Robust Distributed Synchronization of Networked Linear Systems with Intermittent Information ⋆

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Abstract

The problem of synchronization of multiple linear time-invariant systems connected over a network with asynchronous and intermittently available communication events is studied. To solve this problem, we propose a controller with hybrid dynamics, namely, the controller utilizes information transmitted to it during discrete communication events and exhibits continuous dynamics between such events. Due to the additional continuous and discrete dynamics inherent to the interconnected networked systems and communication structure, we use a hybrid systems framework to model and analyze the closed-loop system. The problem of synchronization is then recast as a set stabilization problem and, by employing Lyapunov stability tools for hybrid systems, sufficient conditions for asymptotic stability of the synchronization set are provided. Furthermore, we show that the property of synchronization is robust to perturbations. Numerical examples illustrating the main results are included.

Key words: Synchronization; Nonlinear control systems; Communication networks; Distributed control

1 Introduction

1.1 Motivation

The topic of synchronization or the notion of multiple dynamical systems converging to evolve together has gained significant traction in recent years due to the wide range of applications in science and engineering. Namely, synchronization is seen in spiking neurons Murthy & Fetz (1996), Phillips & Sanfelice (2014), formation control and flocking maneuvers Fax & Murray (2004), Olfati-Saber & Murray (2002), distributed sensor networks Olfati-Saber & Shamma (2005), and satellite constellation formation Sarlette et al. (2007), to name a few. In this paper, we are interested in the topic of synchronization of continuous-time linear time-invariant interconnected systems coupled over a general graph where communication between connected agents occurs only at intermittent time instances. We are interested in designing a distributed hybrid controller that only employs such impulsive information which drives the states of the agents in the network to converge to synchronization. The problem comes with many challenges due to the interconnection between each agent being impulsive, which, under the effect of the hybrid controller, results in a hybrid system. Some of the main challenges in designing a control algorithm for synchronization in such a setting include:

- *Asynchronous and heterogenous communication events at unknown times*: the time instances at which each agent receives information are not synchronized and do not necessarily occur periodically. Namely, each agent may receive information from its neighbors at different and unknown time instances. Furthermore, the amount of ordinary time elapsed between successive communication events for each agent may not be constant or known a priori; for example, one agent may receive information at a much faster "rate" than others.
- *Instability of nominal dynamics:* each of the systems may not be stable, potentially leading to unbounded trajectories in each system. In particular, the individual dynamics of the agents to be synchronized could be such that their origin is marginally stable or unstable, in which case the state trajectories of the agents need to converge to each other while potentially es-

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caping to infinity.

- *Perturbations in the dynamics, parameters, and measurements:* unknown dynamics in the model make it difficult to design an algorithm that guarantees exact synchronization. Synchronization algorithms that are not robust to perturbations on the transmitted information and on the times at which such information arrives could prevent the state trajectories of the agents to converge to nearby values.

1.2 Related Work

The wide applicability of synchronization in science and engineering has promoted a rich set of theoretical results for a variety of classes of dynamical systems using a diverse tools. The study of convergence and stability of synchronization come through the use of systems theory tools such as Lyapunov functions Belykh et al. (2006), Hui et al. (2007), contraction theory Slotine et al. (2004), and incremental input-to-state stability Angeli (2002), Cai et al. (2015). Results for asymptotic synchronization with continuous coupling between agents exist in both the continuous-time domain and the discrete-time domain; see, e.g., Scardovi & Sepulchre (2009), Moreau (2004), Olfati-Saber et al. (2007), where the latter is a detailed survey of coordination and consensus for integrator dynamics, in both continuous-time and discrete-time. Likewise, in Scardovi & Sepulchre (2009), a dynamic control law is shown to guarantee that the solutions to a closed-loop system of linear time-invariant systems converges to that of an homogeneous system with the same dynamics. In Moreau (2004), the author provides a brief survey on the convergence to synchronization through Lyapunov and set convexity analysis. As pointed out therein, a typical approach to guarantee that the interconnected agents converge to synchronization is to leverage the properties of the graph structure inherent in the connection of multiple agents. Namely, the approach is to use the properties of the graph Laplacian matrix to show that every agent converges exponentially to the synchronization manifold. Typically only convergence is considered for such systems and *stability* is usually omitted in definitions of synchronization; see, e.g., Belykh et al. (2006), Scardovi & Sepulchre (2009), Olfati-Saber et al. (2007).

Synchronization in continuous-time systems where communication coupling occurs at discrete events is an emergent area of study. In Cai et al. (2015), the authors study a case of synchronization where agents have nonlinear continuous-time dynamics with continuous coupling and impulsive perturbations. In Lu et al. (2010), the authors use Lyapunov-like analysis to derive sufficient conditions for the synchronization of continuously coupled nonlinear systems with impulsive resets on the difference between neighboring agents. Similar to impulsive systems, synchronization in systems where feedback controllers are designed as state-triggered discrete events appeared in Liu et al. (2013), Demir & Lunze (2012), Wu et al. (2016). In Liu et al. (2013), a distributed event-triggered control strategy was developed to drive the outputs of the agents in a network to synchronization.

An observer-based event policy was developed in Demir & Lunze (2012) for a network of linear time-invariant systems where communication events occur when the distance between the local state and its estimate is larger than a threshold. Using a sample-and-hold self-triggered controller policy, a practical synchronization result was established in Persis & Frasca (2012) for the case of first-order integrator dynamics. In He et al. (2015), an impulsive control algorithm to achieve approximate synchronization of multiple connected agents with a leader-follower architecture is designed. In Liu et al. (2017), a controller to achieve consensus for multi-agent systems is proposed for the case when each agent transmits information to their neighbors continuously. In He et al. (2017), an algorithm for leader-follower consensus under the presence of delays is designed by appropriately choosing the sampling period and the coupling strength. The algorithms designed in this paper achieve synchronization (in the limit) using measurements and communicated information, do not rely on a leader-follower architecture, but, instead, are decentralized, and allow communication events occurring aperiodically.

To the best of our knowledge, methods for the design of algorithms that guarantee *robust synchronization* of multi-agent systems with information arriving at impulsive, asynchronous time instances are not available. Namely, many control design methods for synchronization make rather strong assumptions on the times when communication occurs. As we will define explicitly in the next section, we only require that the times at which communication occurs are upper and lower bounded by positive constants. This implies that successive communication events may occur any time within this interval of time.

1.3 Problem Formulation

We consider the problem of robustly synchronizing (in terms of both exponential attractivity and stability) $N > 1$ continuous-time agents with linear dynamics (under nominal conditions) from intermittent measurements of functions of their outputs over a network. Namely, we consider the following differential equation modeling the evolution of the state of the i-th agent:

$$\dot{x}_i = Ax_i + Bu_i + \Delta_i(x_i, t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is the nominal system matrix, $B \in \mathbb{R}^{n\times p}$ is the input matrix, u_i is the control input, $\Delta_i : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ models unknown and possibly heterogeneous dynamics, and $t \geq 0$ denotes ordinary time. The i-th agent in the network measures its local information y_i and receives information from its neighbors y_k at times $t \in \{t_s^i\}_{s=1}^{\infty}$. Moreover, at such event times, the output of each agent is given by

$$y_i = Hx_i + \varphi_i(x_i, t) \tag{2}$$

where H is the output matrix and φ_i is an unknown function modeling communication noise. The event times t_s^i are independently defined for each agent (as the index i denotes); the only restriction imposed on communication times is that they must satisfy

$$t_{s+1}^i - t_s^i \in [T_1^i, T_2^i] \quad \forall s \in \{1, 2, \dots\}, \quad t_1^i \leq T_2^i \tag{3}$$

where, for each $i \in \mathcal{V}$, the positive scalars T_1^i and T_2^i satisfy $T_2^i \geq T_1^i$ and define the lower and upper bounds on the communication rate, respectively. Namely, these parameters (which are known but may be different for each agent) govern the amount of time allowed to elapse between consecutive communication events. The parameter T_2^i is often referred to as the *maximum allowable time interval* (MATI).

Motivated by the challenges outlined in Section 1.1, we propose a distributed hybrid controller capable of asymptotically synchronizing the state of each agent over the network, with stability and robustness, by only exchanging information among neighbors at independent communication events t_s^i. In the nominal case, the algorithm proposed here guarantees global exponential stability of the set characterizing synchronization, called the synchronization set, and when projected to the state space of all agents, the synchronization set is the set of points $x = (x_1, x_2, \dots, x_N)$ such that $x_1 = x_2 = \cdots = x_N$. Moreover, in the presence of small enough general perturbations, the proposed algorithm guarantees that the stability properties are preserved, semiglobally and practically. Under the perturbation effect of measurement noise, we also show that the system is input-to-state stable (in the hybrid sense).

1.4 Outline of Proposed Solution

The distributed controller has internal state variables (η_i for each $i \in \mathcal{V}$) which have hybrid dynamics; i.e., the internal states are updated both continuously and, at times, are impulsively updated. In this way, we assign the state of the controller, denoted as η_i, to the input of the i-th system u_i, namely, we consider $u_i = \eta_i$. In general terms, the continuous dynamics of the controller state are given by a differential equation of the form

$$\dot{\eta}_i = f_{ci}(y_i, \eta_i), \tag{4}$$

when no new information is available, where y_i is the output of itself. When new information arrives, i.e., when $t \in \{t_s\}_{s=1}^{\infty}$, the internal states are updated according to

$$\eta_i^+ = \sum_{k=1}^{N} g_{ik} G_{ci}^k(\eta_i, \eta_k, y_i, y_k) \tag{5}$$

where $\mathcal{V} := \{1, 2, \dots, N\}$ defines the set of all agents; g_{ik} models the connection between agents i and k, namely, $g_{ik} = 1$ if the k-th agent can share information to agent i and $g_{ik} = 0$ otherwise. The map f_{ci} defines the continuous evolution of the controller state and the map G_{ci}^k defines the impulsive update law when new information is collected from each connected agent. Furthermore, as we will show, the hybrid model in Section 3.1 is time invariant and is able to capture all possible event times $\{t_s^i\}_{s=1}^{\infty}$ satisfying the constraints (3). Then, η_i is injected into the continuous-time dynamics of the i-th agent's input u_i and, at communication events, updates its internal state impulsively. Due to the intermittent nature of communication in many engineering disciplines, the dynamic controller defined in (4) – (5) may have numerous applications, such as, smart decentralized power systems Blaabjerg et al. (2006), formation control of aerial vehicles and satellites Fax & Murray (2004), Olfati-Saber & Murray (2002), Sarlette et al. (2007), and in distributed sensor networks Olfati-Saber & Shamma (2005). Following the hybrid systems framework in Biemond et al. (2012), we model the continuous dynamics of each agent, the communication events, and the distributed hybrid controller as a closed-loop hybrid model in Section 3.1.

