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ABSTRACT

A formal definition of a continuation of partial-wave amplitudes from physical ℓ values is given, and shown to be unique. The cases of potential scattering and of a relativistic S-matrix theory satisfying the Mandelstam representation are then considered in detail.

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1. -Introduction

The usefulness of considering partial-wave amplitudes as analytic functions of ℓ, with certain assumed properties, for an understanding of the S-matrix theory of strong interactions has recently been shown by several authors ([1]). Little attention, however, has so far been paid to the problems of defining the continuation from physical ℓ values and of establishing its properties. In this paper we first define a particular continuation, which we show to be unique and to satisfy a generalized unitarity condition, and then we consider the case of potential scattering ([2]), and of relativistic S-matrix theory. Complications due to spin are ignored throughout.

2. -The continuation from physical ℓ

The partial-wave amplitudes for physical ℓ are defined by

$$a_\ell = \frac{1}{2} \int_{-1}^{+1} A(z)\, P_\ell(z)\, dz, \qquad \ell = +\text{ve integer}, \tag{1}$$

where $A(\cos\theta)$ is the scattering amplitude as a function of θ, the center-of-mass scattering angle. We assume, and will prove below for certain cases, that there exists a function $a(\ell)$ with the properties

(i) $a(\ell) = a_\ell$, $\qquad \ell = +\text{ve integer}$,

(ii) $a(\ell)$ is analytic in $\operatorname{Re}\ell > L$, for some $L > -1/2$,

apart possibly from a finite number of poles. The restriction to $L > -1/2$ is merely for convenience and is not serious since if (ii) is satisfied for a given L it is certainly satisfied for any larger L.

It follows that there are, in fact, many such functions $a(\ell)$, so to specify the function uniquely we add a third condition:

(iii) It is possible to make the Sommerfeld-Watson transform [3]. We formulate this condition precisely as follows. From eq. (1), and for z in some region including the physical values z real and $-1 < z < 1$,

$$A(z) = \sum_{\ell = +\text{ve integers}} (2\ell + 1)\, a_\ell\, P_\ell(z). \qquad (2)$$

Thus, introducing $a(\ell)$ satisfying (i) and (ii), we have

$$A(z) = -\frac{1}{2i} \int_C (2\ell + 1)\, a(\ell)\, \frac{P_\ell(-z)}{\sin \pi \ell} + \sum_{\substack{\ell = +\text{ve integers} \\ < L}} (2\ell + 1)\, a_\ell\, P_\ell(z), \qquad (3)$$

where C is a contour enclosing all the positive integer > L (see Fig. 1), provided there are no poles of $a(\ell)$ inside [4] C and that the integral exists. Then, if $a(\ell)$ has suitable behavior for $|\ell| \to \infty$, $\mathrm{Re}\,\ell > L$, we can write

$$A(z) = -\frac{1}{2i} \int_{\ell = L - i\infty}^{\ell = L + i\infty} (2\ell + 1)\, a(\ell)\, \frac{P_\ell(-z)}{\sin \pi \ell} - \pi \sum_{\substack{i \\ \mathrm{Re}\,\alpha_i > L}} (2\alpha_i + 1)\, \beta_i\, \frac{P_{\alpha_i}(-z)}{\sin \pi \alpha_i}$$

$$+ \sum_{\substack{\ell = +\text{ve integers} \\ < L}} (2\ell + 1)\, a_\ell\, P_\ell(z), \qquad (4)$$

where the integral is taken along a line parallel to the imaginary axis, α_i are the positions of the poles of $a(\ell)$, and β_i are the corresponding residues. Condition (iii) then means that eq. (4) should be valid at least for a region of z which includes the neighborhood of the parts of the real axis

$$-\sqrt{3}/2 < z < \sqrt{3}/2 \quad \text{and} \quad z > 3/2\sqrt{2}.$$

The origin of these limits is explained in the Appendix.

