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Al'brekht's Method in Infinite Dimensions

Arthur J. Krener

Abstract— In 1961 E. G. Albrekht presented a method for the optimal stabilization of smooth, nonlinear, finite dimensional, continuous time control systems. This method has been extended to similar systems in discrete time and to some stochastic systems in continuous and discrete time. In this paper we extend Albrekht's method to the optimal stabilization of some smooth, nonlinear, infinite dimensional, continuous time control systems whose nonlinearities are described by Fredholm integral operators.

Keywords: Infinite Dimensional Optimal Stabilization, Infinite Dimensional Linear Quadratic Regulation, Fredholm Integral Operators

I. INTRODUCTION

A fundamental control engineering problem is to find a feedback law that stabilizes a plant to an operating point. Suppose the plant can be modeled by a finite dimensional system of nonlinear differential equations

$$\dot{x} = f(z, u) \qquad (1)$$

and the operating point is $z = 0, u = 0$ where $f(0, 0) = 0$. We assume that the state z is n dimensional and control u is m dimensional. We also assume that $f(z, u)$ is smooth around the operating point,

$$f(z, u) = Fz + Gu + O(z, u)^2$$

Then posing and solving a Linear Quadratic Regulator (LQR) problem will yield a locally stabilizing linear feedback. We chose an $(n + m) \times (n + m)$ nonnegative definite matrix $[Q, S'; S, R] \geq 0$ with $R > 0$ positive definite. We seek to minimize

$$\frac{1}{2} \int_0^\infty z'Qz + 2z'Su + u'Ru \; dt \qquad (2)$$

subject to the linear dynamics

$$\dot{z} = Fz + Gu$$

and a given initial condition $z(0) = z^0$.

Under the standard assumptions of stabilizability and detectability, the optimal cost exists and is of the form $\frac{1}{2}(z^0)'Pz^0$ and the optimal feedback exists and is of the form $u(t) = Kz(t)$ where the $n \times n$ nonnegative definite

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matrix $P \geq 0$ and the $m \times n$ matrix K satisfy the familiar LQR equations

$$
\begin{aligned}
0 &= F'P + PF + Q - (PG + S)R^{-1}(PG + S)' \\
K &= -R^{-1}(PG + S)'
\end{aligned}
$$
$$(3)$$

The first equation is called the Algebraic Riccati Equation (ARE).

The function $z'Pz$ is a local Lyapunov function for the closed loop nonlinear dynamics using the optimal linear feedback,

$$\dot{z} = f(z, Kz)$$

$$
\begin{aligned}
\frac{d}{dt}z'(t)Pz(t) &= ((F + GK)z + O(z)^2)'Pz \\
&\quad + z'P((F + GK)z + O(z)^2) \\
&= -z'\left(Q + (PG + S)R^{-1}(PG + S)'\right)z \\
&\quad + O(z)^3
\end{aligned}
$$

Consider the problem of minimizing a more general criterion

$$\int_0^\infty l(z, u) \; dt \qquad (4)$$

subject to the nonlinear dynamics (1) where the Lagrangian is smooth

$$l(z, u) = \frac{1}{2}\left(z'Qz + 2z'Su + u'Ru\right) + O(z, u)^3$$

The higher degree terms in $l(z, u)$ could be penalty terms to ensure that state and control constraints are satisfied. They might destroy its even symmetry. The higher degree terms in $f(z, u)$ might destroy its odd symmetry.

Given that $z(0) = z^0$ the optimal cost $\pi(z^0)$ if it exists and is smooth and if the optimal feedback $u(t) = \kappa(z(t))$ exists then they satisfy the familiar Hamilton-Jacobi-Bellman (HJB) equations

$$
\begin{aligned}
0 &= \min_u \left\{ \frac{\partial \pi}{\partial z}(z)f(z, u) + l(z, u) \right\} \\
\kappa(z) &= \operatorname{argmin}_u \left\{ \frac{\partial \pi}{\partial z}(z)f(z, u) + l(z, u) \right\}
\end{aligned}
$$
$$(5)$$

If the quantity to be minimzed in these equations is a smooth function then the HJB equations can be simplified to the sHJB equations

$$
\begin{aligned}
0 &= \frac{\partial \pi}{\partial z}(z)f(z, \kappa(z)) + l(z, \kappa(z)) \\
0 &= \frac{\partial \pi}{\partial z}(z)\frac{\partial f}{\partial u}(z, \kappa(z)) + \frac{\partial f}{\partial u}(z, \kappa(z))
\end{aligned}
$$
$$(6)$$

Assuming $f(z, u)$ and $l(z, u)$ are sufficiently smooth Al'brekht [1] showed how to compute the Taylor polynomials of $\pi(z^0)$ and $\kappa(z)$ degree by degree. At the lowest degrees

he obtained the familiar LQR equations. At higher degrees he obtained a sequence of linear equations for the higher degree coefficients of $\pi(z^0)$ and $\kappa(z)$. The purpose of this paper is to show that Al'brekht's method can be extended to some infinite dimensional control problems. Navasca extended Al'brekht's method to discrete time problems [16]. We have extended it to some stochastic problems in both continuous [11] and discrete time [12].

