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SANTA CRUZ

TOPICS IN THE STANDARD MODEL AND BEYOND

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

PHYSICS

by

Di Xu

June 2018

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Table of Contents

| | |
|---|-------------|
| List of Figures | v |
| List of Tables | vii |
| Abstract | viii |
| Acknowledgments | ix |
| 1 Introduction | 1 |
| 2 SUSYQCD | 18 |
| 2.1 Introduction | 18 |
| 2.2 Conjectured Behaviors of QCD at large N | 18 |
| 2.3 Large N Scaling of the Gaugino Condensate | 23 |
| 2.4 θ and the η' Potential in SQCD | 25 |
| 2.4.1 Supersymmetric $SU(N)$ Gauge Theory Without Matter | 25 |
| 2.4.2 $N_f \ll N$ in supersymmetric QCD: A model for the η' | 26 |
| 2.5 Phases with General $N_f < N$ | 28 |
| 2.6 Matter as a Perturbation | 29 |
| 2.7 Instantons at Large N | 32 |
| 2.7.1 Heuristic treatment of instantons: the infrared cutoff | 32 |
| 2.7.2 Scaling of Reliable Instanton Computations with N | 33 |
| 2.7.3 Further circumstantial evidence for the role of instantons | 36 |
| 2.8 Speculations on Real QCD | 36 |
| 2.8.1 Spontaneous breaking of an explicitly broken discrete symmetry | 37 |
| 2.8.2 Stability of Branches | 39 |
| 2.8.3 't Hooft's Picture of Confinement: A Candidate Setting for Branched Structure | 40 |
| 2.9 Summary | 41 |
| 3 AxionCosmology | 44 |
| 3.1 Introduction | 44 |
| 3.2 Theoretical Uncertainties on the Instanton Contribution to the Free Energy | 46 |
| 3.2.1 The Standard Computation | 46 |

| | | |
|----------|--|------------|
| 3.2.2 | UV-Sensitive Corrections, Heavy Quarks, and Parametric Uncertainties . . . | 48 |
| 3.2.3 | Corrections to the IR Cutoff on ρ and Infrared Sensitivity | 51 |
| 3.3 | $\chi(T)$ At Intermediate Temperatures | 55 |
| 3.4 | Axion Relic Density from Misalignment | 57 |
| 3.5 | Conclusions | 61 |
| 4 | DebyeMass | 63 |
| 4.1 | Introduction | 63 |
| 4.2 | g^2 Expansion of the Free Energy | 70 |
| 4.3 | Perturbative Computation of the Greens function at Large Distances and its Limitations | 71 |
| 4.4 | Defining the Debye Mass | 78 |
| 4.5 | Calculating the Debye Mass | 79 |
| 4.5.1 | Non-Relativistic Effective Theory (NRET) | 79 |
| 4.5.2 | Calculation of NRET Analysis to Conventional Perturbation Theory | 81 |
| 4.5.3 | Implications of the Debye Mass Calculation for the Topological Susceptibility | 82 |
| 4.6 | Infrared Sensitivity in the Instanton Computation | 83 |
| 4.6.1 | Instantons as a Perturbation at Large Distances | 83 |
| 4.6.2 | Subtleties at Two Loops | 86 |
| 4.7 | Conclusions | 87 |
| 5 | AnthropicTheta | 89 |
| 5.1 | Introduction | 89 |
| 5.2 | Anthropic Axions | 95 |
| 5.3 | Models Which Achieve a θ Discretuum | 96 |
| 5.3.1 | Prelude: The Irrational Axion | 97 |
| 5.3.2 | Models with a single axion | 99 |
| 5.3.3 | Models with Multiple Axions | 101 |
| 5.3.4 | A Stringy Variant | 103 |
| 5.4 | Canceling the Cosmological Constant in Different Parameter Ranges | 103 |
| 5.4.1 | Fine $\Delta\Lambda$ | 104 |
| 5.4.2 | Coarse $\Delta\Lambda$ | 105 |
| 5.5 | Conclusions | 108 |
| 6 | Conclusions | 111 |
| | Bibliography | 113 |

List of Figures

| | | |
|-----|--|----|
| 1.1 | Triangle diagram contribution to the anomaly | 2 |
| 1.2 | One loop diagram which contributes to EDM | 5 |
| 3.1 | Axion relic density from misalignment in the post-inflationary scenario. Colors correspond to different models for the temperature-dependent free energy between the dilute gas at high temperatures and chiral perturbation theory at low temperatures. Specifically, the blue (green) band sets the anchor point for ChPT at $T_2 = 100$ (500) MeV. The width of each band reflects the uncertainty in the instanton computation of the free energy used as an anchor at $T = 1.5$ GeV, $F_0 \rightarrow (1/20, 20) \times F_0$, c.f. Eqs. (3.10),(3.15). The dashed line corresponds to the value of $F_0(1.5)$ obtained in the lattice calculation of Ref. [1]. | 59 |

| | | |
|-----|---|-----|
| 3.2 | Axion relic density from misalignment in the pre-inflationary scenario. The curve shows the misalignment angle needed to obtain $\Omega = 0.258$. The band reflects the uncertainty in the instanton computation of the free energy, Eq. (3.15), used as an anchor at $T = 1.5$ GeV, and the anchor point for ChPT has been fixed to $T_2 = 140$ MeV. Left panel: log-log axes over a broad range of axion masses. Right panel: linear axes over a range of axion masses in reach of current and next generation ADMX [2]. | 60 |
| 4.1 | The leading contribution to the ϕ self-energy, $\Sigma(p)$ | 73 |
| 4.2 | One class of diagrams singular in the limit $\delta \rightarrow 0$ | 74 |
| 4.3 | A second class of diagrams singular in the limit $\delta \rightarrow 0$, with actual infrared divergences. | 75 |
| 4.4 | One class of diagrams singular in the limit $\delta \rightarrow 0$ | 75 |
| 4.5 | One class of diagrams singular in the limit $\delta \rightarrow 0$ | 76 |
| 4.6 | Infrared divergent contributions to χ at three loops. | 86 |
| 5.1 | Left: The shaded region is the portion of the region in (θ, y) space that produces an anthropically allowed value for the c.c., as determined by eqn. (5.30). The red shading is the portion of this region for which $\theta < 10^{-10}$ [not drawn to scale]. This shows that values of θ of order one are favored by the c.c. anthropic selection. Right: zoomed in to very small θ , drawn to scale. | 105 |

