The Growth of Number Representation: Successive Levels of Schematic Learning

Lauren B. Resnick
University of Pittsburgh

In a companion paper*, Greeno has outlined a theory of learning as the successive construction of new schemata. In this paper we explore the effects of cumulating schemata on performance and competence in the domain of number knowledge. We begin with a brief consideration of the number representation assumed to be available to children before they have learned place value and decimal notation. Then we outline several stages in the acquisition of decimal place-value knowledge. Finally, we consider the implications of successive stages of number representation for a theory of the understanding of cardinality.

Early Number Representation

Varied databases on early counting abilities (Gelman & Gallistel, 1978; Fuson, in press), simple mental arithmetic (e.g., Groen & Resnick, 1977), and story problem solution (Vergnaud, in press; Carpenter & Moser, in press; Nesher, in press) have provided the basis for formal models of preschool and early school mathematical performances (e.g., Riley, Greeno, & Heller, 1978; Greeno, Gelman, & Riley, 1978; Briars & Larkin, 1981). These models converge on the following features of pre-decimal number representation:

1. Children possess an ordered string of "count words", linked by "next" and "backward next" relationships. Each position in the string has come to stand for a quantity. The string can be used to solve problems via counting. It can also be used as an analog representation of quantity. For example, the positions of two target quantities can be found and compared for relative "largeness" (Sekuler & Mierkiewicz, 1977).

2. Children can interpret small numbers in terms of a Part/Whole schema (Figure 1) such that any number can be interpreted either as a whole composed of two smaller numbers or as a part in a larger whole. The Part/Whole schema includes the

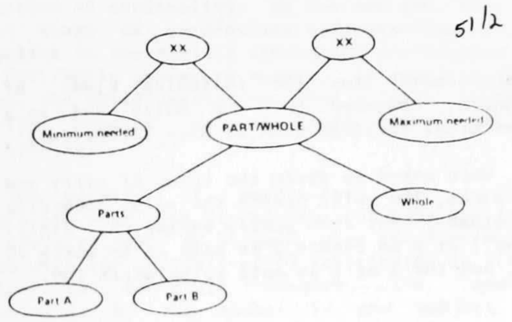


Figure 1

The Part/Whole Schema

---

*Greeno, J. G. Meaningful learning. Paper presented at the meeting of the Cognitive Science Society, Berkeley, CA, August 1981.

constraint that the combined parts neither exceed nor fall short of the whole quantity. The schema has been shown to function in successful solution of certain story problems (e.g., those with the unknown in the first position) that younger children find very difficult. It seems likely that the schema also permits children to discover the complementarity of addition and subtraction, leading to a particularly efficient--and mathematically elegant--solution to subtraction problems. In this procedure, which has been observed in children as young as 7 years (Woods, Resnick, & Groen, 1975; Svenson & Hedenborg, 1979), children either count up from the smaller number or count down from the larger, whichever requires fewer counts.

Acquisition of Place-Value Schemata

In the course of learning the decimal number system, the string of count words is gradually reorganized to reflect an understanding that multidigit numbers are compositions of units and tens (later also hundreds, thousands, etc.). This is accomplished through successive elaborations of the Part/Whole schema. Several stages in the acquisition of this compositional interpretation of multidigit numbers can be identified in a program (MOLLY) that simulates the performances of a 9-year-old girl (Molly) as she acquired new knowledge through special remedial instruction in multidigit subtraction.

Four stages in MOLLY'S knowledge of place value can be distinguished:

Stage 1: Unique partitioning of multidigit numbers. At the earliest stage of place-value knowledge, MOLLY has a knowledge structure which organizes conventional information about the structure of multidigit written numbers (Figure 2).

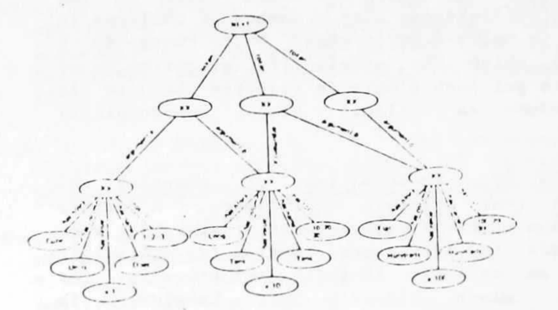


Figure 2

This structure identifies columns according to their positional relationship to each other (thus the centrality of the Next relationship). Attached to each column is a block shape (these refer to the shapes of blocks used in teaching base arithmetic), a counting string, a value, a column name, etc. Using the Next structure, MOLLY can:

1. construct a block display for any written number. This is done by identifying which column the number is in, finding the block shape that match that column, and displaying the number of blocks specified by the digit in the column.

2. *"read" a block display.* This is done by starting with the largest block shape, finding the counting string that matches it, enumerating the blocks using the appropriate string, and then iterating through successively smaller block shapes.

3. interpret a written numeral as "x hundreds, y tens, and z ones."

4. compare block displays on the basis of the highest-valued block only (e.g., for 347 v. 734, compare only the hundreds blocks).