As mentioned in Section 1.2, there are numerous approaches available in the literature to solve such problems. In particular, in He et al. (2015) and He et al. (2017), the authors address the problem of quasi-synchronization for a leader (and its followers) via an impulsive controller that instantaneously updates the state x_i at events. In this paper, we solve the problem of synchronization without resetting the state of the systems. More precisely, we design a *continuous-discrete* (namely, hybrid) control strategy that does not require resetting the state x_i, but rather, instantaneously updates a controller variable. This feature might be more suitable for multi-agent systems where the dynamics of x_i describe the evolution of physical variables of the individual systems, which typically cannot be impulsively changed.

1.5 Contributions and Organization

The main contribution of this work lay on the establishment of sufficient conditions for nominal and robust synchronization over networks with intermittent information availability. In fact, the proposed design conditions guarantee the states of each agent converge to synchronization with an exponential rate when information is only available at, possibly, asynchronous and non-periodic time instance. Precisely, as shown in Section 3.3 through an appropriate choice in coordinates we utilize Lyapunov arguments for hybrid systems to establish sufficient conditions that assure global exponential stability of the synchronization set. An in-depth robustness analysis and design procedure are presented in Section 3.5, wherein we establish several key robustness properties. In part, this is enabled by the proposed hybrid controller which is designed to satisfy certain regularity conditions that, under nominal conditions has uniform global asymptotic stability of the synchronization set, guarantees robustness to small enough perturbations. In Section 3.5.1, we provide results on robustness with respect to perturbations emerging from unmodeled dynamics, skewed clocks, as well as communication noise. In Section 3.5.2, results on robustness in the form of an input-to-state stability (ISS) property with respect to communication noise is provided, for which an explicit ISS bound is given.

In Section 4, we provide numerical simulations to illustrate our results. We consider the case of asynchronous update times where the dynamics of the agents have harmonic oscillator dynamics under different scenarios. Namely, we consider the case of six such systems on a ring graph under nominal conditions as well as subjected to communication noise and packet dropouts. Moreover, we consider such harmonic dynamics on a large-scale system ($N = 100$) representing a small-world network as in Watts & Strogatz (1998).

This paper extends our preliminary work in our conference papers Phillips & Sanfelice (2016) about synchronization. In Phillips & Sanfelice (2016), we consider the case of N-dimensional linear time-invariant systems under general directed graphs where communication events are synchronously triggered throughout the network. In Phillips et al. (2016), we consider asynchronous communication for agents with scalar integral dynamics. This paper not only generalizes these results, but also provides complete proofs, which were not available in Phillips & Sanfelice (2016), includes new results (Theorem 3.6, Proposition 3.11 and Theorem 3.13 are new), and numerous new illustrations via examples in Section 4.

2 Notation and Preliminaries on Graph Theory

2.1 Basic Notation

Given a matrix A, the set $\text{eig}(A)$ contains all eigenvalues of A and $|A| := \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \text{eig}(A^\top A)\}$. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and notation $[u^\top \; v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t\geq 0} |m(t)|$. $\mathbb{Z}_{\geq 1}$ denotes the set of positive integers, i.e., $\mathbb{Z}_{\geq 1} := \{1, 2, 3, \dots\}$. $\mathbb{N}$ denotes the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \lambda \in \text{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \lambda \in \text{eig}(P)\}$. Given matrices A, B with proper dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product; $\text{diag}(A, B)$ denotes a 2×2 block matrix with A and B being the diagonal entries; and $A * B$ defines the Khatri-Rao product between A and B. Given $N \in \mathbb{Z}_{\geq 1}$, $I_N \in \mathbb{R}^{N\times N}$ defines the identity matrix and $\mathbf{1}_N$ is the vector of N ones. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the largest integer that is smaller than or equal to s. The graph of a set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined as $\text{gph}\, G = \{(x, y) : x \in \mathbb{R}^n, y \in G(x)\}$.

2.2 Preliminaries on Graph Theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \dots, N\}$, and the edges are the pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can receive information from agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} \in \mathbb{R}^{N\times N}$, where its (i, k)-th entry g_{ik} is equal to one if $(i, k) \in \mathcal{E}$ and zero otherwise. A digraph is undirected if $g_{ik} = g_{ki}$ for all $i, k \in \mathcal{V}$. Without loss of generality, we assume that $g_{ii} = 0$ for all $i \in \mathcal{V}$. The in-degree and out-degree of agent i are defined by $d_i^{in} = \sum_{k=1}^N g_{ik}$ and $d_i^{out} = \sum_{k=1}^N g_{ki}$. The in-degree matrix $\mathcal{D}$ is the diagonal matrix with entries $D_{ii} = d_i^{in}$ for all $i \in \mathcal{V}$. The Laplacian matrix of the graph Γ, denoted by $\mathcal{L} \in \mathbb{R}^{N\times N}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{G}$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (i, k) \in \mathcal{E}\}$. A directed graph is said to be *strongly connected* if and only if any two distinct nodes of the graph can be connected via a path that traverses the directed edges of the digraph. It is considered *undirected* if communication between every distinct node is bidirectional, namely, for each edge (i, k) in the edge set $\mathcal{E}$, the edge (k, i) is also in the edge set. Let the digraph be strongly connected and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ be the eigenvalues of $\mathcal{L}$.[1] Then, $\lambda_1 = 0$ is a simple eigenvalue of $\mathcal{L}$ associated with the eigenvector 1_N; $\mathcal{L}$ is positive semi-definite and, therefore, there exists an orthonormal matrix $\Psi \in \mathbb{R}^{N\times N}$ such that $\Psi \mathcal{L} \Psi^\top = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. The digraph is undirected if and only if the Laplacian is symmetric. Inspired by Liu et al. (2012) if the Laplacian is symmetric then we have the following properties. We define $\widetilde{\Psi} = (\psi_2, \psi_3, \dots, \psi_N) \in \mathbb{R}^{N\times N-1}$ with $\psi_i = (\psi_{i1}, \psi_{i2}, \dots, \psi_{iN})$ being the orthonormal eigenvector corresponding to the nonzero eigenvalue λ_i, $i \in \{2, 3, \dots, N\}$, which satisfies $\sum_{k=1}^N \psi_{ik} = 0$. Moreover, $\widetilde{\Psi}$ satisfies the following:

$$\widetilde{\Psi}\widetilde{\Psi}^\top = \frac{1}{N} \begin{bmatrix} N-1 & -1 & \dots & -1 \\ -1 & N-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & N-1 \end{bmatrix} =: U \tag{6}$$

$\widetilde{\Psi}^\top \widetilde{\Psi} = I$, $U^2 = U$, $\Lambda := \widetilde{\Psi}^\top \mathcal{L} \widetilde{\Psi} = \text{diag}(\lambda_2, \lambda_3, \dots, \lambda_N)$. Note that $\widetilde{\Psi}$ has smaller dimension than Ψ, namely, $\widetilde{\Psi}$ does not contain the eigenvector associated to the zero eigenvalue of the Laplacian.

[1] See Godsil & Royle (2001) for more information on algebraic graph theory.

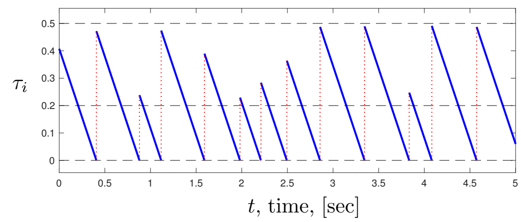

Fig. 1. A trajectory of a single timer τ_i with dynamics in (8) from $\tau(0,0) = 0.41$ where $T_1^i = 0.2$ and $T_2^i = 0.5$.

3 Robust Global Synchronization with Intermittent Information

3.1 Hybrid Modeling

A hybrid system $\mathcal{H}$ has data (C, f, D, G) and is given by

$$\begin{aligned} \dot{\xi} &= f(\xi) \quad \xi \in C, \\ \xi^+ &\in G(\xi) \quad \xi \in D, \end{aligned} \tag{7}$$

where $\xi \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The map G defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. The notation ξ^+ is to represent the value of ξ after a jump. More information about this hybrid system framework can be found in Goebel et al. (2012).

Consider N agents where, for each $i \in \mathcal{V} := \{1, 2, \ldots, N\}$, the i-th agent has dynamics in (1) and connected over a communication network modeled by a graph Γ. Due to the impulsive nature of the communication structure outlined in (3), we define an autonomous hybrid system to model the the communication times. Therefore, we define a positive and decreasing timer (with state $\tau_i \in [0, T_2^i]$) such that, when the timer reaches zero, we say that agent i receives information from its connected agents. The value of the timer decreases linearly with respect to ordinary time and, upon reaching zero, is reset to a point within the interval $[T_1^i, T_2^i]$. Then, each timer state can be considered to be an autonomous hybrid system as in (7) with the following dynamics:

$$\begin{aligned} \dot{\tau}_i &= -1 =: f_{\tau_i} & \tau_i &\in [0, T_2^i] =: C_{\tau_i}, \\ \tau_i^+ &\in [T_1^i, T_2^i] =: G_{\tau_i} & \tau_i &= 0 =: D_{\tau_i}. \end{aligned} \tag{8}$$

Solutions to a general hybrid system $\mathcal{H}$ as in (7) can evolve continuously and/or discretely according to the differential and difference equations/inclusions (and the sets where those apply) that describe the hybrid dynamics. A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \bigcap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\bigcup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_{J+1}$. The t_j's with $j > 0$ define the flow time instants when the state of the hybrid system jumps and j counts the number of jumps.