List of Tables

| | | |
|-----|--|----|
| 1.1 | Charges of fields under different $U(1)$ | 12 |
| 3.1 | The instanton-induced free energy in units of $-10^{-14} \text{ GeV}^{-4}$ at $\theta = 0$ and $T = 1.5$. Rows correspond to a variety of computations: (3F,4F) = three or four light flavors; (1L,2L) = one-loop complete or partial two-loop; (T, $\sqrt{\pi}$ T, π T) = renormalization scale. | 50 |
| 3.2 | Axion relic density as a function of model parameters as computed with the approxi- mate formula (3.29). Here $m_a(0) = 30 \mu\text{eV}$, χ_0 is given in units of $3.7 \cdot 10^{-14} \text{ GeV}^4$, and the misalignment angle is set to the value appropriate for post-inflationary breaking of the Peccei-Quinn symmetry, $\theta_0 = 2.16$ | 58 |

Abstract

Topics in the Standard Model and Beyond

by

Di Xu

We consider a set of questions in the Standard Model, motivated by questions about QCD. These questions take us to issues beyond the Standard Model physics, especially dark matter and axions. Even our studies of Standard Model physics in some cases involve tools developed for Beyond the Standard Model Physics.

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Chapter 1

Introduction

The Standard Model has been a great success which leads to a fantastic agreement between theoretical predictions and experimental data in particle physics. Based on quantum field theory, it unifies three of the four fundamental forces, as well as classifying all known elementary particles. Within this framework, quantum chromodynamics (QCD) sector defines the interactions between quarks and gluons, with SU(3) symmetry and three generations.

But there are still unanswered questions. For example, the weak interaction breaks CP, so it's natural to ask why QCD should respect this symmetry. There is a term allowed in QCD Lagrangian that is able to break CP-symmetry:

$$\theta_{QCD} \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^a F_{\alpha\beta}^a \quad (1.1)$$

This term is closely related to the anomaly effect under which the symmetry of a classical theory is not preserved in the quantum theory. In QCD, there is an approximate global chiral symmetry, under which left- and right-handed quarks transform with opposite charge. The Noether

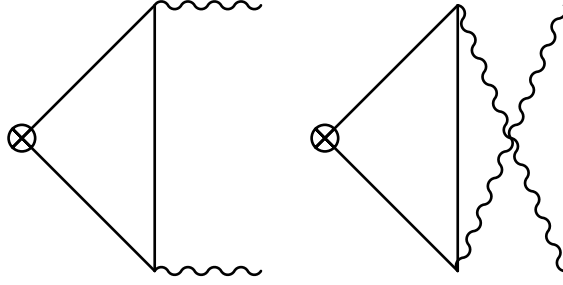


Figure 1.1: Triangle diagram contribution to the anomaly

current associated with this symmetry is

$$J^{\mu 5} = \bar{\psi} \gamma^{\mu} \gamma^5 \psi \quad (1.2)$$

But this symmetry is not conserved quantum mechanically. One can understand this by calculating the triangle diagrams which shows that this current is not conserved at loop level, and leads to anomaly equation

$$\partial_{\mu} J^{\mu 5} = \frac{g^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad (1.3)$$

Another way to understand this is through the path integral. Fujikawa first showed that anomalies arise when symmetry of the action is not respected in the functional measure in the path integral. For simplicity, start with the path integral including fermion charged under a U(1) gauge group

$$\int D\bar{\psi} D\psi D A \exp[i \int d^4x (-\frac{1}{4} F_{\mu\nu}^2 + i\bar{\psi} \not{D} \psi)] \quad (1.4)$$

Now one can do a global chiral transformation $\psi \rightarrow e^{i\alpha\gamma^5} \psi$, which not only changes the action by a boundary term, but also generate a Jacobian factor.

$$D\bar{\psi} D\psi \rightarrow |J|^{-2} D\bar{\psi} D\psi \quad (1.5)$$

Usually, $|J|^2 = 1$ and the measure is invariant. But it is not necessarily true for the chiral symmetry.

We can regulate the divergence and calculate the finite answer. Then we get that under chiral rotation

$$\int D\bar{\psi}D\psi DAexp[i \int d^4x(-\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}\not{D}\psi)] \quad (1.6)$$

$$\rightarrow \int D\bar{\psi}D\psi DAexp[i \int d^4x(-\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}\not{D}\psi - J_\mu^5\partial_\mu\alpha + \alpha\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta})] \quad (1.7)$$

This shows that a chiral symmetry rotation creates an $F\tilde{F}$ term, which is equivalent to the anomaly equation.

Now with the anomaly in QCD, one can use a chiral rotation $\psi \rightarrow e^{i\gamma_5\theta}\psi$ to remove the θ angle. This then induces phases in the mass terms $m\bar{q}qe^{i\theta}$, whose effect can be described in a basis-independent way by considering $\arg \det m$.