In the next section we review Al'brekht method for smooth finite dimensional problems. Rather than rely on the sHJB equations we will use a technique which is a conceptually simpler called completing the square. Completing the square is frequently used to derive the LQR equations. We shall use it to find the higher degree terms of the optimal cost. In Section 4 we extend Al'brekht method to controlled reaction-diffusion systems. Section 5 contains an example of such a system.

We are not the first to use Al'brekht's method on infinite dimensional systems, see the works of Kunisch and coauthors [2], [3], [14]. Krstic and coauthors have had great success stabilizing infinite dimensional systems where the the nonlinearites are expressed by Volterra integral operators of increasing degrees using backstepping techniques, [13], [17]. In our extension of Al'brekht we assume that the nonlinearites are expressed by Fredholm integral operators of increasing degrees.

II. Al'brekht Method in Finite Dimension

Al'brekht assumed that $f(z,u)$ and $l(z,u)$ are sufficiently smooth to have the Taylor polynomial expansions

$$
\begin{aligned}
f(z,u) &= Fz + Gu + f^{[2]}(z,u) + \ldots + f^{[d]}(z,u) \\
&\quad + O(z,u)^{d+1} \\
l(z,u) &= \frac{1}{2}\left(z'Qz + 2z'Su + u'Ru\right) + l^{[3]}(z,u) + \ldots \\
&\quad + l^{[d+1]}(z,u) + O(z,u)^{d+2}
\end{aligned}
$$

for some degree $d > 1$ where $^{[k]}$ indicates terms of homogeneous degree k in z, u.

He also assumed that the optimal cost $\pi(z)$ and the optimal feedback have similar Taylor polynomial expansions

$$
\begin{aligned}
\pi(z) &= \frac{1}{2}z'Pz + \pi^{[3]}(z) + \ldots + \pi^{[d+1]}(z) + O(z)^{d+2} \\
\kappa(z) &= Kz + \kappa^{[2]}(z) + \ldots + \kappa^{[d]}(z) + O(z)^{d+1}
\end{aligned}
$$

First we complete the square for the LQR problem. Let P be any symmetric $n \times n$ matrix. For any control trjectory $u(t)$ that results in a state trajectory $z(t)$ that goes to zero we have

$$
\begin{aligned}
0 &= \frac{1}{2}z'(0)Pz(0) + \frac{1}{2}\int_0^\infty \frac{d}{dt}z'(t)Pz(t)\,dt \\
&= \frac{1}{2}z'(0)Pz(0) \\
&\quad + \frac{1}{2}\int_0^\infty \left[\begin{array}{c} z \\ u \end{array}\right]' \left[\begin{array}{cc} F'P+PF & PG \\ G'P & 0 \end{array}\right] \left[\begin{array}{c} z \\ u \end{array}\right]\,dt
\end{aligned}
$$

We add this zero quantity to the criterion to be minimized to get a new criterion to be minimized

$$
\begin{aligned}
&\frac{1}{2}z'(0)Pz(0) \\
&+ \frac{1}{2}\int_0^\infty \left[\begin{array}{c} z \\ u \end{array}\right]' \left[\begin{array}{cc} F'P+PF+Q & PG+S \\ G'P+S' & R \end{array}\right] \left[\begin{array}{c} z \\ u \end{array}\right]\,dt
\end{aligned}
$$

We want to choose P and an $m \times n$ matrix K so the integrand is a perfect square

$$
\begin{aligned}
&\left[\begin{array}{c} z \\ u \end{array}\right]' \left[\begin{array}{cc} F'P+PF+Q & PG+S \\ G'P+S' & R \end{array}\right] \left[\begin{array}{c} z \\ u \end{array}\right] \\
&= (u-Kz)'R(u-Kz)
\end{aligned}
$$

This will be true iff P and K satisfy the LQR equations (3). Clearly then P is the kernel of the the optimal cost and K is the optimal feedback gain.