As long as the Next structure alone is used to interpret numbers, each written number can have only one block representation--a "canonical" representation, with no more than 9 blocks per column. This means that there is no basis for a semantic interpretation of the operations of carrying and borrowing. The next stage provides the earliest basis for this interpretation.

<u>Stage 2: Multiple partitionings arrived at empirically.</u> At this stage the Part/Whole schema is elaborated to include a special restriction, applied to two-digit numbers, that one of the parts be a multiple of 10. Application of Part/Whole permits multiple partitionings, and therefore multiple block representations, for any written number. Any specific partitions, however, must be arrived at through a counting solution. For example, to "show 47 with more ones," MOLLY first applies Part/Whole in a global fashion and then concludes that if the whole is to stay the same but more ones are to be shown, there must be fewer tens. It therefore reduces the tens by a single block. The schema is next instantiated with 47 in the Whole slot, and 30 in one of the Parts. The remaining Part is found by adding ones blocks and counting up until 47 is reached.

Two important concepts have been added to the number representation at this stage. First, the equivalence of several partitionings has been recognized. Second, the possibility of having more than 9 of a particular block size has been admitted. This is crucial for understanding "borrowing", where--temporarily--more than 9 of a given denomination must be understood to be present, without changing the total value of the quantity. Interviews with a number of children in addition to Molly make it clear that there is a stage in which the possibility of borrowing or trading to get more blocks is rejected because it will produce an "illegal" (i.e., noncanonical) display.

<u>Stage 3: Preservation of quantity by exchanges that maintain equivalence.</u> A further elaboration of Part/Whole appears at Stage 3, when MOLLY adds to its representation for multidigit numbers an explicit 10-for-1 relationship for adjacent block sizes. This knowledge is represented by a Trade schema (Figure 3) which specifies a class of legal exchanges among blocks.

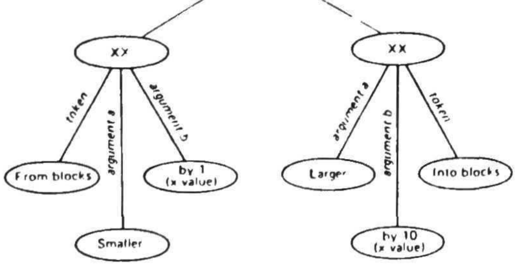
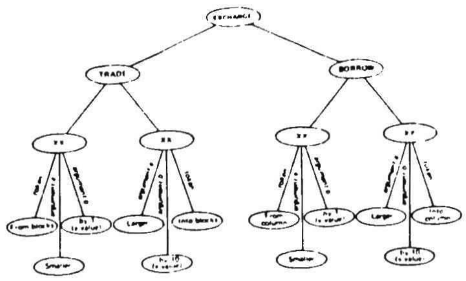


Figure 3

The schema specifies that there is a "from" pile of blocks, from which blocks are removed. This pile becomes smaller by one block. There is also an "into" pile of blocks that becomes larger by 10 blocks. The <u>value</u> of the blocks in the "from" and "into" piles is established by multiplying the number of blocks removed by the value of the block shape (as specified in the Next structure). Thus, when trades are made between adjacent block sizes, the schema specifies that both the "into" and the "from" values will be 10. Applied as an elaboration of the Part/Whole schema, the Trade schema allows MOLLY to conclude--without having to count up--that there has been no change in the whole quantity.

<u>Stage 4: Application to written addition and subtraction.</u> MOLLY also provides a theory of how the various levels of quantity representation discussed above can come to be applied to written numerals. MOLLY simulates the learning sequence achieved as a result of instruction that forced attention to the details of a mapping between the operations of written borrowing and those of block trading (Resnick, in press). Figure 4 shows the result of constructing this mental mapping: an <u>abstraction</u> that treats the two procedures as expressions of the same exchange principles, with analogous elements in the Trade and Borrow schemata. Evidence for this level of understanding



Analogous Borrow and Trade Schemata

Figure 4

of number comes from the following kinds of performances, observed both in Molly and in a number of other children whom we have interviewed:

1. When asked to state the value of carry and borrow marks, the child states the value according to the column rather than simply naming the digit. Thus the 1 at <u>a</u> in Figure 5 is said to be worth 10 (not 1), and the 1 at <u>b</u> is said to be worth 100.

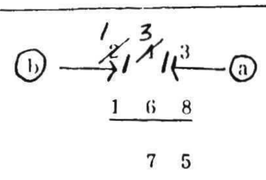


Figure 5

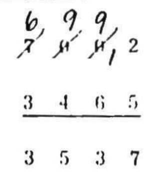


Figure 6

2. When asked to show the blocks that represent a given carry or borrow mark, the child selects blocks according to value. A ten-block is selected for a, a hundred-block for b, and so forth.

