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Character Formulas for 2-Lie Algebras

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Abstract of the Dissertation

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Professor Raphaël Rouquier, Chair

Part I of this thesis lays the foundations of categorical Demazure operators following the work of Anthony Joseph. In Joseph's work, the Demazure character formula is given a categorification by idempotent functors that also satisfy the braid relations. This thesis defines 2-functors on a category of modules over a half 2-Lie algebra and shows that they indeed categorify Joseph's functors. These categorical Demazure operators are shown to also be idempotent and are conjectured to satisfy the braid relations as well as give a further categorification of the Demazure character formula.

Part II of this thesis gives a presentation of localized affine and degenerate affine Hecke algebras of arbitrary type in terms of weights of the polynomial subalgebra and varied Demazure-BGG type operators. The definition of a graded algebra is given whose category of finite-dimensional ungraded nilpotent modules is equivalent to the category of finite-dimensional modules over an associated degenerate affine Hecke algebra. Moreover, unlike the traditional grading on degenerate affine Hecke algebras, this grading factors through central characters, and thus gives a grading to the irreducible representations of the associated degenerate affine Hecke algebra. This paper extends the results of Rouquier, Brundan and Kleschev on the affine and degenerate affine Hecke algebras for GL_n which are shown to be related to quiver Hecke algebras in type A, and also secretly carry a grading.

The dissertation of Robert Arthur Denomme is approved.

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Table of Contents

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Publications

Denomme R., Savin, G., *Elliptic curve primality tests for Fermat and related primes.* Journal of Number Theory, **128** (8) Pages 23982412 (2008)

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Introduction

This thesis is divided into two parts, both having some relation to the categorification of Lie algebras, found for instance in [Rou08]. Categorifications are a technique championed by [CF94] in the context of mathematical physics and which has had major impact in fields as disparate as modular representation theory of symmetric groups and topological invariants of knots. The technique of categorification takes a traditional algebraic object built with sets and functions and replaces those with categories and functors. Equations must be replaced with natural isomorphisms. These natural transformations satisfy their own structural equations which are then called the *higher coherence relations* of the categorification. These natural transformations with their higher coherence relations are often a recognizable and interesting classical algebraic structure which becomes linked to the starting object. One may use symmetries, formulas, and theorems about the original object to inspire and create analogous higher versions involving the coherence structures, often enriching and adding to our understanding of both.

In higher representation theory [Rou08], Lie algebras and their associated quantum groups are categorified with coherence relations coming from graded versions of affine Hecke algebras or quiver Hecke algebras, and the quantum variable coming from grading shift. The crystal bases of representations of quantum Lie algebras now have an interpretation via simple modules of affine Hecke algebras and cyclotomic Hecke algebras [LV11], and certain derived equivalences between blocks of affine Hecke algebras can be viewed as a categorification of the Weyl group action on finite representations of Lie algebras, [CR08]. This thesis explores the role of these and other Hecke algebras in higher representation theory, particularly motivated by the following two questions:

- One consequence of the Demazure character formula for Lie algebras is a formula for the characters of finite dimensional simple modules of a simple Lie algebra $\mathfrak{g}$ in terms of Demazure operators, Δ_α. These operators are interesting in that they satisfy the braid relations, but are not invertible as they also satisfy the quadratic relation

$\Delta_\alpha^2 = \Delta_\alpha$. In [Jos85] this character formula is taken one step further by giving a construction of simple modules of $\mathfrak{g}$ using functors $\mathscr{D}_\alpha$ that categorify the Demazure operators. Separately, a crystal version of the character formula was proved in [Kas93]. Given the connection between crystals and categorification can one pull all these works together by providing categorical Demazure operators which further categorify Joseph's functors?

- The coherence relations in higher representation theory are related to affine Hecke algebras specifically of type A, [Rou08, Thm 3.16, 3.19]. This connection has been used to give a constructive proof of the classification of their irreducible modules [McN12], [HMM12], [KR11] as well as study the homological properties of such algebras, [BKM14]. Much as the case with symmetric groups [Kle10] this connection gives Hecke algebras of type A a secret *grading*, and their graded modules *graded characters*. In light of the impact this has for type A, what can be said about affine Hecke algebras of types other than A?

Part I of this thesis answers the first question in the positive by constructing a further categorification $\boldsymbol{\mathfrak{D}}_\alpha^*$ of Joseph's functors $\mathscr{D}_\alpha$. Along with the decategorification Theorem I.44 for $\boldsymbol{\mathfrak{D}}_\alpha^*$, it is shown show these satisfy a 2-functor analogue of the quadratic relation $\mathscr{D}_\alpha^2 \cong \mathscr{D}_\alpha$. This relation already has many consequences for higher representation theory, including the fully faithful lemma, Lemma I.38. Section 3.5 offers the construction of an object in Corollary I.40 which under the assumption of Conjecture I.42 provides the categorification of the Demazure character formula for a simple module over a Lie algebra. The theory of these categorical Demazure operators is still new and underdeveloped. We pose in Conjecture I.48 that these operators $\boldsymbol{\mathfrak{D}}_\alpha^*$ also satisfy the braid relations, perhaps the most significant property the 2-functors could posses. There is currently nowhere in the literature a 2-categorical version of braid relations. There is also the original motivating question which remains to be answered: may the crystal version of the Demazure character formula be proved or interpreted using the categorical Demazure operators?

In Part II, a separate work also available at [Den13], significant progress on the second question is made by showing that degenerate affine Hecke algebras and their irreducible modules in all types are secretly graded, and have a presentation closely resembling the presentation of quiver Hecke algebras in higher representation theory. Though these algebras themselves are not involved with the categorification of Lie algebras, it is hoped that such a framework will develop our understanding of affine Hecke algebras so as to obtain algebraic classifications of their simple modules, as well as obtain graded character formulas for them.

Part I

Character Formulas for 2-Lie Algebras

CHAPTER 1

Joseph's Functors

1.1 Notation and the Demazure character formula

Recall some basic results on complex semisimple Lie algebras and their finite dimensional representations. Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ Cartan and Borel subalgebras. Let $X \subset \hom_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ be the weight lattice and $(X, R, \check{R}, \Pi)$ be the root datum of $\mathfrak{g}$ associated to these choices. Thus $\Pi \subset R \subset X$ is a choice of simple roots. Let X^+ be the collection of weights λ in X for which $\langle \lambda, \check{\alpha} \rangle \geq 0$ for each $\alpha \in \Pi$. Let ρ be the half sum of the positive roots, which is also the sum of the fundamental weights. It is defined by the property $\langle \rho, \check{\alpha} \rangle = 1$ for each $\alpha \in \Pi$.

For each $\alpha \in R$ define the reflection $s_\alpha \in \mathrm{Aut}(X)$ by the formula, $s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha$. Let $W \subset \mathrm{Aut}(X)$ be the finite subgroup generated by the collection $\{s_\alpha\}_{\alpha \in \Pi}$. The generators $\{s_\alpha\}_{\alpha \in \Pi}$ give rise to the length function ℓ on W. Let w_0 be the unique longest element in W. Define the dot action of W on X by $w.x = w(x + \rho) - \rho$. Also define the negative dot action by $w \cdot x = w(x - \rho) + \rho$. Neither of these dot actions are linear on X. In particular, $s_\alpha.x = s_\alpha(x) - \alpha$ and $s_\alpha \cdot x = s_\alpha(x) + \alpha$.

For each positive weight $\lambda \in X^+$ there is a unique simple $\mathfrak{g}$-module $L(\lambda)$ with highest weight λ. It is a finite-dimensional representation. The spaces $\{L(\lambda)_{w(\lambda)}\}_{w \in W}$ are called extremal weight spaces and are all one dimensional. The extremal weight space $L(\lambda)_{w_0(\lambda)}$ generates $L(\lambda)$ as a $\mathfrak{b}$-module.

The Borel subalgebra $\mathfrak{b}$ is generated by $\mathfrak{h}$ and generators $\{E_\alpha\}_{\alpha \in \Pi}$ which satisfy the Serre relations. The algebra $\mathfrak{g}$ is generated by $\mathfrak{b}$ and generators $\{F_\alpha\}_{\alpha \in \Pi}$ which also satisfy the Serre relations. For each simple root $\alpha \in \Pi$ let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the parabolic subalgebra of $\mathfrak{g}$ generated by $\mathfrak{b}$ and F_α. We let $\mathfrak{g}-\mathrm{fmod}, \mathfrak{b}-\mathrm{fmod}, \mathfrak{g}_\alpha-\mathrm{fmod} \ldots$ stand for the categories of finite dimensional modules. Given V a $\mathfrak{b}$-module, let V^{hw} be the subspace of V of highest weight vectors, $V^{hw} = \cap_{\alpha \in \Pi} \ker(E_\alpha)$. Given V a $\mathfrak{g}$-module, let V^{lw} be the subspace of V of lowest weight vectors, $V^{lw} = \cap_{\alpha \in \Pi} \ker(F_\alpha)$.

Let $\mathfrak{g}$ be an arbitrary complex reductive Lie algebra with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ Cartan and Borel subalgebras. Given $\alpha \in \Pi$, let $\mathfrak{s}_\alpha \subset \mathfrak{g}_\alpha$ be the reductive subalgebra of semisimple rank 1 generated by $\mathfrak{h}$ and the generators E_α, F_α. Let $\mathfrak{b}_\alpha = \mathfrak{b} \cap \mathfrak{s}_\alpha$, a Borel subalgebra of $\mathfrak{s}_\alpha$. Let $\mathfrak{n}_{(\alpha)} \subset \mathfrak{g}_\alpha$ be the nilpotent radical of $\mathfrak{g}_\alpha$. This is an ideal with a decomposition

$$\mathfrak{g}_\alpha = \mathfrak{s}_\alpha \oplus \mathfrak{n}_{(\alpha)}.$$

Let $\mathbb{Z}[X] = \mathbb{Z}[e^x]_{x \in X}/(e^x e^y = e^{x+y})$, a subring of the character ring of $\mathfrak{h}$. Given $V \in \mathfrak{h}-$fmod, semisimple with $V_\lambda = 0, \lambda \notin X$, we define the character of V by

$$\text{char}(V) = \sum_{x \in X} dim(V_x) e^x.$$

Definition. Let $\Delta_\alpha : \mathbb{Z}[X] \to \mathbb{Z}[X]$ be the $\mathbb{Z}$-linear function defined by the following action on basis elements

$$\Delta_\alpha : e^x \mapsto \frac{e^x - e^{s_\alpha \cdot x}}{1 - e^{-\alpha}}.$$

Similarly let $\Delta^*_\alpha : \mathbb{Z}[X] \to \mathbb{Z}[X]$ be defined by the formula

$$\Delta^*_\alpha : e^x \mapsto \frac{e^x - e^{s_\alpha \cdot x}}{1 - e^{\alpha}}.$$

The operators Δ_α and Δ^*_α are called Demazure operators. The following two relations are classical.

Claim I.1. *Given $\alpha, \beta \in \Pi$ let $m_{\alpha,\beta}$ be the order in W of $s_\alpha s_\beta$. The operators $\Delta_\alpha, \Delta^*_\alpha$ satisfy the following relations:*

1.

$$\Delta_\alpha{}^2 = \Delta_\alpha,$$
$$\Delta^{*\,2}_\alpha = \Delta^*_\alpha$$

2.

$$\Delta_\alpha \Delta_\beta \cdots = \Delta_\beta \Delta_\alpha \ldots,$$
$$\Delta^*_\alpha \Delta^*_\beta \cdots = \Delta^*_\beta \Delta^*_\alpha \ldots,$$

with $m_{\alpha,\beta}$ terms on each side.

Proposition I.2 (Demazure character formula)**.** *Let $w \in W$ and $w = s_{\alpha_1} \dots s_{\alpha_k}$ a reduced decomposition. Let $\lambda \in X^+$ and let $V(w)$ be the $\mathfrak{b}$-submodule of $L(\lambda)$ generated by the space $L(\lambda)_{w(\lambda)}$. Then*

$$\operatorname{char} V(w) = \Delta_{\alpha_1} \dots \Delta_{\alpha_k}(e^{\lambda}).$$

This chapter will only be concerned with the case of $w = w_0$, for the which the formula gives the character:

$$\operatorname{char} L(\lambda) = \Delta_{\alpha_1} \dots \Delta_{\alpha_n}(e^{\lambda}).$$

where $w_0 = s_{\alpha_1} \dots s_{\alpha_n}$ is a reduced decomposition of the longest element $w_0 \in W$.

Remark I.3. Note that in the the full Demazure character formula is more general than the Weyl character formula, as it gives the characters of other submodules of the simple modules. The Weyl formula also has a categorification via the BGG-resolution which gives a resolution of a simple module by Verma modules. In the framework of 2-Lie algebras there is no analogue of a Verma module, so there is little hope of giving the kind of categorification sought in this thesis to the Weyl character formula.

1.2 Definitions

This section defines and discusses Joseph's functors on the category of $\mathfrak{b}$-modules. While the original paper [Jos85] uses properties of category $\mathcal{O}$ we shall simply use properties of adjoints to the restriction functor $\operatorname{res} : \mathfrak{g}_\alpha-\text{fmod} \to \mathfrak{b}-\text{fmod}$.

Definition. Let $\operatorname{res} : \mathfrak{g}_\alpha-\text{fmod} \to \mathfrak{b}-\text{fmod}$ be the restriction functor. Define the functor

$$\mathscr{D}_\alpha : \mathfrak{b}-\text{fmod} \to \mathfrak{g}_\alpha-\text{fmod}$$

as the left adjoint of res. Similarly, define

$$\mathscr{D}_\alpha^* : \mathfrak{b}-\text{fmod} \to \mathfrak{g}_\alpha-\text{fmod}$$

as the right adjoint of res. The functors $\mathscr{D}_\alpha$ are referred to as induction functors and $\mathscr{D}_\alpha^*$ as a coinduction functors.

It will be shown in Lemma I.17 that $\mathscr{D}_\alpha$ categorifies Δ_α and $\mathscr{D}_\alpha^*$ categorifies Δ_α^*.

Remark I.4. In this chapter nearly every result about the functors $\mathscr{D}_\alpha$ is proved in [Jos85]. The functors $\mathscr{D}_\alpha^*$ do not appear there, and neither does the language of adjunctions, both of which are needed for the categorical interpretation given in Chapter 3. Thus these results are reproved in this chapter using adjunctions along with analogous results for $\mathscr{D}_\alpha^*$.

Remark I.5. Let $V \in \mathfrak{b}-\mathrm{fmod}$ be a finite dimensional $\mathfrak{b}$-module. Then the infinite-dimensional $\mathfrak{g}_\alpha$-module $U(\mathfrak{g}_\alpha) \otimes_{U(\mathfrak{b})} V$ lies in the BGG category $\mathcal{O}$. By standard properties of category $\mathcal{O}$ one may deduce that this infinite dimensional module has a unique maximal finite dimensional quotient. Let W be another finite dimensional $\mathfrak{g}_\alpha$-module. As every $\mathfrak{g}_\alpha$-morphism $U(\mathfrak{g}_\alpha) \otimes_{U(\mathfrak{b})} V \to W$ factors through the maximal finite dimensional quotient there is a functorial equivalence

$$\hom_{\mathfrak{b}}(V, \operatorname{res} W) \cong \hom_{\mathfrak{g}_\alpha}(U(g_\alpha) \otimes_{U(\mathfrak{b})} V, W) \cong \hom_{\mathfrak{g}_\alpha}(\mathscr{D}_\alpha(V), W)$$

By the Yoneda lemma, this maximal finite dimensional quotient must in fact be $\mathscr{D}_\alpha(V)$. One can give a similar explicit construction of $\mathscr{D}_\alpha^*(V)$.

1.3 Adjunctions

Recall that $\mathscr{D}_\alpha, \mathscr{D}_\alpha^*$, being defined as left and right adjoints both come with unit and counit natural transformations. The following standard notation denotes the unit of an adjunction with η and counit by ε. In this notation $(\eta, \varepsilon) : \mathscr{D}_\alpha \vdash \operatorname{res}$ and $(\eta, \varepsilon) : \operatorname{res} \vdash \mathscr{D}_\alpha^*$ does not distinguish the unit and counit (η, ε) for $\mathscr{D}_\alpha$ and the unit and counit (η, ε) for $\mathscr{D}_\alpha^*$, so one must determine this simply from the context we are using. To be concrete, for $V \in \mathfrak{g}_\alpha-\mathrm{fmod}$ there is a $\mathfrak{g}_\alpha$-module morphism

$$\mathscr{D}_\alpha \operatorname{res}(V) \xrightarrow{\varepsilon} V,$$

and for $W \in \mathfrak{b}-\mathrm{fmod}$ there is a $\mathfrak{b}$-module morphism

$$W \xrightarrow{\eta} \operatorname{res} \mathscr{D}_\alpha(W).$$

These morphisms satisfy the so-called zig-zag equalities. That is, the following morphisms are the identity:

$$\begin{aligned} &\operatorname{res}(V) \xrightarrow{\eta \operatorname{res}} \operatorname{res} \mathscr{D}_\alpha \operatorname{res}(V) \xrightarrow{\operatorname{res} \varepsilon} \operatorname{res}(V), \\ &\mathscr{D}_\alpha(W) \xrightarrow{\mathscr{D}_\alpha \eta} \mathscr{D}_\alpha \operatorname{res} \mathscr{D}_\alpha(W) \xrightarrow{\varepsilon \mathscr{D}_\alpha} \mathscr{D}_\alpha(W). \end{aligned}$$

1.4 An $\mathfrak{sl}_2$ example

Consider the case $\mathfrak{g} = \mathfrak{sl}_2$, where $X = \mathbb{Z}$ which has the simple root $\alpha = 2$. The subalgebra $\mathfrak{b} \subset \mathfrak{sl}_2$ is spanned by H and E. By Jordan normal form for the nilpotent operator E, every finite $\mathfrak{b}$-module is a direct sum of so-called string modules, defined as follows. Let $n \leq m$ be two integers and $S(n,m)$ the $\mathfrak{b}$-module which is one dimensional in each weight space λ with $n \leq \lambda \leq m$, and on which E is injective between non-zero weight spaces. Given $\lambda \geq 0$ there is the unique simple $\mathfrak{sl}_2$-module $L(\lambda)$ with highest weight λ. These simple finite modules exhaust all the simple modules in the semisimple category $\mathfrak{sl}_2$−fmod.

Define the representation $\mathbb{C}_n$ to be the one dimensional $\mathfrak{b}$-module on which H acts as multiplication by n and E acts as multiplication by 0.

Claim I.6. *Let $c \in \mathbb{C}_n \backslash \{0\}$. There is an isomorphism,*

$$\hom_{\mathfrak{b}}(\mathbb{C}_n, V) \xrightarrow{\sim} V_n^{hw}, \tag{1.1}$$

given by $f \mapsto f(c)$.

Corollary I.7. *For $n \geq 0$, there is an isomorphism $\mathscr{D}_\alpha(\mathbb{C}_n) \cong L(n)$. For $m \leq 0$ there is an isomorphism $\mathscr{D}^*_\alpha(\mathbb{C}_m) \cong L(-m)$.*

Lemma I.8. *As* $\mathfrak{sl}_2$−fmod *is a semisimple category with irreducible objects $\{L(n)\}_{n \geq 0}$ there are natural isomorphisms*

$$\begin{aligned} \mathscr{D}_\alpha(V) &\cong \bigoplus_{n \geq 0} L(n) \otimes_{\mathbb{C}} \hom_{\mathfrak{b}}(V, L(n))^*, \\ \mathscr{D}^*_\alpha(V) &\cong \bigoplus_{n \geq 0} L(n) \otimes_{\mathbb{C}} \hom_{\mathfrak{b}}(L(n), V). \end{aligned}$$

1.5 Fully faithful Lemma

Let $\mathfrak{g}$ be an arbitrary complex reductive Lie algebra with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ Cartan and Borel subalgebras. Recall that given $\alpha \in \Pi$, the subalgebra $\mathfrak{s}_\alpha \subset \mathfrak{g}_\alpha$ is the reductive subalgebra of semisimple rank 1 generated by $\mathfrak{h}$ and the generators E_α, F_α. Also, $\mathfrak{b}_\alpha = \mathfrak{b} \cap \mathfrak{s}_\alpha$, is a Borel subalgebra of $\mathfrak{s}_\alpha$, and $\mathfrak{n}_{(\alpha)} \subset \mathfrak{g}_\alpha$ is the nilpotent radical of $\mathfrak{g}_\alpha$. Given $W \in \mathfrak{b}-$fmod we may restrict W to $\mathfrak{b}_\alpha-$fmod. Denote by $\overline{\mathscr{D}_\alpha}$ the left adjoint of the restriction functor $\mathrm{res} : \mathfrak{s}_\alpha-\mathrm{fmod} \to \mathfrak{b}_\alpha-\mathrm{fmod}$, so that $\overline{\mathscr{D}_\alpha}(W) \in \mathfrak{s}_\alpha-$fmod, and denote by $\overline{eta}$ the unit of the adjunction. The $\mathfrak{b}$-module morphism $\eta : W \to \mathscr{D}_\alpha(W)$ gives by the universal mapping property of $\overline{\mathscr{D}_\alpha}$ a $\mathfrak{s}_\alpha$-module morphism, $y \to \mathscr{D}_\alpha(W)$ making the following diagram commute:

$$\begin{array}{ccc} W & \xrightarrow{\overline{\eta}} & \overline{\mathscr{D}_\alpha}(W) \\ & \searrow^{\eta} & \downarrow y \\ & & \mathscr{D}_\alpha(W). \end{array}$$

Lemma I.9. *The morphism $y : \overline{\mathscr{D}_\alpha}(W) \to \mathscr{D}_\alpha(W)$ is an isomorphism.*

Proof. As $\mathfrak{s}_\alpha-$fmod is semisimple the morphism y is split. The module $\mathscr{D}_\alpha(W)$ is generated by the image of η as a $\mathfrak{s}_\alpha$-module (using the fact that $\mathfrak{n}_{(\alpha)}$ is an ideal), thus the morphism y is an isomorphism.

□

Claim I.10. *Let (η, ε) be the unit and counit of the adjunction $\mathscr{D}_\alpha \vdash \mathrm{res}$ as in Section 1.3. Let $V \in \mathfrak{g}-\mathrm{fmod}$ and consider the counit morphism*

$$\mathscr{D}_\alpha \operatorname{res}(V) \xrightarrow{\varepsilon} V.$$

This morphism is an isomorphism with inverse given by the counit morphism $\eta : \mathrm{res}(V) \to \mathrm{res}\, \mathscr{D}_\alpha \operatorname{res}(V)$, which is a $\mathfrak{g}_\alpha$-module morphism and not just a $\mathfrak{b}$-morphism. Analogously consider the counit morphism

$$V \xrightarrow{\eta} \mathscr{D}^*_\alpha \operatorname{res}(V).$$

This morphism is an isomorphism with inverse given by the unit morphism $\eta : \operatorname{res} \mathscr{D}_\alpha^* \operatorname{res}(V) \to \operatorname{res}(V)$*, which is a* $\mathfrak{g}_\alpha$*-module morphism and not just a* $\mathfrak{b}$*-morphism.*

Proof. First, suppose that $\mathfrak{g} = \mathfrak{sl}_2$. As $\mathfrak{sl}_2-$fmod is a semisimple category it suffices to prove that for each simple module $L(n)$, $n \in \mathbb{Z}_{\geq 0}$ the morphism

$$\mathscr{D}_\alpha \operatorname{res} L(n) \xrightarrow{\varepsilon} L(n),$$

is an isomorphism. This follows from the fact that,

$$\hom_{\mathfrak{b}}(L(n), L(m)) \cong \begin{cases} \mathbb{C} & n = m, \\ 0 & n \neq m \end{cases}$$

and Lemma I.8.

It follows that η is an isomorphism if and only if $\overline{\eta}$ is an isomorphism. The claim then follows from the case of $\mathfrak{g} = \mathfrak{sl}_2$.

One can prove the analogous claim for $\mathscr{D}_\alpha^*$ similarly. □

Remark I.11. If one abuses the notation and drops res from notation notice the above Claim shows that $\mathscr{D}_\alpha$ applied to the $\mathfrak{b}$-module $\mathscr{D}_\alpha(W)$ for $W \in \mathfrak{b}-$fmod is isomorphic to $\mathscr{D}_\alpha(W)$. Thus, ${\mathscr{D}_\alpha}^2(W) \cong \mathscr{D}_\alpha(W)$. It follows that the endofunctor $\mathscr{D}_\alpha : \mathfrak{b}-\text{fmod} \to \mathfrak{b}-\text{fmod}$ is an idempotent functor. Similarly, $\mathscr{D}_\alpha^*$ is an idempotent endofunctor in the same sense.

Corollary I.12. *The restriction functor* $\operatorname{res} : \mathfrak{g}_\alpha \to \mathfrak{b}$ *is fully faithful. Given* $V, W \in \mathfrak{g}_\alpha-$fmod*:*

$$\hom_{\mathfrak{g}_\alpha}(V, W) \cong \hom_{\mathfrak{b}}(V, W).$$

Proof. From the adjunction there is an isomorphism

$$\hom_{\mathfrak{b}}(V, W) \cong \hom_{\mathfrak{g}_\alpha}(V, \mathscr{D}_\alpha^*(W)).$$

As $\mathscr{D}_\alpha^*(W) \cong W$ the corollary is proved. □

Remark I.13. It is classical that the above corollary is equivalent to the fact that $\mathfrak{D}^*$ is a localization functor [GZ67, Prop. 1.3, pp.7]. This fits in nicely with the fact that the traditional definition of Zuckerman functors is via localization [MS07].

1.6 Simple modules

Let $\mathfrak{g}$ be a finite complex simple Lie algebra. For each non-negative weight $\lambda \in X^+$ there is unique finite dimensional irreducible representation $L(\lambda)$ with highest weight λ. Let $V \in \mathfrak{g}-$fmod. Let $e \in L(\lambda)_\lambda$ be non-zero. The following fact is well known,

$$\hom_{\mathfrak{g}}(L(\lambda), V) \xrightarrow{\sim} V_\lambda^{hw},$$
$$f \mapsto f(e).$$

Let $\mathbb{C}_\lambda$ be the one dimensional representation of $\mathfrak{b}$ with weight λ and on which E_α acts by zero for each $\alpha \in \Pi$. There is also an isomorphism

$$\hom_{\mathfrak{b}}(\mathbb{C}_\lambda, V) \cong V_\lambda^{hw}.$$

Let $s_1, s_2, \cdots \in W$ be a sequence of simple reflections associated to $\alpha_1, \alpha_2, \cdots \in \Pi$. Let $\mathscr{D}_1, \mathscr{D}_2, \cdots$ be the functors associated to $\alpha_1, \alpha_2, \ldots$, but considered as functors from $\mathfrak{b}-$fmod $\to \mathfrak{b}-$fmod via restriction.

By the above remarks, the adjunction $\mathscr{D}_\alpha \vdash$ res and repeated applications of Proposition I.12,

$$\begin{aligned} V_\lambda^{hw} &\cong \hom_{\mathfrak{b}}(\mathbb{C}_\lambda, V), \\ &\cong \hom_{\mathfrak{g}_{\alpha_n}}(\mathscr{D}_n(\mathbb{C}_\lambda), V), \\ &\cong \hom_{\mathfrak{b}}(\mathscr{D}_n(\mathbb{C}_\lambda), V), \\ &\cong \hom_{\mathfrak{g}_{\alpha_{n-1}}}(\mathscr{D}_{n-1}\,\mathscr{D}_n(\mathbb{C}_\lambda), V), \\ &\cong \hom_{\mathfrak{b}}(\mathscr{D}_{n-2}\ldots\mathscr{D}_n(\mathbb{C}_\lambda), V), \\ &\;\vdots \end{aligned}$$

Given $c \in \mathbb{C}_\lambda \backslash \{0\}$ and $e \in L(\lambda)_\lambda \backslash \{0\}$ there is a nonzero $\mathfrak{b}$-morphism $\mathbb{C}_\lambda \to L(\lambda)$ sending $c \mapsto e$. The above formula gives for each k a $\mathfrak{b}$-morphism $\mathscr{D}_k \cdots \mathscr{D}_n(\mathbb{C}_\lambda) \to L(\lambda)$.

If the $\mathfrak{b}$-module $\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda)$ has the structure of a $\mathfrak{g}$-module then

$$hom_{\mathfrak{b}}(\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda), V) \cong hom_{\mathfrak{g}}(\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda), V)$$

We could thus Identify the functors $\hom_{\mathfrak{g}}(L(\lambda), V) \cong V_\lambda^{hw}$ and $\hom_{\mathfrak{g}}(\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda), V) \cong \hom_{\mathfrak{b}}(\mathbb{C}_\lambda, V) \cong V_\lambda^{hw}$. By Yoneda's lemma, we could conclude the $\mathfrak{b}$-morphism $\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda) \to L(\lambda)$ is an isomorphism. This is summarized in the following Proposition.

Proposition I.14. *If the $\mathfrak{b}$-module $\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda)$ has the structure of a $\mathfrak{g}$-module then there is an isomorphism $\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda) \to L(\lambda)$.*

Proposition I.15. *Let $\lambda \in X^+$ and let $w_0 \in W$ be the longest element of the Weyl group, $w_0 = s_1 s_2 \dots s_n$ a reduced decomposition. Then the $\mathfrak{b}$-morphism constructed before the previous proposition,*

$$\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda) \to L(\lambda),$$

is surjective.

Proof. Let $e_n \in L(\lambda)_\lambda$ be the image of $1 \in \mathbb{C}_\lambda$ under a non-zero $\mathfrak{b}$-morphism $\mathbb{C}_\lambda \to L(\lambda)$. Let $e_{n-1} \in L(\lambda)_{s_n(\lambda)}$ be the element

$$e_{n-1} = F^{\langle \lambda, \check{\alpha_n} \rangle} e_n.$$

Inductively define e_i in the extremal weight space $L(\lambda)_{s_{i+1} \dots s_n(\lambda)}$ by

$$e_i = F^{\langle s_{i+2} \dots s_n(\lambda), \check{\alpha}_{i+1} \rangle} e_{i+1}.$$

For each i there is a surjection

$$\mathscr{D}_i \dots \mathscr{D}_n(\mathbb{C}_\lambda) \to U(\mathfrak{g}_{\alpha_i}) U(\mathfrak{b}) e_{i+1} \subset L(\lambda),$$

given by the canonical morphism from the previous proposition. As $e_0 \in L(\lambda)_{w_0(\lambda)}$ generates $L(\lambda)$ as a $\mathfrak{b}$-module, we find that the canonical morphism

$$\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda) \to L(\lambda),$$

is surjective. □

Proposition I.16. *Keep the setup of the previous proposition. The canonical morphism*

$$\mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n(\mathbb{C}_\lambda) \to L(\lambda),$$

is an isomorphism.

Proof. See [Jos85]. A similar result holds for $\mathscr{D}_\alpha^*$ with $\lambda \in -X^+$. This is a categorification of part of the Demazure character formula, Proposition I.2. □

1.7 Characters

Recall $\mathbb{Z}[X] = \mathbb{Z}[e^x]_{x \in X}/(e^x e^y = e^{x+y})$. Given $V \in \mathfrak{h}-$fmod semisimple with $V_\lambda = 0, \lambda \notin X$, its character is defined by

$$\mathrm{char}(V) = \sum_{x \in X} dim(V_x) e^x.$$

Lemma I.17. *Let $V \in \mathfrak{b}-$fmod and let $\overline{V} = im(V \xrightarrow{\eta} \mathrm{res}\, \mathscr{D}_\alpha(V))$ be the image of the unit η. The following formula gives the character of $\mathscr{D}_\alpha(V)$:*

$$\mathrm{char}\, \mathscr{D}_\alpha(V) = \Delta_\alpha\, \mathrm{char}(\overline{V}).$$

Similarly, let $\hat{V}$ be the image of the counit $\mathrm{res}\, \mathscr{D}_\alpha^*(V) \xrightarrow{\varepsilon} V$. The following formula gives the character of $\mathscr{D}_\alpha^*(V)$:

$$\mathrm{char}\, \mathscr{D}_\alpha^*(V) = \Delta_\alpha^*\, \mathrm{char}(\hat{V}).$$

Proof. To begin with, let $V \xrightarrow{p} \overline{V} \xrightarrow{i} \mathrm{res}\, \mathscr{D}_\alpha(V)$ be the projection and inclusion morphisms. Then $\mathscr{D}_\alpha(p) : \mathscr{D}_\alpha(V) \to \mathscr{D}_\alpha(\overline{V})$ is an isomorphism with inverse given by the composition,

$\mathscr{D}_\alpha \overline{V} \xrightarrow{\mathscr{D}_\alpha(i)} \mathscr{D}_\alpha \operatorname{res} \mathscr{D}_\alpha(V) \xrightarrow{\varepsilon \mathscr{D}_\alpha} \mathscr{D}_\alpha(V)$. Analogously for $j : \hat{V} \to V$, there is an isomorphism $\mathscr{D}_\alpha(\hat{V}) \xrightarrow{\mathscr{D}_\alpha(j)} \mathscr{D}_\alpha(V)$. It follows that to determine the character of $\mathscr{D}_\alpha(V)$ we can assume that V is a $\mathfrak{b}$-submodule of a $\mathfrak{g}_\alpha$-module, and for $\mathscr{D}^*_\alpha(V)$ we can assume that V is a $\mathfrak{b}$-quotient module of a $\mathfrak{g}_\alpha$-module.

By Lemma I.9 it will suffice to show the claims for $\mathfrak{g} = \mathfrak{sl}_2$. In that case $\mathfrak{b}$ is a 2-dimensional Lie algebra, $\mathfrak{b} = H\mathbb{C} \oplus E\mathbb{C}$ with commutativity relation $[H, E] = 2E$. By Jordan normal form for the nilpotent operator E, every finite $\mathfrak{b}$-module is a direct sum of string modules $S(n, m)$, see Section 1.4. If $S(n, m)$ is a submodule of a $\mathfrak{g}$-module then $0 \leq m$ and $-m \leq n$. If $S(n, m)$ is a quotient of a $\mathfrak{g}$-module then $n \leq 0$ and $m \leq -n$.

