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Limits Under Conjugacy of the Diagonal Cartan Subgroup in SL(n,R)

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Limits Under Conjugacy of the Diagonal Cartan Subgroup in $SL_n(\mathbb{R})$

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in

Mathematics

by

Arielle Leitner

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by

Arielle Leitner

For Daryl Cooper

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Abstract

Limits Under Conjugacy of the Diagonal Cartan Subgroup in $SL_n(\mathbb{R})$

Arielle Leitner

A *conjugacy limit group* is the limit of a sequence of conjugates of the positive diagonal Cartan subgroup, $C \leq SL_3(\mathbb{R})$. In chapter 6, we prove a variant of a theorem of Haettel, and show that up to conjugacy in $SL_3(\mathbb{R})$, the positive diagonal Cartan subgroup has 5 possible conjugacy limit groups. Each conjugacy limit group is determined by a nonstandard triangle. We give a criterion for a sequence of conjugates of C to converge to each of the 5 conjugacy limit groups.

In chapter 8, we give a quadratic lower bound on the dimension of the space of conjugacy classes of subgroups of $SL_n(\mathbb{R})$ that are limits under conjugacy of the positive diagonal subgroup. We give the first explicit examples of abelian $(n-1)$-dimensional subgroups of $SL_n(\mathbb{R})$ which are *not* such a limit, however all such abelian groups are limits of the positive diagonal group iff $n \leq 4$.

In chapter 4, we classify all subgroups of $PGL_4(\mathbb{R})$ isomorphic to $(\mathbb{R}^3, +)$, up to conjugacy, and Haettel shows each is a limit of the positive diagonal Cartan subgroup. By taking subgroups of these groups satisfying certain properties, we show there are 4 possible families of generalized cusps up to projective equivalence in dimension 3, and describe each cusp.

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Chapter 1

Summary of Results

This dissertation concerns the study of transitions between different *homogeneous spaces,* G/H, associated with a fixed Lie group, G, obtained by taking limits of conjugates of the subgroup H. The idea of geometric transition may be studied from the perspectives of geometry, topology, algebraic geometry, and dynamics. Types of questions we discuss in this dissertation include:

- How does one geometry transition to another at infinity?
- What are the possible cusps at infinity on a manifold?
- Under what conditions does one group limit to another, in the Chabauty topology on the space of all closed subgroups of a group?

- What properties characterize limits of the diagonal subgroup of the general linear group?

Imagine blowing up a ball with air so that eventually the ball is so large, it looks like the earth. Locally, the ball looks flat. This example is given in [16]: a sequence of spheres tangent to a plane, with increasing radius, will limit to the tangent plane in the Hausdorff topology on closed sets. Such a process is an example of a *geometric transition*: a continuous path of geometric structures that abruptly changes type in the limit.

There are several ways of making the idea of inflating a ball mathematically precise. Envision the curvature of the ball approaching zero. Or, choose coordinates on the ball, and parametrize the radius increasing. A sphere is *intrinsically* different from the plane. On a sphere, the angles in a triangle will sum up to more than 180 degrees, since the edges bulge outwards. In the plane, the angles in a triangle sum to exactly 180 degrees. This property about triangles is *intrinsic* to the geometry of the space, and will hold true no matter how large or small the triangle is. Blowing up a ball is an example of a transition between two different kinds of geometry: spherical and Euclidean.

This dissertation uses the unfamiliar technique of working with *hyperreal numbers* [23]. These are an extension of the real numbers, but include numbers that are infinitesimally small and others that are infinitely large. They were discovered in the 1960s by a logician: Abraham Robinson. The hyperreals provide a convenient method of imagining and describing phenomena that appear after an *infinite* amount of time, by giving a precise way to measure the infinities involved. The way certain groups transition can be described by *triangles* with infinitesimal sides and angles, and such geometric transitions are controlled by very precise measurements of these quantities. In three dimensions, there are exactly 5 types of triangles, corresponding to the 5 conjugacy classes of limit groups that arise. In four dimensions there are 15 conjugacy classes of limit groups, and in dimension 7 and larger, there are infinitely many non-conjugate limit groups!

There was a famous conjecture of Thurston (recently proved by Perelman) that every compact 3-dimensional manifold is composed of pieces, each of which has one of 8 kinds of 3-dimensional geometry, two of which are spherical and Euclidean, [57]. These *Thurston geometries* are (almost) subgeometries of real projective geometry, and one may study geometric transitions in this context as paths of conjugacies, [16]. We will study geometric transitions given by *conjugacy limits*.

Definition 1.0.1. Let G be a Lie group. A subgroup, $H \leq G$, limits under conjugacy to another subgroup, $L \leq G$, if there is a sequence of elements, (p_n), such that $p_n H p_n^{-1} \to L$ in the Hausdorff topology.

In chapters 6 and 8 we build on work of Haettel, [26], to study geometric transitions of the diagonal Cartan subgroup in $SL_n(\mathbb{R})$, (the group of positive diagonal matrices). For example, when $n = 3$, a diagonal matrix with distinct eigenvalues determines a projective triangle, since each eigenvector of the matrix designates a vertex of the triangle. The limits of the positive diagonal group are determined by identifying some of the vertices or edges of the triangle to obtain a *degenerate* triangle.

Two degenerate triangles are *equivalent* if they have the same number of points and lines. Let G be a group. A degenerate triangle, T, is *characteristic for G* if:

1. G preserves every point and line of T

2. T is maximal in the partial order given by inclusion, subject to this condition.

We will show in chapter 6 in Theorems 6.0.64, and 6.0.65:

Theorem 1.0.2. *1. Any subgroup of $SL_3(\mathbb{R})$ isomorphic to $\mathbb{R}^2$ is conjugate to exactly one of the following groups:*

$$\begin{matrix} C & F & N_1 & N_2 & N_3 \\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix} & \begin{pmatrix} a & t & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix} & \begin{pmatrix} 1 & s & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & s & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

where $a, b > 0$ and $s, t \in \mathbb{R}$.

2. *Each of these groups is a conjugacy limit of the Cartan subgroup.*

3. *There is a bijection, θ, from conjugacy class of limit groups to equivalence classes of characteristic degenerate triangles, where $\theta(G) = T$ if and only if T is characteristic for G.*

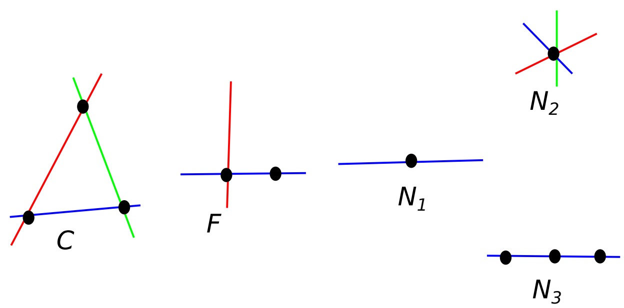


Figure 1.1: The 5 equivalence classes of characteristic degenerate triangles

Parts 1 and 2 are due to Haettel, [26]. In general, it is an open problem to classify conjugacy limits of the Cartan subgroup in $SL_n(\mathbb{R})$. In this dissertation, we give a classification for $n \leq 4$.

To study degenerate triangles, we use the *hyperreals*, a non-Archimedean ordered field containing the reals. The hyperreals are formed by taking equivalence classes of sequences of real numbers, much in the way that the reals are formed by taking Cauchy sequences of rational numbers. However, in this case, the equivalence relation is given by a non-principal ultra-filter, which requires the axiom of choice. The motivation for working over the hyperreals is to eliminate the need to use sequences and take limits going to infinity.

A sequence of matrices determines a single hyperreal matrix, which represents a hyperreal projective transformation taking the standard basis (triangle for $n = 3$), to a nonstandard basis (infinitesimal triangle for $n = 3$). Instead of taking the limit of images of the diagonal group under conjugation by a sequence of matrices, conjugate the hyperreal diagonal group by a single hyperreal matrix, and take the shadow: this is like like projecting back down into the real numbers, and eliminating the infinitesimal information. Using this approach, we will classify nonstandard triangles, and show in Theorem 6.0.66:

Theorem 1.0.3. *A limit group of the Cartan subgroup in $SL_3(\mathbb{R})$ is determined by two hyperreals: the length of the longest side of the nonstandard triangle, and the ratio of the largest infinitesimal angle to the largest side.*

A nonstandard triangle is in one of five equivalence classes, corresponding to the 5 limit groups. As a result, we classify the conjugacy limit of the positive diagonal Cartan subgroup under *any* sequence of matrices. We show also:

Theorem 1.0.4. *There are precisely 15 conjugacy classes of subgroups of $SL_4(\mathbb{R})$ isomorphic to $\mathbb{R}^3$. Each is a limit of the Cartan subgroup.*

Let G be a group, and $\mathcal{S}(G)$ be the set of all closed subgroups of G. Then $\mathcal{S}(G)$ is a compact Hausdorff topological space with the *Chabauty topology* on closed sets: [3], [13], [22], [26]. We define two closed subspaces of $\mathcal{S}(SL_n(\mathbb{R}))$: $\widehat{Ab}(n)$ is the space of all subgroups isomorphic to $\mathbb{R}^{n-1}$; and (following notation in [31]), $\widehat{Red}(n)$ is the closure of all conjugates of the diagonal Cartan subgroup. The quotient by the conjugacy action of $SL_n(\mathbb{R})$ on $\mathcal{S}(SL_n(\mathbb{R}))$ gives two spaces: $Ab(n)$ and $Red(n)$. These are typically *not* Hausdorff. Since limit groups of the Cartan subgroup are isomorphic to $\mathbb{R}^{n-1}$, then $Red(n) \subset Ab(n)$.

Suprenko and Tyshkevitch, [56], have classified conjugacy classes of maximal commutative nilpotent subalgebras over $\mathbb{C}$, for $n \leq 6$. Their results imply $Ab(5)$ is finite, and so $Red(5)$ is finite. Iliev and Manivel, [31], ask if $Red(n)$ is finite when $n \geq 6$. We will give a partial answer to this question in Theorem 8.0.90 using an invariant that shows:

Theorem 1.0.5. *If $n \geq 7$, then $\frac{n^2-8n+8}{8} \leq dim(Red(n)) \leq n^2 - n$.*

The case $n = 6$ is open. When $n \leq 4$, we know $Ab(n) = Red(n)$. When $n \geq 7$, [31] shows $\dim Red(n) < \dim Ab(n)$, but there were no known explicit examples of abelian groups which are not limits. In chapter 8 we give the first explicit example of an element of $Ab(5) - Red(5)$. This allows us to show in Theorem 8.0.91:

Theorem 1.0.6. *$Ab(n) = Red(n)$ if and only if $n \leq 4$. $Red(n) \subsetneq Ab(n)$ if and only if $n \geq 5$.*

In chapter 8 we give two more necessary properties satisfied by elements of $Red(n)$, but there is not yet a sufficient criterion for deciding when an abelian group is a limit group. In the future, I hope to find an invariant of abelian groups that distinguishes limit groups. Optimistically, perhaps we can characterize when a group is a limit of the Cartan subgroup. Even more optimistically, can one classify all limit groups in higher dimensions?

Let $B \subset GL_n(\mathbb{R})$ be the Borel subgroup. The subspace of conjugates of the diagonal group in $\mathcal{S}(B)$ has closure which is a (semi-algebraic) variety $\mathcal{V}$, called the *Chabauty compactification* of the associated homogeneous space. These methods give information about the dynamics of the action of $GL_n(\mathbb{R})$ on $\mathcal{V}$.

For $n = 3$, Haettel shows $\mathcal{V}$ is a CW complex with 2-skeleton the wedge sum of $\mathbb{R}P^2$ and S^2, see [26]. The cells of the CW complex correspond to conjugacy classes of groups, $[G]$. Let $N_B(G)$ denote the normalizer of G in B. Then $\dim cell(G) = \dim B - \dim N_B(G)$. If L is a conjugacy limit of G, then $\dim N_B(G) < \dim N_B(L)$, so $\dim cell(L) < \dim cell(G)$. Moreover, the boundary of $cell(G)$ is glued onto $cell(L)$, so the attaching maps give information about conjugacy limits.

Another application of these ideas is to study *generalized cusps* on convex projective manifolds (see [1], [18], [19]). A *convex projective manifold* is the quotient of convex subset of projective space by a discrete group of projective transformations. A *generalized cusp* in dimension 3 is a properly convex manifold, M, with ∂M strictly convex, and M is diffeomorphic to $T^2 \times [0, \infty)$. The holonomy of a generalized cusp centralizes a 1 parameter subgroup of $PGL_4(\mathbb{R})$. Using the classification of $Red(4)$, we classify all generalized cusps in dimension 3. We show in chapter 4:

Theorem 1.0.7. *A generalized cusp on a properly convex projective 3-dimensional manifold is projectively equivalent to one of 4 possible families of cusps.*

In the future, I will try to extend these results to higher dimensions. Generalized cusps on projective manifolds also give rise to affine structures on the torus (see [2], [47]).

Geometric transitions may be used to understand the interplay between different types of geometry, by describing the range of possible geometries within some given parameters. For instance, geometric transitions give a sense in which some geometries are more unipotent than the original. Furthermore, diagonal subgroups and matrices are central to much of mathematics. A classification of limits of the diagonal group might be useful to many different areas of research.

Understanding geometric transitions provides a new viewpoint for questions about 3-manifolds, dynamics, and algebraic geometry. Applying the hyperreal techniques developed in this dissertation might give new insight into some older questions. For example, how many components are in the variety that is the Zariski closure of the space of conjugates of the Cartan subgroup? Or, what types of cusps may appear on a convex projective manifold? Geometric transitions may lead to studying types of geometry that have not previously been considered.

The idea of continuously deforming one kind of geometry into another appears in many areas of mathematics and physics. For instance, the theory of Inönü-Wigner contractions in physics (see [10]). Physicists use deformations of Lie algebras in several ways, for example the "classical limit" in relativity where the speed of light, $c \to \infty$; and in transitioning from quantum mechanics to Newtonian mechanics, when $\hbar \to 0$. Representations of Lie groups describe different particles (for example the fundamental representation of SU(2) describes the electron). Using geometric transitions may help us to understand the ways in which systems of particles can collapse. These ideas appear in string theory, and in the study of gauge symmetries.

Introduction to the Hyperreals, ${}^*\mathbb{R}$

One of the main innovations in this dissertation is the use of the hyperreals. We present a brief introduction to them here. For a more thorough exposition, see [23].

The hyperreal numbers, ${}^*\mathbb{R}$, are an extension of the real numbers which includes numbers that are infinitesimally small and infinitely large. The hyperreals provide a framework for imagining and describing phenomena that appear after an infinite amount of time, by giving a precise way to measure the infinities involved.

Definition 1.0.8. A nonzero number $\varepsilon \in {}^*\mathbb{R}$ is *infinitely small*, or *infinitesimal*, if $|\varepsilon| < \frac{1}{n}$, for all $n \in \mathbb{N}$. The reciprocal, $\omega = \frac{1}{\varepsilon}$ is *infinitely large* or *infinite*, meaning $|\omega| > n$ for all $n \in \mathbb{N}$.

The hyperreals are a non-Archimedean ordered field containing the reals. The hyperreals are formed by taking equivalence classes of arbitrary sequences of real numbers, much in the way that the reals are formed by taking Cauchy sequences of rational numbers. However, in this case, the equivalence relation is finer and is given by taking a non-principal ultra-filter (which requires the axiom of choice).

To construct the hyperreals, we will define a *non-principal ultra filter*. This filter should capture some of the way that sequences converge- for example, $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}...\}$ and $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}...\}$ both converge to 0, but the second sequence converges twice as fast. The non-principal ultra filter gives a finer equivalence relation than Cauchy sequences, and encodes this extra information. The goal is to define a relation so that if two sequences agree on a 'large' number of places, then they have the same limit.

Definition 1.0.9 (Goldblatt, [23], p.18)**.** Let I be a nonempty set, and denote by $\mathcal{P}(I)$ the power set of I. A *filter* on I is a nonempty collection $\mathcal{F} \subset \mathcal{P}(I)$ satisfying the following axioms:

- Intersections: If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

- Supersets: If $A \in \mathcal{F}$ and $A \subset B \subset I$, then $B \in \mathcal{F}$.

A filter contains the empty set $\emptyset$ if and only if $\mathcal{F} = \mathcal{P}(I)$. A filter $\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$. Every filter contains I, and in fact $\{I\}$ is the smallest filter on I.

An *ultrafilter* is a proper filter that satisfies:

- For any $A \subset I$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ where $A^c = I - A$.

If $B \subset I$, then the *principal filter generated by* B is $\mathcal{F}^B = \{A \subset I : A \supset B\}$.

One may check (see [23] p. 19) that if an ultrafilter contains a finite set then it contains a one element set and is principal. Hence a *non-principal ultra filter* contains all co-finite sets. Furthermore, using the axiom of choice, any infinite set has a non-principal ultra filter on it ([23], Corollary 2.6.2).

To construct the hyperreals, take the ring of all real valued sequences $\mathbb{R}^{\mathbb{N}}$, with component wise addition and multiplication. Let $\mathcal{F}$ be a fixed non principal ultrafilter on $\mathbb{N}$, and define a relation, $\equiv$, on $\mathbb{R}^{\mathbb{N}}$ by $\{r_n\} \equiv \{s_n\}$ if and only if $\{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$. When the relation holds, we say the two sequences agree almost everywhere. The *hyperreals*, ${}^*\mathbb{R} = \{[r] : r \in \mathbb{R}^{\mathbb{N}}\}$ are the quotient ring. Then ${}^*\mathbb{R}$ is an ordered field. The absolute value is defined in any ordered field.

Identify a real number $r \in \mathbb{R}$ with the constant sequence $\{r, r, r, r, r, ...\}$, so the map $r \mapsto {}^*r$ is an order preserving field isomorphism from $\mathbb{R}$ into ${}^*\mathbb{R}$ ([23], Theorem 3.7.1).

Definition 1.0.10 ([23], p.49)**.** A hyperreal number b is

- *limited* or *finite* if $r < b < s$ for some $r, s, \in \mathbb{R}$;
- *unlimited* if $|b| > r$ for all $r \in \mathbb{R}$;
- *infinitesimal* if $0 < |b| < r$ for all $r \in \mathbb{R}_{>0}$;
- *appreciable* if it is limited but not infinitesimal, i.e., $r < |b| < s$ for some $r, s \in \mathbb{R}^+$.

Let $\mathbb{I}$ denote the set of infinitesimals, and $\mathbb{L}$ the set of limited numbers. Since arithmetic in ${}^*\mathbb{R}$ works in the expected way (see [23], p. 50), $\mathbb{I}$ is an ideal and $\mathbb{L}$ is a subring of ${}^*\mathbb{R}$.

Definition 1.0.11 ([23], p.52)**.** The hyperreals b, c are a *limited distance* apart if $b - c$ is limited. Denote the *galaxy* of b by $\text{Gal}(b) = \{c \in {}^*\mathbb{R} : b - c \text{ is limited}\}$. The ε-*galaxy* of b is $\text{Gal}_\varepsilon(b) = \{c \in {}^*\mathbb{R} : |b - c| \leq k \cdot \varepsilon \text{ for some } k \in \mathbb{R}\}$.

Every limited hyperreal b is infinitesimally close to exactly one real number, called the *shadow* or *standard part* of b, and denoted $sh(b)$, ([23], Theorem 5.6.1).

The map $sh : \mathbb{L} \to \mathbb{R}$ is an order preserving epimorphism. Notice $\ker(sh) = \mathbb{I}$, and the quotient ring $\mathbb{L}/\mathbb{I} \cong \mathbb{R}$, ([23], Theorem 5.6.3).

A finite hyperreal may be either appreciable or infinitesimal. Clearly Gal and Gal_ε define equivalence relations on ${}^*\mathbb{R}$. In chapter 6, we denote hyperreal objects in script $\mathscr{G}$ or $\mathscr{L}$, and denote their standardizations G or L. We use the usual inner product, $\langle x, y\rangle = x \cdot y$ for $x, y \in \mathbb{R}^n$ or ${}^*\mathbb{R}^n$.

The transfer principle says a statement in first order logic is true over the real numbers if and only if the transferred statement is true for the hyperreal numbers, [23] p.44. Thus we may prove facts about real numbers in the hyperreal setting. In fact, convergence of sequence and series, continuous functions, differentiation, and Reimmanian integration may all be defined in the hyperreal setting. For example, proving a real function is differentiable may be done without the use of any limits in the hyperreal setting!

Chapter 2

Lie Groups, Lie Algebras, Homogeneous Spaces and Symmetric Spaces

Lie Groups

Lie theory provides a beautiful perspective on the interplay of topology and group theory. We provide only a brief overview here, but some excellent texts on the subject include: [12], [27] [48], [49], [55], and [59].

Definition 2.0.12. Let K be a field (usually $\mathbb{R}$ or $\mathbb{C}$). A *Lie Group* over K is a group, G, which also carries the structure of a differentiable manifold over K so that the multiplication map

$$\mu : G \times G \to G, \qquad \text{where} \qquad \mu((x, y)) = xy$$

and inversion map

$$\iota : G \to G, \qquad \text{where} \qquad \iota(x) = x^{-1}$$

are differentiable.

A Lie group over $\mathbb{R}$ is a *real Lie group* and a Lie group over $\mathbb{C}$ is a *complex Lie group*.

Some examples of commonly known Lie groups include the real line, $\mathbb{R}$; the circle, $\mathbb{S}^1$; the general linear group, $GL_n(K)$; and the special linear group, $SL_n(K)$.

Let G be a Lie group. A subgroup $H \leq G$ is a *Lie subgroup* if it is a submanifold and a closed (topological) subset of the manifold G. For example, $SL_n(K)$ is a Lie subgroup of $GL_n(K)$.

Let G and G' be Lie groups. A map $\phi : G \to G'$ is a *Lie group homomorphism* if it is both an abstract group homomorphism and a differentiable map. A homomorphism is an *isomorphism* if it has an inverse which is differentiable. The usual theorems on homomorphisms and isomorphisms are true in the context of Lie groups (see [48]).

Let G be a Lie group, and X a manifold with group of diffeomorphisms $\mathrm{Diff}(X)$. A G-*action* on X is a homomorphism $\alpha : G \to \mathrm{Diff}(X)$ where the map

$$G \times X \to X \qquad \text{given by} \qquad (g, x) \mapsto \alpha(g)x$$

is differentiable. Common group actions of G on itself are left-multiplication, right-multiplication, and conjugation.

Let $x \in X$ be a point. Consider the map

$$\alpha_x : G \to X \qquad \text{where} \qquad \alpha_x : g \to \alpha(g)x.$$

The image of α_x is the *orbit* of the point x. For any $x \in X$, the derivative of α_x has constant rank, k. If the orbit is a submanifold in X, then $\dim \alpha(G)x = k$. Note that any orbit of a compact Lie group is a closed submanifold.

LIE ALGEBRAS

Every Lie group has a Lie algebra, and the structure of the Lie algebra determines many of the properties of the Lie group. Given a Lie group, G, the *Lie algebra* of G is the tangent space at the identity, $\mathfrak{g} := T_e(G)$. The Lie algebra is endowed with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket. For matrix Lie groups, $[x, y] = xy - yx$. If G is a commutative Lie group, then the Lie algebra has zero bracket.

If $f : G \to H$ is a Lie group homomorphism then $d_e f : T_e(G) \to T_e(H)$ is a homomorphism of Lie algebras. Many properties of Lie groups correspond to properties of the Lie algebras. For example, if H is a normal subgroup of G, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

A *one parameter subgroup* is a homomorphism $\phi : \mathbb{R} \to G$, sometimes thought of as the image subgroup $\phi(\mathbb{R})$. This subgroup need not be closed. For any differentiable path $t \mapsto g(t)$ in G, define a path $t \mapsto \zeta(t)$ in the Lie algebra by

$$\frac{dg(t)}{dt} = \zeta(t)g(t) \qquad \text{with} \qquad \exp(\zeta) = g_\zeta(1). \tag{2.1}$$

For any $\zeta \in \mathfrak{g}$, there is a one parameter subgroup, $g_\zeta(t)$, defined by (2.1), where $\zeta(t) \equiv \zeta$. The *exponential map*, $\exp : \mathfrak{g} \to G$ is defined by $\exp(t\zeta) := g_\zeta(t)$. The exponential map is a diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G$.

ABELIAN LIE GROUPS AND LIE ALGEBRAS

When the Lie group is commutative, the exponential map has additional nice properties. If G is a connected commutative Lie group, then $\exp \mathfrak{g} = G$, see [48] p.29. Therefore any n-dimensional connected commutative Lie group over a field, K, is isomorphic to K^n/Γ where $\Gamma \subset K^n$ is a discrete subgroup. Thus any n-dimensional connected commutative real Lie group is isomorphic to $(\mathbb{S}^1)^k \times \mathbb{R}^{n-k}$. The part isomorphic to $(\mathbb{S}^1)^k$ is the *compact factor*. For example, $SO(2) \cong \mathbb{S}^1$.

If G_1 and G_2 are isomorphic commutative Lie groups, then there is an isomorphism of their tangent algebras which maps the kernel of the homomorphism $\exp : \mathfrak{g}_1 \to G_1$ to the kernel of the homomorphism $\exp : \mathfrak{g}_2 \to G_2$.

Definition 2.0.13. A *full flag* in a vector space, V, is a chain of vector subspaces $V_1 \subset V_2 \subset \cdots \subset V_n = V$, where $\dim V_i = i$.

Let $T_n(K)$ denote the subgroup of upper triangular matrices in $GL_n(K)$, and denote by $\mathfrak{t}_n(K)$ the respective subalgebra of $\mathfrak{gl}_n(K)$. Elements of $T_n(K)$ (respectively $\mathfrak{t}_n(K)$), are operators preserving a full flag. The group $T_n(K)$ and Lie algebra $\mathfrak{t}_n(K)$ are solvable.

Theorem 2.0.14 (Lie, [49] p.8)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra, and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a complex linear representation. There is a full flag in V invariant under $\rho(\mathfrak{g})$.*

A corresponding theorem of Engel is true for nilpotent Lie algebras (groups):

Theorem 2.0.15 ([49] p.11)**.** *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation over a field K. Suppose that for all $x \in \mathfrak{g}$, the operator $\rho(x)$ is nilpotent. Then there is a basis for V such that every operator $\rho(x)$ may be written with respect to that basis as an upper triangular matrix with zeros on the diagonal.*

As a consequence of Theorem 2.0.15, every solvable complex matrix Lie algebra (or Lie group) has a basis with respect to which every element is an upper triangular matrix. In particular, any abelian complex matrix Lie group is conjugate to a group of upper triangular matrices.

Recall that a Lie algebra is *simple* if it is not abelian and the only ideals are 0 and itself. A Lie algebra is *semisimple* if it is a direct sum of simple Lie algebras. A connected Lie group is semisimple if its Lie algebra is semisimple.

Definition 2.0.16. Let $\mathfrak{g}$ be a Lie algebra. A *Cartan subalgebra of* $\mathfrak{g}$ is a nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that is self normalizing: if $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$, then $y \in \mathfrak{h}$.

For example, in $\mathfrak{g} = M_n(\mathbb{R})$ the set of a diagonal matrices is a Cartan subalgebra. See [49] chapter 1.9.3 for the following results: If $\mathfrak{g}$ is a finite dimensional Lie algebra over an infinite field, then it has a Cartan subalgebra. If the field is algebraically closed and of characteristic zero, then all Cartan algebras are conjugate. However, $\mathfrak{sl}_2(\mathbb{R})$ has two non-conjugate Cartan subalgebras: the diagonal one and $\mathfrak{so}(2)$. Any Cartan subalgebra of $\mathfrak{g}$ is abelian if $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over an algebraically closed field.

Definition 2.0.17. Let $\mathfrak{g}$ be a Lie algebra. Given $x \in \mathfrak{g}$, the *adjoint map* is

$$ad(x) : \mathfrak{g} \to \mathfrak{g} \qquad \text{where} \qquad ad(x)(y) = [x, y].$$

The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad \text{where} \qquad B(x, y) = \text{trace}(ad(x)ad(y)).$$

The Killing form has several nice properties: It is invariant under automorphisms of $\mathfrak{g}$, that is, $B(\phi(x), \phi(y)) = B(x, y)$ for $\phi \in \mathrm{Aut}(\mathfrak{g})$. If $\mathfrak{g}$ is a simple Lie algebra then any invariant symmetric bilinear form on $\mathfrak{g}$ is a scalar multiple of B. A Lie algebra is semisimple if and only if B is non-degenerate.

The Cartan Decomposition

The Cartan decomposition of a real semisimple Lie group is analogous to the polar decomposition of an invertible linear operator, $A = PK$ where $K \in O(n)$ and P is positive semi-definite symmetric. The polar decomposition is the same as Cartan decomposition with the global Cartan involution $\theta(A) = \mathrm{transpose}(A^{-1})$.

If G is a semisimple real linear algebraic Lie group, the polar decomposition and Cartan decomposition are the same. The Cartan decomposition gives information on the conjugacy of maximal compact subgroups of a connected Lie group, and is an important ingredient in the classification of connected semisimple Lie groups.