We can see the effects of θ in the presence of quarks using the chiral Lagrangian. The chiral Lagrangian is an effective field theory which describe QCD at an energy scale below Λ_{QCD} . In the limit that quarks are massless, QCD presents a global $U(3)_L \times U(3)_R$ symmetry. The quark condensation $\langle \bar{q}_L q_R \rangle = \Lambda_{QCD}^3$ spontaneously breaks $U(3)_L \times U(3)_R \rightarrow U(3)_{diagonal}$ and the theory hence would be expected to obtain 9 massless goldstone bosons which are three pions, four kaons, the η and η' particle. One can describe fields in a scalar fields $U(x) \equiv exp[i\frac{\pi^a\lambda^a}{F_\pi}]$ and write down the Lagrangian invariant under $SU(3) \times SU(3)$

$$L = F_\pi^2 Tr[(D_\mu U)(D^\mu U)^\dagger] + A Tr[(D_\mu U)(D^\mu U)^\dagger]^2 + \dots \quad (1.8)$$

where F_π is the pion decay constant.

In reality, quarks are massive and mesons pick up mass. We can add the mass term

$$L_M = \Lambda_{QCD}^3 Tr[M_q U + M_q^\dagger U^\dagger] \quad (1.9)$$

where M is the quark mass matrix. Now one can expand the field U in powers of the meson fields and find that

$$m_{ab}^2 F_\pi^2 = \Lambda_{QCD}^3 \text{Tr}[T^a M_q T^b] \quad (1.10)$$

From the experiments, the lightest two neutral mesons are $\pi^0(135\text{MeV})$ and $\eta(549\text{ MeV})$, while the other neutral Goldstone boson $\eta'(957\text{ MeV})$, is much heavier. The puzzle of why the η' is so heavy and its mass is not given by a similar formula as pion is called the U(1) problem. And the key to this question lies in the anomaly. There is a factor of $U(1)$ inside the chiral symmetry which is anomalous. So there shouldn't be a Goldstone boson corresponding to it.

One important consequence one can derive from current algebra is that neutron picks up an electric dipole moment proportional to $\bar{\theta}$. [6] The neutron and proton doublet couples to pion in the form of

$$\mathcal{L}_{\pi NN} = \pi^a \bar{\Psi} (i\gamma^5 g_{\pi NN} + \bar{g}_{\pi NN}) \tau^a \Psi \quad (1.11)$$

where Ψ is the neutron and proton doublet. The second term break CP and its coefficient is proportional to θ

$$g = \frac{2m_u m_d}{f_\pi (m_u + m_d) (2m_s - m_u - m_d)} (M_\Xi - M_N) \theta \approx 0.04\theta \quad (1.12)$$

The neutron electric dipole moment can be calculated from a one loop diagram with a virtual charged pion coupled to a photon. The result is UV divergent and could be cut off at m_N

$$d_N = \frac{m_N}{4\pi^2} g_{\pi NN} \bar{g}_{\pi NN} \log \frac{m_N}{m_\pi} \quad (1.13)$$

The log becomes arbitrarily large as the quark masses become arbitrarily small, so this term becomes more and more important in the chiral limit. The current bound on the neutron EDM gives that $\bar{\theta} < 10^{-10}$. The smallness of $\bar{\theta}$ is known as the strong CP problem.

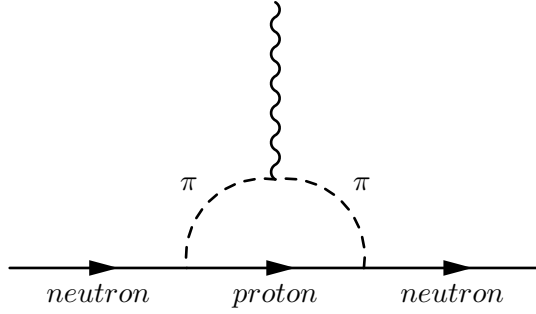


Figure 1.2: One loop diagram which contributes to EDM

One way to resolve this puzzle is to introduce a particle known as axion. The idea is to add fields to the Standard Model and require that there is a new global U(1) symmetry called the Peccei-Quinn symmetry. This is a chiral symmetry which is broken quantum mechanically by an anomaly. Furthermore, if this U(1) symmetry is also spontaneously broken, it will generate a nearly massless Goldstone boson a , with coupling to QCD

$$\mathcal{L}_{axion} = (\partial_\mu a)^2 + \frac{a/f_a + \theta}{32\pi^2} F\tilde{F} \quad (1.14)$$

where f_a is known as the axion decay constant. Now with axion, the vacuum energy is given by

$$E = F_\pi^2 m_\pi^2 \cos\left(\bar{\theta} - \frac{a(x)}{f_a}\right) \quad (1.15)$$

At this point, θ can be absorbed into a redefinition of a and the strong CP problem is solved.

It is quite hard to analyze these problems in QCD because the effective QCD coupling is not small. This has motivated consideration of alternative approaches. One practical way is to use lattice gauge theory, where one discretizes spacetime and the path integral becomes a finite

dimensional integral. This makes it possible to perform the calculation using techniques such as the Monte Carlo method.

Besides lattice gauge theory, there are analytical methods which can be applied to some problems. One way is to think about global effects in field theory. In QCD, the $F\tilde{F}$ term is equal to a boundary term in the action.

$$F\tilde{F} = \partial_\mu K^\mu = \partial_\mu \epsilon_{\mu\nu\rho\sigma} (A_\nu^a F_{\rho\sigma}^a - \frac{2}{3} f^{abc} A_\nu^a A_\rho^b A_\sigma^c) \quad (1.16)$$

So it has no influence in perturbation theory. But one could expect non-perturbative configurations to have non-trivial effects. And actually they do exist, which corresponds to topologically non-trivial gauge configurations which are called instantons. For these configurations, A 's only fall off as $1/r$, so one cannot neglect the surface term. Instantons are solutions to the Euclidan Equation of Motion which carry topological charges and saturate the Bogomolnyi bound $F = \pm\tilde{F}$. In the path integral, these are the leading terms which generate θ -dependence. Now the functional integral, in the presence of a θ term and instanton background, looks like

$$Z_{inst} = e^{-\frac{8\pi^2}{g^2} + i\theta} \int [DA][Dq][D\bar{q}] \exp\left(\frac{\delta^2 S}{\delta\Phi^2} \delta\Phi^2\right) \quad (1.17)$$

where Φ corresponds to all the fields in the theory, $\delta\Phi$ is the quantum fluctuation around the classical instanton field.

