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MODAL PROPERTIES AND EQUIPMENT RESPONSE OF AN EQUIPMENT-CONTINUOUS STRUCTURE SYSTEM

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Abstract

An efficient method for the dynamic analysis of an equipment-structure system, modeled here as a simply supported beam to which some concentrated masses are attached, is developed. By using matrix perturbation, the modal properties of the combined system are obtained in terms of the known properties of the individual subsystems. A closed form result for the estimate of the maximum response of the equipment, including the cases when the equipment frequency is tuned or nearly tuned to a natural frequency of the beam, is obtained in terms of the response spectrum describing the impact force applied to the beam. A numerical example shows that the modal propeties of the combined system and the maximum response of the equipment obtained by the proposed method is in good agreement with the results obtained from a direct numerical integration of the governing equation of motion.

Introduction

Many systems encountered in engineering practice can be characterized as equipment-structure systems. For example, an aircraft with devices mounted inside it, which are light compared to the aircraft, may be considered to be such a system. In the recent past, modal properties and equipment response of equipment-structure systems have been studied by many investigators. To our knowledge, in most of these studies the results derived for the equipment response are based on modeling the structure as a discrete lumped mass subsystem.[2,4,7,8] In many practical instances, it might be necessary to use a continuous model of the structure, especialy if the modal properties of the continuous model are readily available. Reference [6] studied modal properties of an equipment-continuous structure system by the Rayleigh quotient functional method. The secondary subsystem considered in that study was limited to be a single one-degree-of-freedom oscillator. The equipment response caused by dynamic excitation of the structure was not studied.

In this paper, a systematic and efficient approach is presented for estimating the modal properties and equipment response of an equipment-continuous structure system, modeled as several concentrated masses attached to a simply supported beam which is subjected to a transverse impact force. The procedure includes the effect of interaction between the equipment and the structure, which is particularly significant in the case of tuning. After setting up the equation of motion of the combined system, a matrix perturbation method is employed to derive the modal properties of the combined system in terms of that of the individual subsystems. Once this has been done, a modal analysis approach is applied to determine the response of the equipment.

In the case of tuning or nearly tuning, the two tuned modes are still coupled after modal transformation. To obtain their contribution to the equipment response, it is found convenient to use the Laplace transform.. An estimate of the maximum response of the equipment items, which turns out to be a quite simple expression, convenient for engineering practice, is obtained in terms of the response spectrum describing the transverse impact on the structure. A numerical example is calculated for both the modal properties and the equipment response, and a comparision is made between the proposed method and results obtained by a direct numerical integration of the governing differential equations of motion using the CAL computer program [9] with the

Newmark integration method. The comparision shows good agreement between the two results.

Analysis

1. Equation of motion of the combined system

Consider an equipment-continuous structure system, modeled as a simply supported uniform beam to which *m* concentrated masses are attached. If we let

$$y(x,t) = \sum_{i=1}^{n} \varphi_i(x)U_i(t) \quad \text{and} \quad u_r(t)$$

be the displacements of the beam and the equipment, respectively, where the $\varphi_i(x)$ are modal functions of the beam and the $U_i(t)$ are the generalized coordinates of the beam, the Lagrangian of the combined system is then given by

$$L = \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}\dot{U}_i(t)\dot{U}_j(t) + \frac{1}{2}\sum_{r=1}^{m} m_r\dot{u}_r^2(t) - \frac{1}{2}\sum_{i,j=1}^{n} K_{ij}U_i(t)U_j(t)$$
$$- \frac{1}{2}\sum_{r=1}^{m} k_r[y(x_r,t) - u_r(t)]^2 \qquad (1.2)$$

where

$$M_{ij} = \int_0^L \rho a\varphi_i(x)\varphi_j(x)dx \qquad K_{ij} = \int_0^L EI\frac{d^2\varphi_i(x)}{dx^2}\frac{d^2\varphi_j(x)}{dx^2}dx$$

In the above, ρa is the linear mass density of the beam, EI is the beam stiffness and x_r is the location of the attachment point to the beam of the r^{th} equipment item.

The generalized forces related to the loading and the damping are, respectively,

$$F_i = p(t)\varphi_i(x_a) - \sum_{r=1}^{m} c_r\varphi_i(x_r) \sum_{j=1}^{n} [\varphi_j(x_r)U_j(t) - u_r(t)] \tag{1.3}$$

$$f_r = -c_r[u_r(t) - \sum_{i=1}^{n} \varphi_i(x_r)U_i(t) \tag{1.4}$$

The beam is taken to be undamped, c_r represents the damping coefficient of the r^{th} equipment item, and $p(t)$ is the transverse impact force applied at the location x_a on the beam.

Substitution of expressions (1.2) and (1.3) into Lagrange's equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_s}\right] - \frac{\partial L}{\partial q_s} = F_s \tag{1.5}$$

results in the differential equation of motion of the combined system:

$$\bar{M}\ddot{Y} + \bar{C}\dot{Y} + \bar{K}Y = P \tag{1.6}$$

Here

$$\bar{M} = \begin{bmatrix} m_r\delta_{rk} & \\ & I \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} c_r\delta_{rk} & [-c_r\varphi_i(x_r)]^T \\ -c_r\varphi_i(x_r) & \sum_{r=1}^{m} c_r\varphi_i(x_r)\varphi_j(x_r) \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} k_r\delta_{rk} & [-k_r\varphi_i(x_r)]^T \\ -k_r\varphi_i(x_r) & \Omega_i^2\delta_{ij} + \sum_{r=1}^{m} k_r\varphi_i(x_r)\varphi_j(x_r) \end{bmatrix}$$

$$Y = (u_r, U_i)^T \qquad\qquad P = (0, p\varphi_i(x_p))^T$$

where δ_{rk} is the Kronecker delta. The superscript T denotes matrix transposition.

Classical matrix perturbation theory, which is the main method used in this paper to determine the modal properties of the combined system, has proved to be a powerful and convenient tool for the analysis of equipment-structure systems. Since the procedures for utilizing this method in our analysis is straightforward, only the main results are given in this paper, without details. The interested reader can find the basic principles of classical matrix perturbation theory elucidated in reference [1].

The eigenvalue problem associated with equation (1.6) is given by

$$-\lambda\bar{M}\psi + \bar{K}\psi = 0 \tag{1.7}$$

Using the linear transformation

$$\psi = \bar{T}\psi' \tag{1.8}$$

and premultiplying equation (1.7) with $\bar{T}^T$, the eqation (1.7) takes the new form suitable for using matix perturbation

$$\lambda\psi' = \bar{E}\psi' \tag{1.9}$$

where

$$\bar{T} = \begin{bmatrix} T_e & 0 \\ 0 & T_s \end{bmatrix}$$

in which T_e is the normalized modal matrix of the fixed base equipment and T_s is an nxn unit matrix.

and

$$\bar{E} = \bar{T}^T\bar{K}\bar{T} = \begin{bmatrix} \omega_r^2\delta_{rk} & \left[-\omega_r^2\gamma_{ri}^{\frac{1}{2}}\right]^T \\ -\omega_r^2\gamma_r^{\frac{1}{2}} & \Omega_i^2\delta_{ij}+\sum_{r=1}^{m}\omega_r^2\gamma_{ri}^{\frac{1}{2}}\gamma_{rj}^{\frac{1}{2}} \end{bmatrix}$$

in which $\gamma_{pq} = 2\dfrac{m_p}{\rho aL}\sin^2\dfrac{q\pi x_p}{L}$ is the effective mass ratio, and ω_r is the undamped natural frequency of the fixed-base r^{th} equipment item.

Attention is now restricted to the case where the mass of each equipment item (m_p, $p=1,2,\ldots,m$) is small compared to the mass of the beam (ρaL), so that all of the $\gamma_{st}^{\frac{1}{2}}$ terms appearing in $\bar{E}$ will be small compared to 1. Assuming that all of the $\gamma_{st}^{\frac{1}{2}}$ are of the same order, namely, $\gamma^{\frac{1}{2}} << 1$, these can be used as the perturbation parameter in the analysis. Classical matrix perturbation establishes that the second order terms in $\bar{E}$ can be neglected in the computation for the eigenvalues λ_i and the associated eigenvectors ψ_i'. Thus, equation (1.9) can be rewritten as

$$\lambda\psi' = [H + W]\psi' \tag{1.10}$$

where

$$H = \begin{bmatrix} \omega_r^2\delta_{rk} & 0 \\ 0 & \Omega_i^2\delta_{ij} \end{bmatrix} \tag{1.11}$$

and

$$W = \begin{bmatrix} 0 & (-\omega_r^2\gamma_{ri}^{\frac{1}{2}})^T \\ -\omega_r^2\gamma_{ri}^{\frac{1}{2}} & 0 \end{bmatrix} \tag{1.12}$$

2. Completely detuned case

For the completely detuned case, none of the ω_i are equal, or nearly equal to the Ω_j. Classical matrix perturbation then yields the following results for the eigenvalues and eigenvectors.

