Title
Almost Global Solutions of the Eikonal Equation

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The "infinity-Laplace" equation

$$\Delta_\infty u = \frac{1}{2}\langle D(|Du|^2), Du\rangle = \sum_{l,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

arises when one attempts to find a real-valued function u which minimizes $\max |Du|$ subject to u having prescribed values on the boundary of a given set. Here $Dv = (v_{x_i}, \dots, v_{x_i})$ is the gradient of v. From the first expression for $\Delta_\infty u$, solutions of the eikonal equation $|Du| = 1$ are "infinity harmonic", that is, they satisfy $\Delta_\infty u = 0$. Particular solutions of the eikonal equation are given by the linear function $u(x) = \langle p, x\rangle$ where $p \in \mathbb{R}^n$, $|p| = 1$, and the distance to a point $z \in \mathbb{R}^n$, $u(x) = |x - z|$. In the latter case, u is a solution on $x \neq z$; it has a single singularity. The distance to a line segment is also a solution of the eikonal equation off the line segment; it has a one dimensional set of singularities.

We will explain a result obtained with L. Caffarelli which states, roughly speaking, that there is nothing between the cone functions and the distance to a line segment. That is, if u is a solution of the eikonal equation $|Du| = 1$ in $\mathbb{R}^n$ except on some set whose 1-dimensional measure is zero, then there are only the two possibilities above: either u is affine or $u(x) = a \pm |x - z|$ for some $a \in \mathbb{R}, z \in \mathbb{R}^n$.

Motivation for the question arose partly from the fact that the "cone functions" are fundamental to theory of the infinity Laplacian, as infinity harmonic functions are exactly those which obey the maximum principle relative to functions of the form $a|x - z|$ where $a \in \mathbb{R}$. One wondered if there might be other important solutions with, say, some finite number of singularities. The answer is no.