Next, we show that solutions to the hybrid system in (8) are complete and, indeed, generate any possible sequence of time instances $\{t_s^i\}_{s=1}^{\infty}$ at which communication events at the i-th agent satisfy (3). More specifically, given a maximal solution [2] ϕ_{τ_i} to the hybrid system (8), the jump times are given by the hybrid times $(t_j, j) \in \operatorname{dom} \phi_{\tau_i}$ and the sequence $\{t_j\}_{j=1}^{\infty}$ of such times satisfies the conditions in (3). Namely, we have the following result.

Lemma 3.1 *Let $0 < T_1^i \leq T_2^i$ be given. Every maximal solution ϕ_{τ_i} to the hybrid system in* (8) *satisfies the following:*

1) *ϕ_{τ_i} is complete, i.e., its domain is unbounded.*
2) *for each jump time $(t_s, s), (t_{s+1}, s) \in \operatorname{dom} \phi_{\tau_i}$ generates a sequence of times $\{t_j\}_{j=1}^{\infty}$ satisfying* (3).

Proof Firstly, we show that each solution to the hybrid system in (8) is complete. Note that (8) satisfies the hybrid basic conditions in Assumption 6.5 in Goebel et al. (2012) [3] . Note that its completeness follows from Proposition 6.10 in Goebel et al. (2012). More specifically, for any $\tau_i \in [0, T_2^i]$, we have that $\mathbb{T}_{C_{\tau_i}}(\tau_i) \cap \{-1\} \neq 0$, where $\mathbb{T}_S(x)$ is the tangent cone of the set S at the point x. Due to the fact that the flow map is constant, finite escape time during flows cannot occur. Furthermore, from the definition of G_{τ_i} we have that $G_{\tau_i} \subset [0, T_2^i]$. Then, from Proposition 6.10 in Goebel et al. (2012), every maximal solution ϕ_{τ_i} to the hybrid system in (8) is complete.

Next, we show item 2). Note that from the definition of maximal solutions to hybrid systems, we consider an arbitrary maximal solution to (8) from $\phi_{\tau_i}(0, 0) \in [0, T_2^i]$. Without loss in generality, we can consider $\phi_{\tau_i} \in C_{\tau_i}$. From the flow map $f_{\tau_i} = -1$, it follows that the initial jump occurs when $\phi_{\tau_i}(t, j) = 0$ at the hybrid time $(t, j) = (t_1, 0)$ such that $t_1 = \phi_{\tau_i} \leq T_2^i$. The jump map $G_{\tau_i} = [T_1^i, T_2^i]$ assigns the value of the solution after the jump as $\phi_{\tau_i}(t, j + 1) = G_{\tau_i}$. By similar arguments using

[2] A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded.

[3] A hybrid system $\mathcal{H} = (C, f, D, G)$ with data in (7) is said to satisfy the *hybrid basic conditions* if the sets C and D are closed, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and the set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, and $D \subset \operatorname{dom} G$. A set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if its graph $\{(x, y) : x \in \mathbb{R}^n, y \in G(x)\}$ is closed; see (Goebel et al. 2012, Lemma 5.10)

the flow map, it follows that time elapsed between jumps in τ_i is $t_{j+1} - t_j \in [T_1^i, T_2^i]$. Therefore, the sequence of jump times induced by the timer $\{t_s\}_{s=1}^{\infty}$ satisfies (3) concluding the proof. ■

Consider the following definitions of the maps in (4) and (5), which yield the particular hybrid dynamics for η_i therein. Namely, we consider the case of η_i having first order dynamics during flows, and, at jumps takes the output y_k for each $k \in \mathcal{N}(i)$ to update the corresponding states. The map $f_{ci} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is defined as

$$f_{ci}(x_i, \eta_i) = E\eta_i \qquad \forall i \in \mathcal{V} \tag{9}$$

and the map $G_{ci}^k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ as

$$\begin{aligned} G_{ci}^k(\eta_i, \eta_k, y_i, y_k) &= K(y_i - y_k) \\ &= KH(x_i - x_k) + K(\varphi_i(x_i) - \varphi_k(x_k)) \end{aligned} \tag{10}$$

for each $i, k \in \mathcal{V}$. The matrices E and K define the tuning parameters of the control algorithm to be designed, namely, through the matrix inequalities in (16). For simplicity, for the remainder of this section, we will consider the nominal hybrid system, i.e., perfect knowledge of the plant dynamics and its output maps. Namely, we assume that $\Delta \equiv 0$ and $\varphi_i \equiv 0$ for all $i \in \mathcal{V}$. The scenario when these perturbations are nonzero is addressed in Section 3.5. Without such perturbations and with the map (10), the impulsive dynamics of η_i in (5) are given by $\eta_i^+ = KH \sum_{k \in \mathcal{N}(i)} (x_i - x_k)$. For the design of our algorithm for synchronization under intermittent information, we employ the change in coordinates [4]

$$\theta_i = KH \sum_{k \in \mathcal{N}(i)} (x_i - x_k) - \eta_i. \tag{11}$$

which leads to $\theta = (\mathcal{L} \otimes KH)x - \eta$ where $x = (x_1, x_2, \dots, x_N)$, $\theta = (\theta_1, \theta_2, \dots, \theta_N)$, $\eta = (\eta_1, \eta_2, \dots, \eta_N)$, and $\mathcal{L}$ is the Laplacian matrix given by the connectivity graph Γ modeling the communication network. Let $\xi = (z, \tau) \in \mathcal{X} := \mathbb{R}^{(n+p)N} \times \mathcal{T}$ where $z = (x, \theta)$, $\tau = (\tau_1, \tau_2, \dots, \tau_N)$, and $\mathcal{T} = [0, T_2^1] \times [0, T_2^2] \times \cdots \times [0, T_2^N]$. Then, a closed-loop hybrid system $\mathcal{H} = (C, f, D, G)$ is defined by taking the collection of all agents with dynamics (1) integrated with controller dynamics (4) and (5) and has jumps triggered by the timer τ in (8). During flows, i.e., for every $\xi \in C := \mathcal{X}$, we have that

$$\dot{\xi} = (A_f z, -\mathbf{1}_N) =: f(\xi) \tag{12}$$

where the flow state matrix A_f is given by

$$A_f = \begin{bmatrix} A_1 & -\widetilde{B} \\ \widetilde{K}A_1 - \widetilde{E}\widetilde{K} & \widetilde{E} - \widetilde{K}\widetilde{B} \end{bmatrix}$$

where $A_1 = I \otimes A + \widetilde{B}\widetilde{K}$, $\widetilde{B} = I \otimes B$, $\widetilde{K} = \mathcal{L} \otimes KH$, and $\widetilde{E} = I \otimes E$. [5] When $\tau_i = 0$, a jump of the i-th agent occurs: the components θ and τ are mapped via $\theta_i^+ = 0$ and $\tau_i^+ \in [T_1^i, T_2^i]$ while x_i remains constant; moreover, for each $k \in \mathcal{V} \setminus \{i\}$ the state components x_k, θ_k and τ_k are held constant. Specifically, for each $\xi \in D := \cup_{i \in \mathcal{V}} D_i$ where $D_i := \{\xi \in \mathcal{X} : \tau_i = 0\}$, we have that

$$\xi^+ \in G(\xi) := \{G_i(\xi) : \xi \in D_i, i \in \mathcal{V}\} \tag{13}$$

where

$$G_i(\xi) = \begin{bmatrix} x \\ (\theta_1, \theta_2, \dots, \theta_{i-1}, 0, \theta_{i+1}, \dots, \theta_N) \\ (\tau_1, \tau_2, \dots, \tau_{i-1}, [T_1^i, T_2^i], \tau_{i+1}, \dots, \tau_N) \end{bmatrix}.$$

Lemma 3.2 *Given positive scalars T_1^i and T_2^i such that $T_1^i \le T_2^i$, the hybrid system $\mathcal{H} = (C, f, D, G)$ with* (12) - (13) *satisfies the hybrid basic conditions.*

Proof By construction, the sets C and D are closed. The flow map f in (12) is continuous. The jump map G is outer semicontinuous since its graph is closed; moreover, it is locally bounded on D. ■

Remark 3.3 *Note that satisfying the hybrid basic conditions implies that the hybrid system $\mathcal{H}$ is well-posed and that asymptotic stability of a compact set as defined in (Biemond et al. 2012, Definition 3.3) is robust to small enough perturbations. See Section 3.5.1 for more information on specific robustness results as a consequence of the hybrid basic conditions.*

3.2 Properties of Maximal Solutions to $\mathcal{H}$

The following properties of the domain of maximal solutions are established by exploiting the fact that a timer variable being zero is the only trigger of jumps in the system.

[4] This change of coordinates was also found useful for the design of observers under intermittent information in Ferrante et al. (2015), Sanfelice & Praly (2012). Therein, the authors proposed a continuous-time observer design to estimate the state of an LTI plant when its output is available only at intermittent time instances. The observer designed therein uses a memory state (akin to the hybrid controller in this work) that is reset when new measurements are available. Using a similar change of coordinates, sufficient conditions for asymptotic stability of the zero estimation error are derived. These results were extended to the network case in Li et al. (2016, 2018).

[5] Through the change of variables θ, the $z = (x, \theta)$ components of the flow dynamics in (12) are given by $\dot{x} = (I \otimes A)x + (I \otimes B)\eta = (I \otimes A)x + (I \otimes B)(\widetilde{K}x - \theta) = A_1 x - \widetilde{B}\theta$ and $\dot{\theta} = \widetilde{K}\dot{x} - \dot{\eta} = \widetilde{K}(A_1 x - \widetilde{B}\theta) - \widetilde{E}(\widetilde{K}x - \theta)$.

Lemma 3.4 *(Li et al. 2018, Lemma 3.5) Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Every maximal solution $\phi \in \mathcal{S}_\mathcal{H}$ satisfies the following:*

(1) ϕ is complete; i.e., $\operatorname{dom}\phi$ is unbounded;
(2) for each $(t,j) \in \operatorname{dom}\phi$, $\left(\frac{j}{N} - 1\right)\underline{T} \leq t \leq \frac{j}{N}\overline{T}$, where $\underline{T} := \min_{i\in\mathcal{V}} T_1^i$ and $\overline{T} := \max_{i\in\mathcal{V}} T_2^i$;
(3) for all $j \in \{1,2,3,\ldots\}$ such that $(t_{(j+1)N}, (j+1)N), (t_{jN}, jN) \in \operatorname{dom}\phi$, $t_{(j+1)N} - t_{jN} \in [\underline{T}, \overline{T}]$.