Suppose $\pi^{[3}(z)$ is any homogeneous polynomial of degree three in z. Again for any control trjectory $u(t)$ that results in a state trajectory $z(t)$ that goes to zero we have

$$
\begin{aligned}
0 &= \frac{1}{2}z'(0)Pz(0) + \frac{1}{2}\int_0^\infty \frac{d}{dt}z'(t)Pz(t)\,dt \\
&\quad + \pi^{[3}(z(0)) + \int_0^\infty \frac{d}{dt}\pi^{[3}(z(t))\,dt \\
&= \frac{1}{2}z'(0)Pz(0) \\
&\quad + \frac{1}{2}\int_0^\infty \left[\begin{array}{c} z \\ u \end{array}\right]' \left[\begin{array}{cc} F'P+PF & PG \\ G'P & 0 \end{array}\right] \left[\begin{array}{c} z \\ u \end{array}\right]\,dt \\
&\quad + \pi^{[3}(z(0)) + \int_0^\infty \frac{\partial \pi^{[3}}{\partial z}(z)(F+GK)z \\
&\quad + z'Pf^{[2]}(z,Kz) + l^{[3]}(z,Kz)\,dt + O(z,u)^4
\end{aligned}
$$

We add this to the criterion (4) to be minimized. We have already matched quadratic terms. The cubic terms are

$$
\begin{aligned}
&\pi^{[3}(z(0)) + \int_0^\infty \frac{\partial \pi^{[3}}{\partial z}(z)(F+GK)z \\
&+ z'Pf^{[2]}(z,Kz) + l^{[3]}(z,Kz)\,dt
\end{aligned}
$$

We choose $\pi^{[3}(z)$ so that the integrand vanishes,

$$
0 = \frac{\partial \pi^{[3]}}{\partial z}(z)(F+GK)z + z'Pf^{[2]}(z,Kz) + l^{[3]}(z,Kz) \quad (7)
$$

then $\pi^{[3]}(z^0)$ is the cubic part of the optimal cost. Notice that the quadratic part $\kappa^{[2}(z)$ of the optimal feedback does not enter in this equation (7).

The solvability of (7) depends on the invertability of the operator

$$
\pi^{[3]}(z) \mapsto \frac{\partial \pi^{[3]}}{\partial z}(z)(F+GK)z \quad (8)
$$

acting on homogeneous polynomials of degree three. The eigenvalues of this operator are the sums $\mu_i + \mu_j + \mu_k$ of three eigenvalues of the linear closed loop dynamics $F+GK$. Under the standard LQR assumptions these eigenvalues are all in the open left half plane and hence no triple sum is zero. So the operator (8) is invertible.

Suppose $\psi_1, \ldots, \psi_n$ are the left row eigenvectors of $F + GK$. Then the corresponding eigenvectors of the operator (8) are of the form $(\psi_i z)(\psi_j z)(\psi_k z)$. Let Ψ be the $n \times n$

matrix whose rows are the ψ_i then the linear change of state coordinates

$$\zeta \;=\; \Psi z \qquad (9)$$

diagonalizes the operator (8) and makes (7) easy to solve.

The quadratic terms in the second sHJB equation (6) are

$$0 \;=\; \frac{\partial \pi^{[3]}}{\partial z}(z)G + z'P\frac{\partial f^{[2]}}{\partial u}(z, Kz)$$
$$+ \frac{\partial l^{[3]}}{\partial u}(z, Kz) + (\kappa^{[2]}(z))'R$$

where $\kappa^{[2]}(z)$ is the quadratic part of the optimal feedback. Since we have assumed R is positive definite this equation uniquely determines $\kappa^{[2]}(z)$.

The higher degrees terms in the optimal cost and optimal feedback are found in a similar fashion. First we solve

$$\frac{\partial \pi^{[k+1]}}{\partial z}(z)\,(F + GK)\,z \;=\; \text{Known Stuff} \qquad (10)$$

and then we solve

$$\kappa^{[k]}(z) = -R^{-1}\left(\frac{\partial \pi^{[k+1]}}{\partial z}(z)G + \text{Known Stuff}\right)'$$

where the Known Stuff consist of terms from the Taylor polynomials of $f(z, u)$ and $l(z, u)$ and previously computed terms from the Taylor polynomials of $\pi(z)$ and $\kappa(z)$. The first equation (10) uniquely determines $\pi^{[k+1]}(z)$ because the eigenvalues of the linear operator

$$\pi^{[k+1]}(z) \mapsto \frac{\partial \pi^{[k+1]}}{\partial z}(z)\,(F + GK)\,z$$

are the sums of $k+1$ eigenvalues of $F+GK$. Again the linear change of state coordinates (9) diagonalizes the equation (10) for every k.

III. Controlled Reaction-Diffusion Equations

In this section we extend Al'brekht's Method to controlled reaction-diffusion equations of the form

$$\frac{\partial z}{\partial t}(x, t) \;=\; \frac{\partial^2 z}{\partial x^2}(x, t) + f(x, z(\cdot, t), u(\cdot, t)) \quad (11)$$

$$\frac{\partial z}{\partial x}(0, t) = \quad 0 \;=\; \frac{\partial z}{\partial x}(1, t) \qquad (12)$$

$$z(x, 0) \;=\; z^0(x) \qquad (13)$$

for $x \in [0, 1]$ with Neumann (no flux) boundary conditions. We wish to stabilize $z(x, t)$ to $z(x) = 0$. Both $z(x, t)$ and $u(x, t)$ could be vector valued but for simplicity of exposition we assume that they are scalar valued.