By Section 1.4 there are natural isomorphisms:

$$\mathscr{D}_\alpha(V) \cong \bigoplus_{n \geq 0} L(n) \otimes_{\mathbb{C}} \hom_{\mathfrak{b}}(V, L(n))^*,$$

$$\mathscr{D}^*_\alpha(V) \cong \bigoplus_{n \geq 0} L(n) \otimes_{\mathbb{C}} \hom_{\mathfrak{b}}(L(n), V).$$

Thus the characters are given by the precursory character formulas:

$$\operatorname{char} \mathscr{D}_\alpha(V) = \sum_{n \geq 0} dim_{\mathbb{C}} \hom_{\mathfrak{b}}(V, L(n)) \cdot \operatorname{char} L(n),$$

$$\operatorname{char} \mathscr{D}_\alpha(V) = \sum_{n \geq 0} dim_{\mathbb{C}} \hom_{\mathfrak{b}}(L(n), V) \cdot \operatorname{char} L(n).$$

By the preliminary remarks it suffices to prove the first character formula for string modules $V = S(n, m)$ with $0 \leq m$ and $-m \leq n$. In that case

$$dim_{\mathbb{C}} \hom_{\mathfrak{b}}(S(n, m), L(\lambda)) = \begin{cases} 1, & |n| \leq \lambda \leq m, \\ 0, & \text{else.} \end{cases}$$

It is clear that:

$$\operatorname{char} L(\lambda) = \Delta_\alpha(e^\lambda),$$

and by the idempotence of Δ_α,

$$\operatorname{char} L(\lambda) = \Delta_\alpha \operatorname{char} L(\lambda).$$

For $n \geq 0$ the contribution of each $L(\lambda)$ to the precursory character formula is equal to $\Delta_\alpha(e^\lambda)$. Thus,

$$\begin{aligned}\operatorname{char} \mathscr{D}_\alpha(S(n,m)) &= \sum_{-n\leq\lambda\leq m} \Delta_\alpha(e^\lambda), \\ &= \Delta_\alpha(\operatorname{char} S(n,m)).\end{aligned}$$

For $n \leq 0$ the contribution of $L(-n)$ to the precursory character formula is equal to $e^n + e^{n+2} + \cdots + e^{-n} = \Delta_\alpha(e^n + e^{n+2} + \cdots + e^{-n})$. The contribution of $L(\lambda)$ for $n < \lambda \leq m$ is equal to $\Delta_\alpha(e^\lambda)$. All told,

$$\begin{aligned}\operatorname{char} \mathscr{D}_\alpha(S(n,m)) &= \Delta_\alpha(e^n + e^{n+2} + \cdots + e^{-n}) + \sum_{-n<\lambda\leq m} \Delta_\alpha(e^\lambda), \\ &= \Delta_\alpha(\operatorname{char} S(n,m)).\end{aligned}$$

The proof for $\mathscr{D}^*_\alpha(S(n,m))$ when $n \leq 0$ and $m \leq -n$ is similar, using the formula $L(\lambda) = \Delta^*_\alpha(e^{-\lambda})$. □

The following lemma is needed for the next chapter.

Lemma I.18. *Let K be a $\mathfrak{b}$-module. Then $\mathscr{D}^*_\alpha(K) = 0$ if and only if for every $\lambda \in X$ with $-n = \langle \lambda, \check{\alpha} \rangle \leq 0$, the morphism*

$$E^{n+1}_\alpha : K_\lambda \to K_{s_\alpha(\lambda)+\alpha}$$

is injective.

Proof. It suffices to show the claim for $\mathfrak{sl}_2$. As $L(n)$ is a cyclic $\mathfrak{b}$-module generated by $L(n)_{-n}$ such that $E^{n+1} : L(n)_{-n} \to L(n)_{n+1}$ is zero, a $\mathfrak{b}$-morphism $L(n) \to K$ is given by a linear morphism $\mathbb{C} \cong L(n)_{-n} \to \ker(E^{n+1} : K_{-n} \to K_{n+1})$. There are no such morphisms precisely when $E^{n+1} : K_{-n} \to K_{n+1}$ is injective. □

Lemma I.19. *Let $K, V \in \mathfrak{b}-\mathrm{fmod}$ and $M \in \mathfrak{g}_\alpha-\mathrm{fmod}$ and suppose there is an exact sequence:*

$$0 \to K \to \operatorname{res} M \xrightarrow{p} V \to 0.$$

By the adjunction $\mathrm{res} \vdash \mathscr{D}_\alpha^*$ *there is an associated* $\mathfrak{g}_\alpha$*-module morphism*

$$M \xrightarrow{q} \mathscr{D}_\alpha^*(V).$$

Then q *is an isomorphism if and only if for every* $\lambda \in X$ *with* $-n = \langle \lambda, \check{\alpha} \rangle \leq 0$*, the morphism*

$$E^{n+1} : K_\lambda \to K_{s_\alpha(\lambda)+\alpha}$$

is an isomorphism.

Proof. As $\mathscr{D}_\alpha^*$ has a left adjoint it is left exact. Thus q is injective if and only if $\mathscr{D}_\alpha^*(K) \cong 0$. Suppose $M \cong \mathscr{D}_\alpha^*(V)$. Then by Lemma I.17, $\mathrm{char}\, M = \Delta_\alpha^*(\mathrm{char}\, V)$. As M is an extension of V by K the following equality holds:

$$\begin{aligned} \mathrm{char}\, K &= \mathrm{char}\, M - \mathrm{char}\, V, \\ &= \Delta_\alpha^*(\mathrm{char}\, V) - \mathrm{char}\, V. \end{aligned}$$

As ${\Delta_\alpha^*}^2 = \Delta_\alpha^*$ it follows, $\Delta_\alpha^*(\mathrm{char}\, K) = 0$, and hence $s_\alpha \cdot \mathrm{char}\, K = \mathrm{char}\, K$. Let $\lambda \in X$ with $-n = \langle x, \check{\alpha} \rangle \leq 0$. As $\mathscr{D}_\alpha^*(K) = 0$ the previous lemma shows that $E_\alpha^{n+1} : K_\lambda \to K_{s_\alpha \cdot \lambda}$ is injective. But $s_\alpha \cdot \mathrm{char}\, K = \mathrm{char}\, K$ so the spaces $K_\lambda, K_{s_\alpha \cdot \lambda}$ have the same dimension and E_α^{n+1} must also be surjective.

Conversely, suppose that each $E_\alpha^{n+1} : K_\lambda \to K_{s_\alpha \cdot \lambda}$ is an isomorphism.

By the previous lemma and the injectivity of $E_\alpha^{n+1} : K_\lambda \to K_{s_\alpha \cdot \lambda}$ it follows $\mathscr{D}_\alpha^*(K) \cong 0$. As M is a $\mathfrak{g}_\alpha$-module, $\Delta_\alpha^*(\mathrm{char}\, M) = \mathrm{char}\, M$. Finally, the isomorphism $E_\alpha^{n+1} : K_\lambda \to K_{s_\alpha \cdot \lambda}$ shows that $s_\alpha \cdot \mathrm{char}\, K = \mathrm{char}\, K$, from which it follows

$$\begin{aligned} 0 &= \Delta_\alpha^* \,\mathrm{char}\, K, \\ &= \Delta_\alpha^*(\mathrm{char}\, M - \mathrm{char}\, V), \\ &= \mathrm{char}\, M - \Delta_\alpha^*(\mathrm{char}\, V). \end{aligned}$$

Lemma I.17 gives $\mathrm{char}\, \mathscr{D}_\alpha^*(V) = \Delta_\alpha^* \,\mathrm{char}\, V$, so that M and $\mathscr{D}_\alpha^*(V)$ have the same character. As $q : M \to \mathscr{D}_\alpha^*(V)$ is injective and they both have the same characters, q must be an isomorphism.

□

Lemma I.20. *Suppose $K \in \mathfrak{b}-\mathrm{fmod}$ and $\mathscr{D}_\alpha^*(K) \cong 0$. Let K^{hw} be the kernel of E_α. Then to show that for each $\lambda \in X$, $-n = \langle \lambda, \check{\alpha} \rangle \leq 0$ the morphism*

$$E_\alpha^{n+1} : K_\lambda \to K_{s_\alpha \cdot (\lambda)},$$

is an isomorphism, it suffices to show that for each such λ, n that

$$K^{hw}_{s_\alpha \cdot \lambda} \subset E_\alpha^{n+1}(K_\lambda)$$

Proof. It suffices to show the claim for $\mathfrak{g} = \mathfrak{sl}_2$. First, break K into string modules, and let $S(-n, m)$, $-n \leq m$ be one of the string submodules. It cannot be the case that $m \leq 0$ as then there would be a non-zero morphism $L(n) \to S(-n, m)$, which contradicts that $\mathscr{D}_\alpha^*(K) \cong 0$. Thus $m > 0$. Let $y \in S(-n, m)$ be non-zero. If $K_m^{hw} \subset E^{m+1}(K_{-m+2})$ then because $E^{m+1} : K_{-m+2} \to K_m$ is also injective there is a *unique* $x \in K_{-m+2}$ with $E^{m+1}x = y$. As K is the direct sum of such string modules it cannot be that $S(-n, m)_{-m+2} \cong 0$ as then the projection of x onto $S(-n, m)$ would be zero, but $E^{m+1}x = y \in S(-n, m)$. It follows that $-n \leq -m + 2 \leq 1$. If $-n = m = 1$ we are done. Otherwise, as $E^{n+1} : S(-n, m)_{-n} \to S(-n, m)_{n-2}$ is injective, it must be that $n - 2 \leq m$. Hence $-n = -m + 2$, and $S(-n, m) = S(-m + 2, m)$ for which the claim is obvious. As this is the case for each string submodule of K, the claim follows.

□

1.8 Braid relations

The following two claims have no categorical analogue yet, so we include them without proof.

Claim I.21. *Let $V \in \mathfrak{b}-\mathrm{fmod}$. Then V has a unique $\mathfrak{g}_\alpha$-module structure if and only if for each $\alpha \in \Pi$, the natural morphism $\mathscr{D}_\alpha^*(V) \to V$ is an isomorphism.*

Proof. See [Jos85, Lemma 2.16]. □

Claim I.22. *Let* $\alpha, \beta \in \Pi$ *and put* $m_{\alpha,\beta}$ *the order of* $s_\alpha s_\beta$ *in* W*. There is an isomorphism:*

$$\mathscr{D}_\alpha^* \mathscr{D}_\beta^* \ldots \cong \mathscr{D}_\beta^* \mathscr{D}_\alpha^* \ldots,$$

where both sides have $m_{\alpha,\beta}$ *terms. Thus, the functors* $\mathscr{D}_\alpha^*$ *give a categorification of the Hecke algebra of* W *having quadratic relations which make the standard generators idempotent.*

Proof. See [Jos85, Proposition 2.15]. □

CHAPTER 2

Background on 2-Lie Algebras

2.1 Monoidal categories and adjunctions

Given a monoidal category $\mathcal{M}, \otimes$, we will use juxtaposition for the tensor product of both objects and morphisms. Thus, if $X, X', Y, Y' \in \mathcal{M}$ are objects and $x : X \to X', y : Y \to Y'$ are morphisms then,

$$xy : XY \to X'Y'.$$

We use $\circ$ to denote the composition of morphisms in $\mathcal{M}$, so that if $a : X \to Y$ and $b : Y \to Z$ then,

$$b \circ a : X \to Z.$$

We use X to both denote the object X and the identity morphism $Id : X \to X$. Thus, for $x \in \mathrm{End}(X)$,

$$X \xrightarrow{x} X \xrightarrow{X} X = X \xrightarrow{x} X.$$

If $X \in \mathcal{M}$, a right adjoint to X is the data of an object $Y \in \mathcal{M}$ along with two morphisms,

$$\varepsilon : XY \to 1,$$
$$\eta : 1 \to YX,$$

satisfying the zig-zag equalities, i.e. the following diagrams commute :

$$\begin{array}{ccc} X & \xrightarrow{X\eta} & XYX \\ & {\scriptstyle X}\searrow & \downarrow{\scriptstyle \varepsilon X} \\ & & X, \end{array}$$

$$\begin{array}{ccc} Y & \xrightarrow{\eta Y} & YXY \\ & {\scriptstyle Y}\searrow & \downarrow{\scriptstyle Y\varepsilon} \\ & & Y. \end{array}$$

It is also said that X is left adjoint to Y and is sometimes denoted $(\varepsilon, \eta) : X \vdash Y$.

2.2 Half Lie algebras

Retain the notation of Chapter 1. This section reproduces the definition of $\mathfrak{A}$ used throughout Chapter 3. It is based on [Rou11].

Let $\Pi \subset R$ be the simple roots, and put $C = (a_{\alpha,\beta})_{\alpha,\beta\in\Pi}$ the Cartan matrix,

$$a_{\alpha,\beta} = \langle \beta, \check{\alpha} \rangle.$$

Put $m_{\alpha,\beta} = -a_{\alpha,\beta}$. Let $t_{\alpha,\beta,s,r}$ be a family of indeterminates with $0 \leq r < m_{\alpha,\beta}$ and $0 \leq s < m_{\beta,\alpha}$ for $\alpha \neq \beta$ and such that $t_{\beta,\alpha,s,r} = t_{\alpha,\beta,r,s}$. Let $\{t_{\alpha,\beta}\}_{\alpha\neq\beta}$ be another family of indeterminates with $t_{\alpha,\beta} = t_{\beta,\alpha}$ if $a_{\alpha,\beta} = 0$.

Let $\boldsymbol{k} = \boldsymbol{k}_C = \mathbb{Z}[\{t_{\alpha,\beta,r,s}\} \cup \{t_{\alpha,\beta}^{\pm 1}\}]$. Define polynomials $Q_{\alpha,\beta} \in \boldsymbol{k}[u,v]$ by $Q_{\alpha,\alpha} = 0, Q_{\alpha,\beta} = t_{\alpha,\beta}$ if $a_{\alpha,\beta} = 0$ and,

$$Q_{\alpha,\beta} = t_{\alpha,\beta} u^{m_{\alpha,\beta}} + \left(\sum_{\substack{0 \leq r < m_{\alpha,\beta} \\ 0 \leq s < m_{\beta,\alpha}}} t_{\alpha,\beta,r,s} u^r v^s \right) + t_{\beta,\alpha} v^{m_{\beta,\alpha}},$$

for $a_{\alpha,\beta} < 0$.

Definition. Let $\mathcal{B} = \mathcal{B}(C)$ be the free strict monoidal $\boldsymbol{k}$-linear category generated by objects E_α, for $\alpha \in \Pi$ and arrows,

$$x_\alpha : E_\alpha \to E_\alpha,$$
$$\tau_{\alpha,\beta} : E_\alpha E_\beta \to E_\beta E_\alpha,$$

for each $\alpha, \beta \in \Pi$ with relations,

1. $\tau_{\alpha,\beta}\tau_{\beta,\alpha} = Q_{\alpha,\beta}(E_\beta x_\alpha, x_\beta E_\alpha)$

2. $\tau_{\beta,\gamma}E_\alpha \circ E_\beta\tau_{\alpha,\gamma} \circ \tau_{\alpha,\beta}E_\gamma - E_\gamma\tau_{\alpha,\beta} \circ \tau_{\alpha,\gamma}E_\alpha \circ \tau_{\beta,\gamma} =$

$$\begin{cases} \dfrac{Q_{\alpha,\beta}(x_\alpha E_\beta, E_\alpha x_\beta)E_\alpha - E_\alpha Q_{\alpha,\beta}(E_\beta x_\alpha, x_\beta E_\alpha)}{x_\alpha E_\beta E_\alpha - E_\alpha E_\beta x_\alpha} E_\alpha & \text{if } \alpha = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

3. $\tau_{\alpha,\beta} \circ x_\alpha E_\beta - E_\alpha x_\beta \circ \tau_{\alpha,\beta} = \delta_{\alpha,\beta}$

4. $\tau_{\alpha,\beta} \circ E_\alpha x_\beta - x_\alpha E_\beta \circ \tau_{\alpha,\beta} = -\delta_{\alpha,\beta}$

For each $\alpha \in \Pi$ and $n \in \mathbb{Z}_{\geq 0}$ there is a chosen idempotent $b_n : E_\alpha^n \to E_\alpha^n$. Let $E_\alpha^{(n)}$ be the image of the idempotent b_n in the idempotent completion $\mathcal{B}^i$ of $\mathcal{B}$. There is an isomorphism $n! \cdot E_\alpha^{(n)} \cong E_\alpha^n$ in $\mathcal{B}^i$.

2.3 2-Lie algebras

Let $\mathcal{B}_1$ be the strict monoidal $\boldsymbol{k}$-linear category obtained from $\mathcal{B}$ by adding F_α right dual to E_α for every $\alpha \in \Pi$. Denote by

$$\varepsilon_\alpha : E_\alpha F_\alpha \to 1,$$

$$\eta_\alpha : 1 \to F_\alpha E_\alpha,$$

the counit and unit of the adjunctions.

Consider the strict 2-category $\mathfrak{A}_1$ with set of objects X and $\hom(\lambda, \lambda')$ the full subcategory of $\mathcal{B}_1$ generated by direct sums of products of E_α, F_α whose $E_\bullet$-term subscripts, summed, minus the $F_\bullet$-term subscripts, summed, give $\lambda' - \lambda$. As a notation we will write $E_\alpha 1_\lambda$ to denote the 1-arrow $E_\alpha : \lambda \to \lambda + \alpha$ in $\mathfrak{A}_1$.

Definition. Let $\mathfrak{A}$ be the strict $\boldsymbol{k}$-linear 2-category deduced from $\mathfrak{A}_1$ by inverting the following 2-arrows:

- when $\langle \lambda, \check{\alpha} \rangle \geq 0$,

$$\rho_{\alpha,\lambda} = \sigma_{\alpha,\alpha} + \sum_{i=0}^{\langle \lambda, \check{\alpha} \rangle - 1} \varepsilon_\alpha \circ (x_\alpha^i F_\alpha) : E_\alpha F_\alpha 1_\lambda \to F_\alpha E_\alpha 1_\lambda \oplus 1_\lambda^{\langle \lambda, \check{\alpha} \rangle}$$

- when $\langle \lambda, \check{\alpha} \rangle \leq 0$,

$$\rho_{\alpha,\lambda} = \sigma_{\alpha,\alpha} + \sum_{i=0}^{-\langle \lambda, \check{\alpha} \rangle - 1} (F_\alpha x_\alpha^i) \circ \eta_\alpha : E_\alpha F_\alpha 1_\lambda \oplus 1_\lambda^{-\langle \lambda, \check{\alpha} \rangle} \to F_\alpha E_\alpha 1_\lambda$$

- $\sigma_{\alpha,\beta} : E_\alpha F_\beta 1_\lambda \to F_\beta E_\alpha 1_\lambda$ for all $\alpha \neq \beta$ and all λ, where

$$\sigma_{\alpha,\beta} = (F_\beta E_\alpha \varepsilon_\beta) \circ (F_\beta \tau_{\beta,\alpha} F_\alpha) \circ (\eta_\beta E_\alpha F_\beta) : E_\alpha F_\beta \to F_\beta E_\alpha.$$

2.4 Representations of 2-Lie algebras

Let k be an algebraically closed field. A representation of $\mathfrak{A}$ on a k-linear category $\mathcal{V}$ is the data of:

- A morphism $\boldsymbol{k} \to k$
- a k-linear category $\mathcal{V}_\lambda$ for every $\lambda \in X$
- functors $E_\alpha : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+\alpha}$ and $F_\alpha : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha}$ along with an adjunction $(\varepsilon_\alpha, \eta_\alpha) : E_\alpha \vdash F_\alpha$.
- natural transformations $x_\alpha : E_\alpha \to E_\alpha$ and $\tau_{\alpha,\beta} : E_\alpha E_\beta \to E_\beta E_\alpha$

such that the relations of Definition 2.2 hold and the morphisms of Definition 2.3 are invertible. We say that the representation is integrable if each of E_α, F_α are locally nilpotent.

Definition. Given $\lambda \in X^-$ we define an integrable, additive representation of $\mathfrak{A}$ which categorifies the simple representation $L(\lambda)$ of lowest weight λ. Let $\mathcal{L}(\lambda)$ be the additive quotient representation

$$\mathcal{L}(\lambda) = {}_\bullet\mathfrak{A}_\lambda/({}_\bullet\mathfrak{A}F_\alpha 1_\lambda).$$

The following three results are needed for Chapter 3.

Lemma I.23. *(See [Rou08, Lemma 4.14]) Let $\alpha \in \Pi$ and $m, n \in \mathbb{Z}_{\geq 0}$. Let $r = m-n+\langle \lambda, \check{\alpha} \rangle$. There is an isomorphism in $\mathfrak{A}^i$:*

$$E_\alpha^{(m)} F_\alpha^{(n)} 1_\lambda \cong \bigoplus_{i \geq 0} \binom{r}{i} F_\alpha^{(n-i)} E_\alpha^{(m-i)}$$

when $r \geq 0$ *and:*

$$F_\alpha^{(n)} E_\alpha^{(m)} 1_\lambda \cong \bigoplus_{i \geq 0} \binom{-r}{i} E_\alpha^{(m-i)} F_\alpha^{(n-i)}$$

when $r \leq 0$.

Let ${}^0H_n$ denote the *nil affine Hecke algebra* of GL_n. Put $P_i = \mathbb{Z}[X_1, \ldots, X_i]$. We denote by $H_{i,n}$ the subalgebra of ${}^0H_n$ generated by $T_1, \ldots T_{i-1}$ and $P_n^{\mathfrak{S}[i+1,n]}$.

Let $\mathfrak{g} = \mathfrak{sl}_2$. Given $n \in \mathbb{Z}, n \geq 0$, let $\tilde{\mathcal{L}}(-n)_\lambda = H_{(n+\lambda)/2,n}$−free for $\lambda \in \{-n, -n + 2, \ldots, n\}$. We define $E = \bigoplus_i^{n-1} \mathrm{Ind}_{H_{i,n}}^{H_{i+1,n}}$ and $F = \bigoplus_{i=0}^{n-1} \mathrm{Res}_{H_{i,n}}^{H_{i+1,n}}$. There is a canonical adjunction $E \vdash F$. Multiplication by X_{i+1} gives an endomorphism of each $\mathrm{Ind}_{H_{i,n}}^{H_{i+1,n}}$ and taking the sum over all i gives an endomorphism x of E. Similarly, multiplication by T_{i+1} gives an endomorphism of $\mathrm{Ind}_{H_{i,n}}^{H_{i+2,n}}$ which gives an endomorphism τ of E^2. This gives the data of a representation of $\mathfrak{A}$ on $\tilde{\mathcal{L}}(-n) = \bigoplus_\lambda \tilde{\mathcal{L}}(-n)_\lambda$.

Proposition I.24. *([Rou08, Proposition 5.15]) There is a canonical isomorphism of -representations of* $\mathfrak{A}$*,*

$$\mathcal{L}(-n) \xrightarrow{\sim} \tilde{\mathcal{L}}(-n)$$

Lemma I.25. *(See [CR08, Remark 5.25]) Let* k *be a base field. Let* $\mathfrak{g} = \mathfrak{sl}_2$ *and let* $\mathcal{V}$ *be an integrable representation of* $\mathfrak{A}$ *on an abelian, Krull-Schmidt category with the property that for any simple object* S *of* $\mathcal{V}$ *the endomorphism ring of* S *is* k*. Let* I_λ *be the set of isomorphism classes of simple objects* $U \in \mathcal{V}_\lambda$ *such that* $F(U) \cong 0$*. There is an isomorphism*

$$\bigoplus_{\substack{\lambda \in X^+ \\ U \in I_\lambda}} \mathbb{Q} \otimes K_0(\mathcal{L}(\lambda)) \xrightarrow{\sim} \mathbb{Q} \otimes K_0(\mathcal{V}),$$

giving a canonical decomposition of $\mathbb{Q} \otimes K_0(\mathcal{V})$ *into simple summands.*

CHAPTER 3

Categorical Demazure Operators

3.1 Definitions

Definition. The 2-category $\mathfrak{B}$ is the strict $\boldsymbol{k}$-linear 2-category with set of objects X, and for which $\hom_{\mathfrak{B}}(\lambda, \lambda')$ is the full subcategory of $\mathcal{B}$ given by direct sums of products of E_α's the sum of whose subscripts is $\lambda' - \lambda$. As notation, we write $E_\alpha 1_\lambda$ for the 1-arrow $E_\alpha : \lambda \to \lambda + \alpha'$

See Chapter 2 for background, notation and general definitions pertaining to 2-Lie algebras used in this chapter. Recall $\mathfrak{B}$ from Definition 3.1 is the 2-category with objects X and 1-morphisms generated only by E_α, and not F_α. The 2-morphisms of $\mathfrak{B}$ are those coming from the category $\mathcal{B}$, so that $\mathfrak{B}$ is not a full subcategory of $\mathfrak{A}$. This is analogous to the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. For each $\alpha \in \Pi$ let $\mathfrak{A}^\alpha$ be the subcategory of $\mathfrak{A}$ with objects X and 1-morphisms generated by all of the $\{E_\beta\}_{\beta\in\Pi}$ and only F_α. This is a categorification of the parabolic lie subalgebra $\mathfrak{g}_\alpha \subset \mathfrak{g}$.

In this section a 2-functor,

$$\mathfrak{D}^*_\alpha : \mathfrak{B}\text{−mod} \to \mathfrak{A}^\alpha\text{−mod}$$

will be defined which mimics coinduction using the $\mathfrak{B} - \mathfrak{A}^\alpha$ bimodule $\mathfrak{H}om_{\mathfrak{A}^\alpha}(-,-)$. The following suggestive abuse of notation will be used for the remainder of the chapter, $\mathfrak{H}om_{\mathfrak{A}^\alpha}(-,-) = {}_\bullet\mathfrak{A}^\alpha{}_\bullet$ in which the bullets are the reverse order of the dashes. For each $\mu \in X$ the right $\mathfrak{A}^\alpha$-module ${}_\mu\mathfrak{A}^\alpha{}_\bullet$ is equal to ${}_\mu\mathfrak{A}^\alpha{}_\lambda = \mathfrak{H}om(\lambda, \mu)$ in weight λ. There is a contravariant action, for $G \in \mathfrak{H}om(\lambda, \lambda')$ i.e., a functor,

$$- \cdot G : {}_\bullet\mathfrak{A}^\alpha{}_{\lambda'} \to {}_\bullet\mathfrak{A}^\alpha{}_\lambda$$

which is thought of as right multiplication. In the 2-categorical language this map comes from,

$$\mathfrak{H}om(G, -) : \mathfrak{H}om(\lambda', -) \to \mathfrak{H}om(\lambda, -).$$

Similarly, for each $\lambda \in X$ the left $\mathfrak{B}$-module ${}_\bullet\mathfrak{A}^\alpha{}_\lambda$ is equal to ${}_\mu\mathfrak{A}^\alpha{}_\lambda = \mathfrak{H}om(\lambda, \mu)$ in weight μ. There is a covariant action, for $G \in \mathfrak{H}om(\mu, \mu')$ i.e., a functor,

$$G \cdot - : {}_\mu\mathfrak{A}^\alpha{}_\bullet \to {}_{\mu'}\mathfrak{A}^\alpha{}_\bullet,$$

which is interpreted as left multiplication. In the 2-categorical language this map comes from the functor,

$$\mathfrak{Hom}(-,G) : \mathfrak{Hom}(\lambda',-) \to \mathfrak{Hom}(\lambda,-).$$

Definition. For any 2-representation $\mathcal{V}$ of $\mathfrak{B}$ define via pullback a 2-representation of $\mathfrak{A}^\alpha$ on the category of $\mathfrak{B}$-morphisms of 2-representations,

$$\begin{aligned}\mathfrak{D}^*(\mathcal{V}) &= \hom_{\mathfrak{B}}(\mathfrak{Hom}(-,-),\mathcal{V}),\\ &= \hom_{\mathfrak{B}}({}_\bullet\mathfrak{A}^\alpha{}_\bullet,\mathcal{V}).\end{aligned}$$

The λ weight space of this representation of $\mathfrak{A}^\alpha$ is given by,

$$\begin{aligned}\mathfrak{D}^*(\mathcal{V})_\lambda &= \hom_{\mathfrak{D}^*_\alpha}(\mathfrak{Hom}(\lambda,-),\mathcal{V}),\\ &= \hom_{\mathfrak{B}}({}_\bullet\mathfrak{A}^\alpha{}_\lambda,\mathcal{V}).\end{aligned}$$

The action of $G \in {}_\mu\mathfrak{A}^\alpha{}_\lambda$ on $\Sigma \in \mathfrak{D}^*(\mathcal{V})_\lambda = \hom_{\mathfrak{B}}({}_\bullet\mathfrak{A}^\alpha{}_\lambda,\mathcal{V})$ is given by

$$(G\cdot\Sigma)(-) = \Sigma(-\cdot G).$$

3.2 Adjunctions

This section gives the categorical analogues of Section 1.3. In particular it is shown that $\mathfrak{D}^*_\alpha$ may be interpreted as a right adjoint of a restriction functor in an even stronger sense than the usual adjunction between 2-functors.

Let $\mathfrak{Res} : \mathfrak{A}^\alpha{-mod} \to \mathfrak{B}{-mod}$ be the restriction 2-functor. Since $\mathfrak{D}^*_\alpha(\mathcal{W}) = \mathfrak{Hom}_{\mathfrak{B}}({}_\bullet\mathfrak{A}_\bullet,\mathcal{W})$, we rely on a categorical tensor – hom adjunction in which the tensor product, "$-\otimes_{\mathfrak{A}}\mathfrak{A} = Id(-)$" functor is trivial.

Claim I.26. *Given $\mathcal{V}$ a k-linear $\mathfrak{A}$-module and $\mathcal{W}$ a k-linear $\mathfrak{B}$-module, there is an equivalence of categories*

$$K : \hom_{\mathfrak{A}^\alpha}(\mathcal{V},\mathfrak{D}^*_\alpha(\mathcal{W})) \overset{\sim}{\leftrightarrows} \hom_{\mathfrak{B}}(\mathfrak{Res}\,\mathcal{V},\mathcal{W}) : K^{-1},$$

where

$$K(\Phi) = (\mathcal{V}_\lambda \ni v \mapsto \Phi(v)(1_\lambda),$$
$$K^{-1}(\Psi) = (v \mapsto (G \mapsto \Psi(G \cdot v))).$$

Furthermore, if $\mathcal{V}, \mathcal{W}$ are abelian categories, restricting to the subcategories of exact morphisms gives an equivalence denoted by,

$$\hom^{ex}_{\mathfrak{A}^\alpha}(\mathcal{V}, \mathfrak{D}^*_\alpha(\mathcal{W})) \xleftrightarrow{\sim} \hom^{ex}_{\mathfrak{B}}(\mathfrak{R}\mathrm{es}\,\mathcal{V}, \mathcal{W})$$

Proof. There is a standard way to make each of K, K^{-1} functorial, and one notes that with these choices they are well defined, i.e. the $\mathfrak{A}$-module map structure on Φ is mapped to a $\mathfrak{B}$-module map structure on $K(\Phi)$ and analogously for K^{-1}. It is also plain that these two functors are inverses of each other.

If $\Phi \in \hom^{ex}_{\mathfrak{A}}(\mathcal{V}, \mathfrak{D}^*_\alpha(\mathcal{W}))$ then $K(\Phi)$ is also an exact functor due to the abelian structure on $\mathfrak{D}^*_\alpha(\mathcal{W})$. For $\Psi \in \hom^{ex}_{\mathfrak{B}}(\mathcal{V}, \mathcal{W})$ one has $K^{-1}(\Psi) : \mathcal{V}_\lambda \ni v \mapsto (G \mapsto \Psi(G \cdot v))$ where G ranges over the objects of $\mathfrak{H}om_{\mathfrak{A}}(\lambda, -)$. It must be shown that if $v' \to v \to v''$ is exact then so is $\Psi(Gv') \to \Psi(Gv) \to \Psi(Gv')$. As the objects of $\mathfrak{H}om_{\mathfrak{A}}$ are generated by E, F which act as exact functors on $\mathcal{V}$, it follows that $Gv' \to Gv \to Gv''$ is exact, and because Ψ is exact it follows that $\Psi(Gv') \to \Psi(Gv) \to \Psi(Gv')$ is as well.

□

The above adjunction comes from the data $(\varepsilon, \eta) : \mathfrak{R}\mathrm{es} \vdash \mathfrak{D}^*_\alpha$ defined as follows. Let $\mathcal{V}$ be an $\mathfrak{A}$-module and let $\eta : \mathcal{V} \to \mathfrak{D}^*_\alpha \mathfrak{R}\mathrm{es}(\mathcal{V})$ be the canonical 2-morphism of $\mathfrak{A}^\alpha$-modules $\mathcal{V} \to \mathfrak{D}^*_\alpha(\mathcal{V})$ given by mapping $v \in \mathcal{V}_\lambda$ to the unique $\mathfrak{A}^\alpha$-morphism, $\Sigma_v : \mathfrak{H}om(\lambda, -) \to \mathcal{V}$ for which $\Sigma_v(1_\lambda) = v$ and $\hom_{\mathfrak{A}}(1_\lambda, G) \mapsto \hom_{\mathcal{V}}(v, Gv)$ is the 'left-multiplication' map.

Let $\mathcal{W}$ be a $\mathfrak{B}$-module and let $\varepsilon : \mathfrak{R}\mathrm{es}\,\mathfrak{D}^*_\alpha(\mathcal{W}) \to \mathcal{W}$ be the evaluation map, for $\Gamma \in \mathfrak{D}^*_\alpha(\mathcal{W})_\lambda$, $\varepsilon(\Gamma) = \Gamma(1_\lambda)$. As Γ is a $\mathfrak{B}$-module map, $\varepsilon(E_\beta\Gamma) = \Gamma(E_\beta 1_\lambda) \cong E_\beta(\Gamma(1_\lambda))$, so that η is also a $\mathfrak{B}$-module map.

One may check the zig-zag relations that say the following two compositions of functors are equal to the identity functor.

$$\mathbf{\mathfrak{D}}_\alpha^*(\mathcal{W}) \xrightarrow{\eta \mathbf{\mathfrak{D}}_\alpha^*} \mathbf{\mathfrak{D}}_\alpha^* \mathfrak{Res}\, \mathbf{\mathfrak{D}}_\alpha^*(\mathcal{W}) \xrightarrow{\mathbf{\mathfrak{D}}_\alpha^* \varepsilon} \mathbf{\mathfrak{D}}_\alpha^*(\mathcal{W}),$$

$$\mathfrak{Res}(\mathcal{V}) \xrightarrow{\mathfrak{Res}\,\eta} \mathfrak{Res}\, \mathbf{\mathfrak{D}}_\alpha^* \mathfrak{Res}(\mathcal{V}) \xrightarrow{\varepsilon\, \mathfrak{Res}} \mathfrak{Res}(\mathcal{V}).$$

Remark I.27. It must be said that in the usual notion of adjunction between 2-morphisms there is only required to be an isomorphism between the identity functor and the composition in the zig-zag relations. These two so-called triangulator isomorphisms are then required to satisfy their own coherence relation called the swallowtail identity. In our case the triangulators are identity maps, and satisfy these identities trivially. One would likely need to define a *lax* version of the functor $\mathbf{\mathfrak{D}}_\alpha^*$ to have a non-trivial 2-adjunction $\mathfrak{Res} \vdash \mathbf{\mathfrak{D}}_\alpha^*$.