Definition 2.0.18. Let $\mathfrak{g}$ be a real semisimple Lie algebra, with Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. A decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into a direct sum of vector spaces is a *Cartan decomposition* if

(1) the map $\theta : \mathfrak{k} \oplus \mathfrak{p} \to \mathfrak{k} \oplus \mathfrak{p}$, where $\theta(k+p) = k - p$ is an automorphism of $\mathfrak{g}$ and

(2) the bilinear form $b_\theta(x, y) := -B(x, \theta y)$ is positive definite on $\mathfrak{g}$, (where B is the Killing form).

Notice that θ is an involution, so the bilinear form b_θ is symmetric. Condition (1) is equivalent to

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

which mirrors multiplication for positive and negative eigenspaces. Condition (2) is equivalent to the bilinear form, b_θ, being positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

The group $K = \exp \mathfrak{k}$ is a maximal compact subgroup of G. Further $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to b_θ, and $\mathfrak{p}$ is the *Cartan subspace* in $\mathfrak{g}$.

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ then $\mathfrak{k} = \mathfrak{so}_n(\mathbb{R})$ and $\mathfrak{p}$ is the subspace of traceless symmetric matrices. The involution is $\theta(X) = -X^T$ for $X \in \mathfrak{g}$. Here are some more properties of the Cartan decomposition proven in [48] section 4.3.2.

Theorem 2.0.19. *Every real semisimple Lie algebra has a Cartan decomposition. Any two Cartan decompositions can be taken to each other by an inner automorphism.*

Proposition 2.0.20. *Suppose* $\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i$ *is a decomposition into simple ideals with Cartan decompositions* $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$*. Then* $\mathfrak{k} := \bigoplus_{i=1}^{s} \mathfrak{k}_i$ *and* $\mathfrak{p} = \bigoplus_{i=1}^{s} \mathfrak{p}_i$ *are a Cartan decomposition of* $\mathfrak{g}$*, and every Cartan decomposition is obtained this way.*

If G is a real semisimple Lie group (not necessarily connected), then the *Cartan decomposition of* G is $G = KP$. Here K is a Lie subgroup of G with Lie algebra $\mathfrak{k}$, and $P = \exp \mathfrak{p}$. The mapping $\Theta : G \to G$ given by $\Theta(kp) = kp^{-1}$, is an involution called the *global Cartan involution*, with $d\Theta = \theta$. This decomposition has several properties (see [48] section 4.3.3):

Corollary 2.0.21.

1. *The group G is diffeomorphic to $K \times \mathbb{R}^m$ where $m =$* $\dim \mathfrak{p}$.
2. *The group $K = G^\Theta = \{g \in G : \Theta(g) = g\}$ is the fixed point set of Θ.*
3. *The group K is self-normalizing: $K = N_G(K)$.*
4. *The center $Z(G) \subset Z(K)$.*
5. *The subgroup K is compact if and only if G has finitely many connected components, and $Z(G)$ is finite.*

Theorem 2.0.22 (4.3.5 in [48])**.** *Let G be a Lie group with finitely many connected components. Then any two maximal compact subgroups of G are conjugate.*

Every matrix $M \in SL_n(K)$, can be written $M = KAN$, where K is orthogonal, N is unipotent, and A is diagonal. This is an example of the *Iwasawa decomposition*:

Theorem 2.0.23 ([49], p. 158)**.** *Let $\mathfrak{g}$ be a real semisimple lie algebra. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is nilpotent, $\mathfrak{a}$ is diagonal, and $\mathfrak{k}$ is a maximal subalgebra containing the center, according to the Cartan decomposition.*

(Affine) Symmetric Spaces

Homogenous spaces and symmetric spaces are related to Lie groups and algebras. A *symmetric space* is a Riemannian manifold (M, g) satisfying: for every point $p \in M$, there is an isometry $\sigma_p : M \to M$ such that $\sigma(p) = p$, and $d\sigma_p = -\mathrm{id}_{T_pM}$. Translation along a geodesic is a composition of involutions, so M is geodesically complete (every maximal geodesic is defined on $\mathbb{R}$). Since any two points are connected by a geodesic, the isometry group $G = \mathrm{Isom}_0^+(M)$ acts transitively on M. Identify $M \cong G/K$, where $K = \{k \in G : k(p) = p\}$ is a point stabilizer. Since M is Reimmanian, K is compact.

Examples of symmetric spaces include Euclidean spaces, spheres, and hyperbolic space. These are the only simply connected symmetric spaces with constant sectional curvature. Riemannian symmetric spaces are classified as one of three types (see [25], chapter 1):

- compact type, M has nonnegative (but not identically 0) sectional curvature
- non-compact type, M has nonpositive (but not identically 0) sectional curvature
- Euclidean type, M has vanishing curvature.

Definition 2.0.24 ([25] I.2.4)**.** Let X be a symmetric space of non-compact type. The *sphere at infinity of* X is the set of geodesics

$$\partial_\infty X = \{r : [0, \infty) \to X\}/\sim$$

where $r_1 \sim r_2$ if

$$\limsup_{t\to\infty} d(r_1(t), r_2(t))) < \infty.$$

By Proposition I.2.5 in [25] the isometric action of G on X extends to an action of G on $\partial_\infty X$. By Proposition I.2.3, $\partial_\infty X$ can be canonically identified with the unit sphere in the tangent space $T_x X$ at any base point x.

Now we generalize to the idea of an *affine symmetric space*, where point stabilizers are no longer required to be compact. Let G be a non-compact semisimple Lie group with finite center. A subgroup $H \leq G$ is *symmetric* if $H = G^\sigma$ is the fixed point set of an involution. $\sigma : G \to G$. Let G_0 denote the identity component of G. More generally, H is symmetric if $G_0^\sigma \subset H \subset G^\sigma$. The *symmetric space for* G is $X = G/H$ where the stabilizer H of a typical point is an open subgroup of G^σ, the fixed point set of an involution $\sigma \in \mathrm{Aut}(G)$.

The coset space G/H is an *affine symmetric space* (see [16]). An affine symmetric space is a symmetric space if and only if H is compact. The structure theory of affine symmetric spaces generalizes the theory for Riemannian symmetric spaces. There exists a Cartan involution, $\Theta : G \to G$, which commutes with

σ. Let $K = G^{\Theta}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Then σ defines a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ into eigenspaces with eigenvalues ± 1.

Affine symmetric spaces have structure theorems analogous to the Cartan decomposition theorems (see [7], [52]). For example:

$$[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}, \qquad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}.$$

There is a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ which commutes with σ. Set $K = G^{\Theta}$, the fixed point group of Θ, and a maximal compact subgroup of G. Since Θ and σ commute, we have the decomposition:

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}.$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra such that $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{q}$ is a maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{q}$. Then $\mathfrak{b}$ is unique up to the action of $H \cap K$. Set $A = \exp(\mathfrak{a})$ a connected subgroup of G, and $B := \exp(\mathfrak{b}) \subset A$. There is a well known factorization theorem (see [28]):

Theorem 2.0.25. *Let $\mathfrak{b} \subset \mathfrak{p} \cap \mathfrak{q}$ be a maximal abelian subalgebra. Then any $g \in G$ may be written $g = kbh$ with $k \in K, b \in B$, and $h \in H$. Moreover, b is unique up to conjugation in the Weyl group $W_{H \cap K} = N_{H \cap K}(\mathfrak{b})/Z_{H \cap K}(\mathfrak{b})$.*

Homogeneous Spaces and Grass-

Homogeneous spaces look the same everywhere, and they form some of the basic objects in the study of geometry, and in this thesis. A *homogeneous space* is a differentiable manifold, X, equipped with a transitive action of a Lie group, G. Any homogeneous space is isomorphic to G/H where $H \subset G$ is a closed Lie subgroup with the induced action, and H fixes a point $x \in X$.

A homogenous space is symmetric if and only if there exists a point with a symmetry σ_p. A Lie group acting on itself by left multiplication is an example of a homogeneous space which is not symmetric.

For example, all of our earlier examples of symmetric spaces are homogeneous: the sphere $\mathbb{S}^n = O(n+1)/O(n)$, Euclidean space $\mathbb{A}^n = E(n)/O(n)$, and hyperbolic space $\mathbb{H}^n = O^+(1,n)/O(n)$.

Other homogeneous spaces include projective space, $\mathbb{P}^n = O(n+1)/(O(n) \times O(1))$, and the *Grassmannians*, which are a generalization of projective space:

$$\mathrm{Grass}(k,n) = O(n)/(O(k) \times O(n-k))$$

This can be identified with the set of all k-dimensional vector subspaces of $\mathbb{R}^n$, and has dimension $k(n-k)$. It is a non-singular affine algebraic variety, ([8] p. 70ff.), because there is a bijection $\mathrm{Grass}(k,n) \leftrightarrow \mathcal{A}(n,k)$, where

$$\mathcal{A}(n,k) = \{A \in M_n(\mathbb{R}) : A^t = A, A^2 = A, \mathrm{trace}(A) = k\}.$$

The space, $\mathcal{P}$, of positive definite symmetric n by n matrices is homogenous under the action of $GL_n(\mathbb{R})$ by $(A, P) \mapsto PAP^t$.

A *lattice* is a subgroup of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}^2$. The space, $\mathcal{L}$, of lattices in $\mathbb{R}^2$, is homogenous under $GL_2(\mathbb{R})$, and the space of *unimodular* lattices (those with base parallelogram having unit area) is homogeneous under the action of $SL_2(\mathbb{R})$.

Chapter 3

Geometric Structures, and Convex Projective Structures

Topological spaces sometimes come equipped with a certain pattern or structure. Often we study a space by finding structures which fit on it. GEOMETRIC STRUCTURES

Definition 3.0.26. (Thurston, [57] p.110) Let G be a Lie group, and X a connected manifold on which G acts transitively. A (G, X)-*manifold* is a manifold, M, with a collection of G-*compatible* coordinate charts whose domains cover M. A coordinate chart is a pair (U_i, ϕ_i) where $U_i \subset M$ is open and $\phi : U_i \to X$ is a homeomorphism onto its image. Compatibility means that whenever two charts

(U_i, ϕ_i) and (U_j, ϕ_j) intersect, the transition map or coordinate change

$$\gamma_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is in G.

For example, the torus is a $(\text{Isom}(\mathbb{R}^2), \mathbb{R}^2)$-manifold, and the sphere is a $(O(n+1), S^n)$-manifold.

Two G-atlases are *compatible* if their union is also a G-atlas. It is easy to check that compatibility is an equivalence relation. A manifold is often defined by a *maximal* G-atlas.

A G*-isomorphism* is a homeomorphism, $\psi : M \to N$ between manifolds with G-structure, such that when expressed in terms of local charts, ψ is given by an element of G. A *local* G*-isomorphism* is a local homeomorphism, which is locally expressible as an element of G. For example, the line $\mathbb{R}$ and the circle $\mathbb{S}^1$ are locally isomorphic, but not globally isomorphic.

Fix a base point $x_0 \in M$ and a chart (U_0, ϕ_0) whose domain contains x_0, and let $\pi : \tilde{M} \to M$ be the universal covering map. Recall $\tilde{M}$ may be thought of as the space of homotopy classes of paths in M starting at x_0. Let $[\alpha]$ be a representative of a path α in $\tilde{M}$. Subdivide α so

$$x_0 = \alpha(t_0), x_1 = \alpha(t_1), ..., x_n = \alpha(t_n)$$

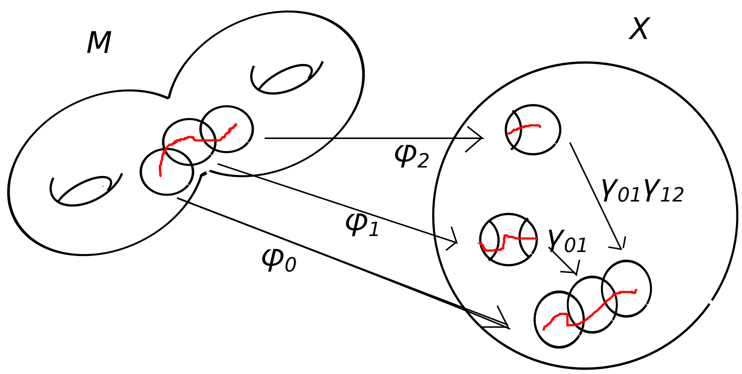


Figure 3.1: The developing map is constructed by analytic continuation.

where $t_0 = 0$, and $t_n = 1$, and each subpath is contained in a single coordinate chart (U_i, ϕ_i). Traverse α, adjusting so each chart ϕ_i agrees in a neighborhood of $x_i \in U_{i-1} \cap U_i$. The adjusted charts are the *analytic continuation* of ϕ_0 along α. The last chart is $\psi = \phi_0^\alpha := \gamma_{01}(x_1)\gamma_{12}(x_2)....\gamma_{n-1,n}(x_{n-1})\phi_n$.

Definition 3.0.27. [Thurston, [57] p.139] For a fixed base point and initial chart ϕ_0, the *developing map* of a (G, X) manifold, M, is the map $D : \tilde{M} \to X$ that agrees with the analytic continuation of ϕ_0 along each path, in a neighborhood of the path's endpoint. So, $D = \psi \circ \pi$ in a neighborhood of $\sigma \in \tilde{M}$.

If $D : \tilde{M} \to X$ is a covering map, then M is a *complete* (G, X)-manifold.

The developing map is a local (G, X)-diffeomorphism between $\tilde{M}$ and X. Since a covering of a simply connected space is a homeomorphism, if M is complete and X is simply connected, then we think of $\tilde{M}$ and X as being identified by D.

Let $\sigma \in \pi_1(M)$. Analytic continuation along a loop representing σ gives a chart, ϕ_0^σ, starting at the same base point as ϕ_0. Let $g_\sigma \in G$ be the element such that $\phi_0^\sigma = g_\sigma \phi_0$. Then g_σ is the *holonomy* of σ. Thus $D \circ T_\sigma = g_\sigma \circ D$, where $T_\sigma : \tau \mapsto \sigma\tau$ is the covering transformation associated to σ. The map $H : \pi_1(M) \to G$, where $H : \sigma \mapsto g_\sigma$, is the *holonomy* of M. It has image the *holonomy group* of M. Notice that H depends on the choices for D: when D changes, H changes by conjugation.

Proposition 3.0.28 ([57] 3.4.5)**.** *If G is a group of analytic diffeomorphisms of a simply connected space, X, any complete (G, X) manifold may be reconstructed from its holonomy group Γ, as the quotient X/Γ.*

For example, the circle, $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, and the Euclidean torus, $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ are constructed as X/Γ. Given a manifold, N, there are often several different possible (G, X) structures on N. We would like to differentiate between them.

Definition 3.0.29. Suppose N is a closed manifold or the interior of a compact manifold with boundary. A *marked* (G, X) structure on N is a pair (M, f) where M is a (G, X) manifold, and $f : N \to M$ is a diffeomorphism, called the marking. Two markings (M, f) and (M', f') are equivalent if $f' \circ f^{-1}$ is isotopic to a G-isomorphism.

One of the most celebrated theorems in geometric topology is Thurston's geometrization theorem, [54], (proved by Perelman). It states that every compact 3-dimensional manifold is composed of pieces, each of which has one of 8 kinds of 3-dimensional geometry, known as the Thurston model geometries.

In some cases, the model geometry restricts the type of structure on the manifold. For example, there is the famous Mostow Rigidity theorem, [53]:

Theorem 3.0.30 (Mostow Rigidity)**.** *Suppose M and N are complete finite-volume hyperbolic n-manifolds with $n \geq 3$. If there exists an isomorphism $f : \pi_1(M) \to \pi_1(N)$, then it is induced by an isometry from M to N.*

Suppose $n \geq 3$, and let M be a finite volume hyperbolic n-manifold. The holonomy, $\rho : \pi_1 M \to \mathrm{Isom}(\mathbb{H}^n)$ is unique up to conjugacy, and we set $\Gamma := \rho(\pi_1(M))$. Then $M \cong \mathbb{H}^n/\Gamma$. The moduli space of hyperbolic structures on M is a single point. Such rigidity theorems are not always true for other types of geometric structures. For manifolds with boundary, we have

Lemma 3.0.31 ([11] p.42)**.** *Let M_0 be a compact n-manifold with boundary and let M be a thickening of M_0, so that $M - M_0$ is a collar neighborhood of ∂M_0. Consider a (G, X) structure on M_0 which extends to M. Then any small deformation of the holonomy representation produces a nearby geometric structure on M_0.*

Stephan Tillmann's notes, [58], provide an excellent background on real convex projective structures. Here we give a brief overview. View projective space as a quotient of $\mathbb{R}^{n+1} - \{\vec{0}\}$ by the relation $v \sim \lambda v$, where $0 \neq \lambda \in \mathbb{R}$. Then $\mathbb{P}(\mathbb{R}^{n+1}) := \mathbb{R}P^n$ is equipped with the quotient topology.

Real Projective Geometry

Let $\{b_1, ..., b_{n+1}\}$ be a basis for a real vector space $V \cong \mathbb{R}^{n+1}$. Projective coordinates for a real projective space $\mathbb{P}(V)$, are given by

$$[t_1 b_1 + \cdots + t_{n+1} b_{n+1}] = [t_1 : \cdots : t_{n+1}] = [\lambda t_1 : \cdots : \lambda t_{n+1}]$$

where $0 \neq \lambda \in \mathbb{R}$. Let $H \subset \mathbb{P}(V)$ be a hyperplane (a co-dimension 1 subspace). Choose coordinates on $\mathbb{P}(V)$ so that

$$H = \{[x_1 : \cdots : x_n : 0] | x_k \in \mathbb{R}, \text{ not all } x_k = 0\}.$$

An *affine patch* is the complement of a hyperplane

$$\mathbb{P}(V) - H = \{[x_1 : \cdots : x_n : 1] | x_k \in \mathbb{R}\} \leftrightarrow \{(x_1, ..., x_n)\},$$

which we identify with $\mathbb{R}^n$. Notice

$$PGL(\mathbb{P}(V) - H) := \{[M] \in PGL_{n+1}(\mathbb{R}) : M = \left(\begin{smallmatrix} A & b \\ 0 & 1 \end{smallmatrix}\right) \text{ where } A \in GL_n(\mathbb{R}), b \in \mathbb{R}^n\}.$$

On the level of groups,

$$PGL(\mathbb{P}(V) - H) \cong \text{Aff}(n) \cong GL_n(\mathbb{R}) \ltimes \mathbb{R}^n,$$

where we embed $\text{Aff}(n) \to PGL_{n+1}(\mathbb{R})$ by $\{x \mapsto Ax + b\} \mapsto \left(\begin{smallmatrix} A & b \\ 0 & 1 \end{smallmatrix}\right)$.

A set of $n+2$ points in $\mathbb{R}P^n$ is in *general position* if any subset of $n+1$ underlying vectors in $\mathbb{R}^{n+1}$ is linearly independent. Such a set of $n+2$ points is a *projective basis*. Given two projective bases $\{p_1, ..., p_{n+2}\}$, and $\{q_1, ..., q_{n+2}\}$, there is a unique linear transformation, $\phi : \mathbb{R}P^n \to \mathbb{R}P^n$, such that $\phi(p_i) = q_i$ for all $1 \leq i \leq n+2$. In particular, there is a unique projective transformation on a projective line taking any three distinct points to any three other distinct points.

The *cross ratio* is a projective invariant defined by 4 points on a line as follows: Map the first three points to $\{0, 1, \infty\}$, and the fourth point to $[x, y, z, w] := \frac{(x-z)(w-y)}{(x-y)(w-z)}$. The symmetric group S_4 acts on quadruples of points. The cross ratio is invariant under the action of pairs of disjoint transpositions. So in general, an unordered quadruple of points has six possible cross ratios.

Real Convex Projective Structures

One type of geometric structure on a manifold is a real convex projective structure. These share many properties with hyperbolic manifolds.

A *real projective structure* on a manifold, M, is a (G, X)-structure in which $G = PGL_{n+1}(\mathbb{R})$, and $X = \mathbb{R}P^n$, as in Definition 3.0.26. There is a developing map $D : \tilde{M} \to \mathbb{R}P^n$, and a holonomy representation $H : \pi_1(M) \to PGL_{n+1}(\mathbb{R})$. Suppose instead of modeling on $X = \mathbb{R}P^n$, we want to use a subset of $\mathbb{R}P^n$ with certain properties.

Definition 3.0.32 (Tillmann, [58])**.** A subset $C \subset \mathbb{P}(V)$ is *convex* if the intersection of any line with C is connected. A convex set $C \subset \mathbb{P}(V)$ is *properly convex* if the closure of C is contained in an affine patch. A point $p \in \partial\overline{C}$ is *strictly convex* if it is not contained in a line segment of positive length in $\partial\overline{C}$. The set C is *strictly convex* if it is properly convex, and strictly convex at every point $p \in \partial\overline{C}$.

A closed projective triangle in $\mathbb{R}P^2$ is convex, but not strictly convex.

Let Ω be a properly convex open set contained in an affine patch. The *Hilbert metric*, d_Ω, on Ω is

$$d_\Omega(a,b) = \log[x,a,b,y] = \log \frac{||b-x|| \cdot ||a-y||}{||b-y|| \cdot ||a-x||},$$

where $x, y \in \partial\Omega$ are the endpoints of a line segment in Ω containing a and b such that a lies between x and b on the line segment.

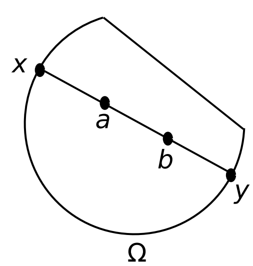


Figure 3.2: The Hilbert metric

Define $\mathrm{Isom}(\Omega)$ to be the set of isometries from Ω to Ω, and $PGL(\Omega) = \{A \in PGL_{n+1}(\mathbb{R}) : A(\Omega) = \Omega\}$. Since projective transformations preserve cross ratio,

$PSL(\Omega)$ is contained in the group of isometries of the Hilbert metric. Projective transformations preserve cross-ratios, and take affine patches to affine patches, and convex sets to convex sets.

Lemma 3.0.33 ([58], 2.5)**.** *Let Ω be a strictly convex domain, and $\gamma \in Isom(\Omega, d_\Omega)$.*

1. *The image under γ of the intersection of a line with Ω is again the intersection of a line with Ω.*

2. *The map γ extends to a homeomorphism $\bar{\Omega} \to \bar{\Omega}$.*

3. *Cross-ratios of collinear points in Ω are preserved by γ.*

Moreover, $PGL(\Omega) = Isom(\Omega, d_\Omega)$.

Suppose X is a locally compact Hausdorff space. The subgroup $H \subset \text{Homeo}(X)$ acts *properly discontinuously* if for every compact $K \subset X$, the set $K \cap hK$ is nonempty for at most finitely many $h \in H$. Let Ω be a properly convex domain and $H \subset PGL(\Omega)$. Then Proposition 3.2 in [58] says H is a discrete subgroup of $PGL(n+1)$ if and only if H acts properly discontinuously on Ω.

Let Ω be a convex set, and $\rho : \Gamma \to PGL_{n+1}(\mathbb{R})$ a discrete and faithful representation whose image preserves Ω. Then $M := \Omega/\rho(\Gamma)$ has a *convex projective structure.*

Recall Mostow rigidity, 3.0.30: If $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ are closed hyperbolic 3-manifolds with $n \geq 3$ and $\Gamma_1 \cong \Gamma_2$, then Γ_1 is conjugate to Γ_2. The notion of a hyperbolic structure may be generalized to a *properly convex projective structure*, requiring only $\Omega \subset \mathbb{R}P^n$ be properly convex, and $M \cong \Omega/\rho(\Gamma)$, where the holonomy, $\rho : \pi_1(M) = \Gamma \to PGL_{n+1}(\mathbb{R})$ is a discrete and faithful representation with image preserving Ω. There is no analog of Mostow rigidity in this setting, so the deformation theory of convex projective structures remains a rich object of study.

When M is closed, Koszul, [36], shows nearby projective structures are properly convex. If M is not compact, Koszul's results no longer hold.

Real Projective Structures on Compact Surfaces

Real projective structures on compact surfaces have been classified by Choi and Goldman, using work of Kuiper, Benzecri, Koszul, Vey, and Kobayashi ([5], [6], [36], [37], [38], [60], [61], [33], [35], [34]). The survey article [14] gives a nice summary of the results.

If M is a convex $\mathbb{R}P^2$ manifold with $\chi(M) < 0$, then the universal cover of M is a strictly convex set $\Omega \subset \mathbb{R}P^2$, with boundary which is a $\mathcal{C}^1$ curve. Then $\partial\Omega$ is either a conic (when the projective structure is hyperbolic), or it is nowhere $\mathcal{C}^{1+\varepsilon}$ for some $\varepsilon > 0$.

Goldman and Choi [14] give a nice example (1.4) of a convex $\mathbb{R}P^2$ structure on an annulus, (with a boundary made up of closed geodesics, each having a geodesically convex collar neighborhood), and whose holonomy has distinct positive eigenvalues. To construct this example, let Δ be the open projective triangle in figure 3.3, and $T \in PGL_3(\mathbb{R})$ be represented by the matrix

$$T = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

with $a > b > c > 0$. The cyclic group $\langle T \rangle$ generated by T is discrete and acts properly and freely on Δ, with quotient space an open annulus.

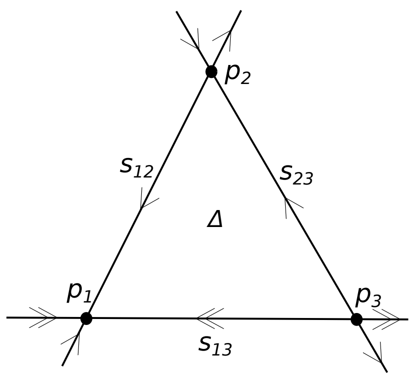


Figure 3.3: The Dynamics of T in $\mathbb{R}P^2$

There are two natural compactifications of $\Delta/\langle T \rangle$:

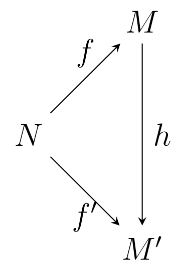
$$A_1 = (\Delta \cup s_{12} \cup s_{13})/\langle T \rangle \qquad \text{and} \qquad A_3 = (\Delta \cup s_{13} \cup s_{23})/\langle T \rangle.$$

Both compactifications A_1 and A_3 are convex $\mathbb{R}P^2$ manifolds with boundary, and have projectively isomorphic interiors. However, the projective isomorphism between the interiors extends to one between A_1 and A_3 only if $ac = b^2$.

Another class of convex $\mathbb{R}P^2$ manifolds consists of hyperbolic manifolds ([14] 1.5). Let $\Omega \subset \mathbb{R}P^2$ be the interior of a conic, and let $G \leq PGL_3(\mathbb{R})$ be the subgroup stabilizing Ω. Then G leaves invariant a Riemannian metric, g, of constant negative curvature. Every isometry of g is realized by a unique projective transformation preserving Ω. Let M be a surface with hyperbolic structure. Composing a developing map $\tilde{M} \to \mathbb{H}^2$ with an isometry $\mathbb{H}^2 \to \Omega$ realizes $M \cong \Omega/\Gamma$ where $\Gamma \subset G$ is a discrete cocompact subgroup.

The Character Variety

Given a manifold N, there are often several possible different projective structures on N. We would like to differentiate between these structures. We generalize Definition 3.0.29 to non-closed manifolds. Suppose N is either closed or the interior of a compact manifold with boundary. A *marked projective structure* on N is a pair (M, f) where M is a projective manifold and $f : N \to M$ is a diffeomorphism, called the marking. Two markings (M, f) and (M', f') are *equivalent* if there is a projective bijection h defined on the complement of a collar neighborhood of ∂M onto the complement of a collar neighborhood of $\partial M'$ such that the following diagram commutes up to isotopy.



Let $\mathbb{R}P(N)$ be the set of equivalence classes of marked projective structures on N. There is a natural identification of $\mathbb{R}P(N)$ with the quotient of the space of isotopy classes of developing maps by the action of $PGL_{n+1}(\mathbb{R})$. Then $\mathbb{R}P(N)$ has the smooth compact open topology on the set of functions $\tilde{N} \to \mathbb{R}P^n$.

Following notation in [1], let $\mathfrak{R}(\pi_1(N), PGL_{n+1}(\mathbb{R}))$ be the set of representations $\rho : \pi_1 N \to PGL_{n+1}(\mathbb{R})$, with the compact open topology. When $\pi_1 N$ is finitely generated, the compact open topology coincides with the notion of pointwise convergence of images of a fixed generating set. The character variety, $\mathfrak{X}(\pi_1(N), PGL_{n+1}(\mathbb{R}))$, is the geometric invariant theory quotient of $\mathfrak{R}(\pi_1(N), PGL_{n+1}(\mathbb{R}))$ by conjugation in $PGL_{n+1}(\mathbb{R})$, with the quotient topology (see [46]). Define

$$\text{hol} : \mathbb{R}P(N) \to \mathfrak{X}(\pi_1(N), PGL_{n+1}(\mathbb{R})) \qquad \text{where } \text{hol}([M, f]) = [\rho_M \circ f_*]$$

and ρ_M is a holonomy for the projective manifold M.