Notice that the reaction term is a functional of $z(\cdot, t), u(\cdot, t)$. We assume that $f(x, z(\cdot, t), u(\cdot, t))$ is given by a sum of Fredholm integral operators of increasing degrees

$$f(x, z(\cdot), u(\cdot))$$
$$= \int_0^1 F^{[1]}(x, x_1)z(x_1) + G^{[1]}(x, x_1)u(x_1) \; dx_1$$
$$+ \int_0^1 \int_0^1 F^{[2]}(x, x_1, x_2)z(x_1)z(x_2) \; dx_1 dx_2$$
$$+ \dots$$

This is not the most general reaction term that we could consider, for example we could have terms quadratic in $u(\cdot)$ or bilinear in $z(\cdot)$ and $u(\cdot)$ or higher degrees in $z(\cdot)$ and $u(\cdot)$. WLOG we assume all the Fredholm kernels are symmetric with respect to the subscripted x_i, e.g.

$$F^{[2]}(x, x_1, x_2) \;=\; F^{[2]}(x, x_2, x_1)$$

Note that $F^{[k]}(x, x_1, \dots, x_k)$ and $G^{[1]}(x, x_1)$ could be generalized functions. For example if $F^{[1]}(x, x_1) = F(x)\delta(x-x_1)$ and $G^{[1]}(x, x_1) = G(x)\delta(x - x_1)$ then the linear part of the reaction term is

$$\int_0^1 F^{[1]}(x, x_1)z(x_1) + G^{[1]}(x, x_1)u(x_1) \; dx_1$$
$$= F(x)z(x) + G(x)u(x) \qquad (14)$$

To keep the notation simple we shall assume that (14) is the case.

To find a linear feedback that locally stabilizes the system to $z(x) = 0$, we pose the infinite dimensional LQR problem of minimizing

$$\frac{1}{2}\int_0^\infty \int_0^1 |z(x,t)|^2 + |u(x,t)|^2 \; dx \; dt \qquad (15)$$
$$= \frac{1}{2}\int_0^\infty \int_0^1 \int_0^1 \delta(x_1 - x_2)z(x_1,t)z(x_2,t)$$
$$+ \delta(x_1 - x_2)u(x_1,t)u(x_2,t) \; dx_1 dx_2 \; dt$$

subject to the linear part of the dynamics. We pose this simple Lagrangian but our method readily extends to more complicated ones in Fredholm form..

We complete the square again. Suppose we have a Fredholm quadratic form in $z^0(x)$

$$\int_0^1 \int_0^1 P^{[2]}(x_1, x_2)z^0(x_1)z^0(x_2) \; dx_1 dx_2$$

and the control trajectory $u(x, t)$ takes $z(x, t)$ to zero as $t \to \infty$. Then

$$0 = \int_0^1 \int_0^1 P^{[2]}(x_1, x_2)z^0(x_1)z^0(x_2) \; dx_1 dx_2$$
$$+ \int_0^\infty \int_0^1 \int_0^1 \frac{d}{dt}\left(P^{[2]}(x_1, x_2)z(x_1,t)z(x_2,t)\right) \; dx_1 dx_2 dt$$

We add this zero quantity to the criterion (15) to get a new expression to be minimized

$\int_0^1 \int_0^1 P^{[2]}(x_1, x_2)z^0(x_1)z^0(x_2) \; dx_1 dx_2$

$+ \int_0^\infty \int_0^1 \int_0^1 P^{[2]}(x_1, x_2)$

$\times \left(\frac{\partial^2 z}{\partial x_1^2}(x_1,t)z(x_2,t) + z(x_1,t)\frac{\partial^2 z}{\partial x_2^2}(x_2,t)\right)$

$+ P^{[2]}(x_1, x_2)\left(F(x_1)z(x_1,t) + G(x_1)u(x_1,t)\right)z(x_2,t)$

$+ P^{[2]}(x_1, x_2)z(x_1,t)\left(F(x_2)z(x_2,t) + G(x_2)u(x_2,t)\right)$

$+ \delta(x_1 - x_2)z(x_1,t)z(x_2,t)$

$+ \delta(x_1 - x_2)u(x_1,t)u(x_2,t) \; dx_1 dx_2 dt$

We have assumed that $z(x,t)$ satisfies Neumann boundary conditions. If we assume that $P^{[2]}(x_1,x_2)$ also satisfies Neumann boundary conditions then when we integrate by parts twice we get

