambda) = u^{-1}(u(x) + \lambda)$ hence

$$f\big(c_Z, \lambda(Z)\big) = 0 \quad \Longrightarrow \quad \lambda(Z) = -u(z_1)g(r_1) - \sum_{i=2}^{n} u(z_i)\left[g\left(\sum_{j=1}^{i} p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right)\right]$$

The function U that represents that preferences $\succeq$ while satisfying $U(\delta_x) = x$ is the function $U(Z) = c_Z$ given above. The function $\psi(x, c_Y)$, the regret of obtaining outcome x in X when the alternative lottery was Y, is given by

[10] We assume that $0 \in \mathrm{int}[\mathcal{D}]$. If this is not the case, then for some $d \in \mathrm{int}[\mathcal{D}]$, let $f(c_Z, \lambda(Z)) = d$ and use the normalization $\psi(x, x) = d$.

[11] Observe that $\psi(x, x) = \psi(x, c_{\delta_x}) = f(x, \lambda(\delta_x)) = 0$.

$$\psi(x, c_Y) = f\big(x, \lambda(Y)\big) = u^{-1}\big(u(x) + \lambda(Y)\big)$$

In other words, the utility u from the regret of receiving x in X when the rejected lottery was Y equals the difference between the utility from x and the (rank-dependent) utility from the lottery Y. This is a meaningful statement in the rank-dependent model where there is a utility function u over outcomes. Most models do not admit such an interpretation.

Not all transitive preferences are consistent. We present next an example that does not satisfy distribution regret and therefore is not consistent.

Example 3. Let $\succeq$ have a linear indifference curve $\mathcal{I}$ such that the preference relation above it is strictly quasi-concave.[12] An example of such a preference relation is provided in Chew, Epstein, and Segal [5].

Let $X, Y, Z \in \mathcal{I}$. Under the supposition that $\succeq$ satisfies distribution regret, we have $V(\Psi(X, c_Z)) = V(\Psi(Y, c_Z)) = 0$, and by the linearity of $\mathcal{I}$, for all $\alpha \in (0, 1)$, $V(\Psi(\alpha X + (1-\alpha)Y, c_Z)) = 0$.

Let Z' be a lottery on a non-linear indifference curve above $\mathcal{I}$ and let X' and Y' be such that

$$\Psi\big(X', c_{Z'}\big) = \Psi(X, c_Z) \quad \text{and} \quad \Psi\big(Y', c_{Z'}\big) = \Psi(Y, c_Z) \tag{5}$$

By continuity of ψ there exist such X', Y', Z' close to $\mathcal{I}$. Distribution regret implies that $X' \sim Z'$ and $Y' \sim Z'$. It follows from (5) that

$$\Psi\left(\frac{1}{2}X' + \frac{1}{2}Y', c_{Z'}\right) = \Psi\left(\frac{1}{2}X + \frac{1}{2}Y, c_Z\right)$$

Hence $V(\Psi(\frac{1}{2}X' + \frac{1}{2}Y', c_{Z'})) = 0$ and yet by the assumption that all indifference curves above $\mathcal{I}$ are strictly quasi-concave it follows that $\frac{1}{2}X' + \frac{1}{2}Y' \nsim Z'$, a contradiction. □

The converse of Proposition 3 is not true. As is demonstrated by the next example, there are distribution-regret preferences that are not consistent.

Example 4. Define the betweenness preferences $\succeq$ by $X \sim \delta_\alpha$ iff $\mathrm{E}[u_\alpha(X)] = u_\alpha(\alpha)$, where

$$u_\alpha(x) = \begin{cases} x & \alpha \leqslant 0 \\ g_\alpha(x) & \alpha > 0 \end{cases} \tag{6}$$

and where for $\alpha > 0$,

$$g_\alpha(x) = \begin{cases} x & x \leqslant 0 \\ (1+\alpha)x & x > 0 \end{cases} \tag{7}$$

Let

$$\mu_X^+ = \sum_{i:x_i>0} p_i x_i$$

and obtain that $\succeq$ can be represented by a function U, given by $X \sim \delta_{\mathrm{E}[X]}$ for X such that $\mathrm{E}[X] \leqslant 0$, and for X with $\mathrm{E}[X] > 0$, $X \sim \delta_\alpha$ where α solves

[12] Preferences are strictly quasi-concave iff $X \succeq Y$ implies $\alpha X + (1-\alpha)Y \succ Y$ for all $\alpha \in (0, 1)$.

$$\mathrm{E}[X] + \alpha\mu_X^+ = (1+\alpha)\alpha \quad \Longrightarrow \quad \alpha = \frac{-(1-\mu_X^+) + \sqrt{(1-\mu_X^+)^2 + 4\mathrm{E}[X]}}{2} \tag{8}$$

These preferences satisfy betweenness, therefore by Proposition 1, they satisfy distribution regret.