2.1 Modal properties

The zeroth order problem of equation (1.10) gives

$$\lambda_k^{(0)} = \omega_k^2 \qquad k=1,2,\ldots,m$$

$$\lambda_{m+i}^{(0)} = \Omega_i^2 \qquad i=1,2,\ldots,n \tag{2.1.1}$$

and

$$\psi_j^{(0)} = e_j \qquad j=1,2,\ldots,m+n \tag{2.1.2}$$

where e_j is a $(m+n)\times 1$ vector whose i^{th} component is one and zero otherwise.

It can be easily verified that $\lambda^{(1)}$, which is the first correction of λ, is equal to zero. Thus the eigenvalues to the first order are given by (2.1.1) and (2.1.2). In other words, the natural frequencies of the combined system can be approximated by the frequencies of the individual subsystems. However, there are certain corrections to the eigenvectors of the combined system which are

$$\psi'^{(1)}_k = \sum_{j=1}^{m+n} c_{kj}\psi'^{(0)}_j = \begin{bmatrix} \bar{0} \\ A_k \end{bmatrix}$$

$$\psi'^{(1)}_{m+s} = \sum_{j=1}^{m+n} c_{m+s,j}\psi'^{(0)}_j = \begin{bmatrix} B_s \\ \hat{0} \end{bmatrix} \tag{2.1.3}$$

where A_k is a $n\times 1$ vector with components $\dfrac{\mu_{k,i}}{\omega_k^2 - \Omega_i^2}$ and B_s is a $m\times 1$ vector with components $\dfrac{\mu_{r,s}}{\omega_r^2 - \Omega_i^2}$. where $\mu_{i,j} = -\omega_i^2\gamma_{i,j}^{\frac{1}{2}}$. The $\bar{0}$ and $\hat{0}$ are m×1 and n×1 zero vectors, respectively. Thus, the eigenvectors are to first order given by

$$\psi'_k = \psi'^{(0)}_k + \psi'^{(1)}_k = \begin{bmatrix} e_k \\ A_k \end{bmatrix}$$

$$\psi'_{m+s} = \psi'^{(0)}_{m+s} + \psi'^{(1)}_{m+s} = \begin{bmatrix} B_s \\ e_s \end{bmatrix} \tag{2.1.4}$$

Putting expressions (2.1.3) and (2.1.4) together and using the transformation (1.8) result in the modal matix of the combined system, which is normalized with respect to the mass

$$\psi = \bar{T}\psi' = \begin{bmatrix} m_r^{-\frac{1}{2}}\delta_{rk} & \dfrac{m_r^{-\frac{1}{2}}\mu_{rj}}{\Omega_j^2 - \omega_r^2} \\ \dfrac{\mu_{ik}}{\omega_k^2 - \Omega_i^2} & I \end{bmatrix} \tag{2.1.5}$$

2.2 Response of the equipment

When the frequencies of the combined system are well spaced, the model can be uncoupled by the modal analysis method directly. Insert the transformation $Y = \psi X$ into equation (1.6) and pre-multiply the resulting equation by ψ^T. We are led to m+n uncoupled equations on the modal coordinates X (to first order). That is,

$$\ddot{x}_r + 2\xi_r\omega_r\dot{x}_r + \omega_r^2 x_r = D_r p(t) \tag{2.2.1}$$

$$\ddot{X}_{m+i} + \Omega_{m+i}^2 X_{m+i} = \varphi_i(x_p)p(t)$$

where $$D_r = \sum_{i=1}^{n} \frac{\varphi_i(x_p)\mu_{ir}}{\omega_r^2 - \Omega_j^2}$$

The solution to equation (2.2.1) can be obtained from ordinary differential equation theory as

$$x_r(t) = \omega_r^{-1} \int_0^t D_r p(\tau) e^{-\xi\omega_r(t-\tau)} \sin\omega_r(t-\tau) d\tau \tag{2.2.2}$$

$$X_{m+i}(t) = \Omega_i^{-1} \int_0^t \varphi_i(x_p) p(\tau) \sin\Omega_i(t-\tau) d\tau \tag{2.2.3}$$

Transforming the response into real space by using $Y = \psi X$, we obtain the displacement response of the equipment as

$$Y_r = \left\{ \sum_{i=1}^{n} \frac{m_r^{-\frac{1}{2}}\varphi_i(x_p)\mu_{ir}}{(\omega_r^2 - \Omega_i^2)\omega_r} e^{-\xi_r\omega_r t}\sin\omega_r t + \sum_{i=1}^{n} \frac{m_r^{-\frac{1}{2}}\varphi_i(x_p)\mu_{ir}}{(\Omega_i^2 - \omega_r^2)\Omega_i}\sin\Omega_i t \right\} * p(t) \tag{2.2.4}$$

where

$$F(t)*p(t) = \int_0^t F(t-\tau)p(\tau)d\tau$$

3. Tuned and nearly tuned case

A particular difficulty arises in the modal analysis of the combined system if closely spaced modes exist. This is due to the existence of a degeneracy in the unperturbed system, which means that certain (or all) eigenvalues are associated with more than one eigenvector. By the so-called subspace method [1], the degeneracy could be broken and the modified eigenvectors could be obtained to construct the modal matrix which is available for uncoupling all the well spaced modes to finally get the equipment responses. An alternate approach, however, is needed to get the tuned equipment response because the modal matrix fails to uncouple the tuned modes. In the proceding analysis, a two-fold degenerate problem is studied. the two tuned modes are set to be ω_k and Ω_l. First, the modal matrix is obtained by a subspace method, which can be applied to get all the responses of the well spaced modes. Then the Laplace transform is employed to determine the response of the tuned modes.

3.1 modal properties

The eigenvalues for the tuned case (to first order) are

$$\lambda_r = \lambda_r^{(0)} + \lambda_r^{(1)} = \omega_r^2 \qquad r=1,2,\dots,m \quad r \neq k \tag{3.1.1}$$

$$\lambda_{m+i} = \lambda_{m+i}^{(0)} + \lambda_{m+i}^{(1)} = \Omega_i^2 \qquad i=1,2,\dots,n \quad i \neq l \tag{3.1.2}$$

and

$$\lambda_k = \lambda_k^{(0)} + \lambda_k^{(1)} = \Omega^2[1 + (\gamma_{kl} + \alpha_{kl}^2)^{\frac{1}{2}}] \tag{3.1.3}$$

$$\lambda_l = \lambda_l^{(0)} + \lambda_l^{(1)} = \Omega^2[1 - (\gamma_{kl} + \alpha_{kl}^2)^{\frac{1}{2}}] \tag{3.1.4}$$

where $\Omega = (\omega_k + \Omega_l)/2$ is the average frequency and $\alpha_{kl} = (\omega_k - \Omega_l)/\Omega$ is the detuning parameter.The eigenvectors are

$$\psi_r = \begin{bmatrix} e_r \\ a_r \end{bmatrix}, \qquad \psi_{m+i} = \begin{bmatrix} B_i \\ e_i \end{bmatrix} \qquad r\neq k,\ i\neq l \tag{3.1.5}$$

and

$$\begin{aligned} \psi_k &= \psi_k^{(0)} + \psi_k^{(1)} = (\psi_1^k,\dots,\psi_m^k,\psi_{m+1}^k,\dots,\ \psi_{m+n}^k)^T \\ \psi_l &= \psi_l^{(0)} + \psi_l^{(1)} = (\psi_1^l,\dots,\psi_m^l,\psi_{m+1}^l,\dots,\ \psi_{m+n}^l)^T \end{aligned} \tag{3.1.6}$$

where

$$\psi_r^k = \frac{m_k^{-\frac{1}{2}}\omega_r^2\gamma_{rl}^{\frac{1}{2}}\zeta}{\Omega^2 - \omega_r^2}, \qquad \psi_{m+i}^k = \frac{\Omega^2\gamma_{kl}^{\frac{1}{2}}}{\Omega^2 - \Omega_i^2}$$

$$\psi_r^l = \frac{m_k^{-\frac{1}{2}}\omega_r^2\gamma_{rl}^{\frac{1}{2}}}{\Omega^2 - \omega_r^2} \qquad \psi_{m+i}^l = \frac{\Omega^2\gamma_{rl}^{\frac{1}{2}}\zeta}{\Omega_i^2 - \Omega^2} \qquad r\neq k, i\neq l \tag{3.1.7}$$

When r=k and i=l, the elements are

$$\psi_k^k = m_k^{-\frac{1}{2}} \qquad \psi_l^k = \zeta$$

$$\psi_l^k = -\zeta m_k^{-\frac{1}{2}} \qquad \psi_l^l = 1 \tag{3.1.8}$$

where

$$\zeta = [(\alpha_{kl}^k + \gamma_{kl})^{\frac{1}{2}} - \alpha\ ]/\gamma_{kl}^{\frac{1}{2}} \tag{3.1.9}$$