$$\int_0^1 \int_0^1 P^{[2]}(x_1,x_2)z^0(x_1)z^0(x_2) \, dx_1 dx_2$$

$$+\int_0^\infty \int_0^1 \int_0^1 \left(\Delta P^{[2]}(x_1,x_2) + \delta(x_1-x_2)\right) z(x_1,t)z(x_2,t)$$

$$+P^{[2]}(x_1,x_2)\left(F(x_1)z(x_1,t)+G(x_1)u(x_1,t)\right)z(x_2,t)$$

$$+P^{[2]}(x_1,x_2)z(x_1,t)\left(F(x_2)z(x_2,t)+G(x_2)u(x_2,t)\right)$$

$$+\delta(x_1-x_2)u(x_1,t)u(x_2,t) \, dt \, dx_1 dx_2 dt$$

where $\Delta P^{[2]}(x_1,x_2)$ is the two dimensional Laplacian of $P^{[2]}(x_1,x_2)$.

We want the integrand of the time integral to be a perfect square of the form

$$\int_0^1 \int_0^1 \left(u(x_1,t) - \int_0^1 K^{[1]}(x_1,x_3)z(x_3,t) \, dx_3\right)\delta(x_1-x_2)$$
$$\times \left(u(x_2,t) - \int_0^1 K^{[1]}(x_2,x_3)z(x_3,t) \, dx_3\right) \, dx_1 dx_2$$

for some not necessarily symmetric $K^{[1]}(x,x_3)$.

This leads to infinite dimensional LQR equations

$$K^{[1]}(x,x_1) = -P^{[2]}(x,x_1)G(x_1) \tag{16}$$

$$\int_0^1 P^{[2]}(x_1,x_3)G(x_3)G(x_3)P^{[2]}(x_3,x_2) \, dx_3$$
$$= \Delta P^{[2]}(x_1,x_2) + \delta(x_1-x_2)$$
$$+F(x_1)P^{[2]}(x_1,x_2) + P^{[2]}(x_1,x_2)F(x_2) \tag{17}$$

The second of these equation is called a Riccati PDE and it is to be interperted in the weak sense. If $\theta(x_1), \theta(x_2)$ are any C^2 functions then

$$\int_0^1 \int_0^1 \int_0^1 \theta(x_1)P^{[2]}(x_1,x_3)G(x_3)$$

$$\times G(x_3)P^{[2]}(x_3,x_2) \, \theta(x_2) \, dx_1 dx_2 dx_3$$

$$= \int_0^1 \int_0^1 \theta(x_1)\left(\Delta P^{[2]}(x_1,x_2) + \delta(x_1-x_2)\right)$$

$$+F(x_1)P^{[2]}(x_1,x_2) + P^{[2]}(x_1,x_2)F(x_2)\right)\theta(x_2) \, dx_1 dx_2$$

Similar equations have appeared in the works of J.L. Lions [15], J. Burns [6], [5], K. Hulsing [8], [4], B. Batten King [9] and others.

If $P^{[2]}(x_1,x_2)$ is a weak solution of the Riccati PDE (17) then it is the kernel of the degree two Fredholm form that is the quadratic part of the optimal cost. The optimal linear feedback gain is given by (16). The closed loop linear dynamics is

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + F(x)z(x,t)$$
$$+ \int_0^1 G(x)K^{[1]}(x,x_1)z(x_1,t) \, dx_1$$

A standard approach to solving the Riccati PDE (17) is to expand $P^{[2]}(x_1,x_2)$ in the eigenfunctions of the diffusion

operator. The eigenvectors and the eigenvalues of $\frac{\partial^2}{\partial x^2}$ subject to Neumann boundary conditions on $x \in [0,1]$ are

$$\begin{array}{ll} \lambda_0 = 0, & \phi_0(x) = 1 \\ \lambda_i = -i^2\pi^2, & \phi_i(x) = \sqrt{2}\cos(i\pi x) \end{array} \tag{18}$$

for $i = 0,1,2,\ldots$. Notice that this is an orthonormal family If we assume that

$$P^{[2]}(x_1,x_2) = \sum_{i,j=0}^\infty \Pi_{i,j}\phi_i(x_1)\phi_j(x_2) \tag{19}$$

Then the Riccati PDE (17) becomes an Algebraic Riccati equation for the infinite dimensional matrix $[\Pi_{i,j}]$.

Let the linear closed loop eigenvalues and left eigenvectors be denoted by

$$\mu_i, \qquad \psi_i(x)$$

With the criterion (15) the LQR is clearly detectable. Under an additional assumption of stabilizability we have that all the μ_i are in the open left half plane [7]. But in general the $\psi_i(x)$ are not orthonormal.