To do this integral, we want to expand the quantum fluctuations in their eigenmodes. Both the bosonic and fermionic fluctuation operators have zero modes which must be treated carefully. For bosonic modes, the zero modes reflect different symmetries of the theory, and the parameters we integrate over are called collective coordinates. For example, the translational invariance requires integrating over the location of the instanton, and same thing happens to rotational invariance. Besides that, classical QCD is scale invariant, so one needs to integrate over all possible sizes of the

instanton ρ , and this integral generally suffers from an infrared divergence and needs to be cut off at Λ_{QCD} . But this implies that the instanton calculation is not reliable at low energy which is not a surprise since at that regime, theory is no more weak coupled and semiclassical methods breaks down.

For QCD with massless fermions, one also has fermion zero modes. For these modes, expanding,

$$q(x) = \sum a_n q_n(x), \quad S = \sum \lambda_n a_n^* a_n \quad (1.18)$$

The fermion part of functional integral now becomes

$$\int [Dq][D\bar{q}] e^{-S} = \prod_n da_n da_n^* \exp(-\sum_{n \neq 0} \lambda_n a_n^* a_n). \quad (1.19)$$

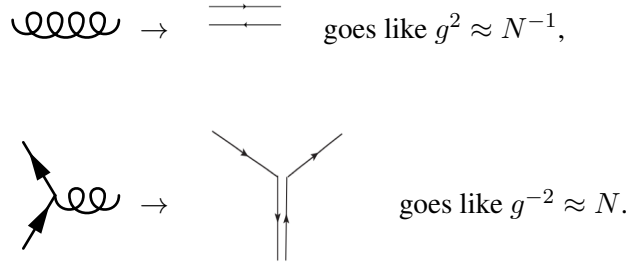
Because the integral over grassman number gives zero, one needs to insert fermionic operators to compensate the intergral over zero modes. So many Green's functions vanish except for ones with certain number of fermion operators. In the presence of small quark masses, one can use the instanton to calculate the vacuum energy. The trick is to use the quark mass term as a perturbation. Expanding this term will provide a different power of the quark field operator and will give non-zero answer.

Instantons are phenomenon in the theory which indicate that QCD has θ dependence and the anomaly leads to symmetry breaking. But the problem is that there is no small parameter which permits a reliable instanton computataion. 't Hooft suggested a strategy to introduce a small parameter into QCD. He pointed out [7]that one can use the number of colors N to act as a free parameter. With an appropriate defination of "large N limit", $N \rightarrow \infty$ and $g^2 N = \lambda$ is fixed , the perturbation theory simplifies (only so-called planar diagrams survive, which is a subset diagrams of the full set) and presents nice features.

To understand the large N limit further, let's start with the counting of powers of N in Feynman diagrams. First, write down the lagrangian of a non-abelian gauge theory with fermions.

$$L = -\frac{1}{2g^2}Tr[F_{\mu\nu}^2] + \bar{\psi}(i\not{D} + A^a T^a - m)\psi, \quad (1.20)$$

where $F_{\mu\nu} = F_{\mu\nu}^a T^a$. One can read off the feynman rules and determine how the gauge coupling appears in propagators. The vertices and propagators have non-trivial g^2 dependence.



Besides, Feynman diagrams will have factors of N resulting from internal color indices.

The fermion propagator reads

$$\langle \psi^i(x) \bar{\psi}^j \rangle = \delta^{ij} S(x - y) \quad (1.21)$$

For gluon propagator

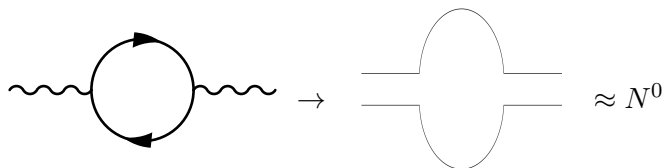
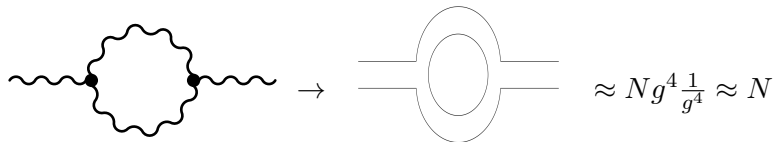
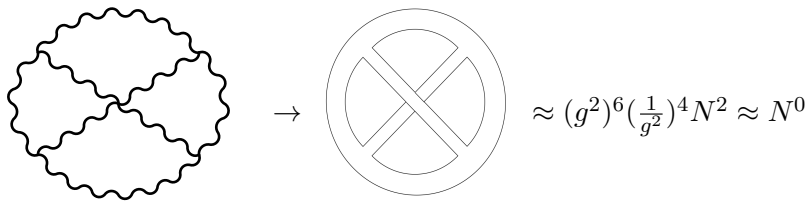
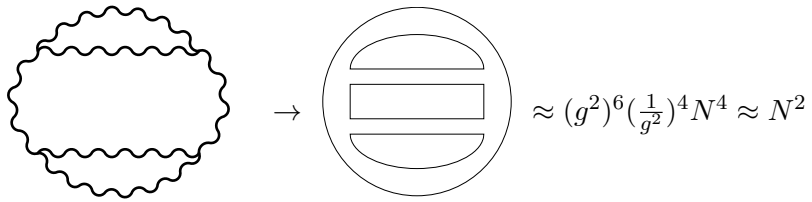
$$\langle A_\mu A_\nu \rangle = (T^a)_j^i (T^b)_k^l D_{\mu\nu}(x - y) \quad (1.22)$$

$$= \left(\frac{1}{2} \delta_j^i \delta_k^l - \frac{1}{2N} \delta_k^i \delta_j^l \right) D_{\mu\nu}(x - y) \quad (1.23)$$