3.2 Response of equipment

Because of the tuning effect between modes k and l, the modal matrix ψ fails to diagonalize the system as occured in the completely untuned case . Although we could uncouple all the detuned modes by applying the transformation $Y = \psi X$, and obtain their response by modal superposition, further work is needed to obtain the response of the tuned modes. This can be done by several alternate approaches. One is to use the Laplace transform method. Applying the transformation $Y = \psi X$ to equation (1.6) and pre-multipling by ψ^T result in a set of uncoupled equations

$$\ddot{X}_r + 2\xi_r\omega_r\dot{X}_r + \omega_r^2X_r = D_r p(t) \qquad r \neq k \tag{3.2.1}$$

$$\ddot{X}_{m+i} + \Omega^2_{m+i}X_{m+i} = \varphi_{m+i}(x_p)p(t) \qquad i \neq l \tag{3.2.2}$$

and two coupled equations

$$\underline{m}\ddot{X} + \underline{c}\dot{X} + \underline{k}X = \underline{p} \tag{3.2.3}$$

where

$$\underline{m} = (1+\zeta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{c} = 2\xi\Omega\begin{bmatrix} 1 & \zeta \\ \zeta & \zeta^2 \end{bmatrix} \tag{3.2.4}$$

$$\underline{k} = \Omega\begin{bmatrix} (1+\zeta)+2\zeta\gamma_{kl}^{\frac{1}{2}}+\alpha_{kl}(1-\zeta^2) & (\zeta^2-1)\gamma_{kl}^{\frac{1}{2}}+2\alpha_{kl}\zeta \\ (\zeta^2-1)\gamma_{kl}^{\frac{1}{2}}+2\alpha_{kl}\zeta & (1+\zeta)-2\zeta\gamma_{kl}^{\frac{1}{2}}-\alpha_{kl}(1-\zeta^2) \end{bmatrix} \tag{3.2.5}$$

and $\underline{p} = p(t)[a_k, a_l]^T$ with

$$a_k = \varphi_l(x_p)\zeta + \sum_{\substack{i=1 \\ i\neq l}}^{n} \frac{\varphi_i(x_p)\Omega^2\gamma_{kl}^{\frac{1}{2}}}{\Omega^2 - \Omega_i^2}$$

$$a_l = \varphi_l(x_p) + \sum_{\substack{i=1 \\ i\neq l}}^{n} \frac{\varphi_i(x_p)\Omega^2\gamma_{kl}^{\frac{1}{2}}\zeta}{\Omega_i^2 - \Omega^2}$$

In order to get the response of the tuned modes, we use the Laplace transformation and residue theory to solve equation (3.2.3). The transformed equation is

$$\begin{aligned} (s^2m_{11} + sc_{11} + k_{11})\bar{X}_1 + (sc_{12} + k_{12})\bar{X}_2 &= a_k\bar{p} \\ (sc_{21} + k_{21})\bar{X}_1 + (s^2m_{22} + sc_{22} + k_{22})\bar{X}_2 &= a_l\bar{p} \end{aligned} \tag{3.2.6}$$

where s is Laplace transform parameter. The solution of equation (3.2.6), by Cramer's rule , is

$$\bar{X}_1(s) = \frac{\Delta_1}{\Delta}\bar{p} \qquad\qquad \bar{X}_2(s) = \frac{\Delta_2}{\Delta}\bar{p} \tag{3.2.7}$$

where Δ, Δ_1, Δ_2 are the usual determinants employed in Cramer's rule. The inversion of expression (3.2.7) directly gives the displacement response of the two tuned modes in the transformed space. That is

$$X_j(t) = \frac{1}{2\pi i}\int_\Gamma \frac{\Delta_j}{\Delta}\bar{p}e^{st}ds \tag{3.2.8}$$

If $\bar{p}(s)$ is taken to be 1, then (3.2.8) yields Green's function X_G for the solution. Thus (3.2.8) can be rewritten as

$$X_j(t) = \int_0^t X_G(t-\tau)\bar{p}(\tau)d\tau \tag{3.2.9}$$

By using residue theory, the Green's function can be obtained as follows (to first order)

$$\begin{aligned}
X_G^k = {} & \frac{\varphi_1(x_p)e^{-\frac{\xi}{2}\Omega t}}{\Omega(1+\zeta^2)(\rho^2+\theta^2)}\Big[-e_{k1}\sinh\frac{\rho\Omega t}{2}\cos\Omega t\cos\frac{\theta\Omega t}{2} - \\
& - e_{k1}\cosh\frac{\rho\Omega t}{2}\sin\Omega t\sin\frac{\theta\Omega t}{2} + \\
& + (e_{k2}\sinh\frac{\rho\Omega t}{2} - e_{k3}\cosh\frac{\rho\Omega t}{2})\sin\Omega t\cos\frac{\theta\Omega t}{2} - \\
& - (e_{k2}\cosh\frac{\rho\Omega t}{2} - e_{k3}\sinh\frac{\rho\Omega t}{2})\cos\Omega t\sin\frac{\theta\Omega t}{2}\Big]
\end{aligned} \tag{3.2.10}$$

$$\begin{aligned}
X_G^l = {} & \frac{\varphi_1(x_p)e^{-\frac{\xi}{2}\Omega t}}{\Omega(1+\zeta^2)(\rho^2+\theta^2)}\Big[-e_{l1}\sinh\frac{\rho\Omega t}{2}\cos\Omega t\cos\frac{\theta\Omega t}{2} - \\
& e_{l1}\cosh\frac{\rho\Omega t}{2}\sin\Omega t\sin\frac{\theta\Omega t}{2} + \\
& +(e_{l2}\sinh\frac{\rho\Omega t}{2} - e_{l3}\cosh\frac{\rho\Omega t}{2})\sin\Omega t\cos\frac{\theta\Omega t}{2} -
\end{aligned}$$

$$-(e_{12}\cosh\frac{\rho\Omega t}{2} - e_{13}\sinh\frac{\rho\Omega t}{2})\cos\Omega t\sin\frac{\theta\Omega t}{2}] \qquad (3.2.11)$$

where

$$e_{k1} = \rho\gamma_{kl}^{\frac{1}{2}} + \zeta(\theta\xi - \alpha\rho) \qquad e_{11} = \rho(\alpha + \zeta\gamma_{kl}^{\frac{1}{2}}) - \theta\xi$$

$$e_{k2} = \theta\gamma_{kl}^{\frac{1}{2}} - \zeta(\rho\xi + \alpha\theta) \qquad e_{12} = \rho\xi + \theta(\alpha + \zeta\gamma_{kl}^{\frac{1}{2}})$$

$$e_{k3} = \zeta(\rho^2 + \theta^2) \qquad e_{13} = -(\rho^2 + \theta^2)$$

and

$$\theta = \frac{1}{\sqrt{2}}\left\{[(\gamma_{kl} - \xi^2 + \alpha^2)^2 + \xi^2\alpha^2]^{\frac{1}{2}} + (\gamma_{kl} - \xi^2 + \alpha^2)\right\}^{\frac{1}{2}} \qquad (3.2.12)$$

$$\rho = \frac{1}{\sqrt{2}}\left\{[(\gamma_{kl} - \xi^2 + \alpha^2)^2 + \xi^2\alpha^2]^{\frac{1}{2}} - (\gamma_{kl} - \xi^2 + \alpha^2)\right\}^{\frac{1}{2}} \qquad (3.2.13)$$

Expression (3.2.9) is the response of the two tuned modes. The response of all the other detuned modes are similar to those given by expressions (2.2.3) and (2.2.4). It follows that the responses of the equipment items of the combined system in real space are, by taking the transformation $Y=\psi X$, given by

$$Y_k = p(t)*\left\{\frac{\varphi_1(x_k)}{\gamma_{kl}^{\frac{1}{2}}}\left[X_G^k(t) + \zeta X_G^l(t)\right] + \right.$$

$$\left.\sum_{\substack{i=1\\i\neq l}}^{n}\frac{\varphi_1(x_p)\varphi_1(x_k)\Omega^2}{(\Omega^2 - \Omega_i^2)\Omega_i}\sin\Omega_i t + \sum_{\substack{i=1\\i\neq l}}^{n}\frac{\varphi_1(x_p)\varphi_1(x_k)\Omega}{(\Omega_i^2 - \Omega^2)}e^{-\frac{\xi}{2}\Omega t}\sin\Omega t\right\} \qquad (3.2.13)$$

$$Y_r = p(t)*\left\{\frac{\varphi_1(x_r)\omega_r^2}{\Omega^2 - \omega_r^2}\left[\zeta X_G^k(t) - X_G^l(t)\right] + \right.$$

$$\left.\sum_{\substack{i=1\\i\neq l}}^{n}\frac{\varphi_1(x_p)\varphi_1(x_r)\omega_r^2}{(\omega_r^2 - \Omega_i^2)\Omega_i}\sin\Omega_i t + \sum_{\substack{i=1\\i\neq l}}^{n}\frac{\varphi_1(x_p)\varphi_1(x_r)\omega_r^2}{(\Omega_i^2 - \omega_r^2)}e^{-\frac{\xi}{2}\omega_r t}\sin\omega_r t\right\} \qquad (3.2.14)$$

We can see from expression (3.2.13) that the first term in the braces is dominant because the second and the third, which are the contribution from the nontuning modes and nondominant terms of the tuned modes, are higher order in comparision with the first one.