Let $P^{[3]}(x_1,x_2,x_3)$ be the kernel of any degree three Fredholm form and suppose that $u(x,t)$ is a control trajectory that results in an asymptotically stable state trajectory $z(x,t)$. Then

$$0 = \tfrac{1}{2}\int_0^1 \int_0^1 P^{[2]}(x_1,x_2)z^0(x_1)z^0(x_2) \, dx_1 dx_2$$

$$+\tfrac{1}{2}\int_0^\infty \int_0^1 \int_0^1 \frac{d}{dt}\left(P^{[2]}(x_1,x_2)z(x_1,t)z(x_2,t)\right) \, dx_1 dx_2 dt$$

$$+\int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1,x_2,x_3)z^0(x_1)z^0(x_2)z^0(x_3) \, dx_1 dx_2 dx_3$$

$$+\int_0^\infty \int_0^1 \int_0^1 \int_0^1 \frac{d}{dt}\left(P^{[3]}(x_1,x_2,x_3)z(x_1,t)z(x_2,t)z(x_3,t)\right)$$

$$dx_1 dx_2 dx_3 dt$$

As before we add this zero quantity to the criterion to be minimized to get a new expression to be minimized. We have already matched up the quadratic terms in this expression. The cubic terms are

$$\int_0^1 \int_0^1 \int_0^1 P^{[3]}(x_1,x_2,x_3)z^0(x_1)z^0(x_2)z^0(x_3) \, dx_1 dx_2 dx_3$$

$$+\int_0^\infty \int_0^1 \int_0^1 \int_0^1 \left(\Delta P^{[3]}(x_1,x_2,x_3)\right.$$

$$+3\int_0^1 P^{[3]}(x_4,x_1,x_2)\left(F(x_4)+G(x_4)K^{[1]}(x_4,x_3) \, dx_4\right)$$

$$+\int_0^1 P^{[2]}(x_4,x_1)F^{[2]}(x_4,x_2,x_3) \, dx_4\right)$$

$$\times z(x_1,t)z(x_2,t)z(x_3,t) \, dx_1 dx_2 dx_3 dt$$

where $\Delta P^{[3]}(x_1,x_2,x_3)$ is the three dimensional Laplacian.

We set the time integrand to zero to get a weak linear elliptic PDE for the symmetric function $P^{[3]}(x_1,x_2,x_3)$

$$0 = \Delta P^{[3]}(x_1,x_2,x_3)$$

$$+3\int_0^1 P^{[3]}(x_4,x_1,x_2)\left(F(x_4)+G(x_4)K^{[1]}(x_4,x_3) \, dx_4\right)$$

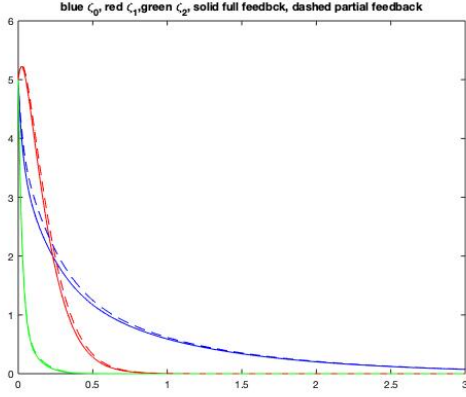$$+\int_0^1 P^{[2]}(x_4,x_1)F^{[2]}(x_4,x_2,x_3) \, dx_4 \tag{20}$$

Fig. 1. First three modes under full (solid) and partial (dashed) cubic feedback

subject to Neumann boundary conditions. By weak we mean if we multiply the right side of this equation by any three C^2 functions $\theta(x_1)\theta(x_2)\theta(x_3)$ and integrate over the unit cube we get zero.

Based on what we saw for finite dimensional systems it is natural to make the linear change of state coordinates

$$\zeta_i(t) = \int_0^1 \psi_i(x)z(x,t)$$

Suppose

$$P^{[3]}(x_1, x_2, x_3) = \sum_{i,j,k=0}^{\infty} \Pi_{i,j,k} \psi_i(x_1)\psi_j(x_2)\psi_k(x_3)$$

Then (20) becomes the triple sequence of equations

$$0 = (\mu_i + \mu_j + \mu_k) \, \Pi_{i,j,k}$$
$$+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 P^{[2]}(x_4, x_1)F^{[2]}(x_4, x_2, x_3) \qquad (21)$$
$$\times \psi_i(x_1)\psi_j(x_2)\psi_k(x_3) \; dx_1 dx_2 dx_3 dx_4$$

This determines $\Pi_{i,j,k}$.