3.3 Integrability and $\mathfrak{sl}_2$ example

For this section, set $\mathfrak{A} = \mathfrak{A}(\mathfrak{sl}_2)$, the 2-Lie algebra associated with $\mathfrak{sl}_2$ and drop α from the notation. Recall from Remark I.5 that when the standard induction functor $U(\mathfrak{g}_\alpha) \otimes_{U(\mathfrak{b})} -$ is applied to a finite $\mathfrak{b}$-module, the resulting $\mathfrak{g}$-module is an infinite dimensional module in the BGG category $\mathcal{O}$. Thus to define $\mathscr{D}_\alpha$ using such an approach one needs to take a maximal finite dimensional quotient and use properties of category $\mathcal{O}$ to study it. We show in this section, Corollary I.30 that for a finite, or more precisely integrable $\mathfrak{B}$-module $\mathcal{W}$, the corresponding $\mathfrak{A}$-module $\mathbf{\mathfrak{D}}_\alpha^*(\mathcal{W})$ is integrable already, and thus needs no such quotient or category $\mathcal{O}$, making $\mathbf{\mathfrak{D}}_\alpha^*$ somewhat simpler to use.

For this section let $\mu_0 \in -X^+$ be fixed, and let $\mathcal{V}$ be a $\mathfrak{B}$-module for which $\mathcal{V}_\mu = 0$ if $\mu < \mu_0$.

Claim I.28. *Let $\lambda \in X$ with $\lambda = \mu_0 + a\alpha$, $a \geq 0$. For every $\Sigma \in \mathbf{\mathfrak{D}}^*(\mathcal{V})_\lambda = \hom_{\mathfrak{B}}({}_\bullet\mathfrak{A}_\lambda, \mathcal{V})$, the map Σ vanishes on ${}_\bullet\mathfrak{A} \cdot F^{a+1} 1_\lambda$.*

Proof. Let $\Sigma \in \hom_{\mathfrak{B}}({}_\bullet\mathfrak{A}_\lambda, \mathcal{V})$, and $G \in \mathfrak{Hom}(\mu_0 - \alpha, \mu)$. It follows from Lemma I.23 that G

is a direct summand of a sum of objects of the form

$$ {}_{\mu}E^{b}_{\mu_0-(c+1)\alpha}F^{c}_{\mu_0-\alpha}. $$

It follows that

$$ \Sigma({}_{\mu}E^{b}_{\mu_0-(c+1)\alpha}F^{c}F^{a+1}_{\lambda}) \cong E^{b}\Sigma({}_{\mu_0-(c+1)\alpha}F^{c+a+1}) \cong 0, $$

as $\Sigma({}_{\mu_0-(c+1)\alpha}F^{c+a+1}) \in \mathcal{V}_{\mu_0-(c+1)\alpha} = \{0\}$. The claim is now shown. $\square$

Claim I.29. *If $\lambda < \mu_0$ then $\mathfrak{D}^*(\mathcal{V})_\lambda = \hom_{\mathfrak{B}}({}_{\bullet}\mathfrak{A}_\lambda, \mathcal{V}) \cong 0$. Likewise, if $\lambda > -\mu_0$ then $\mathfrak{D}^*(\mathcal{V})_\lambda \cong 0$.*

Proof. For the first claim, consider $\lambda < \mu_0$. As above, every $G \in \mathfrak{H}om(\lambda, \mu)$ is a direct summand of a sum of terms of the form ${}_{\mu}E^{b}F^{c}_{\lambda}$. Then for $\Sigma \in \mathcal{H}om_{\mathfrak{B}}(\mathfrak{H}om(\lambda, -), \mathcal{V})$, we have

$$ \Sigma(E^{b}F^{c}_{\lambda}) \cong E^{b}\Sigma({}_{\lambda-c\alpha}F^{c}) \cong 0. $$

For the second claim, consider $\lambda > -\mu_0$. Let $\mu = \lambda - n\alpha$ and suppose $\mu \geq \mu_0$. Let m be the smallest positive integer such that $\mu - m\alpha < \mu_0$. Consider the element ${}_{\mu}E^{m}F^{n+m}_{\lambda}$. As in Lemma I.23 set $r = \lambda - n$. Note that $r > 0$ if and only if $\mu > -\lambda$. In particular, $r > 0$ since $\mu \geq \mu_0$. There is an isomorphism (*loc. cit.*),

$$ {}_{\mu}E^{m}F^{n+m}1_{\lambda} \cong \bigoplus_{\ell=0}^{m} F^{n+m-\ell}E^{m-\ell}1_{\lambda} \otimes_k k^{\frac{(m+n)!m!}{(m+n-\ell)!(m-\ell)!}\binom{r}{\ell}}. $$

In particular, for $\ell = m$ we find that ${}_{\mu}F^{n}_{\lambda}$ is a direct summand of ${}_{\mu}E^{m}F^{n+m}_{\lambda}$. Thus, for $\Sigma \in \mathcal{H}om_{\mathfrak{B}}(\mathfrak{H}om(\lambda, -), \mathcal{V})$, $\Sigma({}_{\mu}F^{n}_{\lambda})$ is a direct summand of $E^{m}\Sigma({}_{\mu-m\alpha}F^{n+m}_{\lambda}) \cong 0$. It follows that $\Sigma \cong 0$. $\square$

Corollary I.30. *Let $\mu_0 \in -X^+$. If $\mathcal{V}$ is a $\mathfrak{B}$-module which is bounded below by μ_0, then $\mathfrak{D}^*(\mathcal{V})$ is bounded below by μ_0 and above by $-\mu_0$.*

Lemma I.31. *Suppose $\mathcal{V}$ is an integrable 2-representation of $\mathfrak{A}$ and $T \in \mathcal{V}_{\mu_0}$ is such that $E^{n+1}(T) = 0$, where $\mu_0 = -n$. Then there exists a canonical $\mathfrak{B}$-morphism $\Sigma : \mathcal{L}(\mu_0) = {}_\bullet\mathfrak{A}_\lambda / {}_\bullet\mathfrak{A} \cdot F1_\lambda \to \mathcal{V}$ with $\Sigma(\overline{1}_\lambda) = T$ and $\hom(\overline{1}_\lambda, \overline{G}) \to \hom_{\mathcal{V}}(T, GT)$ for each G a 1-morphism in $\mathfrak{B}$ coming from left multiplication of $\mathfrak{A}$ on $\mathcal{V}$. Moreover, Σ is also a morphism of $\mathfrak{A}$-modules.*

Proof. See [Rou08, section 5.1.2] □

Consider now an example analogous to the one from Corollary I.7. Let $\mu_0 \leq 0$ be a weight, and $k-\text{fmod}_{\mu_0}$ the $\mathfrak{B}$-module which is $k-\text{fmod}$ in weight μ_0 and zero on every other weight. The following claim computes the 2-representation $\mathfrak{D}^*(k-\text{fmod}_{\mu_0})$.

Claim I.32. *Consider the right $\mathfrak{A}$-module $(E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A}_\bullet)\backslash_{\mu_0}\mathfrak{A}_\bullet$. Define a representation of $\mathfrak{A}$ via pullback on*

$$\hom_k((E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A}_\bullet)\backslash_{\mu_0}\mathfrak{A}_\bullet, k-\text{fmod}),$$

the space of k-linear functors from the category $(E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A}_\bullet)\backslash_{\mu_0}\mathfrak{A}_\bullet$ to $k-\text{fmod}$. There is an equivalence of $\mathfrak{A}$-modules,

$$\mathfrak{D}^*(k-\text{fmod}_{\mu_0}) \cong \hom_k((E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A}_\bullet)\backslash_{\mu_0}\mathfrak{A}_\bullet, k-\text{fmod}).$$

Proof. The categorical tensor–hom adjunction, Claim I.26 gives an equivalence

$$\hom^{ex}_{\mathfrak{A}}(\mathcal{V}, \mathfrak{D}^*(k-\text{fmod}_{\mu_0})) \cong \hom^{ex}_{\mathfrak{B}}(\mathcal{V}, k-\text{fmod}_{\mu_0}).$$

There is an equivalence

$$\hom^{ex}_{\mathfrak{B}}(\mathcal{V}, k-\text{fmod}_{\mu_0}) \xrightarrow{\sim} \{\Sigma \in \hom_k(\mathcal{V}_{\mu_0}, k-\text{fmod}) \mid \Sigma(E_\alpha(\mathcal{V}_{\mu_0-\alpha})) \cong 0\},$$

$$\Phi \mapsto \Phi\mid_{\mathcal{V}_{\mu_0}}$$

which may be written suggestively as

$$\hom^{ex}_{\mathfrak{B}}(\mathcal{V}, k-\text{fmod}_{\mu_0}) \cong \hom^{ex}_k((E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A}_\bullet)\backslash_{\mu_0}\mathfrak{A}_\bullet \otimes_{\mathfrak{A}} \mathcal{V}, k-\text{fmod}).$$

Now, consider the *right* $\mathfrak{A}$-module $(E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A})\backslash_{\mu_0}\mathfrak{A}$. There is a similar equivalence

$$\hom_k^{ex}((E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A})\backslash_{\mu_0}\mathfrak{A} \otimes_{\mathfrak{A}} \mathcal{V}, k\text{−fmod}) \cong \hom_{\mathfrak{A}}^{ex}(\mathcal{V}, \hom_k((E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A})\backslash_{\mu_0}\mathfrak{A}, k\text{−fmod})),$$

also given by using another categorical tensor – hom adjunction. Putting this together gives the chain of equivalences

$$\begin{aligned}\hom_{\mathfrak{A}}^{ex}(\mathcal{V}, \mathfrak{D}^*(k\text{−fmod}_{\mu_0})) &\cong \hom_{\mathfrak{B}}^{ex}(\mathcal{V}, k\text{−fmod}_{\mu_0}) \\ &\cong \hom_k^{ex}(\mathcal{V}_{\mu_0}/E_\alpha(\mathcal{V}_{\mu_0-\alpha}), k\text{−fmod}) \\ &\cong \hom_{\mathfrak{A}}^{ex}(\mathcal{V}, \hom_k((E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A})\backslash_{\mu_0}\mathfrak{A}, k\text{−fmod})).\end{aligned}$$

Using the Yoneda lemma this gives an equivalence between $\mathfrak{A}$-modules,

$$\mathfrak{D}^*(k\text{−fmod}_{\mu_0}) \cong \hom_k((E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A})\backslash_{\mu_0}\mathfrak{A}, k\text{−fmod}).$$

□

Remark I.33. Chevalley duality gives an equivalence between $\mathfrak{A}$-modules and $\mathfrak{A}^{\text{rev}}$-modules, where the superscipt rev stands for the reverse 2-category as defined in [Rou08, Section 2.2.2, Section 4.2.1]. Under this equivalence the $\mathfrak{A}^{\text{rev}}$-module $(E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A})\backslash_{\mu_0}\mathfrak{A}$ corresponds to the $\mathfrak{A}$-module $\mathfrak{A}_{\mu_0}/\mathfrak{A}_{\mu_0-\alpha} \cdot F_\alpha \cong \mathcal{L}(\mu_0)$. Writing a superscript $\vee$ for this Chevalley duality, $(E_\alpha \cdot {}_{\mu_0-\alpha}\mathfrak{A})\backslash_{\mu_0}\mathfrak{A} = \mathcal{L}(\mu_0)^\vee$, and rewriting the above equivalence gives,

$$\mathfrak{D}^*(k\text{−fmod}_{\mu_0}) \cong \hom_k(\mathcal{L}(\mu_0)^\vee, k\text{−fmod}).$$

3.4 Fully faithful lemma

Let $\mathcal{V}$ be an $\mathfrak{A}$-module. Recall the $\mathfrak{A}$-module unit map $\eta : \mathcal{V} \to \mathfrak{D}_\alpha^* \mathfrak{R}\text{es}(\mathcal{V})$ from Section 3.2.

Lemma I.34. *The $\mathfrak{A}$-morphism unit functor $\eta : \mathcal{V} \to \mathfrak{D}_\alpha^*(\mathcal{V})$, $v \mapsto \Sigma_v$ is fully faithful.*

Proof. Since $\Sigma_v, \Sigma_w : {}_\bullet\mathfrak{A}_\lambda \to \mathcal{V}$ are $\mathfrak{A}$-morphisms, given $v, w \in \mathcal{V}_\lambda$ let $\hom_{\mathfrak{A}}(\Sigma_v, \Sigma_w)$ be the set of morphisms $\sigma : \Sigma_v \to \Sigma_w$ in $\hom_{\mathfrak{A}}(\Sigma_v, \Sigma_w)$. Thus, $\hom_{\mathfrak{A}}(\Sigma_v, \Sigma_w) \cong \hom_{\mathcal{V}}(v, w)$ by a standard universal property of the $\mathfrak{A}$-module ${}_\bullet\mathfrak{A}_\bullet$. Similarly, let $\hom_{\mathfrak{B}}(\Sigma_v, \Sigma_w)$ be the set of morphisms $\sigma : \Sigma_v \to \Sigma_w$ in $\mathfrak{D}^*_\alpha(\mathcal{V})$.

There is a split surjection $\hom_{\mathfrak{B}}(\Sigma_v, \Sigma_w) \to \hom_{\mathfrak{A}}(\Sigma_v, \Sigma_w)$ given by mapping σ in the first set to $\sigma(1) \in \hom_{\mathcal{V}}(v, w) \cong \hom_{\mathfrak{A}}(\Sigma_v, \Sigma_w)$ in the latter. The splitting is given by the canonical inclusion $\hom_{\mathfrak{A}}(\Sigma_v, \Sigma_w) \hookrightarrow \hom_{\mathfrak{B}}(\Sigma_v, \Sigma_w)$. It follows that the morphism in the lemma is faithful.

To be more explicit, let $v, w \in \mathcal{V}_\lambda$ and consider $\sigma : \Sigma_v \to \Sigma_w$ any morphism from Σ_v to Σ_w. Then σ is a $\mathfrak{B}$-morphism if and only if each diagram of the following form is commutative:

$$\begin{array}{ccc}
\Sigma_v(EG) & \xrightarrow{\sim} & E\Sigma_v(G) \\
{\scriptstyle \sigma(EG)}\downarrow & & \downarrow{\scriptstyle E\sigma(G)} \\
\Sigma_w(EG) & \xrightarrow{\sim} & E\Sigma_w(G).
\end{array}$$

Note that the horizontal arrows are part of the data for Σ_v, Σ_w to be $\mathfrak{A}$-morphisms. In fact, by definition $\Sigma_v(EG) := EG(v) \in \mathcal{V}$, so they are in actuality identity maps in $\mathcal{V}$.

To show the map is full, we must show that if σ is a $\mathfrak{B}$-morphism then it must also be an $\mathfrak{A}$-morphism, which is to say every diagram of the following form is also commutative:

$$\begin{array}{ccc}
\Sigma_v(FG) & \xrightarrow{\sim} & F\Sigma_v(G) \\
{\scriptstyle \sigma(FG)}\downarrow & & \downarrow{\scriptstyle F\sigma(G)} \\
\Sigma_w(FG) & \xrightarrow{\sim} & F\Sigma_w(G).
\end{array}$$

Here again, $\Sigma_v(FG) = FG(v) \in \mathcal{V}$, and the isomorphisms are identity maps. Let $\sigma : \Sigma_v \to \Sigma_w$ be a $\mathfrak{B}$-morphism. Consider the following diagram:

$$\begin{array}{ccccccc}
\Sigma_v(FG) & \xrightarrow{\eta\Sigma_v(FG)} & FE\Sigma_v(FG) & \xrightarrow{\quad\sim\quad} & F\Sigma_v(EFG) & \xrightarrow{F\Sigma_v(\epsilon G)} & F\Sigma(G) \\
{\scriptstyle \sigma(FG)}\downarrow & & {\scriptstyle FE\sigma(FG)}\downarrow & & {\scriptstyle F\sigma(EFG)}\downarrow & & {\scriptstyle F\sigma(G)}\downarrow \\
\Sigma_w(FG) & \xrightarrow{\eta\Sigma_w(FG)} & FE\Sigma_w(FG) & \xrightarrow{\quad\sim\quad} & F\Sigma_w(EFG) & \xrightarrow{F\Sigma_w(\epsilon G)} & F\Sigma_w(G).
\end{array}$$

The first square commutes as η is a morphism of functors. The second square commutes as σ, a $\mathfrak{B}$-morphism, commutes with E. The third square commutes as σ is a functor. It follows that if $\sigma : \Sigma_v \to \Sigma_w$ is a $\mathfrak{B}$-morphism, then σ is also an $\mathfrak{A}$-morphism.

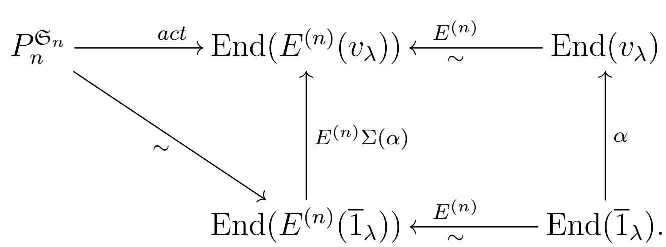
□

Proposition I.35. *If $\Sigma \in \mathfrak{D}^*(\mathcal{V})$ is a lowest weight object, then there is a canonical extension $\Sigma' \in \hom_{\mathfrak{A}}({}_\bullet\mathfrak{A}_\bullet, \mathcal{V})$ and a canonical $\mathfrak{B}$-isomorphism $\Sigma' \cong \Sigma$.*

Proof. Suppose that $\Sigma \in \mathfrak{D}^*(\mathcal{V})_\lambda$ is a lowest weight object, $F \cdot \Sigma = 0$. If $\lambda > 0$ then 1_λ divides EF, and hence $\Sigma = 0$. In that case, the proposition is clear. For the remainder of the proof we assume $\lambda \leq 0$.

More precisely, $F \cdot \Sigma = 0$ means that for every $G \in \mathfrak{H}om(\lambda - \alpha, -)$, we have $\Sigma(GF_\lambda) \cong 0$. This is the equivalent to the condition for Σ to factor through the representation $\mathcal{L}(\lambda)$, defined above. As an $\mathfrak{A}$-morphism $\Sigma : \mathcal{L}(\lambda) \to \mathcal{V}$ is determined by the value $\Sigma(1_\lambda)$ alone,we must show that a $\mathfrak{B}$-module map $\Sigma : \mathcal{L}(\lambda) \to \mathcal{V}$ is determined by the value $\Sigma(\overline{1}_\lambda)$ alone.

By the description [Rou08, Proposition 5.15] of $\mathcal{L}(\lambda)$, a $\mathfrak{B}$-module map $\Sigma : \mathcal{L}(\lambda) \to \mathcal{V}$ is determined by the value $v_\lambda = \Sigma(\overline{1}_\lambda)$, and by the given algebra map $\alpha : \mathrm{End}(\overline{1}_\lambda) \to \mathrm{End}(v_\lambda)$. All other data is determined as the algebra $H_{a,n}$ is generated by $X_1, \ldots X_a, T_1, \ldots T_{a-1} \in \mathrm{End}(E^a)$ over $P_n^{\mathfrak{S}_n}$. The following commutative diagram shows that the morphism α is determined by the image, $\Sigma(\overline{1}_\lambda)$ alone:



Let $v_\lambda = \Sigma(\overline{1}_\lambda) \in \mathcal{V}$. Let $\lambda = -n$, so that $E^{n+1}v_\lambda = 0$ in $\mathcal{V}$. As F_λ divides $F^{n+2}E_\lambda^{n+1}$, it follows that $F(v_\lambda) = 0$ in $\mathcal{V}$. Thus, there is a unique $\mathfrak{A}$-morphism $\Sigma' = \Sigma_{v_\lambda} : \mathcal{L}(\lambda) \to \mathcal{V}$, which is isomorphic to Σ as a $\mathfrak{B}$-morphism.

□

The following Corollary is a categorification of Claim I.10.

Corollary I.36. *If $\mathcal{V}$ is a $\mathfrak{A}$-module, the unit map $\eta : \mathcal{V} \to \mathfrak{R}\mathrm{es}\,\mathfrak{D}^*\,\mathfrak{R}\mathrm{es}(\mathcal{V})$ is a fully faithful map which is an equivalence on the full subcategories of lowest weight objects, $\mathcal{V}^{lw} \xrightarrow{\sim} \mathfrak{D}^*(\mathcal{V})^{lw}$.*

If the additive quotient $\mathfrak{D}^*(\mathcal{V})/\mathcal{V}$ is integrable and $\mathcal{V}$ is an abelian category then the inclusion $\mathcal{V} \hookrightarrow \mathfrak{D}^*(\mathcal{V})$ is an equivalence. In particular, if $\mathcal{V}$ is an integrable abelian category, $\mathcal{V}$ is canonically equivalent to $\mathfrak{D}^*(\mathcal{V})$, via η and the inverse equivalence is given by the counit map, $\varepsilon : \Sigma \mapsto \Sigma(1_\lambda)$ for $\Sigma \in \mathfrak{D}^*(\mathcal{V})_\lambda$. It follows that there is an essentially unique way to extend the $\mathfrak{B}$-module structure to an $\mathfrak{A}$-module structure on an integrable $\mathfrak{A}$-module, namely by the above evaluation isomorphism $\mathfrak{R}\mathrm{es}\,\mathfrak{D}^*(\mathcal{V}) \to \mathcal{V}$.

Proof. The first statement follows from the above proposition. For the second, consider the following fact: If $\Gamma \in \mathfrak{D}^*(\mathcal{V})$ is an extension of Σ_w by Σ_v, and $F(w) \cong 0$, then $\Gamma \cong \Sigma_{\Gamma(1)} \in \mathcal{V}$. To see this let $j > 0$ and note the exact sequence,

$$0 \to F^j v \to \Gamma(F^j) \to F^j w \cong 0 \to 0,$$

shows that $\Gamma(F^j) \cong F^j v$. By the exactness of F, the following sequence is also exact,

$$0 \to F^j v \to F^j \Gamma(1) \to F^j w \cong 0 \to 0.$$

This shows that $F^j v \cong F^j \Gamma(1)$. It follows that for any i, j we have $\Gamma(E^i F^j) \cong E^i \Gamma(F^j) \cong E^i F^j \Gamma(1)$, so that $\Gamma \cong \Sigma_{\Gamma(1)}$.

Now let $\Gamma \in \mathfrak{D}^*(\mathcal{V})$ with $F \cdot \Gamma \in \mathcal{V}$. Consider the following map,

$$EF\Gamma \xrightarrow{\epsilon} \Gamma.$$

After applying F, the map is a split surjection, with the splitting given by,

$$FEF\Gamma \xleftarrow{\eta F} F\Gamma.$$

As $\mathfrak{D}^*(\mathcal{V})$ is also an abelian category, the following sequence is exact,

$$EF\Gamma \xrightarrow{\epsilon} \Gamma \to \Gamma/EF\Gamma \to 0.$$

This stays exact after applying F, hence $F(\Gamma/EF\Gamma) \cong 0$, which by the above proposition implies that $\Gamma/EF\Gamma \cong \Sigma_w$ for some $w \in \mathcal{V}$ with $Fw \cong 0$. We claim that $\ker(\epsilon) \cong \Sigma_v$ for some $v \in \mathcal{V}$, hence $EF\Gamma/\ker(\epsilon) \in \mathcal{V}$, from which the short exact sequence,

$$0 \to EF\Gamma/\ker(\epsilon) \xrightarrow{\epsilon} \Gamma \to \Gamma/EF\Gamma \to 0,$$

shows that $\Gamma \in \mathcal{V}$.

To show that $\ker(\epsilon) \in \mathcal{V}$, let $\Gamma' \xrightarrow{\psi} FEF\Gamma$ be the kernel of the split surjection $FEF\Gamma \xrightarrow{F\epsilon} F\Gamma$. As $FEF\Gamma \in \mathcal{V}$ by assumption, and $\mathcal{V}$ is idempotent complete, $\Gamma' \cong \Sigma_{v'}$ for some $v' \in \mathcal{V}$. By the adjunction (E, F) there is a canonical map $\psi' : E\Gamma' \to EF\Gamma$ given by the composition $E\Gamma' \xrightarrow{E\psi} EFEF\Gamma \xrightarrow{\epsilon EF} EF\Gamma$. Moreover, the interchange law gives the commutativity of the following diagram,

$$\begin{array}{ccc} EFEF\Gamma & \xrightarrow{\epsilon EF} & EF\Gamma \\ \downarrow{\scriptstyle EF\epsilon} & & \downarrow{\scriptstyle \epsilon} \\ EF\Gamma & \xrightarrow{\epsilon} & \Gamma \end{array},$$

which shows that the composition $E\Gamma' \xrightarrow{\psi'} EF\Gamma \xrightarrow{\epsilon} \Gamma$ is zero. This gives the following complex,

$$0 \to E\Gamma' \xrightarrow{\psi'} EF\Gamma \xrightarrow{\epsilon} \Gamma \to \Gamma/EF\Gamma \to 0.$$

Now, the coimage of $E\Gamma' \to \ker(\epsilon)$ is isomorphic to the coimage of $E\Gamma' \xrightarrow{\psi'} EF\Gamma$, hence is in $\mathcal{V}$. It is also the case that F applied to this coimage is also isomorphic to Γ', and applying F to the short exact sequence,

$$0 \to \operatorname{coim}(E\Gamma' \to \ker(\epsilon)) \to \ker(\epsilon) \to \ker(\epsilon)/E\Gamma' \to 0,$$

shows that $F(\ker(\epsilon)/E\Gamma') \cong 0$, hence $\ker(\epsilon)/E\Gamma' \cong \Sigma_w$ for some $w \in \mathcal{V}$ with $Fw \cong 0$ as well. As above, since $\operatorname{coim}(E\Gamma' \to \ker(\epsilon)) \cong \Sigma_{v'}$ for some $v' \in \mathcal{V}$, and $Fw \cong 0$, we have that $\ker(\epsilon) \cong \Sigma_v$ for some $v \in \mathcal{V}$.

It follows that the 2-representation on the additive categorical quotient $\mathfrak{D}^*(\mathcal{V})/\mathcal{V}$ has no lowest weight objects. If $\mathfrak{D}^*(\mathcal{V})/\mathcal{V}$ is integrable, as in the case that $\mathcal{V}$ is integrable, this shows that $\mathcal{V} \hookrightarrow \mathfrak{D}^*(\mathcal{V})$ is an equivalence of categories.

□

Remark I.37. Just as in Remark I.11 the idempotence relation ${\mathfrak{D}_\alpha^*}^2 \cong \mathfrak{D}_\alpha^*$ follows from the above claim. From this one can also draw the following analogue of Corollary I.12, the fully faithful lemma.

Lemma I.38. *Let $\mathcal{V}, \mathcal{W} \in \mathfrak{A}^\alpha - mod$. The restriction functor $\mathfrak{R}$es : $\mathfrak{A}$−mod → $\mathfrak{B}$−mod is fully faithful on integrable modules. Given $\mathcal{V}, \mathcal{W} \in \mathfrak{A}$−mod with $\mathcal{W}$ integrable, restriction gives an equivalence*

$$\hom_{\mathfrak{A}^\alpha}(\mathcal{V}, \mathcal{W}) \cong \hom_{\mathfrak{B}}(\mathcal{V}, \mathcal{W}).$$

Proof. See the proof of Corollary I.12.

□

3.5 Simple modules

Now consider the 2 Kac-Moody algebra $\mathfrak{A}$ associated to a higher rank finite semi-simple Lie algebra. For $\lambda \in X^-$ define the $\mathfrak{A}$-module $\mathcal{L}^*(\mu_0)$ as follows:

$$\mathcal{L}^*(\mu_0) = \hom_k((1_{\mu_0}\langle E_i \mathfrak{A}\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}, k\text{−fmod}),$$

where again, $\hom_k$ denotes the category of k-linear functors. Let $w_0 = s_1 \cdots s_n$ be a reduced decomposition of the longest element of the Weyl group associated with $\mathfrak{A}$. Put $w_i = s_{i+1} \cdots s_n$ and let ${\mathfrak{D}^*}_{w_i} = {\mathfrak{D}^*}_{s_{i+1}} \circ \cdots \circ {\mathfrak{D}^*}_{s_n}$ be the composition of categorical demazure operators.

Claim I.39. *Let $\mathcal{V}$ be an $\mathfrak{A}$-module. There is a natural equivalence*

$$\hom^{ex}_{\mathfrak{B}}(\mathcal{V}, {\mathfrak{D}^*}_{w_0}(k\text{−fmod}_{\mu_0})) \cong \hom^{ex}_k(\mathcal{V}_{\mu_0}/\langle E_i(\mathcal{V})\rangle_i, k\text{−fmod}).$$

Proof. Putting $s_1 = s_\alpha$ and given that $\mathcal{V}$ is an $\mathfrak{A}_\alpha$-module, the categorical tensor – hom adjunction gives an equivalence,

$$\hom^{ex}_{\mathfrak{B}}(\mathcal{V}, \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})) \cong \hom^{ex}_{\mathfrak{A}_\alpha}(\mathcal{V}, \mathfrak{D}^{*\,2}_\alpha \circ \mathfrak{D}^*{}_{w_1}(k\text{−fmod}_{\mu_0})),$$

As $\mathfrak{D}^{*\,2}_\alpha \cong \mathfrak{D}^*_\alpha$, there is an equivalence

$$\begin{aligned}\hom^{ex}_{\mathfrak{B}}(\mathcal{V}, \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})) &\cong \hom^{ex}_{\mathfrak{A}_\alpha}(\mathcal{V}, \mathfrak{D}^{*\,2}_\alpha \circ \mathfrak{D}^*{}_{w_1}(k\text{−fmod}_{\mu_0})),\\ &\cong \hom^{ex}_{\mathfrak{A}_\alpha}(\mathcal{V}, \mathfrak{D}^*_\alpha \circ \mathfrak{D}^*{}_{w_1}(k\text{−fmod}_{\mu_0})).\end{aligned}$$

Using the tensor – hom adjunction again gives

$$\hom^{ex}_{\mathfrak{B}}(\mathcal{V}, \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})) \cong \hom^{ex}_{\mathfrak{B}}(\mathcal{V}, \mathfrak{D}^*{}_{w_1}(k\text{−fmod}_{\mu_0}))$$

Continuing in this fashion for $s_2, s_3, \dots$ gives an equivalence,

$$\begin{aligned}\hom^{ex}_{\mathfrak{A}}(\mathcal{V}, \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})) &\cong \hom^{ex}_{\mathfrak{B}}(\mathcal{V}, k\text{−fmod}_{\mu_0})\\ &\cong \hom^{ex}_{k}(\mathcal{V}_{\mu_0}/\langle E_i(\mathcal{V})\rangle_i, k\text{−fmod}).\end{aligned}$$

□

The following is a categorification of Proposition I.14.

Corollary I.40. *If the* $\mathfrak{B}$*-module* $\mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})$ *can be given the structure of an* $\mathfrak{A}$*-module extending the* $\mathfrak{B}$*-action, then there is an* $\mathfrak{A}$*-module equivalence,*

$$\mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0}) \cong \mathcal{L}^*(\mu_0)$$

Proof. If $\mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})$ can be extended to an $\mathfrak{A}$-module, such a structure would necessarily be unique according to Section 3.4. If $\mathcal{V}$ is an $\mathfrak{A}$-module, Corollary I.38 and the previous claim give,

$$\begin{aligned}\hom^{ex}_{\mathfrak{A}}(\mathcal{V}, \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})) &\cong \hom^{ex}_{\mathfrak{B}}(\mathcal{V}, \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0}))\\ &\cong \hom^{ex}_{k}(\mathcal{V}_{\mu_0}/\langle E_i(\mathcal{V})\rangle_i, k\text{−fmod}).\end{aligned}$$

The module $\mathcal{L}^*(\mu_0)$ is defined so that the tensor – hom adjunction gives,

$$\begin{aligned}\hom^{ex}_{\mathfrak{A}}(\mathcal{V}, \mathcal{L}^*(\mu_0)) &= \hom^{ex}_{\mathfrak{A}}(\mathcal{V}, \hom_k((1_{\mu_0}\langle E_i\mathfrak{A}\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}, k\text{−fmod}))\\ &\cong \hom^{ex}_k(\mathcal{V}_{\mu_0}/\langle E_i(\mathcal{V})\rangle_i, k\text{−fmod}).\end{aligned}$$

Thus, the Yoneda Lemma gives an explicit equivalence $\mathcal{L}^*(\mu_0) \cong \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})$. □

Given that $\mathcal{L}^*(\mu_0)$ is an $\mathfrak{A}$-module one can construct maps from it to $\mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})$ by computing elements of,

$$\begin{aligned}\hom^{ex}_{\mathfrak{B}}(\mathcal{L}^*(\mu_0), \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})) &\cong \hom^{ex}_k(\mathcal{L}^*(\mu_0)_{\mu_0}/\langle E_i(\mathcal{L}^*(\mu_0))\rangle_i, k\text{−fmod})\\ &\cong \hom^{ex}_k(\mathcal{L}^*(\mu_0)_{\mu_0}, k\text{−fmod})\\ &\cong \hom^{ex}_k(\hom_k((1_{\mu_0}\langle E_i\mathfrak{A}\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{\mu_0}, k\text{−fmod}), k\text{−fmod})\end{aligned}$$

Of course the category $(1_{\mu_0}\langle E_i\mathfrak{A}\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{\mu_0}$ maps into this double dual category and we will focus on the image of the object $1_{\mu_0} \in (1_{\mu_0}\langle E_i\mathfrak{A}\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{\mu_0}$, which produces a morphism $\Sigma_{can} : \mathcal{L}^*(\mu_0) \to \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})$.

Remark I.41. Abusing the tensor – hom analogy would give the equivalence,

$$\mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0}) \cong \hom_k(((\langle E_i\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{s_1} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} \mathfrak{A}_{s_n}, k\text{−fmod}).$$

Again, using double duality we would be able to construct maps $\mathcal{L}^*(\mu_0) \to \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0})$ by constructing right $\mathfrak{B}$-module maps

$$(1_{\mu_0}\langle E_i\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{s_1} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} \mathfrak{A}_{s_n} \to (1_{\mu_0}\langle E_i\mathfrak{A}\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{\mu_0}$$

On the decategorified level one may actually construct an isomorphism between the analogues of the above two objects using a PBW basis. We would like to mimic that construction by instead constructing a special element of $(1_{\mu_0}\langle E_i\mathfrak{A}\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{\mu_0}$ which maps into $\hom^{ex}_{\mathfrak{B}}(\mathcal{L}^*(\mu_0), \mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0}))$.