4. Analysis of response spectrum

The results obtained in the previous sections can be employed for equipment response analysis in the time domain if a specified time history were available. In most cases, however, the quantity of interest is the maximum response of the equipment not the response as a function of time. Because the design spectrum usually is readily available, the following section will focus on the procedures to derive an estimate of the maximum response of the equipment in terms of the design response spectrum. The maximum response, for the completely detuned case, can be expressed by the conventional square root of the sum of the square procedure in the form

$$Y_{max} = \left\{ \sum_{i=1}^{n} \left[\frac{\varphi_1(x_p)\varphi_1(x_r)}{\omega_r^2 - \Omega_i^2}\omega_r^2 S_D(\omega_r, \xi_r)\right]^2 + \right.$$

$$\left. + \sum_{i=1}^{n} \left[\frac{\varphi_1(x_p)\varphi_1(x_r)}{\Omega_i^2 - \omega_r^2}\omega_r^2 S_D(\Omega_i, 0)\right]^2 \right\}^{\frac{1}{2}} \tag{4.1.1}$$

where $S_D(\omega, \xi)$ is the given design response spectrum evaluated at frequency ω and damping ξ. For the tuned or nearly tuned case, the maximum response of the equipment given in expression (3.2.13) consists of two parts. One is dominant and takes a fairly long time to attain its maximum value because it is governed by energy transfer from the structure to equipment through beating. The other one, representing the nondominant contribution, is conventional and would achieve its peak during the excitation or shortly thereafter. In the perfectly tuned case, the nondominant terms are much smaller than the dominant one and could be neglected. Thus expression (3.2.13) can then be simplified as

$$Y_k \approx p(t)*\left[\frac{\varphi_1(x_k)}{\gamma_{k1}^{\frac{1}{2}}}(X_G^k(t) + \zeta X_G^1(t)\right]$$

$$= B\int_0^t e^{-\xi(t-\tau)/2}\Big[k_1\sinh\mu(t-\tau)\cos\Omega(t-\tau)\cos\lambda(t-\tau) +$$

$$+ k_1\cosh\mu(t-\tau)\sin\Omega(t-\tau)\sin\lambda(t-\tau) +$$

$$+ k_2\sinh\mu(t-\tau)\sin\Omega(t-\tau)\cos\lambda(t-\tau) -$$

$$- k_2\cosh\mu(t-\tau)\cos\Omega(t-\tau)\sin\lambda(t-\tau)\Big]p(\tau)d\tau \tag{4.1.2}$$

where here $\mu = \xi\Omega/2$ and $\lambda = \theta\Omega/2$ and

$$B = \frac{\varphi_1(x_p)\varphi_1(x_k)}{\Omega\gamma_{k1}^{\frac{1}{2}}(1 + \zeta^2)(\rho^2 + \theta^2)}$$

$$k_1 = -\rho\gamma_{k1}^{\frac{1}{2}}(1 + \zeta^2) \qquad\qquad k_2 = \theta\gamma_{k1}^{\frac{1}{2}}(1 + \zeta^2)$$

Following a procedures similar to that presented in [7], expression (4.1.2) can further simplified as

$$Y_k \approx \frac{\varphi_1(x_p)\varphi_1(x_k)}{\Omega(\rho^2 + \theta^2)}\int_0^t e^{-\Omega\xi(t-\tau)/2}\Big[-\rho\sinh\mu t\cos\lambda t\cos\Omega(t-\tau) -$$

$$- \rho\cosh\mu t\sin\lambda t\sin\Omega(t-\tau) + \theta\sinh\mu t\cos\lambda t\sin\Omega(t-\tau) -$$

$$- \theta\cosh\mu t\sin\lambda t\cos\Omega(t-\tau)\Big]p(\tau)d\tau \tag{4.1.3}$$

Because we are trying to find the maximum value of equipment response, it could be concluded that the response should achieve its peak within time T which is the beat period of a tuned or nearly tuned system. Suppose t' is the time when the response of the equipment attains its peak, than we have $t' < T = 2\pi/\lambda$ and $\mu t' = 2\pi\mu/\lambda = 2\pi\rho/\theta$. If we consider the special cases $\xi=0$, or $\alpha=0$, then $\rho/\theta = 0$. If $\alpha\xi << \gamma-\xi^2+\alpha^2$ then it is easy to see that ρ/θ is of higher order, i.e., $\rho/\theta << 1$. This leads us to anticipate that in general, for parameter ranges of interest, $\rho/\theta << 1$. This is illustrated in Figs. 2a and 2b where ρ/θ is shown for a range of values of the parameters α, ξ and γ. (For example, ρ/θ is less than .04 when $\gamma = 0.01$ and $\xi = 0.01$) Thus, expression (4.1.3) can be approximated as

$$Y_k \approx \frac{-\varphi_1(x_p)\varphi_1(x_k)}{\Omega\theta} \sin\lambda t \int_0^t e^{-\Omega\xi(t-\tau)/2} \cos\Omega(t-\tau)p(\tau)d\tau \tag{4.1.4}$$

To express (4.1.4) in terms of the response spectrum of the excitation, the reader is referred to [7] in which the derivation is given in detail.

The estimate of the maximum displacement of the equipment is

$$\left| Y_k \right|_{max} = \frac{e^{-\Gamma}\varphi_1(x_p)\varphi_1(x_k)}{\Omega(\theta^2 + \xi^2)^{\frac{1}{2}}} S_{rv}(\Omega,\xi/2) \tag{4.1.5}$$

where

$$\Gamma = \mathrm{arctg}(\lambda/\mu)/(\lambda/\mu) = \mathrm{arctg}(\theta/\xi)/(\theta/\xi)$$

and $S_v(\Omega,\xi/2)$ is the relative velocity response spectrum for a lightly damped single-degree-freedom oscillator of frequency Ω and damping factor $\xi/2$ subjected to the impact force p(t). The derivation gives the result naturally in the form of the relative velocity spectrum, but design information is generally provided in the form of a pseudo-velocity spectrum. However, the pseudo-velocity response spectrum is nearly equal to the relative velocity spectrum for systems with moderate or high frequencies and differs only for very low-frequency systems. Thus, for most cases, S_{rv} in Eq.(4.1.5) can be replaced by S_v, the pseudo-velocity response spectrum. Recalling that

$$\Omega^2 S_d = \Omega S_v = S_a \tag{4.1.6}$$

we have

$$\left| Y_k \right|_{max} = \frac{\varphi_1(x_p)\varphi_1(x_k)e^{-\Gamma}}{(\theta^2 + \xi^2)^{\frac{1}{2}}} S_d(\Omega, \xi/2) \tag{4.1.7}$$

5. Numerical example

The results obtained in the previous section are applicable to a system with m equipment items. For simplicity and with no loss of generality, however, a system with a single equipment item is studied as a numerical example to test the effectiveness of the results developed in this paper.

5.1 Modal properties

The modal frequencies and mode shapes are first computed. Two examples are considered, one with the equipment on the beam at $x_1 = L/4$ and the other with equipment at $x_2 = L/2$. The effective mass ratios are set to be $\gamma = 0.001$, 0.01 and 0.1, respectively. For each of these cases, results are obtained for the equipment tuned to the first mode of the beam (Table 1), and then for tuning to the third mode (Table 2). The modal properties calculated by expressions (3.1.1) - (3.1.8) are compared with the results obtained by the CAL program which represents an 'exact' result. To describe the error in the

i^{th} mode shape, a difference function $e_i = \varphi_i^{cal} - \varphi_i^{apx.}$ [8] is introduced where φ_i^{cal} is the exact mode shape normalized with respect to the mass of the combined system and φ_i^{apx} is the approximate mode shape computed according to the approximate formula developed in this paper. Then the % error in the mode shape is calculated as $e_i^T M e_i$. It can be seen from Table 1 and Table 2 that the frequencies by the approximate formula are quite accurate in comparision with the exact one. Even for the tuned modes there is only a slight error. We can see that the maximum error when the equipment is tuned to the first mode of the beam does not exceed about 3% . The error in the mode shape, just as is true for the error in the frequency, is concentrated at the tuned modes. An interesting point to observe is that the error in the mode shape is essentially independant of the mass ratio for the range of parameters considered in this study.