The quadratic part of the optimal feedback is then found from the second sHJB equation,

$$K^{[2]}(x, x_1, x_2) = -3P^{[3]}(x, x_1, x_2)G$$

The higher degree terms are found in a similar fashion

IV. EXAMPLE

We close with a simple example that is a quadratic modification of Example 6.2 of Curtain and Zwart [7]. Consider a rod of length one with distributed heating/cooling, no flux boundary conditions and a quadratic nonlinearity.

$$\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) + u(x, t) + z(x, t)^2$$
$$\frac{\partial z}{\partial x}(0, t) = 0 = \frac{\partial z}{\partial x}(1, t)$$
$$z(x, 0) = z^0(x)$$

Then

$$F(x) = 0, \qquad G(x) = 1$$
$$F^{[2]}(x, x_1, x_2) = \delta(x - x_1)\delta(x - x_2)$$

and all the other higher degree terms are zero.

To find a feedback that stabilizes this system to $z(x) = 0$ we pose the optimal control problem of minimizing (15) subject to this dynamics.

The open loop linear eigenvalues and eigenvectors are (18) so if we assume that (19) then the Riccati PDE becomes

$$\sum_{k=0}^{\infty} \Pi_{i,k}\Pi_{k,j} = (\lambda_i + \lambda_j)\Pi_{i,j} + \delta_{i,j}$$

This is identical to equation (6.81) of [7] and their solution is

$$\Pi_{i,j} = \delta_{i,j}\left(\lambda_i + \sqrt{\lambda_i^2 + 1}\right)$$

The linear part of the optimal feedback is

$$u(x, t) = \int_0^1 K^{[1]}(x, x_1)z(x_1, t) \; dx_1$$
$$= -\int_0^1 P^{[2]}(x, x_1)z(x_1, t) \; dx_1$$

The closed loop eigenvalues are $\mu_i = -\sqrt{\lambda_i^2 + 1}$. The closed loop eigenvectors are still $\phi_i(x)$. But the basin of asymptotic stability of the nonlinear system closed by the linear feedback is not very large. If $z^0(x) = 1 + \epsilon$ for $\epsilon > 0$ then the linearly closed loop nonlinear system diverges.

The Fredholm kernel of cubic part of the optimal cost is of the form

$$P^{[3]}(x_1, x_2, x_3) = \sum_{i,j,k=0}^{\infty} \Pi_{i,j,k} \; \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)$$

and (21) becomes

$$0 = (\mu_i + \mu_j + \mu_k)\Pi_{i,j,k} + \frac{1}{2}\left(\Pi_{i,j+k} + \Pi_{i,j-k}\right)$$

The solutions $\Pi_{i,j,k}$ of these equations are not symmetric in the indices i, j, k. They need to be symmetrized. The Fredholm kernel of the quadratic part of the optimal feedback is given by

$$K^{[2]}(x, x_1, x_2) = -3P^{[3]}(x, x_1, x_2)$$

The Fredholm kernel of quartic part of the optimal cost is of the form

$$P^{[4]}(x_1, x_2, x_3, x_4) = \sum_{i,j,k,l=0}^{\infty} \Pi_{i,j,k,l} \; \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)$$

where the coefficients satisfy the equations

$$0 = (\mu_i + \mu_j + \mu_k + \mu_l)\,\Pi_{i,j,k,l} + \frac{3}{2}\left(\Pi_{i,j,k+l} + \Pi_{i,j,k-l}\right)$$
$$- \frac{9}{2}\sum_{r=0}^{\infty} \Pi_{i,j,r}\Pi_{r,k,l}$$

Again the solutions $\Pi_{i,j,k,l}$ of these equations are not symmetric in the indices i, j, k, l. They need to be symmetrized. The Fredholm kernel of the cubic part of the optimal feedback is given by

$$K^{[3]}(x, x_1, x_2, x_3) = -4P^{[3]}(x, x_1, x_2, x_3)$$

The linear closed loop poles go to $-\infty$ quite fast, for example $\mu_3 \approx -88.8321$ so we did a Galerkin projection on the first three eigenfunctions. Let $\zeta_i(t) = <\phi_i(x), z(x, t)>$ and $\nu_i(t) = <\phi_i(x), u(x, t)>$ for $i = 0, 1, 2$. The Galerkin projection is the three dimensional nonlinear control system

$$\dot{\zeta}_0 = \lambda_0\zeta_0 + \nu_0 + \zeta_0^2 + \frac{1}{2}\zeta_1^2 + \frac{1}{2}\zeta_2^2$$
$$\dot{\zeta}_1 = \lambda_1\zeta_1 + \nu_1 + 2\zeta_0\zeta_1 + \zeta_1\zeta_2$$
$$\dot{\zeta}_2 = \lambda_2\zeta_2 + \nu_2 + 2\zeta_0\zeta_2 + \frac{1}{2}\zeta_1^2$$