Since, for Re $\alpha > -\frac{1}{2}$,

$$P_\alpha(z) \sim (z)^\alpha, \quad |z| \to \infty, \tag{5}$$

it follows that

$$A(z) \sim (z)^{\alpha_1}, \qquad |z| \to \infty, \tag{6}$$

where α_1 is the position of the pole which lies farthest to the right. This property is not obviously preserved if we take $L < -\frac{1}{2}$ (assuming that the three conditions are still satisfied) and consider Re $\alpha_1 < -\frac{1}{2}$. However, Mandelstam (private communication) has suggested an alternative representation which shows that this behavior does indeed persist to the left of Re $\ell = -\frac{1}{2}$, provided

$$a(\ell) = a(-\ell - 1), \qquad \ell = \text{half odd integer}. \tag{7}$$

We can now easily see that conditions (i), (ii), (iii) define $a(\ell)$ uniquely. For, if $a(\ell)$ and $a'(\ell)$ satisfy the conditions, then so does

$$a(\ell) + \frac{a(\ell) - a'(\ell)}{\ell - \ell_0}$$

for any ℓ_0. It follows from eqs. (4) and (5) that, for Re $\ell_0 >$ Re α_1,

$$A(z) \sim \frac{a(\ell_0) - a'(\ell_0)}{\sin \pi \ell_0} (2\ell_0 + 1) z^{\ell_0}, \quad |z| \to \infty, \tag{8}$$

unless $a(\ell_0) = a'(\ell_0)$. Comparing eqs. (8) and (6), we see that

$$a'(\ell_0) = a(\ell_0), \qquad \text{Re } \ell_0 > \text{Re } \alpha_1,$$

from which the uniqueness follows. It is worth mentioning that the existance of an $a(\ell)$ satisfying the three conditions imposes severe restrictions on the a_ℓ. We give below an example where the required $a(\ell)$ does not exist. If instead of (iii) we had imposed the more restrictive condition that $a(\ell)$ should not have an essential singularity at infinity, it is almost certain that we would have excluded "physics" from our discussion [see below eq. (23)].

We have been able to obtain this property without an explicit discussion of the properties of $P_\ell(z)$ for large $|\ell|$. However, such a discussion can be given (see Appendix), and it is easily seen that condition (iii) is equivalent to the requirement that $a(\ell)$ is bounded by a power of ℓ as $|\ell| \to \infty$, $-\pi/2 < \arg \ell < +\pi/2$, and that $a(\ell) \to 0$ at least as fast as $(\ell)^{-3/2}$, as $\ell \to \pm i\,\infty$. It follows that the function

$$a(\ell)\, a^*(\ell^*) - \frac{1}{2i}\left[a(\ell) - a^*(\ell^*)\right]$$

satisfies conditions (ii) and (iii), provided $a(\ell)$ does. Thus, since by unitarity this function is zero for ℓ = integer, the above uniqueness proof can be used to show that this function is zero in $\mathrm{Re}\,\ell > L$. Therefore, conditions (i), (ii), (iii) imply unitarity for all ℓ.

Finally, we remark that, since there are in general infinitely many continuations of a_ℓ away from physical integers, any discussions of the positions and residues of poles of $a(\ell)$ as a function of ℓ which do not involve using the condition (iii) should be treated with caution.

3. -Nonrelativistic potential scattering

Regge (2) and Bottino et al. (2) have discussed this problem at length, and have considered a specific continuation from physical ℓ, namely that defined by the solution of the Schrodinger equation

$$\frac{d^2\phi}{dr^2} + k^2\phi - \frac{\ell(\ell+1)}{r^2}\,\phi - V(r)\,\phi = 0, \qquad (9)$$

for all ℓ, using the usual definition of the S matrix,

$$\phi(r) \to e^{-ikr} - e^{i\pi\ell} S(\ell)\, e^{ikr}, \quad |r| \to \infty, \qquad (10)$$

and the partial-wave scattering amplitude,

$$a(\ell) = \frac{S(\ell) - 1}{2i}. \tag{11}$$

Regge has shown that, for a wide class of potentials $V(r)$, $a(\ell)$ defined as above satisfies condition (ii) with $L = -\frac{1}{2}$, and further that, when the potential is a sum of Yukawa potentials,

$$r\,V(r) = \int_{\mu_0}^{\infty} \sigma(\mu)\, e^{-\mu r} d\mu, \tag{12}$$

it also satisfies condition (iii).