5.2 Response of equipment

The maximum response of the equipment for a range of the detuning parameter was computed by both the formula developed in this paper and by the use of the CAL program (employing Newmark step-by-step integration). The beam used in this example had the properties, $EI = 49.59 MNm^2$ (1.728×10^{10} lb.inch^2), $\rho A = 2.571\times10^4$kg/m (0.3729 lb.sec.^{2}/inch^2), L = 3.658m (120 inch), resulting in a fundamental period $T_f = 0.0426$ sec. The equipment was located at $x_e = L/4$, and the mass ratios $\gamma = 0.001$, 0.01 and 0.1 were used. Damping was set at $\xi = 0.001$, and the impact force history was taken to be a rectangular pulse of duration 0.05 seconds applied at $x_p = 3L/4$. From Figs. 3a,b and c, we see good agreement between the exact and approximate results for $\gamma = 0.001$. Eq.(4.1.7) was used for the tuned case and Eq.(4.1.1) was used for the detuned cases. It is seen that as the mass ratio increases, the error increases, especially when the equipment is nearly tuned. But for the case of perfect tuning, which is the most important one, the error is still within acceptable accuracy. Another interesting fact is that all of the maximum responses estimated by the response spectrum method are seem to be upper bounds on that obtained by the Newmark integration algorithm. Fig.4 shows the displacement response history (computed from Eq.(3.2.13)) for a system with $\alpha = 0$, $\xi = 0.05$ and $\gamma = 0.01$. It is seen that the response history represents a damped beating phenomenon with a beat period of approximately 0.5 seconds.

Conclusion

In this paper, we present an approximate method for determining the response of an equipment-continuous structure system. The modal properties of the combined system are obtained based on the modal properties of the individual subsystems by the use of classical matrix perturbation theory. By employing a modal analysis approach and the Laplace transform, the response of the equipment is obtained. An estimate of the maximum response of the equipment is presented in terms of the response spectrum of the excitation. A numerical example shows that the modal properties of the combined system are in close agreement with the exact results. The maximum of the transient as estimated by the results of this paper also show good agreement with exact results, when the mass ratio is small. As the mass ratio increases, the accuracy of the estimation deteriorates, as is to be expected.

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Appendix 1

Notation

A_k n×1 vector (see (2.1.4))

B_s m×1 vector (see (2.1.4))

c_r damping of equipment

$\bar{C}$ damping matrix of combined system

D_r factor (see (2.2.1))

e_i (m+n)×1 vector

$\bar{E}$ matrix (see (1.9))

H diagonal matrix of fixed base natural frequencies

k_r stiffness of equipment

K_{ij} generalized stiffness of beam

$\bar{K}$ stiffness matrix of combined system

m_r mass of equipment

M_{ij} generalized mass of combined system

$\bar{M}$ mass matrix of combined system

P(t) impact force

S_D response spectrum for impact excitation

$\bar{T}$ transformatiom matrix (see (1.8))

u_r displacement of equipment

U_i displacement of beam

x_r displacement of equipment in transformed space

X_G^1 Green's function

Y_r displacement of equipment in real space

α detuning parameter

γ mass ratio

δ_{ij} Kronecker delta

Δ determinant

ζ factor (see (3.1.9))

θ factor (see (3.2.12))

φ_i mode shape of beam

ψ_i eigenvector of combined system in real space

ψ_i' eigenvector of combined system in transformed space

ω_r natural frequency of equipment

Ω_i natural frequency of beam

	mode	concentrated mass attached at x=l/2 exac fre. (rad./s)	error in fre. (%)	error in mode shape(%)	concentrated mass attached at x=l/4 exact fre. (rad./s)	error in fre. (%)	error in mode shape(%)
	0	9.715	-0.0	8.89	9.714	-0.0	17.32
	1	10.027	-0.03	8.27	10.027	-0.03	17.31
	2	39.478	0.0	0.0	39.481	-0.0	0.0
	3	88.827	0.0	0.0	88.827	-0.0	0.0
.001	4	157.914	0.0	0.0	157.914	0.0	0.0
	5	246.740	0.0	0.0	246.740	0.0	0.0
	6	355.306	0.0	0.0	355.306	0.0	0.0
	7	483.611	0.0	0.0	483.611	0.0	0.0
	8	631.655	0.0	0.0	631.655	0.0	0.0
	9	799.438	0.0	0.0	799.438	0.0	0.0
	0	9.388	-0.27	9.73	9.385	-0.23	17.28
	1	10.375	-0.23	7.65	10.371	-0.19	17.28
	2	39.478	0.0	0.0	39.478	-0.07	0.0
	3	88.832	-0.0	0.0	88.832	-0.0	0.0
.01	4	157.914	0.0	0.0	157.914	0.0	0.0
	5	246.742	-0.0	0.0	246.742	-0.0	0.0
	6	355.305	0.0	0.0	355.309	-0.0	0.0
	7	483.611	0.0	0.0	483.612	-0.0	0.0
	8	631.655	0.0	0.0	631.655	0.0	0.0
	9	799.438	0.0	0.0	799.438	0.0	0.0
	0	8.429	-3.17	12.92	8.405	-2.90	18.38
	1	11.548	-1.95	6.40	11.503	-1.56	18.33
	2	39.478	0.0	0.0	39.742	-0.06	2.12
	3	88.881	-0.07	0.0	88.882	-0.0	0.20
.1	4	157.914	0.0	0.0	157.914	0.0	0.0
	5	246.760	-0.0	0.0	246.760	-0.0	0.0
	6	355.305	0.0	0.0	355.333	-0.0	0.0
	7	483.621	-0.0	0.0	483.621	-0.0	0.0
	8	631.655	0.0	0.0	631.655	0.0	0.0
	9	799.444	-0.0	0.0	799.444	-0.0	0.0

Table 1. Comparision of natural frequencies and mode shapes of combined system with exact values (mass tuned to first mode of beam)

	mode	concentrated mass attached at x=1/2 exac fre. (rad./s)	error in fre. (%)	error in mode shape(%)	concentrated mass attached at x=1/4 exact fre. (rad./s)	error in fre. (%)	error in mode shape(%)
	1	9.865	-0.0	0.02	9.865	-0.0	0.0
	2	39.478	-0.0	0.0	39.429	0.12	0.0
	0	87.451	-0.05	8.64	87.502	-0.10	17.31
	3	90.260	-0.04	8.64	90.313	-0.10	17.28
.001	4	157.914	0.0	0.0	157.914	0.0	0.0
	5	246.758	0.0	0.0	246.758	0.0	0.0
	6	355.306	0.0	0.0	355.329	0.0	0.0
	7	483.619	0.0	0.0	483.619	0.0	0.0
	8	631.655	0.0	0.0	631.655	0.0	0.0
	9	799.443	0.0	0.0	799.443	0.0	0.0
	1	9.820	0.50	0.17	9.820	0.50	0.0
	2	39.478	-0.0	0.0	39.003	1.22	0.08
	0	84.675	-0.48	8.85	85.150	-1.04	18.50
	3	93.554	-0.42	8.41	94.094	-0.99	18.31
.01	4	157.914	0.0	0.0	157.914	0.0	0.0
	5	246.924	0.07	0.02	246.924	0.07	0.0
	6	355.306	0.0	0.0	355.543	-0.07	0.0
	7	483.695	0.02	0.0	483.695	-0.0	0.0
	8	631.655	0.0	0.0	631.655	0.0	0.0
	9	799.487	0.0	0.0	799.488	0.0	0.0
	1	9.405	4.93	1.47	9.399	5.00	0.0
	2	39.478	0.0	0.0	35.784	10.32	3.54
	0	77.610	-5.36	9.34	81.089	-9.42	31.84
	3	105.563	-3.46	9.69	110.639	-7.89	27.24
.1	4	157.914	0.0	0.0	157.914	0.0	0.0
	5	248.609	-0.75	0.15	248.607	-0.75	0.14
	6	355.306	0.0	0.0	357.726	-0.68	0.12
	7	484.461	-0.18	0.05	484.477	-0.18	0.02
	8	631.655	0.0	0.0	631.655	0.0	0.0
	9	799.964	-0.06	0.0	799.939	-0.06	0.0

Table 2. Comparision of natural frequencies and mode shapes of combined system with exact values (mass tuned to third mode of beam)