Observe that

$$\mathrm{E}[X] > 0 \quad \text{and} \quad \Pr(X < 0) > 0 \quad \Longrightarrow \quad X \succ \delta_{\mathrm{E}[X]} \tag{9}$$

This follows because $\mu_X^+ > E[X] > 0$ and if $X \sim \delta_\alpha$, then by Eq. (8) we have $\alpha > E[X]$.

Suppose that the preference relation $\succeq$ is consistent. Let $-1 < s, t < 0$, and λ_0 be such that $f(-1, \lambda_0) = s$ and $f(t, \lambda_0) = 0$. For every $z \geqslant t$ we have

$$\mathrm{E}\left[\left(z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z}\right)\right] = t, \quad \text{hence} \quad \left(z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z}\right) \sim (t, 1)$$

Consistency implies that

$$\left(f(z, \lambda_0), \frac{1+t}{1+z}; s, \frac{z-t}{1+z}\right) \sim (0, 1) \tag{10}$$

As the certainty equivalent in Eq. (10) is not greater than 0, it follows from the definition of $\succeq$ that

$$\frac{1+t}{1+z} f(z, \lambda_0) + \frac{s(z-t)}{1+z} = 0 \quad \Longrightarrow \quad f(z, \lambda_0) = \frac{s(t-z)}{1+t}$$

From Eq. (6) we have $(-1, \frac{1}{2}; 1, \frac{1}{2}) \sim \delta_0$, hence by consistency and monotonicity

$$\left(s, \frac{1}{2}; f(1, \lambda_0), \frac{1}{2}\right) \sim \delta_{f(0,\lambda_0)} \tag{11}$$

The expected value of the last lottery is

$$\frac{s}{2} + \frac{s}{2}\frac{t-1}{1+t} = \frac{st}{1+t} = f(0, \lambda_0) > 0$$

which together with Eq. (11) contradicts Eq. (9). □

We have seen so far that betweenness and consistent preferences satisfy distribution regret (Propositions 1 and 3). There are betweenness preferences that are not consistent (Example 4), there are consistent preferences that are not betweenness (e.g., rank dependent, see Example 2c), and there are betweenness preferences that are consistent (e.g., expected utility or weighted utility, see Example 5 below). The next question is whether there are distribution-regret preferences that are neither consistent nor betweenness. We do not know the answer to this question for the general case, but suppose that in addition to distribution regret, the regret function has the following property:

Definition 6. The regret function ψ is *commutative* if for all x, x', y, y',

$$\psi(x, x') = \psi(y, y') \quad \Longrightarrow \quad \psi(x, y) = \psi(x', y') \tag{12}$$

It turns out that if the regret function is commutative, then the converse of Proposition 3 holds.

Proposition 4. *If the preference relation $\succeq$ satisfies distribution regret with a commutative regret function ψ, then it is consistent.*

Proof. Let $d \in \Re$ be such that $\psi(x,x) = d$ for all $x \in \mathcal{D}$. (Definition 4a implies that d exists.) Define $f(x,\lambda) = y$ where $\psi(x,y) = d - \lambda$. That is, for all x and λ,

$$\psi\big(x, f(x,\lambda)\big) = d - \lambda \tag{13}$$

Let $X \succeq Y$, hence by distribution regret, $V(\Psi(X, c_Y)) \geqslant 0$. As by Eq. (13)

$$\psi\big(c_X, f(c_X,\lambda)\big) = \psi\big(c_Y, f(c_Y,\lambda)\big) = d - \lambda$$

we have from Eq. (12) that $\psi(c_X, c_Y) = \psi(f(c_X,\lambda), f(c_Y,\lambda))$, hence

$$\Psi(\delta_{c_X}, c_Y) = \Psi\big(\delta_{f(c_X,\lambda)}, f(c_Y,\lambda)\big)$$

Thus, $\delta_{c_X} \succeq \delta_{c_Y}$ and distribution regret together imply that $\delta_{f(c_X,\lambda)} \succeq \delta_{f(c_Y,\lambda)}$.