If we consider only potentials for which $r\,V(r)$ is regular at the origin, which is true for eq. (12) provided $\sigma(\mu) \to 0$ at least as fast as an exponential for $\mu \to \infty$, we can show that $a(\ell)$ is analytic, apart from poles, for all $\ell \neq \infty$. To see this we consider the solution of eq. (9), $\phi(\ell, k, r)$, which, for Re $\ell > -1$, is regular at $r = 0$,

$$\phi(\ell, k, r) = r^{\ell+1} - \frac{1}{2\ell + 1} \int_0^r \left(\frac{r'^{\ell+1}}{r^{\ell}} - \frac{r^{\ell+1}}{r'^{\ell}} \right) [V(r') - k^2] \phi(\ell, k, r') \tag{13}$$

[see, for instance, Bottino et al.(2), eq. (1, 2)]. With

$$[r\,V(r) - k^2 r] = \sum_{n=0}^{\infty} \alpha_n r^n, \qquad |r| < R, \tag{14}$$

we can expand $\phi(\ell, k, r)$ about $r = 0$ and obtain

$$\phi(\ell, k, r) = r^{\ell+1} \sum_0^{\infty} a_n\, r^n, \tag{15}$$

with

$$a_n = \frac{1}{(2\ell + 1 + n)} \; \frac{1}{n} \sum_{m=0}^{n-1} \alpha_m\, a_{n-1-m}, \quad n \geq 1, \tag{16}$$

and

$$a_0 = 1. \tag{17}$$

(The idea of making this expansion was suggested to me by a letter from M. L. Goldberger to G. F. Chew.)

The radius of convergence of the series for ϕ can be shown to be at least as large as that for $r\,V(r)$. Hence for $|r| < R$, $\phi(\ell, k, r)$ is analytic in ℓ apart from poles. Now [Bottino et al. (2)] the S matrix is given by

$$S(\ell) = - \frac{W[f(\ell, k, r),\ \phi(\ell, k, r)]}{W[f(\ell, -k, r),\ \phi(\ell, k, r)]}\ e^{i\pi\ell} \tag{18}$$

where W indicates the Wronskian of the two functions, and $f(\ell, k, r)$ is that solution of eq. (9) which tends to e^{-ikr} for large $|r|$. Since the Wronskian can be formed for any r, in particular for $|r| < R$, the meromorphic property of $S(\ell)$ follows from this equation when we recall that $f(\ell, k, r)$ is analytic in ℓ. [This follows from a theorem due to Poincare (5), and the fact that the boundary condition which specifies $f(\ell, k, r)$ is independent of ℓ; the result is proved explicitly by Regge (2)].

The function $\phi(\ell, k, r)$ has poles at 2ℓ equal to a negative integer, but these do not affect the S matrix since they cancel in eq. (18). Specifically, to find $S(\ell)$ for $2\ell \to -p - 1$ $(p = 0, 1, 2, \ldots)$, we consider

$$\operatorname*{Lt}_{2\ell \to -p-1} \frac{(2\ell + p + 1)\,\phi(\ell, k, r)}{\frac{1}{p}\sum_{m=0}^{p-1} a_m\, a_{p-1-m}} = r^{-\frac{p}{2}+1} \sum_{n=0}^{\infty} a_n\, r^{p+n}, \tag{19}$$

where we have used eqs. (16) and (17). Thus (6)

$$\operatorname*{Lt}_{2\ell \to -p-1} (2\ell + p + 1) \frac{\phi(\ell, k, r)}{\frac{1}{p}\sum_{m=0}^{p-1} a_n\, a_{p-1-m}} = r^{(p/2)+1} \sum_{n=0}^{\infty} a_n\, r^n$$

$$= \phi(p/2, k, r),$$

$$p = \text{integer}. \tag{20}$$

Now $f(\ell, k, r)$ has the property

$$f(\ell, k, r) = f(-\ell - 1, k, r), \text{ for all } \ell, \tag{21}$$

since the Schrodinger equation is clearly unaffected by the substitution $\ell \rightarrow -(\ell + 1)$. Hence the S matrix has the following symmetry properties:

$$S(\ell) = S(-\ell - 1), \qquad \ell = \text{half odd integer} \tag{22}$$

and

$$S(\ell) = -S(-\ell), \qquad \ell = \text{integer}. \tag{23}$$

The first equation is the Mandelstam symmetry (eq. 7). It follows from these equations that $a(\ell)$ has an essential singularity at infinity.