Conjecture I.42. *The $\mathfrak{B}$-morphism,*

$$\mathfrak{D}^*{}_{w_0}(k\text{−fmod}_{\mu_0}) \to \hom_k(((\langle E_i\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}, k\text{−fmod}).$$

is an isomorphism. This gives a construction of the $\mathfrak{A}$*-module* $\hom_k((\langle E_i\rangle_i)\backslash 1_{\mu_0}\mathfrak{A}_{s_1}\otimes_{\mathfrak{B}}\cdots\otimes_{\mathfrak{B}}\mathfrak{A}_{s_n}, k-\mathrm{fmod})$ *and gives a categorification of Proposition I.16. This is a 2-categorical analogue of the original Demazure character formula.*

3.6 Decategorification

Let $\mathcal{V}$ be a $\mathfrak{B}$-module on an abelian category which is Artinian and Noetherian. The counit map $\mathfrak{Res}\,\mathfrak{D}_\alpha^*(\mathcal{V})\xrightarrow{\varepsilon}\mathcal{V}$ is exact and thus its kernel $\mathcal{K}$ is a Serre subcategory of $\mathfrak{Res}\,\mathfrak{D}_\alpha^*(\mathcal{V})$.

Definition. Define $\hat{\mathcal{V}}$ to be the abelian quotient category of $\mathfrak{Res}\,\mathfrak{D}_\alpha^*(\mathcal{V})$ by the kernel of ε, and let $j:\hat{\mathcal{V}}\to\mathcal{V}$ and $k:\mathfrak{Res}\,\mathfrak{D}_\alpha^*(\mathcal{V})\to\hat{\mathcal{V}}$ be the canonical functors which factor $\mathfrak{Res}\,\mathfrak{D}_\alpha^*(\mathcal{V})\xrightarrow{\varepsilon}\mathcal{V}$.

Claim I.43. *The following map,*

$$\mathfrak{D}_\alpha^*(j):\mathfrak{D}_\alpha^*(\hat{\mathcal{V}})\to\mathfrak{D}_\alpha^*(\mathcal{V})$$

is an equivalence with inverse given by the composition

$$\mathfrak{D}_\alpha^*(\mathcal{V})\xrightarrow{\eta\mathfrak{D}_\alpha^*}\mathfrak{D}_\alpha^*\,\mathfrak{Res}\,\mathfrak{D}_\alpha^*(\mathcal{V})\xrightarrow{\mathfrak{D}_\alpha^* k}\mathfrak{D}_\alpha^*(\hat{\mathcal{V}})$$

The category $\hat{\mathcal{V}}$ *satisfies the following universal property. Given* $\mathcal{W}\in\mathfrak{A}_\alpha-\mathrm{mod}$ *there is an equivalence*

$$\hom_{\mathfrak{B}}^{ex}(\mathfrak{Res}(\mathcal{W}),\mathcal{V})\cong\hom_{\mathfrak{B}}^{ex}(\mathfrak{Res}(\mathcal{W}),\hat{\mathcal{V}})$$

Proof. This is a standard exercise in using the zig-zag equalities and the tensor – hom adjunction referred to earlier. □

The decategorification Theorem may now be stated and proved.

Theorem I.44 (Decategorification Theorem)**.** *The decategorification of the operator* $\mathfrak{D}_\alpha^*$ *is given by* $\mathscr{D}_\alpha^*$ *in the following sense. The map* $k:\mathfrak{D}_\alpha^*(\mathcal{W})\to\hat{\mathcal{W}}$ *gives rise to a map* $[k]:\mathbb{C}\otimes K_0(\mathfrak{D}_\alpha^*(\mathcal{W}))\to\mathbb{C}\otimes K_0(\hat{\mathcal{W}})$*. Where defined, there is a natural isomorphism,*

$$\mathbb{C}\otimes K_0(\mathfrak{D}_\alpha^*(\mathcal{W}))\cong\mathscr{D}_\alpha^*(\mathbb{C}\otimes K_0(\hat{\mathcal{W}})).$$

This theorem will be proved in two steps, showing first that the map is injective, Claim I.46 and second that it is surjective, Claim I.47.

Lemma I.45. *Let* $\lambda \in X^-$ *and* $\Gamma, \Sigma \in \boldsymbol{\mathfrak{D}}^*_\alpha(\hat{\mathcal{V}})^{lw}_\lambda$. *Then,*

$$\hom_{\boldsymbol{\mathfrak{D}}^*_\alpha(\hat{\mathcal{V}})}(\Gamma, \Sigma) \cong \hom_{\hat{\mathcal{V}}}(k\Gamma, k\Sigma).$$

Proof. Both $\Gamma, \Sigma : \boldsymbol{\mathfrak{A}}_\lambda \to \mathcal{V}$ factor through $\mathcal{L}(\lambda) \cong \boldsymbol{\mathfrak{A}}_\lambda / \boldsymbol{\mathfrak{A}} F_\alpha 1_\lambda$. The description [Rou08, section 5.2] of $\mathcal{L}(\lambda)$ shows that as a $\boldsymbol{\mathfrak{B}}$-module, $\mathcal{L}(\lambda)$ is generated by the image of $\overline{1}_\lambda \in {}_\bullet\boldsymbol{\mathfrak{A}}_\lambda / {}_\bullet\boldsymbol{\mathfrak{A}} F_\alpha 1_\lambda$. Any subobject Ω of Γ also factors through $\boldsymbol{\mathfrak{A}}_\lambda / \boldsymbol{\mathfrak{A}} F_\alpha 1_\lambda$ and so if $\varepsilon(\Omega) = \Omega(1_\lambda) \cong 0$, i.e. $\Omega \in \mathcal{K} = \ker(\varepsilon)$, then $\Omega \cong 0$. Similarly, if a quotient object Ω of Σ were in $\mathcal{K}$, it would factor through $\boldsymbol{\mathfrak{A}}_\lambda / \boldsymbol{\mathfrak{A}} F_\alpha 1_\lambda$ and thus if $\varepsilon(\Omega) = \Omega(1_\lambda) \cong 0$ then $\Omega \cong 0$. It follows that in the abelian quotient category $\hat{\mathcal{V}}$ the set of morphisms between Γ and Σ is the same as the set of morphisms between Γ and Σ in $\mathcal{V}$:

$$\hom_{\boldsymbol{\mathfrak{D}}^*_\alpha(\hat{\mathcal{V}})}(\Gamma, \Sigma) \cong \hom_{\hat{\mathcal{V}}}(k\Gamma, k\Sigma).$$

□

Claim I.46. *Let* $\mathcal{V}$ *be a* $\boldsymbol{\mathfrak{B}}$*-module and* $\hat{\mathcal{V}}$ *as above. As* $\mathbb{C} \otimes K_0(\boldsymbol{\mathfrak{D}}^*_\alpha(\hat{\mathcal{V}}))$ *is a* $\mathfrak{g}_\alpha$*-module and*

$$[k] : \mathbb{C} \otimes K_0(\boldsymbol{\mathfrak{D}}^*_\alpha(\hat{\mathcal{V}})) \to \mathbb{C} \otimes K_0(\hat{\mathcal{V}}),$$

is a $\mathfrak{b}$*-module morphism, the adjunction* $\operatorname{res} \vdash \mathscr{D}^*_\alpha$ *gives a* $\mathfrak{g}_\alpha$*-module morphism*

$$\mathbb{C} \otimes K_0(\boldsymbol{\mathfrak{D}}^*_\alpha(\hat{\mathcal{V}})) \to \mathscr{D}^*_\alpha(\mathbb{C} \otimes K_0(\hat{\mathcal{V}})).$$

This map is injective.

Proof. The following is a commutative diagram,

$$\begin{array}{ccc} \mathbb{C} \otimes K_0(\boldsymbol{\mathfrak{R}}\mathrm{es}\, \boldsymbol{\mathfrak{D}}^*_\alpha(\hat{\mathcal{V}})) & \longrightarrow & \mathscr{D}^*_\alpha(\mathbb{C} \otimes K_0(\hat{\mathcal{V}})) \\ & \searrow^{[k]} & \downarrow \\ & & \mathbb{C} \otimes K_0(\hat{\mathcal{V}}). \end{array}$$

It remains to show that the top row is injective. As the top row is a map of $\mathfrak{g}_\alpha$-modules it suffices to show that it is injective on lowest weight spaces. Commutativity of the diagram shows that it is enough to prove that the restriction,

$$\mathbb{C} \otimes K_0(\mathfrak{Res}\,\mathfrak{D}^*_\alpha(\hat{\mathcal{V}}))^{lw} \xrightarrow{[k]} \mathbb{C} \otimes K_0(\hat{\mathcal{V}})$$

is injective. As $\mathfrak{D}^*_\alpha(\hat{\mathcal{V}})$ is an abelian categorification on an Artinian and Noetherian category then by Lemma I.25,

$$\mathbb{C} \otimes K_0(\mathfrak{D}^*_\alpha(\hat{\mathcal{V}}))^{lw} \cong \mathbb{C} \otimes K_0(\mathfrak{D}^*_\alpha(\hat{\mathcal{V}})^{lw}).$$

By the previous lemma, the restriction,

$$k : \mathfrak{D}^*_\alpha(\hat{\mathcal{V}})^{lw} \to \hat{\mathcal{V}}$$

is fully faithful. Because $\mathfrak{D}^*_\alpha(\hat{\mathcal{V}})^{lw}$ is the kernel of the exact functor F, it is a Serre subcategory of the Artinian and Noetherian category $\mathfrak{D}^*_\alpha(\hat{\mathcal{V}})$. Finally, as k is fully faithful and $\hat{\mathcal{V}}$ is also an Artinian and Noetherian category this implies

$$[k] : \mathbb{C} \otimes K_0(\mathfrak{Res}\,\mathfrak{D}^*_\alpha(\hat{\mathcal{V}}))^{lw} \to \mathbb{C} \otimes K_0(\hat{\mathcal{V}})$$

is injective and the claim follows.

□

Claim I.47. *Keep the setup of Theorem I.44. The map*

$$\mathbb{C} \otimes K_0(\mathfrak{D}^*_\alpha(\mathcal{V})) \to \mathscr{D}^*_\alpha(\mathbb{C} \otimes K_0(\hat{\mathcal{V}})).$$

is surjective.

Proof. Let $K = \mathbb{C} \otimes K_0(\mathcal{K})$. There is an exact sequence of $\mathfrak{b}$-modules,

$$0 \to K \to \mathbb{C} \otimes K_0(\mathfrak{D}^*_\alpha(\hat{\mathcal{V}})) \to \mathbb{C} \otimes K_0(\hat{\mathcal{V}}) \to 0.$$

Also, $K_0(\mathfrak{D}^*_\alpha(\hat{\mathcal{V}}))$ is a $\mathfrak{g}_\alpha$-module, and the associated map

$$K_0(\mathfrak{D}^*_\alpha(\hat{\mathcal{V}})) \to \mathscr{D}^*_\alpha(\mathbb{C} \otimes K_0(\hat{\mathcal{V}}))$$

is injective by Lemma I.46, thus by Lemma I.19, i.e. by the left exactness of $\mathscr{D}_\alpha^*$ there is an isomorphism,

$$\mathscr{D}_\alpha^*(K) \cong 0.$$

To show the claim, Lemma I.20 implies that it suffices to show that for each $\lambda \in X$, $-n = \langle \lambda, \check{\alpha} \rangle \leq 0$, that

$$K^{hw}_{s_\alpha \cdot \lambda} \subset E_\alpha^{n+1}(K_\lambda).$$

Since $\mathbf{\mathfrak{D}}_\alpha^*(\mathcal{V})$ is an abelian $\mathfrak{sl}_2$-categorification on a k-linear Artinian and Noetherian category, Lemma I.25 shows that $K^{hw}_{s_\alpha \cdot \lambda}$ is spanned by the classes of objects in $\mathcal{K}^{hw} \subset \mathbf{\mathfrak{D}}_\alpha^*(\mathcal{V})^{hw}$. Let $\lambda \in X$, $-n = \langle \lambda, \check{\alpha} \rangle \leq 0$ and let $\Gamma \in \mathcal{K}^{hw}_{s_\alpha \cdot \lambda}$ be non-zero. Then,

$$E^{(n+2)} F^{(n+2)} \Gamma \cong \Gamma.$$

The claim would be proved if $EF^{(n+2)}\Gamma \in \mathcal{K}$, i.e. if $\Gamma(EF_\lambda^{(n+2)}) \cong 0$, though this is not the case in general. Since $\Gamma \in \mathcal{K}^{hw}_{s_\alpha \cdot \lambda}$ it follows that $E_\alpha \cdot \Gamma \cong 0$. By [Rou08, Lemma 4.6] it is also the case that $F \cdot F^{n+2} \cdot \Gamma \cong 0$, thus $F^{n+2} \cdot \Gamma \in \mathbf{\mathfrak{D}}_\alpha^*(\mathcal{V})^{lw}_{\lambda-\alpha}$. It will be shown that there is a subobject $\Sigma \hookrightarrow EF^{n+2}\Gamma$ with $E^{n+1}\Sigma \cong 0$ and $((EF^{(n+2)}\Gamma)/\Sigma)(1_\lambda) \cong 0$, i.e. $(EF^{(n+2)}\Gamma/\Sigma \in \mathcal{K}$. As E_α is exact, and $E^{n+1}\Sigma \cong 0$ there is an isomorphism,

$$\begin{aligned} E^{n+1}((EF^{n+2} \cdot \Gamma)/\Sigma) \cong& (E^{n+2}F^{n+2} \cdot \Gamma)/E^{n+1}\Sigma, \\ \cong& E^{n+2}F^{n+2}\Gamma. \end{aligned}$$

Now, $E^{n+2}F^{n+2}\Gamma$ is a non-zero multiple of $E^{(n+2)}F^{(n+2)}\Gamma \cong \Gamma$. It follows that the class of Γ is in the image of $E^{n+1} : K_\lambda \to K_{s_\alpha \cdot \lambda}$.

To construct $\Sigma \hookrightarrow EF^{n+2} \cdot \Gamma$ note that $F^{n+2} \cdot \Gamma \in \mathbf{\mathfrak{D}}_\alpha^*(\mathcal{V})^{lw}_{\lambda-\alpha}$, and so $F^{n+2} \cdot \Gamma : \mathfrak{A}_{\lambda-\alpha}$ factors through $\mathfrak{A}_{\lambda-\alpha}/\mathfrak{A}F_{\lambda-\alpha}$, which may be written,

$$F^{n+2} \cdot \Gamma : \mathcal{L}(\lambda - \alpha) \to \mathcal{V}.$$

Recall that $\mathcal{L}(\lambda-\alpha) \cong P_{n+2}^{\mathfrak{S}_{n+2}}-$free, where P_{n+2} is the polynomial algebra on $n+2$ variables and $\mathfrak{S}_{n+2}$ is the action of the symmetric group. In particular, under this identification

$F^{n+2} \cdot \Gamma(P_{n+2}^{\mathfrak{S}_{n+2}}) = (F^{n+2} \cdot \Gamma)(1_{\lambda-\alpha})$ with the action of $P_{n+2}^{\mathfrak{S}_{n+2}}$ coming from the Hecke algebra action $H_{n+2} \to End(F^{n+2})$ and the inclusion $P_{n+2} \hookrightarrow H_{n+2}$. Include $P_n \hookrightarrow P_{n+2}$ using the 'middle' n variables. This gives a morphism $P_n^{\mathfrak{S}_n} \to End(EF^{n+2} \cdot \Gamma(1_\lambda))$. Let $\Sigma : \mathcal{L}(\lambda) \to \mathcal{V}$ be defined on $P_n^{\mathfrak{S}_n}$ − free by the above map. As $\Gamma \in \mathcal{K}$, we have $E^{n+1}((EF^{n+2} \cdot \Gamma)(1_\lambda)) \cong 0$, which shows that indeed Σ extends to a well defined morphism $\mathcal{L}(\lambda) \to \mathcal{V}$. Moreover, such a morphism necessarily has $F \cdot \Sigma \cong 0$, so because Σ has weight λ, $E^{n+1}\Sigma \cong 0$.

□

The claims of Section I.22 suggest the following Conjecture.

Conjecture I.48. *The 2-functors $\mathbb{D}_\alpha^*$ satisfy braid relations.*

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Part II

Affine Hecke Algebras and Quiver Hecke Algebras

Abstract

We give a presentation of localized affine and degenerate affine Hecke algebras of arbitrary type in terms of weights of the polynomial subalgebra and varied Demazure-BGG type operators. We offer a definition of a graded algebra $\mathcal{H}$ whose category of finite-dimensional ungraded nilpotent modules is equivalent to the category of finite-dimensional modules over an associated degenerate affine Hecke algebra. Moreover, unlike the traditional grading on degenerate affine Hecke algebras, this grading factors through central characters, and thus gives a grading to the irreducible representations of the associated degenerate affine Hecke algebra. This paper extends the results [Rou11, Theorem 3.11], and [BK09, Main Theorem] where the affine and degenerate affine Hecke algebras for GL_n are shown to be related to quiver Hecke algebras in type A, and also secretly carry a grading.

Introduction

The representation theory of affine and degenerate affine Hecke algebras has a rich and continuing history. The work of Kazhdan and Lusztig in [KL87] (c.f. also [Gin98]) gives a parametrization and construction of irreducible modules over an affine Hecke algebra with equal parameters which aren't a root of unity. This parametrization is in the spirit of the Langlands program, and is carried out by constructing a geometric action of the Hecke algebra on equivariant (co)homology and K-theory of various manifolds related to the flag variety. Moreover, character formulas for irreducible representations are deduced from this theory, giving a satisfactory geometric understanding of the representation theory of such algebras. Unfortunately, for unequal parameters the geometric approach has not yielded as much progress.

More recently, the categorification of quantum groups has given a renewed interest to the theory of affine Hecke algebras of type A. It is shown in [BK09, Main Theorem], [Rou08, Theorem 3.16, 3.19] that a localization of the affine Hecke algebra $\mathscr{H}_n$ of GL_n at a maximal central ideal is isomorphic to a localization of a quiver Hecke algebra H_λ associated to a finite or affine type A Cartan matrix at a corresponding central ideal. This coincidence has been used in [McN12], [KR11] to give a new approach to classifying the irreducible representations of these algebras for parameters q which aren't a root of unity, as well as understanding the homological algebra of their representation categories. This algebraic approach replaces the set-up of the Langlands program with the main categorification result, that quiver Hecke algebras categorify quantum groups [KL09; Rou08]. Even further, this work shows that the affine Hecke algebras of type A carry a secret grading which recovers the quantum variable in the decategorification. It is a natural, and important question to ask if these techniques may be used for affine Hecke algebras in other types.

This paper defines a graded algebra $\mathcal{H}$ associated to any simply connected semisimple root datum and arbitrary parameters whose category of (ungraded) finite modules with a nilpotence condition is equivalent to the category of finite modules over the associated degenerate affine Hecke algebra. Thus degenerate affine Hecke algebras in other types and with unequal parameters are secretly graded as are those of type A. The presentation of $\mathcal{H}$ is a natural analogue of a quiver Hecke algebra, but with the symmetric group $\mathfrak{S}_n$ replaced with the Weyl group of the root datum, thus we refer to $\mathcal{H}$ as a quiver Hecke algebra as well. In this way we generalize [Rou08, Theorem 3.16], and give a grading on finite-dimensional irreducible representations of degenerate affine Hecke algebras. It is unclear if the algebras $\mathcal{H}$ are in fact related to the geometry of quiver-type varieties, but the considerable applications of the the theory of quiver Hecke algebras gives cause for their study. Moreover, there are many natural questions to be asked about the algebra $\mathcal{H}$. Could the graded characters of an irreducible module could be computed from the geometric standpoint mentioned above? Are there natural graded cyclotomic quotients of $\mathcal{H}$? Could one give an algebraic parametrization of the irreducible $\mathcal{H}$-modules and offer an algebraic construction of them following the work of [McN12]?

It should be noted that this paper uses localizations where other authors, [Lus89], [Hof+12] use completions. Our approach is rooted in finding a graded version of the degenerate affine Hecke algebras, which does not need the machinery of completions, and is perhaps a simpler approach in the first place.

We now briefly summarize the results of the paper. Fix a simply connected semisimple algebraic group over a field k, and let $h_0 \in k$ be a parameter. In this paper we define a locally unital localized quiver Hecke algebra $\mathscr{H}^{h_0}(G)$ associated to a data G with generators and relations. The algebra is given as a direct sum over Weyl group orbits in the dual to a maximal torus when $h_0 \neq 0$ and over Weyl group orbits in the dual space to a Cartan subalgebra when $h_0 = 0$:

$$\mathscr{H}^{h_0}(G) = \bigoplus_{\Lambda \in \mathscr{T}^{h_0}/W} \mathscr{H}^{h_0}(G)_\Lambda.$$

We associate a data G and parameter $h_0 \in k$ to the affine and degenerate affine Hecke algebras $\mathscr{H}, \mathbb{H}$ associated with this group where $h_0 = 0$ in the degenerate case. We then define a non-unital localization, $\dot{\mathscr{H}}^{h_0}$ of $\mathscr{H}, \mathbb{H}$ and produce an isomorphism,

$$\mathscr{H}^{h_0}(G) \xrightarrow{\sim} \dot{\mathscr{H}}^{h_0},$$

which generalizes [Rou11, Theorem 3.11, 3.12]. The graded version of a degenerate affine Hecke algebra, $\mathcal{H}$, is defined as a subalgebra of $\mathscr{H}^{h_0}(G)$ with $h_0 = 0$, and we give a separate presentation of this algebra with generators and relations.

In the last section 3, we define the *quiver Hecke algebra* $\mathcal{H}$ associated to a degenerate affine Hecke algebra $\mathbb{H}$. This is a graded algebra whose category of finite-dimensional ungraded nilpotent representations is equivalent to that of $\mathbb{H}$. In fact, we show in this section that every irreducible ungraded representation of $\mathcal{H}$ has a grading whose graded character is invariant under inverting the grading. In section 3.4 we study the representation theory of $\mathscr{H}^{h_0}(G)$ one weight space at a time. A crucial tool is the PBW-basis given in Theorem II.15. Using this basis along with a few simple results on the action of the Weyl group on the torus we provide two algebraic constructions of all irreducible representations which have a non-zero eigenspace V_λ with λ a standard parabolic weight, both in the equal and unequal parameters case. A highlight of this study is the structure of the so-called *weight*

Hecke algebra, ${}_\lambda\mathscr{H}_\lambda^{h_0}$, which turns out to be a matrix ring for λ a standard parabolic. This recovers and extends a well known result of Rodier in the case that λ is W-invariant, as well as a result of Bernstein-Zelevinsky in the case that λ is regular. This chapter includes an example algebraic computation of the graded characters of each irreducible representation of a degenerate affine Hecke algebra of type SL_3 with a specific central character.

We hope our construction can be used to obtain new and algebraic insights on representations of (degenerate) affine Hecke algebras at unequal parameters, where geometric methods are missing.

CHAPTER 1

Affine Hecke algebras

1.1 Bernstein's presentation

We recall the Bernstein presentation of affine Hecke algebras, following [Lus89]. Let $(X, Y, R, \check{R}, \Pi)$ be a (reduced) simply connected semisimple root datum. Thus X, Y are finitely generated free abelian groups in perfect pairing we denote by $\langle , \rangle$. Further, the finite subsets $R \subset X$, $\check{R} \subset Y$ of roots and coroots are in a given bijection $\alpha \mapsto \check{\alpha}$. The set R is invariant under the simple reflections, $s_\alpha \in \mathrm{GL}(X)$, which are given by,

$$s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha.$$

Similarly, it is required that $\check{R}$ be invariant under $s_{\check{\alpha}}$, defined by,

$$s_{\check{\alpha}}(y) = y - \langle y, \alpha \rangle \check{\alpha}.$$

Denote by $W \subset GL(X)$ the finite Weyl group of the system, and $\Pi \subset R$ a root basis. As the root system is reduced, the only multiples of a root α which are also roots are $\pm\alpha$. Lastly, the root datum being simply connected means X contains the fundamental weights $\{\omega_\alpha\}_{\alpha \in \Pi}$ defined as follows. Given $\alpha \in \Pi$ let $\omega_\alpha \in \mathbb{Q} \cdot R \subset \mathbb{Q} \otimes_{\mathbb{Z}} X$ be defined by $\langle \omega_\alpha, \check{\beta} \rangle = \delta_{\alpha,\beta}$ for all $\beta \in \Pi$. The assumption that the root system is simply connected simplifies a number of the formulas in [Lus89], in particular $\check{\alpha} \notin 2Y$ for any $\alpha \in \Pi$. This also simplifies the W-module structure of the group ring of X, as we shall see.

Let $\mathscr{A}$ be the group ring of X,

$$\mathscr{A} = \mathbb{Z}[e^x]_{x \in X} / (e^x e^{x'} = e^{x+x'}),$$

which is a domain. Finally, fix a parameter set given by a collection $q_{s_\alpha} = q_\alpha$ of indeterminates indexed by $\alpha \in \Pi$ such that $q_\alpha = q_\beta$ whenever the order $m_{\alpha,\beta}$ of $s_\alpha s_\beta$ in W is odd. As convention, we put $m_{\alpha,\alpha} = 2$.

With this data we associate the affine Hecke algebra, $\mathscr{H}$, which appears naturally in the complex, admissible representation theory of the associated algebraic group over p-adic fields.

Definition. Let $\mathscr{H}^f$ denote the finite Hecke algebra associated to Weyl group of the root datum. This is the $\mathbb{Z}[q_\alpha]_{\alpha\in\Pi}$-algebra generated by symbols $T_\alpha = T_{s_\alpha}, \alpha \in \Pi$, with the relations:

i. $\cdots T_\beta T_\alpha = \cdots T_\alpha T_\beta, \quad$ for $\alpha \neq \beta$, with $m_{\alpha,\beta}$ terms on both sides,

ii. $(T_\alpha + 1)(T_\alpha - q_\alpha) = 0, \quad \alpha \in \Pi.$

Denote by $\mathscr{H}$ the affine Hecke algebra of the root system. As an additive group,

$$\mathscr{H} = \mathscr{H}^f \otimes_{\mathbb{Z}} \mathscr{A}.$$

Let $\mathscr{H}^f$ and $\mathscr{A}$ be subrings, with the indeterminates q_α central and give $\mathscr{H}$ the following commutativity relation between $T_\alpha \in \mathscr{H}^f$, $f \in \mathscr{A}$:

$$T_\alpha f - s_\alpha(f)T_\alpha = (q_\alpha - 1)\frac{f - s_\alpha(f)}{1 - e^{-\alpha}}. \tag{1.1}$$

We remark that while $(1 - e^{-\alpha})^{-1} \notin \mathscr{A}$, the fraction appearing on right side of the above commutativity formula is in $\mathscr{A}$. For example,

$$\frac{e^x - e^{s_\alpha(x)}}{1 - e^{-\alpha}} = \begin{cases} e^x + e^{x-\alpha} + \cdots + e^{s_\alpha(x)+\alpha} & \langle x, \check{\alpha}\rangle > 0 \\ -(e^{x+\alpha} + e^{x+2\alpha} + \cdots + e^{s_\alpha(x)}) & \langle x, \check{\alpha}\rangle < 0 \\ 0 & \langle x, \check{\alpha}\rangle = 0. \end{cases}$$

1.2 Degenerate and interpolating affine Hecke algebras

The degenerate affine Hecke algebra $\mathbb{H}$ is introduced in this section, along with an algebra $\mathscr{H}^h$ which interpolates the affine and degenerate affine Hecke algebras. Let $(X, Y, R, \check{R}, \Pi)$ be a simply connected semisimple root datum. A set of parameters for the degenerate affine Hecke algebra, $\mathbb{H}$, is a collection of indeterminates c_α such that $c_\alpha = c_\beta$ when $m_{\alpha,\beta}$ is odd. Let $\mathcal{C} = \mathbb{Z}[c_\alpha]_{\alpha\in\Pi}$ be the parameter ring. Let $\mathbb{A} = S_{\mathbb{Z}}(X)$ the symmetric algebra of X over $\mathbb{Z}$, a polynomial algebra over $\mathbb{Z}$ with variables given by a basis of X.

Definition. As an additive group let

$$\mathbb{H} = \mathbb{Z}[W] \otimes_{\mathbb{Z}} \mathcal{C} \otimes_{\mathbb{Z}} \mathbb{A}.$$

Here $\mathbb{Z}[W]$ denotes the group ring of the Weyl group, generated by the simple reflections $s_\alpha \in W, \alpha \in \Pi$. Let $\mathbb{C}[W], \mathcal{C}$ and $\mathbb{A}$ be subrings of $\mathbb{H}$, the parameters c_α be central and give $\mathbb{H}$ the following commutativity relation between $s_\alpha \in \mathbb{Z}[W]$ and $x \in X \subset \mathbb{A}$:

$$s_\alpha \cdot x - s_\alpha(x) \cdot s_\alpha = c_\alpha \frac{x - s_\alpha(x)}{\alpha}.$$

We stop here to remark that as before the fraction on the right of the above formula does define an element of $\mathbb{A}$. Indeed,

$$\frac{x - s_\alpha(x)}{\alpha} = \langle x, \alpha \rangle.$$

The interpolating Hecke algebra, $\mathscr{H}^h$, is an algebra defined using a parameter h such that the specialization at zero gives, $\mathscr{H}^h \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(h) \cong \mathbb{H}$ whereas the specialization away from zero gives, $\mathscr{H}^h \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h^{\pm 1}] \cong \mathscr{H} \otimes_{\mathbb{Z}} \mathbb{Z}[h^{\pm 1}]$, where the parameters q_s, c_s for $\mathscr{H}$ and $\mathbb{H}$ are related by $q_s = 1 + hc_s$.

Consider the polynomial ring, $\hat{\mathscr{A}}^h$, over $\mathbb{Z}[h]$ with generators $\{P_x \mid x \in X\}$. The symmetric algebra, $\mathbb{A} = S_{\mathbb{Z}}(X)$, of X over $\mathbb{Z}$ is the quotient of $\hat{\mathscr{A}}^h$ by the relations $P_x + P_y = P_{x+y}$, and $h = 0$. Let $\mathscr{A}^h$ be the quotient of $\hat{\mathscr{A}}^h$ by the relations

$$P_x + P_y + hP_xP_y = P_{x+y},$$

$$P_0 = 0.$$

It is noted in [Hof+12] that $\mathscr{A}^h$ is a rational form of the formal group ring of the multiplicative formal group law over the abelian group X. Upon specializing at $h = 0$, $\mathscr{A}^h \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(h) \cong S_{\mathbb{Z}}(X)$. Let,

$$U_x = 1 + hP_x.$$

Notice that,

$$\begin{aligned} U_x U_y =& h^2 P_x P_y + h(P_x + P_y) + 1, \\ =& hP_{x+y} + 1, \\ =& U_{x+y}. \end{aligned}$$

It follows that $\mathscr{A}^h$ is isomorphic to the $\mathbb{Z}[h]$-subalgebra of $\mathscr{A} \otimes_{\mathbb{Z}} \mathbb{Z}[h^{\pm 1}]$ generated by $\{P_x = h^{-1}(e^x - 1)\}_{x \in X}$. Hence, $\mathscr{A}^h[h^{-1}]$ is isomorphic to the group ring of X over $\mathbb{Z}[h^{\pm 1}]$. Note that W acts $\mathbb{Z}[h]$-linearly on $\mathscr{A}^h$ and this action specializes to the action of W on $\mathscr{A}$ and $\mathbb{A}$.

Let $\mathcal{C}$ be the parameter ring, $\mathcal{C} = \mathbb{Z}[c_\alpha]_{\alpha \in \Pi}$ and let $q_\alpha = 1 + hc_\alpha \in \mathcal{C}[h]$ be parameters for the finite Hecke algebra $\mathscr{H}^f$ over $\mathcal{C}[h]$. We now define the *interpolating hecke algebra*, $\mathscr{H}^h$.

Definition. As an additive group let $\mathscr{H}^h$ be the tensor product,

$$\mathscr{H}^h = \mathscr{H}^f \otimes_{\mathbb{Z}[h]} \mathscr{A}^h.$$

Let $\mathscr{A}^h$ and $\mathscr{H}^f$ be subalgebras, let h, c_α be central and give $\mathscr{H}^h$ the following commutativity relation:

$$T_\alpha P_x - P_{s_\alpha(x)} T_\alpha = \begin{cases} 0 & \text{if } \langle x, \check{\alpha} \rangle = 0, \\ c_\alpha \left(\langle x, \check{\alpha} \rangle + h(P_x + P_{x-\alpha} + \dots P_{s_\alpha(x)+\alpha}) \right) & \text{if } \langle x, \check{\alpha} \rangle > 0, \\ c_\alpha \left(\langle x, \check{\alpha} \rangle - h(P_{x+\alpha} + P_{x+2\alpha} + \dots P_{s_\alpha(x)}) \right) & \text{if } \langle x, \check{\alpha} \rangle < 0. \end{cases}$$

Proposition II.1. *We have canonical identifications $\mathscr{H}^h \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(h) \cong \mathbb{H}$, and $\mathscr{H}^h \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h^{\pm 1}] \cong \mathscr{H} \otimes_{\mathbb{Z}} \mathbb{Z}[h^{\pm 1}]$.*

Proof. From the above relations we find $\mathscr{H}^h \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(h) \cong \mathbb{H}$ by sending P_x to the associated element x of the symmetric algebra. For the specialization with $\mathcal{C}[h^{\pm 1}]$, we note that for $\langle x, \check{\alpha} \rangle > 0$, the number of terms in the sum $P_x + P_{x-\alpha} + \cdots + P_{s_\alpha(x)+\alpha}$ is precisely $\langle x, \check{\alpha} \rangle$. Similarly, for $\langle x, \check{\alpha} \rangle < 0$ the number of terms in $P_{x+\alpha} + P_{x+2\alpha} + \cdots + P_{s_\alpha(x)}$ is precisely $-\langle x, \check{\alpha} \rangle$. Using the fact that $U_y = 1 + hP_y$, as well as $q_\alpha - 1 = hc_\alpha$ we see,

$$T_\alpha U_x - U_{s_\alpha(x)} T_\alpha = \begin{cases} 0 & \text{if } \langle x, \check{\alpha} \rangle = 0, \\ (q_\alpha - 1)\left(U_x + U_{x-\alpha} + \ldots U_{s_\alpha(x)+\alpha}\right) & \text{if } \langle x, \check{\alpha} \rangle > 0, \\ -(q_\alpha - 1)\left(U_{x+\alpha} + U_{x+2\alpha} + \ldots U_{s_\alpha(x)}\right) & \text{if } \langle x, \check{\alpha} \rangle < 0. \end{cases}$$

These are nothing more than the commutativity relations for the affine Hecke algebra $\mathscr{H}$. □

1.3 Demazure operators and polynomial representations

Now we define BGG operators and a variant of Demazure operators to discuss the representations of Hecke algebras on their commutative subalgebras. Define $\Delta_\alpha : S_{\mathbb{Z}}(X) \to S_{\mathbb{Z}}(X)$ by the following formula:

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}.$$

As usual, the right side of the formula actually lies in $S_{\mathbb{Z}}(X)$. Note that the commutativity relation for $\mathbb{H}$ may be written

$$s_\alpha \cdot f - s_\alpha(f) \cdot s_\alpha = c_\alpha \Delta_\alpha(f),$$

for any $f \in \mathbb{A}$. Define $D_\alpha : \mathscr{A} \to \mathscr{A}$ by the following formula:

$$D_\alpha(f) = \frac{f - s_\alpha(f)}{1 - e^{-\alpha}}.$$

Note that the commutativity relation for $\mathscr{H}$ may be written

$$T_\alpha f - s_\alpha(f) T_\alpha = (q_\alpha - 1) D_\alpha(f),$$

for any $f \in \mathscr{A}$

Recall that the algebra $\mathscr{A}^h$ is a subalgebra of the localization $\mathscr{A} \otimes_{\mathbb{Z}} \mathbb{Z}[h^{\pm 1}]$ by the inclusion which maps P_x to $h^{-1}(e^x - 1)$. We compute:

$$\begin{aligned} hD_\alpha(P_x) &= D_\alpha(e^x - 1) \\ &= D_\alpha(e^x) \\ &= \begin{cases} 0 & \text{if } \langle x, \check{\alpha} \rangle = 0, \\ \langle x, \check{\alpha} \rangle + h(P_x + P_{x-\alpha} + \dots P_{s_\alpha(x)+\alpha}) & \text{if } \langle x, \check{\alpha} \rangle > 0, \\ \langle x, \check{\alpha} \rangle - h(P_{x+\alpha} + P_{x+2\alpha} + \dots P_{s_\alpha(x)}) & \text{if } \langle x, \check{\alpha} \rangle < 0, \end{cases} \end{aligned}$$

and see that $D_\alpha : \mathscr{A}^h \to h^{-1}\mathscr{A}^h$. To put it informally, D_α is singular at $h = 0$. Nonetheless we have a well defined operator $hD_\alpha : \mathscr{A}^h \to \mathscr{A}^h$. This is summarized by the following.