3.3 Sufficient Conditions for Synchronization

As mentioned in Section 1.3, asymptotic synchronization is where every solution, starting from some arbitrary initial conditions, converges to the set of points $x = (x_1, x_2, \ldots, x_N)$ such that $x_1 = x_2 = \cdots = x_N$ with stability. In this section, we recast asymptotic synchronization as a set stability problem. Our goal is to stabilize the set of points ξ such that each component of x and θ are synchronized. In particular, given a complete solution $\phi = (\phi_x, \phi_\theta, \phi_\tau)$ to the hybrid system $\mathcal{H}$, the goal is to assure that $\lim_{t+j\to\infty} |\phi_{x_i}(t,j) - \phi_{x_k}(t,j)| = 0$ and $\lim_{t+j\to\infty} |\phi_{\theta_i}(t,j) - \phi_{\theta_k}(t,j)| = 0$ for each $i,k \in \mathcal{V}$ where $\phi_x = (\phi_{x_1}, \phi_{x_2}, \ldots, \phi_{x_N})$ and $\phi_\theta = (\phi_{\theta_1}, \phi_{\theta_2}, \ldots, \phi_{\theta_N})$. To obtain such a property by solving a set stability problem, we define the synchronization set as

$$\begin{aligned}\mathcal{A} = \{\xi = (z,\tau) \in \mathcal{X} : x_1 = x_2 = \cdots = x_N,\\ \theta_1 = \theta_2 = \cdots = \theta_N\},\end{aligned} \tag{14}$$

for the hybrid system $\mathcal{H}$.

We consider the following global exponential stability notion of closed sets $\mathcal{A}$ for general hybrid systems $\mathcal{H}$.

Definition 3.5 *(Teel et al. (2013)) Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be* globally exponentially stable *for the hybrid system $\mathcal{H}$ if every maximal solution to $\mathcal{H}$ is complete and there exist strictly positive scalars κ and r such that for each solution ϕ to $\mathcal{H}$, $|\phi(t,j)|_\mathcal{A} \leq \kappa \exp(-r(t+j))|\phi(0,0)|_\mathcal{A}$ for all $(t,j) \in \operatorname{dom}\phi$.*

In light of the form of $\mathcal{A}$ in (14) and Definition 3.5, global exponential stability of $\mathcal{A}$ is equivalent to global synchronization of the closed-loop system in (12) – (13). Note that, in this context, global asymptotic synchronization is not merely convergence, but also requires stability of the diagonal for the state component z.

Next, we establish a sufficient condition that guarantees the synchronization property via stability analysis of $\mathcal{A}$ in (14). We establish such a result by using a Lyapunov function. An appropriate choice of V must satisfy $V(\xi) = 0$ for each $\xi \in \mathcal{A}$, while for any $\xi \in \mathcal{X} \setminus \mathcal{A}$, $V(\xi) > 0$. To simplify notation, we introduce the average of the timers of $\mathcal{H}$ given by $\bar{\tau} = \frac{1}{N}\sum_{i=1}^N \tau_i$. Inspired by Liu et al. (2012), we define the Lyapunov function candidate as

$$V(\xi) = z^\top \bar{\Psi} R(\tau) \bar{\Psi}^\top z, \tag{15}$$

where $\bar{\Psi} = \operatorname{diag}(\widetilde{\Psi} \otimes I_n, \widetilde{\Psi} \otimes I_p)$ with $\widetilde{\Psi}$ defined in Section 2.2, $R(\tau) = \operatorname{diag}(P, Q\exp(\sigma\bar{\tau}))$, $P = \operatorname{diag}(P_2, P_3, \ldots, P_N)$, $Q = \operatorname{diag}(Q_2, Q_3, \ldots, Q_N)$, $P_i = P_i^\top > 0$, and $Q_i = Q_i^\top > 0$ for each $i \in \{2,3,\ldots,N\}$. The Lyapunov function V in (15) satisfies (Goebel et al. 2012, Definition 3.16), which makes it a suitable Lyapunov function candidate for asymptotic stability of $\mathcal{A}$ in (14). The following result shows that, under certain conditions, for each $\xi \in C$, V decreases during flows, however, at jumps, may have a nonnegative change. Therefore, to guarantee exponential stability of the synchronization set, we exploit (Goebel et al. 2012, Proposition 3.29) which balances the change of V during flows with the change of V during jumps to maintain an overall decreasing V along any maximal solutions. This property guarantees that such solutions converge to the desired set. The Lyapunov function V in (15) is inspired by Li et al. (2018) where we focus on distributed estimation and utilize a similar construction of V which decreases during flows and has a non-positive change during jumps.

Theorem 3.6 *Given $0 < T_1^i \leq T_2^i$ for each $i \in \mathcal{V}$ and an undirected connected graph Γ, the set $\mathcal{A}$ in (14) is globally exponentially stable for the hybrid system $\mathcal{H}$ with data in (12) and (13) if there exist scalars $\sigma > 0$, $\varepsilon \in (0,1)$, matrices $K \in \mathbb{R}^{n\times p}$ and $E \in \mathbb{R}^{p\times p}$, and positive definite symmetric matrices P_i, Q_i for each $i \in \{2,3,\ldots,N\}$, satisfying*

$$M(\nu) = \begin{bmatrix} \operatorname{He}(P\bar{A}) & -P\bar{B} + \exp(\sigma\nu)(\bar{K}\bar{A} - \bar{E}\bar{K})^\top Q \\ \star & \operatorname{He}(\exp(\sigma\nu)Q(\bar{E} - \bar{K}\bar{B} - \frac{\sigma}{2}I)) \end{bmatrix} < 0 \tag{16}$$

for each $\nu \in [0, \overline{T}]$, where $\bar{A} = I\otimes A + \Lambda \otimes BKH$, $\bar{B} = I \otimes B$, $\bar{E} = I \otimes E$, $\bar{K} = \Lambda \otimes KH$, $P = \operatorname{diag}(P_2, P_3, \ldots, P_N)$, $Q = \operatorname{diag}(Q_2, Q_3 \ldots, Q_N)$, $\Lambda = \operatorname{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N)$ where λ_i are the nonzero eigenvalues of $\mathcal{L}$, and

$$(1-\varepsilon)\underline{T} - \frac{\alpha_2 \sigma \overline{T}}{\beta} > 0. \tag{17}$$

where $\underline{T} := \min_{i\in\mathcal{V}} T_1^i$, $\overline{T} := \max_{i\in\mathcal{V}} T_2^i$,

$$\beta = -\max_{\nu\in[0,\overline{T}]} \bar{\lambda}(M(\nu))$$
$$\alpha_2 = \max\{\overline{\lambda}(P), \overline{\lambda}(Q)\exp(\sigma\overline{T})\}.$$

Moreover, every $\phi \in \mathcal{S}_\mathcal{H}$ satisfies

$$|\phi(t,j)|_\mathcal{A} \leq \kappa \exp\left(-r(t+j)\right)|\phi(0,0)|_\mathcal{A} \tag{18}$$

for all $(t,j) \in \operatorname{dom}\phi$*, where* $\kappa = \sqrt{\frac{\alpha_2}{\alpha_1}}\exp\left(\frac{\beta(1-\varepsilon)\underline{T}}{2\alpha_2}\right)$ *and* $r = \frac{\beta}{2\alpha_2 N}\min\left\{\varepsilon N, (1-\varepsilon)\underline{T} - \frac{\alpha_2\sigma\overline{T}}{\beta}\right\}$*, and* $\alpha_1 = \min\{\underline{\lambda}(P), \underline{\lambda}(Q)\}$.

Proof Consider the Lyapunov function V in (15). Note that, due to the definition of $\bar{\Psi}$, the distance of ξ to the set $\mathcal{A}$ is equivalent to the distance of $\bar{\Psi}^\top z$ to the origin due to the domain of the timer states. More specifically, $|\xi|^2_{\mathcal{A}} = |\bar{\Psi}z|^2$. Furthermore, from V it follows that

$$\alpha_1|\xi|^2_{\mathcal{A}} \le V(\xi) \le \alpha_2|\xi|^2_{\mathcal{A}} \tag{19}$$

where α_1 and α_2 are given below (16).[6] During flows, the change in V is given by $\langle\nabla V(\xi), f(\xi)\rangle$ for each $\xi \in C$. To compute such inner product, define $\tilde{R}(\tau) = \operatorname{diag}(0, Q\exp(\sigma\bar{\tau}))$ and note that $\dot{\bar{\tau}} = -1$. Then, it follows that

$$\begin{aligned}\langle\nabla V(\xi), f(\xi)\rangle &= 2z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f z - \sigma z^\top\bar{\Psi}\tilde{R}(\tau)\bar{\Psi}^\top z\\ &= 2z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f\bar{\Psi}\bar{\Psi}^\top z\\ &\quad + 2z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f(I-\bar{\Psi}\bar{\Psi}^\top)z\\ &\quad - \sigma z^\top\bar{\Psi}\tilde{R}(\tau)\bar{\Psi}^\top z\end{aligned} \tag{20}$$

where we use the property that $\dot{z}^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top z = z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top\dot{z}$. Recall from Section 2.2 that $\Psi\Psi^\top = U$, $U\mathcal{L} = \mathcal{L}U$ and $\Psi^\top 1 = 0_{N-1\times N}$, which leads to $\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f(I-\bar{\Psi}\bar{\Psi}^\top) = 0$, which reduces (20) to

$$\begin{aligned}\langle\nabla V(\xi), f(\xi)\rangle &= z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f\bar{\Psi}\bar{\Psi}^\top z\\ &\quad + z^\top\bar{\Psi}\bar{\Psi}^\top A_f^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top z\\ &\quad - \sigma z^\top\bar{\Psi}\tilde{R}(\tau)\bar{\Psi}^\top z\\ &= z^\top\bar{\Psi}(R(\tau)\bar{\Psi}^\top A_f\bar{\Psi} + \bar{\Psi}^\top A_f^\top\bar{\Psi}R(\tau)\\ &\quad - \sigma\tilde{R}(\tau))\bar{\Psi}^\top z.\end{aligned} \tag{21}$$

Due to the definition of $\widetilde{\Psi}\mathcal{L}\widetilde{\Psi}^\top = \Lambda = \operatorname{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N)$, we have that

$$\bar{\Psi}^\top A_f^\top\bar{\Psi} = \begin{bmatrix}\bar{A} & -\bar{B}\\ \bar{K}\bar{A}-\bar{E}\bar{K} & \bar{E}-\bar{K}\bar{B}\end{bmatrix} =: \bar{A}_f$$

where $\bar{A} = I\otimes A + \Lambda\otimes BKH$, $\bar{B} = I\otimes B$, $\bar{E} = I\otimes E$, and $\bar{K} = \Lambda\otimes KH$. Therefore, from (21), it follows that $\langle\nabla V(\xi), f(\xi)\rangle = z^\top\bar{\Psi}M(\tau)\bar{\Psi}^\top z$ where M is defined in (16). From (16) it follows that

$$\langle\nabla V(\xi), f(\xi)\rangle = z^\top\bar{\Psi}M(\tau)\bar{\Psi}^\top z \le -\beta|\xi|^2_{\mathcal{A}} \le -\frac{\beta}{\alpha_2}V(\xi) \tag{22}$$

where $\beta = -\max_{\nu\in\mathcal{T}}\overline{\lambda}(M(\nu))$ and α_2 is defined in (17).