For the square well potential, depth V and radius a, the S matrix defined by eqs. (9) and (10) satisfies

$$S(\ell) = - \frac{y J_{\ell+3/2}(y) \; Q^{+}_{\ell+1/2} - x J_{\ell+1/2}(y) \; Q^{+}_{\ell+3/2}}{y J_{\ell+3/2}(y) \; Q^{-}_{\ell+1/2} - x J_{\ell+1/2}(y) \; Q^{-}_{\ell+1/2}} , \tag{24}$$

where

$$Q^{\pm}_{\nu} = e^{\pm i \pi \nu} J_{\nu}(x) - J_{-\nu}(x), \tag{25}$$

$$x = ka, \tag{26}$$

and

$$y = a\sqrt{k^2 + V} \tag{27}$$

The resulting $a(\ell)$ is clearly meromorphic in ℓ, for all ℓ. However, since

$J(z) \rightarrow (\frac{z}{a})^{\nu} \frac{1}{\Gamma(\nu+1)}$, $\quad |\nu| \rightarrow \infty$, and $\Gamma(\nu + 1)$ can be expressed for large $|\nu|$ by means of Stirling's formula, it follows that

$$S(\ell) \sim e^{\pi |\ell|}, \qquad \ell \rightarrow -i\infty, \tag{28}$$

so that condition (iii) is violated.

Now there are clearly many other continuations that satisfy condition (i), and which are meromorphic [for example we could replace $e^{i\pi\ell}$ in eq. (24) by $\cos \pi\ell$ or $(\cos \pi\ell)^{-1}$]. However, there is in fact none which satisfies all three conditions, for if there were then eq. (4) would show that $A(z)$ is bounded by a power of $|z|$ for large (z), which is false for the square well potential. (I am indebted to J. Charap for pointing out that the search for a suitable continuation for the square well potential was doomed to failure.)

4. -S-matrix theory

Here we do not have the Schrodinger equation but, following a suggestion made by Froissart (unpublished, talk at La Jolla conference, 1961), we can prove that if $A(z)$ satisfies a dispersion relation then a function $a(\ell)$ with the required properties exists.

We note first that the right-hand side of eq. (1), continued to all ℓ, is analytic everywhere (except $\ell = \infty$), and so is unlikely to be the required continuation. In fact, the asymptotic behavior of P_ℓ for large ℓ [see eq. (A1)] will in general make the function violate condition (iii). However, if the amplitude satisfies the Mandelstam representation we can write

$$A(s, z) = \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{A_t(s, t')}{t' - t} + \frac{1}{\pi} \int_{u_0}^{\infty} du' \frac{A_u(s, u')}{u' - u}, \tag{29}$$

where s, t, and u are the squares of the invariant energies in the three channels, and A_t and A_u are the absorptive parts in the t and u channels, respectively. In terms of the center-of-mass momentum in the s channel (q) we have, for equal masses,

$$s = 4(q^2 + m^2), \tag{30}$$

$$z = (1 + \frac{t}{2q^2}) = -(1 + \frac{u}{2q^2}). \tag{31}$$

Thus

$$A(s,z) = \frac{1}{\pi} \int_{1+(t_0/2q^2)}^{\infty} dz' \, \frac{A_t[s, 2q^2(z'-1)]}{z'-z}$$

$$+ \frac{1}{\pi} \int_{-[1+(u_0/2q^2)]}^{-\infty} dz' \, \frac{A_u[s, -2q^2(z'+1)]}{(z'-z)} . \qquad (32)$$

Actually some subtractions will in general be necessary in the integrals of eq. (29), but these will not be indicated explicitly. We insert eq. (32) into eq. (1) and change the order of integration, which is valid for sufficiently large ℓ, thus obtaining (7)

$$a_\ell(s) = \frac{1}{\pi} \int_{1+(t_0/2q^2)}^{\infty} dz' \, A_t \, [s, 2q^2(z'-1)] \, Q_\ell(z')$$

$$- \frac{1}{\pi} \int_{1+(u_0/2q^2)}^{\infty} dz' \, A_u \, [s, 2q^2(z'-1)] \, Q_\ell(-z'), \qquad (33)$$

for ℓ sufficiently large.