Claim II.2. *Let $f \in \mathscr{A}^h$ and consider the operator $hD_\alpha : \mathscr{A}^h \to \mathscr{A}^h$. The commutativity relation for the interpolating Hecke algebra may be written,*

$$T_\alpha f - s_\alpha(f) T_\alpha = c_\alpha h D_\alpha(f).$$

In the specialization $\mathscr{A}^{h_0} \otimes_{\mathbb{Z}[h]} \mathbb{Z}[h]/(h) \cong S(X)$, the operator hD_α specializes to Δ_α.

We also remark that the classical Demazure operators, $\widetilde{D}_\alpha : \mathscr{A} \to \mathscr{A}$, given by

$$\widetilde{D}_\alpha : f \mapsto \frac{f - e^{-\alpha} s_\alpha(f)}{1 - e^{-\alpha}},$$

may be expressed in terms of D_α. Let $\rho \in X$ be defined by $\langle \rho, \check{\alpha} \rangle = 1, \alpha \in \Pi$, and recall that $\widehat{e^\rho} : \mathscr{A} \to \mathscr{A}$ is the invertible operator given by multiplication by e^ρ. We claim that,

$$\widetilde{D}_\alpha = \widehat{e^{-\rho}} \circ D_\alpha \circ \widehat{e^\rho}.$$

Indeed, $s_\alpha(e^\rho) = e^{-\alpha} e^\rho$, hence,

$$\begin{aligned} \widehat{e^{-\rho}} \circ D_\alpha \circ \widehat{e^\rho}(f) =& e^{-\rho} \cdot \frac{e^\rho f - s_\alpha(e^\rho f)}{1 - e^{-\alpha}}, \\ =& \frac{f - e^{-\alpha} s_\alpha(f)}{1 - e^{-\alpha}}. \end{aligned}$$

As the classical Demazure operators satisfy the braid relations, so do the D_α

$$\cdots D_\beta D_\alpha = \cdots D_\alpha D_\beta, \quad m_{\alpha,\beta} \text{ terms.}$$

It is also important to note that D_α does not specialize to Δ_α when $h \to 0$, but rather hD_α specializes to Δ_α. This proves that the Δ_α also satisfy the braid relations. Further, the quadratic relation $D_\alpha^2 = D_\alpha$ gives $(hD_\alpha)^2 = h(hD_\alpha)$, which specializes to 0 when $h \to 0$, showing $\Delta_\alpha^2 = 0$.

We remark that for $w \in W$, the operator D_w is uniquely defined by taking a reduced decomposition $w = s_1 \cdots s_r$ and setting $D_w = D_{s_1} \cdots D_{s_r}$. The assumption that $(X, Y, R, \check{R}, \Pi)$ is simply connected implies by the Pittie-Steinberg theorem that W forms a basis of $\mathrm{End}_{(\mathscr{A}^h)^W}(\mathscr{A}^h)$ as a left $\mathscr{A}^h$-module. Extending scalers from $\mathscr{A}^h$ to the fraction field $ff(\mathscr{A}^h)$, we see by a triangular base change that $\{D_w \mid w \in W\}$ forms a basis of $\mathrm{End}_{ff(\mathscr{A}^h)^W}(ff(\mathscr{A}^h))$.

Proposition II.3. *Consider the representation of the finite Hecke algebra $\mathscr{H}^f$ on $\mathcal{C}[h]$ given by sending $T_\alpha \mapsto q_\alpha$. There is an induced representation of $\mathscr{H}^h$ on $\mathscr{H}^h \otimes_{\mathscr{H}^f} \mathcal{C}[h] \cong \mathscr{A}^h \otimes_{\mathbb{Z}[h]} \mathcal{C}[h]$. Write $\widehat{T_\alpha}, \widehat{q_\alpha}, \cdots : \mathscr{A}^h \otimes_{\mathbb{Z}[h]} \mathcal{C}[h] \to \mathscr{A}^h \otimes_{\mathbb{Z}[h]} \mathcal{C}[h]$ for the action of $T_\alpha, q_\alpha, \ldots$ as operators on $\mathscr{A}^h \otimes_{\mathbb{Z}[h]} \mathcal{C}[h]$. Then,*

$$\widehat{T_\alpha - q_\alpha} : P_x \mapsto (c_\alpha + q_\alpha P_{-\alpha}) \cdot hD_\alpha(P_x). \tag{1.2}$$

Proof. The claim is obvious for $\langle x, \check{\alpha} \rangle = 0$. We show the case $\langle x, \check{\alpha} \rangle > 0$, the other case being nearly identical. By the commutativity relation for $\mathscr{H}^h$ we find:

$$\begin{aligned}
\widehat{T_\alpha - q_\alpha}(P_x) =&P_{s_\alpha(x)}T_\alpha \otimes 1 + (q_\alpha - 1)D_\alpha(P_x) - q_\alpha P_x, \\
=&q_\alpha P_{s_\alpha(x)} - q_\alpha P_x + c_{s_\alpha} h D_\alpha(P_x) \\
=&q_\alpha(P_{s_\alpha(x)} - P_x) + (q_\alpha - 1)(h^{-1}\langle x, \check{\alpha}\rangle + P_x + \cdots + P_{s_\alpha(x)+\alpha}) \\
=&q_\alpha \left(h^{-1}\langle x, \check{\alpha}\rangle + P_{x-\alpha} + \cdots + P_{s_\alpha(x)}\right) - \left(h^{-1}\langle x, \check{\alpha}\rangle + P_x + \cdots + P_{s_\alpha(x)+\alpha}\right) \\
=&(q_\alpha(1 + hP_{-\alpha}) - 1)D_\alpha(P_x) \\
=&(q_\alpha - 1 + q_\alpha h P_{-\alpha})D_\alpha(P_x) \\
=&(c_{s_\alpha} + q_\alpha P_{-\alpha})hD_\alpha(P_x) \\
=&(c_{s_\alpha} + P_{-\alpha} + hc_{s_\alpha}P_{-\alpha})hD_\alpha(P_x).
\end{aligned}$$

Here we have used that $D_\alpha(P_x) = h^{-1}D_\alpha(U_x)$, and that $1 + hP_{-\alpha} = U_{-\alpha}$ which satisfies the relation, $U_{-\alpha}U_y = U_{y-\alpha}$. □

1.4 Weight spaces of $\mathscr{H}^{h_0}$-modules

For this section fix k a field and fix a parameter $h_0 \in k$.

Definition. Let $\mathscr{A}^{h_0} = \mathscr{A}^h \otimes_{\mathbb{Z}[h]} k[h]/(h - h_0)$, $\mathscr{T}^{h_0} = \mathrm{Hom}_{k-alg}(\mathscr{A}^{h_0}, k)$. We call $\mathscr{T}^{h_0}$ the space of weights for the algebra $\mathscr{A}^{h_0}$, and given $\lambda \in \mathscr{T}^{h_0}$ and $f \in \mathscr{A}^{h_0}$ we let $f(\lambda) \in k$ denote the evaluation of λ at f.

As $\mathscr{A}^{h_0} \otimes_{\mathbb{Z}[h]} k[h]/(h - h_0)$ is isomorphic to the symmetric algebra of X over k for $h_0 = 0$ and the group ring of X over k for $h_0 \neq 0$, we find that $\mathscr{T}^{h_0} \cong Y \otimes_{\mathbb{Z}} k$ for $h_0 = 0$ and $\mathscr{T}^{h_0} \cong Y \otimes_{\mathbb{Z}} k^*$ for $h_0 \neq 0$. Motivated by the following paragraph we will call $\mathscr{T}^{h_0}$ the space of weights for the algebra $\mathscr{A}^{h_0}$ with parameter h_0.

Let V be an $\mathscr{A}^{h_0}$-module which is finite dimensional as a k-vector space. If k is algebraically closed, there is canonical generalized eigenspace (weight space) decomposition:

$$V = \bigoplus_{\lambda \in \Omega} V_\lambda,$$

where for $\lambda \in \mathscr{T}^{h_0}$, $V_\lambda = \{v \in V \mid (f - \lambda(f))^n v = 0 \text{ for all } f \in \mathscr{A}^{h_0}, n \gg 0\}$ and $\Omega = \Omega(V) \subset \mathscr{T}^{h_0}$ is the finite subset of λ such that $V_\lambda \neq 0$.

In discussing the weight spaces, V_λ, it is convenient to introduce a non-unital localization $\dot{\mathscr{A}}^{h_0}$ of $\mathscr{A}^{h_0}$. We set

$$\dot{\mathscr{A}}^{h_0} = \bigoplus_{\lambda \in \mathscr{T}^{h_0}} \mathscr{A}^{h_0}_\lambda,$$

where $\mathscr{A}^{h_0}_\lambda = \mathscr{A}^{h_0}[f^{-1} \mid \lambda(f) \neq 0]$ with the unit element denoted by 1_λ. We have an equivalence from the category of finite dimensional $\mathscr{A}^{h_0}$-modules with eigenvectors in k to the category of unital (with 1_λ as the projection onto V_λ) finite dimensional $\dot{\mathscr{A}}^{h_0}$-modules, sending $V \mapsto \bigoplus_{\lambda \in \Omega} V_\lambda$.

Now fix some set of parameters for $\mathscr{H}^f$ in k, in other words, fix an algebra morphism $\mathcal{C}[h] \to k$ for which $h \to h_0$. Define $\mathscr{H}^{h_0} = \mathscr{H}^h \otimes_{\mathcal{C}} k$. As we will see, if V is a finite dimensional representation of $\mathscr{H}^{h_0}$ and $\lambda \in \Omega(V)$ is a weight of the subalgebra $\mathscr{A}^{h_0} \subset \mathscr{H}^{h_0}$ which is not invariant under s_α, then T_α does not preserve the weight space V_λ, nor does it permute the weight spaces. In fact $T_\alpha(V_\lambda) \subset V_\lambda \oplus V_{s_\alpha(\lambda)}$. There is, however, a relation $1_\lambda T_\alpha 1_\lambda = f_\alpha 1_\lambda$, in $\mathrm{End}_{\mathbb{C}}(V)$, where $f_\alpha \in \mathscr{A}^{h_0}_\lambda$ will be made explicit. In this work we describe generators and relations of $\mathscr{H}^{h_0}$ which permute the weight spaces of finite representations. These generators are based on the elements $1_{s_\alpha(\lambda)} T_\alpha 1_\lambda$.

Before we define the localized Hecke algebra $\dot{\mathscr{H}}^{h_0}$, note that $(-P_{-\alpha}) \cdot hD_\alpha(f) = f - s_\alpha(f)$. This relation may be used to extend hD_α to an operator on $\dot{\mathscr{A}}^{h_0}$ as follows. Let $f \in \mathscr{A}^{h_0}_\lambda$ be given. If $s_\alpha(\lambda) \neq \lambda$, then $P_{-\alpha}$ is invertible in both $\mathscr{A}^{h_0}_\lambda$ and $\mathscr{A}^{h_0}_{s_\alpha(\lambda)}$, and we set,

$$hD_\alpha(f) = (-P_{-\alpha})^{-1} f - (-P_{-\alpha})^{-1} s_\alpha(f) \in \mathscr{A}^{h_0}_\lambda \oplus \mathscr{A}^{h_0}_{s_\alpha(\lambda)}.$$

If $s_\alpha(\lambda) = \lambda$, we appeal to the description of $\mathscr{A}^{h_0}$ as one of $\mathscr{A}$ or $\mathbb{A}$, where the action of $hD_\alpha(f)$ may be written as a fraction lying in $\mathscr{A}^{h_0}_\lambda$. This gives a well defined operator $hD_\alpha : \dot{\mathscr{A}}^{h_0} \to \dot{\mathscr{A}}^{h_0}$.

Definition. As an additive group let $\dot{\mathscr{H}}^{h_0} = \mathscr{H}^{h_0} \otimes_{\mathscr{A}^{h_0}} \dot{\mathscr{A}}^{h_0} \cong \bigoplus_{\lambda \in \mathscr{T}^{h_0}} \mathscr{H}^f \otimes_{\mathbb{C}} \mathscr{A}^{h_0}_\lambda$. Let the algebra $\dot{\mathscr{A}}^{h_0}$ be a subalgebra of $\dot{\mathscr{H}}^{h_0}$. For $\Lambda \in \mathscr{T}^{h_0}/W$, put $1_\Lambda = \sum_{\lambda \in \Lambda} 1_\lambda$ and

$T_\alpha^\Lambda = T_\alpha \otimes 1_\Lambda \in \dot{\mathscr{H}}^{h_0}$. For each such Λ let the inclusion of the finite Hecke algebra,

$$\mathscr{H}^f \hookrightarrow \bigoplus_{\lambda \in \Lambda} \mathscr{H}^{h_0} \otimes_{\mathscr{A}^{h_0}} \mathscr{A}_\lambda^{h_0} \subset \dot{\mathscr{H}}^{h_0},$$
$$T_\alpha \mapsto T_\alpha^\Lambda.$$

be an algebra morphism. The following decomposition,

$$\dot{\mathscr{H}}^{h_0} = \bigoplus_{\Lambda \in \mathscr{T}^{h_0}/W} \dot{\mathscr{H}}^{h_0} 1_\Lambda.$$

gives $\dot{\mathscr{H}}^{h_0}$ the structure of a locally unital algebra with the following commutativity relation:

$$T_\alpha^\Lambda f - s_\alpha(f) T_\alpha^\Lambda = c_\alpha h D_\alpha(f),$$

for any $f \in \dot{\mathscr{A}}^{h_0} 1_\Lambda$.

First note that the unital algebra $\dot{\mathscr{H}}^{h_0} 1_\Lambda$ in the direct sum above is a subalgebra of the completion of $\mathscr{H}^{h_0}$ at the kernel of the associated central character $\Lambda : (\mathscr{A}^{h_0})^W \to k$ [Lus89]. It is in fact the subalgebra generated by $\mathscr{H}^{h_0}$ and the localizations $\mathscr{A}_\lambda^{h_0} 1_\lambda$ in the completion of $\mathscr{A}^{h_0}$ with respect to this ideal.

We show that a finite dimensional representation of $\mathscr{H}^{h_0}$ for which the eigenvectors for $\mathscr{A}^{h_0}$ are in k indeed gives rise to a representation of $\dot{\mathscr{H}}^{h_0}$, where 1_λ acts as the projection onto the λ weight space.

Lemma II.4. *Let V be a finite dimensional representation of $\mathscr{H}^{h_0}$ for which the eigenvalues of $\mathscr{A}^{h_0}$ are in k. Let $1_\lambda \in \mathrm{End}_{\mathbb{C}}(V)$ be the projection onto V_λ. If $\alpha \in \Pi$ with $s_\alpha(\lambda) = \lambda$ then $T_\alpha(\lambda) \subset V_\lambda$, hence,*

$$\begin{aligned} T_\alpha 1_\lambda =& 1_\lambda T_\alpha \\ =& 1_\lambda T_\alpha + c_\alpha h D_\alpha(1_\lambda). \end{aligned}$$

Moreover, if $s_\alpha(\lambda) \neq \lambda$, then $(P_{-\alpha})^{-1} \in \mathscr{A}_\lambda^{h_0}, \mathscr{A}_{s_\alpha(\lambda)}^{h_0}$. Thus the eigenvalues of $P_{-\alpha}$ are non-zero and hence $P_{-\alpha}$ is an invertible operator on $V_\lambda, V_{s_\alpha(\lambda)}$. Also, $T_\alpha(V_\lambda) \subset V_\lambda \oplus V_{s_\alpha(\lambda)}$

and the following commutativity relation holds,

$$
\begin{aligned}
T_\alpha 1_\lambda - 1_{s_\alpha(\lambda)} T_\alpha =& c_\alpha(-P_{-\alpha})^{-1}(1_\lambda - 1_{s_\alpha}),\\
=& c_\alpha h D_\alpha(1_\lambda).
\end{aligned}
$$

Proof. Let $x \in X$ and $z \in k$. With the simple identity, $a^N - b^N = (a-b)\sum_{0\le i\le N-1} a^i b^{N-i-1}$ we find

$$
\begin{aligned}
(-P_{-\alpha})hD_\alpha((P_x - z)^N) =& (P_x - z)^N - (P_{s_\alpha(x)} - z)^N,\\
=& (P_x - P_{s_\alpha(x)}) \sum_{0\le i\le N-1} (P_x - z)^i (P_{s_\alpha(x)} - z)^{N-i-1},\\
=& ((-P_{-\alpha})hD_\alpha(P_x)) \cdot \sum_{0\le i\le N-1} (P_x - z)^i (P_{s_\alpha(x)} - z)^{N-i-1}.
\end{aligned}
$$

As $\mathscr{A}^{h_0}$ is a domain,

$$
(P_x - z)^N T_\alpha = T_\alpha (P_{s_\alpha(x)} - z)^N + hD_\alpha(P_x) \sum_{0\le i\le N-1} (P_x - z)^i (P_{s_\alpha(x)} - z)^{N-i-1}.
$$

Let $\lambda \in \mathscr{T}^{h_0}$ with $s_\alpha(\lambda) = \lambda$ and suppose N, z are such that $(P_x - z)^{\lfloor N/2\rfloor}(V_\lambda) = (P_{s_\alpha(x)} - z)^{\lfloor N/2\rfloor}(V_\lambda) = 0$. In this case, the above expression shows that $(P_x - z)^N T_\alpha 1_\lambda v = 0$ for $v \in V$. Thus, $T_\alpha(V_\lambda) \subset V_\lambda$ and $T_\alpha 1_\lambda = 1_\lambda T_\alpha$.

Now, suppose $s_\alpha(\lambda) \neq \lambda$ so that $\lambda(P_{-\alpha}) \neq 0$. Thus, $(P_{-\alpha})^{-1} \in \mathscr{A}^{h_0}_\lambda, \mathscr{A}^{h_0}_{s_\alpha(\lambda)}$ and $(P_{-\alpha})^{-1}$ may be considered as an operator on $V_\lambda, V_{s_\alpha(\lambda)}$ (as the operator $P_{-\alpha}$ has a lone eigenvalue which is non-zero). We also suppose that N, z are picked so that $(P_x - z)^N(V_\lambda) = 0$. Then,

$$
\begin{aligned}
(P_{s_\alpha(x)} - z)^N (T_\alpha 1_\lambda - c_\alpha(-P_{-\alpha})^{-1} 1_\lambda) v =& T_\alpha (P_{s_\alpha(x)} - z)^N 1_\lambda v\\
&+ c_\alpha(-P_{-\alpha})^{-1}(P_{s_\alpha(x)} - z)^N 1_\lambda v,\\
=& 0.
\end{aligned}
$$

In particular, it follows that $T_\alpha(V_\lambda) \subset V_\lambda \oplus V_{s_\alpha(\lambda)}$, and moreover,

$$
\begin{aligned}
T_\alpha 1_\lambda - 1_{s_\alpha(\lambda)} T_\alpha =& c_\alpha(-P_{-\alpha})^{-1}(1_\lambda - 1_{s_\alpha}),\\
=& c_\alpha h D_\alpha(1_\lambda).
\end{aligned}
$$

in $\mathrm{End}_{\mathbb{C}}(V)$.

□

To show that a finite dimensional representation of $\mathscr{H}^{h_0}$ lifts to a finite dimensional representation of $\dot{\mathscr{H}}^{h_0}$ we note that the ring of W-invariants, $(\mathscr{A}^{h_0})^W$, is the center of $\mathscr{H}^{h_0}$ and each finite representation splits into a direct sum of generalized eigenspaces V_Λ of the center of $\mathscr{H}^{h_0}$ according to the central characters $\Lambda \in \mathscr{T}^{h_0}/W$:

$$V = \sum_{\Lambda \in \mathscr{T}^{h_0}/W} V_\Lambda.$$

We can decompose the operators $T_\alpha = \sum_\Lambda T_\alpha^\Lambda$, $f = \sum_\lambda f_\lambda$. The above lemma shows that these operators give an action of $\dot{\mathscr{H}}^{h_0}$.

CHAPTER 2

Localized Quiver Hecke Algebras

2.1 Weyl group orbits on tori

We will need the following lemmas to study the localized quiver Hecke algebras. Let W be the Weyl group of the reduced root datum, $(X, Y, R, \check{R}, \Pi)$. Recall that $\Pi \subset R$ defines a length function ℓ on W.

Definition. Let $W' \subset W$ be a subgroup. We say that W' is a standard parabolic subgroup if there is a subset $\Pi^\lambda \subset \Pi$ so that W' is the subgroup generated by $\{s_\alpha \in W \mid \alpha \in \Pi^\lambda\}$. We call a subgroup parabolic if it is W-conjugate to a standard parabolic subgroup.

Let $\lambda \in \mathscr{T}^{h_0}$. We say that λ is standard parabolic, if there is a subset $\Pi^\lambda \subset \Pi$ so that the stabilizer of λ in the Weyl group W is the standard parabolic subgroup generated by $\{s_\alpha \in W \mid \alpha \in \Pi^\lambda\}$. We call a weight parabolic (resp. parabolic with respect to W^P) if it is in the W-orbit (resp. W^P-orbit) of a standard parabolic weight (resp. standard parabolic with respect to W^P).

Lemma II.5. *Let $w = s_{\alpha_n} \cdots s_{\alpha_1} = s_{\beta_n} \cdots s_{\beta_1}$ be two reduced expressions. Then there is a permutation p of $\{1, \cdots, n\}$ so that whenever $p(i) = j$,*

$$s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i) = s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j),$$
$$s_{\alpha_n} \cdots s_{\alpha_{i+1}}(\alpha_i) = s_{\beta_n} \cdots s_{\beta_{j+1}}(\beta_j).$$

Proof. This is simply a restatement of the following standard theorem, see [Hum90, Section 1,7].

Let $w = s_{\alpha_1} \cdots s_{\alpha_n}$ be a reduced decomposition. Put $\gamma_i = s_{\alpha_n} \cdots s_{\alpha_{i+1}}(\alpha_i)$. Then the roots $\gamma_1, \ldots, \gamma_n$ are all distinct and the set $\{\gamma_1, \ldots, \gamma_n\}$ equals $R^+ \cap w^{-1}R^-$, which is the set of $\gamma \in R^+$ such that $w(\gamma) \in R^-$.

□

Let $(X, Y, R, \check{R}, \Pi)$ be a root data associated to the k-split semisimple algebraic group G. Associated to this data is a maximal torus, $\mathscr{T} = Y \otimes_{\mathbb{Z}} k^*$ and a dual torus, $\mathscr{T}^* = X \otimes_{\mathbb{Z}} k^*$, both with actions of the Weyl group W. The purpose of this section is to collect some results on the stabilizers of elements of $\mathscr{T}$ in the Weyl group W.

Let $\lambda \in \mathscr{T}$ and denote by $\langle\lambda\rangle$ the smallest closed subgroup of $\mathscr{T}$ containing λ.

Claim II.6. *The subgroup $\langle\lambda\rangle$ is the direct sum of a cyclic subgroup generated by $\zeta \in \mathscr{T}$ with finite order and a torus $\mathscr{S} \subset \mathscr{T}$.*

Proof. The subgroup $\langle\lambda\rangle$ has finitely many components. Let $\mathscr{S}$ be the identity component of $\langle\lambda\rangle$. The morphism from $\langle\lambda\rangle \to \langle\lambda\rangle/\mathscr{S}$ is a split surjection, as the category of diagonalizable groups is dual to the category of finitely generated abelian groups. Considering that the set of components of $\langle\lambda\rangle$ containing the powers $\lambda^i, i \in \mathbb{Z}$ is a closed subgroup of $\mathscr{T}$, we find that the group of components is a cyclic group. It follows that $\langle\lambda\rangle$ is the direct sum of a cyclic subgroup generated by $\zeta \in \mathscr{T}$ with finite order and a torus $\mathscr{S} \subset \mathscr{T}$. □

Definition. Let $\tilde{\alpha} \in R^+$ be the highest root. Define an augmented standard parabolic subgroup of W to be a subgroup W' such that there is a subset $I \subset \Pi$ for which W' is generated by $\{s_\alpha\}_{\alpha \in I} \cup \{s_{\tilde{\alpha}}\}$. A subgroup is called an augmented parabolic subgroup if it is W-conjugate to an augmented standard parabolic subgroup.

Corollary II.7. *Assume $(X, Y, R, \check{R}, \Pi)$ to be simply connected semisimple. Given $\lambda \in \mathscr{T}$, the centralizer of λ in the Weyl group W is the intersection of a parabolic subgroup with an augmented parabolic subgroup.*

Proof. As the Weyl group acts by algebraic automorphisms of $\mathscr{T}$, the stabilizer of λ in W is equal to the intersection of the stabilizer in W of $\mathscr{S}$ with the stabilizer of ζ. As $\mathscr{S} \subset G$

is a torus, the centralizer $Z_G(\mathscr{S})$ is a connected, reductive subgroup [Hum75, section 26.2]. Moreover the centralizer $Z_G(\mathscr{S})$ is the Levi subgroup of a parabolic subgroup of G (see [DM91, Proposition 1.22]) and hence the centralizer $Z_W(\mathscr{S})$ is a parabolic subgroup.

It remains only to show that the centralizer of an element $\zeta \in \mathscr{T}$ of finite order is an augmented parabolic subgroup. We may restrict our attention to the group $Y \otimes_{\mathbb{Z}} \langle\zeta\rangle$. If ζ has order n, then this is isomorphic to the group $Y \otimes_{\mathbb{Z}} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. Lift $\zeta \in Y \otimes_{\mathbb{Z}} \langle\zeta\rangle$ to an element in the Euclidean space $z \in Y \otimes_{\mathbb{Z}} \frac{1}{n}\mathbb{Z} \subset Y \otimes_{\mathbb{Z}} \mathbb{R}$, and note that for $w \in W$ we have $w(\zeta) = \zeta$ if and only if $w(z) - z \in Y \otimes \mathbb{Z} \subset Y \otimes_{\mathbb{Z}} \mathbb{R}$. This is the same as asserting that there exists $t \in Y$ with $w(z) - y = z$. For $w \in W, y \in Y$ fixed, the transformation $x \mapsto w(x) - y$ is an element of the affine group W_a, which is the semidirect product of the Weyl group with the translation group Y. As Y has a basis given by $\{\check{\alpha}\}_{\alpha\in\Pi}$, W_a is a Coxeter group with Coxeter generators $\{s_\alpha\}_{\alpha\in\Pi} \cup \{s_{\tilde{\alpha},1}\}$ where $s_{\tilde{\alpha},1}(x) = s_{\tilde{\alpha}}(x) + \check{\alpha}$ (see [Hum90]). Examining the Weyl group stabilizers of elements of the fundamental alcove $\{x \in Y \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle x, \alpha\rangle > 0\alpha \in \Pi, \langle x, \tilde{\alpha}\rangle = 1\}$, whose walls are the hyperplanes orthogonal to $\alpha, \alpha \in \Pi$ and $\tilde{\alpha}$, we find that the stabilizer of $\zeta \in \mathscr{T}$ is conjugate to an augmented parabolic subgroup. $\square$

Remark II.8. Let $(X, Y, R, \check{R}, \Pi)$ be any root data for which X contains the fundamental weights $\{\omega_\alpha\}_{\alpha\in\Pi} \subset \mathbb{Q} \cdot R \subset X \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $Y' = \mathbb{Z} \cdot \check{R}$ be the $\mathbb{Z}$-span of the coroots in Y. We have a natural inclusion $i : Y' \hookrightarrow Y$. Define the projection $p : Y \to Y'$ by the following formula,

$$p(y) = \sum_{\alpha\in\Pi} \omega_\alpha(y)\check{\alpha}.$$

Then p is a split surjection, $i \circ p(y') = y'$ for $y' \in Y'$. Moreover, p, i are both W-equivariant. Abusing notation, let $p : Y \otimes_{\mathbb{Z}} k^*$ and $i : Y' \otimes_{\mathbb{Z}} k^*$ denote the corresponding maps on tori. Again, $p : Y \otimes_{\mathbb{Z}} k^* \to Y' \otimes_{\mathbb{Z}} k^*$ is a split W-equivariant surjection, and hence is a bijection on W-orbits.

We compute the stabilizers of weights in rank 2, which by the above remark is equivalent to computing $W^{\alpha,\beta}$-orbits on general tori associated to simply connected root data. Given a rank 2 root system let $(X, Y, R, \check{R}, \Pi)$ be a root data for which X has the basis $\{\omega_\alpha\}_{\alpha\in\Pi}$ and Y has the basis $\{\check{\alpha}\}_{\alpha\in\Pi}$. We compute the stabilizers of elements $\lambda \in \mathscr{T} = Y \otimes_{\mathbb{Z}} k^*$.

In type $A_1 \times A_1, A_2$ we find that the stabilizer of any element is already a parabolic subgroups of W.

In types B_2, G_2, consider the action of W on the vector space $Y \otimes_{\mathbb{Z}} k$. The only non-parabolic, augmented standard parabolic subgroup of W is generated by the reflection $s_{\tilde{\alpha}}$ and one other reflection. As the two reflections have distinct 1-dimensional eigenspaces we find that the subspace of fixed points on $Y \otimes_{\mathbb{Z}} k = \mathrm{Lie}(Y \otimes_{\mathbb{Z}} k^*)$ is the zero subspace. It follows that the group of fixed points on $\mathscr{T}$ of a non-parabolic, augmented standard parabolic subgroup of W is a finite torsion subgroup of $\mathscr{T}$.

Consider, as above, the real vector space $E = Y \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that the torsion subgroup of $\mathscr{T}$ may be embedded as a subgroup of $Y \otimes_{\mathbb{Z}} \mathbb{Q} \subset E/Y$. It is plain to check that the set of elements λ in the fundamental alcove of E which are stabilized by $s_{\tilde{\alpha},1}$ and one other Coxeter reflection actually lie in $\frac{1}{2}Y$, which gives us the following corollary.

Corollary II.9. *If $(X, Y, R, \check{R}, \Pi)$ is a simply connected semisimple root system of rank 2, and $\lambda \in \mathscr{T} = Y \otimes_{\mathbb{Z}} k^*$ satisfies $\mathrm{stab}_W(\lambda)$ is a non-parabolic subgroup of W, then λ satisfies $\lambda^2 = 1 \in \mathscr{T}$, and for $\alpha \in \Pi$,*

$$\langle \alpha, \lambda \rangle = \begin{cases} 1, \text{ if } s_\alpha(\lambda) = \lambda, \\ -1, \text{ if } s_\alpha(\lambda) \neq \lambda. \end{cases}$$

2.2 The datum G and its conditions

Remark II.10. We now define an abstract set of datum on which our definition of localized quiver Hecke algebra depends. The definition of localized quiver Hecke algebra we give extends that of [Rou08], [Rou11]. We briefly paraphrase the set-up of [Rou11] where actually two quivers are given.

Fix a finite set I as well as a Cartan matrix $C = (a_{i,j})_{i,j\in I}$. This determines the first quiver, whose vertices are identified with I, and with $-a_{i,j}$ arrows from i to j. Next, consider the action of the symmetric group $\mathfrak{S}_n$ on the set I^n. We form the quiver $\Psi_{I,n}(\Lambda)$ whose vertices are identified with a fixed $\mathfrak{S}_n$-orbit Λ in I^n. There are two types of arrows for the quiver $\Psi_{I,n}(\Lambda)$. The first type is an arrow $x_i : \lambda \to \lambda$ for each $1 \leq i \leq n$, and second type is an arrow $\tau_i : \lambda \to s_i(\lambda)$, where $s_i \in \mathfrak{S}_n$ is the element $(i, i+1)$.

Finally, the first quiver determines a set of polynomials $Q = (Q_{i,j}(u,v))_{i,j \in I}$ with which one constructs for each $\mathfrak{S}_n$-orbit in I^n the *quiver Hecke algebra*, $H(Q)_\Lambda$. The algebra $H(Q)_\Lambda$ is a quiver algebra with relations on the second quiver $\Psi_{I,n}$, with relations determined by the polynomial data Q. For Γ of type A the algebras $H(Q)_\Lambda$ may be identified with localizations of affine Hecke algebras of type A. The quiver Hecke algebras defined in this paper do not give a way of generalizing the first quiver, but instead generalize the data Q and second quiver $\Psi_{I,n}(\Lambda)$ to give a quiver Hecke algebra presentation of affine Hecke algebras of any type.

Fix $h_0 \in k$. Then $\mathscr{A}^{h_0}$ is the specialization $\mathscr{A}^{h_0} = \mathscr{A}^h \otimes_{\mathbb{Z}[h]} k[h]/(h - h_0)$ of the interpolating ring $\mathscr{A}^h$ at $h \to h_0$, and hD_α as the specialization of the Demazure operator. Thus, either $h_0 = 0$ in which case $\mathscr{A}^{h_0}$ is isomorphic to the symmetric algebra and hD_α is the BGG operator Δ_α, or $h_0 \neq 0$ in which case $\mathscr{A}^{h_0}$ is isomorphic to the group ring $k[X]$ and hD_α is a scaler multiple of the Demazure operator. Recall that $\mathscr{T}^{h_0} = \mathrm{Hom}_{k-alg}(\mathscr{A}^{h_0}, k)$ is the set of k-algebra homomorphisms from $\mathscr{A}^{h_0}$ to k, which is either isomorphic to $\mathfrak{h} = Y \otimes_{\mathbb{Z}} k$ for $h_0 = 0$, or isomorphic to $\mathscr{T} = Y \otimes_{\mathbb{Z}} k^*$, for $h_0 \neq 0$. We have

$$\dot{\mathscr{A}}^{h_0} = \bigoplus_{\lambda \in \mathscr{T}^{h_0}} \mathscr{A}^{h_0}_\lambda,$$

the localized, non-unital algebra associated to $\mathscr{A}^{h_0}$, where $\mathscr{A}^{h_0}_\lambda$ is the localization of $\mathscr{A}^{h_0}$ at $\lambda \in \mathscr{T}^{h_0}$.