Theorem 3.0.34 (Thurston [11] p.45)**.** *If N is the interior of a compact smooth manifold, then $hol : \mathbb{R}P(N) \to \mathfrak{X}(\pi_1(N), PGL_{n+1}(\mathbb{R}))$ is a local homeomorphism.*

Elements of $\mathbb{R}P(N)$ are locally parametrized by $\mathfrak{X}(\pi_1(N), PGL_{n+1}(\mathbb{R}))$. Let $\mathfrak{B}(N) \subset \mathfrak{X}(\pi_1(N), PGL_{n+1}(\mathbb{R}))$ be the set of isotopy classes with a properly convex representative. So $[(M, f)] \in \mathfrak{B}(N)$ if and only if there is some $(M', f') \in [(M, f)]$ where M' is a properly convex projective manifold. When N is closed Koszul [36] shows $\mathfrak{B}(N)$ is open in $\mathbb{R}P(N)$, and Benoist [4] shows $\mathfrak{B}(N)$ is closed. If N is not compact, there might exist sequences of non-discrete representations $\pi_1(N) \to PGL_{n+1}(\mathbb{R})$ which converge to the holonomy of a properly convex projective structure on N. Thus in general, $\mathfrak{B}(N)$ is not an open subset of $\mathbb{R}P(N)$.

In the special case when S is a closed surface with $\chi(S) < 0$, the holonomy representation defines a local homeomorphism:

$$\text{hol} : \mathbb{R}P^2(S) \to \mathfrak{X}(S).$$

Then the deformation space $\mathbb{R}P^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$, see [24].

Euclidean Structures on a Torus

Recall the notion of a *complete Euclidean structure*, which is a quotient of the Euclidean plane by a discrete group of translations, see Definition 3.0.27 and Proposition 3.0.28. The following is explained in the introduction of [2]. A Euclidean structure on a torus up to scaling is a conformal structure. Conformal structures on the 2-torus are elliptic curves, which have been classified by a moduli space that is the quotient of the upper half plane, $\mathbb{H}^2$, by the modular group $SL_2(\mathbb{Z})$. Equivalence classes of elliptic curves correspond to orbits of $SL_2(\mathbb{Z})$ on $\mathbb{H}^2$. However, $\mathbb{H}^2/SL_2(\mathbb{Z})$ is not a smooth manifold (it is an orbifold with 2 cone points), so we often study elliptic curves in terms of the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}^2$. The action of $SL_2(\mathbb{Z})$ on $\mathbb{H}^2$ is proper, and the quotient is Hausdorff.

Affine Structures on a Torus

An *affine structure* on a manifold is a maximal atlas with coordinate transformations in the group of affine transformations. Nagano and Yagi [47] describe the set $\mathcal{A}$ of all affine structures on T^2 modulo $\mathrm{Diff}(T^2)_0$, where $\mathrm{Diff}(T^2)_0$ is the group of diffeomorphisims homotopic to the identity. The space $\mathcal{A}/\mathrm{Diff}(T^2)_0$ endowed with the correct topology may be seen as an affine version of Teichmüller space.

Nagano and Yagi show affine structures on the torus are determined by their holonomy groups, and completely describe the space $\mathcal{A}/\mathrm{Diff}(T^2)_0$. The torus is the only orientable surface which admits an affine structure (see [5]).

Since the holonomy groups are images of $\pi_1(T^2)$, they are abelian groups with at most 2 generators. To classify holonomy groups, Nagano and Yagi classify 2-dimensional abelian Lie subgroups of $\mathbb{A}(2)$, the group of affine transformations of the plane. They show every 2-dimensional abelian subgroup of the affine group is conjugate to one of 7 groups. Let

$$\begin{pmatrix} a & b & u \\ c & d & v \end{pmatrix} \qquad \text{denote} \qquad (x,y) \mapsto (ax+by+u, cx+dy+v)$$

as a transformation given in affine coordinates.

Theorem 3.0.35 ([47], Proposition 2.5)**.** *Every maximal abelian subgroup of* $\mathbb{A}(2)$ *is conjugate to one of the following:*

$$\begin{pmatrix} a & b & 0 \\ 0 & a & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ -b & a & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & v \end{pmatrix},$$

$$\begin{pmatrix} 1 & b & u \\ 0 & 1 & b \end{pmatrix}, \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \end{pmatrix}, \begin{pmatrix} 1 & b & v \\ 0 & 1 & 0 \end{pmatrix}.$$

Later we will see 5 of these groups are limits of the Cartan subgroup in $SL_3(\mathbb{R})$, which are classified in chapter 6. Nagano and Yagi show every affine structure on the torus is characterized as G/Γ where G is one of their seven groups and $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$ is a lattice in G.

Baues and Goldman [2] study deformations of affine structures on the real 2-torus. They show that the standard linear action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2$ is *not* proper and the deformation space of *complete* affine structures is *not* Hausdorff. They give coordinates for the deformation space of complete affine structures on T^2, and show there is a differentiable structure on the deformation space. With this structure, the deformation space is diffeomorphic to $\mathbb{R}^2$, and "the action of the mapping class group of T^2 is equivalent in these coordinates with the standard linear action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2$." They also study the dynamics of the action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2$, and describe how these affine structures arise.

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Generalized Cusps

Is it possible to deform convex projective structures to get new convex projective structures? Cooper, Long, and Tillman, [19], have given a sufficient condition for when small deformations of the holonomy of a properly convex structure remain the holonomy of a properly convex projective structure. The ends of the manifold must have the structure of *generalized cusps*, defined as follows (see [19]):

A *radial flow* on $\mathbb{R}P^n$ is a one-parameter subgroup of $PGL_{n+1}(\mathbb{R})$ which fixes each point in a projective hyperplane. There is a point p such that the orbit of every point is contained in a line though p. Let $\Omega \subset \mathbb{R}P^n$ be a properly convex subset in the complement of a hyperplane, and Γ be a subgroup of $PGL_{n+1}(\mathbb{R})$ which acts freely and properly discontinuously on Ω. A convex projective n-manifold $M = \Omega/\Gamma$ *admits a radial flow* if Γ centralizes some radial flow. In particular, M is foliated by lines which develop into lines intersecting at p. A *generalized cusp* is a properly convex projective n-manifold, $B = [0, \infty) \times \partial B$

contained in M, with the product structure given by the flowlines, and ∂B is strictly convex, and the holonomy of B has a finite index subgroup that preserves a complete flag (see [19]). For example, a cusp on a hyperbolic manifold is a generalized cusp.

Let N be a properly convex real projective manifold with compact, strictly convex boundary and which is a union of a compact part and finitely many ends that are generalized cusps. Then the set of holonomies of such structures is an open subset in the representation variety, defined by the condition that the holonomy takes the right form on each cusp. In particular, Cooper, Long, and Tillman show

Theorem 3.0.36 ([19] Theorem 4.3). *Suppose $M = A \cup \mathcal{B}$ is a properly convex n-manifold with holonomy ρ, and ∂M is strictly convex, and A is a compact manifold with boundary $\partial A = A \cap \mathcal{B} = \partial \mathcal{B}$, and each component B of $\mathcal{B}$ is a generalized cusp. Then there is a neighborhood $\mathcal{U} \subset Hom(\pi_1(M), PGL_{n+1}, \mathbb{R}))$ of ρ with the property that if $\rho^\varepsilon \in \mathcal{U}$, and for each component $B \subset \mathcal{B}$ there is a convex projective structure on B, which is a generalized cusp with holonomy $\rho^\varepsilon|\pi_1(B)$, then there is a convex projective structure M^ε on M with holonomy ρ^ε, and ∂M^ε is strictly convex.*

Ballas [1], uses Theorem 3.0.36 to show an open set of the representations of the fundamental group of figure-eight knot complement are the holonomies of a family of finite volume properly convex projective structures on the figure-eight knot complement. In particular:

Theorem 3.0.37 ([1], Theorem 1.1)**.** *Let M be the complement in $\mathbb{S}^3$ of the figure-eight knot. There exists ε such that for each $s \in (-\varepsilon, \varepsilon)$, ρ_s is the holonomy of a finite volume properly convex projective structure on M. Furthermore, when $s \neq 0$ this structure is not strictly convex.*

To do so, he constructs a generalized cusp later classified in chapter 4 as F, and shows it deforms to the standard cusp N. He shows these cusps are foliated by *horospheres*, defined below.

Recall the upper half space model of hyperbolic space gives a coordinate system with a *point at infinity.* A generalization of these coordinates for properly convex domains is introduced in [18]. Let Ω be a properly convex domain, p a point in $\partial\Omega$, and H a supporting hyperplane containing p. There is an identification of the affine patch $\mathbb{R}P^n - H$ with $\mathbb{R}^n$ in which lines through p not contained in H are parallel to the x_1 axis. This is achieved by applying a projective change of coordinates which sends $p \mapsto [e_1]$ and H to the projective hyperplane dual to $[e_{n+1}]$. The x_1 direction is called the *vertical direction.* A set of coordinates with this property is called *parabolic coordinates centered at* (H, p), or just parabolic coordinates if H and p are clear from the context.

Algebraic horospheres are defined using parabolic coordinates as follows: Let $t > 0$, and define $\mathcal{S}_t$ as the translation of the part of $\partial\Omega$ that does not contain any line segments through p by the vector te_1. These sets are *algebraic horospheres centered at* (p, H). See [18] for more on algebraic horospheres.

Chapter 4

A Classification of Generalized Cusps on Convex Projective 3-Manifolds

Suppose M is a manifold of dimension greater than 2. Recall Mostow-Prasad rigidity, 3.0.30, tells us that a finite volume hyperbolic structure on M is unique up to isometry. The notion of a hyperbolic structure may be generalized to a *properly convex projective structure* as follows. Suppose $\Omega \subset \mathbb{R}P^n$ is properly convex, and the holonomy, $\rho : \pi_1(M) = \Gamma \to PGL_{n+1}(\mathbb{R})$ is a discrete and faithful representation with image preserving Ω, and $M \cong \Omega/\rho(\Gamma)$. As we have seen in section 3, there is no analog of Mostow rigidity in this setting, so the deformation theory of convex projective structures remains a rich object of study.

When M is closed, Koszul, [36], shows small perturbations in the holonomy of properly convex projective structures give properly convex structures. If M is not compact, Koszul's results no longer hold. Cooper, Long and Tillman, [19], show that if M is the interior of a compact manifold, then a small perturbation remains properly convex if the holonomy of each boundary component preserves a complete flag.

A *generalized cusp* in dimension 3 is a properly convex 3 manifold, M, diffeomorphic to $T^2 \times [0, \infty)$, and ∂M is strictly convex. This chapter makes progress in classifying generalized cusps in dimension 3 up to projective equivalence. It is closely related to work of [1] and [40].

It is shown in [40] that the holonomy of a generalized cusp is a lattice in a unique upper triangular subgroup $\mathbb{R}^2 \cong H \subset PGL_4(\mathbb{R})$, called a *cusp Lie group*. The groups H are characterized by the property that there is a point $x \in \mathbb{R}P^3$ such that $H \cdot x \subset \mathbb{R}P^3$ is a strictly convex surface. Moreover there is a one parameter subgroup $\Phi_t = \exp(tA)$ with $rank(A) = 1$, and H and Φ_t commute, and $H \cdot \Phi \cong \mathbb{R}^3$.

We classify cusp Lie groups for 3 manifolds:

Theorem 4.0.38. *Each cusp Lie group in $PGL_4(\mathbb{R})$ is conjugate to exactly one of the following groups:*

$$C([r:s:t]) = \left\{ \begin{pmatrix} e^a & 0 & 0 & 0 \\ 0 & e^b & 0 & 0 \\ 0 & 0 & e^c & 0 \\ 0 & 0 & 0 & e^{-(a+b+c)} \end{pmatrix} : \begin{matrix} a,b,c \in \mathbb{R} \\ ar+bs+ct=0 \end{matrix} \right\}, \quad \begin{matrix} [r:s:t] \in \mathbb{R}P^2 \\ r \geq s \geq t > 0 \end{matrix}$$

$$E(s) = \left\{ \begin{pmatrix} e^{b-a} & 0 & 0 & 0 \\ 0 & e^a & e^a(bs+a) & 0 \\ 0 & 0 & e^a & 0 \\ 0 & 0 & 0 & e^{-a-b} \end{pmatrix} : a,b \in \mathbb{R} \right\}, \textit{where } 0 < s < 1/2$$

$$F = \left\{ \begin{pmatrix} e^a & e^a b & \frac{1}{2}e^a(b^2+2a) & 0 \\ 0 & e^a & e^a b & 0 \\ 0 & 0 & e^a & 0 \\ 0 & 0 & 0 & e^{-3a} \end{pmatrix} : a,b \in \mathbb{R} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & a & b & \frac{1}{2}(a^2+b^2) \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} : a,b \in \mathbb{R} \right\}.$$

Let $\Omega \subset \mathbb{R}P^3$ be a properly convex set in the complement of a hyperplane, and let $M \cong \Omega/\Gamma$ be a generalized cusp, where $\Gamma = \mathrm{Hol}(\pi_1(M)) \cong \mathbb{Z} \oplus \mathbb{Z}$. A *radial flow* on $\mathbb{R}P^n$ is a one-parameter subgroup of $PGL_n(\mathbb{R})$ which fixes each point in a projective hyperplane (see section 3). We will use the remainder of this chapter to prove Theorem 4.0.38.

Sketch of proof of Theorem 4.0.38. It follows from Theorem 3.0.36, that if $M \cong \Omega/\Gamma$ is a generalized cusp, there exists a radial flow $\Phi_t \subset PGL_4(\mathbb{R})$, which centralizes Γ, and $\Phi_t(\Omega) \subset \Omega$ for $t < 0$. Therefore, to classify generalized cusps in dimension 3, we want to find all conjugacy classes of subgroups, $H \leq PGL_4(\mathbb{R})$, satisfying:

(a) $H \cong (\mathbb{R}^2, +)$

(b) There is a point $p \in \mathbb{R}P^3$ with orbit $H.p$ that is strictly convex

(c) There exists a radial flow centralized by H.

It follows that H and Φ generate a subgroup of $PGL_4(\mathbb{R})$ isomorphic to $\mathbb{R}^3$, which contains H. All subgroups of $PGL_4(\mathbb{R})$ isomorphic to $(\mathbb{R}^3, +)$ are classified in Proposition 4.0.39, and they are all closed.

Proposition 4.0.44 lists which of these groups contains a 2-dimensional subgroup with convex orbit. Proposition 4.0.45 completes the proof of Theorem 4.0.38 by determining which of the 2 dimensional groups are conjugate. □

Subgroups of $PGL_4(\mathbb{R})$ Isomorphic to $(\mathbb{R}^3,+)$

As a first step, we classify all the subgroups of $PGL_4(\mathbb{R})$ isomorphic to $(\mathbb{R}^3,+)$. In higher dimensions, some subgroups of $SL_n(\mathbb{R})$ isomorphic to $(\mathbb{R}^{n-1},+)$ are not (topologically) closed. An example can be made using a subgroup containing a line of irrational slope in $\mathbb{S}^1 \times \mathbb{S}^1$.

The classification of 3-dimensional abelian subalgebras in $\mathfrak{gl}_4(\mathbb{C})$ is given in [56], p.134, and in [31], section 3.1. The classification of maximal abelian subalgebras of $\mathfrak{sl}_4(\mathbb{R})$ is given as the main result of [63], but there are some of dimension larger than 3. However, the author was unable to find a classification of 3-dimensional abelian subalgebras over $\mathbb{R}$.

Let $G \leq PGL_{n+1}(\mathbb{R})$ be a group, and $p \in \mathbb{R}P^n$. The *orbit* of p under G is the set of images $\{g.p : g \in G\}$. The orbits of G acting on $\mathbb{R}P^n$ give a partition of $\mathbb{R}P^n$. An *orbit closure of* G is the closure of an orbit of G.

Proposition 4.0.39. *In $PGL_4(\mathbb{R})$ there are precisely 15 conjugacy classes of subgroups isomorphic to $(\mathbb{R}^3,+)$:*

$$\underset{C}{\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & \frac{1}{abc} \end{pmatrix}} \quad \underset{E_1}{\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & \frac{1}{ab^2} \end{pmatrix}} \quad \underset{F_0}{\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \frac{1}{a} & c \\ 0 & 0 & 0 & \frac{1}{a} \end{pmatrix}} \quad \underset{F_1}{\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & \frac{1}{a^3} \end{pmatrix}}$$

$$F_2 \quad \begin{pmatrix} a & b & c & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & \frac{1}{a^3} \end{pmatrix} \qquad F_3 \quad \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & \frac{1}{a^3} \end{pmatrix} \qquad N_1 \quad \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_2 \quad \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$N_3 \quad \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_4 \quad \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_4' \quad \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_5 \quad \begin{pmatrix} 1 & 0 & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$N_6 \quad \begin{pmatrix} 1 & a & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_7 \quad \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_8 \quad \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where each matrix represents a group by taking the union over all possible $a, b, c \in \mathbb{R}$ *or* $\mathbb{R}_+$*, as appropriate.*

Proof. We use Haettel's classification, [26] Proposition 6.1, in which he proves every 3-dimensional abelian Lie algebra of $SL_4(\mathbb{R})$ is one of 10 types, up to conjugacy in the Borel group. The proof has two parts. Step 1: exponentiate each of

Haettel's algebras into the Lie group, and consider conjugacy in $SL_4(\mathbb{R})$, rather than just the Borel subgroup. We determine how many conjugacy classes of subgroups are contained in each of Haettel's 10 types. Step 2: show none of the groups are conjugate.

Step 1: We exponentiate Haettel's algebras.

Type 1: The Cartan subalgebra

$$\mathfrak{a} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -a-b-c \end{pmatrix}$$

has $\exp(\mathfrak{a}) = C$.

Type 2: These are algebras with three distinct weights and one off diagonal entry, which consist of matrices of the forms:

$$\mathfrak{i}_\alpha = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -2a-c \end{pmatrix}, \mathfrak{i}_\beta = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -2b-a \end{pmatrix},$$

$$\mathfrak{i}_\gamma = \begin{pmatrix} a & & 0 & 0 \\ 0 & -2b-a & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & b \end{pmatrix}, \mathfrak{i}_{\alpha+\beta} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -2a-b \end{pmatrix},$$

$$\mathfrak{i}_{\beta+\gamma} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & -2b-a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \mathfrak{i}_{\alpha+\beta+\gamma} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & b & 0 & 0 \\ 0 & 0 & -2a-b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

The $\mathfrak{i}_\delta$ are all conjugate in $\mathfrak{sl}_4(\mathbb{R})$ by permutation matrices, and $\exp(\mathfrak{i}_\beta) = E_1$.

Types 3 and 5: These are algebras with 2 distinct weights, that decompose as a direct sum of a 3-dimensional and 1-dimensional algebra. Let $[x : y] \in \mathbb{R}P^1$ be fixed. Types 3 and 5 are algebras consisting of matrices of the forms:

$$\mathfrak{i}^{\alpha,\beta}_{[x:y]} = \begin{pmatrix} c & ax & b & 0 \\ 0 & c & ay & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -3c \end{pmatrix}, \mathfrak{i}^{\beta,\gamma}_{[x:y]} = \begin{pmatrix} -3c & 0 & 0 & 0 \\ 0 & c & ax & b \\ 0 & 0 & c & ay \\ 0 & 0 & 0 & c \end{pmatrix}.$$

The algebras $\mathfrak{i}^{\alpha,\beta}_{[x:y]}$ and $\mathfrak{i}^{\beta,\gamma}_{[x:y]}$ are conjugate in $\mathfrak{sl}_4(\mathbb{R})$ by a permutation matrix. Note $\exp(\mathfrak{i}^{\alpha,\beta}_{[0:1]}) = F_3$, $\exp(\mathfrak{i}^{\alpha,\beta}_{[1:0]}) = F_2$, and $\exp(\mathfrak{i}^{\alpha,\beta}_{[x:y]}) = F_1$, for $(x, y) \neq (0, 0)$.

Type 4: This algebra has 2 distinct weights, and decomposes as a sum of two 2-dimensional algebras. It consists of matrices of the form

$$\mathfrak{i}^{\alpha,\gamma} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & c \\ 0 & 0 & 0 & -a \end{pmatrix},$$

and $\exp(\mathfrak{i}^{\alpha,\gamma}) = F_0$.

Type 6: Let $[x : y : z] \in \mathbb{R}P^2$ be fixed, with $x, z \neq 0$, and consider the algebra consisting of matrices of the form

$$\mathfrak{i}_{[x:y:z]} = \begin{pmatrix} 0 & ax & bx & c \\ 0 & 0 & ay & bz \\ 0 & 0 & 0 & az \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

When $y \neq 0$, the group $\exp(\mathfrak{i}_{[x:y:z]})$ is conjugate to N_1, and $\exp(\mathfrak{i}_{[x:0:z]})$ is conjugate to N_4 by a diagonal matrix.

Type 7: Let $(y,t) \in \mathbb{R}^2$ be fixed, and consider the algebra consisting of matrices of the form

$$\mathfrak{i}_{\alpha,y,t} = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & ay & at \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $(y,t) \neq (0,0)$, the group $\exp(\mathfrak{i}_{\alpha,y,t})$ is conjugate to N_2, and $\exp(\mathfrak{i}_{\alpha,0,0}) = N_8$.

Type 8: Let $(y,t) \in \mathbb{R}^2$ be fixed, and consider the algebra consisting of matrices of the form

$$\mathfrak{i}_{\gamma,y,t} = \begin{pmatrix} 0 & 0 & at & c \\ 0 & 0 & ay & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $(y,t) \neq (0,0)$, the group $\exp(\mathfrak{i}_{\gamma,y,t})$ is conjugate to N_3, and $\exp(\mathfrak{i}_{\gamma,0,0}) = N_7$.

Type 9: Let $[x:y:z:t] \in \mathbb{R}P^3$ be fixed, and consider the algebra consisting of matrices of the form

$$\mathfrak{i}_{[x:y:z:t]} = \left\{ \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & a & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| ax + by + cz + dt = 0 \right\}.$$

By computing orbit closures, it is easy to check $\exp(\mathfrak{i}_{[x:y:z:t]})$ is conjugate to N_6 if $[x:y:z:t] \in \{[0:0:1:t],[1:y:0:0],[1:0:z:0],[0:1:0:t]\}$, and $\exp(\mathfrak{i}_{[x:y:z:t]})$ is conjugate to N_5 otherwise.

Type 10: Let $(x,y) \in \mathbb{R}^2$ be fixed, and consider the algebra consisting of matrices of the form

$$\mathfrak{i}_{x,y} = \begin{pmatrix} 0 & a & by & c \\ 0 & 0 & 0 & ax \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $sign(x) = sign(y)$ then $\exp(\mathfrak{i}_{x,y})$ is conjugate to N_4, and if $sign(x) = -sign(y)$, then $\exp(\mathfrak{i}_{x,y})$ is conjugate to N_4'. Finally, $\exp(\mathfrak{i}_{0,0}) = N_6$, $\exp(\mathfrak{i}_{x,0}) = N_2$ for $x \neq 0$, and $\exp(\mathfrak{i}_{0,y}) = N_3$ for $y \neq 0$.

Thus we have shown that every one of the groups is the image under the exponential map of one of Haettel's algebras.

Step 2: We show none of the groups are conjugate.

The action of each group on $\mathbb{R}P^3$ partitions $\mathbb{R}P^3$ into orbit closures. We prove none of the 15 groups in the list are conjugate, by showing they have orbit closures which are not projectively equivalent. Every orbit closure is a projective subspace.

Let $\{e_1, ...e_4\}$ be the usual basis for $\mathbb{R}^4$, and let $\{[e_1], ...[e_4]\}$ be the projective images in $\mathbb{R}P^3$. Here is a list of the orbit closures of each group.

Orbit closures of C: The orbit closures of C form a projective tetrahedron. The points $[e_1], [e_2], [e_3], [e_4]$, are fixed. The lines $\langle e_1, e_2\rangle, \langle e_1, e_3\rangle, \langle e_2, e_3\rangle, \langle e_1, e_4\rangle$, $\langle e_2, e_4\rangle$, $\langle e_3, e_4\rangle$, are orbit closures, as are the planes $\langle e_1, e_2, e_3\rangle, \langle e_1, e_2, e_4\rangle$, $\langle e_1, e_3, e_4\rangle$, $\langle e_2, e_3, e_4\rangle$. The orbit closure of any point not in one of these subspaces is all of $\mathbb{R}P^3 \cong \langle e_1, e_2, e_3, e_4\rangle$.

Orbit closures of E_1: The group E_1 fixes the points $[e_1], [e_2], [e_4]$. The lines $\langle e_1, e_2\rangle, \langle e_2, e_3\rangle, \langle e_1, e_4\rangle, \langle e_2, e_4\rangle$, and the planes $\langle e_1, e_2, e_3\rangle, \langle e_1, e_2, e_4\rangle, \langle e_2, e_3, e_4\rangle$ are are orbit closures of E_1. The orbit closure of any point not in one of these subspaces is all of $\mathbb{R}P^3$.

Orbit closures of F_0: The group F_0 fixes $[e_1]$ and $[e_3]$. The lines $\langle e_1, e_3\rangle, \langle e_1, e_2\rangle$, and $\langle e_3, e_4\rangle$, and the planes $\langle e_1, e_2, e_3\rangle$ and $\langle e_1, e_3, e_4\rangle$ are orbit closures. The orbit closure of any point not in one of these subspaces is all of $\mathbb{R}P^3$.

Orbit closures of F_1: The group F_1 fixes $[e_1]$ and $[e_4]$. The lines $\langle e_1, e_2\rangle$ and $\langle e_1, e_4\rangle$, and the planes $\langle e_1, e_2, e_3\rangle$ and $\langle e_1, e_2, e_4\rangle$ are orbit closures of F_1. The orbit closure of any point not in one of these subspaces is all of $\mathbb{R}P^3$.

Orbit closures of F_2: The group F_2 fixes $[e_1]$ and $[e_4]$. Every line through $[e_1]$ contained in the plane $\langle e_1, e_2, e_3\rangle$ is an orbit closure. The line $\langle e_1, e_4\rangle$ and the plane $\langle e_1, e_2, e_4\rangle$ are also orbit closures of F_2. The orbit closure of any point not in one of the subspaces in this list is all of $\mathbb{R}P^3$.

Orbit closures of F_3: The group F_3 fixes $[e_4]$, and fixes every point on the line $\langle e_1, e_2\rangle$. The line $\langle e_1, e_4\rangle$ and the planes $\langle e_1, e_2, e_4\rangle, \langle e_1, e_2, e_3\rangle$ are orbit closures of F_3. The orbit closure of any point not in one of these subspaces is all of $\mathbb{R}P^3$.

Orbit closures of N_1: The group N_1 has orbit closures that are a full flag: $[e_1], \langle e_1, e_2\rangle, \langle e_1, e_2, e_3\rangle$, and $\langle e_1, e_2, e_3, e_4\rangle$.

Orbit closures of N_2: The group N_2 fixes $[e_1]$. Every line through $[e_1]$ in the plane $\langle e_1, e_2, e_4\rangle$ is an orbit closure of N_2. The orbit closure of any other point is a projective plane containing $\langle e_1, e_2\rangle$.

Orbit closures of N_3: The group N_3 fixes every point on the line $\langle e_1, e_2\rangle$. Every line through $[e_2]$ in the plane $\langle e_1, e_2, e_3\rangle$ is an orbit closure of N_3. The orbit closures of any point not in one of these subspaces is all of $\mathbb{R}P^3$.

Orbit closures of N_4 and N_4': The groups N_4 and N_4' have the same orbit closures. Both fix the point $[e_1]$. Every line through $[e_1]$ in the plane $\langle e_1, e_2, e_3\rangle$ is an orbit closure. The orbit closure of any other point is all of $\mathbb{R}P^3$. Lemma 4.0.40 shows N_4 and N_4' are not conjugate.

Orbit closures of N_5: The group N_5 fixes every point on the line $\langle e_1, e_2\rangle$, and the orbit of any other point is a projective plane containing $\langle e_1, e_2\rangle$.

Orbit closures of N_6: The group N_5 fixes every point on the line $\langle e_1, e_3 \rangle$, and every line through the point $[e_1]$ in the plane $\langle e_1, e_2, e_3 \rangle$ is an orbit closure of N_6. The orbit of any other point is a projective plane containing $\langle e_1, e_3 \rangle$.

Orbit closures of N_7: The group N_7 fixes every point in the plane $\langle e_1, e_2, e_3 \rangle$ and the orbit of any other point is all of $\mathbb{R}P^3$.

Orbit closures of N_8: The group N_8 fixes $[e_1]$ and every line through $[e_1]$ is an orbit closure of N_8.

None of the orbit closures of the groups in the list are projectively equivalent, except N_4 and N_4', which are shown not to be conjugate in Lemma 4.0.40. Thus these are all conjugacy classes of subgroups of $SL_4(\mathbb{R})$ isomorphic to $\mathbb{R}^3$. □

Lemma 4.0.40. *The groups N_4 and N_4' are not conjugate in $PGL_4(\mathbb{R})$, but they are conjugate in $PGL_4(\mathbb{C})$.*

Proof. Consider the respective Lie algebras:

$$\mathfrak{N}_4 = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathfrak{N}_4' = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We will consider images of subalgebras under the exponential map to show $\mathfrak{N}_4$ and $\mathfrak{N}_4'$ are non-isomorphic Lie algebras. Notice $\exp(\mathfrak{N}_4)$ has $c + \frac{ab}{2}$ in the upper right corner, and $\exp(\mathfrak{N}_4')$ has $c + \frac{a^2+b^2}{2}$ in the upper right. The subalgebra of $\mathfrak{N}_4$ with $a = 0$ is a 2-dimensional subalgebra which exponentiates linearly. There is no 2-dimensional Lie subalgebra of $\mathfrak{N}_4'$ which exponentiates linearly (since $a^2 + b^2$ is positive definite as a real quadratic form).