List of Figures

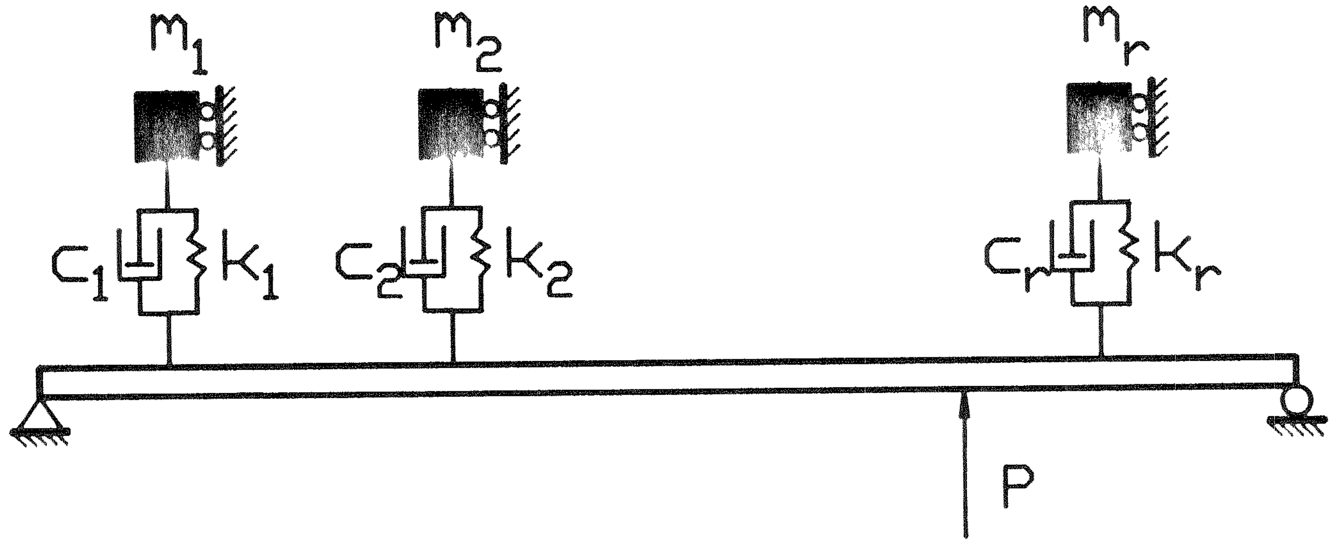
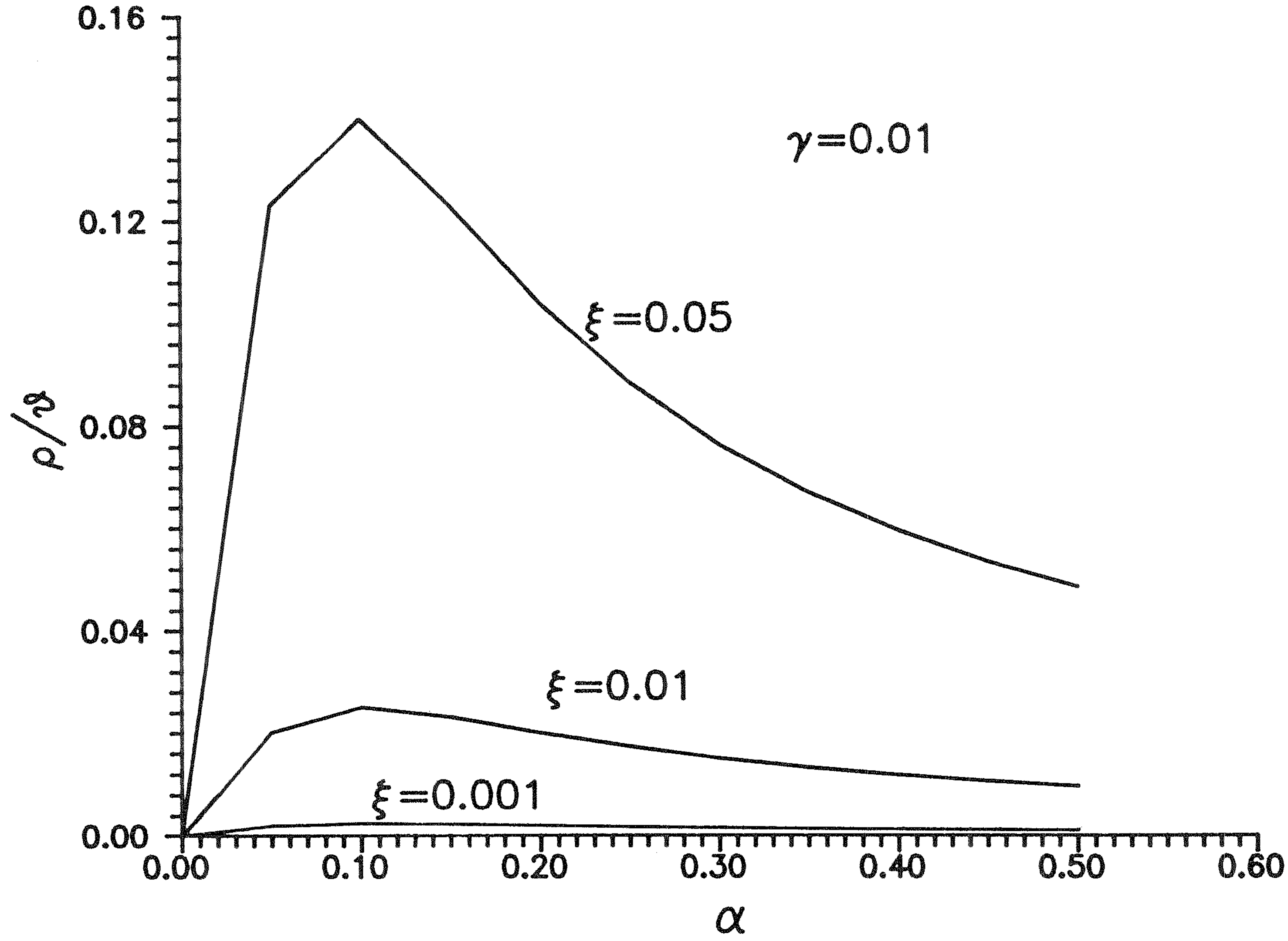


Fig.1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0.12
0.08
0.04
0.00
ρ/ϑ
γ=0.01
ξ=0.05
ξ=0.01
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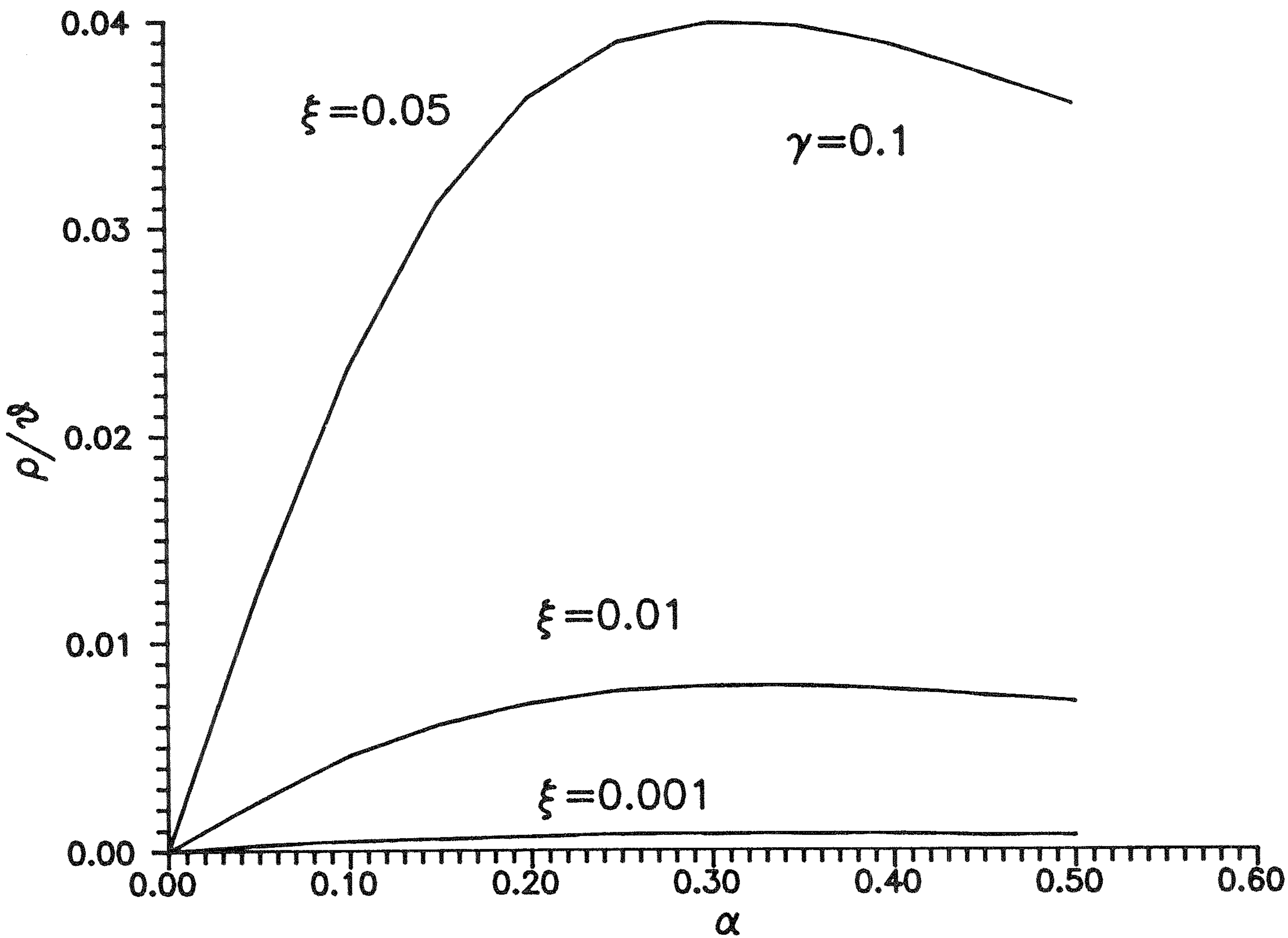
0.00
0.10
0.20
0.30
0.40
0.50
0.60
α