Let $G = (G^\lambda_\alpha)_{\lambda \in \mathscr{T}^{h_0}, \alpha \in \Pi}$ be a collection of non-zero rational functions, $G^\lambda_\alpha \in \mathscr{A}^{h_0}_\lambda \backslash \{0\} = \dot{\mathscr{A}}^{h_0} 1_\lambda \backslash \{0\}$. We list now a few conditions that this data is required to satisfy before we can define the localized quiver Hecke algebra $\mathscr{H}^{h_0}(G)$.

For notational purposes, we need the following lemma.

Lemma II.11. *Let $\alpha, \beta \in \Pi$. We put $W^{\alpha,\beta} = \langle s_\alpha, s_\beta \rangle$, the subgroup of W generated by s_α, s_β. We also put $m = m_{\alpha,\beta}$ as the order of $s_\alpha s_\beta$ when $\alpha \neq \beta$ and 2 otherwise, and finally we write $w_{\alpha,\beta}$ for the longest element in $W^{\alpha,\beta}$. We have,*

$$w_{\alpha,\beta}(\alpha) = \begin{cases} -\alpha & \text{if } m \text{ even,} \\ -\beta & \text{if } m \text{ odd.} \end{cases}$$

Instead of using cases, we will simply write $w_{\alpha,\beta} s_\alpha(\alpha)$ which is equal to α for $m_{\alpha,\beta}$ even, and β for $m_{\alpha,\beta}$ odd.

Definition. Let $\lambda \in \mathscr{T}^{h_0}$ $\alpha \in \Pi$. The weight λ is said to be α-exceptional if there exists $\beta \in \Pi$ with λ not parabolic with respect to $W^{\alpha,\beta}$ and $s_\alpha(\lambda) \neq \lambda$. If β is as above, then it is automatic that $m_{\alpha,\beta} = 4, 6$.

Let $G = (G_\alpha^\lambda)_{\lambda \in \mathscr{T}^{h_0}, \alpha \in \Pi}$ and assume that G satisfies the following conditions.

1. For λ, α as above,

   $$s_\alpha(G_\alpha^\lambda) = G_\alpha^{s_\alpha(\lambda)}.$$

   We shall refer to this as the *associative relation* on G.

2. If $s_\alpha(\lambda) = \lambda$ then $G_\alpha = 1$.

3. For any $\alpha, \beta \in \Pi$,

   $$w_{\alpha,\beta} s_\alpha(G_\alpha^\lambda) = G_{w_{\alpha,\beta} s_\alpha(\alpha)}^{w_{\alpha,\beta} s_\alpha(\lambda)}$$

   where, again, $w_{\alpha,\beta} \in W^{\alpha,\beta}$ is the longest element. We will refer to this relation as the *braid relation* on G. Note that in the case $\alpha = \beta$, we have that $w_{\alpha,\alpha} = s_\alpha$ and the condition is vacuous.

4. If λ is α-exceptional then $G_\alpha^\lambda = 1$.

2.3 The localized quiver Hecke algebra

Let $G = (G_\alpha^\lambda)_{\lambda \in \mathscr{T}^{h_0}, \alpha \in \Pi}$ be a collection satisfying the conditions in section 2.2. We define the localized quiver Hecke algebra $\mathscr{H}^{h_0}(G)$ associated to this choice in analogy with a quiver algebra with relations over the ring $\mathscr{A}$, see [Rou11]. Underlying this construction of a quiver algebra with relations is the quiver with vertices $\mathscr{T}^{h_0}$, and arrows $f : \lambda \to \lambda$, $r_\alpha^\lambda : \lambda \to s_\alpha(\lambda)$, whenever $f \in \mathscr{A}_\lambda^{h_0}, \alpha \in \Pi$. We remark that the arrows r_α^λ give precisely the Cayley graph of the action of $\{s_\alpha \mid \alpha \in \Pi\}$ on $\mathscr{T}^{h_0}$.

Definition. First, define $\widetilde{\mathscr{H}^{h_0}(G)}$ as the non-unitary algebra given by adjoining generators r_α^λ to $\dot{\mathscr{A}}^{h_0}$ which satisfy the following relations.

- $r_\alpha^\lambda 1_\nu = 1_{s_\alpha(\nu)} r_\alpha^\lambda = \delta_{\lambda,\nu} r_\alpha^\lambda$.
- $r_\alpha^{s_\alpha(\lambda)} r_\alpha^\lambda = \begin{cases} G_\alpha^\lambda & \text{if } s_\alpha(\lambda) \neq \lambda, \\ h_0 r_\alpha^\lambda & \text{if } s_\alpha(\lambda) = \lambda. \end{cases}$
- For $f \in \mathscr{A}_\lambda$,
$$r_\alpha^\lambda f - s_\alpha(f) r_\alpha^\lambda = \begin{cases} 0 & \text{, if } s_\alpha(\lambda) \neq \lambda, \\ hD_\alpha(f) & \text{, if } s_\alpha(\lambda) = \lambda. \end{cases}$$
- For $\alpha, \beta \in \Pi$ distinct and $\lambda \in \mathscr{T}^{h_0}$ a *standard parabolic weight* with respect to $\{\alpha, \beta\}$, or λ which is not parabolic with respect to $W^{\alpha,\beta}$, but fixed by one of s_α, s_β:
$$r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_1}^{\lambda_1} = r_{\beta_m}^{\mu_m} \cdots r_{\beta_1}^{\mu_1},$$
where $m = m_{\alpha,\beta}$ is the order of $s_\alpha s_\beta$ in W,
$$\alpha_i = \begin{cases} \alpha & \text{, } i \text{ odd,} \\ \beta & \text{, } i \text{ even,} \end{cases}$$
$$\beta_i = \begin{cases} \beta & \text{, } i \text{ odd,} \\ \alpha & \text{, } i \text{ even,} \end{cases}$$

and $\lambda_i = s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda)$, $\mu_i = s_{\beta_{i-1}} \cdots s_{\beta_1}(\mu)$. This is known as the braid relation.

Finally, let $\mathscr{I}_\lambda$ be the right $\mathscr{A}_\lambda^{h_0}$-module consisting of elements $I \in \widetilde{\mathscr{H}^{h_0}(G)}1_\lambda$ such that there is $f \in \mathscr{A}_\lambda^{h_0} \backslash \{0\}$ with $I \cdot f = 0$. We define $\mathscr{H}^{h_0}(G) = \widetilde{\mathscr{H}^{h_0}(G)} / \bigoplus_\lambda I_\lambda$. Thus, there is no right polynomial torsion in $\mathscr{H}^{h_0}(G)$.

Given simply connected root datum we will produce in section 3.1 a family G and an isomorphism $\mathscr{H}^{h_0}(G) \to \dot{\mathscr{H}}^{h_0}$.

2.4 The PBW property and a faithful representation

This section analyzes the structure of $\mathscr{H}^{h_0}(G)$. We start by defining a filtration $(\mathscr{F}^n)$ on $\mathscr{H}^{h_0}(G)$, letting $\mathscr{F}^n \subset \mathscr{H}^{h_0}(G)$ be the right $\dot{\mathscr{A}}^{h_0}$-linear span of all products $r_{\alpha_k}^{\lambda_k} \cdots r_{\alpha_1}^{\lambda_1}$ with up to n terms in them. We see $\mathscr{F}^n \cdot \mathscr{F}^m \subset \mathscr{F}^{n+m}$.

Lemma II.12. *Fix some $\lambda \in \mathscr{T}^{h_0}$ and let $\mathfrak{B}_1 = (\alpha_n, \cdots, \alpha_1)$, $\mathfrak{B}_2 = (\beta_n, \cdots, \beta_1)$, $\alpha_i, \beta_i \in \Pi$, be two ordered collections of simple roots with the same cardinality such that $s_{\alpha_n} \cdots s_{\alpha_1} = s_{\beta_n} \cdots s_{\beta_1}$. Then,*

$$r_{\alpha_n}^{\lambda_n} \cdots r_{\alpha_1}^{\lambda_1} - r_{\beta_n}^{\mu_n} \cdots r_{\beta_1}^{\mu_1} \in \mathscr{F}^{n-1},$$

where, $\lambda_i = s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda)$, $\mu_i = s_{\beta_{i-1}} \cdots s_{\beta_1}(\lambda)$.

Proof. We prove the assertion by induction on n. The cases of $n = 0, 1$ are trivial.

First, put $w = s_{\alpha_n} \cdots s_{\alpha_1}$. Suppose that $\ell(w) < n$. By the deletion condition, there exists, $1 \leq i < j \leq n$ with,

$$s_{\alpha_j} \cdots s_{\alpha_{i+1}} = s_{\alpha_{j-1}} \cdots s_{\alpha_i}.$$

By induction, we may assume

$$r_{\alpha_j}^{\lambda_j} \cdots r_{\alpha_{i+1}}^{\lambda_{i+1}} - r_{\alpha_{j-1}}^{\lambda'_j} \cdots r_{\alpha_i}^{\lambda'_{i+1}} \in \mathscr{F}^{j-i-1},$$

with the appropriately chosen $\lambda'_k = \lambda_k, 1 \le k \le i+1, j+1 \le k \le n$. Thus,

$$r_{\alpha_n}^{\lambda_n} \cdots r_{\alpha_{j+1}}^{\lambda_{j+1}} \left(r_{\alpha_j}^{\lambda_j} \cdots r_{\alpha_{i+1}}^{\lambda_{i+1}} \right) r_{\alpha_i}^{\lambda_i} \cdots r_{\alpha_1}^{\lambda_1} - r_{\alpha_n}^{\lambda'_n} \cdots r_{\alpha_{j+1}}^{\lambda'_{j+1}} \left(r_{\alpha_{j-1}}^{\lambda'_j} \cdots r_{\alpha_i}^{\lambda'_{i+1}} \right) r_{\alpha_i}^{\lambda'_i} \cdots r_{\alpha_1}^{\lambda'_1} \in \mathscr{F}^{n-1}.$$

Because $r_{\alpha_i}^{\lambda'_{i+1}} r_{\alpha_i}^{\lambda'_i} \in \mathscr{F}^1$, the second term is in $\mathscr{F}^{n-1}$, hence $r_{\alpha_n}^{\lambda_n} \cdots r_{\alpha_1}^{\lambda_1} \in \mathscr{F}^{n-1}$. The claim now follows for non-reduced expressions.

Now we show that the assertion is true in the case of a braid relation. Let $\alpha, \beta \in \Pi$ be distinct, and let $W^{\alpha,\beta}$ be the dihedral subgroup of W generated by s_α, s_β. Let $m = m_{\alpha,\beta}$ be the order of $s_\alpha s_\beta$, and set

$$\alpha_i = \begin{cases} \alpha, & \text{if } i \text{ odd}, \\ \beta, & \text{if } i \text{ even}. \end{cases}$$

$$\beta_i = \begin{cases} \beta, & \text{if } i \text{ odd}, \\ \alpha, & \text{if } i \text{ even}. \end{cases}$$

We will show,

$$r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_1}^{\lambda_1} - r_{\beta_m}^{\mu_m} \cdots r_{\beta_1}^{\mu_1} \in \mathscr{F}^{m-1}. \tag{2.1}$$

First, suppose λ is not parabolic with respect to $W^{\alpha,\beta}$. By analyzing the four simply connected semisimple groups of rank 2 in lemma II.9 we see the only such λ have $m_{\alpha,\beta} = 4, 6$. In the case $m_{\alpha,\beta} = 4$, we may assume that β is longer than α and $\langle \lambda, \beta \rangle, \langle \lambda, \alpha \rangle = \pm 1$, as λ has order two. Checking the four elements satisfying that requirement in the torus of the associated simply connected semisimple group, the $W^{\alpha,\beta}$-orbit of λ must be of order exactly two, with the two elements being $\pi = \check{\beta} \otimes -1$ and $s_\alpha(\pi) = (\check{\alpha} \otimes -1) \cdot (\check{\beta} \otimes -1)$. We find $s_\beta(\pi) = \pi$ and $s_\beta(s_\alpha(\pi)) = s_\alpha(\pi)$. We have the following picture of the Cayley graph of the orbit:

$${}^{s_\beta} \circlearrowleft \pi \overset{s_\alpha}{\longleftrightarrow} s_\alpha(\pi) \circlearrowright {}^{s_\beta}.$$

Both of these weights are β-exceptional, hence the braid relation holds for both $\pi, s_\alpha(\pi)$, thus the difference in question is in fact zero.

In the case G_2 we also assume that β is the longer root. Again, $\langle \lambda, \beta \rangle, \langle \lambda, \alpha \rangle = \pm 1$. In this case the order of the $W^{\alpha,\beta}$-orbit must be exactly three, with the three elements given by $\pi = \check{\alpha} \otimes -1$, $s_\beta(\pi) = (\check{\alpha} \otimes -1) \cdot (\check{\beta} \otimes -1)$, and $s_\alpha s_\beta(\pi) = \check{\beta} \otimes -1$. We find that π is s_α-invariant, and $s_\alpha s_\beta(\pi)$ is s_β invariant, so these two weights are exceptional. We have the following picture of the Cayley graph of this orbit:

$$^{s_\alpha} \circlearrowleft \pi \overset{s_\beta}{\longleftrightarrow} s_\beta(\pi) \overset{s_\alpha}{\longleftrightarrow} s_\alpha s_\beta(\pi) \circlearrowright {}^{s_\beta}.$$

We will show that the braid relation for $\pi, s_\alpha s_\beta(\pi)$ implies the braid relation for $s_\beta(\pi)$, which is the only weight not fixed by one of s_α, s_β. This will also demonstrate the technique used in the general case, that the braid relation for a standard parabolic weight implies a (different) braid relation for the other weights in its $W^{\alpha,\beta}$-orbit. With this in mind, recall that we wish to calculate the difference,

$$r_\beta^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} r_\alpha^{s_\beta(\pi)} - r_\alpha^{s_\alpha s_\beta(\pi)} \cdots r_\alpha^\pi r_\beta^{s_\beta(\pi)}.$$

We simply multiply this difference on the right by $r_\alpha^{s_\alpha s_\beta(\pi)}$:

$$r_\beta^\pi r_\alpha^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} r_\alpha^{s_\beta(\pi)} r_\alpha^{s_\alpha s_\beta(\pi)} - r_\alpha^{s_\alpha s_\beta(\pi)} r_\beta^{s_\alpha s_\beta(\pi)} \cdots r_\alpha^\pi r_\beta^{s_\beta(\pi)} r_\alpha^{s_\alpha s_\beta(\pi)}.$$

The first term simplifies as, $r_\alpha^{s_\beta(\pi)} r_\alpha^{s_\alpha s_\beta(\pi)} = G_\alpha^{s_\alpha s_\beta(\pi)}$. As $s_\alpha s_\beta(\pi)$ is α-exceptional we have $G_\alpha^{s_\alpha s_\beta(\pi)} = 1$. The last six elements in the product in the second term may be substituted by the braid relation for the β-exceptional weight $s_\alpha s_\beta(\pi)$:

$$r_\alpha^{s_\alpha s_\beta(\pi)} r_\beta^{s_\alpha s_\beta(\pi)} \cdots r_\alpha^\pi r_\beta^{s_\beta(\pi)} r_\alpha^{s_\alpha s_\beta(\pi)} = r_\alpha^{s_\alpha s_\beta(\pi)} r_\alpha^{s_\beta(\pi)} \cdots r_\alpha^{s_\alpha s_\beta(\pi)} r_\beta^{s_\alpha s_\beta(\pi)}$$

Notice that the first two terms of the product on the right side of the equality simplify;

$$r_\alpha^{s_\alpha s_\beta(\pi)} r_\alpha^{s_\beta(\pi)} = G_\alpha^{s_\beta(\pi)}.$$

Again, $s_\beta(\pi)$ is α-exceptional so $G_\alpha^{s_\beta(\pi)} = 1$. It follows that the difference in question is equal to:

$$\begin{aligned}
&r_\beta^\pi r_\alpha^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} r_\alpha^{s_\beta(\pi)} r_\alpha^{s_\alpha s_\beta(\pi)} - r_\alpha^{s_\alpha s_\beta(\pi)} r_\beta^{s_\alpha s_\beta(\pi)} \cdots r_\alpha^\pi r_\beta^{s_\beta(\pi)} r_\alpha^{s_\alpha s_\beta(\pi)} \\
&= r_\beta^\pi r_\alpha^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} G_\alpha^{s_\alpha s_\beta(\pi)} - s_\alpha w_{\alpha,\beta}(G_\alpha^{s_\alpha s_\beta(\pi)}) r_\beta^\pi r_\alpha^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} \\
&= r_\beta^\pi r_\alpha^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} - r_\beta^\pi r_\alpha^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} \\
&= 0.
\end{aligned}$$

Multiplying again on the right by $r_\alpha^{s_\beta(\pi)}$ and noting that $r_\alpha^{s_\alpha s_\beta(\pi)} r_\alpha^{s_\beta(\pi)} = G_\alpha^{s_\beta(\pi)} = 1$, we find the braid relation for $s_\beta(\pi)$:

$$r_\beta^\pi \cdots r_\beta^{s_\alpha s_\beta(\pi)} r_\alpha^{s_\beta(\pi)} - r_\alpha^{s_\alpha s_\beta(\pi)} \cdots r_\alpha^\pi r_\beta^{s_\beta(\pi)}.$$

This concludes the claim for exceptional weights.

We move on, and assume λ is parabolic with respect to $W^{\alpha,\beta}$. If the stabilizer of λ has 1 element, or is $W^{\alpha,\beta}$ itself, then λ was a standard parabolic weight with respect to $\{\alpha, \beta\}$ and we are done, as the braid relation shows that the difference in (2.1) is zero.

Thus, assume that λ is a parabolic weight, but not a standard parabolic weight. Then there is a unique $1 \leq t < m$ so that $\lambda_{t+1} = s_{\alpha_t} \cdots s_{\alpha_1}(\lambda)$ is a standard parabolic weight with $s_{\alpha_{t+1}}(\lambda_{t+1}) = \lambda_{t+1}$. We will swap α, β if it happens that $t \geq \frac{m}{2}$, which has the effect of changing t to $m - t - 1$. From now on, we assume $t < \frac{m}{2}$.

Define $\lambda_{-i} = \lambda_{i+1}, \mu_{-i} = \mu_{i+1}$, and multiply the difference in (2.1) on the right by $r_{\alpha_1}^{\lambda_{-1}} \cdots r_{\alpha_t}^{\lambda_{-t}}$. The two terms that appear are grouped as follows:

$$\begin{aligned}
&r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}} (r_{\alpha_t}^{\lambda_t} \cdots r_{\alpha_1}^{\lambda_1} r_{\alpha_1}^{\lambda_{-1}} \cdots r_{\alpha_t}^{\lambda_{-t}}) - \\
&r_{\beta_m}^{\mu_m} \cdots r_{\beta_{m-t+1}}^{\mu_{m-t+1}} (r_{\beta_{m-t+1}}^{\mu_{m-t+1}} \cdots r_{\beta_1}^{\mu_1} r_{\alpha_1}^{\lambda_{-1}} \cdots r_{\alpha_t}^{\lambda_{-t}}).
\end{aligned}$$

As $\beta_1 \neq \alpha_1$, the last m entries of the second term alternate between α and β, and start at the parabolic weight $\lambda_{-t} = \lambda_{t+1}$. Thus, they may be switched using the braid relation to the following:

$$r_{\alpha_m}^{\lambda_m}\cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}(r_{\alpha_t}^{\lambda_t}\cdots r_{\alpha_1}^{\lambda_1}r_{\alpha_1}^{\lambda-1}\cdots r_{\alpha_t}^{\lambda-t})-$$
$$r_{\beta_m}^{\mu_m}\cdots r_{\beta_{m-t+1}}^{\mu_{m-t+1}}(r_{\beta_{m-t+1}}^{\mu-(m-t+1)}\cdots r_{\beta_m}^{\mu-m}r_{\alpha_m}^{\lambda_m}\cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}).$$

We combine the last $2t$ entries in the first term and the first $2t$ entries in the second term to simplify this expression,

$$r_{\alpha_m}^{\lambda_m}\cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}\cdot P-$$
$$P'\cdot r_{\alpha_m}^{\lambda_m}\cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}$$

where,

$$P=\prod_{i=1}^{t}s_{\alpha_t}\cdots s_{\alpha_{i+1}}(G_{\alpha_i}^{\lambda-i}),$$
$$P'=\prod_{j=m-t+1}^{m}s_{\beta_m}\cdots s_{\beta_{j+1}}(G_{\beta_j}^{\mu-j}).$$

Using the commutativity relation between r_α^λ and elements of $\dot{\mathscr{A}}^{h_0}$ we find that the above expression is equal to,

$$s_{\alpha_m}\cdots s_{\alpha_{t+1}}(P)\cdot r_{\alpha_m}^{\lambda_m}\cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}+$$
$$s_{\alpha_m}\cdots s_{\alpha_{t+2}}(hD_{\alpha_{t+1}}(P))\cdot r_{\alpha_m}^{\lambda_m}\cdots r_{\alpha_{t+2}}^{\lambda_{t+2}}-$$
$$P'\cdot r_{\alpha_m}^{\lambda_m}\cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}.$$

We claim that $s_{\alpha_m}\cdots s_{\alpha_{t+1}}(P)=P'$. We can use the permutation p from lemma II.5 to show that the i-th term in the product expression for $s_{\alpha_m}\cdots s_{\alpha_{t+1}}(P)$ is the same as the j-th term in the expression for P', where $j=p(i)$. In fact, let $j=m-i+1=p(i)$. Then the corresponding terms are exactly,

$$s_{\alpha_m}\cdots s_{\alpha_{i+1}}G_{\alpha_i}^{\lambda-i},$$
$$s_{\beta_m}\cdots s_{\beta_{j+1}}G_{\beta_j}^{\mu-j}.$$

The *braid relation* for G axiomatizes the above equality.

Now, $\mu_{-j} = w_{\alpha,\beta}(\lambda_{-i})$ and $\beta_j = w_{\alpha,\beta}(\alpha_i)$. All in all, the difference in (2.1) simplifies to,

$$s_{\alpha_m} \cdots s_{\alpha_{t+2}}(hD_{\alpha_{t+1}}(P)) \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{t+2}}^{\lambda_{t+2}}.$$

The above expression has $m-t-1$ terms of the form r_α^λ, and we can replace t of them (using the assumption that $t < \frac{m}{2}$) after we multiply on the right by $r_{\alpha_t}^{\lambda_{-t}} \cdots r_{\alpha_1}^{\lambda_{-1}}$.

To summarize, let's define the following non-zero element of $\mathscr{A}_\lambda^{h_0}$:

$$R = r_{\alpha_1}^{\lambda_{-1}} \cdots r_{\alpha_t}^{\lambda_{-t}} r_{\alpha_t}^{\lambda_{t-1}} \cdots r_{\alpha_1}^{\lambda_1}.$$

We have shown that the following relation holds in $\mathscr{H}^{h_0}(G)$:

$$\begin{aligned} &\left(r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_1}^{\lambda_1} - r_{\beta_m}^{\mu_m} \cdots r_{\beta_1}^{\mu_1}\right) R \\ =&s_{\alpha_m} \cdots s_{\alpha_{t+2}}(hD_{\alpha_{t+1}}(P)) \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{2t+2}}^{\lambda_{2t+2}} \cdot R. \end{aligned}$$

The relation that $\mathscr{H}^{h_0}(G)$ has no right $\dot{\mathscr{A}}^{h_0}$-torsion implies:

$$\begin{aligned} &r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_1}^{\lambda_1} - r_{\beta_m}^{\mu_m} \cdots r_{\beta_1}^{\mu_1} \\ =&s_{\alpha_m} \cdots s_{\alpha_{t+2}}(hD_{\alpha_{t+1}}(P)) \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{2t+2}}^{\lambda_{2t+2}}, \end{aligned}$$

where again,

$$P = \prod_{i=1}^{t} s_{\alpha_t} \cdots s_{\alpha_{i+1}}(G_{\alpha_i}^{\lambda_{-i}}).$$

Finally, we show the assertion for reduced expressions. Let $w \in W$ with $\ell(w) = n$, and take two expressions $w = s_{\alpha_n} \cdots s_{\alpha_1} = s_{\beta_n} \cdots s_{\beta_1}$ of minimal length. We show:

$$r_{\alpha_n}^{\lambda_n} \cdots r_{\alpha_1}^{\lambda_1} - r_{\beta_n}^{\mu_n} \cdots r_{\beta_1}^{\mu_1} \in \mathscr{F}^{n-1},$$

by reducing to a smaller length case, or by using a braid relation. We need to apply the following lemma, which follows directly from lemma II.5, possibly many times.

Lemma II.13. *Let $u \in W$ with $\ell(u) = m$ and consider two reduced expressions $u = s_{\delta_m} \cdots s_{\delta_1} = s_{\gamma_m} \cdots s_{\gamma_1}$ in W. Then there is a unique $1 \leq i_0 \leq m$ with*

$$\begin{aligned} \delta_1 =& s_{\gamma_1} \cdots s_{\gamma_{i_0-1}}(\gamma_{i_0}), \\ s_{\gamma_{i_0-1}} \cdots s_{\gamma_1} s_{\delta_1} =& s_{\gamma_{i_0}} \cdots s_{\gamma_1}. \end{aligned}$$

Applying the lemma directly to the two expressions we have for w, we see if $i_0 < n$, then by induction we have

$$r_{\beta_{i_0}} \cdots r_{\beta_1} - r_{\beta_{i_0-1}} \cdots r_{\beta_1} r_{\alpha_1} \in \mathscr{F}^{i_0-1}.$$

Though we drop the weights μ_i, λ_i the reader may check this does no harm.

By the inductive hypothesis,

$$r_{\alpha_n} \cdots r_{\alpha_2} - r_{\beta_n} \cdots r_{\beta_{i_0+1}} r_{\beta_{i_0-1}} \cdots r_{\beta_1} \in \mathscr{F}^{n-2}.$$

Thus,

$$\begin{aligned} &\left(r_{\alpha_n} \cdots r_{\alpha_1} - r_{\beta_n} \cdots r_{\beta_{i_0+1}} r_{\beta_{i_0-1}} \cdots r_{\beta_1} r_{\alpha_1}\right) + \\ &\left(r_{\beta_n} \cdots r_{\beta_{i_0+1}} r_{\beta_{i_0-1}} \cdots r_{\beta_1} r_{\alpha_1} - r_{\beta_n} \cdots r_{\beta_1}\right) \in \mathscr{F}^{n-1}. \end{aligned}$$

We assume $i_0 = n$, or

$$\begin{aligned} \alpha_1 &= s_{\beta_1} \cdots s_{\beta_{n-1}}(\beta_n), \\ s_{\beta_{n-1}} \cdots s_{\beta_1} s_{\alpha_1} &= s_{\beta_n} \cdots s_{\beta_1} = w. \end{aligned}$$

Similar to the argument above, we have by induction,

$$r_{\beta_{n-1}} \cdots r_{\beta_1} - r_{\alpha_n} \cdots r_{\alpha_2} \in \mathscr{F}^{n-2}.$$

Thus, the following two assertions are equivalent,

$$\begin{aligned} &r_{\beta_n} \cdots r_{\beta_1} - r_{\alpha_n} \cdots r_{\alpha_1} \in \mathscr{F}^{n-1}, \\ &r_{\beta_n} \cdots r_{\beta_1} - r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} \in \mathscr{F}^{n-1}. \end{aligned}$$

We now apply the lemma above to the second expression, finding an i_0 with,

$$\beta_1 = s_{\alpha_1} s_{\beta_1} \cdots s_{\beta_{i_0-2}}(\beta_{i_0-1}).$$

Again, either $i_0 < n$ in which case we apply the induction to show the claim, or we show the following two assertions are equivalent,

$$r_{\beta_n} \cdots r_{\beta_1} - r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} \in \mathscr{F}^{n-1}$$
$$r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} - r_{\beta_{n-2}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} \in \mathscr{F}^{n-1}.$$

Using the same trick we show either the second claim or the equivalence of the following two assertions,

$$r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} - r_{\beta_{n-2}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} \in \mathscr{F}^{n-1},$$
$$r_{\beta_{n-3}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} r_{\alpha_1} - r_{\beta_{n-2}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} \in \mathscr{F}^{n-1}.$$

At this point, if $m_{\alpha,\beta} = 3$ we are done due to the above proof for the braid relation. If $m_{\alpha,\beta}$ is larger, we keep applying this algorithm to eventually find a braid relation.

This finishes the proof of the lemma. □

Corollary II.14. *Let B be a set of reduced expressions $r_{\alpha_n} \cdots r^{\lambda}_{\alpha_n}$ so that every $w \in W$ is represented exactly once. Then B generates $\mathscr{H}^{h_0}(G)1_\lambda$ as a right $\mathscr{A}^{h_0}_\lambda$-module.*

Let $gr\mathscr{H}^{h_0}(G)$ be the graded algebra associated to the filtration $(\mathscr{F}^n)$. We wish to describe the structure of $gr\mathscr{H}^{h_0}(G)$.

Let ${}^0\mathscr{H}^f$ be the finite nil-Hecke algebra. This is the algebra with generators $r_\alpha, \alpha \in \Pi$, satisfying:

$$r_\alpha^2 = 0,$$
$$\cdots r_\beta r_\alpha = \cdots r_\alpha r_\beta, \text{with } m_{\alpha,\beta} \text{ terms}.$$

We form the wreath product algebra

$$\dot{\mathscr{A}}^{h_0} \wr {}^0\mathscr{H}^f,$$

which as a k vector space is given by the tensor product, $\dot{\mathscr{A}}^{h_0} \otimes_k {}^0\mathscr{H}^f$. We give the multiplication by setting,

$$1 \otimes r_{s_\alpha} \cdot f \otimes 1 = s_\alpha(f) \otimes r_{s_\alpha}.$$

There is a natural surjective morphism

$$\dot{\mathscr{A}}^{h_0} \wr {}^0\mathscr{H}^f \to gr\mathscr{H}^{h_0}(G).$$

We say that $\mathscr{H}^{h_0}(G)$ has the PBW property if this morphism is an isomorphism.

Theorem II.15. *The following assertions hold:*

- $\mathscr{H}^{h_0}(G)$ *satisfies the PBW property.*
- *For every* $\lambda \in \mathscr{T}^{h_0}$, $\mathscr{H}^{h_0}(G)1_\lambda$ *is a free right* $\mathscr{A}_\lambda^{h_0}$*-module with basis* B.

Proof. The first two assertions are equivalent thanks to the generating family B mentioned in the above corollary.

Lemma II.16. *Given a family* G *satisfying the conditions of section 2.2, there exists a splitting family* $F = (F_\alpha^\lambda)$, $F_\alpha \in \mathscr{A}_\lambda^{h_0}$, *which satisfy the following conditions:*

1. *One of* F_α^λ *or* $F_\alpha^{s_\alpha(\lambda)}$ *is equal to 1.*
2. $F_\alpha^\lambda \cdot s_\alpha(F_\alpha^{s_\alpha(\lambda)}) = G_\alpha^\lambda$.

Remark II.17. This lemma takes the place of the splitting $Q_{i,j}(u, u') = P_{i,j}(u, u')P_{j,i}(u', u)$ in [Rou11, Section 3.2.3].

Proof. Fix $\lambda \in \mathscr{T}^{h_0}, \alpha, \beta \in \Pi$ distinct. If $s_\alpha(\lambda) = \lambda$, we put $F_\alpha^\lambda = F_\alpha^{s_\alpha(\lambda)} = 1$. Note, in this case, $G_\alpha^\lambda = 1$. We see that $w_{\alpha,\beta}s_\alpha(\lambda) = w_{\alpha,\beta}(\lambda)$, and because $s_{w_{\alpha,\beta}s_\alpha(\alpha)}w_{\alpha,\beta} = w_{\alpha,\beta}s_\alpha$, we have $F_{w_{\alpha,\beta}s_\alpha(\alpha)}^{w_{\alpha,\beta}s_\alpha(\lambda)} = F_{w_{\alpha,\beta}s_\alpha(\alpha)}^{w_{\alpha,\beta}(\lambda)}$, so this choice is consistent with the braid relation.

Assume $s_\alpha(\lambda) \neq \lambda$, and set $F_\alpha^\lambda = G_\alpha^\lambda, F_\alpha^{s_\alpha(\lambda)} = 1$. Consider the set,

$$\{\lambda, s_\alpha(\lambda), w_{\alpha,\beta}s_\alpha(\lambda), w_{\alpha,\beta}(\lambda)\}. \tag{2.2}$$

As $s_\alpha(\lambda) \neq \lambda$, we have $w_{\alpha,\beta}s_\alpha(\lambda) \neq w_{\alpha,\beta}(\lambda)$. In accordance with the braid relations, we set

$$F^{w_{\alpha,\beta}s_\alpha(\lambda)}_{w_{\alpha,\beta}s_\alpha(\alpha)} := w_{\alpha,\beta}s_\alpha(F^\lambda_\alpha) = w_{\alpha,\beta}s_\alpha(G^\lambda_\alpha), \tag{2.3}$$
$$F^{w_{\alpha,\beta}(\lambda)}_{w_{\alpha,\beta}s_\alpha(\alpha)} := w_{\alpha,\beta}s_\alpha(F^{s_\alpha(\lambda)}_\alpha) = 1.$$

If $m_{\alpha,\beta}$ is odd, then $w_{\alpha,\beta}s_\alpha(\beta) = \beta$ and the four pairs

$$\{(\lambda,\alpha),(s_\alpha(\lambda),\alpha),(w_{\alpha,\beta}(\lambda),w_{\alpha,\beta}s_\alpha(\alpha)),(w_{\alpha,\beta}s_\alpha(\lambda),w_{\alpha,\beta}s_\alpha(\alpha))\}, \tag{2.4}$$

are distinct, so we have not defined any element of F twice. If $m_{\alpha,\beta}$ is even and $\lambda = w_{\alpha,\beta}s_\alpha(\lambda)$, then the two sides of (2.3) are already equal by the braid relation for $G^\lambda_\alpha = F^\lambda_\alpha$. Thus, we have defined the two elements, $F^\lambda_\alpha, F^{s_\alpha(\lambda)}_\alpha$ twice, but with the same values each time. If $m_{\alpha,\beta}$ is even and $\lambda = w_{\alpha,\beta}(\lambda)$, then because $w_{\alpha,\beta}$ has even length, it is not a reflection, thus λ is not a parabolic weight. This means that λ is α-exceptional so all four values in (2.3) are 1.