Next, we consider the case $\xi \in D$ and $g \in G(\xi)$. In particular, $\xi \in D$ if there exists at least one component of τ, say, the i-th component, such that $\tau_i = 0$. From the definition of G in (13), x is updated by its identity, $\theta_i^+ = 0$ and $\tau_i^+ \in [T_1^i, T_2^i]$. Moreover, for each $k \in \mathcal{V}\setminus\{i\}$, the k-th component of θ is updated by its identity, i.e., $\theta_k^+ = \theta_k$. Therefore, it follows that during jumps we have that $(\theta^+)^\top\theta^+ \le \theta^\top\theta$ due to the i-th component being updated to zero when $\tau_i = 0$. Likewise, after the jump of the i-th timer τ_i, we have that τ_i^+ is reset to a point $\nu \in [T_1^i, T_2^i]$. It follows that $\exp(\sigma\bar{\tau}^+) = \exp(\sigma\bar{\tau})\exp(\sigma\frac{\nu}{N})$. Then, the function V after a jump is given by

$$V(g) \le \exp\left(\sigma\frac{\overline{T}}{N}\right)V(\xi). \tag{23}$$

Note that the quantity $\exp(\sigma\frac{\overline{T}}{N}) - 1$ may be positive.

Next, we evaluate V over a solution to ensure that the distance of the solution ϕ to the set $\mathcal{A}$ converges to zero in the limit as $t+j$ approaches infinity. Pick $\phi \in \mathcal{S}_{\mathcal{H}}$ and any $(t,j) \in \operatorname{dom}\phi$. Let $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_{j+1} \le t$ satisfy $\operatorname{dom}\phi\bigcup([0,t_{j+1}]\times\{0,1,2,\ldots,j\}) = \bigcup_{s=0}^{j}([t_s,t_{s+1}]\times\{s\})$ for each $s \in \{0,1,2,\ldots,j\}$ and almost all $r \in [t_s, t_{s+1}], \phi(r,s) \in C$. Then, (22) implies that, for each $s \in \{0,1,2,\ldots,j\}$ and for almost all $r \in [t_s,t_{s+1}]$, $\frac{d}{dr}V(\phi(r,s)) \le -\frac{\beta}{\alpha_2}V(\phi(r,s))$. Integrating both sides of this inequality yields

$$V(\phi(t_{s+1},s)) \le \exp\left(-\frac{\beta}{\alpha_2}(t_{s+1}-t_s)\right)V(\phi(t_s,s)) \tag{24}$$

for each $s \in \{0,1,\ldots,j\}$. Similarly, for each $s \in \{1,2,\ldots,j\}, \phi(t_s,s-1) \in D$, and using (23), we get $V(\phi(t_s,s)) \le \exp\left(\sigma\frac{\overline{T}}{N}\right)V(\phi(t_s,s-1))$. It follows, from the previous two inequalities, for each $(t,j) \in \operatorname{dom}\phi$,

$$V(\phi(t,j)) \le \exp\left(-\frac{\beta}{\alpha_2}t + \sigma\frac{\overline{T}}{N}j\right)V(\phi(0,0)) \tag{25}$$

By virtue of (19) and Lemma 3.4, it follows that (24) becomes

$$\begin{aligned}|\phi(t,j)|_{\mathcal{A}} \le \sqrt{\frac{\alpha_2}{\alpha_1}}\exp\left(\frac{\beta(1-\varepsilon)\underline{T}}{2\alpha_2}\right)\exp\Bigg(-\frac{\beta\varepsilon}{2\alpha_2}t\\ + \left(\frac{\sigma\overline{T}}{2N} - \frac{\beta(1-\varepsilon)\underline{T}}{2\alpha_2 N}\right)j\Bigg)|\phi(0,0)|_{\mathcal{A}}\end{aligned}$$

where we used the property that there exists $\varepsilon \in (0,1)$ such that $t = \varepsilon t + (1-\varepsilon)t \ge \varepsilon t + (1-\varepsilon)\left(\frac{j}{N}-1\right)\underline{T}$. Moreover, from (17), and due to every maximal solution to $\mathcal{H}$ being complete, it follows that the bound on

[6] Note that $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ are the maximum and minimum eigenvalues, respectively.

$|\phi(t,j)|_{\mathcal{A}}$ implies that $\mathcal{A}$ is globally exponentially stable for the hybrid system $\mathcal{H}$. ■

Remark 3.7 *The matrix inequality in* (16) *comes from the asymptotic stability analysis in the proposed new coordinates* $\xi = (x, \theta, \tau)$*, namely, the analysis during flows; see* (22). *This approach introduces some conservativeness as the reset of* θ_i *to zero when* $\tau_i = 0$ *is not being exploited. This is due to the multiplication of* θ *by* $\widetilde{\Psi} \otimes I_p$ *in* V. *In fact, it is not straightforward to ensure a nonpositive change in* V *during jumps. If such a change could be guaranteed, then the conditions in Theorem 3.6 could be relaxed. Though it exists due to converse theorems, at this time we do not have a Lyapunov function that satisfies the decreasing properties on both jumps and flows.*

Remark 3.8 *Note that, although the graph associated to the network in Theorem 3.6 is assumed to be undirected for technical reasons, the information transmitted between agents* i *and* k *and between agents* k *and* i *do not necessarily occur at the same time. Namely, when* $\tau_i = 0$*, agent* i *receives information from its neighbors (e.g., agent* k*), while when* $\tau_k = 0$ *may not occur at the same time.*

Note that the matrix inequality in (16) must be satisfied for an infinite number of points, i.e., $\nu \in [0, \overline{T}]$. Moreover, it can be noted that (16) may be a large matrix in general, which could make finding feasible solutions difficult. It turns out that (16) can be decomposed into $N-1$ matrices due to the fact that each block in the matrix is block diagonal. This leads to the following result.

Proposition 3.9 *Let* $0 < T_1^i \le T_2^i$ *be given for all* $i \in \mathcal{V}$. *Inequality* (16) *holds if there exist scalar* $\sigma > 0$ *and matrices* $P_i = P_i^\top > 0$ *and* $Q_i = Q_i^\top > 0$ *for each* $i \in \{2, 3, \ldots, N\}$ *satisfying* $\overline{M}_i(0) < 0$ *and* $\overline{M}_i(\overline{T}) < 0$ *where*

$$\overline{M}_i(\nu) := \begin{bmatrix} \mathrm{He}(P_i \bar{A}_i) & -P_i B + \exp(\sigma\nu)(\bar{K}_i \bar{A}_i - E\bar{K}_i)Q_i \\ \star & \mathrm{He}(\exp(\sigma\nu) Q_i (E - \bar{K}_i B - \frac{\sigma}{2} I)) \end{bmatrix} \tag{26}$$

for each $\lambda_i \in \lambda(\mathcal{L}) \setminus \{0\}$*, where* $\bar{A}_i = A + \lambda_i BKH$ *and* $K_i = \lambda_i KH$.

The proof follows similar to the proof of Proposition 3.9 in Li et al. (2018).

Remark 3.10 *Note that conditions* $\overline{M}_i(0) < 0$ *and* $\overline{M}_i(\overline{T}) < 0$ *are nonconvex in* P, Q, K, E, *and* σ. *At this time, there is no clear way to reduce the matrices in the conditions into a convex form. In fact, the matrices are bilinear in these variables; therefore, to solve* (26) *one should use a BMI solver such as YALMIP and BMILAB.*

3.4 Time to Synchronize

Due to its properties along solutions shown in Theorem 3.6, the proposed Lyapunov function can be further exploited to provide a bound on the time to converge to a neighborhood about the synchronization set $\mathcal{A}$. As expected, this time depends on the initial distance to the set $\mathcal{A}$ and the parameters of the hybrid system.

Proposition 3.11 *Given* $0 < T_1^i \le T_2^i$ *for each* $i \in \mathcal{V}$ *and an undirected connected graph* Γ*, if there exist scalars* $\sigma > 0$ *and* $\varepsilon \in (0,1)$*, matrices* $K \in \mathbb{R}^{n\times p}$ *and* $E \in \mathbb{R}^{p \times p}$*, and positive definite symmetric matrices* P_i, Q_i *for each* $i \in \{2, 3, \ldots, N\}$, (16) *and* (17)*, then for each* $c_0 > c_1 > 0$ *every maximal solution* ϕ *to* $\mathcal{H}$ *with initial condition*[7] $\phi(0,0) \in \mathcal{X} \cap L_V(c_0)$ *is such that* $\phi(t,j) \in L_V(c_1)$ *for each* $(t,j) \in \mathrm{dom}\,\phi$, $t + j \ge \bar{r}$*, where* $\bar{r} = (\frac{N}{\underline{T}} + 1)\Omega + 1$, $\Omega = \frac{\ln\left(\frac{c_1}{c_0}\right) - \sigma\frac{\overline{T}}{N}}{\frac{-\beta}{\alpha_2} + \frac{\sigma\overline{T}}{\underline{T}}}$*, and* $\underline{T}$, $\overline{T}$, β *and* α_2 *are given below* (17).