Fig. 2a



0.04
0.03
0.02
0.01
0.00
ρ/ϑ
0.00
0.10
0.20
0.30
0.40
0.50
0.60
α
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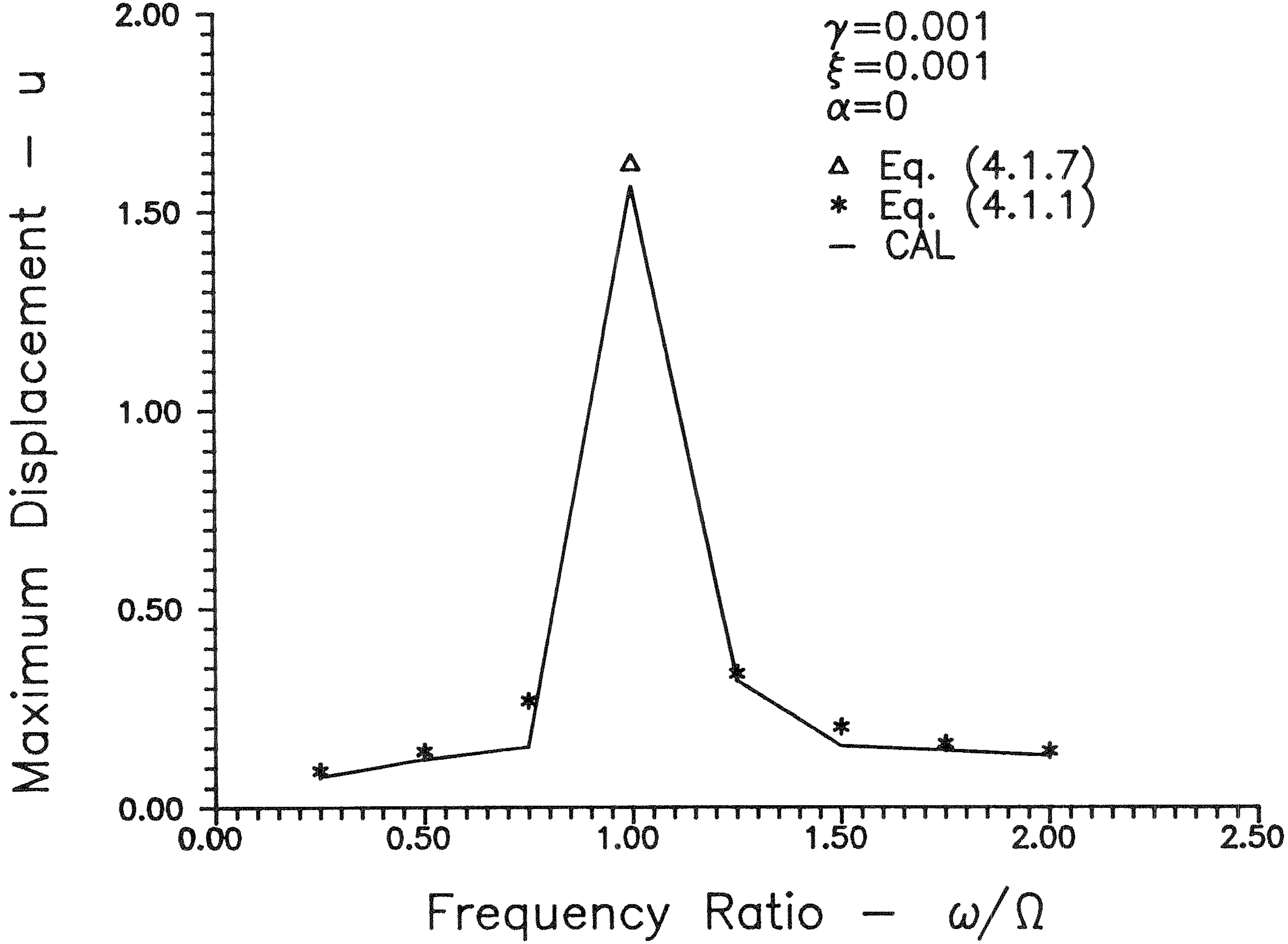
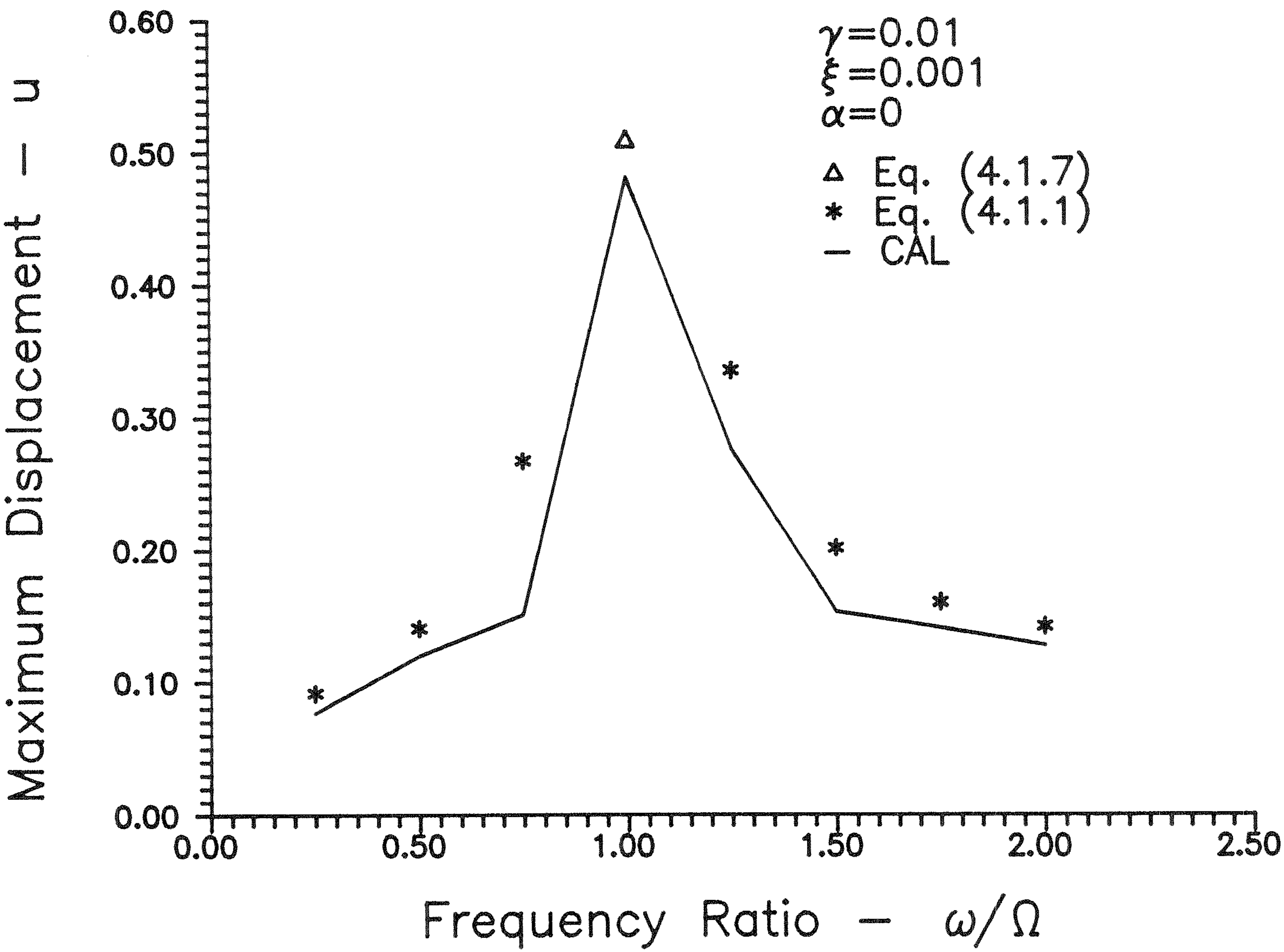
γ=0.1
ξ=0.01
ξ=0.001