The groups N_4 and N_4' are conjugate by a complex matrix, but *not* a real matrix. Over $\mathbb{C}$, the algebra $\mathfrak{N}_4'$ has a 2-dimensional subalgebra which exponentiates linearly, when $a = ib$. Iliev and Manivel prove there are 14 conjugacy classes of 3-dimensional abelian subalgebras in $\mathfrak{sl}_4(\mathbb{C})$, see [31] section 3.1. Their list is the same as in Proposition 4.0.39, with only one representative for the conjugacy class $\{\mathfrak{N}_4, \mathfrak{N}_4'\}$ over $\mathbb{C}$. □

Description of cusp Lie subgroups of E_1

We must determine which 2-dimensional subgroups of the groups in Proposition 4.0.39 have a strictly convex orbit. In this section, we do the case of E_1 in detail. We produce a 2 parameter family, $E(r, s)$, of cusp Lie groups. In Proposition 4.0.45, we show they are all conjugate to the groups $E(s)$ in Theorem 4.0.38.

Ballas describes the cusps arising from N (the standard cusp), and F in [1]. Gye-Seon Lee has described the family of cusps arising from $C([r : s : t])$. We will follow the notation and ideas outlined in [1].

Recall the *second fundamental form* is a symmetric bilinear form on the tangent plane of a smooth surface in three-dimensional Euclidean space (see [62]). It is given explicitly for the graph of a twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^3$ which is tangent to the xy plane at the origin by

$$\mathrm{II}(f) = \frac{\partial^2 f}{\partial x^2} dx^2 + 2\frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2.$$

This gives the curvature of f at the origin.

The *Gauss curvature*, G, is the determinant of $\mathrm{II}(f)$, see [45] p.13. Let p be a point in a twice differentiable surface $f(x, y) \subset \mathbb{R}^3$. Proposition 3.5 in [45] says the second fundamental form at p, written $\mathrm{II}(f)_p$, is similar to

$$g^{-1} \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} \cdot \vec{n} & \frac{\partial^2 f}{\partial x \partial y} \cdot \vec{n} \\ \frac{\partial^2 f}{\partial x \partial y} \cdot \vec{n} & \frac{\partial^2 f}{\partial y^2} \cdot \vec{n} \end{bmatrix},$$

where $\vec{n}$ is the normal vector to f at p, and g is the metric. Proposition 3.5 in [45] also implies the sign of the curvature at p depends only on the sign of $\det \mathrm{II}(f)_p$. Therefore if $\det \mathrm{II}(f)_p$ is positive, then f is convex at p.

Suppose there is a transitive affine group action on the graph of f. Since affine maps preserve convexity, the graph of f is convex if there is one point at which f is convex. In particular, there is a transitive group action on the graph of f if it is the orbit of a group action. We summarize this as:

Proposition 4.0.41. *Suppose the graph of $f(x, y) \subset \mathbb{R}^3$ is the orbit of an affine group action, and $p \in f(x, y)$. If $\det \mathrm{II}(f)_p$ is positive, then f is convex.*

Given $[r:s] \in \mathbb{R}P^1$, define

$$E(r,s) := \left\{ \begin{pmatrix} e^{a-b} & 0 & 0 & 0 \\ 0 & e^b & e^b(ar+bs) & 0 \\ 0 & 0 & e^b & 0 \\ 0 & 0 & 0 & e^{-b-a} \end{pmatrix} : a,b \in \mathbb{R} \right\}.$$

Then $E(r,s) \cong (\mathbb{R}^2,+)$ is a subgroup of E_1.

Proposition 4.0.42. *The cusp Lie groups contained in E_1 are the subgroups $E(r,s)$ with $|s| < |r|/2$.*

Proof. Every 2-dimensional Lie subalgebra of $\mathrm{Lie}(E_1)$ is defined by an equation $ar + bs + ct = 0$, where $r, s, t \in \mathbb{R}$ are fixed. After a coordinate permutation, we may assume $t \neq 0$, and take $t = -1$, so $c = ar + bs$. Exponentiating gives the Lie group $E(r,s)$.

Let $p = [x_0 : y_0 : z_0 : 1] \in \mathbb{R}P^3$. The orbit of p under $E(r,s)$ is the surface

$$S := \{[e^{a-b}x_0 : e^b y_0 + e^b(ar+bs)z_0 : e^b z_0 : e^{-a-b}] : a,b \in \mathbb{R}\}.$$

Scale by e^{-a-b} so

$$S = \{[\frac{e^{a-b}x_0}{e^{-a-b}} : \frac{e^b y_0 + e^b(ar+bs)z_0}{e^{-a-b}} : \frac{e^b z_0}{e^{-a-b}} : 1] : a,b \in \mathbb{R}\},$$

and S is in the affine patch that is the complement of the hyperplane $[* : * : * : 0]$. Moreover, elements $E(r,s)$ preserve the 3-dimensional affine subspace

$\{[x_1 : x_2 : x_3 : 1]\}$, so view $E(r, s)$ as affine transformations on $\mathbb{R}^3$. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(a, b) = (e^{2a}, e^{a+2b} + e^{a+2b}(ar + bs), e^{a+2b}).$$

Then S is the graph of f. Perform the coordinate change $A = e^{2a}, B = e^{a+2b}$. So

$$S = \{(A, B(1 + r(\frac{1}{2}\ln A) + s(\frac{-1}{4}\ln A + \frac{1}{2}\ln B)), B)|A, B \in \mathbb{R}_{>0}\}.$$

Then S is the graph of $f(A, B) = B(1 + r(\frac{1}{2}\ln A) + s(\frac{-1}{4}\ln A + \frac{1}{2}\ln B)) \subset \mathbb{R}^3$.

The determinant of the second fundamental form is $\det \mathrm{II}(f) = \frac{(r^2-4s^2)}{16A^2}$, which is positive when $|s| < |r|/2$. So by Proposition 4.0.41, $E(r, s)$ has a convex orbit if $|s| < |r|/2$. $\square$

Algebraic horospheres were defined using parabolic coordinates in section 3.

Proposition 4.0.43. *If $|s| < |r|/2$, the cusp Lie group $E(r, s)$ acts on a convex set foliated by algebraic horospheres, each of which is a convex surface preserved by the action of $E(r, s)$.*

Proof. Notice

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3|x_1, x_3 > 0, (1 + \frac{1}{4}(2r - s)\ln x_1 + \frac{1}{2}s\ln x_3) > x_2)\}$$

is a convex set preserved by the action of $E(r, s)$. Let $\mathcal{H}_k$ be the orbit of $(0, k, 0)$ under $E(r, s)$. So, $\mathcal{H}_k$ is the graph of the strictly convex function

$$(1 + \frac{1}{4}(2r - s) + \frac{1}{2}s)\ln k = \frac{1}{4}(4 + 2r + s)\ln k.$$

Then $\bigcup_{k>0} \mathcal{H}_k$ is a foliation of Ω by horospheres around the point $(0, 1, 0)$.

Let Γ be a lattice in Ω. Then Ω/Γ is a generalized cusp, diffeomorphic to $T^2 \times [0, \infty)$, by a diffeomorphism which sends $\mathcal{H}_k/\Gamma \to T^2 \times \{k\}$. The map $[x_1 : x_2 : x_3 : 1] \to (x_1, x_3)$ restricted to $\mathcal{H}_k$ is a developing map for an affine structure on $\mathcal{H}_k/\Gamma$. $\square$

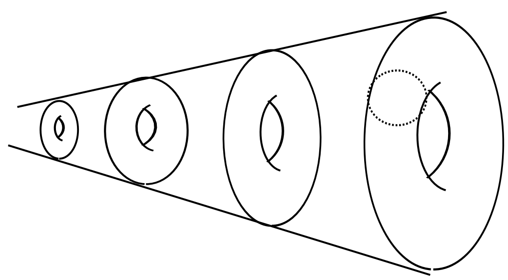


Figure 4.1: A representation of $E(s)$

Convex Orbits

To finish the proof of Theorem 4.0.38, it remains to decide which of the 15 groups in Proposition 4.0.39 have a subgroup that is a cusp Lie group, and to determine conjugacies between these cusp Lie groups. We prove Proposition 4.0.44 using the methods of Proposition 4.0.42.

Suppose $\theta : \mathbb{R}^3 \to \mathfrak{g}$ is an isomorphism of Lie algebras. Given $[r : s : t] \in \mathbb{R}P^2$, define the subalgebra $\mathfrak{g}[r : s : t] := \theta\{(a, b, c) \in \mathbb{R}^3 : ra + sb + tc = 0\}$. Every 2-dimensional subalgebra of $\mathfrak{g}$ is obtained this way. Set $G[r : s : t] = \exp \mathfrak{g}[r : s : t]$.

Proposition 4.0.44. *Suppose G is one of the groups in Proposition 4.0.39, and H is a cusp Lie subgroup of G. Then G is one of C, E_1, F_1 or N_4', and H is conjugate in $PGL_4(\mathbb{R})$ to one of*

- $C[r:s:t]$ *with* $rst(r+s+t) > 0$,
- $E_1[r:s:-1] = E(r,s)$ *with* $|s| < |r|/2$,
- $F_1[r:s:-1]$ *with* $r > 0$
- $N_4'[r:s:-1]$.

Proof. Since every 2-dimensional subalgebra of a 3-dimensional Lie algebra is obtained as $\mathfrak{g}[r:s:t]$, all possible subgroups of the groups in Proposition 4.0.39 isomorphic to $(\mathbb{R}^2, +)$ are realized as $G[r:s:t]$. In all cases except $G = C$, we may perform a change of basis in the Lie algebra so that $t \neq 0$. So unless $G = C$, we assume without loss of generality that $t \neq 0$, and take $t = -1$ so $c = ar + bs$. Set $H := G[r:s:t] = \exp \mathfrak{g}[r:s:t] \cong (\mathbb{R}^2, +)$.

Let $p = [x_0 : y_0 : z_0 : 1] \in \mathbb{R}P^3$, and let $S := \{h.p : h \in H\}$ be the orbit of p. The elements of H preserve the affine patch $\{x_1 : x_2 : x_3 : 1\}$, so we regard H as a set of affine transformations of $\mathbb{R}^3$, and S as a surface that is the image of an orbit of H. As in Proposition 4.0.42, use projective equivalence to scale S to be in the complement of the hyperplane $[* : * : * : 0]$, and dehomogenize (and perhaps perform a change of coordinates) so S is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}^3$. By

Proposition 4.0.41, if $\det \mathrm{I\!I}(f)_p > 0$, then f is convex. We wrote a Mathematica program to execute this process, and ran it on all subgroups isomorphic to $(\mathbb{R}^2, +)$, of the groups in Proposition 4.0.39.

The second fundamental form depends on r, s. The only groups which give rise to positive definite second fundamental forms are C, E_1, F_1, and N_4'. Below is a chart showing the results of the computations for each of the groups. The computations are similar to those for E_1 given in Proposition 4.0.42.

Group	Orbit of Generic Point $p = [x_0 : y_0 : z_0 : 1]$ $\det \mathrm{I\!I}(f)_p$ [up to scaling] : r, s, t are fixed and a, b, c are variables
C	$(e^{\frac{ar}{-ar-bs-ct}} x_0, e^{\frac{bs}{-ar-bs-ct}} y_0, e^{\frac{ct}{-ar-bs-ct}} z_0)$ $rst(r+s+t)x_0^2 y_0^2 z_0^2$
E_1	$(e^a x_0, e^b y_0 + e^b(ar+bs)z_0, e^b z_0, e^{-a-2b})$ $16(r-2s)(r+2s)x_0^2 z_0^4$
F_0	$(e^a x_0 + be^a y_0, e^a y_0, e^{-a} z_0 + e^{-a}(ar+bs), e^{-a})$ $-16r^2 y_0^4$
F_1	$(e^a x_0 + e^a(bs+ar)z_0, e^a y_0 + be^a z_0, e^a z_0, e^{-3a})$ $64rz_0^6$
F_2	$(e^a x_0 + e^a b y_0 + e^a(ar+bs)z_0, e^a y_0, e^a z_0, e^{-3a})$ 0
F_3	$(e^a x_0 + be^a y_0 + \frac{1}{2}e^a(b^2+2bs+2ar)z_0, e^a y_0 + be^a z_0, e^a z_0, e^{-3a})$ 0
N_1	$(x_0 + ay_0 + (\frac{a^2}{2}+b)z_0 + (\frac{a^3}{6}+bs+a(b+s), y_0 + az_0(\frac{a^2}{2}+b), z_0+a, 1)$ -1
N_2	$(x_0 + ay_0 + (\frac{a^2}{2}+b)z_0 + ar + bs, y_0 + az_0, z_0, 1)$ 0

N_3	$(x_0 + (ar + bs)z_0, y_0 + az_0 + (\frac{a^2}{2} + b), z_0 + a, 1)$ 0
N_4	$(x_0 + ay_0 + bz_0 + (ar + bs)ab, y_0 + b, z_0 + a, 1)$ -1
N_4'	$(x_0 + ay_0 + bz_0 + (ar + bs)\frac{1}{2}(a^2 + b^2), y_0 + a, z_0 + b, 1)$ 1
N_5	$(x_0 + az_0 + ar + bs, y_0 + bz_0, z_0, 1)$ 0
N_6	$(x_0 + ay_0 + ar + bs, y_0, z_0 + b, 1)$ 0
N_7	$(x_0 + ar + bs, y_0 + b, z_0 + a, 1)$ 0
N_8	$(x_0 + ay_0 + bz_0 + ar + bs, y_0, z_0, 1)$ 0

□

Notice Theorem 4.0.44 provides an alternate proof that N_4 and N_4' are not conjugate: N_4' has a 2-dimensional subgroup with convex orbit, and N_4 does not.

Since convexity of the cusp Lie group $F[r : s : -1]$ depends only on r, define $F(r) := F[r : 0 : -1]$.

The cusp Lie group $C([r:s:t])$ has a convex orbit if $rst(r+s+t)>0$. The convexity condition is shown in figure 4.2 in an affine patch where $t=1$.

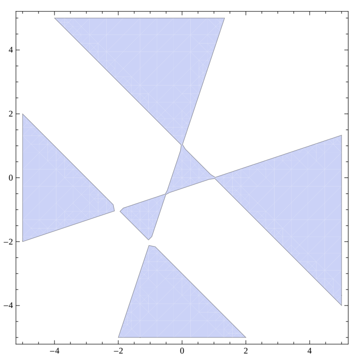


Figure 4.2: $rs(1+r+s)>0$ if and only if (r,s) is in a shaded region.

Conjugacy Classification of Cusp Lie Groups

Proposition 4.0.39 shows the groups C, E_1, F_1, and N_4' are not conjugate since they have orbit closures which are not projectively equivalent. It is easy to see the cusp Lie groups $C([r:s:t])$, $E(r,s)$, $F(r)$, and N have orbit closures which are not projectively equivalent, and so the cusp Lie groups are non-conjugate for any values of r,s,t. In this section, we give conditions for when two groups in the same family are conjugate, and parametrize conjugacy classes in each family by subsets of projective space. This will conclude the proof of Theorem 4.0.38.

Proposition 4.0.45. *1. Every cusp Lie group $C([r:s:t])$ is conjugate to exactly one cusp Lie group where $r\geq s\geq t>0$.*

2. *Every cusp Lie group $E(r,s)$ is conjugate to exactly one cusp Lie group with $1/2 > s > 0$ and $r = 1$.*

3. *If $r \neq 0$, then $F(r)$ is conjugate to $F(1)$.*

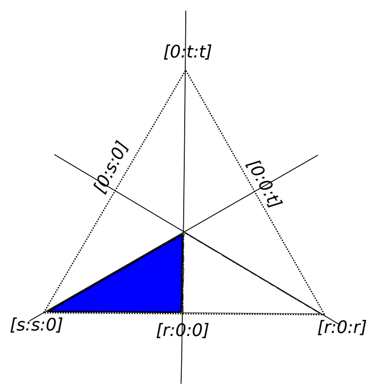


Figure 4.3: A fundamental domain for $S_4 \curvearrowright \mathbb{R}P^2$, where $r \geq s \geq t > 0$

Proof. **Conjugacy in** $C([r:s:t])$: Work in the setting of Lie algebras. Let $\mathfrak{a} \cong \mathbb{R}^3$ be the Cartan subalgebra and $\mathfrak{a}^*$ be the dual. So

$$\mathfrak{a} = \left\{ \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix} \middle| x_i \in \mathbb{R}, \quad \Sigma x_i = 0 \right\}.$$

Let $\phi \in \mathfrak{a}^*$ be a linear functional. Then $\ker\phi$ is a 2-dimensional subalgebra of $\mathfrak{a}$. Notice $\ker\phi$ is unchanged by scaling ϕ. There is a bijection between points of $\mathbb{R}P^2 \cong \mathbb{P}(\mathfrak{a}^*)$ and 2-dimensional subalgebras of $\mathfrak{a}$. We will find the subset of $\mathbb{R}P^2$ which parametrizes conjugacy classes of 2-dimensional subalgebras. Given $[r:s:t] \in \mathbb{R}P^2$, recall the Lie algebra

$$\mathfrak{c}([r:s:t]) = \left\{ \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & -(x_1+x_2+x_3) \end{pmatrix} \middle| rx_1 + sx_2 + tx_3 = 0 \right\}.$$

By Proposition 4.0.44, $C([r:s:t])$ is convex when

$$rst(r+s+t) > 0. \tag{4.1}$$

We will describe a fundamental domain in $\mathbb{R}P^2$ which parametrizes convex subgroups $C([r:s:t])$. Conjugacy must permute the weight spaces, so $C([r:s:t])$ is conjugate to $C([r':s':t'])$ only if there is some $P \in GL_4(\mathbb{R})$ which is a signed permutation of the standard basis of $\mathbb{R}^4$, and $C([r:s:t]) = PC([r':s':t'])P^{-1}$.

Let $\{e_1, e_2, e_3, e_4\}$ be the coordinate vectors in $\mathbb{R}^4$, then S_4 acts on this set by permutations. Thus S_4 preserves the Cartan algebra, $\langle e_1+e_2+e_3+e_4\rangle^\perp = \mathfrak{a} \cong \mathbb{R}^3$. Define $f_i \in \mathbb{R}^3$ to be the orthogonal projection of e_i onto $\mathfrak{a}$. So $f_1+f_2+f_3+f_4 = 0$. The f_i are the vertices of a regular tetrahedron, T, centered at the origin. The action of S_4 permutes $\{f_1, f_2, f_3, f_4\}$. So S_4 acts on $\mathfrak{a} \cong \mathbb{R}^3$ as the group of symmetries of T. Therefore the subset of $\mathbb{R}P^2$ for which $C([r : s : t])$ is convex is divided into 24 fundamental domains under the action of S_4.

Perform a change of coordinates:

$$\alpha_1 = r, \alpha_2 = s, \alpha_3 = t, \alpha_4 = -(r + s + t)$$

then the convexity condition becomes

$$\alpha_1\alpha_2\alpha_3\alpha_4 < 0. \tag{4.2}$$

Notice

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0, \tag{4.3}$$

and (4.3) is preserved under the action of S_4 on $\mathbb{R}^4$. In fact, this is the only linear equation preserved by the action of S_4.

Now S_4 acts transitively on $\{e_1, ..., e_4\}$, so S_4 acts transitively on the projection of $\{e_1, ..., e_4\}$ in the plane (4.3). This divides the double cover of $\mathbb{P}(\mathfrak{a}^*) \cong \mathbb{R}P^2$ into 14 regions as follows. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, with the tiling of a cubeoctahedron shown in figure 4.4. There are 8 triangular regions where $rst(r + s + t) > 0$ and $C([r : s : t])$ is convex; and 6 square regions where $C([r : s : t])$ is not convex. The great circles correspond to $r = 0, s = 0, t = 0$, and $r + s + t = 0$.

The group of symmetries of a cubeoctahedron is the same as the group of symmetries of the cube: signed 4×4 permutation matrices. Projecting

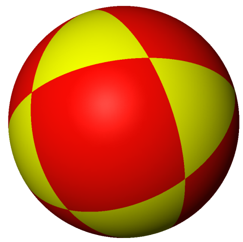


Figure 4.4: A cubeoctahedron.

down to $\mathbb{R}P^2$, the group S_4 acts transitively on the triangles in figure 4.2, and a fundamental domain for the action is pictured in figure 4.3, where $r \geq s \geq t > 0$. Therefore every cusp Lie group $C([r:s:t])$ is conjugate to a cusp Lie group with $r \geq s \geq t > 0$.

Conjugacy in $E(r,s)$ **:** If $(r,s) \neq (0,0)$, then $E(r,s)$ has 3 eigenvectors, $\{e_1, e_2, e_4\}$, so any conjugacy must permute them. The dimension of the generalized eigenspace associated to e_2 is 2, which is larger than the dimension of the generalized eigenspaces associated to e_1 and e_4. So, any conjugacy must fix e_2, permute $\{e_1, e_4\}$, and sends e_3 to any vector in the the generalized eigenspace spanned by $\langle e_2, e_3 \rangle$. Thus any conjugacy is by Q or QP where

$$Q = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & 0 & \alpha_5 \end{pmatrix}, \text{ and } P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

with $\alpha_1, ..., \alpha_5 \in \mathbb{R}$. Since $QE(r,s)Q^{-1} = E(\frac{\alpha_2}{\alpha_4}r, \frac{\alpha_2}{\alpha_4}s)$, and $[r:s] = [\frac{\alpha_2}{\alpha_4}r, \frac{\alpha_2}{\alpha_4}s] \in \mathbb{R}P^1$, conjugating by Q does not change the group, and any conjugacy must be by P.

Notice $E(0,0)$ is conjugate to $C([-2:-1:0])$, so assume $(r,s) \neq (0,0)$, and $[r:s] \in \mathbb{R}P^1$. Finally $E(r,s)$ is conjugate to $E(-r,s)$ by P. Since $(r-2s)(r+2s) > 0$, every $E(r,s)$ is conjugate to a group where $r = 1$ and $1/2 > s > 0$.

Conjugacy in $F(r)$: Given $r, s \in \mathbb{R}$, the group

$$F(r,s) = \begin{pmatrix} e^a & e^a b & \frac{1}{2}e^a(b^2 + 2ar + 2bs) & 0 \\ 0 & e^a & e^a b & 0 \\ 0 & 0 & e^a & 0 \\ 0 & 0 & 0 & e^{-3a} \end{pmatrix},$$

is the image of $\mathfrak{f}_1[r:s:-1]$ under the exponential map. First check $F(r,s)$ is conjugate to $F(r)$ by conjugating by S.

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} \frac{1}{\sqrt{r}} & 1 & 1 & 0 \\ 0 & 1 & \sqrt{r} & 0 \\ 0 & 0 & \sqrt{r} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $r \neq 0$, then $F(r)$ is conjugate to $F(1)$ by R. Recall from Proposition 4.0.44 that $F(r)$ has a convex orbit if and only if $r > 0$. Thus there is only one cusp in this family.

$\square$

Chapter 5

Geometric Transitions

A geometric transition is a continuous path of geometries which abruptly changes type in the limit. An intuitive example is given in chapter 1 of a sequence of balls with increasing radius, which limits to a plane. This chapter defines these ideas more precisely, and gives an overview of some results on geometric transitions by Haettel [26], Iliev and Manivel [31], Cooper, Danciger and Weinhard [16], and others.

To describe limits of sequences of groups, we will use the Chabauty topology, which is defined on the set of all closed subgroups of a locally compact group (see [22], [9], [3], [11]). CHABAUTY TOPOLOGY

Definition 5.0.46 (Chabauty Topology, see [22])**.** Let X be a topological space, and let 2^X denote the set of closed subsets of X. For a compact subset K, and a nonempty open subset U of X, set

$$\mathcal{O}_K = \{F \in 2^X | F \cap K = \emptyset\} \text{ and } \mathcal{O}'_U = \{F \in 2^X | F \cap U \neq \emptyset\}.$$

The finite intersections $\mathcal{O}_{K_1} \cap \cdots \cap \mathcal{O}_{K_m} \cap \mathcal{O}'_{U_1} \cap \cdots \cap \mathcal{O}'_{U_n}$, for $m, n \geq 0$, form a basis of the *Chabauty topology* on 2^X.

Proposition 2 in [22], shows for any space X that 2^X with the Chabauty topology is compact. Convergence in 2^X is given by the following.

Theorem 5.0.47. *[[11] p.60] A sequence (F_j) of closed subsets of X converges to a closed subset, F, if and only if the following 2 conditions hold:*

- *for all $x \in F$, there exists for all $i \geq 1$ a point $x_i \in F_i$ such that $x_i \to x$*
- *for all strictly increasing sequences (i_j) and for all sequences x_{i_j} such that $x_{i_j} \in F_{i_j}$ and $x_{i_j} \to x \in X$ then $x \in F$.*

If $X = G$ is a locally compact group, and $\mathcal{S}(G) \subset 2^G$ is the space of closed subgroups, then $\mathcal{S}(G)$ is a closed subspace of 2^G, and therefore compact (see [11] p.61). A basis of neighborhoods for a closed subgroup $C \in \mathcal{S}(G)$ is given by

$$\mathcal{N}_{K,U}(C) = \{D \in \mathcal{S}(G) | D \cap K \subset CU \text{ and } C \cap K \subset DU\}.$$

For example, if $G = \mathbb{R}$, then $\mathcal{S}(\mathbb{R})$ is homeomorphic to a compact interval $[0, \infty]$. The points $0, \lambda$ and ∞ correspond respectively to the subgroups $\{0\}$, $\frac{1}{\lambda}\mathbb{Z}$ and $\mathbb{R}$. If $G = \mathbb{Z}$, then $\mathcal{S}(\mathbb{Z})$ is homeomorphic to $\{\frac{1}{n}\}_{n \geq 1} \cup \{0\} \subset [0, 1]$, with $\frac{1}{n}$ corresponding to $n\mathbb{Z}$, and 0 to $\{0\}$, see [22].

Proposition 5.0.48 ([3] E.1.3)**.** *If X is a compact metric space, then the Hausdorff distance induces the Chabauty topology on 2^X.*

For more information on the Chabauty topology, see [13], [3], and [26].

LIMITS OF GEOMETRIES

Cooper, Danciger and Weinhard, [16], discuss limits of geometries embedded in a larger ambient geometry, and classify the *geometric limit* of any geometry with an isometry group that is a symmetric subgroup of $PGL_n(\mathbb{R})$. As an application, they classify which Thurston geometries are limits of hyperbolic geometry inside of projective geometry.

A geometry (H, Y) is a *subgeometry* of (G, X), written $(H, Y) \subset (G, X)$, if H is a closed subgroup of G, and Y is an open subset of X on which H acts transitively. For example, spherical and Euclidean geometry are both subgeometries of real projective geometry, and we study transitions between them in this context.

Definition 5.0.49. [[16]] Let (H, Y) and (L, Z) be subgeometries of the ambient geometry (G, X). Then (L, Z) is a *limit* of (H, Y), if there exists a sequence $c_n \in G$ so that

- the sequence of conjugates $c_n H c_n^{-1} \to L$ in the Chabauty topology,
- there exists $z \in Z \subset X$ so that $z \in c_n Y$ for n sufficiently large.

There are several other notions of limit discussed in [16], among them: connected geometric limit, local geometric limit, expansive limit, Lie algebra limit, and intrinsic limit. We will focus on the notion of *geometric limit* given in Definition 5.0.49. The main theorem of [16] shows limits of symmetric subgroups of semisimple Lie groups are 1-parameter limits:

Theorem 5.0.50. *Let H be a symmetric subgroup of a semisimple Lie group, G, with finite center. Then any limit L' of H in G is the limit under conjugacy by a one parameter subgroup. More precisely, there exists $X \in b$ such that the limit $L = \lim_{t\to\infty} \exp(tX) H \exp(-tX)$ is conjugate to L'. Furthermore,*

$$L = Z_H(X) \ltimes N_+(X)$$

where $Z_H(X)$ is the centralizer of X in H, and $N_+(X)$ is the connected nilpotent subgroup

$$N_+(X) := \{g \in G : \lim_{t\to\infty} \exp(tX)^{-1} g \exp(tX) = 1\}.$$

In the case $G = PGL_n(\mathbb{R})$, Theorem 5.0.50 implies limits of symmetric subgroups have a block matrix form. Let $H \leq PGL_n(\mathbb{R})$ be a symmetric subgroup, and L be a limit of H. Then there is a basis of $\mathbb{R}^n$ with respect to which L has the form:

$$\begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ * & A_2 & 0 & \dots & 0 \\ * & * & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & A_k \end{pmatrix},$$

where the blocks $*$ are arbitrary, and $\text{diag}(A_1, ..., A_k) \in H$. The symmetric subgroups of $PGL_n(\mathbb{R})$ are $P(GL_p(\mathbb{R}) \oplus GL_q(\mathbb{R})), PO(p,q)$ where $p + q = n$, or $P(GL_m(\mathbb{C})), P(Sp(2m, \mathbb{R}))$ where $n = 2m$, and their limits are classified in [16]. They also show there are finitely many limits of any symmetric subgroup.

Properties of Geometric Transitions

Several important properties of geometric transitions are proven in [16], and since they will be used later in chapters 6 and 8, we present them here. Cooper, Danciger and Weinhard give several examples (3.2), of limits with unexpected properties. For example, the geometric limit of a connected group might not be connected, and there is a one-dimensional group with a two-dimensional conjugacy limit.