Now for $2q^2$ positive and not too large, we can use the asymptotic form for $Q_\ell(z)$ for $z > 3/2\sqrt{2}$, $|\ell| \to \infty$, given in the Appendix. Apart from irrelevant constants, we find

$$Q_\ell(z) \to \exp\left(-\frac{1}{2} \log |\ell|\right) \frac{\exp\{(\ell+\frac{1}{2}) \log[z-(z^2-1)^{1/2}]\}}{(z^2-1)^{1/4}} . \qquad (34)$$

Hence for large ℓ the first term of eq. (33) tends to

$$\exp\left[-\frac{1}{2}\log(\ell)\right] \int_{1+(t_0/2q^2)}^{\infty} dz' \, A_t \, [s, 2q^2(z'-1)] \, \frac{\exp\left\{(\ell+\frac{1}{2})\log[z-(z^2-1)^{1/2}]\right\}}{(z^2-1)^{1/4}} .$$

No difficulty arises in using eq. (34) in the integral since the convergence is obviously uniform as $z \to \infty$ (see Appendix). Also, since $|z-(z^2-1)^{1/2}| < 1$ in the region of integration, the integral above is bounded independently of ℓ, provided Re ℓ is sufficiently large. Thus the first term of eq. (33) certainly satisfies conditions (ii) and (iii). Now the second term of eq. (33) is similar to the first with the addition of a factor $(-1)^\ell$, which has a bad behavior as $|\ell| \to \infty$, and in general will prevent condition (iii) from being satisfied. In particular, if $A_t = A_u$ (i.e., as in the scattering of identical bosons), then $a_\ell = 0$ for odd ℓ. It follows from our uniqueness proof (since the odd integers are effectively equivalent to the integers for this purpose) that such an a_ℓ can have no continuation satisfying (i), (ii), and (iii).

We therefore form two amplitudes

$$a_\ell^{\pm}(\ell) = \frac{1}{\pi} \int_{z_0}^{\infty} dz' Q_\ell(z') \, (A_t \pm A_u) \, , \tag{35}$$

both of which satisfy conditions (ii) and (iii). Also, from eqs. (33) and (35),

$$\begin{aligned} a^{+}(\ell) &= a_\ell \, , \qquad \text{for } \ell = \text{even integer,} \\ a^{-}(\ell) &= a_\ell \, , \qquad \text{for } \ell = \text{odd integer.} \end{aligned} \tag{36}$$

We refer to these two amplitudes as the even and odd "j-parity" amplitudes, respectively. Note that, for example, the even j-parity amplitude has no physical significance when ℓ is an odd integer. In eq. (35), z_0 is the minimum of $\left(1 + \frac{t_0}{2q^2}\right)$ and $\left(1 + \frac{u_0}{2q^2}\right)$.

We further define

$$A^{\pm}(z) = \sum_{\ell = +\text{ve integers}} (2\ell + 1)\, a^{\pm}(\ell)\, P_{\ell}(z), \qquad (37)$$

and perform the Sommerfeld-Watson transform on each of these amplitudes separately. Finally, then, we have—from eqs. (2), (36), and (37):

$$A(z) = \frac{1}{2}\,[A^{+}(z) + A^{+}(-z) + A^{-}(z) - A^{-}(-z)]. \qquad (38)$$

The right-hand side of eq. (35) is only meaningful for Re ℓ greater than some particular value, in which region $a^{\pm}(\ell)$ can have no poles. It is at present a conjecture that, as one moves farther to the left in the ℓ plane, poles—with positions depending on the energy—will appear, and that it will be possible to move the integral parallel to the imaginary axis farther to the left. From eq. (38), poles in the even or odd j-parity amplitudes, at $\ell = \alpha^{\pm}$, respectively, will contribute terms of the form (1)

$$\frac{\beta^{\pm}(2\alpha^{\pm} + 1)[P_{\alpha^{\pm}}(-z) \pm P_{\alpha^{\pm}}(z)]}{\sin \pi\alpha^{\pm}}.$$