Fig.2b



γ=0.001
ξ=0.001
α=0
Eq. (4.1.7)
Eq. (4.1.1)
CAL
Maximum Displacement – u
Frequency Ratio – ω/Ω
0.00
0.50
1.00
1.50
2.00
2.50

Fig.3a



0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.00
0.50
1.00
1.50
2.00
2.50
Maximum Displacement — u
Frequency Ratio — ω/Ω
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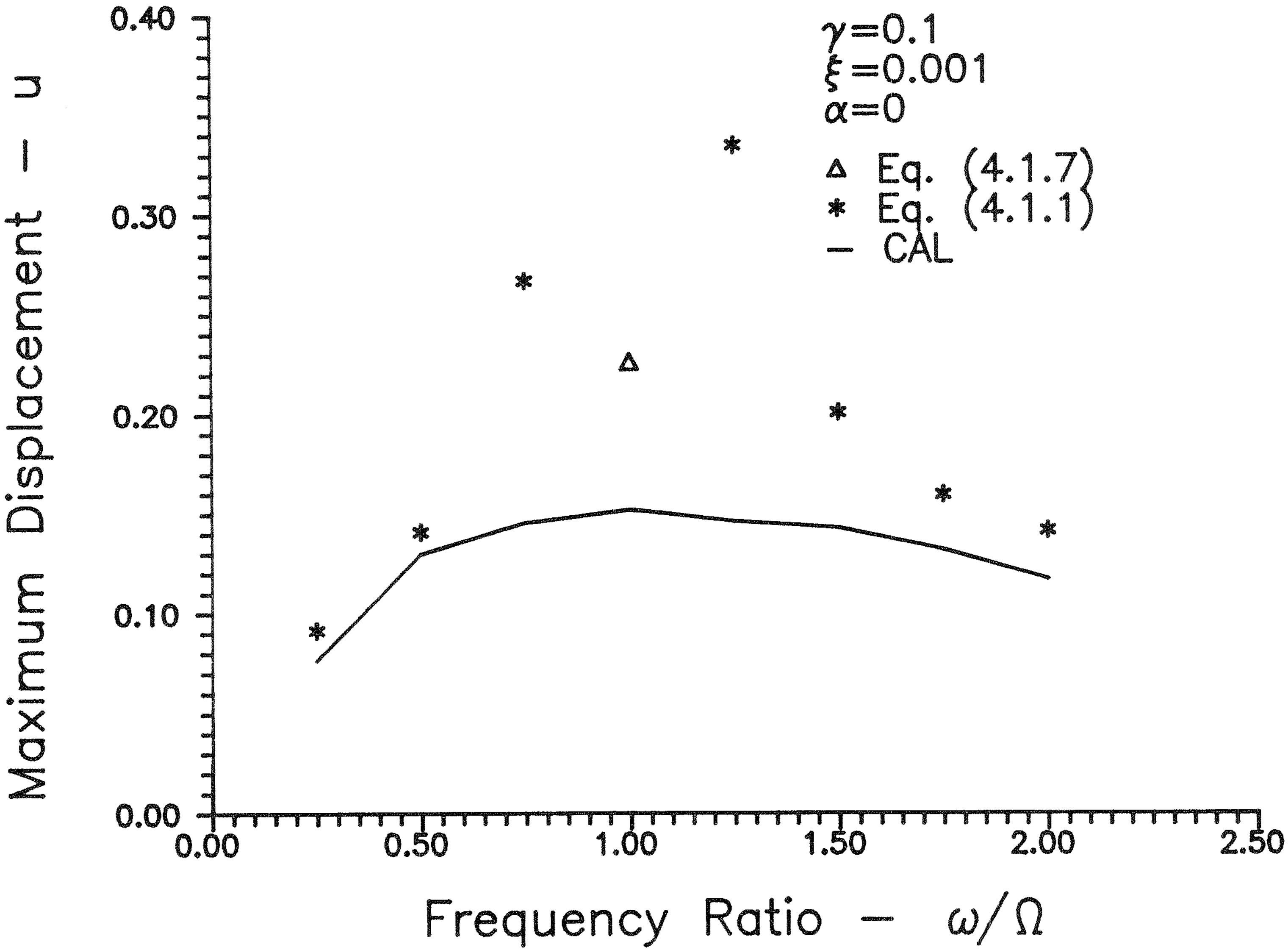
ξ=0.001
α=0
Δ Eq. (4.1.7)
* Eq. (4.1.1)
— CAL

Fig.3b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α=0
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* Eq. (4.1.1)
— CAL
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0.00
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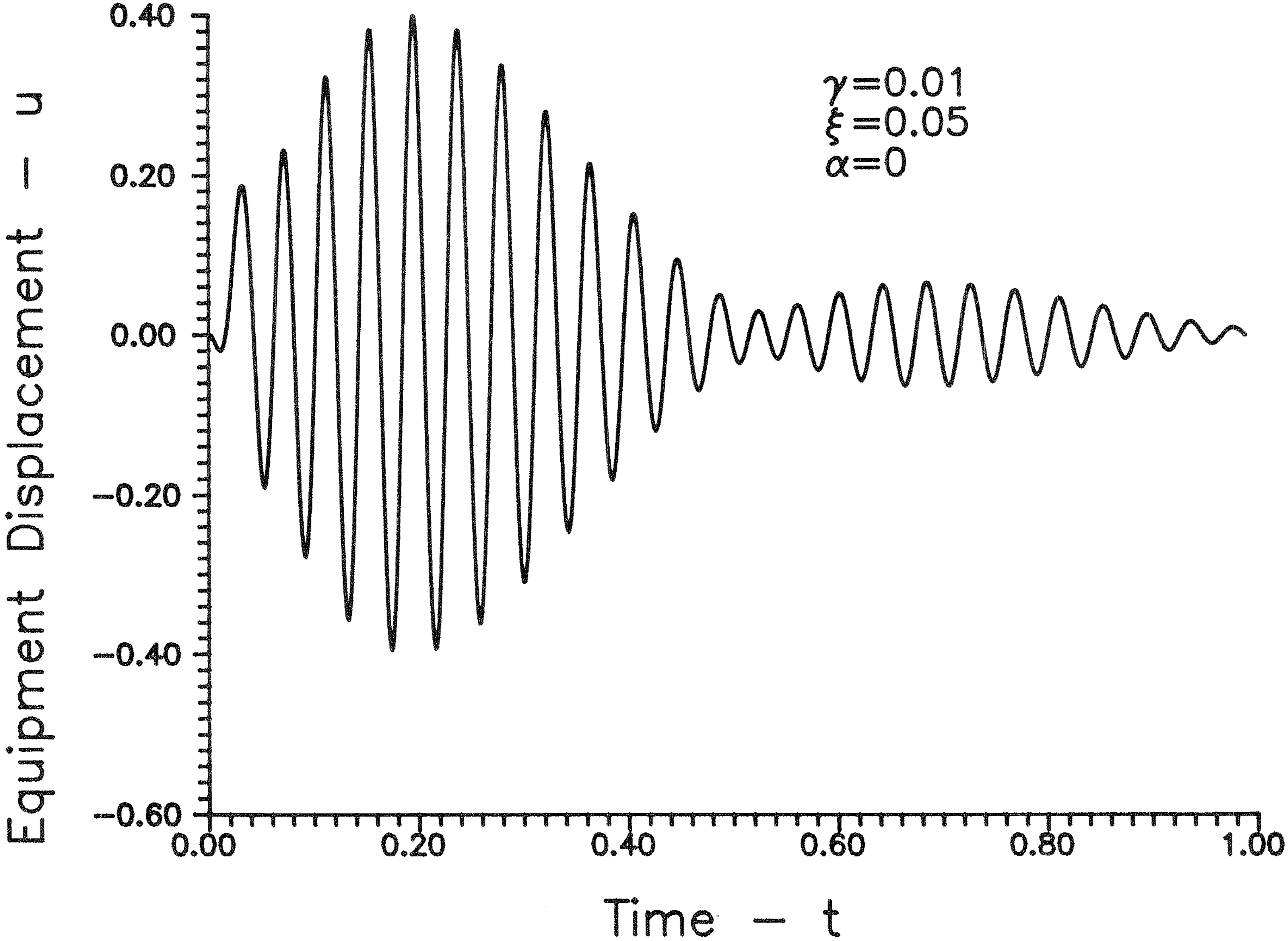
1.00
1.50
2.00
2.50
Maximum Displacement — u
Frequency Ratio — ω/Ω

Fig.3c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ξ=0.05
α=0
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0.20
0.00
−0.20
−0.40
−0.60
0.00
0.20
0.40
0.60
0.80
1.00
Equipment Displacement − u
Time − t

Fig.4