Proof Let $\phi_0 = \phi(0,0)$ and pick a maximal solution $\phi \in \mathcal{S}_{\mathcal{H}}(\phi_0)$. From the proof of Theorem 3.6, we have that, for each $(t,j) \in \mathrm{dom}\,\phi$, (25) holds. Namely, for each $(t,j) \in \mathrm{dom}\,\phi$, V satisfies $V(\phi(t,j)) \le \exp\left(-\frac{\beta}{\alpha_2}t + \sigma\frac{\overline{T}}{N}j\right)V(\phi_0)$. We want to find $(T, J) \in \mathrm{dom}\,\phi$ such that $V(\phi(T,J)) \le c_1$ when $\phi(0,0) \in L_V(c_0)$. Considering the worst case for $V(\phi_0)$, it follows that $c_1 \le \exp\left(-\frac{\beta}{\alpha_2}T + \sigma\frac{\overline{T}}{N}J\right)c_0$ which implies that $\ln\left(\frac{c_1}{c_0}\right) \le -\frac{\beta}{\alpha_2}T + \sigma\frac{\overline{T}}{N}J$. Then, from Lemma 3.4, we have that for $(T,J) \in \mathrm{dom}\,\phi$, it follows that $J \le N\left(\frac{T}{\underline{T}} + 1\right)$ which implies that $T \le \Omega$ where $\Omega = \frac{\ln\left(\frac{c_1}{c_0}\right) - \sigma\frac{\overline{T}}{N}}{\frac{-\beta}{\alpha_2} + \frac{\sigma\overline{T}}{\underline{T}}}$. Then, after $t + j \ge T + J$, the solution is at least c_1 close to the set $\mathcal{A}$. Defining $\bar{r} = T + J$, we have that $\bar{r} = \left(\frac{N}{\underline{T}} + 1\right)\Omega + 1$. ■

3.5 Robustness of Synchronization

In this section, we consider the effect of general perturbations and unmodeled dynamics on the agents in the network. In such a setting, the perturbed model of each agent is given in (1) and the output generated by each agent is given by (2), where the functions $\Delta_i : \mathbb{R}^n \times \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ and $\varphi_i : \mathbb{R}^n \to \mathbb{R}^p$ are unknown functions that may capture the unmodeled dynamics, as well as the disturbances and communication noise, respectively. In particular, due to the disturbances on the output, the values of y_k transmitted to agent i at communication times $t = t_s$ from agent k (where $k \in \mathcal{N}(i)$) may be affected by some communication channel noise, specifically, $y_k(t_s) = Hx_k(t_s) + \varphi_k(x_k(t_s), t_s)$.

[7] A sublevel set of V, denoted as $L_V(\mu)$, is given by $L_V(\mu) := \{x \in \mathcal{X} : V(x) \le \mu\}$.

Adding the perturbations to (9) and (10), we have that the continuous dynamics of the distributed controllers do not change, but the discrete dynamics become

$$\eta_i^+ = KH \sum_{k\in\mathcal{N}(i)} (x_i - x_k) + K\tilde{\varphi}_i(x,t) \quad (27)$$

where $\tilde{\varphi}_i(x,t) = \sum_{k\in\mathcal{N}(i)}(\varphi_i(x_i,t) - \varphi_k(x_k,t))$. For simplicity, henceforth, we will drop the arguments of some of the perturbations. We consider the model in the θ coordinates in Section 3.1 for the study of robustness. Then, following the definition of θ_i in (11), the resulting perturbed hybrid system $\widetilde{\mathcal{H}}$ has data $(C, \widetilde{f}, D, \widetilde{G})$ and state $\xi = (z,\tau) \in \mathcal{X}$, $z = (x,\theta)$. The perturbed data is given by

$$\widetilde{f}(\xi) = f(\xi) + (\Delta(x,t), \widetilde{K}\Delta(x,t), 0) \quad \forall \xi \in C \quad (28)$$

where $\Delta(t,x) = (\Delta_1(x_1,t), \Delta_2(x_1,t), \dots, \Delta_N(x_1,t))$ and $\widetilde{K} = (\mathcal{L} \otimes KH)$. Moreover, when $\xi \in D$,

$$\widetilde{G}(\xi,\varphi) := \{\widetilde{G}_i(\xi,\delta) : \xi \in \widetilde{D}_i, i \in \mathcal{V}\} \quad \forall \xi \in D \quad (29)$$

and

$$\widetilde{G}(\xi,\varphi) := \begin{bmatrix} x \\ (\theta_1, \theta_2, \dots, \theta_{i-1}, -K\tilde{\varphi}_i, \theta_{i+1}, \dots, \theta_N) \\ (\tau_1, \tau_2, \dots, \tau_{i-1}, [T_1^i, T_2^i], \tau_{i+1}, \dots, \tau_N) \end{bmatrix}. \quad (30)$$

3.5.1 General Robustness on Compact Sets

In this section, we focus on the generic robustness property to small perturbations. To apply standard robustness results for hybrid systems, the set that is asymptotically stable must be compact. Note that the set $\mathcal{A}$ given by (14) is unbounded: the points $x_1 = x_2 = \cdots = x_N$ and $\theta_1 = \theta_2 = \cdots = \theta_N$ can be any value in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. Therefore, we restrict the state space to the compact set $S \times \mathcal{T}$. While this set restricts the state space of the hybrid system, it can easily be considered to be arbitrarily large. The price to pay is that, due to the fact that the state space is now bounded, it is not guaranteed that maximal solutions to the hybrid system are complete. We consider the hybrid system $\widetilde{\mathcal{H}} = (C, \widetilde{f}, D, \widetilde{G})$ as in Section 3.1 with flow and jumps sets given by $\widetilde{C} = C \cap (S\times\mathcal{T})$ and $\widetilde{D} = D \cap (S\times\mathcal{T})$ where $S \subset \mathbb{R}^{N(n+p)}$ is compact. Moreover, the set of interest is given by $\widetilde{\mathcal{A}} = \mathcal{A} \cap (S\times\mathcal{T})$. We have the following result.

Theorem 3.12 *Let $0 < T_1^i \le T_2^i$ be given for all $i \in \mathcal{V}$. Suppose that the hybrid system satisfies the conditions in Theorem 3.6 for the unperturbed hybrid system $\mathcal{H}$ with data in* (12) *and* (13)*. Then, there exists $\beta \in \mathcal{KL}$ such that, for every compact set $S \subset \mathbb{R}^{N(n+p)}$ and $\varepsilon > 0$, there exists $\rho^\star \ge 0$ such that if $\max\{\bar{\Delta}, \bar{\varphi}\} \le \rho^\star$ where $\bar{\Delta} = \sup_{(x,t)\in\mathcal{X}\cap(S\times\mathcal{T})\times\mathbb{R}_{\ge 0}} |\Delta(x,t)|$ and $\bar{\varphi} = \sup_{(x,t)\in\mathcal{X}\cap(S\times\mathcal{T})\times\mathbb{R}_{\ge 0}} |\widetilde{\varphi}(x,t)|$ then, every $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}(S\times\mathcal{T})$ satisfies $|\phi(t,j)|_{\widetilde{\mathcal{A}}} \le \beta(|\phi(0,0)|_{\widetilde{\mathcal{A}}}, t+j) + \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$.*[8]

Proof Consider the hybrid system $\widetilde{\mathcal{H}}$ and a continuous function $\rho : \mathbb{R}^{nN} \times \mathbb{R}^{pN} \times \mathcal{T} \to \mathbb{R}_{\ge 0}$, the ρ-perturbation of $\widetilde{\mathcal{H}}$, denoted $\widetilde{\mathcal{H}}_\rho$, is the hybrid system

$$\begin{cases} \xi \in \widetilde{C}_\rho & \dot{\xi} \in F_\rho(\xi) \\ \xi \in \widetilde{D}_\rho & \xi^+ \in G_\rho(\xi) \end{cases} \quad (31)$$

where

$$\begin{aligned} \widetilde{C}_\rho &= \{\xi \in \widetilde{C} \cup \widetilde{D} : (\xi + \rho(\xi)\mathbb{B}) \cap \widetilde{C} \ne 0\} \\ F_\rho(\xi) &= \overline{\text{con}} f((\xi + \rho(\xi)\mathbb{B}) \cap C = +\rho(\xi)\mathbb{B} \quad \forall \xi \in \widetilde{C} \cap \widetilde{D} \\ \widetilde{D}_\rho &= \{\xi \in \widetilde{C} \cup \widetilde{D} : (\xi + \rho(\xi)\mathbb{B}) \cap \widetilde{D} \ne 0\} \\ G_\rho(\xi) &= \{v \in \widetilde{C} \cap \widetilde{D} : v \in g + \rho(g)\mathbb{B}, g \in G(\xi + \rho(\xi)) \cap \widetilde{D}\} \\ & \qquad \forall \xi \in \widetilde{C} \cap \widetilde{D} \end{aligned}$$

Since the set $\mathcal{A}$ is GES for $\mathcal{H}$, it is also UGAS for $\mathcal{H}$. Since ρ is continuous and $\mathcal{H}$ satisfies the hybrid basic conditions, by (Goebel et al. 2012, Theorem 6.8), $\widetilde{\mathcal{H}}_\rho$ is nominally well-posed and, moreover, by (Goebel et al. 2012, Proposition 6.28) is well-posed. Then, (Goebel et al. 2012, Theorem 7.20) implies that $\mathcal{A}$ is semiglobally practically robustly $\mathcal{KL}$ pre-asymptotically stable for $\widetilde{\mathcal{H}}$. Namely, for every compact set $S\times\mathcal{T} \subset \mathbb{R}^{N(n+p)} \times \mathcal{T}$ and every $\varepsilon > 0$, there exists $\widetilde{\rho} \in (0,1)$ such that every maximal solution ϕ to $\mathcal{H}_{\widetilde{\rho}\rho}$ from $S \times \mathcal{T}$ satisfies $|\phi(t,j)|_{\mathcal{A}\cap(S\times\mathcal{T})} \le \beta(|\phi(0,0)|_{\mathcal{A}\cap(S\times\mathcal{T})}, t+j) + \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$. Then, the result follows by picking $\rho^* > 0$ such that $\max\{1, |\widetilde{K}|, |K|\}\rho^* \le \widetilde{\rho}$ and relating solutions to $\widetilde{\mathcal{H}}$ and solutions to $\mathcal{H}_{\widetilde{\rho}\rho}$. ∎

3.5.2 Robustness to Communication Noise

In this section, we consider the hybrid system $\mathcal{H}$ in Section 3.5 when communication noise is present. Namely, φ_i reduces to a function $m_i(t) = \widetilde{\varphi}_i(x_i,t)$ for all $t \in \mathbb{R}_{\ge 0}$ and $i \in \mathcal{V}$. We have the following result.