The even and odd j-parity amplitudes both satisfy the generalized unitarity relation

$$a^{\pm}(\ell)\, a^{\pm}(\ell^{*})^{*} = \frac{1}{2i}\,[a^{\pm}(\ell) - a^{\pm}(\ell^{*})^{*}], \qquad (39)$$

as follows from the arguments in Section 2. In general we do not expect them to have the same poles, from which it follows that we expect the integrals $\int_{z_0}^{\infty} dz'\, Q_{\ell}(z')\, A_{t,u}$ to have poles at the same values of ℓ, and with equal residues, apart possibly from a (-1) factor.

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I am indebted to many members of the high-energy physics group at the Lawrence Radiation Laboratory for introducing me to the subject of complex angular momenta and for numerous discussions.

APPENDIX

I have not been able to locate an asymptotic expansion of $P_\nu(z)$, for large $|\nu|$, which has sufficient generality for our purposes, in any book, so I outline here the derivation of the required expression.

Apart from trivial constants the result we obtain is

$$\frac{P_\nu(\cos\theta)}{\sin\pi\nu} \sim \nu^{-1/2} \frac{\exp(|\mathrm{Im}\nu\, \mathrm{Re}\,\theta| + |\mathrm{Re}\,\nu\, \mathrm{Im}\,\theta|)}{\exp(\pi|\mathrm{Im}\,\nu|)}, \quad |\nu| \to \infty, \qquad \text{(A1)}$$

which is already given, for example, in Bottino et al. [2]. We shall see that this is valid at least for the following regions of $\cos\theta$ and ν:

(I) θ real, $0 < \theta < \pi$, $\quad \mathrm{Re}\,\nu \to \infty$.

(II) A connected region of $z = \cos\theta$, containing portions of the real axis $-\frac{\sqrt{3}}{2} < z < \frac{\sqrt{3}}{2}$, $z < -\frac{3}{2\sqrt{2}}$, $z > \frac{3}{2\sqrt{2}}$; $|\arg\,\nu| \neq 0$ or π.

We use region (i) in replacing the sum over integral values of ℓ by the contour integral, and then in distorting the contour to run along rays with $|\arg\,\nu| \neq 0$, for z physical. Then, using region (II), we distort the contour to run parallel to the imaginary axis, for z real, $-\frac{\sqrt{3}}{2} < z < \frac{\sqrt{3}}{2}$; and finally we move continuously in region II to the real axis $z < -\frac{3}{2\sqrt{2}}$, $z > \frac{3}{2\sqrt{2}}$. The necessary and sufficient properties of $a(\ell)$, given below eq. (8), then follow from eq. (A1).

To prove eq. (A1) in region I we use [8] B142, eq. (21)

$$(2\pi)^{1/2}(z^2-1)^{1/4} \frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu+1)} P_\nu(z)$$

$$= [z+(z^2-1)^{1/2}]^{\nu+\frac{1}{2}} F(\tfrac{1}{2}, \tfrac{1}{2}, \nu+\tfrac{3}{2}, \zeta_1)$$

$$+ i\,[z-(z^2-1)^{1/2}]^{\nu+1/2} F(\tfrac{1}{2}, \tfrac{1}{2}, \nu+\tfrac{3}{2}, \zeta_2^{+}), \qquad \text{(A2)}$$

with

$$\zeta_{1,2} = \frac{\pm z + (z^2 - 1)^{1/2}}{2(z^2 - 1)^{1/2}}, \qquad \text{(A3)}$$

the square roots being real and positive when z is real and >1. We obtain eq. (A1) provided that the F in eq. A2 tends to 1 when $|\nu| \to \infty$. From B76 eqs. (11) and (12), this is certainly true provided $\text{Re}\,\nu \to \infty$ and $|\arg(1 - \zeta_{1,2})| < \pi$. The latter condition rules out all of the real axis of z except $-1 < z < +1$. For region II we use first B128, eq. (26) [see also B123, eq. (9)]:

$$P_\nu(z) = (2\pi)^{-1/2} \frac{\Gamma(-\frac{1}{2} - \nu)}{\Gamma(-\nu)} \frac{[z - (z^2 - 1)^{1/2}]^{\nu + 1/2}}{(z^2 - 1)^{1/4}} F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2} - \nu, \zeta_2)$$

$$+ (2\pi)^{-1/2} \frac{\Gamma(\frac{1}{2} + \nu)}{\Gamma(1 + \nu)} \frac{[z - (z^2 - 1)^{1/2}]^{-\nu - 1/2}}{(z^2 - 1)^{1/4}} F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} - \nu, \zeta_2). \qquad \text{(A4)}$$

The advantage of this expression is that the F's contain only ζ_2. Thus from B76, eq. (10) they both tend to one as $|\nu| \to \infty$, provided $|\arg \nu| \neq 0$ or π, and $|\zeta_2| < 1$. In terms of $z = \cos\theta$,

$$\zeta_2 = \frac{i}{1 - e^{\pm 2i\theta}}, \qquad \text{(A5)}$$

where we take the upper sign if $0 < \text{Re}\,\theta < \pi/2$, $\text{Im}\,\beta < 0$, or $\pi/2 < \text{Re}\,\theta < \pi$, $\text{Im}\,\beta > 0$; otherwise we take the lower sign. In the θ plane this enables us to prove the validity of eq. (A1), except for the region shaded in Fig. 2. This region includes $-\sqrt{3}/2 < z < \sqrt{3}/2$ and $z > 3/2\sqrt{2}$. We can then enlarge the region on the right by using B128, eq. (30), which contains ζ_1 rather than ζ_2, and so effectively reflects Fig. 2 about $\text{Re}\,\theta = \pi/2$.

The expression for Q required in Section 4 is obtained from B137, eq. (44) [see also B123, eq. (10)],

$$Q_\nu(z) = (2\pi)^{-1/2} \frac{\Gamma(1+\nu)}{\Gamma(\frac{3}{2}+\nu)} \frac{[z-(z^2-1)^{1/2}]^{\nu+(1/2)}}{(z^2-1)^{1/4}} F(\frac{1}{2}, \frac{1}{2}, \nu + \frac{3}{2}, \zeta_2). \tag{A6}$$

As before, for z real and $z > \frac{3}{2\sqrt{2}}$, $|\zeta_2| < 1$ — so eq. (34) follows from eq. (A6) and B76, eq. (10). Note also that the asymptotic expression for F improves as $\zeta_2 \to 0$, i.e., as $z \to \infty$.

FOOTNOTES AND REFERENCES

(*) Work was done under the auspices of the U. S. Atomic Energy Commission.

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(2) T. Regge, Nuovo cimento 14, 951 (1959) and 18, 947 (1960); A. Bottino, A. M. Longoni, and T. Regge, University of Torino, preprint (1961).

(3) A. Sommerfeld, Partial Differential Equations in Physics (Academic Press, Inc., New York, 1954).

(4) For ℓ a positive integer, $a(\ell)$ is not singular so we can always modify the contour to satisfy this condition.

(5) See, for example, footnote 8 of R. G. Newton, J. Math. Phys. 1, 319 (1960).

(6) In obtaining eq. (20) we require $\sum_{m=0}^{p=1} a_n a_{p-1-m} \neq 0$, which will be true in general if $a_0 \neq 0$. If, on the other hand, all a_n are zero for n even, then this condition is only satisfied for p an odd integer. This occurs, for example, for the square well potential, and for the unperturbed solution with V = 0. In the latter case eq. (20) then expresses a well-known symmetry property for Bessel functions of integral argument.

(7) M. Froissart, Phys. Rev. 123, 1053 (1961).

(8) References are to page and equation number of Higher Transcendental Functions, Bateman Manuscript Project, Vol. I, (McGraw-Hill, New York, 1954).

FIGURE CAPTIONS

Fig. 1. Contour C for the integral in eq. (3).