And we can neglect second term in the large N limit. One can see for fermion propagator, one index flows in one direction for a particle, the opposite for an antiparticle, while for the gluon propagator one has two flowing in opposite directions. These can be represented by t Hoofts double line notation as showed in the figure. Every Feynman graph in the original theory can then be written as a sum of graphs with single and double lines. Each double line graph gives a particular

color index contraction of the original diagram. Now we can count powers of N by counting the number of internal color loops in double line graphs and then add the power of gauge couplings from vertex and propagators.

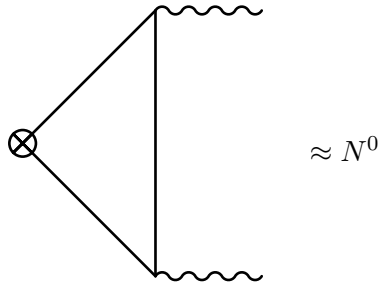
Here are some examples.



These examples illustrate two general rules in large N expansion:

- non-planar diagrams (in double line notation, these are diagrams that could not be drawn without line crossings) are suppressed by factor of $\frac{1}{N^2}$
- diagrams with internal quark loop are suppressed by factor of $\frac{1}{N}$

In the 1970's Witten [8] suggested that in the large N limit, the chiral anomaly could be treated as a perturbation, and the η' is a Goldstone boson in this limit. One could understand this by counting power of N in triangle diagram



which is one power of N lower than $\frac{1}{g^2} F_{\mu\nu} F^{\mu\nu}$.

In particular, a Green's function with n insertions of $F\tilde{F}$ behaves as

$$\left\langle \left(\int F\tilde{F} \right)^n \right\rangle \sim N^{-n+2} . \quad (1.24)$$

which now could be easily understood by counting powers of N. Now to understand the mass of η' , start from

$$\partial_\mu j_5^\mu = m_{\eta'}^2 f_\pi \eta' . \quad (1.25)$$

The divergence of the current can be written in terms of $F\tilde{F}$ using the anomaly equation. So the mass of the η' is proportional to the two point correlation function at zero momentum of $F\tilde{F}$ and $1/f_\pi^2$. Now since f_π^2 is order of N, one can see that the mass of η' goes to zero in large N limit.

Furthermore, the vacuum energy is order of N^2 and could be generally written as

$$E(\theta) = N^2 h(\theta/N) \quad (1.26)$$

The requirement that physics is periodic in θ and the formula above means that $E(\theta)$ is not smooth. This suggests that the pure gauge theory might exhibit a “branched” behavior.

Real QCD is difficult to study, but there are a variety of known theories that are similar but more tractable than QCD, including supersymmetric QCD (SQCD). Supersymmetry has long been an attractive idea for physics beyond the Standard Model. It could explain the hierarchy problem (the huge gap between weak scale and planck scale), gauge coupling unification and provide natural dark matter candidates. With such appealing properties, it is widely believed that at some scale below planck scale, supersymmetry should be a good approximation, and physics should be described by a supersymmetric field theory. Thus, we have the reasons to explore supersymmetric Yang-Mills theory.

Supersymmetric QCD includes $SU(N_c)$ gauge group with N_f flavors of quark superfields in the fundamental representation of the gauge group and same number of superfields in the anti-fundamental representation of the gauge group. The lagrangian reads

$$\mathcal{L} = \int d^4\theta (Q_i^\dagger e^V Q_i + \bar{Q}_i e^V \bar{Q}_i^\dagger) - \frac{i}{16\pi} \int d^2\theta \tau W^{\alpha a} W_\alpha^a + h.c \quad (1.27)$$

with $i = 1, \dots, N_f$ and no superpotential. τ is the combination of the gauge coupling and the θ parameter which could be explained by some spurion with some non-zero vacuum expectation value

$$\langle \tau \rangle = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2} \quad (1.28)$$

The theory has two anomalous global symmetries, the R symmetry and $U(1)$ flavor symmetry. One can combine them to form an anomaly-free symmetry $U(1)_{AF}$. Then the full global symmetry of the theory is

$$SU(N_f) \times SU(N_f) \times U_B(1) \times U_{AF}(1) \quad (1.29)$$

| | $U(1)_B$ | $U(1)_A$ | $U(1)_R$ | $U(1)_{AF}$ |
|--------------------|----------|----------|----------|-------------------------|
| Q_i | +1 | +1 | 0 | $\frac{N_f - N_c}{N_f}$ |
| ψ_{Q_i} | +1 | +1 | 0 | $-\frac{N_c}{N_f}$ |
| \bar{Q}_i | -1 | +1 | -1 | $\frac{N_f - N_c}{N_f}$ |
| $\bar{\psi}_{Q_i}$ | -1 | +1 | -1 | $-\frac{N_c}{N_f}$ |
| λ | 0 | 0 | +1 | +1 |

Table 1.1: Charges of fields under different $U(1)$

where the first $U(1)_B$ corresponds to baryon number conservation. The charges of fields under different $U(1)$ are given in the table, where Q_i and ψ_{Q_i} are the scalar and fermion component of the quark superfields, λ is the gaugino.