Now, let $\gamma \in \Pi$ be distinct from α, β and define $F^{w_{\alpha,\gamma}s_\alpha(\lambda)}_{w_{\alpha,\gamma}s_\alpha(\alpha)}, F^{w_{\alpha,\gamma}(\lambda)}_{w_{\alpha,\gamma}s_\alpha(\alpha)}$ as above. If either of $m_{\alpha,\beta}$ or $m_{\alpha,\gamma}$ are odd, then the values of the F-terms are well defined. Assuming that $m_{\alpha,\beta}, m_{\alpha,\gamma}$ are both even we see that if $w_{\alpha,\gamma}(\lambda) = w_{\alpha,\beta}(\lambda$ then as $F^{s_\alpha(\lambda)}_\alpha = 1$, we indeed have $w_{\alpha,\beta}s_\alpha(F^{s_\alpha(\lambda)}_\alpha) = w_{\alpha,\gamma}s_\alpha(F^{s_\alpha(\lambda)}_\alpha)$.

In case $w_{\alpha,\beta}s_\alpha(\lambda) = w_{\alpha,\gamma}s_\alpha(\lambda)$ we have the braid relation:

$$w_{\alpha,\beta}s_\alpha(G^\lambda_\alpha) = G^{w_{\alpha,\beta}s_\alpha(\lambda)}_\alpha = G^{w_{\alpha,\gamma}s_\alpha(\lambda)}_\alpha = w_{\alpha,\gamma}s_\alpha(G^\lambda_\alpha).$$

Thus, to show that no contradiction forms from these choices it is enough to consider the case

$$w_{\alpha,\gamma}(\lambda) = w_{\alpha,\beta}s_\alpha(\lambda), \tag{2.5}$$
$$w_{\alpha,\gamma}s_\alpha(\alpha) = w_{\alpha,\beta}s_\alpha(\alpha) = \alpha. \tag{2.6}$$

In the case $m_{\alpha,\beta} = m_{\alpha,\gamma} = 2$ we find that if $s_\alpha s_\gamma(\lambda) = s_\beta(\lambda)$ then $s_\alpha s_\beta s_\gamma(\lambda) = \lambda$. As $s_\gamma(\omega_\alpha) = s_\beta(\omega_\alpha) = \omega_\alpha$ we find,

$$\omega_\alpha(\lambda) = \omega_\alpha(s_\alpha s_\beta s_\gamma(\lambda))$$
$$= \omega_\alpha(\lambda)^{-1}.$$

It follows that $\omega_\lambda(\lambda) = \pm 1$. As $m_{\alpha,\beta} = m_{\alpha,\gamma} = 2$ this shows that λ is in fact s_α-invariant, i.e. this case never happens.

We are left to consider the rank 3 root systems with the following cases $m_{\alpha,\beta} = 2, m_{\alpha,\gamma} = 4, 6$ and $m_{\alpha,\beta} = 4, 6$, $m_{\alpha,\gamma} = 2$. We must show that $F_\alpha^{w_{\alpha,\gamma}(\lambda)} = F_\alpha^{w_{\alpha,\beta}s_\alpha(\lambda)}$, which by definition means we must show that,

$$w_{\alpha,\gamma}s_\alpha(F_\alpha^{s_\alpha(\lambda)}) = w_{\alpha,\beta}s_\alpha(F_\alpha^\lambda).$$

As we have already defined $F_\alpha^{s_\alpha(\lambda)} = 1$, we must show that $F_\alpha^\lambda = G_\alpha^\lambda = 1$.

When $m_{\beta,\gamma} = 2$, we see ω_γ is both s_α and s_β invariant. Thus $s_\gamma s_\alpha(\lambda) = w_{\alpha,\beta}s_\alpha(\lambda)$ implies that $\omega_\gamma(\lambda)^{-1} = \omega_\gamma(\lambda)$, and hence $s_\gamma(\lambda) = \lambda$. From this we deduce that $w_{\alpha,\beta}(\lambda) = \lambda$. As the length of $w_{\alpha,\beta}$ is even and $s_\alpha(\lambda) \neq \lambda$ we deduce that λ is α-exceptional and so $G_\alpha^\lambda = 1$ as we desired.

The other cases arise from the simply connected root datum associated with B_3 and C_3. Consider the simply connected root datum associated to B_3. Let $\Pi = \{\alpha, \beta, \gamma\}$, where α is the short root, $m_{\alpha,\beta} = 4$, and $m_{\alpha,\gamma} = 2$. By an explicit calculation with the element $\lambda = (x, y, z) \in (k^*)^3$ corresponding to $(\check{\alpha} \otimes x) \cdot (\check{\beta} \otimes y) \cdot (\check{\gamma} \otimes z) \in \mathscr{T}^{h_0} \otimes k^*$, we find that the only elements satisfying $s_\gamma s_\alpha(\lambda) = s_\beta s_\alpha s_\beta(\lambda)$ and $s_\alpha(\lambda) \neq \lambda$ are of the form $(i, 1, -1)$ where i is a square root of (-1). In that case the Cayley graph of the action of $s_\alpha, s_\beta, s_\gamma$ looks like,

$$^{s_\beta, s_\gamma} \circlearrowleft (i, 1, -1) \overset{s_\alpha}{\longleftrightarrow} (-i, 1, -1) \circlearrowright {}^{s_\beta, s_\gamma}.$$

As $m_{\alpha,\beta} = 4$ it is clear that λ is α-exceptional and so $G_\alpha^\lambda = 1$ as desired.

A similar calculation for the simply connected root datum associated to C_3 shows that every weight λ with $w_{\alpha,\gamma}(\lambda) = w_{\alpha,\beta}s_\alpha(\lambda)$ are in fact s_α-invariant.

This shows that we may define F_α^λ consistently. $\square$

Now take a splitting family $F_\alpha^\lambda \in \mathscr{A}_\lambda^{h_0}$ for G.

For $s_\alpha(\lambda) = \lambda$ we let r_α^λ act as $hD_\alpha 1_\lambda$. Otherwise we let r_α^λ act as $s_\alpha F_\alpha^\lambda 1_\lambda$. To show this representation is well defined we only need to check the relations.

The only difficult relation is the braid relation in the case where λ is a parabolic weight with respect to $W^{\alpha,\beta}$, but is fixed by only one of the weights.

For this case, suppose λ is s_α invariant and not s_β invariant. We set

$$\alpha_i = \begin{cases} \alpha, & \text{if } i \text{ odd,} \\ \beta, & \text{if } i \text{ even} \end{cases}$$

$$\beta_i = \begin{cases} \alpha, & \text{if } i \text{ even,} \\ \beta, & \text{if } i \text{ odd} \end{cases} .$$

The relevant relation we must show is equivalent to

$$s_{\alpha_m} s_{\alpha_{m-1}} \cdots s_{\alpha_2} D_{\alpha_1} = D_{\beta_m} s_{\beta_{m-1}} \cdots s_{\beta_1}.$$

If we consider D_α as given by the fraction, $\dfrac{1 - s_\alpha}{1 - e^{-\alpha}}$, then the relation

$$D_{\alpha_m} = (w_\ell s_\alpha) D_{\alpha_1} (w_\ell s_\alpha)^{-1}$$

makes the desired relation above obvious.

The above morphism defines a faithful representation of $\mathscr{H}^{h_0}(G)$ on $\dot{\mathscr{A}}^{h_0}$.

The image of the set $B \subset \mathscr{H}^{h_0}(G)1_\lambda$ gets mapped to $(\dot{\mathscr{A}}^{h_0} \wr W)1_\lambda$, and is linearly independent over $1_\lambda \dot{\mathscr{A}}^{h_0} 1_\lambda$, just as in [Rou11, Proposition 3.8], with $k[X_1, \cdots X_n]$ replaced by $\mathscr{A}^{h_0}$. □

2.5 Isomorphism class of $\mathscr{H}^{h_0}(G)$

The main result of this section shows that the isomorphism class of $\mathscr{H}^{h_0}(G)$ is invariant under multiplying the data G by invertible functions which also satisfy braid and reflexive relations.

Theorem II.18. *Let $G = (G_\alpha^\lambda)$ and $H = (H_\alpha^\lambda)$ be datum satisfying the conditions from section 2.2. Suppose $(g_\alpha^\lambda)_{\lambda \in \mathscr{T}^{h_0}, \alpha \in \Pi}$ is the set of functions $g_\alpha^\lambda = H_\alpha^\lambda / G_\alpha^\lambda$, and suppose that the*

g_α^λ *are invertible rational functions,* $g_\alpha^\lambda \in (\mathscr{A}_\lambda^{h_0})^*$*. Then there is an isomorphism,*

$$\mathscr{H}^{h_0}(H) \to \mathscr{H}^{h_0}(G)$$

Proof. Suppose G and H are sets of datum satisfying the conditions from section 2.2. Suppose, further, that $g_\alpha^\lambda = H_\alpha^\lambda / G_\alpha^\lambda$ is a unit in $\mathscr{A}_\lambda^{h_0}$. By the splitting lemma, II.16, there exists a splitting family $(F) = (F_\alpha^\lambda)$ for (g_α^λ). As each F_α^λ is either 1 or g_α^λ, we find that F_α^λ is also invertible in $\mathscr{A}_\lambda^{h_0}$. Consider the elements,

$$\tau_\alpha^\lambda = r_\alpha^\lambda F_\alpha^\lambda \in \mathscr{H}^{h_0}(G).$$

From the proof of Theorem II.15, we find the elements τ_α^λ satisfy the same relations as $r_\alpha^\lambda \in \mathscr{H}^{h_0}(H)$. Moreover they generate, along with $\dot{\mathscr{A}}^{h_0}$ the algebra $\mathscr{H}^{h_0}(G)$. This proves our claim. □

CHAPTER 3

Applications

3.1 Affine Hecke algebras and localized quiver Hecke algebras

Given a root datum $(X, Y, R, \check{R}, \Pi)$ and set of parameters $c_\alpha \in \mathbb{C}^*$, $\alpha \in \Pi$, with a fixed $h_0 \in \mathbb{C}$, we construct datum G satisfying the properties above, and an isomorphism $\mathscr{H}^{h_0}(G) \to \dot{\mathscr{H}}^{h_0}$.

If $s_\alpha(\lambda) = \lambda$ or λ is α-exceptional, let $G_\alpha^\lambda = 1$. Otherwise, for λ with $s_\alpha(\lambda) \neq \lambda$ let

$$G_\alpha^\lambda = (c_\alpha + q_\alpha P_{-\alpha})(P_{-\alpha} - c_\alpha)(-P_{-\alpha})^{-2}$$

Theorem II.19. *The data G constructed above satisfies the conditions of section 2.2, so $\mathscr{H}^{h_0}(G)$ is well defined. Consider the map $\mathscr{H}^{h_0}(G) \to \dot{\mathscr{H}}^{h_0}$ which is the identity on $\dot{\mathscr{A}}^{h_0}$, and on generators is given by:*

$$r_\alpha^\lambda \mapsto \begin{cases} (c_\alpha + q_\alpha P_{-\alpha})^{-1}(T_{s_\alpha} - q_{s_\alpha})1_\lambda, & \text{if } s_\alpha(\lambda) = \lambda, \\ \left(\dfrac{P_{-\alpha}}{c_\alpha + P_{-\alpha} + hc_\alpha P_{-\alpha}}\right) 1_{s_\alpha(\lambda)} T_{s_\alpha} 1_\lambda, & \text{if } \lambda \text{ is } \alpha\text{-exceptional.}, \\ 1_{s_\alpha(\lambda)} T_{s_\alpha} 1_\lambda, & \text{else.} \end{cases}$$

This map is well defined and it is an isomorphism.

Proof. We easily see that G_α^λ satisfies the associative property, and the braid relation follows from the Weyl group lemmas. It follows that $\mathscr{H}^{h_0}(G)$ is well defined. To check that the above map is well defined we must check the 4 relations from section 2.3 on the generators, and confirm that there is no right $\mathscr{A}_\lambda^{h_0}$-torsion in $\dot{\mathscr{H}}^{h_0} 1_\lambda$.

Abusing notation, we use r_α^λ for its image in $\dot{\mathscr{H}}^{h_0}$. From the definition of $\dot{\mathscr{H}}^{h_0}$ we have that $1_{s_\alpha(\nu)} r_\alpha^\lambda = r_\alpha^\lambda 1_\nu = \delta_{\lambda,\nu} r_\alpha^\lambda$.

We now check the quadratic relation,

$$r_\alpha^{s_\alpha(\lambda)} r_\alpha^\lambda = \begin{cases} G_\alpha^\lambda & \text{if } s_\alpha(\lambda) \neq \lambda, \\ h r_\alpha^\lambda & \text{if } s_\alpha(\lambda) = \lambda. \end{cases}$$

First, suppose $s_\alpha(\lambda) \neq \lambda$. We have the quadratic relation $T_{s_\alpha}^2 = (q_{s_\alpha} - 1)T_{s_\alpha} + q_{s_\alpha}$. We multiply on the left and right by 1_λ to obtain,

$$\begin{aligned} 1_\lambda T_{s_\alpha}^2 1_\lambda =& (q_{s_\alpha} - 1) 1_\lambda T_{s_\alpha} 1_\lambda + q_{s_\alpha} 1_\lambda, \\ =& (q_{s_\alpha} - 1) c_\alpha (-P_{-\alpha})^{-1} 1_\lambda + q_\alpha. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} 1_\lambda T_{s_\alpha}^2 1_\lambda =& 1_\lambda T_{s_\alpha} (1_\lambda + 1_{s_\alpha(\lambda)}) T_{s_\alpha} 1_\lambda, \\ =& 1_\lambda T_{s_\alpha} 1_{s_\alpha(\lambda)} T_{s_\alpha} 1_\lambda + c_\alpha^2 (-P_{-\alpha})^{-2}. \end{aligned}$$

Equating the two expressions yields the equality:

$$1_\lambda T_{s_\alpha} 1_{s_\alpha(\lambda)} T_{s_\alpha} 1_\lambda = (c_\alpha + q_\alpha P_{-\alpha})(P_{-\alpha} - c_\alpha)(-P_{-\alpha})^{-2}.$$

Consequently, we find that $1_{s_\alpha(\lambda)} T_{s_\alpha} 1_\lambda$ is invertible when $\lambda(P_{-\alpha}) \neq c_\alpha, -q^{-1} c_\alpha$.

Next, we verify the commutativity relation:

$$r_\alpha^\lambda f - s_\alpha(f) r_\alpha^\lambda = \begin{cases} 0 & \text{, if } s_\alpha(\lambda) \neq \lambda, \\ h D_\alpha(f) & \text{, if } s_\alpha(\lambda) = \lambda. \end{cases}$$

First, suppose $s_\alpha(\lambda) \neq \lambda$. We simply multiply the original commutativity relation,

$$T_{s_\alpha} f - s_\alpha(f) T_{s_\alpha} = c_\alpha h D(f),$$

on the left by $1_{s_\alpha(\lambda)}$ and on the right by 1_λ. Since $1_\lambda 1_{s_\alpha(\lambda)} = 0$, the claim follows.

Now suppose $s_\alpha(\lambda) = \lambda$. We check directly:

$$\begin{aligned} (T_\alpha - q_\alpha) f - s_\alpha(f)(T_\alpha - q_\alpha) =& c_\alpha h D_\alpha(f) - q_\alpha(f - s_\alpha(f)), \\ =& (c_\alpha + q_\alpha P_{-\alpha}) h D_\alpha(f). \end{aligned}$$

One could also expand the expression,

$$(c_\alpha + q_\alpha P_{-\alpha})^{-1}(T_\alpha - q_\alpha)(c_\alpha + q_\alpha P_{-\alpha})^{-1}(T_\alpha - q_\alpha)$$

and verify the quadratic relation, $(r_\alpha^\lambda)^2 = hr_\alpha^\lambda$, but we will use the induced representation of $\mathscr{H}^{h_0}$ on $\mathscr{A}^{h_0}$ for this and the braid relations.

Finally we verify the braid relations. The only standard parabolic subgroups of the Coxeter group $(W^{\alpha,\beta}, \{s_\alpha, s_\beta\})$ are $W^{\alpha,\beta}, \langle e\rangle, \langle s_\alpha\rangle, \langle s_\beta\rangle$.

Suppose that the stabilizer of λ is the trivial group $\langle e\rangle$. The element r_α^λ is given by $1_{s_\alpha(\lambda)}T_{s_\alpha}1_\lambda$. In this case, with $\lambda' = \cdots s_\alpha s_\beta s_\alpha(\lambda)$, we have

$$1_{\lambda'}\cdots T_{s_\alpha}T_{s_\beta}T_{s_\alpha}1_\lambda = 1_{\lambda'}\cdots T_{s_\alpha}1_{s_\beta s_\alpha(\lambda)}T_{s_\beta}1_{s_\alpha(\lambda)}T_{s_\alpha}1_\lambda,$$

and similarly for $\cdots T_{s_\beta}T_{s_\alpha}T_{s_\beta}$. Thus the braid relation for $T_{s_\alpha}, T_{s_\beta}$ yields the braid relation between r_α, r_β.

Consider, now, the case where the stabilizer of λ is s_α. In this case, we also have

$$1_{\lambda'}\cdots T_{s_\alpha}T_{s_\beta}T_{s_\alpha}1_\lambda = 1_{\lambda'}\cdots T_{s_\alpha}1_{s_\beta s_\alpha(\lambda)}T_{s_\beta}1_{s_\alpha(\lambda)}T_{s_\alpha}1_\lambda.$$

Replacing the rightmost T_{s_α} with $(c_\alpha + q_{s_\alpha}P_{-\alpha})^{-1}(T_{s_\alpha} - q_{s_\alpha})$ and using the commutativity relation yields the desired result.

Finally, suppose that $\mathrm{stab}_{W^{\alpha,\beta}}(\lambda) = W^{\alpha,\beta}$. We will use the Demazure-Lusztig representation of $\mathscr{H}^{h_0}$ from section 1.3.

Recall equation (1.2), which gives the formula for the action of $\widehat{T_\alpha - q_\alpha}$ on $\mathscr{A}^{h_0}$:

$$\widehat{T_\alpha - q_\alpha} : f \mapsto (c_\alpha + q_\alpha P_{-\alpha})hD_\alpha(f).$$

We extend the action of $\mathscr{H}^{h_0}$ on $\mathscr{A}^{h_0}$ to an action of $\dot{\mathscr{H}}^{h_0}1_\lambda$ on $\mathscr{A}_\lambda^{h_0}$, and find $r_\alpha^\lambda = hD_\alpha$. As the Demazure-Lusztig representation is faithful this shows the braid relation between $r_\alpha^\lambda, r_\lambda^\beta$, as well as the quadratic relation $r_\alpha^\lambda r_\alpha^\lambda = hr_\alpha^\lambda$.

From the structure theory of $\dot{\mathscr{H}}^{h_0}$ we see it has no polynomial torsion, and the same PBW basis, by the same Demazure-Lusztig representation, thus the map in question is an isomorphism.

$\square$

3.2 Quiver Hecke algebras

In this section we define quiver Hecke algebras attached to simply connected semisimple root data as a subalgebra of $\mathscr{H}^{h_0}(G)$ defined in the previous section. Let $h_0 = 0$, then $P_\alpha + P_\beta = P_{\alpha+\beta}$. Pick a choice of parameters $c_\alpha \in k^*$. Recall that there are no exceptional weights in this case, as every weight is conjugate to a standard parabolic weight.

Recall the data G associated to a simply connected semisimple root datum $(X, Y, R, \check{R}, \Pi)$:

$$G_\alpha^\lambda = \begin{cases} 1 & \text{if } s_\alpha(\lambda) = \lambda, \\ (P_\alpha - c_\alpha)(P_\alpha + c_\alpha)(P_\alpha)^{-2} & \text{else.} \end{cases}$$

Suppose the characteristic of k is not 2. Define the data H and g as follows:

$$H_\alpha^\lambda = \begin{cases} 1 & \text{if } s_\alpha(\lambda) = \lambda, \\ c_\alpha - P_\alpha & \text{if } \langle \lambda, \alpha \rangle = c_\alpha, \\ c_\alpha + P_\alpha & \text{if } \langle \lambda, \alpha \rangle = -c_\alpha, \\ 1 & \text{else,} \end{cases}$$

$$g_\alpha^\lambda = \begin{cases} 1 & \text{if } s_\alpha(\lambda) = \lambda, \\ (P_\alpha + c_\alpha)^{-1}(P_\alpha)^2 & \text{if } \langle \lambda, \alpha \rangle = c_\alpha, \\ (P_\alpha - c_\alpha)^{-1}(P_\alpha)^2 & \text{if } \langle \lambda, \alpha \rangle = -c_\alpha, \\ (P_\alpha - c_\alpha)^{-1}(P_\alpha + c_\alpha)^{-1}(P_\alpha)^2 & \text{else,} \end{cases}$$

In the case that the characteristic of k is 2, define H and g instead as,

$$H_\alpha^\lambda = \begin{cases} 1 & \text{if } s_\alpha(\lambda) = \lambda, \\ (c_\alpha - P_\alpha)^2 & \text{if } \langle \lambda, \alpha \rangle = c_\alpha, \\ 1_\lambda & \text{else,} \end{cases}$$

$$g_\alpha^\lambda = \begin{cases} 1_\lambda & \text{if } s_\alpha(\lambda) = \lambda, \\ (P_\alpha)^2 & \text{if } \langle \lambda, \alpha \rangle = c_\alpha, \\ (P_\alpha - c_\alpha)^{-1}(P_\alpha + c_\alpha)^{-1}(P_\alpha)^2 & \text{else,} \end{cases}$$

Proposition II.20. *The data H satisfies the assumptions of section 2.2. Moreover, the algebras $\mathscr{H}^{h_0}(G)$ and $\mathscr{H}^{h_0}(H)$ are isomorphic.*

Proof. Simply apply theorem II.18 to the datum G, H, g. □

The advantage of the datum H_α^λ is that it is contained in the image of $k[\mathfrak{h}] \hookrightarrow k[\mathfrak{h}]_\lambda$. This allows us to define the following subalgebra of the algebras $\mathscr{H}^{h_0}(H)$.

Definition. Suppose $H = (H_\alpha^\lambda)_{\alpha \in \Pi, \lambda \in \mathfrak{h}}$ is a datum which satisfies the conditions of section 2.2, and for which H_α^λ is in the image of the inclusion $k[\mathfrak{h}] \hookrightarrow \mathscr{A}_\lambda$. Let $\mathcal{H}$ be the subalgebra of $\mathscr{H}^{h_0}(H)$ generated by the image of $k[\mathfrak{h}] \hookrightarrow \mathscr{A}_\lambda$, for each $\lambda \in \mathfrak{h}$ and $r_\alpha^\lambda, \alpha \in \Pi, \lambda \in \mathfrak{h}$.

Proposition II.21. *The inclusion $\mathcal{H} \hookrightarrow \mathscr{H}^{h_0}(G)$ gives, via pullback, an equivalence from the category of finite representations of $\mathscr{H}^{h_0}(G)$ and the category of representations of $\mathcal{H}$ for which $(P_\alpha - \langle \lambda, \alpha \rangle)^n 1_\lambda$ acts by 0 for large enough n.*

Proof. It is plain that a $\mathbb{A}$-module V_λ with the property $(P_\alpha - \langle \lambda, \alpha \rangle)^n$ acts by zero for large enough n has a unique extension to a module over $\mathbb{A}_\lambda$ by letting elements $f^{-1} \in \mathbb{A}_\lambda$ with $f \in \mathbb{A}$ with $f(\lambda) \neq 0$ act via the inverse of the action of f, which has only non-zero eigenvalues on V_λ. The claim the follows, as any module over $\mathcal{H}$ with the above property has a unique lift to a module over $\mathscr{H}^{h_0}(H)$. □

We do one more change of variables to get the presentation of $\mathcal{H}$ that we need to define a grading. Let $\psi_\lambda : k[\mathfrak{h}] \to k[\mathfrak{h}]$ be the W-equivariant map given by,

$$\psi_\lambda(P_\alpha) = P_\alpha + \langle \lambda, \alpha \rangle.$$

After this change of variables we use the notation x_α for the variable P_α, thus for $f \in k[\mathfrak{h}]$, a polynomial in $\{P_\alpha\}_{\alpha\in\Pi}$, we consider $\psi_\lambda(f)$ a polynomial in $\{x_\alpha\}_{\alpha\in\Pi}$.

Let $\tilde{R}$ be the algebra with generating set $\{1_\lambda\}_{\lambda\in\mathfrak{h}} \cup \{x_\alpha^\lambda\}_{\alpha\in\Pi,\lambda\in\mathfrak{h}} \cup \{\tau_\alpha^\lambda\}_{\alpha\in\Pi,\lambda\in\mathfrak{h}}$ and relations:

$$1_\lambda 1_{\lambda'} = 1_\lambda \delta_{\lambda,\lambda'},$$

$$x_\alpha^\lambda 1_{\lambda'} = 1_{\lambda'} x_\alpha^\lambda = \delta_{\lambda,\lambda'} x_i^\lambda,$$

$$\tau_\alpha^\lambda 1_{\lambda'} = 1_{s_\alpha(\lambda')}\tau_\alpha^\lambda = \delta_{\lambda,\lambda'}\tau_\alpha^\lambda,$$

$$x_\alpha^\lambda x_\beta^\lambda = x_\beta^\lambda x_\alpha^\lambda,$$

$$\tau_\alpha^\lambda x_\beta^\lambda = x_{s_\alpha(\beta)}^{s_\alpha(\lambda)}\tau_\alpha^\lambda, \text{ for } s_\alpha(\lambda) \neq \lambda,$$

$$\tau_\alpha^\lambda x_\beta^\lambda - x_{s_\alpha(\beta)}^\lambda \tau_\alpha^\lambda = c_\alpha \langle \beta, \check{\alpha} \rangle, \text{ for } s_\alpha(\lambda) = \lambda$$

along with the following quadratic relations,

$$\tau_\alpha^{s_\alpha(\lambda)}\tau_\alpha^\lambda = \begin{cases} 0 & \text{if } s_\alpha(\lambda) = \lambda, \\ \psi_\lambda(H_\alpha^\lambda) & \text{else.} \end{cases}$$

Also give $\tilde{R}$ the following braid relation between τ_α and τ_β for λ a standard parabolic weight. For $\alpha, \beta \in \Pi$ distinct and $\lambda \in \mathfrak{h}$ a *standard parabolic weight* with respect to $\{\alpha, \beta\}$,

$$\tau_{\alpha_m}^{\lambda_m} \cdots \tau_{\alpha_1}^{\lambda_1} = \tau_{\beta_m}^{\mu_m} \cdots \tau_{\beta_1}^{\mu_1},$$

where $m = m_{\alpha,\beta}$ is the order of $s_\alpha s_\beta$ in W,

$$\alpha_i = \begin{cases} \alpha & , i \text{ odd}, \\ \beta & , i \text{ even}, \end{cases}$$

$$\beta_i = \begin{cases} \beta & , i \text{ odd}, \\ \alpha & , i \text{ even}, \end{cases}$$

and $\lambda_i = s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda)$, $\mu_i = s_{\beta_{i-1}} \cdots s_{\beta_1}(\mu)$. Let R be the quotient of $\tilde{R}$ by right polynomial torsion. Then R has no polynomial torsion.

Proposition II.22. *Suppose $(X, Y, R, \check{R}, \Pi)$ is a simply connected semisimple root datum. Let H be any datum satisfying the conditions of 2.2 and which is included in the image $k[\mathfrak{h}] \hookrightarrow k[\mathfrak{h}]_\lambda$, so that the algebras $\mathcal{H}, R$ are defined. Then the map on generators $1_\lambda \mapsto 1_\lambda$, $x_\alpha^\lambda \mapsto P_\alpha - \langle \lambda, \alpha \rangle$, $\tau_\alpha^\lambda \mapsto r_\alpha^\lambda$ is an isomorphism of algebras.*

Moreover, suppose for each $\alpha \in \pi, \lambda \in \mathfrak{h}$ that $\psi_\lambda(H_\alpha^\lambda)$ is a homogeneous polynomial in $\{x_\alpha\}_{\alpha\in\Pi}$. Then let deg *be defined on generators as* $\deg(1_\lambda) = 0$, $\deg(x_\alpha^\lambda) = 2$, *and*

$$\deg(\tau_\alpha^\lambda) = \begin{cases} -2 & \text{if } s_\alpha(\lambda) = \lambda, \\ \frac{1}{2}\deg(\psi_\lambda(H_\alpha^\lambda)) & \text{else.} \end{cases}$$

Then deg *extends to a grading on the algebra R, hence on the quiver Hecke algebra $\mathcal{H}$.*

Corollary II.23. *Let H be the data associated above to a degenerate affine Hecke algebra as above. Then the quadratic relations for R are given by,*

$$\tau_\alpha^{s_\alpha(\lambda)}\tau_\alpha^\lambda = \begin{cases} 0 & \text{if } \langle \alpha, \lambda \rangle = 0, \\ x_\alpha^\lambda & \text{if } \langle \alpha, \lambda \rangle = c_\alpha, \\ -x_\alpha^\lambda & \text{if } \langle \alpha, \lambda \rangle = -c_\alpha \\ 1_\lambda & \text{else,} \end{cases},$$

whereas, for characteristic 2 let,

$$\tau_\alpha^{s_\alpha(\lambda)}\tau_\alpha^\lambda = \begin{cases} 0 & \textit{if } \langle \alpha, \lambda\rangle = 0, \\ (x_\alpha^\lambda)^2 & \textit{if } \langle \alpha, \lambda\rangle = c_\alpha, , \\ 1_\lambda & \textit{else}, \end{cases}$$

The grading is as follows: $\deg(1_\lambda) = 0$, $\deg(x_\alpha^\lambda) = 2$, *for characterstic of* k *not* 2,

$$\deg(\tau_\alpha^\lambda) = \begin{cases} -2 & \textit{if } s_\alpha(\lambda) = \lambda, \\ 1 & \textit{if } \langle \lambda, \alpha\rangle = \pm c_\alpha, \\ 0 & \textit{else}, \end{cases}$$

whereas for characteristic of k *equal to* 2,

$$\deg(\tau_\alpha^\lambda) = \begin{cases} -2 & \textit{if } s_\alpha(\lambda) = \lambda, \\ 2 & \textit{if } \langle \lambda, \alpha\rangle = c_\alpha, \\ 0 & \textit{else}. \end{cases}$$

Proof. The presentation of the algebra R is simply a recollection of the relations of $\mathscr{H}^{h_0}(G)$, restricted to $\mathcal{H}$ and with the above change of variables.

To show that the above grading on generators extends to a grading on $\mathcal{H}$ we need to show that all the relations are graded. This is trivial with the exception of the braid relation and the relations pertaining to polynomial torsion. In the case of the braid relation, we have

$$\langle \lambda_i, \alpha_i\rangle = \langle \lambda, s_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i)\rangle.$$

Applying lemma II.5 to the expression $s_{\alpha_m}\cdots s_{\alpha_1} = s_{\beta_m}\cdots s_{\beta_1}$ we find a permutation p of the set $\{1,\cdots,m\}$ with,

$$\langle \lambda_i, \alpha_i\rangle = \langle \lambda, s_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i)\rangle = \langle \lambda, s_{\beta_1}\cdots s_{\alpha_{p(i)-1}}(\beta_{p(i)})\rangle = \langle \mu_{p(i)}, \beta_{p(i)}\rangle$$

It follows that the braid relations are graded.

As for the polynomial torsion, section 2.4 shows that the polynomial torsion relation may be replaced with the following relations for general $\lambda \in \mathfrak{h}$ and $\alpha, \beta \in \Pi$ distinct. For λ not fixed by either s_α, s_β, then we may pick t as in section 2.4. In that case, following the notation of 2.4, we have the following braid-like relation:

$$\tau_{\alpha_m}^{\lambda_m} \cdots \tau_{\alpha_1}^{\lambda_1} - \tau_{\beta_m}^{\mu_m} \cdots \tau_{\beta_1}^{\mu_1} = s_{\alpha_m} \cdots s_{\alpha_{t+2}}(\Delta_{\alpha_{t+1}}(P))\tau_{\alpha_m} \cdots \tau_{\alpha_{2t+2}},$$

where,

$$\begin{aligned} P =& \prod_{i=1}^{t} s_{\alpha_t} \cdots s_{\alpha_{i+1}}(G_{\alpha_i}^{\lambda_{-i}}), \\ =& \tau_{\alpha_t}^{\lambda_t} \cdots \tau_{\alpha_1}^{\lambda_1} \tau_{\alpha_1}^{\lambda_{-1}} \cdots \tau_{\alpha_t}^{\lambda_{-t}}. \end{aligned}$$

Here, $\lambda_{-i} = s_{\alpha_i}(\lambda_i)$. The two terms on the left side of the relation have the same degree by the above description of $\langle \lambda_i, \alpha_i \rangle$. The term $\Delta_{\alpha_{t+1}}(P)$ has the same degree as $\tau_{\alpha_{2t+1}}^{\lambda_{2t+1}} \cdots \tau_{\alpha_1}$ as the product expression for P shows that it has the same terms with the exception of $\tau_{\alpha_t}^{\lambda_t}$, which has degree -2. As the operator Δ_α has degree -2, it follows that the above braid-like relation is graded.

□

3.3 Graded characters of irreducible representations

Let $\mathcal{H}$ be the quiver Hecke algebra with grading defined above. We define an anti-involution $\iota : \mathcal{H} \to \mathcal{H}^{opp}$ as follows:

$$\begin{aligned} \iota(1_\lambda) =& 1_\lambda \\ \iota(x_\alpha^\lambda) =& x_\alpha^\lambda \\ \iota(\tau_\alpha^\lambda) =& \tau_\alpha^{s_\alpha(\lambda)}. \end{aligned}$$

The only difficulty in showing that ι is well defined is in showing that the braid relations are preserved under ι. Indeed, to show that

$$\tau_{\alpha_1}^{\lambda_2} \cdots \tau_{\alpha_m}^{\lambda_{m+1}} - \tau_{\beta_1}^{\mu_2} \cdots \tau_{\beta_m}^{\mu_{m+1}} = s_{\alpha_{2t+1}} \cdots s_{\alpha_{t+2}} \Delta_{\alpha_{t+1}}(P) \tau_{2t+2}^{\lambda_{2t+3}} \cdots r_{\alpha_m}^{\lambda_{m+1}},$$

it suffices to multiply the left side of the equation on the left by $\tau_{\alpha_1} \cdots \tau_{\alpha_t} \tau_{\alpha_t} \cdots \tau_{\alpha_1}$ and simplify as in section 2.4.