3. The child can construct justifications for the various markings in written subtraction. For example, Molly explained that in the subtraction problem shown in Figure 6 above, she had borrowed one thousand from the 7. When asked where she had put the thousand, she was puzzled at first, because there is no place where one can "see" 1000 in the markings given. However, she then said, "It is divided up. Nine hundred of it is here (indicating the hundreds column), and the other hundred is here (indicating the tens and the ones columns together). You see, this 9 is really 90 and this 1 is really 10 and that adds up to 100!" MOLLY is able to construct this explanation by first calling on the Exchange (Borrow) schema, which specifies that if a quantity of 1000 has been taken from a column it must be put into another column. Unable to find a column with 1000 put into it, MOLLY calls on the Part/Whole schema, sets the Whole slot equal to 1000 and looks for two Parts that add up to 1000. It finds 900, but cannot find a column with the 100 necessary for completing the Whole. MOLLY iterates through the Part/Whole schema again, this time setting the Whole equal to 100 and now finding 90 and 10 as the Parts.

An Interpretation of Cardinality

This characterization of children's developing number knowledge permits us to give a more precise psychological meaning to the understanding of "cardinality" than has heretofore been possible. Gelman and Gallistel (1978) included in their principles of counting a cardinality principle, which specifies that the final count word reached when a set of objects is being enumerated is the total number in the set--i.e., the set's cardinality. For the preschool child, who has not yet come to interpret quantity in terms of the Part/Whole schema, this is the only meaning of cardinality available. This criterion of understanding cardinality has been criticized, however, (e.g., Comiti, 1980) as too weak, and in particular as not reflecting the Piagetian definition of cardinality. We can now see that a higher stage of cardinality understanding can be recognized in the child's subsequent application of the Part/Whole schema to number. In applying this schema, the child understands that a total (whole) quantity remains the same even under variant partitionings.

The meaning of cardinality is further elaborated when the place-value schemata outlined here are acquired. At Stage 2, when the Part/Whole schema with the multiple-of-10 restriction is first applied to two-digit numbers, the amount represented by the number is now subject to multiple partitionings without a change in quantity. This is exactly parallel to the new understanding of cardinality for smaller numbers that was achieved when the Part/Whole schema was applied to them. Without application of the Part/Whole schema the cardinality of a number resides in the specific display set and the number attached to it through legal counting procedures. With Part/Whole, cardinality resides in the total quantity, no matter how it is displayed or partitioned.

The Trade stage of multidigit number representation represents yet a higher level of understanding of cardinality. Now it is recognized that cardinality is not altered by a specified set of legal exchanges. An analogy can be drawn with an earlier recognition of quantity as unchanged under various physical transformations (such as spreading out a display of objects-- the classic Piagetian test of conservation). However, the transformations produced under control of the Trade schema do in fact involve a change in the actual number of objects present. Thus, recognition that the value of the total quantity remains unchanged requires a level of abstraction concerning the nature of cardinality that was not required for earlier stages of understanding.

References

Briars, D. J., & Larkin, J. H. An integrated model of skill in solving elementary word problems. Unpublished manuscript, Carnegie-Mellon University, 1981.

Carpenter, T., & Moser, J. The development of addition and subtraction problem solving skills. In T. Carpenter, J. Moser, & T. Romberg (Eds.) Addition and subtraction: A developmental perspective, Hillsdale, NJ: Lawrence Erlbaum Associates, in press.

Comiti, C. Le concept de nombre chez l'enfant. Communication presented at the Quatrieme Congres International sur l' Enseignement des Mathematiques, Berkeley, CA, August 1980.

Fuson, K. An analysis of the counting-on solution procedure in addition. In T. Romberg, T. Carpenter, & J. Moser (Eds.) Addition and subtraction: A developmental perspective. Hillsdale, NJ: Lawrence Erlbaum Associates, in press.

Gelman, R., & Galistel, C. R. The child's understanding of number. Cambridge, MA: Harvard University Press, 1978.

Greeno, J. G. Gelman, R. & Riley, M. Young children's counting and understanding. Paper presented at the annual meeting of the Psychonomic Society, San Antonio, TX, November 1978.

Groen, G. J., & Resnick, L. B. Can preschool children invent addition algorithms? Journal of Educational Psychology, 1977, 69, 645-652.

Nesher, P. Levels of description in the analysis of addition and subtraction word problems. In T. Carpenter, J. Moser, & T. Romberg (Eds.) Addition and subtraction: A developmental perspective. Hillsdale, NJ: Lawrence Erlbaum Associates, in press.

Resnick, L. B. Syntax and semantics in learning to subtract. In T. Carpenter, J. Moser, & T. Romberg (Eds.), Addition and subtraction: A developmental perspective. Hillsdale, NJ: Lawrence Erlbaum Associates, in press.

Riley, M. S., Greeno, J. G., & Heller, J. I. Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg, (Ed.), The development of mathematical thinking. New York: Academic Press, in press.

Sekuler, R., & Mierkiewicz, D. Children's judgments of numerical inequality. Child Development, 1977 48, 630-633.

Svenson, O., & Hedenborg, M. L. Strategies used by children when solving simple subtractions. Acta Psychology, 1979 43, 1-13.

Vergnaud, G. A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. Carpenter, J. Moser, & T. Romberg (Eds.), Addition and subtraction: A developmental perspective. Hillsdale, NJ: Lawrence Erlbaum Associates, in press.

Woods, S., Resnick, L. B., & Groen, G. J. An experimental test of five process models for subtraction. Journal of Educational Psychology, 67(1), 17-21.