In SQCD, the strongly interacting gaugino could undergo pair condensation like quarks in QCD. Notice that $\langle \lambda^{\alpha a} \lambda_{\alpha}^a \rangle$ is also the scalar component of the superfield $W^{\alpha a} W_{\alpha}^a$. So one can calculate $\langle \lambda^{\alpha a} \lambda_{\alpha}^a \rangle$ by taking the derivative of the path integral with respect to F term in τ .

$$\langle \lambda^{\alpha a} \lambda_{\alpha}^a \rangle = 16\pi \frac{\partial}{\partial F_{\tau}} \log Z = 16\pi i \frac{\partial}{\partial \tau} W_{eff} \quad (1.30)$$

where W_{eff} is the effective superpotential after integrating out the gauge field.

In Chapter 2, we study the large N θ dependence and the η' potential in supersymmetric QCD where SUSY-breaking terms act as a probe of the non-SUSY limit. We find the presence of branched behavior, in quite good agreement with the conjectured large N behavior of QCD, but we also found that in certain range of parameters, instanton effects are calculable and don't fall off exponentially with N.

Now to switch the topic back to axion physics. One appealing property of the axion is

that it remains a promising candidate for the dark matter. In the early universe, the axion field is unlikely to start out at the minimum of its potential and evolves according to its equation of motion

$$\ddot{a} + 3H\dot{a} + m_a(T)^2 f_a \sin \frac{a}{f_a} = 0 \quad (1.31)$$

The axion field begins to oscillate coherently when its thermal mass, $m_a(T)$, becomes comparable to the Hubble scale. The axion mass is related to the QCD topological susceptibility by

$$m_a^2(T) f_a^2 = \chi(T), \quad \chi = \partial_\theta^2 F(\theta, T) \quad (1.32)$$

where, in the last expression, $F(\theta, T)$ is the θ -dependent free energy. So to estimate the relic density of axion, one need to know how to calculate the free energy at finite temperature.

Field theory at finite temperature shows interesting features. Here, one is interested in the partition function.

$$Z = \text{Tr} e^{-\beta H} = \int D\phi \langle \phi | e^{-\beta H} | \phi \rangle \quad (1.33)$$

where H is the hamiltonian of the theory, and $\beta = \frac{1}{kT}$. One can define an imaginary time $\beta = i\tau$.

Then the partition function becomes equivalent to a path integral of quantum mechanics in Euclidean spacetime, with a periodic boundary condition $x(\tau = 0) = x(\tau = -i\beta)$, so one comes to:

$$Z[\beta] = N \int_{\phi(\beta)=\phi(0)} D\phi e^{\int_0^\beta d\tau \int d^3x L} \quad (1.34)$$

Because of the periodic boundary condition, when doing a Fourier transformation of the fields, instead of an integral over frequencies, one sums over discrete frequencies which are multiples of T ,

$$\phi = N \sum_n \int d^3p e^{i(p \cdot x + \omega_n \tau)} \phi_n(p) \quad (1.35)$$

where $\omega_n = 2\pi nT$. The path integral for a free field now reads

$$Z = \int \prod d\phi_n d\phi_n^* \exp\left(-\sum_n \frac{1}{2}(\omega_n^2 + \omega^2) |\phi_n|^2\right) \quad (1.36)$$

After performing the Gaussian integrals, we have the results for Z

$$Z = \prod \frac{1}{\omega^2 + \omega_n^2} \quad (1.37)$$

Now we have learned that the finite temperature field theory is equivalent to a Euclidean theory with one compact dimension, And one can expand fields into Fourier modes.

$$\phi(x, \tau) = \sum_n \phi_n(x) \exp(i\omega_n^b \tau) \quad (1.38)$$

$$\psi(x, \tau) = \sum_n \psi_n(x) \exp(i\omega_n^f \tau) \quad (1.39)$$

where $\omega_n^b = 2n\pi T$ for bosons and $\omega_n^f = (2n + 1)\pi T$ for fermions. Now the time direction integral could be performed and one gets an effective field theory in three dimension with infinite number of degrees of freedom. At distance larger than β , the heavy modes could be integrated out and only zero modes remain.

$$Z \rightarrow \int [DA] \exp\left(-\frac{1}{g^2 T} \int d^3x \mathcal{L}_{eff}\right) \quad (1.40)$$

We can use this to study the free energy which is now a Euclidean problem with discrete energies. The gauge coupling should now be evaluated at temperature T . At high temperature, there is a phase transition to a phase with quarks and gluons with a weak coupling and it is believed that theory becomes perturbative again. The free energy should now be a power series in $g(T)$. In this regime, semiclassical approaches like instanton becomes valid. It is believed that at some high temperature T_c , QCD undergoes a phase transition and become deconfined. Above this scale, the degrees of freedom are quarks and gluons, with interactions controlled by $g^2(T)$ and the theory exhibits a mass scale called Debye Mass. One can use perturbative and semiclassical methods to study the theory, and particularly to determine the θ dependence. The leading contribution at high

temperature is given by instanton, behaving as

$$F(\theta) = \prod_f m_f \Lambda^{b_0} T^{-b_0+n_f-4} \cos \theta \quad (1.41)$$

In the case of axion, one can just replace θ with a/f_a . At temperature below Λ , QCD becomes confined, and theory is described by the Chiral Perturbation Theory and one can use the potential $m_\pi^2 f_\pi^2 \cos \theta$ as zero temperature limit.