Recall a *linear algebraic group* is a subgroup of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ that is defined by polynomial equations.

Proposition 5.0.51 (3.1 in [16])**.** *Let G be a linear algebraic group (defined over $\mathbb{C}$ or $\mathbb{R}$). Suppose that H is an algebraic subgroup and L a conjugacy limit of H. Then L is algebraic and $\dim L = \dim H$.*

Note that [16] proves Theorem 5.0.51 in the more general setting of algebraic groups, but in this thesis, we shall only apply it to linear algebraic groups. Denote by H_0 the component of H containing the identity.

Proposition 5.0.52 (3.2 in [16])**.** *Let G be an algebraic Lie group (defined over $\mathbb{C}$ or $\mathbb{R}$), let H be an algebraic subgroup and let L be any limit of H. Then $\dim N_G(H_0) \leq \dim N_G(L_0)$ with equality if and only if L and H are conjugate.*

This says the dimension of the normalizer must always increase in a limit. Thus, the length of a chain of limits is finite. As a corollary, if $H \to L$ and $L \to H$, then L is conjugate to H.

Theorem 5.0.53 (3.3 in [16])**.** *Let G be an algebraic Lie group. The relation of being a connected geometric limit induces a partial order on the connected, algebraic, subgroups of G. Moreover the length of every chain is at most $\dim G$.*

If L is a limit of H, then the eigenvalues of L are closely related to those of H. In a limit, eigenvalues are either unchanged, or become equal. Let $\mathfrak{h}$ denote the Lie algebra, and $\mathrm{Char}(\mathfrak{h}) \subset \mathbb{R}[x]$ be the set of characteristic polynomials of all elements in $\mathfrak{h}$. It is easy to check $\mathrm{Char}(\mathfrak{h})$ is closed and invariant under conjugation.

Proposition 5.0.54 (3.4 in [16])**.** *Suppose H is a closed algebraic subgroup of $GL_n(\mathbb{R})$, and L is a conjugacy limit of H. Then $Char(\mathfrak{l}) \subset Char(\mathfrak{h})$, where $\mathfrak{h}, \mathfrak{l} \subset \mathfrak{gl}(n)$ denote the Lie algebras of H and L respectively.*

Haettel: Compactification of the Space of Limits

Haettel, [26], studies the homogeneous space of Cartan subgroups of a Lie group G. The *Chabauty compactification* is the closure of the set of all closed subgroups of G endowed with the Chabauty topology. In the case when G has rank 1, or when $G = SL_n(\mathbb{R})$ for $n = 3, 4$, the Chabauty compactification is the same as the set of all closed connected abelian subgroups with dimension the real rank of G. In the case of $SL_3(\mathbb{R})$, Haettel shows the Chabauty compactification is a simply connected cell complex.

Let G be a real semisimple Lie group with a finite number of connected components, finite center, and no compact factors. The space $\mathcal{S}(G)$ of closed subgroups of G is naturally equipped with the Chabauty topology, and is compact. Let $\overline{\text{Cartan}(G)}^{\mathcal{S}} \subset \mathcal{S}(G)$ denote the closure of the space of Cartan subgroups of G. This is the *Chabauty compactification* of Cartan(G). Let $B \subset G$ denote the Borel subgroup. Define $\mathcal{A}(G)$ to be the set of closed connected abelian subgroups of B with dimension the real rank of G. Notice $\overline{\text{Cartan}(G)}^{\mathcal{S}} \subset \mathcal{A}(G) \subset \mathcal{S}(G)$. Haettel shows the following fact about the orbits of $G \curvearrowright G$ by conjugation:

Theorem 5.0.55. *Suppose G is a semi direct product of real rank 1 groups or of copies of $SL_3(\mathbb{R})$ and $SL_4(\mathbb{R})$. Then $\overline{Cartan(G)}^{\mathcal{S}} = \mathcal{A}(G)$ and is the union of a finite number of orbits of G.*

For $SL_3(\mathbb{R})$ and $SL_4(\mathbb{R})$, Haettel classifies these orbits. In the case $n \geq 7$, Haettel uses a dimension count to show the inclusion $\overline{\text{Cartan}(SL_n(\mathbb{R}))}^{\mathcal{S}} \subset \mathcal{A}(SL_n(\mathbb{R}))$ is proper. Since this will be of importance in chapter 8, we present Haettel's proof here.

Lemma 5.0.56 (3.4 in [26])**.** *When $m \geq 7$, there is an abelian subalgebra of dimension $m - 1$ contained in the Borel subalgebra, which is not the limit of a Cartan subalgebra.*

Proof. Haettel adapts Iliev and Manivel's proof in [31] to the real setting. Let $V \subset \mathbb{R}^m$ be a vector subspace of dimension $p = \lfloor \frac{m}{2} \rfloor$. The set of endomorphisms

$$X_V = \{f \in \text{End}(\mathbb{R}^m) : \text{Im}(f) \subset V \subset \text{Ker}(f)\}$$

is a vector subspace of dimension $p(m-p)$, and it is an abelian Lie subalgebra of $\mathfrak{sl}_m(\mathbb{R})$ contained in the Borel subalgebra. Every $(m-1)$-dimensional vector subspace of X_V is an $(m-1)$-dimensional abelian subalgebra of $\mathfrak{sl}_m(\mathbb{R})$ contained in a Borel subalgebra, so it is an element of $\mathcal{A}(\mathfrak{sl}_m(\mathbb{R}))$. Any generic $(m-1)$-dimensional subspace of X_V uniquely determines V. The set of $m-1$ dimensional vector subspace of X_V, as V runs over all p dimensional subspaces of $\mathbb{R}^m$, is a subvariety of $\mathcal{A}(\mathfrak{sl}_m(\mathbb{R}))$ of dimension

$$\dim \mathrm{Grass}(m-1, p(m-p)) + \dim \mathrm{Grass}(p, m) = (m-1)(p(m-p)-m+1) + p(m-p),$$

which is strictly larger than $m(m-1) = \dim \mathrm{Cartan}(\mathfrak{sl}_m(\mathbb{R}))$ as soon as $m \geq 7$. By [8], 2.8.13, $\mathrm{Cartan}(\mathfrak{sl}_m(\mathbb{R}))$ is a proper subset of $\mathcal{A}(\mathfrak{sl}_m(\mathbb{R}))$. $\square$

Thus the inclusion $\overline{\mathrm{Cartan}(SL_n(\mathbb{R}))}^{\mathcal{S}} \subset \mathcal{A}(SL_n(\mathbb{R}))$ is proper when $n \geq 7$, and $\overline{\mathrm{Cartan}(SL_n(\mathbb{R}))}^{\mathcal{S}} = \mathcal{A}(SL_n(\mathbb{R}))$ when $n = 3, 4$. It is a natural question to ask what happens for $n = 5, 6$.

Following notation in section 2, let X be the symmetric space of non-compact type associated to the Lie group, G, and let $\partial_\infty X$ be the sphere at infinity defined in section 2. In the case G has rank 1, Haettel has completely determined the topology of $\overline{\mathrm{Cartan}(G)}^{\mathcal{S}}$:

Theorem 5.0.57. *Suppose G has real rank 1. Denote by $\partial_\infty X^{(2)}$ the space of pairs of points that are identified in $\partial_\infty X$. Then $\overline{Cartan(G)}^{\mathcal{S}}$ is naturally G-homeomorphic to the blow up of $\partial_\infty X^{(2)}$ along the diagonal.*

When $G = SL_3(\mathbb{R})$, Haettel shows

Theorem 5.0.58. *The space $\overline{Cartan(G)}^{\mathcal{S}}$ is simply connected.*

Recall the definition of convergence in the Chabauty topology 5.0.47 on the level of groups. Haettel shows convergence in the Lie algebra is equivalent to convergence in the Lie group:

Proposition 5.0.59. *The map*

$$\eta : \overline{Cartan(G)}^{\mathcal{S}} \to \overline{Cartan(\mathfrak{g})}^{\mathcal{S}} \qquad by \qquad H \mapsto Lie(H)$$

is a G-equivariant homeomorphism. Here $\overline{Cartan(\mathfrak{g})}^{\mathcal{S}}$ is a subspace of the Grassmannian, $Grass(r, \mathfrak{g})$, and G acts on $Cartan(\mathfrak{g})$ and on $\overline{Cartan(G)}$ with the adjoint action.

Let A be a fixed Cartan subgroup of G, and denote by $N_G(A)$ the normalizer of A in G. Recall Cartan(G) is homeomorphic to the homogenous space $G/N_G(A)$.

Proposition 5.0.60. *The maps*

$$\Phi : G/N_G(A) \to \mathcal{S}(G) \qquad and \qquad \phi : G/N_G(A) \to Grass(r, \mathfrak{g})$$

$$gN_G(A) \mapsto gAg^{-1} \qquad\qquad gN_G(A) \mapsto Ad(g(\mathfrak{a}))$$

are G-equivariant homeomorphisms.

Haettel gives several more properties of the Chabauty compactification cited from [31] and [39]. Denote by $\overline{\text{Cartan}(\mathfrak{g})}^{\text{Zar}}$ the Zariski closure of Cartan($\mathfrak{g}$) in $\text{Grass}_r(\mathfrak{g})$.

Theorem 5.0.61. *Let G be an algebraic real semi-simple Lie group with a finite number of connected components, finite center, and no compact factors.*

1. *Every Lie subalgebra of $\mathfrak{g}$ belonging to $\overline{Cartan(\mathfrak{g})}^{Zar}$ is the Lie algebra of an algebraic subgroup of G.*

2. *If the G orbit of a Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}$ belonging to $\overline{Cartan(\mathfrak{g})}^{Zar}$ is closed, then $\mathfrak{l}$ contains only nilpotent operators.*

3. *The variety $\overline{Cartan(\mathfrak{sl}_3(\mathbb{R}))}^{Zar}$ is smooth.*

Haettel goes on to classify 2-dimensional subalgebras of $\mathfrak{sl}_3(\mathbb{R})$, and shows there are 8 2-dimensional subalgebras up to conjugacy in the Borel group. He shows each is a limit of the Cartan subgroup using the convergence of various sequences under the adjoint action. He does the same for $\mathfrak{sl}_4(\mathbb{R})$.

Iliev and Manivel: Va-

Iliev and Manivel, [31], study varieties of reductions associated to the variety of rank 1 matrices in $\mathfrak{gl}_n$. Throughout, they study varieties over a normed algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ (reals, complexes, quaternions, octonions). Denote by $\widehat{Red}(n)^0$ the space of Cartan subalgebras of $\mathfrak{gl}_n$. Let $\mathfrak{a}$ be a fixed Cartan subalgebra of $\mathfrak{sl}_n$. There is an isomorphism $\widehat{Red}(n)^0 \cong PGL_n/N(\mathfrak{a})$. Then $\widehat{Red}(n)$ is the Zariski closure of $\widehat{Red}(n)^0$ in the Grassmannian, $\text{Grass}(n-1, \mathfrak{sl}_n)$. Iliev and Manivel call $\widehat{Red}(n)$ the *variety of reductions*. It is easy to see that $\widehat{Red}(n)$ is isomorphic to Haettel's $\overline{\text{Cartan}(G)}^{\mathcal{S}}$. Notice $\widehat{Red}(n)$ is a subvariety of the space $\widehat{Ab}(n)$ of abelian $(n-1)$-dimensional subalgebras of $\mathfrak{sl}_n$.

Iliev and Manivel show $\widehat{Ab}(n) = \widehat{Red}(n)$ for $n = 3, 4$, and that in general the variety $\widehat{Red}(n)$ is an irreducible component of $\widehat{Ab}(n)$. They show $Red(n) \neq Ab(n)$ for large n (when $n \geq 7$) as in Lemma 5.0.56. Some questions in [31] are:

Question 5.0.62. 1. Does $\widehat{Red}(n) = \widehat{Ab}(n)$ for $n = 5, 6$?

2. What properties characterize points of $\widehat{Red}(n)$ in $\widehat{Ab}(n)$?

They show the action of PGL_n on $\widehat{Ab}(n)$ has finitely many orbits only for $n \leq 5$. In particular, this implies the action of PGL_n on $\widehat{Red}(n)$ has finitely many orbits if $n \leq 5$. They also ask

Question 5.0.63. Does $\widehat{Red}(n)$ contain infinitely many orbits when $n \geq 6$?

Iliev and Manivel describe several orbits of $\widehat{Red}(n)$. For example, if an algebra $\mathfrak{a} \in \widehat{Ab}(n)$ can be described as the centralizer of two of its elements, then $\mathfrak{a} \in \widehat{Red}(n)$. They go on to prove properties of the variety $\widehat{Red}(n)$: for instance, $\widehat{Red}(n)$ contains a smooth codimension one orbit, but $\widehat{Red}(n)$ is always singular for $n \geq 4$. For any n, the variety $\widehat{Red}(n)$ is an irreducible component of $\widehat{Ab}(n)$.

The last section classifies all of the orbits in $\widehat{Red}(4)$ over $\mathbb{C}$, and shows there are precisely 14 orbits. They give a graph showing the degeneracies of the orbits, which corresponds to a digraph of conjugacy limits of groups. Several of Iliev and Manivel's questions are answered in [43].

Regeneration Degenerations, Deformations, and Contrac-

Geometric transitions and regenerations have also been studied extensively by Porti, Danciger, Kerckhoff, Hodgson and others in [20], [21], [30], [32], [50], [51], [17], [15], [29], [1], [10], [32]. The idea of a limit is related to degeneration, regeneration, deformation, and contraction.

Geometric transitions appear in physics as *contractions* of a Lie algebra. A Lie algebra is determined by the values of $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on a basis, and hence by *structure constants*. We may continuously change the structure constants as long as $[\cdot,\cdot]$ still determines a Lie algebra. Changing the basis changes theses constants. A limit under such a change of basis given by the adjoint action is a contraction.

This is related to theory of Inönü-Wigner contractions in physics (see [10], [32]). Note this construction is independent of embedding $\mathfrak{g}$ in a larger Lie algebra. Physicists use deformations of Lie algebras in several ways, for example the classical limit in relativity where the speed of light, $c \to \infty$; and in transitioning from quantum mechanics to Newtonian mechanics, when $\hbar \to 0$.

The simplest example of regeneration and degeneration is studied in [29] and [50]: a hyperbolic structure which may be collapsed to a point, and then rescaled to a Euclidean metric. A spherical structure may be collapsed to a point as well, and rescaled to a Euclidean one, so this provides a continuous path from a hyperbolic structure to a spherical structure. Figure 5.1 shows spherical structures on a sphere with three cone points (of equal angle) which collapse down to a point as the cone angles increase to $\frac{2\pi}{3}$. After rescaling the metric, the structures limit to a Euclidean cone sphere and then transition to hyperbolic cone structures.

The process of rescaling to the structure is called *regeneration*, and the collapsing process is called *degeneration*.

Porti and collaborators have studied regenerating hyperbolic structures in [30], [50], [51]. They study a sequence of hyperbolic cone structures on a manifold, parametrized by the cone angle, which collapse to a fibration. They then rescale

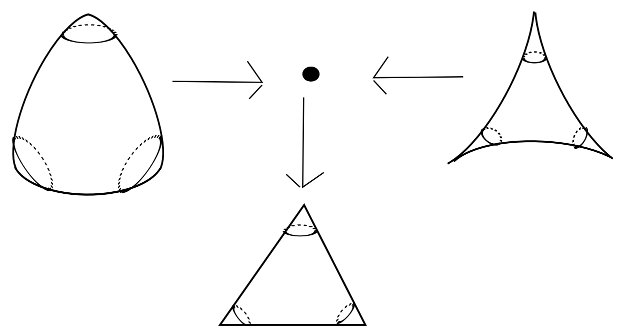


Figure 5.1: A transition between spherical and hyperbolic structures.

the metric to a sol structure [30]. In [50], [51] Porti studies limits of hyperbolic structures on orbifolds which regenerate (respectively) to nil and Euclidean structures. These results are based on Hodgson's work in his thesis, [29].

Some of these ideas are used in [17] in the discussion of cone manifolds. Deformations of hyperbolic cone manifolds lead to new geometric structures. Hyperbolic cone manifolds are classified in [17].

Deformations of hyperbolic structures inside projective structures are studied in [15]. In the case of some smoothness conditions, Cooper, Long, and Thistlethwaite show that such deformations of real hyperbolic structures exist if and only if the deformation may be considered as one of a complex hyperbolic structure. The image of such deformations inside $\mathrm{Isom}(\mathbb{C}H^n) \cong PU(n,1)$ is discrete and faithful. Manifolds which deform projectively are *flexible*. They conjecture that all closed hyperbolic 3-manifolds have finite sheeted covers which are flexible.

Danciger [20] constructs a geometric transition from 3-dimensional hyperbolic geometry to Anti de Sitter (AdS) geometry, and produced a new intermediate geometry, called *half pipe* geometry, known as $X((1,2)(1))$ in the classification of [16], which arises in the following way. A limit of conjugates of 3 -dimensional hyperbolic geometry may collapse onto a hyperbolic plane, $\mathcal{P}$, but $\mathbb{H}^3$ may be rescaled to prevent collapse. This rescaling preserves $\mathcal{P}$ but stretches transverse to $\mathcal{P}$. As the hyperbolic structures degenerate, the collapsing direction is rescaled so that the structures converge to half-pipe geometry. Interpreting the hyperbolic structures as projective structures, this rescaling is a projective change of coordinates. If Σ is a knot in a closed manifold N, then [20] shows how to construct a path of hyperbolic structures on $N - \Sigma$ which degenerate to AdS structures, for some $\Sigma \subset N$.

Danciger shows in [21] how to construct the aforementioned transverse hyperbolic foliations by gluing ideal tetrahedra (as prescribed by Thurston's gluing equations) in which some of the tetrahedra are collapsed. He describes when such transverse hyperbolic foliations can be regenerated to give hyperbolic or AdS structures. Gluing constructions of ideal tetrahedra and triangulations give new examples of geometric transitions as paths of projective structures, [21].

Chapter 6

Conjugacy Limits of the Cartan Subgroup in $SL_3(\mathbb{R})$

This chapter classifies conjugacy limits of the positive diagonal Cartan subgroup $C \leq SL_3(\mathbb{R})$ using hyperreal techniques. Let G be a Lie group and H a closed subgroup. Recall Definition 5.0.49: a sequence of subgroups H_n of G *converges* to H if the following two conditions are satisfied:

(a) For every $h \in H$ there is a sequence $h_n \in H_n$ converging to h

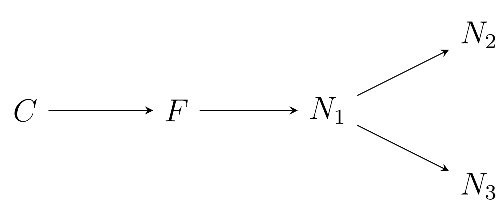
(b) For every sequence $h_n \in H_n$, if there is a subsequence which converges to h, then $h \in H$.

This definition is logically equivalent to convergence in the Chabauty topology, 5.0.47. A subgroup $L \leq G$ is a *conjugacy limit* of a subgroup $H \leq G$ if there is a sequence of elements $P_n \in G$ such that $P_n H P_n^{-1}$ converges to L. The identity component of a conjugate of C is the stabilizer of the vertices of a triangle in $\mathbb{R}P^2$. We will show in Theorem 6.0.65 that a conjugacy limit of C is characterized in terms of the stabilizer of a *configuration*, which is a set whose elements are points and lines in $\mathbb{R}P^2$.

Let $\mathcal{Q} = \{C, F, N_1, N_2, N_3\}$ be the set consisting of the following 5 subgroups of $SL_3(\mathbb{R})$ with $a, b > 0$,

$$\underset{C}{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}}, \quad \underset{F}{\begin{pmatrix} a & t & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}}, \quad \underset{N_1}{\begin{pmatrix} 1 & s & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}}, \quad \underset{N_2}{\begin{pmatrix} 1 & s & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \underset{N_3}{\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}}.$$

Let Γ be the directed graph shown below whose vertices are the elements of $\mathcal{Q}$.



Theorem 6.0.64. *1. Every subgroup of $SL_3(\mathbb{R})$ isomorphic to $\mathbb{R}^2$ is conjugate to exactly one element of $\mathcal{Q}$.*

2. If $G_1 \neq G_2 \in \mathcal{Q}$ then G_2 is a conjugacy limit of G_1 if and only if there is a directed path in Γ from G_1 to G_2.

3. If G is a conjugacy limit of C, then G is a 1-parameter limit of C. Moreover every directed path in Γ is a 1-parameter path of conjugacies.

A *configuration* is a finite set T each element of which is a point or projective line in $\mathbb{R}P^2$. Define $\mathcal{T}Q = \{TC, TF, TN_1, TN_2, TN_3\}$ to be the set of 5 elements which are the configurations shown below.

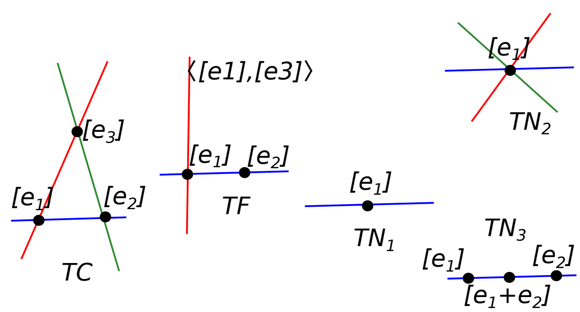


Figure 6.1: The 5 configurations in $\mathcal{T}\mathcal{Q}$

There is a partial order on $\mathcal{T}Q$ given by inclusion.

Theorem 6.0.65. *There is a bijection $\theta : \mathcal{Q} \longrightarrow \mathcal{T}Q$ defined by $\theta(G) = T$ if and only if*

1. G preserves each element of T

2. T is maximal (in the partial order) subject to this condition

The *hyperreals* ${}^*\mathbb{R}$ are an ordered field that contains $\mathbb{R}$. (See section 1.) A *non-standard triangle* is a subset $\mathscr{T} \subset {}^*\mathbb{R}P^2$ consisting of three points $p, q, x \in {}^*\mathbb{R}P^2$

in general position, and the three lines between p, q and x. Let $\mathscr{G} = \mathscr{G}(p, q, x) \leq SL_3({}^*\mathbb{R})$ be the conjugate of the positive diagonal subgroup which fixes p, q, and x . The *shadow* of $\mathscr{G}$ is the subgroup $sh(\mathscr{G}) \leq SL_3(\mathbb{R})$ consisting of those matrices whose entries differ by an infinitesimal from some matrix in $\mathscr{G}$.

The next theorem determines exactly when a sequence $P_n \in SL_3(\mathbb{R})$ has the property $P_n C P_n^{-1}$ converges, and if it does, the conjugacy class of the limit under *any* sequence of matrices. See section 6 for the definition of α.

Theorem 6.0.66. *Let $\mathscr{T}$ be a nonstandard triangle. Then $sh(\mathscr{G}(p, q, x))$ is conjugate to a group in $\mathcal{Q}$, determined by the following table, where α is a particular hyperreal that depends on $\mathscr{T}$.*

# infinitesimal angles	*# infinitesimal sides*	α	*nonstandard triangle*
0	0	*appreciable*	C
1	1	*finite*	F
2	3	*appreciable*	N_1
$*$	3	*infinitesimal*	N_2
2	$*$	*infinite*	N_3

This result is used to determine conjugacy limits of the positive diagonal group $C \leq SL_3(\mathbb{R})$ as follows. The columns of a matrix in $SL_3(\mathbb{R})$ determine a triangle in $\mathbb{R}P^2$. Given a sequence of conjugating matrices P_n we choose a subsequence so the corresponding triangles $P_n(TC)$ converge to a nonstandard triangle $\mathscr{T} \subset {}^*\mathbb{R}P^2$. Theorem 6.0.66 determines the conjugacy class of the limit $P_nCP_n^{-1}$.

Most of the work in this chapter follows Haettel, [26]. The cells correspond to the strata in the digraph Γ. The dimension of a cell decreases as the dimension of the normalizer of a group increases. To find limits of Lie algebras, Haettel works with sequences under the adjoint action. This chapter follows his work, but from the perspective of Lie groups, and introduces the geometric notions of characteristic degenerate triangles, nonstandard triangles, and a maximal configuration preserved by a group.

Sections 6 and 6 prove Theorems 6.0.65 and 6.0.64, respectively. The second part of the chapter is dedicated to the hyperreal Theorem 6.0.66. We first explain the lower dimensional case, $SL_2(\mathbb{R})$, and then build the theory of projective geometry over the hyperreals, to prove Theorem 6.0.66 in Proposition 6.0.84.

Degenerate Triangle Configurations:

This section proves Theorem 6.0.65. We derive the 5 configurations in $\mathcal{TQ}$, and explain how each configuration in $\mathcal{TQ}$ determines a conjugacy limit group in $\mathcal{Q}$.

Recall a *configuration* is a set with elements which are points and lines in $\mathbb{R}P^2$. A configuration, T, is a *limit* of a configuration, S, if there is a sequence of projective transformations, P_n, such that for every $x \in T$, and n sufficiently large, then $x \in P_n S$. (See [16] Definition 2.6). Write $P_n S \to T$. A *projective triangle* consists of three points and three line segments connected in the usual way. A *triangle configuration* in $\mathbb{R}P^2$ is the configuration of three lines in general position, and their three intersection points, obtained by extending the lines of a projective triangle in the natural way. A *degenerate triangle configuration* is a configuration that is a limit of a triangle configuration. It has at most three points and three lines. Two degenerate triangle configurations are in the same equivalence class if they have the same number of points and lines.

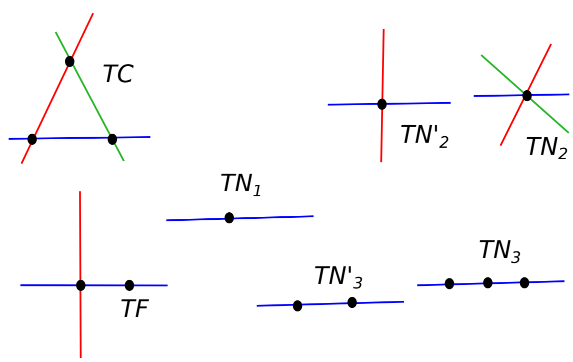


Figure 6.2: Degenerate triangle configurations

Since it is not possible to make a degenerate triangle configuration with 3 lines and 2 points, or 2 lines and 3 points, figure 6.2 shows a representative from every equivalence class of degenerate triangle configurations. Paths of projective transformations from TC to the degenerate triangle configurations in $\mathcal{TQ}$ are given in Proposition 6.0.68.

Proposition 6.0.67. *Let C be the positive diagonal Cartan subgroup of $SL_3(\mathbb{R})$, and $S \subset \mathbb{R}P^2$, be the projective triangle configuration preserved by C. Let $P_t \in PSL_3(\mathbb{R})$ be a sequence of projective transformations, such that S converges to $S_\infty = \lim_{t\to\infty} P_t S$, a degenerate triangle configuration, and G has conjugacy limit $G_\infty = \lim_{t\to\infty} P_t G P_t^{-1}$. Then G_∞ preserves S_∞.*

Proof. Let $G_t := P_t C P_t^{-1}$, so that G_t preserves $S_t := P_t S$ for all t. Suppose for contradiction that G_∞ does not preserve S_∞. Then there is some $x \in S_\infty$, and $g \in G_\infty$ such that $gx \notin S_\infty$, or $d(gx, S_\infty) > 0$. Take a sequence $g_t \in G_t$ such that $\lim_{t\to\infty} g_t = g$. Pick a point $x_0 \in S$ so that $\lim_{t\to\infty} P_t x_0 = x$, so $P_t x_0 \in S_t$. But then $\lim_{t\to\infty} d(g_t(P_t x_0), S_t) = d(gx, S_\infty) > 0$, contrary to $d(g_t(P_t x_0), S_t) = 0$. □

The set of degenerate triangle configurations is partially ordered by inclusion. Given a group, $G \leq SL_3(\mathbb{R})$, a degenerate triangle configuration, T, is *characteristic* for G, if G maps each point or line of T to itself, and T is maximal subject to the partial order. There are degenerate triangle configurations which are not characteristic for any group in $\mathcal{Q}$. They are shown in figure 6.2: TN_2' and TN_3'.

Proof. of Theorem 6.0.65: The notation has been chosen so that for every $G \in \mathcal{Q}$, it is easy to check $\theta(G) = TG \in \mathcal{TQ}$ is a characteristic degenerate triangle configuration for G. That is, G preserves every point or line of $\theta(G)$, and $\theta(G)$ is maximal in the partial order. $\square$

Given a group $G \leq SL_3(\mathbb{R})$, a *characteristic triangle class*, $[T]$ for G, is an equivalence class of degenerate triangle configurations such that each $S \in [T]$ is characteristic for a group conjugate to G. Figure 6.1 shows the set $\mathcal{TQ}$, which consists of one representative of each characteristic triangle class. By Theorem 6.0.64, $\mathcal{Q}$ contains one representative of each conjugacy class of conjugacy limit group of C. Therefore, θ induces a bijection from conjugacy classes of conjugacy limit groups of C to characteristic triangle classes. This map is well defined: If G_1 and G_2 are conjugacy limits of C, and T_1, T_2 are their respective characteristic degenerate triangle configurations, and $P \in GL_3(\mathbb{R})$, then $PG_1P^{-1} = G_2$ if and only if $PT_1 = T_2$.