As $\deg(\tau_\alpha^\lambda) = \deg(\tau_\alpha^{s_\alpha(\lambda)})$, we see that ι is a graded anti-involution. Let V be a graded $\mathcal{H}$-module. Then $V^\vee := \hom_k(V, k)$ is naturally a $\mathcal{H}^{opp}$-module, which we consider as a $\mathcal{H}$-module via the map ι. We see that the graded character of $V^\vee$ is given by switching v and v^{-1} in the graded character of V.

Proposition II.24. *Let $\mathcal{H}$ be the quiver Hecke algebra associated to a degenerate affine Hecke algebra as above. Let V be an irreducible graded representation of $\mathcal{H}$. There is a grading shift $V\{\ell\}$ of V such that the graded character of $V\{\ell\}$ is invariant under the substitution $v \mapsto v^{-1}$.*

Proof. By Schur's lemma, a non-graded irreducible $\mathcal{H}$-module can have at most one grading, up to grading shift. As an irreducible $\mathcal{H}$-module is determined up to isomorphism by its ungraded character we find that V and $V^\vee$ are isomorphic as ungraded modules. It follows that the graded character of V is v^k times the graded character of $V^\vee$ for some k. First, we claim that k must be even. It suffices to consider the proposition for the graded character of V_λ as an $1_\lambda\mathcal{H}1_\lambda$-module, where V_λ is some non-zero generalized eigenspace. Let $v \in V_\lambda$ be an element of highest degree. As $\deg(x_\alpha^\lambda) > 0$, we see that $x_\alpha^\lambda v = 0$ for all $\alpha \in \Pi$. We have $V_\lambda = 1_\lambda\mathcal{H}1_\lambda v$, so by the PBW-theorem for $\mathcal{H}$ we find that V_λ is the k-span over elements $\tau_{\alpha_m}^{\lambda_m} \cdots \tau_{\alpha_1}^{\lambda_1}$ with $\lambda = \lambda_1 = s_{\alpha_m}(\lambda_m)$, and the other λ_i defined as usual by $\lambda_{i+1} = s_{\alpha_i}(\lambda_i)$. We claim that the degree of such an element $\tau_{\alpha_m}^{\lambda_m} \cdots \tau_{\alpha_1}^{\lambda_1}$ must be even. In characteristic 2 this is automatic, as every τ_α^λ has even degree. Otherwise, the claim is equivalent to showing that the number of i for which $\langle \lambda_i, s_{\alpha_i} \rangle = \pm c_{\alpha_i}$ is even. There exists $u \in W$ with $\lambda' = u^{-1}(\lambda)$ a standard parabolic weight. Given a decomposition $u = s_{\beta_n} \cdots s_{\beta_1}$, we easily see that the element,

$$\tau_{\beta_1} \cdots \tau_{\beta_n}^\lambda \tau_{\alpha_k}^{\lambda_k} \cdots \tau_{\alpha_1}^\lambda \tau_{\beta_n} \cdots \tau_{\beta_1}^{\lambda'},$$

has the same degree as $\tau_{\alpha_m}^{\lambda_m} \cdots \tau_{\alpha_1}^{\lambda_1}$ modulo 2, as the former element has for every $\tau_{\beta_i}^\mu$ term, a term of the form $\tau_{\beta_i}^{s_{\beta_i}(\mu)}$, and these two degrees add up to 2.

Thus, we are left to show that for λ a parabolic weight and $s_{\alpha_m}\cdots s_{\alpha_1}(\lambda)=\lambda$ we must have the number of i with $\langle\lambda_i,\alpha_i\rangle=\pm c_{\alpha_i}$ is even. If $s_{\alpha_m}\cdots s_{\alpha_1}$ is a reduced expression then each $\lambda_i=\lambda$, so that $\langle\lambda_i,\alpha_i\rangle=0$ and the claim follows.

Note that

$$\langle\lambda_i,\alpha_i\rangle=\langle\lambda,s_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i)\rangle.$$

If $s_{\alpha_m}\cdots s_{\alpha_1}$ is not reduced, then there exists $i<j$ with

$$\begin{aligned}s_{\alpha_j}\cdots s_{\alpha_i}=&s_{\alpha_{j-1}}\cdots s_{\alpha_{i+1}},\\ s_{\alpha_j}\cdots s_{\alpha_{i+1}}=&s_{\alpha_{j-1}}\cdots s_{\alpha_i},\end{aligned}$$

both of which are reduced expressions. It follows for this choice of i,j that,

$$\alpha_i=s_{\alpha_i}\cdots s_{\alpha_{j-1}}(\alpha_j),$$

and hence,

$$\begin{aligned}\langle\lambda_i,\alpha_i\rangle=&\langle\lambda s_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i)\rangle\\ =&\langle\lambda s_{\alpha_1}\cdots s_{\alpha_{j-1}}(\alpha_j)\rangle\\ =&\langle\lambda_j,\alpha_j\rangle.\end{aligned}$$

Moreover, as the equation $s_{\alpha_j}\cdots s_{\alpha_{i+1}}=s_{\alpha_{j-1}}\cdots s_{\alpha_i}$ is an equality of reduced expressions, there is a bijection $p:\{i+1,\cdots j\}\xrightarrow{\sim}\{i,\cdots,j-1\}$ with the property that,

$$\begin{aligned}s_{\alpha_{i+1}}\cdots s_{\alpha_{k-1}}(\alpha_k)=&s_i\cdots s_{\alpha_{p(k)-1}}(\alpha_{p(k)}),\\ p(j)=&i.\end{aligned}$$

It follows that modulo 2, the degree of $\tau_{\alpha_m}^{\lambda_m}\cdots\tau_{\alpha_1}^{\lambda_1}$ is the same as the degree of the $\tau_{\beta_{m-2}}^{\mu_{m-2}}\cdots\tau_{\beta_1}^{\mu_1}$ where the sequence $\beta_1,\cdots,\beta_{m-2}$ is the same as the sequence $\alpha_1,\cdots\hat{\alpha}_i,\cdots,\hat{\alpha}_j,\cdots,\alpha_m$, where the hat denotes ommision, and the μ_i are defined as usual, $\mu_1=\lambda$, $\mu_{i+1}=s_{\beta_i}(\mu_i)$.

By induction it follows that the degree of any sequence $\tau_{\alpha_m}^{\lambda_m}\cdots\tau_{\alpha_1}^{\lambda_1}$ with $\lambda_1=s_{\alpha_m}(\lambda_m)$ is even. We have proved that for any irreducible graded $\mathcal{H}$-module V with $V_\lambda\neq 0$, the degree of any two elements in V_λ differ by a multiple of two. In particular the highest degree and lowest degree element in the graded character of V_λ differ by v^k where k is even. Put $\ell = \frac{k}{2}$. We then have that $V\{-\ell\}$ is a module whose graded character is invariant under the substitution $v \mapsto v^{-1}$. □

Remark II.25. As one can see, the result is true more generally if $\deg(\tau_\alpha^\lambda)$ depends only on $\langle \lambda, \alpha \rangle$.

Corollary II.26. *The category of finite representations of a degenerate affine Hecke algebra* $\mathbb{H}$ *associated to a simply connected semi-simple root data is Morita equivalent to the category of ungraded finite representations of the associated quiver Hecke algebra* $\mathcal{H}$ *for which the elements* x_α^λ *act nilpotently for all* $\alpha \in \Pi, \lambda \in \mathfrak{h}$*. An irreducible representation* V *of* $\mathbb{H}$ *is associated to a unique graded irreducible representation of* $\mathcal{H}$*, whose graded character is invariant under the substitution* $v \mapsto v^{-1}$*, and for which the substitution* $v \mapsto 1$ *yields the character of* V*.*

Proof. We must show that every ungraded irreducible representation of $\mathcal{H}$ has a grading compatible with the action of $\mathcal{H}$. As the center $Z(\mathcal{H})$ of $\mathcal{H}$ is given by the space of invariants $(\bigoplus_{\lambda\in\mathfrak{h}} k[\mathfrak{h}]1_\lambda)^W$, it is a graded ideal of $\mathcal{H}$. Moreover, the action of $\mathcal{H}$ on an irreducible representation factors through the finite dimensional graded algebra $\mathcal{H}/Z(\mathcal{H})$. Then the claim follows from [NV04, Theorem 4.4.4]. □

3.4 A pair of adjoint functors

Fix $\lambda \in \mathscr{T}^{h_0}$ and consider the algebra ${}_\lambda\mathscr{H}_\lambda^{h_0} := 1_\lambda \dot{\mathscr{H}}^{h_0} 1_\lambda$, which we refer to as the *weight Hecke algebra.* There is a functor on finite dimensional representations which we will call the λ-weight restriction functor,

$$\begin{aligned} wRes_\lambda : \mathscr{H}^{h_0} - \quad \text{mod} &\longrightarrow {}_\lambda\mathscr{H}_\lambda^{h_0} - \quad \text{mod}\ , \\ V &\mapsto \qquad V_\lambda. \end{aligned}$$

The weight restriction functor admits a left adjoint, which we will call $wInd_\lambda$, or the λ-weight induction functor,

$$\begin{aligned} wInd_\lambda : {}_\lambda\mathscr{H}_\lambda^{h_0} - \quad \text{mod} &\longrightarrow \quad \mathscr{H}^{h_0} - \quad \text{mod} \ , \\ V_\lambda &\longrightarrow \quad \dot{\mathscr{H}}^{h_0} 1_\lambda \otimes_{{}_\lambda\mathscr{H}_\lambda^{h_0}} V_\lambda. \end{aligned}$$

Proposition II.27. *Let $\lambda \in \mathscr{T}^{h_0}$ be a weight. The following gives a construction of all irreducible representations V of $\mathscr{H}^{h_0}$ for which $V_\lambda \neq 0$, in terms of irreducible representations of ${}_\lambda\mathscr{H}_\lambda^{h_0}$.*

1. *Let V_λ be a non-zero, irreducible ${}_\lambda\mathscr{H}_\lambda^{h_0}$-module. Then, $wInd_\lambda(V_\lambda)$ has a unique irreducible quotient, $L(wInd_\lambda(V_\lambda))$.*

2. *Conversely, suppose that V is an irreducible representation of $\mathscr{H}^{h_0}$ with $V_\lambda \neq 0$. Then, V_λ is an irreducible ${}_\lambda\mathscr{H}_\lambda$-module, and the counit of the adjunction, non-zero map*

$$wInd_\lambda(V_\lambda) \to V,$$

   is non-zero and identifies V with $L(wInd_\lambda(V_\lambda))$.

3. *Finally, the kernel of the above map is the largest $\mathscr{H}^{h_0}$-submodule U of $wInd_\lambda(V_\lambda)$ for which $U_\lambda = 0$. This kernel may be computed in terms of the ${}_\lambda\mathscr{H}_\lambda^{h_0}$-module structure on V_λ.*

Proof. For the first claim, suppose V_λ is an irreducible ${}_\lambda\mathscr{H}_\lambda^{h_0}$-module and let $U \subsetneq wInd_\lambda(V_\lambda)$ be a proper ${}_\lambda\mathscr{H}_\lambda^{h_0}$-submodule. We claim that $U_\lambda = 0$. If not, V_λ irreducible implies that $U_\lambda = V_\lambda$. But, V_λ generates $\dot{\mathscr{H}}^{h_0} 1_\lambda \otimes_{{}_\lambda\mathscr{H}_\lambda^{h_0}} V_\lambda$ as an $\mathscr{H}^{h_0}$-module, so we would have $U = wInd_\lambda(V_\lambda)$, a contradiction.

Now we note that the interior sum of two submodules $U, U' \subset wInd_\lambda(V_\lambda)$ with $U_\lambda = U'_\lambda = 0$ is a submodule, $U + U'$, with $(U + U')_\lambda = 0$. So there is a unique maximal proper submodule, categorized as the sum of all $\mathscr{H}^{h_0}$-submodules U with $U_\lambda = 0$.

For the second claim, we have the easy fact, $wInd_\lambda(V_\lambda)_\lambda = 1_\lambda \dot{\mathscr{H}}^{h_0} 1_\lambda \otimes_{{}_\lambda\mathscr{H}_\lambda^{h_0}} V_\lambda \cong V_\lambda$. If there were a non-trivial submodule $U_\lambda \subset V_\lambda$, then the image U' of the composition of maps

$$wInd_\lambda(U) \to wInd_\lambda V_\lambda \to V$$

would be a submodule of V with $0 \subsetneq U'_\lambda \subsetneq V_\lambda$. Thus, U' would be a non-trivial $\mathscr{H}^{h_0}$-submodule of V.

The last claim follows from the proof of the first claim. We show how to describe the maximal proper submodule.

Let $U_{\lambda'}$ be the left $\mathscr{A}^{h_0}$-span of the elements of the form $r \otimes v \in 1_{\lambda'} \dot{\mathscr{H}}^{h_0} 1_\lambda \otimes V_\lambda$ for which $1_\lambda \dot{\mathscr{H}}^{h_0} 1_{\lambda'}(r \otimes v) = 0$. Then $U_{\lambda'}$ is clearly the λ'-weight space of the maximal proper submodule. We may describe this set as $1_{\lambda'} \dot{\mathscr{H}}^{h_0} 1_\lambda \otimes V_\lambda^{\lambda'}$, where $V_\lambda^{\lambda'}$ is the kernel of the action of $1_\lambda \dot{\mathscr{H}}^{h_0} 1_{\lambda'} \dot{\mathscr{H}}^{h_0} 1_\lambda \subset {}_\lambda\mathscr{H}_\lambda^{h_0}$ on V_λ. □

3.5 The Demazure algebra

For simply connected root datum, $(X, Y, R, \check{R}, \Pi)$, we may identify the algebra, $End_{(\mathscr{A}^{h_0})^W}(\mathscr{A}^{h_0})$ with the Demazure algebra, ${}^{h_0}\mathscr{H}$, an interpolating version of the affine nil-Hecke algebra ${}^0H$ of [Rou11, Section 2.1]. As a vector space this algebra is equal to a tensor product

$$\mathscr{A}^{h_0} \otimes_{\mathbb{C}} {}^{h_0}\mathscr{H}^f$$

of $\mathscr{A}^{h_0}$ with the finite Demazure algebra, ${}^{h_0}\mathscr{H}^f$ of W. the latter algebra is the algebra with generators $\tau_\alpha, \alpha \in \Pi$, which satisfy the braid relation between τ_α, τ_β, as well as the quadratic relation

$$\tau_\alpha^2 = h_0 \tau_\alpha.$$

The algebra structure of ${}^{h_0}\mathscr{H}$ is given by letting $\mathscr{A}^{h_0}$ and ${}^{h_0}\mathscr{H}^f$ be subalgebras, and giving the commutativity relation,

$$\tau_\alpha f - s_\alpha(f)\tau_\alpha = hD_\alpha(f).$$

The following theorem is an algebraic link between the Demazure algebra and the weight Hecke algebra, ${}_\lambda\mathscr{H}_\lambda^{h_0}$, for certain $\lambda \in \mathscr{T}^{h_0}$.

It is clear that for a weight, $\lambda \in \mathscr{T}^{h_0}$, the subalgebra of ${}_\lambda\mathscr{H}_\lambda^{h_0}$ generated by $\mathscr{A}_\lambda^{h_0}$ and r_α^λ with $s_\alpha(\lambda) = \lambda$ is isomorphic to a Demazure algebra with possibly smaller root datum, $(X, Y, R^\lambda, \check{R}^\lambda, \Pi^\lambda)$, $\Pi^\lambda = \{\alpha \in \Pi \mid s_\alpha(\lambda) = \lambda\}$. There is a special case when this subalgebra is the entirety of $1_\lambda \mathscr{H} 1_\lambda$.

We can use simple Weyl group lemmas and the structure theorem of the quiver Hecke algebra $\mathscr{H}(G)$ to give the solution to this question.

Theorem II.28. *Suppose $\lambda \in \mathscr{T}^{h_0}$ is a standard parabolic weight,* i.e. *the stabilizer of λ in W is a standard parabolic subgroup of W. Then the weight-Hecke algebra ${}_\lambda\mathscr{H}_\lambda^{h_0}$ is isomorphic to the Demazure algebra associated to the root data, $(X, Y, R^\lambda, \check{R}^\lambda, \Pi^\lambda)$.*

Corollary II.29. *If $\lambda \in \mathscr{T}^{h_0}$ is a parabolic weight, then there is, up to isomorphism, only one irreducible representation V of $\mathscr{H}^{h_0}$ with $V_\lambda \neq 0$.*

Proof. By corollary II.14, we see that ${}_\lambda\mathscr{H}_\lambda^{h_0}$ is spanned by products $r_{\alpha_n} \cdots r_{\alpha_1}$ with $w = s_{\alpha_n} \cdots s_{\alpha_1}$ a reduced expression for $w \in W$, a Weyl group element which stabilizes λ. By assumption the stabilizer is generated by $s_\alpha, \alpha \in \Pi$ fixing λ, and a reduced expression will use only these terms $s_\alpha, \alpha \in \Pi^\lambda$.

Now, for $(X, Y, R, \check{R}, \Pi)$ simply connected, a parabolic subgroup corresponding to Π^λ will also be simply connected. Thus, the subalgebra ${}_\lambda\mathscr{H}_\lambda^{h_0} \cong {}^{h_0}\mathscr{H}$ will be a matrix algebra over $\mathscr{A}^{W^\lambda}$. Modulo the kernel of the central character corresponding to the irreducible representation, the algebra is a matrix algebra over $\mathbb{C}$. Thus, the weight Hecke algebra ${}_\lambda\mathscr{H}_\lambda^{h_0}$ has only one irreducible representation with a non-zero weight λ. In fact, it's dimension is $\#W^\lambda$, the cardinality of the stabilizer of λ. □

3.6 Example computation

Let us take $(X, Y, R, \check{R}, \Pi)$ the standard root datum for SL_3, and $\mathcal{H}$ the quiver Hecke algebra with the grading given above. Let v stand for the grading shift, so that characters of finite $\mathcal{H}$ modules are in the group ring $\mathbb{Z}[v^{\pm1}][\mathfrak{h}]$. Let k be a field with characteristic not 2. We have the roots, $\alpha = (1, -1, 0), \beta = (0, 1, -1)$, which span the vector space $X \otimes_{\mathbb{Z}} k$. Then $\mathbb{A} = S_k(X)$, the symmetric algebra is a polynomial algebra in the variables α, β. Pick a parabolic weight $\lambda = (1, 1, 0) \in \mathrm{Hom}_{alg}(\mathbb{A}, \mathbb{C})$ and let Λ be the $\mathfrak{S}_2$-orbit of λ. We compute the graded characters of each irreducible representation of $\mathcal{H}$ whose associated representation of $\mathbb{H}$ have central character Λ.

First, suppose V is a finite irreducible $\mathcal{H}$-module with non-zero λ-weight space. Since λ is a standard parabolic weight, there is only one such representation up to isomorphism and we may construct it as follows. By the previous section, $1_\lambda \mathcal{H} 1_\lambda$ is isomorphic to the nil-affine Hecke algebra for the root system $(X, Y, \{\alpha\}, \{\check{\alpha}, \{\alpha\})$. There are two ways of constructing the irreducible $1_\lambda \mathcal{H} 1_\lambda$-module, V_λ. Let k be the trivial $\mathbb{A}$-module with x_α, x_β acting by 0. Then $1_\lambda \mathcal{H} 1_\lambda \otimes_{\mathbb{A}1_\lambda} k$ has the correct dimension, and hence must be the unique irreducible representation of $1_\lambda \mathcal{H} 1_\lambda$ with x_α, x_β acting nilpotently.

Alternatively we could induce from the finite nil Hecke algebra. Let $\mathbb{A}^{\langle s_\alpha \rangle}$ be the functions which are invariant under the action of s_α. Let J_0 be the positively graded elements of this subalgebra. Then J_0 is a central ideal of $1_\lambda \mathcal{H} 1_\lambda$, and we can form the representation $1_\lambda \mathcal{H} 1_\lambda \otimes_{{}^0\mathscr{H}^f} k$, where k is the trivial nil-Hecke ${}^0\mathscr{H}^f$-module with τ_α^λ acting by 0. Again, this representation has the correct dimension and so must be isomorphic to the unique irreducible nilpotent representation.

It is clear that the graded character of the first representation is $1 + v^2$, whereas the graded character of the second one is $v^{-2} + 1$. We may shift the grading so that the graded character of this module is $v + v^{-1}$, which is invariant under the substitution $v \mapsto v^{-1}$. Let $L(V_\lambda)$ be the irreducible quotient of the module weight-induced from V_λ. Then the graded character of $L(V_\lambda)_{s_\beta(\lambda)}$ is simply 1, and the character of $L(V_\lambda)_{s_\alpha s_\beta(\lambda)}$ is simply 0. It is worth noting that the graded character of $L(V_\lambda)$ is invariant under the substitution $v \mapsto v^{-1}$.

The case of irreducible representations with non-zero $s_\alpha s_\beta(\lambda)$-weight space is identical as that is also a standard parabolic weight. We are left to compute the irreducible modules $V_{s_\beta(\lambda)}$ over $1_{s_\beta(\lambda)}\mathcal{H}1_{s_\beta(\lambda)}$ whose associated irreducible $\mathcal{H}$-module $L(V_{s_\alpha})$ has no $\lambda, s_\alpha s_\beta(\lambda)$-weight space. This is equivalent to $\tau_\beta^\lambda \tau_\beta^{s_\beta(\lambda)} = x_\beta$, and $\tau_\alpha^{s_\alpha s_\beta(\lambda)} \tau_\alpha^{s_\beta(\lambda)} = -x_\alpha$ acting by 0 on $V_{s_\beta(\lambda)}$. By the commutativity relation between $\tau_\gamma := \tau_\alpha \tau_\beta \tau_\alpha^{s_\beta(\lambda)}$ and $x_\gamma := x_{\alpha+\beta}$, $\tau_\gamma x_\gamma + x_\gamma \tau_\gamma = 2$, we see that the constant map 2 must be zero on $V_{s_\beta(\lambda)}$. It follows that there is no such non-zero irreducible representation.

3.7 Appendix: quiver Hecke algebras in type A

We note that in type A the (graded) quiver Hecke algebras appearing in [Rou11] are related to the algebra $\mathscr{H}^{h_0}(G)$ we have defined here with the standard root datum GL_n. We first present the quiver Hecke algebra $H_n(\Gamma)$ from [Rou11] which is shown there to be related to the degenerate affine Hecke algebras of type A.

Definition. Define the quiver Γ with vertices $I = k$ and arrows $a \to a+1$, $a \in k$. Denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in k^n$ with $\lambda_i \in I, i = 1, 2, \ldots, n$. Then $H_n(\Gamma)$ is the algebra generated by idempotents $\{1_\lambda\}_{\lambda \in \Lambda}$, by variables $\{x_i^\lambda\}_{i=1}^n$, and $\{\tau_i^\lambda\}_{i \in I}$ subject to the following relations.

$$1_\lambda 1_{\lambda'} = 1_\lambda \delta_{\lambda,\lambda'},$$

$$x_i^\lambda 1_{\lambda'} = 1_{\lambda'} x_i^\lambda = \delta_{\lambda,\lambda'} x_i^\lambda,$$

$$\tau_i^\lambda 1_{\lambda'} = 1_{s_i(\lambda')} \tau_i^\lambda = \delta_{\lambda,\lambda'} \tau_i^\lambda,$$

$$x_i^\lambda x_j^\lambda = x_j^\lambda x_i^\lambda,$$

$$\tau_i x_j^\lambda - x_{s_i(j)}^{s_i(\lambda)} \tau_i^\lambda = \begin{cases} -1_\lambda & \text{if } s_i(\lambda) = \lambda \text{ and } j = i, \\ 1_\lambda & \text{if } s_i(\lambda) = \lambda \text{ and } j = i+1, \\ 0 & \text{else,} \end{cases}$$

$$\tau_i^{s_i(\lambda)} \tau_i^\lambda = \begin{cases} 0 & \text{if } \lambda_i = \lambda_j \\ x_{i+1}^\lambda - x_i^\lambda & \text{if } \lambda_j = \lambda_i + 1 \\ x_i^\lambda - x_{i+1}^\lambda & \text{if } \lambda_j = \lambda_i - 1 \\ 1_\lambda & \text{else,} \end{cases}$$

$$\tau_i^{s_j(\lambda)} \tau_j^\lambda - \tau_j^{s_i(\lambda)} \tau_i^\lambda = 0 \text{ if } \mid i - j \mid > 1,$$

$$\tau_i^{s_{i+1}s_i(\lambda)} \tau_{i+1}^{s_i(\lambda)} \tau_i^\lambda - \tau_{i+1}^{s_i s_{i+1}(\lambda)} \tau_i^{s_{i+1}(\lambda)} \tau_{i+1}^\lambda = 0 \text{ if } \lambda_i = \lambda_{i+1} = \lambda_{i+2} \text{ or } \lambda_i \neq \lambda_{i+2}.$$

There is also a relation stating that $H_n(\Gamma)$ have no polynomial torsion (see [Rou11, Section 3.2.2]), which is equivalent to the missing braid like relations from [Rou08, Section 3.2.1].

Let $(\mathbb{Z}^n, \mathbb{Z}^n, R, \check{R}, \Pi)$ be a root data for GL_n with standard basis $\{e_i\}_{i=1}^n$ of $\mathbb{Z}^n$ and simple roots $\alpha_i = e_{i+1} - e_i$, $i = 1, \ldots, n-1$. Let $c_\alpha = 1$ be a set of parameters, and let $\mathbb{H}$ be the associated degenerate affine Hecke algebra, with $T_i = T_{\alpha_i}$. We remark that although the weight lattice for GL_n does not contain the fundamental weights, it is the case that every weight is conjugate to a standard parabolic weight. Thus, our construction of $\mathscr{H}^{h_0}(G), \mathcal{H}$ may be carried out with no change.

Finally, let V be a representation of the degenerate affine Hecke algebra $\mathbb{H}$. In this case we identify $P_i = P_{e_i}$, $\mathbb{A} = k[P_1, \ldots, P_n]$, and $\lambda \in (k^*)^n$ a weight of $\mathbb{A}$ via $\lambda(P_i) = \lambda_i$. By

[Rou08, Theorem 3.11], we can turn V into an $H_n(\Gamma)$-module with 1_λ the projection onto the λ-generalized-eigenspace for $\mathbb{A}$, x_i^λ acting by $(P_i - \lambda_i)1_\lambda$, and τ_i^λ acting by,

$$\tau_i^\lambda \mapsto \begin{cases} (P_i - P_{i+1} + 1)^{-1}(T_i - 1)1_\lambda & \text{if } s_i(\lambda) = \lambda, \\ ((P_i - P_{i+1})T_i + 1)1_\lambda & \text{if } \lambda_{i+1} = \lambda_i + 1, \\ \frac{P_i - P_{i+1}}{P_i - P_{i+1} + 1}(T_i - 1)1_\lambda + 1_\lambda & \text{else.} \end{cases}$$

Using our notation for α_i, as well as the relation commutativity relation for T_i and 1_λ, we find that this is equivalent to,

$$\tau_i^\lambda \mapsto \begin{cases} (1 - P_{\alpha_i})^{-1}(T_i - 1)1_\lambda & \text{if } s_i(\lambda) = \lambda, \\ (-P_{\alpha_i})1_{s_i(\lambda)}T_i1_\lambda & \text{if } \lambda_{i+1} = \lambda_i + 1, \\ (\frac{-P_{\alpha_i}}{1 - P_{\alpha_i}})1_{s_i(\lambda)}(T_i)1_\lambda & \text{else.} \end{cases}$$

Using the isomorphism $\mathscr{H}^{h_0}(G) \to \dot{\mathscr{H}}^{h_0}$ of the previous section we find the following:

Theorem II.30. *There is a map $H_n(\Gamma) \to \mathscr{H}^{h_0}(G)$ given by $x_i^\lambda \mapsto (X_i - \lambda_i)1_\lambda \in \dot{\mathscr{A}}^{h_0}$, and*

$$\tau_i^\lambda \mapsto \begin{cases} r_{\alpha_i}^\lambda & \textit{if } s_i(\lambda) = \lambda, \\ (-\alpha_i)r_{\alpha_i}^\lambda & \textit{if } \lambda_{i+1} = \lambda_i + 1 \\ (\frac{-\alpha_i}{1-\alpha_i})r_{\alpha_i}^\lambda & \textit{else.} \end{cases}$$

Moreover, this map gives a graded isomorphism of $H_n(\Gamma)$ onto $\mathcal{H}$.

We can do something analogous for localized affine Hecke algebras, but only in the simply laced case.

Definition. Let $q_\alpha = q \in k^*$. Define the quiver Γ of [Rou08, Section 3.2.5] with vertices $I = k^*$ and arrows $a \to q \cdot a$. Denote $\lambda = (\lambda_1, \ldots, \lambda_n) \in (k^*)^n$. Then $H_n(\Gamma)$ is the algebra generated by idempotents $\{1_\lambda\}_{\lambda\in(k^*)^n}$, variables $\{x_i^\lambda\}_{i=1}^n$ and $\{\tau_i^\lambda\}_{i=1}^n$ with relations:

$$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'}1_\lambda,$$

$$x_i^\lambda 1_{\lambda'} = 1_{\lambda'}x_i^\lambda = \delta_{\lambda,\lambda'}x_i^\lambda,$$

$$\tau_i^\lambda 1_{\lambda'} = 1_{s_i(\lambda')}\tau_i^\lambda = \delta_{\lambda,\lambda'}\tau_i^\lambda,$$

$$x_i^\lambda x_j^\lambda = x_j^\lambda x_i^\lambda,$$

$$\tau_i^\lambda x_j^\lambda - x_{s_i(j)}^{s_i(\lambda)}\tau_i^\lambda = \begin{cases} -1_\lambda & \text{if } s_i(\lambda)=\lambda \text{ and } j=i, \\ 1_\lambda & \text{if } s_i(\lambda)=\lambda \text{ and } j=i+1, \\ 0 & \text{else,} \end{cases}$$

$$\tau_i^{s_i(\lambda)}\tau_i^\lambda = \begin{cases} 0 & \text{if } \lambda_i=\lambda_j \\ x_{i+1}^\lambda - x_i^\lambda & \text{if } \lambda_j = \lambda_i\cdot q \\ x_i^\lambda - x_{i+1}^\lambda & \text{if } \lambda_j=\lambda_i\cdot q^{-1} \\ 1_\lambda & \text{else,} \end{cases}$$

$$\tau_i^{s_j(\lambda)}\tau_j^\lambda - \tau_j^{s_i(\lambda)}\tau_i^\lambda = 0 \text{ if } \mid i-j\mid > 1,$$

$$\tau_i^{s_{i+1}s_i(\lambda)}\tau_{i+1}^{s_i(\lambda)}\tau_i^\lambda - \tau_{i+1}^{s_is_{i+1}(\lambda)}\tau_i^{s_{i+1}(\lambda)}\tau_{i+1}^\lambda = 0 \text{ if } \lambda_i=\lambda_{i+1}=\lambda_{i+2} \text{ or } \lambda_i\neq\lambda_{i+2}.$$

Again, there is also a relation (see [Rou11]) in $H_n(\Gamma)$ saying that $H_n(\Gamma)$ contains no polynomial torsion which accounts for the missing braid like relations in [Rou08].

Let $c_\alpha = 1, q_\alpha = q \in k^*\backslash\{\pm 1\}$ and $h_0 = q-1$. Again, let $(\mathbb{Z}^n, \mathbb{Z}^n, R, \check{R}, \Pi)$ be a root datum for GL_n. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{Z}^n$, and let the root basis be defined by $\alpha_i = e_{i+1}-e_i$. With this data we associate the affine Hecke algebra $\mathscr{H}$ and its localized version $\dot{\mathscr{H}}^{h_0}$. We identify $\mathscr{A}$ with the ring $k[X_1^{\pm1},\dots,X_n^{\pm1}]$, with X_i corresponding to the exponential of e_i. By [Rou11, Theorem 3.12], there is a map $H_n(\Gamma)\to\dot{\mathscr{H}}^{h_0}$ given by mapping

$x_i^\lambda \mapsto X_i \lambda_i^{-1} 1_\lambda$, and:

$$\tau_i^\lambda \mapsto \begin{cases} \lambda_i X_{i+1}^{-1}(qX_iX_{i+1}^{-1} - 1)^{-1}(T_i - q)1_\lambda & \text{if } s_i(\lambda) = \lambda, \\ q^{-1}\lambda_i^{-1} X_{i+1}(X_iX_{i+1}^{-1} - 1)1_{s_i(\lambda)}T_i 1_\lambda & \text{if } \lambda(\alpha) = q, \\ \frac{X_iX_{i+1}^{-1}-1}{qX_iX_{i+1}^{-1}-1} 1_{s_i(\lambda)}T_i 1_\lambda & \text{else.} \end{cases}$$

Using our notation for U_x the exponential of $x \in \mathbb{Z}^n$ in $\mathscr{A}$ the group ring of $\mathbb{Z}^n$, we find the above mapping to be:

$$\tau_i^\lambda \mapsto \begin{cases} \lambda_i X_{i+1}^{-1}(qU_{-\alpha_i} - 1)^{-1}(T_i - q)1_\lambda & \text{if } s_i(\lambda) = \lambda, \\ q^{-1}\lambda_i^{-1} X_{i+1}(U_{-\alpha_i} - 1)1_{s_i(\lambda)}T_i 1_\lambda & \text{if } \lambda(\alpha) = q, \\ \frac{U_{-\alpha_i}-1}{qU_{-\alpha_i}-1} 1_{s_i(\lambda)}T_i 1_\lambda & \text{else.} \end{cases}$$

It follows from the polynomial representations of $\dot{\mathscr{H}}^{h_0}, \mathscr{H}^{h_0}(G)$ that there is a mapping from $H_n(\Gamma)$ to $\mathscr{H}^{h_0}(G)$ given by mapping $x_i^\lambda \mapsto X_i^\lambda \lambda_i^{-1} 1_\lambda$, and:

$$\tau_i^\lambda \mapsto \begin{cases} \lambda_i X_{i+1} r_{\alpha_i}^\lambda & \text{if } s_i(\lambda) = \lambda, \\ q^{-1}\lambda_i^{-1} X_{i+1}(U_{-\alpha_i} - 1) r_\alpha^\lambda & \text{if } \lambda(\alpha) = q, \\ \frac{U_{-\alpha_i}-1}{qU_{-\alpha_i}-1} r_\alpha^\lambda & \text{else.} \end{cases}$$

Notice now, that the braid relation for τ_α^λ is *different* than the braid relation for r_α^λ.

Remark II.31. It should be noted that in [Rou11, Theorem 3.11], a grading is also given to the affine Hecke algebras of type GL_n by use of an algebraic W-equivariant map $\mathbb{A} \to \mathscr{A}$. There is an algebraic obstruction to this approach in other types and we do not give gradings to affine Hecke algebras in this paper. One could give irreducible representations of affine Hecke algebras gradings by using the correspondence given by [Lus89] between their representation categories.

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