In recent years, several lattice papers have reported calculations of the topological susceptibility. In some cases, the free energies differ by about an order of magnitude from the semiclassical result. In Chapter 3, we explore instanton calculations on axion relic density at temperatures of order a few GeV, and estimate the sensitivity to QCD uncertainties. At finite temperature, it has been argued that the Debye mass provides the infrared cutoff on the instanton scale size integration, so there is the potential for large corrections. This claim has been used to argue that lattice computations are essential to determine the behavior of hypothetical axions in the early universe. In Chapter 4, We show that there is a reliable, infrared finite computation through one loop of this length, and moreover, analyze the problem at higher loop order. We then use this to provide a numerical estimate on the uncertainties in the semiclassical computation.

In particle physics, there are many other questions remained unanswered. One of the greatest problems is the smallness of Cosmological Constant. In Einsteins' equation, one could principally add a cosmological constant term which corresponds to a term in T_{00} arising from the vacuum energy.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} + \Lambda g_{\mu\nu} \quad (1.42)$$

From quantum mechanics, the vacuum should have quantum fluctuations which add into this term. This vacuum energy can easily be calculated in quantum field theory and generally turns out to be

quartically divergent. It is given by the form

$$\sum_i (-1)^F \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m_i^2} \quad (1.43)$$

where the sum is over all particles and $(-1)^F$ is 1 for bosons and -1 for fermions. A natural cutoff would be the Planck scale at 10^{19} GeV, which gives a result of order $\Lambda \approx 10^{76} GeV^4$. While we know from facts of our universe that the cosmological constant must be very small, $\Lambda \approx 10^{-47} GeV^4$, which is roughly 120 orders of magnitude smaller than theoretical prediction. One could cancel this with a counterterm, but this counterterm would have to be remarkably fine tuned.

S. Weinberg proposed what is presently the only viable theoretical solution to this problem which involves the anthropic principle in an essential way. There are different versions of the anthropic principle. Weinberg made use of the so-called weak anthropic principle which says the law of physics should be restricted to allow the existence of observers. He imagined that the laws of nature might allow a huge number of possible ground states each with a different value of Λ and then [9] pointed out that for a large Λ , the universe enters an exponentially expanding de Sitter phase. This will interfere with the formation of gravitationally bound states. Therefore the anthropic principle provides a prediction that Λ must be small enough to allow structure formation. For a simple estimate on the order of magnitude, quasars exist with red shifts up to about $z = 4.4$, when gravitational condensation had already begun. We want the cosmological constant to be small compared to the mass density at that point so that it didn't influence the structure formation. So $\Lambda < (1+z)^3 \rho_0 \approx 10^2 \rho_0$, where ρ_0 is the mass density at present. This gives a rough estimate of Λ which is few orders of magnitude larger than the observed value. In some more refined arguments, these two numbers come even closer.

But if the anthropic principle really does play an important role in deciding the laws of

physics, then the parameters of the Standard Model should be either determined by anthropic considerations or should be random numbers. Among these is the θ parameters. It seems unlikely to be determined anthropically, so its extremely small value is a mystery. Kaloper and Terning [10] suggest that the value of θ is correlated with the value of the cosmological constant through its contribution to the QCD vacuum energy. In Chapter 5, we will study their proposal and ask where in the parameter space, the smallness of θ could be explained by anthropic principle.

Chapter 2

SUSYQCD

2.1 Introduction

Following the discussion in Chapter 1, we will study the large N θ dependence and η' potential in supersymmetric QCD and see if this model could reproduce the brached structure and other properties proposed in the large N story.

2.2 Conjectured Behaviors of QCD at large N

In [11], Witten suggested that instantons fail to provide even a qualitative picture of the θ dependence of QCD and the solution of the $U(1)$ problem. Instead, he advanced strong arguments that the large N approximation was a much more useful tool. Particularly remarkable was his observation that in large N , the anomaly can be treated as a perturbation and the η' understood as a pseudogoldstone boson.

The large N picture for the physics of θ and the η' rests on the assumption that correlation

functions of $F\tilde{F}$ at zero momentum behave with N as similar correlation functions *at non-zero* momentum in perturbation theory. In particular, a Green's function with n insertions of $F\tilde{F}$ behaves as

$$\langle \left(\int F\tilde{F} \right)^n \rangle \sim N^{-n+2} . \quad (2.1)$$

With this assumption, and the requirement of 2π periodicity in θ , the vacuum energy must behave, to leading order in $1/N$, as

$$E(\theta) = \min_k c(\theta + 2\pi k)^2 \quad (2.2)$$

The minimization over k reflects a branched structure in the theory, and ensures that θ is a periodic variable [11, 12, 13]. The branches are characterized by a constant background topological charge density,

$$\langle F\tilde{F} \rangle_k \propto (\theta + 2\pi k) , \quad (2.3)$$

and are smoothly traversed under $\theta \rightarrow \theta + 2\pi$. A dual description of the branches in a higher dimensional gravity theory was analyzed in [13].

With $N_f \ll N$, the fermions are expected to be a small perturbation of the large N pure gauge theory. In particular, the axial anomaly can be treated as a perturbation [14, 12]. The mass of the η' is an $\mathcal{O}(\frac{1}{N})$ effect, and a pseudo-Goldstone boson, the η' , should be included in chiral perturbation theory in order to nonlinearly realize the approximate axial symmetry. To leading order in $1/N$ and in the chiral limit, its potential is obtained by the replacement

$$\theta \rightarrow \theta + \frac{N_f \eta'}{f_\pi} \quad (2.4)$$

in the vacuum energy. This form is fixed by the axial anomaly.¹ Including the branch label,

$$V_k(\eta') = c \cdot \Lambda^4 \left(\theta + 2\pi k + \frac{N_f \eta'}{f_\pi} \right)^2. \quad (2.5)$$

Taking $N_f = 1$ as an example, under $\eta' \rightarrow \eta' + 2\pi f_\pi$, the state passes from one branch to another.