Given a group $G \leq SL_3(\mathbb{R})$, the *maximal configuration* preserved by G is the set whose elements are the points fixed by G and the lines preserved by G. It might be infinite. For example, the maximal configuration for N_2 consists of every line through a point, and the maximal configuration for N_3 is every point on a line.

In configuration TN_3, three points on the line are fixed, and any group which acts on $\mathbb{R}P^2$ and fixes three points on a projective line must fix every point on the line. In configuration TN_2, three lines in the link of the vertex are preserved, and similarly, every line in the link is preserved. In this way, each configuration in figure 6.1 determines a maximal configuration. These ideas will be investigated further with the hyperreal viewpoint.

The Digraph of Conjugacy Limit Groups: Proof of Theorem 6.0.64

In this section we prove Theorem 6.0.64. Part 1 follows easily from work of Haettel [26], or the classification over $\mathbb{C}$ of Suprenko and Tyshkevitch, [56], p.134. Next we prove part 3: every element of $\mathcal{Q}$ is a conjugacy limit of C. If each element of the sequence of conjugating matrices lies in the same one parameter subgroup of $SL_n(\mathbb{R})$, then G is a conjugacy limit *under a 1-parameter path of conjugacies.*

Proposition 6.0.68. *Each conjugacy limit group is a limit under a 1-parameter path of conjugacies.*

Proof. Each conjugacy limit group is a limit under the 1 parameter path of conjugacies (as $n \to \infty$):

$$\begin{array}{cccc} C \to F & C \to N_1 & C \to N_2 & C \to N_3 \\ \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & n & \frac{n^2}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & n & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}. \end{array}$$

□

To finish the proof of Theorem 6.0.64. We must show:

(1) there is a 1-parameter path of conjugacies for every arrow in the digraph, and

(2) no other arrows are possible.

The path $C \to F$ is given in Proposition 6.0.68. For the rest of the arrows, use the paths:

$$\begin{array}{ccc} F \to N_1 & N_1 \to N_2 & N_1 \to N_3 \\ \begin{pmatrix} 1 & n & \frac{n^2}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{n} \end{pmatrix} & \begin{pmatrix} \frac{1}{n} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{array}$$

as $n \to \infty$. This shows every directed path in Γ is realized by a 1-parameter limit. To show these are all possible arrows, use Proposition 5.0.52.

As a corollary, if we have both conjugacy limits $A \to B$ and $B \to A$, then A and B are conjugate. Computing the dimension of the normalizer shows there are no arrows going backwards in the digraph: $\dim N_G(C) = 2, \dim N_G(F) = 3, \dim N_G(N_1) = 4$, and $\dim N_G(N_2) = \dim N_G(N_3) = 5$, where $G = SL_3(\mathbb{R})$.

It remains to show neither N_2 or N_3 is a limit of the other. Since $\dim N_G(N_2) = \dim N_G(N_3)$, if $N_2 \to N_3$, then Proposition 5.0.52 would imply N_2 and N_3 are conjugate, which is false. Therefore, neither N_2 or N_3 can limit to the other. This finishes the proof of Theorem 6.0.64. □

Remark 6.0.69. Notice N_3 fixes every point on a line, and N_2 preserves every line through a fixed point. The maximal configurations for N_2 and N_3 are dual. Moreover, duality induces an automorphism of the digraph of conjugacy limits.

Suppose $H, L \leq SL_n(\mathbb{R})$ are not conjugate, and $H \to L$. The limit from H to L is *decompressed* if there is an orbit closure of H which is a union of infinitely many orbit closures of L. The conjugacy limit $N_1 \to N_2$ is decompressed, since every line through the fixed point $[e_1]$ is preserved under the action of N_2. This is the maximal configuration for N_2: a single point is fixed and every line through the point is preserved. The conjugacy limit $N_1 \to N_3$ is decompressed, since every point on the line $\langle e_1, e_2 \rangle$ is fixed under the action of N_3. This is the maximal configuration for N_3: a line with every point on the line fixed.

We concluded this section with a final property satisfied by real conjugacy limits, which we will use in Corollary 6.0.86.

Lemma 6.0.70. *Suppose H is a subgroup of $SL_3(\mathbb{R})$, and P_n is a sequence of conjugating matrices such that $P_n H P_n^{-1}$ has conjugacy limit $L \leq SL_3(\mathbb{R})$. There is a sequence of upper triangular matrices, P'_n such that $P'_n H {P'_n}^{-1}$ converges to a conjugate of L.*

Proof. Recall the Iwasawa decomposition of a matrix, $P = KNA$, where K is orthogonal, N is unipotent, and A is diagonal. Writing each P_n in this way,

$P_n H P_n^{-1} = K_n(N_n A_n H A_n^{-1} N_n^{-1}) K_n^{-1}$. The orthogonal group is compact, so every sequence has a convergent subsequence, and in particular, every sequence $K_n H' K_n^{-1}$ converges to a conjugate of H'. Thus we may assume $P_n = N_n A_n$, so P_n is upper triangular. $\square$

$PGL_n({}^*\mathbb{R})$ AND ${}^*\mathbb{R}P^n$

The second part of this chapter is devoted to proving Theorem 6.0.66. In this section we prove some properties about the action of the group $PGL_n({}^*\mathbb{R})$ on ${}^*\mathbb{R}P^n$. Section 1 provided an introduction to ${}^*\mathbb{R}$.

Definition 6.0.71. 1. A *projective basis* for $\mathbb{R}P^n$ (or ${}^*\mathbb{R}P^n$) consists of $n+2$ equivalence classes of vectors, such that any $n+1$ vectors form a basis for the underlying vector space. The word basis means either vector space basis or projective basis, depending on the context.

2. The *usual basis* for ${}^*\mathbb{R}^{n+1}$ (or $\mathbb{R}^{n+1}$) is $\{e_0, e_1, ..., e_n\}$. The *usual basis* for ${}^*\mathbb{R}P^n$ (or $\mathbb{R}P^n$) is $\{[e_0], ..., [e_n], [e_0 + e_1 + \cdots + e_n]\}$.

3. The *shadow* map is $sh : {}^*\mathbb{R}P^n \to \mathbb{R}P^n$ where $sh([v]) = [sh(\frac{v}{||v||})]$, and we take the shadows of the coordinates. The *shadow* of a basis $\mathscr{B} \subset {}^*\mathbb{R}P^n$ is $sh(\mathscr{B}) = \{[sh(v)] | v \in \mathscr{B}\} \subset {}^*\mathbb{R}P^n$.

4. A hyperreal projective basis $\mathscr{B}$ is *appreciable* if $sh(\mathscr{B})$ is a projective basis for $\mathbb{R}P^n$. A hyperreal projective transformation is *appreciable* if the image of some appreciable basis is an appreciable basis.

Notice the shadow of a hyperreal basis may not be a real basis, since shadows of basis elements could be the same! A hyperreal transformation $[A] \in PGL_n({}^*\mathbb{R})$ is called finite if and only if there exists $[B] \in PGL_n(\mathbb{R})$ and $\lambda \in {}^*\mathbb{R}$ such that $B - \lambda A$ is infinitesimal. Every finite hyperreal projective transformation differs from a real projective transformation infinitesimally.

Definition 6.0.72. Given $\mathscr{G} \leq SL_n({}^*\mathbb{R})$, the *finite part*, $\mathrm{Fin}(\mathscr{G})$, is the subset of all elements that have finite entries. The *subset of infinitesimal elements*, $\mathscr{I}$, is the set of matrices that are the identity matrix plus a matrix with infinitesimal entries.

Lemma 6.0.73. *Fin*($\mathscr{G}$) *and* $\mathscr{I}$ *are subgroups of* $\mathscr{G}$.

Proof. Sums and products of finite hyperreals are finite, so $\mathrm{Fin}(\mathscr{G})$ is closed under multiplication. Let $A \in \mathrm{Fin}(\mathscr{G}) \leq SL_n({}^*\mathbb{R})$. Since sums and products of finite hyperreal numbers are finite, $\mathrm{Adj}A$ is finite. Since $\det A = 1$, then $A^{-1} = \frac{\mathrm{Adj}A}{\det A} \in \mathrm{Fin}(\mathscr{G})$. Thus $\mathrm{Fin}(\mathscr{G})$ is a group. Similarly, $\mathscr{I}$ is a group. ∎

Definition 6.0.74. Given $\mathscr{G} \leq SL_n({}^*\mathbb{R})$ and $\mathscr{A} \in \mathrm{Fin}(\mathscr{G})$, the *shadow*, $sh(\mathscr{A})$, of $\mathscr{A}$, has entries that are the shadows of entries of $\mathscr{A}$. The *standard part* or *shadow* of $\mathscr{G}$ is $\mathrm{sh}(\mathscr{G}) := \{sh(\mathscr{A}) | \mathscr{A} \in \mathrm{Fin}(\mathscr{G})\}$.

Lemma 6.0.75. *$sh(Fin(\mathscr{G})) \cong Fin(\mathscr{G})/(\mathscr{I} \cap Fin(\mathscr{G}))$.*

Proof. Note $sh : \mathrm{Fin}(\mathscr{G}) \to sh(\mathrm{Fin}(\mathscr{G}))$ is a homomorphism since the map is defined by taking the shadow of each entry, and $sh : \mathbb{F} \to \mathbb{R}$ is a ring homomorphism. The kernel is $\mathscr{I} \cap \mathrm{Fin}(\mathscr{G})$. The map sh is surjective since $sh(r) = r$ for any $r \in \mathbb{R} \subset \mathbb{F}$ (see section 1). Apply the first isomorphism theorem. □

Instead of writing this discussion in terms of matrices, we could have considered nonstandard projective transformations in the context of *appreciable* (vector space or projective) bases. It is easy to show a hyperreal projective transformation is appreciable if and only if the image of every appreciable basis is an appreciable basis.

A *nonstandard triangle configuration* consists of three lines in general position in ${}^*\mathbb{R}P^2$, which intersect in three distinct points. Let $\mathscr{P}$ be a matrix of hyperreals, and $\mathscr{T}$ be the nonstandard triangle configuration which is the image of a projective triangle configuration under $\mathscr{P}$. Let $\mathscr{C} \leq SL_3({}^*\mathbb{R})$ be the group of positive diagonal (hyperreal) matrices. A conjugate of $\mathscr{C}$ by $\mathscr{P}$ is uniquely determined as the stabilizer of the vertices of $\mathscr{T}$.

Theorem 6.0.76. *Suppose $P_n \in GL_3(\mathbb{R})$ and $P_n C P_n^{-1}$ converges to L. Define $\mathscr{P} := [P_n] \in GL_3({}^*\mathbb{R})$, and $\mathscr{G} := \mathscr{P}\mathscr{C}\mathscr{P}^{-1}$. Then*

1. *$sh(Fin(\mathscr{G})) = L$*

2. *$\mathscr{G}$ is the conjugate of $\mathscr{C}$ that preserves the points $\mathscr{P}([e_1], [e_2], [e_3]) \subset {}^*\mathbb{R}P^2$.*

3. *the nonstandard triangle configuration $\mathscr{T}$ with vertices $\mathscr{P}([e_1], [e_2], [e_3])$ has shadow $T = sh(\mathscr{T})$ a degenerate triangle, and L preserves T.*

Proof. (1) Conjugating $\mathscr{C}$ by $\mathscr{P}$, gives $({}^*\mathbb{R})^2 \cong \mathscr{G} \le SL_3({}^*\mathbb{R})$. For dimension reasons, (see Theorem 5.0.53), $sh(\text{Fin}(\mathscr{G})) \cong \mathbb{R}^2$. Every subgroup of $SL_3(\mathbb{R})$ isomorphic to $\mathbb{R}^2$ is a conjugacy limit of C, so $sh(\text{Fin}(\mathscr{G}))$ is a conjugacy limit of C. By assumption, $[P_n] = \mathscr{P}$, therefore $sh(\text{Fin}(\mathscr{G})) = L$.

(2) Since $\mathscr{C}$ preserves $([e_1], [e_2], [e_3])$, then $\mathscr{G} = \mathscr{P}\mathscr{C}\mathscr{P}^{-1}$ preserves $\mathscr{P}([e_1], [e_2], [e_3])$.

(3) This follows from Proposition 6.0.67. □

Conjugacy Limits of the Diagonal

In this section, we show how to view a limit of the positive diagonal group in $SL_2(\mathbb{R})$ as the shadow of the finite part of a conjugate of the positive diagonal group in $SL_2({}^*\mathbb{R})$.

There are three conjugacy classes of 1-parameter subgroups in $SL_2(\mathbb{R})$: elliptic, parabolic, and hyperbolic (see [12]). Elliptic subgroups are isomorphic to $\mathbb{S}^1$, so any subgroup isomorphic to $\mathbb{R}$ must be *hyperbolic* in which case it is conjugate to $\{\left(\begin{smallmatrix} a & 0 \\ 0 & \frac{1}{a} \end{smallmatrix}\right) | a > 0\}$, or *parabolic* in which case it is conjugate to $\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) | t \in \mathbb{R}\}$.

Theorem 6.0.77. *The Cartan subgroup* $\{\left(\begin{smallmatrix} a & 0 \\ 0 & \frac{1}{a} \end{smallmatrix}\right) | a > 0\} \leq SL_2(\mathbb{R})$*, has two conjugacy limits: the Cartan subgroup and the parabolic group* $\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) | t \in \mathbb{R}\}$*. The Cartan subgroup preserves the maximal configuration consisting of two fixed points on a projective line, and the parabolic group preserves the maximal configuration consisting of a projective line with one fixed point.*

Proof. Conjugate by the sequence of projective transformations as $n \to \infty$:

$$\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & \frac{1}{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} a & n(a-\frac{1}{a}) \\ 0 & \frac{1}{a} \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right).$$

Since we want the conjugacy limit to be finite, i.e., we want $n(a - \frac{1}{a})$ to converge to some $t \in \mathbb{R}$, we need $a \to 1$. Since $a \in \mathbb{R}$ is arbitrary, the limit is a group where t is any real number.

The diagonal Cartan subgroup preserves the maximal configuration of 2 fixed points an appreciable distance apart on a projective line. Applying this sequence of projective transformations identifies the points in the limit, so that the limit configuration consists of one point on a projective line, which is the maximal configuration preserved by the parabolic group. □

Conjugate the Cartan subgroup (given in the usual basis $\{[1:0],[0:1]\}$), by a hyperreal transformation to change to the basis $\{[1:0],[1:\delta]\}$. Define

$$\mathscr{G}(\delta) := \{ \begin{pmatrix}1&1\\0&\delta\end{pmatrix}\begin{pmatrix}a&0\\0&\frac{1}{a}\end{pmatrix}\begin{pmatrix}1&1\\0&\delta\end{pmatrix}^{-1} = \begin{pmatrix}a&\frac{1}{\delta}(a-\frac{1}{a})\\0&\frac{1}{a}\end{pmatrix}\}.$$

Set $G(\delta) = sh(\mathrm{Fin}(\mathscr{G}(\delta))) \leq SL_2(\mathbb{R})$.

Corollary 6.0.78. *If δ is appreciable, then $G(\delta)$ is hyperbolic. If δ is infinitesimal, then $G(\delta)$ is parabolic.*

Proof. If δ is infinitesimal, the finite part of $\mathscr{G}(\delta)$ has a is infinitesimally close to 1, so that the upper right entry is finite. Then $sh(a) = 1$, so $G(\delta)$ is conjugate to the parabolic group, and $G(\delta)$ acts on $\mathbb{R}P^1$ with one fixed point.

If δ is appreciable, then $G(\delta)$ is conjugate to a group of hyperbolic projective transformation by a real matrix, and $G(\delta)$ acts on $\mathbb{R}P^1$ with two fixed points. □

Non-Standard Triangles: Proof of Theorem 6.0.66

So far, we have shown every conjugacy limit group of C is conjugate to an element of $\mathcal{Q}$, and determined by a characteristic triangle class in $\mathcal{TQ}$ (Theorem 6.0.65). In this section, we establish the bijection between conjugacy limit groups and *equivalence classes of nonstandard triangles*, given as a partition in the table in Theorem 6.0.66. Before proving the table gives a partition, we define α.

A nonstandard triangle may be built from a nonstandard 1-simplex as follows. Consider ${}^*\mathbb{R}P^2$ with the positive scalar curvature metric inherited from the sphere. This metric is not preserved by projective transformations. Three non collinear points, $p, q, x \in {}^*\mathbb{R}P^2$, determine a nonstandard triangle, $\Delta(p,q,x)$. Assume p, q, x satisfy the following labeling conditions. The length of the shortest altitude is measured from the point x. Let $\mathscr{H} \cong {}^*\mathbb{R}P^1$ be the line containing p and q, and $y \in \mathscr{H}$ be the foot of the altitude measured from x. Assume $d(y,p) \leq d(y,q)$, and without loss of generality, assume p, q, x, y have coordinates $p = [1:0], q = [1:\delta], y = [1:\varepsilon] \in \mathscr{H}$ and $x = [1:\varepsilon:\eta]$. The shortest altitude is measured from x, so $0 \leq |\eta| \leq |\delta|$ and $0 \leq |\varepsilon| \leq |\delta|$. In the remainder of this section, we denote by $\mathscr{G} := \mathscr{G}(\delta) \leq SL_2({}^*\mathbb{R})$ the group preserving p and q, and by $\hat{\mathscr{G}} := \hat{\mathscr{G}}(p,q,x) \leq SL_3({}^*\mathbb{R})$ the group preserving p, q and x. Set $G = sh(\text{Fin}(\mathscr{G}))$ and $\hat{G} = sh(\text{Fin}(\hat{\mathscr{G}}))$

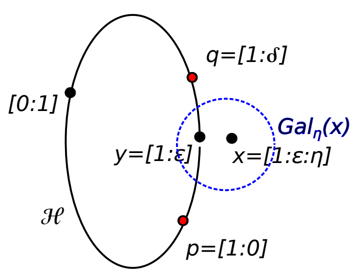


Figure 6.3: Adding a point to a 1-simplex

Lemma 6.0.79. *The table in Theorem 6.0.66 is a partition on the set of nonstandard triangles, where* $\alpha = \frac{\varepsilon\delta}{\eta}$.

Proof. We show that every nonstandard triangle is in exactly one row of the table. Consider the first two columns of the table: the number of infinitesimal sides and angles. A nonstandard triangle has $0, 1$ or 2 infinitesimal angles, and $0, 1$ or 3 infinitesimal sides. If a nonstandard triangle has exactly one infinitesimal side (angle), then it must have at least one infinitesimal angle (side). It is straightforward to see all 7 possibilities for a nonstandard triangle with 0,1 or 2 infinitesimal angles, and 0,1, or 3 infinitesimal sides are listed in the table. Since $|\delta| \geq |\eta|$, if the number of infinitesimal angles and sides is not 2 and 3 respectively, it is easy to check the order of $\frac{\varepsilon\delta}{\eta}$ is determined by the number of infinitesimal sides and angles, these details appear as part of Proposition 6.0.84.

It remains to show that there is no overlap in the rows. Examining the first two columns, we see the only repeat is a nonstandard triangle with 3 infinitesimal sides and 2 infinitesimal angles, which appears three times. The row for a nonstandard triangle with 2 infinitesimal angles and three infinitesimal sides is determined by whether $\frac{\varepsilon\delta}{\eta}$ is appreciable, infinitesimal, or infinite. □

Recall a projective transformation is appreciable if it maps an appreciable basis to an appreciable basis. Let $\mathscr{X} \subset {}^*\mathbb{R}P^n$ be a subspace, and extend the usual basis of $\mathscr{X}$ to the usual basis of ${}^*\mathbb{R}P^n$. A group $\mathscr{F}$ *acts finitely* on $\mathscr{X}$, if $f|_{\mathscr{X}}$ is a finite transformation, for all $f \in \mathscr{F}$.

Let $[v] \in {}^*\mathbb{R}P^n$. The *link* of $[v]$ is $\mathscr{L}(v) \cong {}^*\mathbb{R}P^{n-1}$, the set of all lines through $[v]$. Given a projective basis $\{[e_0], [e_1], ..., [e_{n+1}]\}$ for ${}^*\mathbb{R}P^n$ with $[e_0] = [v]$, a projective basis for $\mathscr{L}(v) \cong {}^*\mathbb{R}P^{n-1}$ consists of lines $\{\langle [v], [e_i] \rangle | 1 \leq i \leq n+1\}$.

Recall a basis, $\mathscr{B}$, for $\mathscr{L}(v)$ is appreciable if $sh(\mathscr{B})$ is a basis for $L(v) := sh(\mathscr{L}(v))$. In ${}^*\mathbb{R}P^2$ a basis for $\mathscr{L}(v)$ is appreciable if and only if the angles between the projective lines in the basis for $\mathscr{L}(v)$ are appreciable.

Let $\mathscr{G} = \mathscr{G}(\delta) \leq SL_2({}^*\mathbb{R})$ act on $\mathscr{H}$, and let $x \in {}^*\mathbb{R}P^2 - \mathscr{H}$ as in figure 6.3. The action of $\mathscr{G}$ on $\mathscr{L}(x)$ is defined as follows. In projective space, every pair of lines intersect in a point, so every line in $\mathscr{L}(x)$ intersects $\mathscr{H}$ in a point, z. Thus $\mathscr{L}(x) = \{\langle x, z\rangle : z \in \mathscr{H}\}$, and the action of $\mathrm{Fin}(\mathscr{G})$ on $\mathscr{L}(x)$ is given as $\langle x, z\rangle \mapsto \langle x, g(z)\rangle$, for $g \in \mathscr{G}$. We will show the conjugacy limit group that preserves the shadow of the nonstandard triangle is controlled by the action on the link of the new point, $\mathscr{L}(x)$.

Recall $\mathrm{Gal}_\eta(x) = \{r \in {}^*\mathbb{R} : x - r \in \eta \cdot \mathbb{R}\}$ from Definition 1.0.11.

Lemma 6.0.80. *Fin($\mathscr{G}$) preserves Gal$_\eta(y)$ if and only if Fin($\mathscr{G}$) acts finitely on $\mathscr{L}(x)$.*

Proof. Let $\{v_1 = [1 : \varepsilon + \eta], v_2 = [1 : \varepsilon - \eta], v_3 = [1 : \varepsilon]\}$ be a basis for $\mathscr{H}$, and set $\mathscr{B} = \{\langle x, v_i\rangle : i = 1, 2, 3\}$, a basis for $\mathscr{L}(x)$. The basis $\mathscr{B}$ is appreciable since the lines $\langle x, v_1\rangle$ and $\langle x, v_3\rangle$ form a 45° isosceles triangle with $\mathscr{H}$, and $\langle x, v_2\rangle$ is perpendicular to $\mathscr{H}$. The distance a point $z \in \mathscr{H}$ is moved by $g \in \mathscr{G}$ is $|z - g.z|$. The distance a point is moved in $\mathscr{L}(x)$ is $\angle(\langle x, z\rangle, \langle x, (g.z)\rangle) \approx \frac{|z-g.z|}{\eta}$.

So $\mathrm{Fin}(\mathscr{G})$ acts finitely on $\mathscr{L}(x)$ if and only if $\mathrm{Fin}(\mathscr{G})$ acts finitely on $\mathscr{B}$, the basis for $\mathscr{L}(x)$. But $\mathrm{Fin}(\mathscr{G})$ acts finitely on $\mathscr{B}$ if and only if $\mathrm{Fin}(\mathscr{G})$ keeps the angles between the lines in $\mathscr{B}$ appreciable, i.e. if $\mathrm{Fin}(\mathscr{G})$ moves the points $v_i \in \mathscr{H}$ a distance of at most order η. Thus $\mathrm{Fin}(\mathscr{G})$ acts finitely on $\mathscr{L}(x)$ if and only if $\mathrm{Fin}(\mathscr{G})$ preserves $\mathrm{Gal}_\eta(y)$. ☐

Two nonzero hyperreals $\alpha, \beta \in {}^*\mathbb{R}$ have the same *order* if and only if $\frac{\alpha}{\beta}$ is appreciable. Denote this by $\alpha \approx \beta$.

Lemma 6.0.81. *$Fin(\mathscr{G})$ moves a point in $Gal_\eta(y)$ a distance of at most order $\varepsilon\delta$.*

Proof. Recall

$$\mathscr{G}(\delta) = \{\left(\begin{smallmatrix} 1 & 1 \\ 0 & \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & \frac{1}{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & \delta \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} a & \frac{1}{\delta}(a-\frac{1}{a}) \\ 0 & \frac{1}{a} \end{smallmatrix}\right)\}.$$

In $\mathrm{Fin}(\mathscr{G})$, we have $\frac{1}{\delta}(a - \frac{1}{a}) = 2t$, a finite hyperreal. The action on $y = [1 : \varepsilon]$ depends on $\eta, \delta, \varepsilon$, and we want to find the subgroup of $\mathrm{Fin}(\mathscr{G})$ that preserves $\mathrm{Gal}_\eta(y)$. So:

$$\left(\begin{smallmatrix} a & \frac{1}{\delta}(a-\frac{1}{a}) \\ 0 & \frac{1}{a} \end{smallmatrix}\right)[1 : \varepsilon] = [a + \frac{\varepsilon}{\delta}(a - \frac{1}{a}) : \frac{\varepsilon}{a}] = [1 : \frac{\frac{\varepsilon}{a}}{a + \frac{\varepsilon}{\delta}(a - \frac{1}{a})}].$$

We want to find the distance y is moved. Using $\frac{1}{a} - a = 2t\delta$ then

$$\left|\frac{\frac{\varepsilon}{a}}{a + \frac{\varepsilon}{\delta}(a - \frac{1}{a})} - \varepsilon\right| = \varepsilon\left|\frac{\frac{1}{a} - a + 2t\varepsilon}{a + 2t\varepsilon}\right| = 2t\varepsilon\left|\frac{\delta + \varepsilon}{a + 2t\varepsilon}\right| \approx 2t\varepsilon|\delta + \varepsilon|$$

since $a \approx 1$ in $\mathrm{Fin}(\mathscr{G})$. Since $0 < \varepsilon \leq \delta$ then $\varepsilon\delta \leq \varepsilon(\delta + \varepsilon) \leq 2\varepsilon\delta$. Therefore

$$2t\varepsilon|\delta + \varepsilon| \approx \varepsilon(\delta + \varepsilon) \approx \varepsilon\delta,$$

and y is moved a distance of order $\varepsilon\delta$. ☐

Corollary 6.0.82. *Fin($\mathscr{G}$) acts finitely on $\mathscr{L}(x)$ if and only if $\frac{\varepsilon\delta}{\eta}$ is finite. Moreover, the action of Fin($\mathscr{G}$) on $\mathscr{L}(x)$ is infinite if $\frac{\varepsilon\delta}{\eta}$ is infinite, and sh(Fin($\mathscr{G}$)) acts as the identity on $L(x)$ if $\frac{\varepsilon\delta}{\eta}$ is infinitesimal.*

Proof. By Lemma 6.0.81, the distance $y \in \mathrm{Gal}_\eta(x)$ is moved in $\mathscr{H}$ is of order $\varepsilon\delta$, and so by the proof of Lemma 6.0.80, Fin($\mathscr{G}$) moves points in $\mathscr{L}(x)$ a distance of order $\frac{\varepsilon\delta}{\eta}$. The result follows. □

We have shown the ratio $\frac{\varepsilon\delta}{\eta}$ determines the action of Fin($\mathscr{G}$) on $\mathscr{L}(x)$. Next we show how to extend $\mathscr{G} \leq SL_2(^*\mathbb{R})$ to $\hat{\mathscr{G}} \leq SL_3(^*\mathbb{R})$.

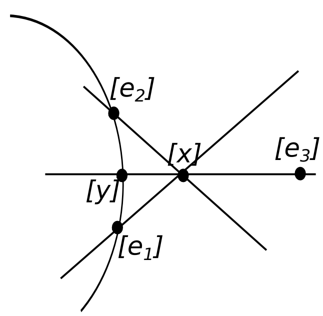


Figure 6.4: Choosing $\mathscr{B}$

Lemma 6.0.83. *Let $\mathscr{G} = \mathscr{G}(\delta) \leq SL_2(^*\mathbb{R})$ act on $\mathscr{H}$. If Fin($\mathscr{G}$) acts finitely on $\mathscr{L}(x)$, define $\hat{\mathscr{G}} \leq SL_3(^*\mathbb{R})$ to be the set of elements which fix x, act finitely on $^*\mathbb{R}P^2$, and for every $\hat{g} \in \hat{\mathscr{G}}$, the restriction $\hat{g}|_{\mathscr{H}} = g$ for some $g \in \mathscr{G}$. Then*

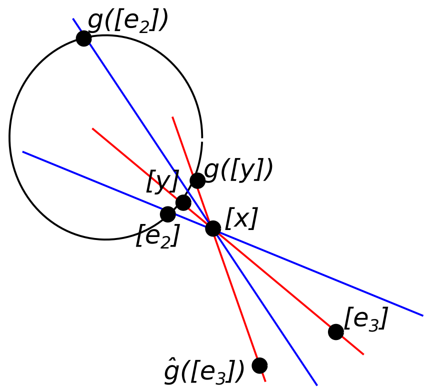


Figure 6.5: The action on $\mathscr{L}(x)$ extended to $^*\mathbb{R}P^2$: the image of $\mathscr{B}$ under $\hat{\mathscr{G}}$.