Theorem 3.13 *Given $0 < T_1^i \le T_2^i$ for each $i \in \mathcal{V}$ and an undirected connected graph Γ, if there exist scalars $\sigma > 0, \varepsilon \in (0,1)$ and matrices $K \in \mathbb{R}^{n\times p}$ and $H \in \mathbb{R}^{p\times p}$, and positive definite symmetric matrices P_i, Q_i for each $i \in \{2,3,\dots,N\}$, satisfying* (16) *for each $\nu \in [0,\overline{T}]$ and* (17) *holds, then the set $\mathcal{A}$ is input-to-state stable for the hybrid system $\widetilde{\mathcal{H}}$ in* (28) *and* (29) *with respect*

[8] The set $\mathcal{S}_\mathcal{H}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_\mathcal{H}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ.

to communication noise $m = (m_1, m_2, \ldots, m_N)$. More specifically, for each $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}$ and for any $(t,j) \in \operatorname{dom}\phi$,

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\left\{\kappa \exp\left(-r(t+j)\right)|\phi(0,0)|_{\mathcal{A}}, \gamma_m |m|_\infty\right\} \quad (32)$$

where $\underline{T}$, $\overline{T}$, α_2, and β are given below (17) and κ, r, α_1, are given below (18), $b = \exp(\sigma\overline{T}/N)\overline{\lambda}(Q)$, $\gamma_m = NS\sqrt{\frac{\alpha_1}{\alpha_2}}\exp(\sigma\overline{T})b|K|^2$ and $S = \frac{\exp(-\epsilon)}{\exp(-\epsilon)-1}$ where $\epsilon \in \left(0, \frac{\alpha_2\sigma\overline{T}}{\beta} - (1-\varepsilon)\underline{T}\right)$.

Proof Consider the Lyapunov function candidate $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ given by (15). It follows that V satisfies (19) for all $\xi \in C \cup D$ where α_1 and α_2 are given in the Proof of Theorem 3.6. Note that communication noise only occurs upon communication events, when $\xi \in D$. Therefore, for each $\xi \in C$, we have that

$$\langle \nabla V(\xi), \widetilde{f}(\xi)\rangle \leq -\beta|\xi|^2_{\mathcal{A}} \leq -\frac{\beta}{\alpha_2}V(\xi) \quad (33)$$

and $\beta = -\overline{\lambda}(M(\nu))$ for each $\nu \in [0,\overline{T}]$ where M is given by (16). Moreover, at jumps, we have that the state is updated by (30), with $m_i(t) = \widetilde{\varphi}_i(x,t)$. It follows that for each $\xi \in D$ and $g \in G(\xi)$, there exists at least one timer reseting, i.e., $\tau_i = 0$, after the jump it follows that $\tau_i^+ = \nu$ where $\nu \in [T_1^i, T_2^i]$ and $\theta_i = -Km_i$. With $\exp(\sigma\bar{\tau}^+) = \exp(\sigma\bar{\tau})\exp(\sigma\nu/N)$, it follows that $V(g) \leq \exp\left(\frac{\sigma\overline{T}}{N}\right)V(\xi) + b|K|^2|m|^2$ where $b = \exp(\sigma\overline{T}/N)\overline{\lambda}(Q)$.

Now pick $\phi \in \mathcal{S}_{\mathcal{H}}$, or any $(t,j) \in \operatorname{dom}\phi$ and let $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{j+1} \leq t$ satisfy $\operatorname{dom}\phi \cup ([0,t_{j+1}] \times \{0,1,\ldots,j\}) = \cup_{s=0}^{j}([t_s,t_{s+1}] \times \{s\})$. For each $s \in \{0,1,\ldots,j\}$ and almost all $r \in [t_s,t_{s+1}]$, $\phi(r,s) \in C$. Then, integrating both sides of (33) implies that for each $s \in \{0,1,\ldots,j\}$ and for almost all $r \in [t_s,t_{s+1}]$, we have that $V(\phi(r,s)) \leq \exp\left(-\frac{\beta}{\alpha_2}\right)V(\phi(t_s,s))$. Similarly, for each $s \in \{1,2,\ldots,j\}$, $\phi(t_s,s-1) \in D$, and using the change in V at jumps, we get $V(\phi(t_s,s-1)) \leq \exp\left(\frac{\sigma\overline{T}}{N}\right)V(\phi(t_s,s-1)) + b|K|^2|m|^2$. Then, it follows that for each $(t,j) \in \operatorname{dom}\phi$, we have $V(\phi(t,j)) \leq \exp\left(\frac{\sigma\overline{T}}{N}j - \frac{\beta}{\alpha_2}t\right)V(\phi(0,0))$ $+\ b|m|^2_\infty \sum_{k=1}^{j}\left(\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t-t_k)\right)\right)$. For $t \geq t_j$, we have $\sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t-t_k)\right) \leq \sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_k)\right)$. Due to the increasing sequence of times $t_1 \leq t_2 \leq \cdots \leq t_j$, there must exist an integer $\tilde{j}$ which defines the maximum multiple of N, i.e., $\tilde{j} = \lfloor\frac{j}{N}\rfloor$. Then, we can group the expression $\sum_{k=1}^{j}\left(\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_k)\right)\right)$ into a double sum as follows:

$$\begin{aligned}
&\sum_{k=1}^{j}\left(\exp\frac{\sigma\overline{T}}{N}k - \frac{\beta}{\alpha_2}(t_j-t_k)\right)\\
&\quad= \sum_{s=0}^{\tilde{j}-1}\sum_{k=1}^{N}\exp\left(\frac{\sigma\overline{T}}{N}(sN+k) - \frac{\beta}{\alpha_2}(t_j-t_{sN+k})\right)\\
&\qquad+ \sum_{k=\tilde{j}N+1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k - \frac{\beta}{\alpha_2}(t_j-t_k)\right)
\end{aligned}$$

Note that for each $s \in \{0,\ldots,\tilde{j}-1\}$, we have

$$\begin{aligned}
&\max_{t_{sN+k},k\in\{1,\ldots,N-1\}}\sum_{k=1}^{N}\exp\left(\frac{\sigma\overline{T}}{N}(sN+k) - \frac{\beta}{\alpha_2}(t_j-t_{sN+k})\right)\\
&\quad= \sum_{k=1}^{N}\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_{(s+1)N}) + \sigma\overline{T}(s+1)\right)\\
&\quad= N\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_{(s+1)N}) + \sigma\overline{T}(s+1)\right)
\end{aligned}$$

which corresponds to the maximizer satisfying $t_{sN+k} = t_{(s+1)N}$ for all $k \in \{1,\ldots,N-1\}$. Therefore, we have

$$\begin{aligned}
&\sup_{\substack{\phi\in\mathcal{S}_{\mathcal{H}}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k - \frac{\beta}{\alpha_2}(t_j-t_k)\right)\\
&\leq \sup_{\substack{\phi\in\mathcal{S}_{\mathcal{H}}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{s=1}^{\tilde{j}-1}\max_{\substack{t_{sN+k}\\k\in\{1,\ldots,N-1\}}}\sum_{k=1}^{N}\exp\left(\frac{\sigma\overline{T}}{N}(sN+k)\right.\\
&\qquad\left.-\frac{\beta}{\alpha_2}(t_j-t_{sN+k})\right)\\
&\qquad+ \sup_{\substack{\phi\in\mathcal{S}_{\mathcal{H}}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=\tilde{j}N+1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k - \frac{\beta}{\alpha_2}(t_j-t_k)\right)\\
&\leq \sup_{\substack{\phi\in\mathcal{S}_{\mathcal{H}}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=1}^{\tilde{j}-1}N\exp\left(-\frac{\beta}{\alpha_2}(t_{\tilde{j}N}-t_{(s+1)N}) + \sigma\overline{T}(s+1)\right)\\
&\qquad+ N\exp(\sigma\overline{T})
\end{aligned}$$

where we use the property that $j - \tilde{j}N < N$. By item 3 in Lemma 3.4, we have that $t_{(j+1)N} - t_{jN} \in [\underline{T},\overline{T}]$ for all $j \geq 0$ such that $(t_{(j+1)N},(j+1)N), (t_{jN},jN) \in \operatorname{dom}\phi$ which implies that for each $s \in \{0,1,\ldots,\tilde{j}-1\}$ $t_{\tilde{j}N} - t_{sN} \in [(\tilde{j}-s)\underline{T}, (\tilde{j}-s)\overline{T}]$. Therefore,

$$\begin{aligned}
&\sup_{\substack{\phi\in\mathcal{S}_{\mathcal{H}}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k - \frac{\beta}{\alpha_2}(t_j-t_k)\right)\\
&\qquad\leq N\exp(\sigma\overline{T})\sum_{s=0}^{\tilde{j}}\exp\left(\left(-\frac{\beta}{\alpha_2}\underline{T} + \sigma\overline{T}\right)s\right)
\end{aligned}$$

Then, it follows that

$$V(\phi(t,j)) \leq \exp\left(-\frac{\beta}{\alpha_2}t + \frac{\sigma\overline{T}}{N}j\right)$$