Using our Nonlinear Systems Toolbox we found the optimal cost $\pi(\zeta)$ to degree 4 for the three dimensional Galerkin approximation. If the initial state is $[\zeta_0; \zeta_1; \zeta_2]$ then

$$
\begin{aligned}
\pi^{[2]}(\zeta) &= 0.5000\ \zeta_0^2 + 0.0253\ \zeta_1^2 + 0.0063\ \zeta_2^2 \\
\pi^{[3]}(\zeta) &= 0.3333\ \zeta_0^3 + 0.0288\ \zeta_0\zeta_1^2 \\
&\quad +0.0066\ \zeta_0\zeta_2^2 + 0.0010\ \zeta_1^2\zeta_2 \\
\pi^{[4]}(\zeta) &= 0.1250\ \zeta_0^4 + 0.0281\ \zeta_0^2\zeta_1^2 + 0.0065\ \zeta_0^2\zeta_2^2 \\
&\quad +0.0012\ \zeta_0\zeta_1^2\ \zeta_2 + 0.0004\ \zeta_1^4 \\
&\quad +0.0002\ \zeta_1^2\zeta_2^2 + 0.0000\ \zeta_2^4
\end{aligned}
$$

By way of comparison if the initial state of the infinite dimensional system is $z^0(x) = \sum_{i=0}^{\infty} \zeta_i \phi_i(x)$ and if $\Pi^{[k]}(\zeta)$ is degree k part of its optimal cost then

$$
\begin{aligned}
\Pi^{[2]}(\zeta) &= 0.5000\ \zeta_0^2 + 0.0253\ \zeta_1^2 + 0.0063\ \zeta_2^2 \\
&\quad +\frac{1}{2}\sum_{i=3}^{\infty} \Pi_{i,i}\zeta_i^2 \\
\Pi^{[3]}(\zeta) &= 0.3333\ \zeta_0^3 + 0.0276\ \zeta_0\zeta_1^2 \\
&\quad +0.0065\ \zeta_0\zeta_2^2 + 0.0010\ \zeta_1^2\zeta_2 \\
&\quad +\sum_{i,j,k=3}^{\infty} \Pi_{i,j,k}\zeta_i\zeta_j\zeta_k \\
\Pi^{[4]}(\zeta) &= 0.1250\ \zeta_0^4 + 0.0279\ \zeta_0^2\zeta_1^2 + 0.0063\ \zeta_0^2\zeta_2^2 \\
&\quad +0.0011\zeta_0\zeta_1^2\ \zeta_2 + 0.0004\ \zeta_1^4 \\
&\quad +0.0002\ \zeta_1^2\zeta_2^2 + 0.0000\ \zeta_2^4 \\
&\quad +\sum_{i,j,k,l=3}^{\infty} \Pi_{i,j,k,l}\zeta_i\zeta_j\zeta_k\zeta_l
\end{aligned}
$$

We started the three dimensional Galerkin approximation at $\zeta_0 = 5, \zeta_1 = 5, \zeta_2 = 5$ which is way outside the basin of asymptotic stability for the linearly closed loop nonlinear system We used two cubic feedbacks. The full cubic feedback was the one computed by the Nonlinear Systems Toolbox [10] using Al'brekht's method on the three dimensional system. But under the linear closed loop dynamics the monomial $\zeta_{i_1} \cdots \zeta_{i_k}$ satisfies

$$
\frac{d}{dt}\zeta_{i_1} \cdots \zeta_{i_k} = (\mu_{i_1} + \cdots + \mu_{i_k})\ \zeta_{i_1} \cdots \zeta_{i_k}
$$

so it may go to zero extremely fast. The partial cubic feedback ignores any monomial $\zeta_{i_1} \cdots \zeta_{i_k}$ where

$$
\mu_{i_1} + \cdots + \mu_{i_k} < -20
$$

and performs almost as well as the full cubic feedback, Figure 1.

Because the linear change of coordinates (9) diagonalizes the equations (10) for every $k \geq 2$ we don't have to solve these equations for all the monomials of degree $k+1$. We can choose to solve them only for the monomials that don't go to zero very fast under the linear closed loop dynamics

V. Conclusion

Al'brekht developed a method to compute the Taylor polynomial expansions for the optimal cost and optimal feedback for smooth, finite dimensional, infinite horizon optimal control problems whose linear quadratic part satisfies the standard LQR conditions. We have shown that Al'brekht's method can be extended to the optimal stabilization of some smooth infinite dimensional controlled reaction-diffusion equations whose linear quadratic part satisfies the infinite dimensional LQR conditions. The crucial steps in the extension are the ability to express the nonlinearities as Fredholm linear operators and the ability to compute a few of the least stable left eigenfunctions of closed loop linear part of the dynamics.

The example that we presented had distributed control. The next step is to apply Al'brekht's merhod to a reaction-diffusion problem under boundary control. We would also like to extend Al'brekht's method to other infinite dimensional problems such as nonlinear wave equations and nonlinear delay equations.

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