$\hat{G} := sh(Fin(\hat{\mathscr{G}}))$ is a conjugacy limit group, and the action of $\hat{G}$ on H coincides with the action of G on H.

Proof. Pick a projective basis $\mathscr{B} = \{[e_1], [e_2], [e_3], [e_1 + e_2 + e_3]\}$ for $^*\mathbb{R}P^2$, and show the image of $\mathscr{B}$ under $\hat{\mathscr{G}}$ is an appreciable basis. By assumption, $\mathscr{G}$ fixes $[e_1] \in \mathscr{H}$. Let $[y] \in \mathscr{H}$ be the point closest to $[x]$. Choose $[e_3] \in \langle [x], [y] \rangle$ an appreciable distance from $[e_1]$. Then the lines $\langle [x], [e_1] \rangle$ and $\langle [x], [e_3] \rangle = \langle [x], [y] \rangle$ are at an appreciable angle, since $[x]$ is infinitesimally close to $\mathscr{H}$. Pick $[e_2] \in \mathscr{H}$ an appreciable distance from $[e_1]$ and $[e_3]$, and such that $\langle [x], [e_2] \rangle$ and $\langle [x], [e_3] \rangle$ are at an appreciable angle. Again, $\langle [x], [e_2] \rangle$ and $\mathscr{H}$ are at an appreciable angle, since $[x]$ is infinitesimally close to $\mathscr{H}$.

If $g \in \text{Fin}(\mathscr{G})$, then $g([e_2])$ is an appreciable distance from $g([e_1]) = [e_1]$. Let $\hat{g} \in \hat{\mathscr{G}}$, so that $\hat{g}|_{\mathscr{H}} = g$. Since $\hat{g}([e_3])$ lies on the line $\langle [x], g([y]) \rangle$, there is a 1-parameter hyperreal family of choice of image of $[e_3]$. Choose $\hat{g}([e_3])$ to be an appreciable distance from $\hat{g}([e_1]) = [e_1]$. The action of $\text{Fin}(\mathscr{G})$ on $\mathscr{L}(x)$ is finite, so $\langle [x], [\hat{g}(e_i)] \rangle$ and $\langle [x], \hat{g}([e_j]) \rangle$ are an appreciable distance apart, for all $i \neq j$. Thus the image of an appreciable basis is an appreciable basis, and $\{p, q, x\}$ is the image of the usual basis under a nonstandard projective transformation. By Theorem 6.0.76, $sh(\mathrm{Fin}(\hat{\mathscr{G}}))$ is a conjugacy limit group, and by construction $\hat{\mathscr{G}}|_{\mathscr{H}}$ is isomorphic to a subgroup of $\mathscr{G}$. $\square$

By Theorem 6.0.64 there are 5 conjugacy classes of conjugacy limit groups, and by Theorem 6.0.76 every conjugacy limit group of C is $sh(\mathrm{Fin}(\mathscr{P}\mathscr{C}\mathscr{P}^{-1}))$. We show next that the conjugacy class of any conjugacy limit group is completely determined by the hyperreal numbers δ, η and $\frac{\varepsilon\delta}{\eta}$. Recall $0 \leq |\eta| \leq |\delta|$, and $0 \leq |\varepsilon| \leq |\delta|$.

Proposition 6.0.84. *Given $p, q, x \in {}^*\mathbb{R}P^2$ in general position, let $\hat{\mathscr{G}}(p, q, x) \leq SL_3({}^*\mathbb{R})$ be the group preserving p, q, x. Set $\hat{G} = sh(\hat{\mathscr{G}}(p, q, x))$.*

1. *If δ is appreciable, and*
   - *(a) η is appreciable, then $\hat{G}$ is conjugate to C and $\frac{\varepsilon\delta}{\eta}$ is appreciable*
   - *(b) η is infinitesimal, and $p \in Gal_\eta(x)$, then $\hat{G}$ is conjugate to F and $\frac{\varepsilon\delta}{\eta}$ is finite*
   - *(c) η infinitesimal, but $p \notin Gal_\eta(x)$, then $\hat{G}$ is conjugate to N_3 and $\frac{\varepsilon\delta}{\eta}$ is infinite*
2. *If δ is infinitesimal, then η is infinitesimal, and*

(a) *if* $\frac{\varepsilon\delta}{\eta}$ *is infinitesimal, then* $\hat{G}$ *is conjugate to* N_2.

(b) *if* $\frac{\varepsilon\delta}{\eta}$ *is appreciable, then* $\hat{G}$ *is conjugate to* N_1.

(c) *if* $\frac{\varepsilon\delta}{\eta}$ *is infinite, then* $\hat{G}$ *is conjugate to* N_3.

Proof. **Case 1a:** The points are an appreciable distance apart, and each point is an appreciable distance from the hyperplane containing the other two points. Since η is appreciable, $\mathrm{Gal}_\eta(x) \cap \mathscr{H} = \mathscr{H}$, so the action of $\mathrm{Fin}(\mathscr{G})$ preserves $\mathrm{Gal}_\eta(x)$, and $\mathscr{G}$ acts finitely on $\mathscr{L}(x)$. The action of $\hat{G}$ on each of the three projective lines is hyperbolic. Thus $\hat{G}$ is conjugate to C, since this is the only group in $\mathcal{Q}$ which acts hyperbolically on more than one line.

Since δ, ε and η are all appreciable, then $\frac{\varepsilon\delta}{\eta}$ is appreciable.

Case 1b: Suppose $p \in \mathrm{Gal}_\eta(x)$. Then p and x are infinitesimally close, and q is an appreciable distance away from $\langle p, x\rangle$. So, $sh(\Delta(p,q,x))$ has two distinct points, $sh(p) = sh(x)$ and $sh(q)$, and two distinct lines, $sh(\langle p,q\rangle) = sh(\langle x,q\rangle)$ and $sh(\langle p,x\rangle)$. The action of $\hat{G}$ on $sh(\langle p,x\rangle)$ is parabolic, since $sh(\langle p,x\rangle)$ contains a unique fixed point. The action of $\hat{G}$ on $H = sh(\langle p,q\rangle)$ is hyperbolic, since H contains two distinct fixed points. Thus $\hat{G}$ has two 0-dimensional invariant subspaces, and two 1-dimensional invariant subspaces, one with a parabolic action, and one with a hyperbolic action. So, $\hat{G}$ is conjugate to F.

Since δ is appreciable, $\frac{\varepsilon\delta}{\eta} \approx \frac{\varepsilon}{\eta}$. Since $p \in \mathrm{Gal}_\eta(x)$, then $|\varepsilon| \leq |\eta|$, which implies $\frac{\varepsilon}{\eta}$ is finite.

Case 1c: Since δ is appreciable, $\frac{\varepsilon\delta}{\eta} \approx \frac{\varepsilon}{\eta}$. Since η is infinitesimal, and $\mathrm{Gal}_\eta(x)$ does not contain p, then $|\varepsilon|$ has larger order than $|\eta|$, which implies $\frac{\varepsilon}{\eta}$ is infinite.

Since $\frac{\varepsilon\delta}{\eta}$ is infinite, then the action of $\mathrm{Fin}(\mathscr{G})$ on $\mathscr{L}(x)$ is infinite by Corollary 6.0.82. The subgroup of $\mathrm{Fin}(\mathscr{G})$ which acts finitely on $\mathscr{L}(x)$ is infinitesimal, and its shadow in G is the identity. Thus $\hat{G}$ is conjugate to N_3, since this is the only group in $\mathcal{Q}$ which acts as the identity on a line.

Case 2a: By assumption δ and η are infinitesimal, so the points p, q, x are infinitesimally close. Therefore $sh(\Delta(p, q, x))$ has one point, $sh(p) = sh(q) = sh(x)$, which is the fixed point under the action of $\hat{G}$. By Lemma 6.0.85 which follows this proof, $\hat{G}$ is unipotent, and so $\hat{G}$ acts parabolically on any line it preserves through $sh(p)$. Since $\frac{\varepsilon\delta}{\eta}$ is infinitesimal, $\hat{G}$ acts as the identity on $L(x)$ by Corollary 6.0.82, and $\hat{G}$ preserves at least two 1-dimensional invariant subspaces. There are two groups in $\mathcal{Q}$ with a single fixed point, N_1 and N_2. But, N_1 preserves only one line, and $\hat{G}$ preserves at least two lines, so $\hat{G}$ is conjugate to N_2.

Case 2b: Since $\frac{\varepsilon\delta}{\eta}$ is appreciable, the action of $\text{Fin}(\mathscr{G})$ on $\mathscr{L}(x)$ is finite by Corollary 6.0.82, and G does not fix $L(x)$ point wise. By Lemma 6.0.83, the action of $\hat{G}$ on H coincides with the nontrivial action of G on H. Since δ and η are infinitesimal, the points p, q, x are infinitesimally close, so $sh(\Delta(p, q, x))$ has one point. By Lemma 6.0.85, $\hat{G}$ acts parabolically on H, since $\hat{G}$ fixes a single point. Since $\hat{G}$ does not fix $L(x)$, then $\hat{G}$ has only one 1-dimensional invariant subspace and one 0-dimensional invariant subspace, so $\hat{G}$ is conjugate to N_1.

Case 2c: Since $\frac{\varepsilon\delta}{\eta}$ is infinite, then $\mathscr{G}$ acts infinitely on $\mathscr{L}(x)$ by Corollary 6.0.82. The subgroup of $\text{Fin}(\mathscr{G})$ which acts finitely on $\mathscr{L}(x)$ is infinitesimal, and its shadow in G is the identity. Thus $\hat{G}$ is conjugate to N_3, since this is the only group in $\mathcal{Q}$ which acts as the identity on a line.

□

Recall from Lemma 6.0.70 that every conjugacy limit group is conjugate to one under a sequence of upper triangular sequence of matrices, and by Theorem 6.0.76 that any conjugacy limit of $C \leq SL_3(\mathbb{R})$ is $\hat{G} := \text{Sh}(\text{Fin}(\mathscr{P}\mathscr{C}\mathscr{P}^{-1}))$, where

$$\mathscr{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \delta & \varepsilon \\ 0 & 0 & \eta \end{pmatrix}. \tag{6.1}$$

Lemma 6.0.85. *If δ is infinitesimal, then $\hat{G} = sh(Fin(\hat{\mathscr{G}}(p,q,x)))$ is conjugate to a unipotent group.*

Proof. Since δ is infinitesimal, and $|\delta| \geq |\eta|$ and $|\delta| \geq |\varepsilon|$, then η and ε are infinitesimal. It is easy to check that all of the weights must be 1 in $sh(\text{Fin}(\mathscr{P}\mathscr{C}\mathscr{P}^{-1}))$. $\square$

This concludes the proof of Theorem 6.0.66, by establishing a bijection between equivalence classes of nonstandard triangles and conjugacy limit groups. We reproduce a table here for completeness. The ratio $\frac{\varepsilon\delta}{\eta} = \alpha$ in Theorem 6.0.66.

δ	η	$\frac{\varepsilon\delta}{\eta}$	Group
appreciable	appreciable	appreciable	C
appreciable	infinitesimal	finite	F
infinitesimal	infinitesimal	appreciable	N_1
infinitesimal	infinitesimal	infinitesimal	N_2
finite	infinitesimal	infinite	N_3

The shadow of $\mathscr{T}$ does not depend on $\frac{\varepsilon\delta}{\eta}$, but $sh(\text{Fin}(\hat{\mathscr{G}}(p,q,x))$ does depend on $\frac{\varepsilon\delta}{\eta}$.

Corollary 6.0.86. *The columns of $\mathscr{P}$ in (6.1) are the coordinates of the vertices of a nonstandard triangle. The equivalence class of nonstandard triangle determines the conjugacy limit group $\hat{G}$, the conjugacy limit of C under the sequence $P_n \leq GL_3(\mathbb{R})$ with sequences in the strictly upper triangular portion, $\delta_n, \epsilon_n, \eta_n$,* *converging to* $\delta, \varepsilon, \eta$. *By Proposition 6.0.84,* $\hat{G}$ *depends only on the relative orders of* δ, η, *and* $\frac{\varepsilon\delta}{\eta}$.

Chapter 7

Interlude: $Ab(4) = Red(4)$

In this chapter, we will show that every 3-dimensional abelian subgroup in Proposition 4.0.39 is a conjugacy limit of the Cartan subgroup in $SL_4(\mathbb{R})$.

Proposition 7.0.87. *$Ab(4) = Red(4)$.*

Proof. We will show every abelian group in Proposition 4.0.39 is a limit of C by exhibiting a path of conjugating matrices. The name of the group, G, appearing above a matrix, P, indicates that $PCP^{-1} \to G$.

$$\overset{E_1}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{F_0}{\begin{pmatrix} 1 & n & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{F_1}{\begin{pmatrix} 1 & n & \frac{n^2}{2} & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{F_2}{\begin{pmatrix} 1 & n & n & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$F_3 \quad \begin{pmatrix} 1 & 0 & n & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad N_1 \quad \begin{pmatrix} 1 & n & \frac{n^2}{2} & \frac{n^3}{6} \\ 0 & 1 & n & \frac{n^2}{2} \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad N_2 \quad \begin{pmatrix} 1 & n & \frac{n^2}{2} & n \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad N_3 \quad \begin{pmatrix} 1 & 0 & 0 & n \\ 0 & 1 & n & \frac{n^2}{2} \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$N_4 \quad \begin{pmatrix} 1 & n & n & n^2 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad N_4' \quad \begin{pmatrix} 1 & n & n & n^2 \\ 0 & 1 & 0 & -n \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad N_5 \quad \begin{pmatrix} 1 & 0 & n & n \\ 0 & 1 & n & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad N_6 \quad \begin{pmatrix} 1 & n & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$N_7 \quad \begin{pmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad N_8 \quad \begin{pmatrix} 1 & n & n & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

Corollary 7.0.88. *Every group in Red(4) except for possibly N_4', is a limit under a 1-parameter path of conjugacies.*

Proof. The paths given in Proposition 7.0.87 are all one parameter limits except the one for N_4'. □

Conjecture 7.0.89. *The group N_4' is* not *a one parameter limit.*

Chapter 8

Conjugacy Limits of the Cartan Subgroup in $SL_n(\mathbb{R})$

Let $C \leq SL_n(\mathbb{R})$ be the group of positive diagonal matrices, a Cartan subgroup. The conjugacy limits of C are classified for $n \leq 4$, in chapters 6 and 7, and in [26], [42], [41]. It is an open problem to classify the conjugacy limits of C when $n \geq 5$ (Questions 5.0.62 and 5.0.63).

The set of all closed subgroups of a group is a Hausdorff topological space with the *Chabauty topology* on closed sets (see section 5). Following notation in [31], the set of all closed abelian subgroups $\widehat{Ab}(n) = \{G \leq SL_n(\mathbb{R}) : G \cong (\mathbb{R}^{n-1}, +)\}$, is then a subspace, as is the set of conjugacy limit groups $\widehat{Red}(n) = \{G \leq SL_n(\mathbb{R}) : G \text{ is a limit of } C\}$. Taking the quotients by conjugacy, there are two topological spaces with the quotient topology: $Ab(n) = \widehat{Ab}(n)/\text{conjugacy}$ and $Red(n) = \widehat{Red}(n)/\text{conjugacy}$. In general these are not Hausdorff. For example, Theorem 6.0.77, shows $Red(2) = \{C, P\}$, where P is the parabolic group. Since $C \to P$, every neighborhood of P contains C.

Since every conjugacy limit of C is isomorphic to $\mathbb{R}^{n-1}$, then $Red(n) \subset Ab(n)$, see [31], Proposition 1. From chapter 6 we know $Ab(3) = Red(3)$, which has 5 points corresponding to 5 conjugacy classes of groups, and from Theorem 7.0.87, $Ab(4) = Red(4)$, which has 15 points, listed in chapter 4. When $n \leq 6$, Suprenko and Tyshkevitch, [56], have classified maximal commutative nilpotent (i.e. ad_x is nilpotent for all $x \in X$) subalgebras of $\mathfrak{sl}_n(\mathbb{C})$. Their results imply $Ab(5)$ has finitely many points, so $Red(5)$ has finitely many points. Iliev and Manivel, [31], ask if $Red(n)$ is finite when $n \geq 6$ (Question 5.0.63). The answer follows for $n \geq 7$ from the main result of this chapter:

Theorem 8.0.90. *If $n \geq 7$, then $\frac{n^2-8n+8}{8} \leq \dim Red(n) \leq n^2 - n$.*

The upper bound is given in [31]. This leaves the case $n = 6$ open. Haettel, and Iliev and Manivel show $\dim Red(n) < \dim Ab(n)$ for $n > 6$. We also give the first explicit examples of elements of $Ab(n) - Red(n)$ for $n = 5, 6, 8$ by describing certain properties of limit groups, which answers Question 5.0.62. In particular, we show

Theorem 8.0.91. *If $n \leq 4$, then $Ab(n) = Red(n)$. If $n \geq 5$, then $Red(n) \subsetneq Ab(n)$.*

A Family of Conjugacy Limit Groups

In this section, we define a family of groups, L_T, and show each is a conjugacy limit of the Cartan subgroup.

Definition 8.0.92. Let T be an m by n matrix, and $\rho_T : \mathbb{R}^{m+n} \to SL_{m+n+1}(\mathbb{R})$ be the homomorphism given by

$$\rho_T((a_1, ..., a_m, b_1, ..., b_n)) = \left(\begin{array}{cc} \underbrace{\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{matrix}}_{(m+1)\times(m+1)} & \begin{matrix} T_{11}a_1 & T_{12}a_1 & \dots & T_{1n}a_1 \\ T_{21}a_2 & T_{22}a_2 & \dots & T_{2n}a_2 \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1}a_m & T_{m2}a_m & \dots & T_{mn}a_m \\ b_1 & b_2 & \dots & b_n \end{matrix} \\ \underbrace{\begin{matrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}}_{n\times(m+1)} & \underbrace{\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{matrix}}_{n\times n} \end{array} \right).$$

The image of ρ_T is a group, $L_T \leq SL_{m+n+1}(\mathbb{R})$.

One may easily check that ρ_T is a homomorphism and L_T is a group, since matrix multiplication is given by

$$\left(\begin{array}{c|c} I & P \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & Q \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} I & P+Q \\ \hline 0 & I \end{array}\right).$$

Lemma 8.0.93. *For any m by n matrix T, with at least one nonzero entry in every row, the group L_T is a conjugacy limit of the positive diagonal Cartan subgroup.*

Proof. Let $C = \text{diag}\langle x_1, ...x_{m+n+1}\rangle \leq SL_{m+n+1}(\mathbb{R})$, be the positive diagonal Cartan subgroup, so $x_1 \cdot x_2 \cdots x_{m+n+1} = 1$. Let $\{P_r\}_{r=0}^{\infty}$ be the sequence of matrices

$$P_r = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & T_{11}r & T_{12}r & \dots & T_{1n}r \\ 0 & 1 & \dots & 0 & 0 & T_{21}r & T_{22}r & \dots & T_{2n}r \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & T_{m1}r & T_{m2}r & \dots & T_{mn}r \\ 0 & 0 & \dots & 0 & 1 & r^2 & r^2 & \dots & r^2 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Conjugating, $P_r C P_r^{-1} = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ where

$$A = \begin{pmatrix} x_1 & 0 & \dots & 0 & 0 \\ 0 & x_2 & \dots 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & x_m & 0 \\ 0 & 0 & \dots & 0 & x_{m+1} \end{pmatrix}, D = \begin{pmatrix} x_{m+2} & 0 & \dots & 0 \\ 0 & x_{m+3} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{m+n+1} \end{pmatrix},$$

$$B = \begin{pmatrix} T_{11}r(x_1 - x_{m+2}) & T_{12}r(x_1 - x_{m+3}) & \dots & T_{1n}r(x_1 - x_{m+n+1}) \\ T_{21}r(x_2 - x_{m+2}) & T_{22}r(x_2 - x_{m+3}) & \dots & T_{2n}r(x_2 - x_{m+n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1}r(x_m - x_{m+2}) & T_{m2}r(x_m - x_{m+3}) & \dots & T_{mn}r(x_m - x_{m+n+1}) \\ r^2(x_{m+1} - x_{m+2}) & r^2(x_{m+1} - x_{m+3}) & \dots & r^2(x_{m+1} - x_{m+n+1}) \end{pmatrix}.$$

The matrix A is $m+1$ by $m+1$, and D is n by n. The 0 matrix is $m+1$ by n and the matrix B is $m+1$ by n.

Assume for simplicity that all entries in the first column of T are non-zero. Given an element $l_T \in L_T$, we will find a sequence of elements in $P_r C P_r^{-1}$ which converges to l_T. Then the definition of convergence implies that L_T is a subgroup of the limit of $P_r C P_r^{-1}$.

Given x_{m+1} for $1 \le i \le n$ define

$$x_{m+1+i} = -r^{-2} b_i + x_{m+1}.$$

This ensures row $m+1$ of l_T and of $P_r C P_r^{-1}$ are equal since

$$r^2(x_{m+1} - x_{m+1+i}) = b_i. \tag{8.1}$$

For $i \leq m$ define x_i in terms of x_{m+1} by

$$x_i = r^{-1}a_i - r^{-2}b_1 + x_{m+1}.$$

It follows that column $m+2$ of l_T and of $P_r C P_r^{-1}$ are equal because

$$x_i - x_{m+2} = (r^{-1}a_i - r^{-2}b_1 + x_{m+1}) - (-r^{-2}b_1 + x_{m+1}) = r^{-1}a_i. \tag{8.2}$$

The determinant condition $x_1 \cdots x_{m+n+1} = 1$ determines x_{m+1}. Observe that $x_i \to x_{m+1}$ as $r \to \infty$, so the determinant is approximately $(x_{m+1})^{m+n+1}$. Thus every $x_i \to 1$ as $r \to \infty$.

We have now determined x_i for $1 \leq i \leq m+n+1$. It remains to show convergence in the remainder of the entries. Using equation (8.1) since $r \to \infty$,

$$r(x_{m+1} - x_{m+1+i}) \to 0.$$

By taking the difference of any two of these terms,

$$r(x_{m+1+j} - x_{m+1+k}) \to 0,$$

and, in particular

$$r(x_{m+2} - x_{m+1+k}) \to 0. \tag{8.3}$$

Consider the $(j, m+1+k)$ entry, for $1 \leq j, k \leq n$. Using (8.2) and (8.3), implies

$$T_{jk}r(x_j - x_{m+k+1}) = T_{jk}r(x_j - x_{m+2}) - T_{jk}r(x_{m+2} - x_{m+1+k}) \to T_{jk}a_j - T_{jk}0 = T_{jk}a_j.$$

This completes the proof when the entries in the first column of T are nonzero. Suppose some entries in the first column of T are zero. By hypothesis, T has a nonzero entry in every row, say T_{jk}. Pick x_i for $1 \leq i \leq m$ so that $T_{jk}m(x_j - x_{m+1+k}) \to a_jT_{jk}$. Since $T_{jk} \neq 0$, proceed as in the rest of the proof. Thus we have found a sequence $\text{diag}\langle x_1, ... x_{m+n+1}\rangle$ such that $P_rCP_r^{-1} \to l_T$.

This shows L_T is contained in the limit of $P_rCP_r^{-1}$. For dimension reasons (Proposition 5.0.51), and since C and L_T are connected and isomorphic to $\mathbb{R}^{m+n}$ (see [31] Proposition 1), then $P_rCP_r^{-1} \to L_T$. □

A Continuum of Conjugacy Classes of Limit

In this section we find some conjugacy invariants of the group L_T and use them to produce a family of conjugacy classes of dimension at least $(n^2 - 8n + 16)/8$ when $n \geq 7$. We first illustrate this when $n = 7$.

A subgroup $G \leq SL_n(\mathbb{R})$ acts on $\mathbb{R}P^{n-1}$. The orbit of a point, $x \in \mathbb{R}P^{n-1}$ is $G.x = \{y \in \mathbb{R}P^{n-1} : g.x = y \text{ for some } g \in G\}$. Denote by $\overline{G.x}$ the *orbit closure* of x.

Lemma 8.0.94. *Suppose $G, H \leq SL_n(\mathbb{R})$ and $Q \in SL_n(\mathbb{R})$ so that $G = QHQ^{-1}$. Then $[Q]$ is a projective transformation taking the orbit closures of G to the orbit closures of H.*

Proof. Since Q conjugates G to H, then Q takes the orbits of G to the orbits of H. Hence Q takes orbit closures of G to orbit closures of H. $\square$

Given $G \leq SL_n(\mathbb{R})$. The *orbit dimension function*, $\mathcal{R}_G : \mathbb{R}P^{n-1} \to \mathbb{N}$, is $\mathcal{R}_G(x) = \dim(\overline{G.x})$. As a corollary of Lemma 8.0.94, $\mathcal{R}_G(Q(x)) = \mathcal{R}_{QGQ^{-1}}(x)$ for all $x \in \mathbb{R}P^{n-1}$.

Next we define some conjugacy invariants of the action of a group on $\mathbb{R}P^{n-1}$. To do this we need an invariant, the *unordered generalized cross ratio*, of a collection of points in general position in projective space, which generalizes the cross ratio of 4 points on a projective line. This invariant is a finite subset of a product of projective spaces. Let $\mathcal{P}(S)$ denote the power set of S.

Let $\{e_1, ..., e_n\}$ be the *standard basis* in $\mathbb{R}^n$. The *standard projective basis* in $\mathbb{R}P^{n-1}$ is $\{[e_1], ..., [e_n], [e_1 + \cdots + e_n]\}$, and an *augmented basis* in $\mathbb{R}P^{n-1}$ is a set of $m \geq n+2$ points in general position, which means every subset of $(n+1)$ points is a projective basis.

Definition 8.0.95. 1. The *ordered generalized cross ratio* is the function, $C : (\mathbb{R}P^{n-1})^m \to (\mathbb{R}P^{n-1})^{m-(n+1)}$ defined as follows. Given any (ordered) augmented basis $(y_1, y_2, ..., y_m)$ in $\mathbb{R}P^{n-1}$, there is unique projective transformation, Q, which maps $(y_1, ..., y_{n+1}) \mapsto ([e_1], ..., [e_n], [e_1 + \cdots + e_n])$. Define $C(y_1, y_2, ..., y_m) := (Q(y_{n+2}), Q(y_{n+3}), ..., Q(y_m))$.

2. Given an (unordered) augmented basis in $\mathbb{R}P^{n-1}$, the *unordered generalized cross ratio*, $\mathcal{UC} : (\mathbb{R}P^{n-1})^m \to \mathcal{P}((\mathbb{R}P^{n-1})^{m-(n+1)})$ is the set of all generalized cross ratio tuples, $\mathcal{UC}(y_1, ..., y_m) := \{C(y_{\sigma(1)}, ..., y_{\sigma(m)}) : \sigma \in S_m\}$.

For example, if $\mathcal{A} = \{[1:0] : [1:1], [1:2], [1:\alpha]\} \subset \mathbb{R}P^1$, then

$$\mathcal{UC}(\mathcal{A}) = \{\frac{2(\alpha-1)}{\alpha}, \frac{\alpha}{2(\alpha-1)}, \frac{\alpha}{2-\alpha}, \frac{2-\alpha}{\alpha}, \frac{2(\alpha-1)}{\alpha-2}, \frac{\alpha-2}{2(\alpha-1)}\} \subset \mathbb{R}P^1.$$

Thus $\mathcal{UC}(\mathcal{A})$ is the set of all possible cross ratios of the points in $\mathcal{A}$. The cross ratio on $\mathbb{R}P^1$ is a special case of the ordered generalized cross ratio.

Proposition 8.0.96. *Let $\{y_1, ..., y_m\}$ and $\{x_1, ..., x_m\}$ be unordered augmented bases in $\mathbb{R}P^{n-1}$, so $m \geq n+2$. Then $\mathcal{UC}(y_1, ..., y_m) = \mathcal{UC}(x_1, ..., x_m)$, if and only if there is a projective transformation, $Q : \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$, such that $Q(\{y_1, ..., y_m\}) = \{x_1, ..., x_m\}$.*

Proof. First, suppose $\mathcal{UC}(y_1, ..., y_m) = \mathcal{UC}(x_1, ..., x_m)$. For the generalized cross ratio tuple coming from the identity permutation, $C(x_1, ..., x_m) \in \mathcal{UC}(x_1, ..., x_m)$, there is some reordering, $\sigma \in S_m$, such that $C(x_1, ..., x_m) = (z_1, ..., z_{m-n-1}) = C(y_{\sigma(1)}, ..., y_{\sigma(m)}) \in \mathcal{UC}(y_1, ..., y_m)$. That is, there exist projective transformations

$Q_1, Q_2 : \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ such that $Q_1((x_1, ..., x_{n+1})) = ([e_1], ..., [e_n], [e_1 + \cdots + e_n])$ and $Q_2((y_{\sigma(1)}, ..., y_{\sigma(n+1)})) = ([e_1], ..., [e_n], [e_1 + \cdots + e_n])$, and also $Q_1(x_{n+1+i}) = z_i = Q_2(y_{\sigma(n+1+i)})$, for $1 \leq i \leq m-(n+1)$. Set $Q := Q_2^{-1}Q_1$, so Q is a projective transformation such that $Q((x_1, ..., x_m)) = (y_{\sigma(1)}, ..., y_{\sigma(m)})$.