Fig. 2. Regions in which ζ_2 satisfied $|\zeta_2| > 1$ (shaded areas).

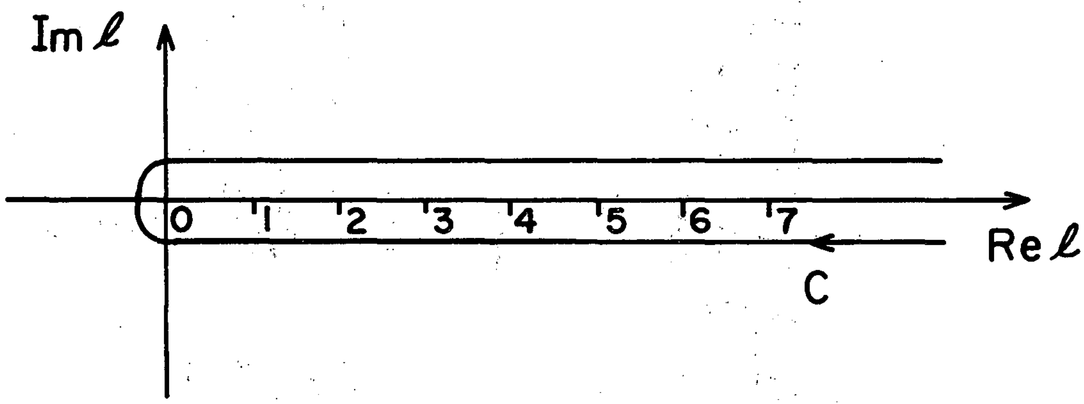

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Fig. 1

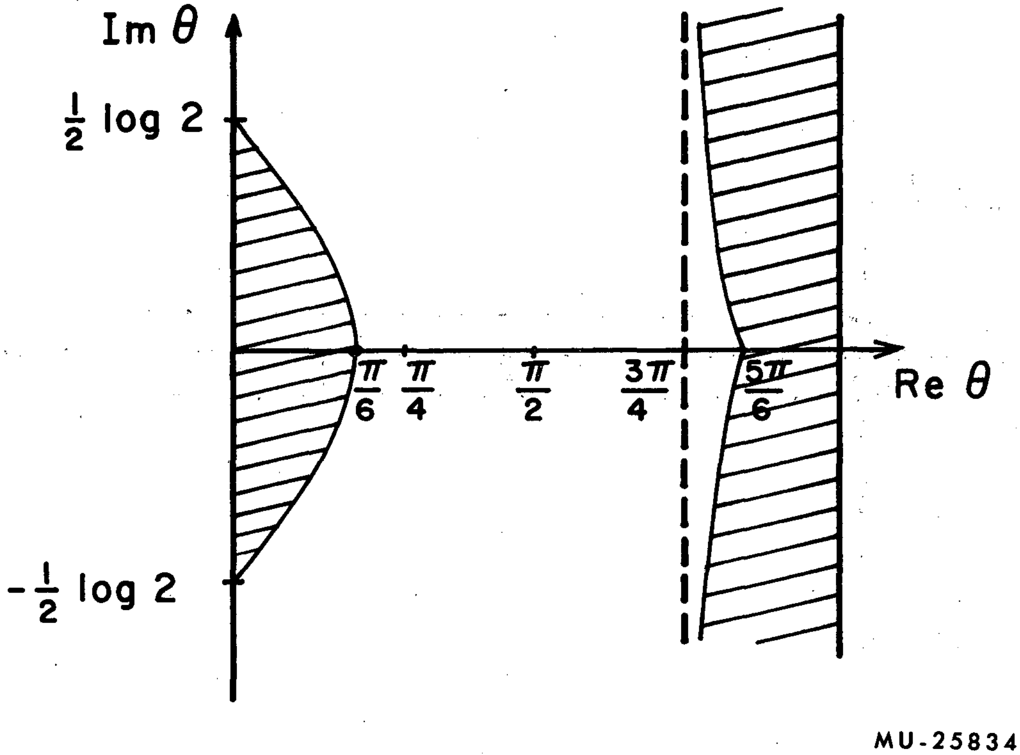

Fig. 2