Next, another application of Eq. (13) and then Eq. (12) implies that

$$\psi(x_i, c_X) = \psi\big(f(x_i,\lambda), f(c_X,\lambda)\big)$$

Therefore, $\Psi(X, c_X) = \Psi(f(X,\lambda), f(c_X,\lambda))$. Hence,

$$V\Big(\Psi\big(f(X,\lambda), f(c_X,\lambda)\big)\Big) = V\big(\Psi(X, c_X)\big) = 0,$$

and $f(c_X,\lambda)$ is the certainty equivalent of $f(X,\lambda)$. Similarly, $f(c_Y,\lambda)$ is the certainty equivalent of $f(Y,\lambda)$. Hence, consistency. □

We gave several instances of consistent (and hence distribution-regret preference) in Example 2. We show next that they are all commutative. For CARA, let $\psi(x,y) = f(x,\lambda(\delta_y)) = x - y$; for CRRA, let $\psi(x,y) = f(x,\lambda(\delta_y)) = x/y$; and for rank-dependent preferences with the utility function u, let $\psi(x,y) = f(x,\lambda(\delta_y)) = u^{-1}(u(x) - u(y))$.

The regret function in Example 2b is also commutative. To see this, define

$$\psi(x,y) = f\big(x, 1/\big(\log_2[y+1]\big)\big) = [x+1]^{\frac{1}{\log_2[y+1]}} - 1$$

The steps in the proof of Proposition 3 imply that this ψ represents the preference relation in Example 2b. Commutativity of ψ is easily verified.

Example 4, which satisfies betweenness, is not consistent. The next example shows that betweenness and consistency do not imply expected utility. It also satisfies Eq. (12). Hence, consistency, commutative ψ, and (non-expected utility) betweenness are compatible.

Example 5. Weighted utility (Chew [3]): Let $X \succeq Y$ iff $\frac{\sum_i \nu(x_i)p_i}{\sum_i \tau(x_i)p_i} > \frac{\sum_j \nu(y_j)q_j}{\sum_j \tau(y_j)q_j}$. Then

$$\begin{aligned} X \succeq Y \quad &\Longleftrightarrow \quad \frac{\sum_i \nu(x_i)p_i}{\sum_i \tau(x_i)p_i} \geq \frac{\nu(c_Y)}{\tau(c_Y)} \\ &\Longleftrightarrow \quad \sum_i \left[\frac{\nu(x_i)}{\tau(x_i)} - \frac{\nu(c_Y)}{\tau(c_Y)}\right] p_i \geq 0. \end{aligned}$$

Let $\psi(x,x') = \frac{\nu(x)}{\tau(x)} - \frac{\nu(x')}{\tau(x')}$. It is readily verified that ψ is commutative. □

5. Discussion

We established that betweenness and consistent preferences are two families of transitive preferences that satisfy distribution regret. Moreover, if the regret function ψ is commutative then transitivity and distribution regret implies consistency, and betweenness and consistent preferences become the *only* two families of transitive preferences that satisfy distribution regret. That weighted utility, which is a special case of betweenness preferences, satisfies distribution regret was known from Machina [10] and Starmer [14].[13] Our first contribution is to show that *all* betweenness preferences satisfy distribution regret. The second family of preferences we discuss, that of consistent preferences, is a new class that may be of independent interest.

We presented in the Introduction two ways in which the regret between two independent random variables X and Y can be analyzed: as in Eq. (1), where regret is represented as a list of the regrets between all possible pairs of outcomes from X and from Y, and as in Eq. (2), where regret is computed between each outcome of X and the random variable Y.

The two are not equivalent. Consider Eq. (2). Let X and Y be two random variables such that all their outcomes are in the interior of $\mathcal{D}$. For Y' sufficiently close to Y there is, by continuity, X' such that for all i, $\psi(x_i', Y') = \psi(x_i, Y)$, hence $\Psi(X', Y') = \Psi(X, Y)$. On the other hand, if we adopt the regret notion of Eq. (1) and take Y' such that $y_1' > y_1$ but $y_2' < y_2$, then by monotonicity $\phi(x_1', y_1') = \phi(x_1, y_1)$ implies $x_1' > x_1$ but $\psi(x_1', y_2') = \psi(x_1, y_2)$ implies $x_1' < x_1$.

The regret functional Eq. (1) is appropriate for correlated lotteries X and Y and, as argued in this paper, the regret functional Eq. (2) is appropriate for statistically independent lotteries X and Y. In Bikhchandani and Segal [2] it was shown that if the alternatives X and Y are fully correlated, then transitive regret (where the regret functional is as in Eq. (1)) implies expected utility. In this paper we find that at the other extreme, where X and Y are statistically independent, a large class of non-expected utility models are compatible with transitive regret of the form in Eq. (2). The intermediate case of not perfectly correlated X and Y is the subject of future research.

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[13] These authors used the additive form of the regret functional in Eq. (1) rather than the distribution regret functional in Eq. (2).