Conversely, suppose there exists a projective transformation $Q_0 : \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ such that $Q_0(\{x_1, ..., x_m\}) = \{y_1, ..., y_m\}$. Recall $\mathcal{UC}(x_1, ...x_m)$ $= \{C(x_{\sigma(1)}, ..., x_{\sigma(m)}) : \sigma \in S_m\}$. Set $Q_\sigma : \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ to be the unique projective transformation such that $Q_\sigma((x_{\sigma(1)}, ..., x_{\sigma(n+1)})) = ([e_1], ..., [e_n], [e_1 + \cdots + e_n])$. Then $\mathcal{UC}(x_1, ...x_m) = \{Q_\sigma(x_{\sigma(m)}) : \sigma \in S_m\}$. Since $Q_\sigma Q_0^{-1}((y_{\sigma(1)}, ..., y_{\sigma(n+1)})) = ([e_1], ..., [e_n], [e_1 + \cdots + e_n])$, and such a projective transformation is unique, so $\mathcal{UC}(y_1, ...y_m) = \{Q_\sigma Q_0^{-1}(y_{\sigma(m)}) : \sigma \in S_m\} = \mathcal{UC}(x_1, ..., x_m)$. □

Proposition 8.0.96 shows that unordered cross ratio of an unordered augmented basis is a complete projective invariant. As a warm-up, we show $Red(7)$ contains a subspace homeomorphic to an interval.

Definition 8.0.97. Let $\alpha \in \mathbb{R} - \{0, 1, 2\}$ be fixed, and let $\rho_\alpha : \mathbb{R}^6 \to SL_7(\mathbb{R})$ be the homomorphism defined by

$$\rho_\alpha((a, b, c, d, s, t)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & 0 & 0 & b & b \\ 0 & 0 & 1 & 0 & 0 & c & 2c \\ 0 & 0 & 0 & 1 & 0 & d & \alpha d \\ 0 & 0 & 0 & 0 & 1 & e & f \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The image of ρ_α is a group, $L_\alpha \leq SL_7(\mathbb{R})$.

An application of Lemma 8.0.93 shows that L_α is a conjugacy limit group. The unordered generalized cross ratio may be used to distinguish conjugacy classes of limit groups.

Proposition 8.0.98. *Given $\alpha, \beta \in \mathbb{R}$, then L_α is conjugate to L_β if and only if $\beta \in \left\{\frac{2(\alpha-1)}{\alpha}, \frac{\alpha}{2(\alpha-1)}, \frac{\alpha}{2-\alpha}, \frac{2-\alpha}{\alpha}, \frac{2(\alpha-1)}{\alpha-2}, \frac{\alpha-2}{2(\alpha-1)}\right\}$.*

Proof. We showed in Lemma 8.0.94 that if two groups are conjugate, there is a projective transformation taking the orbit closures of the first group to the orbit closures of the second. The group L_α partitions $\mathbb{R}P^6$ into orbit closures, and we will use the cross ratio to give an invariant of such a partition.

Let $\{e_1, ...e_7\}$ be the standard basis for $\mathbb{R}^7 := V$. Let $U = \langle e_1, .., e_5\rangle$, and $W = \langle e_6, e_7\rangle$. Then $V = U \oplus W$, and denote the quotient map $q : V \to V/U \cong W$. Given $[te_6 + e_7] \in \mathbb{P}(W)$, define the 5-dimensional projective subspace $\mathcal{H}_t := \mathbb{P}\langle e_1, ..., e_5, te_6 + e_7\rangle = \mathbb{P}\langle q^{-1}(te_6 + e_7)\rangle$. We show the orbit closure of a typical point $x \in \mathbb{R}P^6$ is $\mathcal{H}_t$, but there are 4 exceptional $\mathcal{H}_t$, which are the pre-images of 4 points in $\mathbb{P}(W)$. The unordered cross ratio gives an invariant of these points in $\mathbb{P}(W) \cong \mathbb{R}P^1$.

For convenience, denote the orbit dimension function for L_α by $\mathcal{R}_\alpha := \mathcal{R}_{L_\alpha}$. Let $x = [x_1 : \cdots : x_7] \in \mathbb{R}P^6$. The action of L_α is given by $L_\alpha.x =$

$$[x_1+ax_6 : x_2+b(x_6+x_7) : x_3+c(x_6+2x_7) : x_4+d(x_6+\alpha x_7) : x_5+ex_6+fx_7 : x_6 : x_7]. \tag{8.4}$$

If $x \in \mathbb{P}(U)$, then $\mathcal{R}_\alpha(x) = 0$, since $\mathbb{P}(U) = \mathrm{Fix}(L_\alpha)$. By (8.4), if $x \in \mathbb{P}(V - U)$, then $\mathcal{R}_\alpha(x) = 5$, unless one or more of the coefficients on a, b, c, d are zero, i.e., x satisfies one of the equations

$$x_6 = 0, \qquad x_6 + x_7 = 0, \qquad x_6 + 2x_7 = 0, \qquad x_6 + \alpha x_7 = 0. \tag{8.5}$$

Since $x \in V - U$, at least one of x_6, x_7 is not zero, and x satisfies at most one equation in (8.5). Consequently,

$$\mathcal{R}_\alpha(x) = \begin{cases} 0 & \text{if } x \in \mathbb{P}(U) \\ 4 & \text{if } x \in \mathcal{H}_t \text{ and } t \in \{0,1,2,\alpha\} \\ 5 & \text{if } x \in \mathcal{H}_t \text{ and } t \notin \{0,1,2,\alpha\}. \end{cases}$$

Then $\mathcal{A} := \{[1:t] \in \mathbb{R}P^1 : t = 0,1,2,\alpha\}$, is an augmented basis in $\mathbb{R}P^1$. The unordered generalized cross ratio of $\mathcal{A}$ is the set of cross ratios of $\mathcal{A}$, permuting the order of the points. Thus

$$\mathcal{UC}(\mathcal{A}) = \{\frac{2(\alpha-1)}{\alpha}, \frac{\alpha}{2(\alpha-1)}, \frac{\alpha}{2-\alpha}, \frac{2-\alpha}{\alpha}, \frac{2(\alpha-1)}{\alpha-2}, \frac{\alpha-2}{2(\alpha-1)}\} \subset \mathbb{R}P^1$$

Therefore L_α is conjugate to L_β if and only if $\beta \in \mathcal{UC}(\mathcal{A})$. □

We have shown the map $\mathbb{R} \to Red(7)$ given by $\alpha \to L_\alpha$ is at most 6 to 1. Therefore $Red(7)$ contains a continuum of non-conjugate limits.

Recall the *covering dimension* of a topological space, X, is smallest number, n, such that any open cover has a refinement in which no point is included in more than $n+1$ sets in the open cover. (See [44]). Denote the covering dimension of X by $\dim X$. Covering dimension is a topological invariant. We will show later that $\dim Red(7) \geq 1$.

The General Case: Bounds for $\dim Red(n)$

In this section, we exploit the unordered generalized cross ratio to obtain bounds on $\dim Red(n)$.

Definition 8.0.99. Let $G \leq SL_n(\mathbb{R})$ and $x \in \mathbb{R}P^{n-1}$. Let $\mathcal{H}$ be a projective subspace of $\mathbb{R}P^{n-1}$.

1. Set $M_G := \max\left\{\mathcal{R}_G(x) : x \in \mathbb{R}P^{n-1}\right\}$.

2. A point, x, is *typical* if $\mathcal{R}_G(x) = M_G$. The subspace $\mathcal{H}$ is *typical* if $\mathcal{H}$ is the orbit closure of a typical point.

3. The point x is *exceptional* if $0 < \mathcal{R}_G(x) < M_G$. The subspace $\mathcal{H}$ is *exceptional* if $\mathcal{H}$ is the union of orbit closures of exceptional points, and $\dim\, \mathcal{H} = M_G$.

Thus there are three types of points: fixed points with $\mathcal{R}_G(x) = 0$, exceptional points when $0 < \mathcal{R}_G(x) < M_G$, and typical points where $\mathcal{R}_G(x) = M_G$. In our previous example, $M_{L_\alpha} = 5$, the dimension of a typical subspace, and $\mathcal{H}_t$ is the orbit closure of a typical point. There are 4 exceptional subspaces $\{\mathcal{H}_t : t = 0, 1, 2, \alpha\}$ that break into orbit closures of smaller dimension. Next we generalize this example.

Definition 8.0.100. An m by n matrix, T, is *generic* if all collections of n row vectors of T are linearly independent. Set $\widehat{\mathcal{T}} := \{T : T \text{ is generic}\}$. When

$m \geq n+2$, the rows of a generic matrix, T, determine an augmented basis, $\tilde{T} \subset \mathbb{R}P^{n-1}$. Define an equivalence relation on $\widehat{\mathcal{T}}$ by $T \sim S$ if $\mathcal{UC}(\tilde{T}) = \mathcal{UC}(\tilde{S})$. Define $\mathcal{T} := \widehat{\mathcal{T}}/\sim$, and denote by $[T]_{\mathcal{T}} \in \mathcal{T}$ the equivalence class of T.

We give $\mathcal{T}$ a topology as follows. Take the subspace topology on $\widehat{\mathcal{T}} \subset \mathbb{R}^{m \times n}$, then $\mathcal{T}$ has the quotient topology. Since $\widehat{\mathcal{T}}$ is an open subset of $\mathbb{R}^{n \times m}$, it follows $\dim \widehat{\mathcal{T}} = nm$.

Proposition 8.0.101. $\dim \mathcal{T} = nm - n^2 - m$.

Proof. Consider the map $\Phi : \widehat{\mathcal{T}} \to (\mathbb{R}P^{n-1})^m$, where $\Phi(T) = \tilde{T} \in (\mathbb{R}P^{n-1})^m$, so Φ projectivizes the rows of T. The unordered generalized cross ratio is the surjective map $\mathcal{UC} : (\mathbb{R}P^{n-1})^m \to (\mathbb{R}P^{n-1})^{m-(n+1)}$. Given $T, S \in \widehat{\mathcal{T}}$, then $T \sim S$ if and only if $\mathcal{UC}(\Phi(T)) = \mathcal{UC}(\Phi(S))$. Therefore,

$$\dim \mathcal{T} = \dim(\mathcal{UC}(\Phi(\widehat{\mathcal{T}}))) = (n-1)(m-n-1) - 1 = nm - n^2 - m.$$

Where we subtract 1 for projectivizing. $\square$

A set of hyperplanes is in *general position* in $\mathbb{R}P^n$, if the set of dual points in the dual projective space to these hyperplanes is in general position. Let $[L]$ denote the conjugacy class of a group L, and T^t denote the transpose of T.

Proposition 8.0.102. *Suppose $m \geq n+2$, and $n \geq 2$. The function $f : \mathcal{T} \to$ Red$(m+n+1)$ given by $f([T]_{\mathcal{T}}) = [L_T]$ is well defined and injective.*

Proof. First we show f is well defined. Suppose $[S]_{\mathcal{T}} = [T]_{\mathcal{T}}$. Then there is a linear map $Q : \mathbb{R}^n \to \mathbb{R}^n$ such that Q maps the rows of T to the rows of S. That is, $Q(T^t) = S^t$, and taking the transpose of both sides, $TQ^t = S$. Set $Q^t = P$. Then L_T is conjugate to L_S by $I_{m+1} \oplus P^{-1}$, because:

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & P^{-1} \end{array}\right)\left(\begin{array}{c|c} I & T \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & P^{-1} \end{array}\right)^{-1} = \left(\begin{array}{c|c} I & TP \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} I & S \\ \hline 0 & I \end{array}\right)$$

So if $[T]_{\mathcal{T}} = [S]_{\mathcal{T}}$ then $[L_T] = [L_S]$. This shows f is well-defined.

To prove f is injective, we show if $[T]_{\mathcal{T}} \neq [S]_{\mathcal{T}}$, then the actions of $[L_T]$ and $[L_S]$ partition $\mathbb{R}P^{m+n}$ into orbit closures which are not projectively equivalent.

Let $\{e_1, ...e_{m+n+1}\}$ be the standard basis for $V = \mathbb{R}^{m+n+1}$. Define $U = \langle e_1, ..., e_{m+1}\rangle$, and $W = \langle e_{m+2}, ..., e_{m+n+1}\rangle$, then $V = U \oplus W$. Let $q : V \to V/U \cong W$ be the quotient map. Given $[v] \in \mathbb{P}(W)$, let $\mathcal{H}_v$ be the $(m+1)$-dimensional projective subspace $\mathcal{H}_v = \mathbb{P}\langle e_1, ..., e_{m+1}, v\rangle = \mathbb{P}\langle q^{-1}(v)\rangle$. We show the orbit closure of a typical point $x \in \mathbb{R}P^{m+n}$ is $\mathcal{H}_v$, and the exceptional subspaces are the pre-image of m hyperplanes in $\mathbb{P}(W)$, which determine an invariant of L_T.

The orbit dimension function for L_T by $\mathcal{R}_T := \mathcal{R}_{L_T}$, has maximum $M_T := M_{L_T}$. The action of L_T on $\mathbb{R}P^{m+n}$ is given by

$$\begin{aligned}
&L_T.[x_1 : \cdots : x_{m+n+1}] = \\
&[x_1 + a_1(\sum_{i=1}^{n} T_{1i}x_{m+1+i}) : x_2 + a_2(\sum_{i=1}^{n} T_{2i}x_{m+1+i}) : \cdots : \\
&x_m + a_m(\sum_{i=1}^{n} T_{mi}x_{m+1+i}) : x_{m+1} + \sum_{i=1}^{n} x_{m+1+i}b_i : x_{m+2} : \cdots : x_{m+n+1}].
\end{aligned} \tag{8.6}$$

Set

$$\phi_j(x_{m+2}, ..., x_{m+n+1}) = \sum_{i=1}^{n} T_{ji}x_{m+1+i}, \qquad 1 \leq j \leq m, \tag{8.7}$$

a collection of linear functionals $\phi_j : \mathbb{R}^n \to \mathbb{R}$. Then we may rewrite

$$\begin{aligned}
&L_T.[x_1 : \cdots : x_{m+n+1}] = \\
&[x_1 + a_1\phi_1(x_{m+2}, ..., x_{m+n+1}) : x_2 + a_2\phi_2(x_{m+2}, ..., x_{m+n+1}) : \cdots : \\
&x_m + a_m\phi_n(x_{m+2}, ..., x_{m+n+1}) : x_{m+1} + \sum_{i=1}^{n} x_{m+1+i}b_i : x_{m+2} : \cdots : x_{m+n+1}].
\end{aligned} \tag{8.8}$$

Since $T \in \mathcal{T}$ is generic, any n rows of T are linear independent, so by (8.8), $M_T = m + 1$. If $x \in \mathbb{P}(U)$, then the group L_T fixes x, so $\mathcal{R}_T(x) = 0$. We want to find the exceptional points. From (8.8) the coefficient on a_i is ϕ_i. Thus $\mathcal{R}_T(x) < m + 1$ if and only if ϕ_i is zero, i.e., $(x_{m+2}, ..., x_{m+n+1}) \in \ker(\phi_i)$.

The set $W_j := \ker(\phi_j) \subset W$ is a hyperplane. Then $\mathcal{R}_T(x) < n + 3$ if and only if $x \in q^{-1}(W_j) = U \oplus W_j$, for some $1 \leq j \leq m$. Thus, the set of exceptional points is the pre-image of the m hyperplanes, $\mathbb{P}(W_j) \subset \mathbb{P}(W) \cong \mathbb{R}P^{n-1}$. Let

$w_j \in \mathbb{P}(W^*)$ denote the point in the dual projective space determined by the hyperplane $W_j \subset W$.

By hypothesis, $T \in \mathcal{T}$ is generic, so these hyperplanes are in general position. The points $\{w_j\}_{j=1}^m$ are in general position, and form an augmented basis,

$$\delta(T) \equiv \{w_1, \cdots, w_m\} \subset \mathbb{P}(W^*) \cong \mathbb{R}P^{n-1} \tag{8.9}$$

We are now able finish the proof that f is injective. Suppose $[T]_{\mathcal{T}}, [S]_{\mathcal{T}} \in \mathcal{T}$ with $f([S]_{\mathcal{T}}) = f([T]_{\mathcal{T}})$. That is, L_S is conjugate to L_T, so Lemma 8.0.94 implies this conjugacy takes the exceptional hyperplanes in the orbit closures of L_T, to the exceptional hyperplanes in the orbit closures of L_S. The dual conjugacy takes the dual augmented basis, $\delta(T)$, to the dual augmented basis, $\delta(S)$. By Proposition 8.0.96, $\mathcal{UC}(\delta(T)) = \mathcal{UC}(\delta(S))$, so there is a projective transformation taking $\delta(T)$ to $\delta(S)$. A row of T determines a dual vector, ϕ_i, with $\ker \phi_i = W_i$, dual to $w_i = [\phi_i] \in \delta(T)$. So the dual transformation takes the (projectivized) rows of T to the (projectivized) rows of S. Thus $[T]_{\mathcal{T}} = [S]_{\mathcal{T}}$, and f is injective. □

Proposition 8.0.102 shows there are infinitely many non-conjugate limits of the positive diagonal Cartan subgroup in $SL_k(\mathbb{R})$ when $k \geq 7$. We want to give bounds for $\dim Red(k)$. In the remainder of the section, set $k = m + n + 1$.

Theorem 8.0.103. *Let $m - 2 \geq n \geq 2$. The function $\hat{f} : \widehat{\mathcal{T}} \to \widehat{Red}(k)$ defined by $\hat{f}(T) = L_T$, is continuous and one to one on an open subset, $X \subset \widehat{\mathcal{T}}$.*

Proof. Recall from Definition 8.0.92 the linear map $\rho_T : \mathbb{R}^{m+n} \to SL_{m+n+1}(\mathbb{R}) \subset \mathrm{End}(\mathbb{R}^{m+n+1})$. Thus $\rho_T \in \mathrm{Hom}(\mathbb{R}^{m+n}, \mathrm{Mat}_{m+n+1})$. Since $T \in \mathbb{R}^{m+n} = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, the map $T \mapsto \rho_T$ is a continuous linear map, as it maps one matrix to a larger one. Remember L_T is defined as the image of ρ_T, so we view $L_T \subset \mathrm{End}(\mathbb{R}^k)$. Since $\hat{f}(T) = L_T$, the image $\hat{f}(\widehat{\mathcal{T}}) \subset \mathrm{End}(\mathbb{R}^k)$, and $\hat{f}$ is continuous.

Define X to be the set containing one representative of each equivalence class $[T]_{\mathcal{T}}$. Since L_T is conjugate to L_S if and only if $\mathcal{UC}(\tilde{T}) = \mathcal{UC}(\tilde{S})$, then $\hat{f} : X \to \widehat{Red}(k)$ is injective. ☐

Corollary 8.0.104. *If $k \geq 7$, then* $\dim Red(k) \geq \frac{k^2-8k+8}{8}$.

Proof. Proposition 8.0.101 says $\dim \mathcal{T} = nm - n^2 - m$, and Theorem 8.0.103 implies $\dim \widehat{Red}(k) \geq \dim X = \dim \mathcal{T}$. Since $k \geq 7$, we may choose $m-2 \geq n \geq 2$.

We may change the size of the m by n matrix (as long as $m - 2 \geq n \geq 2$), so $\dim Red(n)$ is bounded below by the maximum of $mn-n^2-m$. Since $m+n+1 = k$, and k is fixed, we want to maximize

$$g(n) = n(k-n-1) - n^2 - (k-n-1) = kn - 2n^2 - k + 1.$$

The maximum occurs at $n = \frac{k}{4}$ and $m = \frac{3k-4}{4}$. Therefore the maximum of $mn - n^2 - m + 1$ is $\frac{k^2-8k+8}{8}$. ☐

In particular, $\frac{k^2-8k+8}{8} \geq 0$ for $k \geq 7$. Below is the proof of an upper bound of $\dim Red(k)$, given in [31] for (Krull) dimension of $Red(n)$.

Theorem 8.0.105. $\dim Red(k) \leq k^2 - k$.

Proof. Let C denote the positive diagonal Cartan subgroup, and let $P \in GL_k(\mathbb{R})$. By [48] Theorem 1, or [59] Theorem 2.9.7, the dimension of the set of all conjugates of C is $k^2 - k$, since $PCP^{-1} = C$ if and only if P is a diagonal matrix or a permutation matrix. Since C is a semi-algebraic set ([8] Proposition 2.1.8), the set of conjugates of C is a semi-algebraic set ([8] Proposition 2.2.7). Thus the set of conjugacy limits of C is the boundary of the Zariski closure of the set of conjugates. Applying Propositions 2.8.2 and 2.8.13 from [8], gives $\dim(Red(k)) \leq k^2 - k$. □

Corollary 8.0.104 and Theorem 8.0.105 imply Theorem 8.0.90.

Abelian Groups which are Not Conjugacy Limit Groups

In this section, we give examples of elements of $Ab(n) - Red(n)$. There are two properties of conjugacy limit groups of C which are not universal amongst abelian groups. The first property is a conjugacy limit group is *flat*, and the second is that it contains a one parameter subgroup with a particular Jordan block structure.

Suppose L is a conjugacy limit of C in $SL_n(\mathbb{R})$. Then we claim L is the intersection of a vector space with $SL_n(\mathbb{R}) \subset \mathrm{End}(\mathbb{R}^n)$, which is a vector space. The positive diagonal Cartan subgroup is of this form, and conjugacy is a linear map, so it preserves this property. Such a group is a *flat group*. Conjugacy limits of C are flat groups.

Definition 8.0.106. Let $\mu_k : \mathbb{R}^{k-1} \to SL_k(\mathbb{R})$ be the representations below for $k = 5, 6$.

$$\mu_5((a,b,c,d)) = \begin{pmatrix} 1 & a & 0 & \frac{a^2}{2} & b \\ 0 & 1 & 0 & a & 0 \\ 0 & 0 & 1 & c & d \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mu_6((a,b,c,d,e)) = \begin{pmatrix} 1 & a & \frac{a^2}{2} & 0 & b & c \\ 0 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & d & e \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Set $M_k \leq SL_k(\mathbb{R})$ to be the respective images of μ_k.

It is easy to check that M_k is an abelian group of dimension $k-1$. Moreover, neither is a limit of C, since they are not flat groups.

Thus we have given examples of elements in $Ab(n) - Red(n)$ for $n = 5, 6$. This shows $Ab(n) \neq Red(n)$ when $n = 5, 6$, which answers Question A in [31]. By Lemma 5.0.56 there is an abelian subalgebra of dimension $n-1$ which is not the conjugacy limit of a Cartan subalgebra. Applying Haettel's result implies $Ab(n) = Red(n)$ if and only if $n \leq 4$. For $n = 5, 6$, we have shown $Red(n) \subsetneq Ab(n)$. Combined with Haettel's result, this completes the proof of Theorem 8.0.91.

We give another property satisfied by conjugacy limit groups of C, and an example of an element of $Ab(8) - Red(8)$, which is a flat group, but does not satisfy this additional property. Thus to determine if a group is a conjugacy limit of C, it is necessary but not sufficient for the group to be a flat group.

Suppose $\mathbb{R}^{n-1} \cong G \leq SL_n(\mathbb{R})$. Define the *rank* of G to be $\text{rank}(G) = \text{rk}(G) := \max_{g \in G} \text{rk}(g - I_n)$. If $\text{rk}(g - I_n) = \text{rk}(G)$, then g is *generic*. In the special case when G is a unipotent group, one may compute the rank from the Jordan Normal Form (JNF) of each group element, by counting the number of off-diagonal entries.

Proposition 8.0.107. *Suppose $G \leq SL_n(\mathbb{R})$ is a unipotent group, and L is a conjugacy limit of G. Then $rk(L) \leq rk(G)$.*

Proof. Proposition 5.0.52 implies that the dimension of the normalizer increases under taking a conjugacy limit. The dimension of the normalizer of an abelian group depends on the size of the blocks of the JNF of a generic element: it has largest dimension when generic elements have JNF closest to the identity, which is when the size of the Jordan blocks is smallest. Thus the size of the Jordan blocks of a generic element must remain constant or decrease. Since the rank of a unipotent group may be computed by counting the sizes of the blocks in the JNF of a typical element, the rank cannot increase. □

Suppose $G \leq SL_n(\mathbb{R})$ is isomorphic to $(\mathbb{R}^{n-1}, +)$. A *flag of subgroups* in G is a collection of subgroups H_i with $1 \leq i \leq n-1$, and $H_{i-1} \leq H_i$.

Corollary 8.0.108. *If L is a conjugacy limit of C, then L contains a flag of subgroups, H_i, with rk$(H_i) \leq i$ for all i. In particular, L contains a 1 parameter subgroup H_1 with rk$(H_1) = 1$.*

Proof. Suppose $PCP^{-1} \to L$ by some $P \in SL_n(\mathbb{R})$. Set $C_1 = \text{diag}\langle a, 1, 1, ..., 1\rangle$, and let L_1 be the conjugacy limit of C_1 by P. By Proposition 8.0.107, $\text{rk}(L_1) \leq \text{rk}(C_1) = 1$. Since $L_1 \cong \mathbb{R}$, then L_1 cannot be the identity group, so $\text{rk}(L_1) = 1$. All of the elements in C_1 are contained in C, and their limits under conjugacy by P are contained in L. Therefore L_1 is a rank 1 subgroup of L.

In general, C has a flag of subgroups with rank $1, ..., n-1$, as more of the entries on the diagonal are allowed to vary. The conjugacy limits of this flag of subgroups of C give a flag of conjugacy limits. □

Set $E \leq SL_8(\mathbb{R})$ to be the image of the representation $\rho : \mathbb{R}^7 \to SL_8(\mathbb{R})$:

$$\rho((a,b,c,d,e,f,g)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & c & g & f \\ 0 & 1 & 0 & 0 & c & b & f & e \\ 0 & 0 & 1 & 0 & b & a & e & d \\ 0 & 0 & 0 & 1 & a & g & d & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that E is an abelian subgroup, since matrix multiplication is given by

$$\left(\begin{array}{c|c} I & A \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & B \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} I & A+B \\ \hline 0 & I \end{array}\right).$$

Proposition 8.0.109. *The group E has no 1 parameter subgroups of rank 1.*

Proof. A matrix has rank 1 if and only if every 2×2 minor is zero. We show that $\rho(a,b,c,d,e,f,g) - I_8$ has rank 1 if and only if $(a,b,c,d,e,f,g) = (0,...,0)$.

Consider the 2×2 minors of

$$\begin{pmatrix} 0 & c & g & f \\ c & b & f & e \\ b & a & e & d \\ a & g & d & 0 \end{pmatrix}.$$

Since the upper left minor must be zero, then $c = 0$. Looking at the minor directly below, implies $b = 0$. Continuing in this fashion, $b = 0, a = 0, d = 0, e = 0, f = 0$ and $g = 0$. (Alternatively, take all of the minors, and check $(0, 0, ..., 0)$ is the only solution.) Thus $\rho(a, b, c, d, e, f) - I_8$ has rank 1 if and only if $(a, b, c, d, e, f, g) = (0, ..., 0)$. But if $(a, b, c, d, e, f, g) = (0, ..., 0)$ then $\rho(0, .., 0) - I_8$ is the zero matrix, with rank 0. Therefore E (the image of ρ) contains no rank 1 subgroups. □

Combining Corollary 8.0.108 and Proposition 8.0.109, shows the abelian group, E, is *not* a conjugacy limit of C. Thus there are two necessary conditions for a group to be a limit group: the group must be a flat group, and contain a rank 1 subgroup.

Chapter 9

Future Work and Applications

Diagonal matrices are central to much of pure and applied mathematics. Understanding limits of groups of diagonal matrices provides a coordinate free approach to studying invariant properties of diagonal matrices.

In future work, I would also like to study limits of other abelian subgroups, or perhaps nilpotent or solvable groups. Such limits may be related to circle factors and tori. I would also like to study limits of other types of symmetric spaces.

Geometric structures often arise as a homogeneous space quotiented by a discrete group of isometries. Perhaps it is possible to find a discrete subgroup inside a conjugacy limit group, and study geometric structures that arise as such quotients. There may be some new types of geometry, giving rise to affine structures on manifolds.

Further, some conjugacy limit groups give rise to affine structures on the torus (in $SL_3(\mathbb{R})$, the limits C, F, N_1 and N_3). Is it possible to find other limit groups which give rise to affine structures on higher dimensional tori? In particular, I have a conjecture that a necessary condition is for the action of the group on $\mathbb{R}P^n$ must have an orbit closure that is $\mathbb{R}P^n$.

There are many more questions we might ask about the spaces $Red(n)$ and $Ab(n)$. For example: are they connected? Does every component of $Ab(n)$ contain a component of $Red(n)$, and is it possible to retract from $Ab(n)$ to $Red(n)$? What properties characterize $Red(n)$ that are not inherited by $Ab(n)$? If an element of $Ab(n)$ is flat and contains a rank 1 subgroup, are these two properties sufficient to show that it is an element of $Red(n)$?

The space $Red(n)$ is finite for $n = 5$. My next project is to write down a classification theorem, and then to work out the case $n = 6$. Are there infinitely many or finitely many conjugacy limits of C when $n = 6$?

I would also like to extend the hyperreal techniques to higher dimensions, and use them to classify conjugacy limit groups. Given a complete list of $(n-1)$-dimensional abelian subgroups in each dimension, it might be possible to classify generalized cusps in higher dimensions.

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