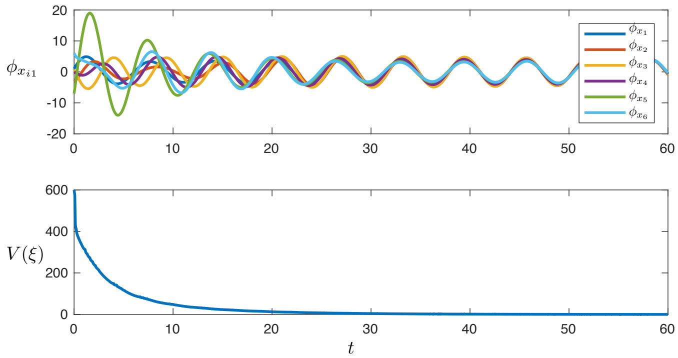

Fig. 2. Numerical solutions of 6 interconnected linear oscillators communicating over a ring graph.

$$+ N\exp(\sigma\overline{T})b|K|^2|m|^2_\infty \sum_{s=0}^{\tilde{j}} \exp\left(\left(-\frac{\beta}{\alpha_2}\underline{T} + \sigma\overline{T}\right)s\right)$$

where $\tilde{j} = \lfloor \frac{j}{N} \rfloor$. Note that by the continuity of (17), there exists small positive scalar ϵ such that $-\frac{\beta}{\alpha_2}\underline{T} + \sigma\overline{T} \leq -\epsilon$. Note that for each $n \in \mathbb{N}$, we have that $\sum_{s=0}^{n} \exp(-\epsilon s) \leq \frac{\exp(-\epsilon)}{\exp(-\epsilon)-1} =: S$. Then, it follows from (19), we have (32). We can conclude the proof using similar arguments as in the proof of Theorem 3.6. ■

4 Numerical Examples

Example 4.1 *Given $T_1^i = 0.7$ and $T_2^i = 0.9$ for each $i \in \mathcal{V}$, we apply Theorem 3.6 to a network of six harmonic oscillators, where each agent has dynamics given by $\ddot{x}_i + x_i = u_i$. We consider the case where each agent is connected only to two neighbors in a circle graph. Moreover, the output of each agent is both position x_1 and velocity x_2 information, i.e, $H = I$. In state space form, we have an LTI system of the form in* (1) *with state matrices $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ connected with a ring graph. It can be shown that the following parameters $K = -\begin{bmatrix} 0.15 & 0.15 \end{bmatrix}$, $E = -1$, $\sigma = 0.9$ and P_i and Q_i can be found such that the matrices satisfy the conditions in* (16) *and* (17).[9] *In Figure 2, a numerical solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ to the hybrid system $\mathcal{H}$ with the above parameters from initial conditions $\phi_x(0,0) = (-5, 1, -2, -3, 5, 0, 0, 0, -18, -7, -4, 6)$, $\phi_\eta = (0.5, 0, 10, -2, 5, -10)$ and $\phi_\tau(0,0) = (0.25, 0.5, 0.86, 0.87, 0.14, 0.1)$ is shown.*

The exponential convergence rate to the synchronization set $\mathcal{A}$ is guaranteed by the sufficient condition in Proposition 3.11. Note the conditions in Theorem 3.6 are not necessary and it may be possible that gains can be found so that solutions still converge to the synchronization set. In Table 1, we compare the convergence time (in flow time, t) of solutions to $\mathcal{H}$ with different gains K and E in (9) *and* (10)*, respectively. Namely, we indicate whether it is possible to satisfy the conditions in Theorem 3.6 for the gains chosen by placing a ✓ if the conditions are satisfied and by placing a × if it is not possible to satisfy the conditions for the selected gain. Moreover, we compare convergence times of solutions to the set $\mathcal{A}$ for different parameter choices. More specifically, we consider a solution ϕ such that $|\phi(0,0)|_\mathcal{A} \approx 50$ and find the time it takes for the solution to converge to and stay in a neighborhood near $\mathcal{A}$ in* (14)*, i.e., we find t^* such that $t^* = \{T \in \mathbb{R}_{\geq 0} : |\phi(t,j)|_\mathcal{A} \leq 0.1 \ \forall (t,j) \in \operatorname{dom}\phi$ s.t. $t \geq T\}$. Due to the nonuniqueness of solutions $\mathcal{H}$ in* (12) *and* (13) *when the network parameters are such that $T_1^i \neq T_2^i$, Table 1 provides an average t^* over 100 solutions.* △

K	E	Theorem 3.6	t^*
$-\begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$	-0.5	✓	165.38
$-\begin{bmatrix} 0.15 & 0.15 \end{bmatrix}$	-1	✓	120.2
$-\begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$	-1.8	×	30.1
$-\begin{bmatrix} 0.6 & 0.6 \end{bmatrix}$	-0.1	×	27.05
$\begin{bmatrix} 0.15 & -0.6 \end{bmatrix}$	-0.1	×	30.66

Table 1
Comparison of convergence times for different gains K and E for the hybrid system $\mathcal{H}$ with asynchronous communication in Section 3.1. The ✓ indicates that the conditions are satisfied, and the × indicates that the conditions are not satisfied but solutions converge to the synchronization set.

A small-world network is a type of sparse network known to model real-world settings such as the world wide web, electric power girds, and networks of brain neurons. In the following example, we use the random graph generator in Watts & Strogatz (1998) to generate the interconnection between 100 agents.

Example 4.2 *In this example, we consider the case of a network of 100 agents with dynamics as in Example 4.1 with $T_1^i = 0.7$ and $T_2^i = 0.9$ for each $i \in \mathcal{V}$. We generated a random graph using the small world generator in Watts & Strogatz (1998) for $N = 100$, the average degree $k = 3$ and special restructuring parameter $\beta = 0.1$. The resulting graph structure is depicted in the upper left of Figure 3. Furthermore, we use the parameters in Example 4.1, namely, $K = -[0.15, 0.15]$ and $E = -1$. The solutions $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ were initialized randomly inside a bounded region, namely, $\phi_{x_i}(0,0) \in [-5,5]^2$, $\phi_{\eta_i}(0,0) \in [-5,5]$ and $\phi_{\tau_i}(0,0) \in \mathcal{T}$ for each $i \in \mathcal{V}$*[10] *The plots in the upper right section of Figure 3 show the evolution of the first component of the plant state x_{1i} for each $i \in \mathcal{V}$. It can be seen that solutions asymptotically converge to synchronization as time progresses: in fact, the bottom plot shows that, indeed, the error converges to zero.* △

Example 4.3 *In this example, we consider the case of $\mathcal{H}$ with measurement noise. Let the system be given by the dynamics in Example 4.1 connected by a network*

[9] Code at `github.com/HybridSystemsLab/LTIAsyncSync`

[10] Code at `github.com/HybridSystemsLab/LTISyncSmallWorld`

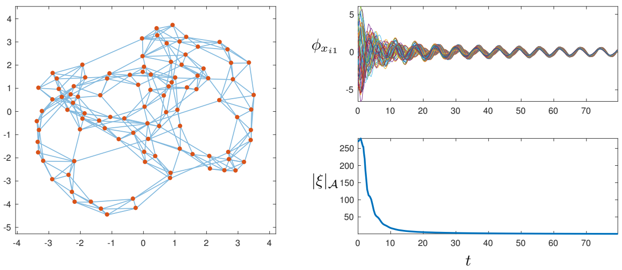

Fig. 3. (left) Randomly generated undirected small-world network containing 100 agents. (upper right) The first component of the states of each agent in the network. Note that over time all agents converge to synchrony. (bottom right) The norm of the relative error over ordinary time converges to zero.

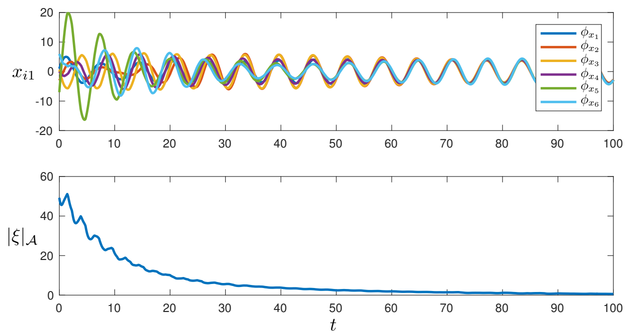

Fig. 4. When communication noise is present, solutions converge to a neighborhood about the synchronization set as indicated by the norm of the relative error converging to an average value of 0.1147.

with a ring graph where $T_1^i = 0.7$ *and* $T_2^i = 0.9$ *for each* $i \in \mathcal{V}$*. Let the output* y_i *in* (2) *be given by a constant bias, i.e.,* $\varphi_i(x_i, t) \equiv m_i$ *for each* $i \in \mathcal{V}$*, where* $m_1 = (0.1, 0.1)$*,* $m_2 = (-0.1, -0.1)$*,* $m_3 = (0, 0)$*,* $m_4 = (0.2, 0.2)$*,* $m_5 = (-0.15, -0.15)$*, and* $m_6 = (0.3, 0.3)$*. Moreover, let* $K = -\begin{bmatrix} 0.15 & 0.15 \end{bmatrix}$ *and* $E = -1$*, which, as was shown in Example 4.1, satisfy Theorem 3.6 therefore the resulting hybrid system* $\mathcal{H}$ *with data given by* (12) *and* (13) *has* $\mathcal{A}$ *exponentially stable. In Figure 4, we show a numerical solution to the hybrid system from the initial conditions in Example 4.1. In this figure, it can be seen that after the transient period, the norm of the relative error* $|\varepsilon|$ *of the solution converges to an average value of* 0.1147 *for this case.* △

5 Conclusion

The problem of synchronization of multiple continuous-time linear time-invariant systems connected over an asynchronous intermittent network was studied. Communications across the network occurs at isolated time events, which, using the hybrid systems framework was modeled using a decreasing timer. Recasting synchronization as a set stability problem, we took advantage of several properties of the graph structure and employed a Lyapunov based approach to certify exponential stability of the synchronization set. Then, in part, as a consequence of the regularity of the hybrid systems data and the aforementioned stability properties, robustness to communication noise, and unmodeled dynamics was characterized in terms of semi-global practical stability. When communication noise was affecting the dynamics, the Lyapunov function candidate chosen certified input-to-state stability for the synchronization set and relative to such noise.

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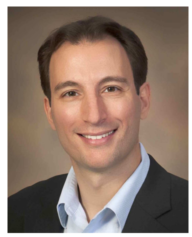

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