Because $f_\pi^2 \propto N$, the η' (mass)² is a $1/N$ effect. Higher order interactions of the η' are suppressed by powers of N , behaving as

$$V_n \sim \Lambda^4 N^2 \left(\frac{\eta'}{N f_\pi} \right)^n. \quad (2.6)$$

In other words, the η' a true Goldstone boson in the large N limit, in the sense that its interactions vanish rapidly as $N \rightarrow \infty$.

Note that θ can be absorbed into the η' . With at least one massless quark, and ignoring terms in the chiral lagrangian associated with high scale (weak or above) physics, the η' potential has a minimum at the CP conserving point.

These expressions for θ dependence and the η' potential are in stark contrast with qualitative expectations from instantons, assumed to be cut off in the infrared in some manner. In this case, one would expect a convergent Fourier series, for example, for $E(\theta)$ in the pure gauge theory:

$$E(\theta) = \Lambda^4 \sum_q c_q \cos(q\theta). \quad (2.7)$$

Correlators of n insertions of $F\tilde{F}$ at zero momentum would scale with N in a manner independent of n , i.e. the extra powers of $1/N$ expected from perturbation theory counting at non-zero momentum would be absent. Likewise this picture makes a distinctive physical prediction for the couplings of

¹Here η'/f_π is normalized as an ordinary angle, valued on $[0, 2\pi)$. In chiral perturbation theory, it is included at leading order in large N by the substitution $\Sigma \rightarrow \Sigma e^{i\eta'/f_\pi}$, where Σ are the $SU(3)$ σ -model fields. The axial symmetry can be realized as $\eta' \rightarrow \eta' + \beta f_\pi$, and the anomaly coefficient is N_f , constraining the potential to have the form (2.4). A different periodicity and anomaly are obtained if the η' is instead introduced with canonically normalized kinetic term.

the η' : the extra powers of $1/N$ in equation 2.6 should be absent. We refer to behavior of the type of Eq. (2.2) as “monodromy” or “branched” behavior, while that of Eq. (2.7) as “instanton” behavior.

Lattice gauge theory is the only framework available in which the conjectured θ dependence of large N QCD can be tested. However, such questions are technically extremely challenging. Some recent progress in testing Eq. (2.1) was recently reported in [15], but concrete tests of the predicted cuspy behavior near $\theta = \pi$, or the existence, lifetime, and other properties of the tower of k branches, remains elusive.²

On the other hand, there are a variety of known theories that are similar but more tractable than QCD, including supersymmetric QCD (SQCD), deformed Yang-Mills, and QCD at large ‘t Hooft coupling, in which the θ dependence and existence of branches can be studied analytically [17, 18, 19, 13, 20, 21]. While differing in the details, these theories largely reflect the behaviors in Eqs. (2.2,2.3).

The case of SQCD will be analyzed in detail in this work. More generally, progress in the understanding of the dynamics of strongly coupled supersymmetric gauge theories [22, 23, 24, 25] led to new studies of ordinary QCD, considering it as a limit of *Softly Broken Supersymmetric QCD* (SBQCD), or SUSY QCD with N_f vectorlike flavors and soft SUSY-breaking masses [26, 27, 17, 28, 29]. We will study aspects of θ dependence in large N SBQCD, including the existence of branches, N scalings, the physics of the η' , the role of instantons, and the sense in which adding matter can be thought of as a perturbation. In Secs. 2.3 and 2.4, we observe a number of properties consistent with the large N conjectures for ordinary QCD, including, as noted previously in [17, 27], branched behavior (associated with the gaugino condensate in the SUSY limit, as well as $F\tilde{F}$ in the presence of soft breakings), and, as noted in [26, 29], a supersymmetric version of the η' with mass

²See discussion in [16].

of order $1/N$ in certain regions of parameter space.

We also make several new observations. The behavior of ordinary QCD is different if the quark mass dominates over the effects of the $U(1)_A$ anomaly and when the anomaly dominates. In the former case, there are N branches, while in the latter limit there are N_f branches. Phase transitions are expected in passing between these regimes. In Sec. 2.5 we show that the same phenomenon arises in SBQCD and we exhibit the phase structure. In Sec. 2.6 we demonstrate that small changes in the number of flavors $\Delta N_f \ll N$ leads to small changes in the physics of different vacua at large N : this provides a concrete realization of “matter as a perturbation.”

Finally, we point out two ways in which the properties of SBQCD *differ* from the conjectured properties of QCD. First, in Sec. 2.7 we return to the fate of instantons in large N : the conjectured exponential suppression of instanton effects in QCD is critical to the large N scaling properties described above. A simple heuristic argument suggests that if IR divergences associated with QCD instantons are cut off at a scale of order Λ_{QCD}^{-1} , there is no exponential suppression. As a counterargument, Ref. [11] emphasized that because of the extreme nature of the power law divergences, the result is extremely sensitive to how the cutoff is chosen, and the notion that such a cutoff computation makes sense, even at a qualitative level, is hard to support. But in SQCD with $N_f = N - 1$, where a systematic instanton computation of holomorphic quantities is possible, we show that the results are not suppressed by e^{-N} , and that the gauge boson mass acts as an infrared cutoff approaching Λ at precisely the required rate. On the other hand, the N -scalings are, in fact, exactly as predicted by perturbative arguments, and the θ -dependence reflects the branched structure! We provide other evidence, in less controlled situations, that a notion of cut-off instantons may survive in supersymmetric theories in large N .

Secondly, in Sec. 2.8, we comment on the role of discrete symmetries. Unlike QCD, branched structure in SBQCD is associated with an approximate Z_N symmetry, and a corresponding set of N quasi-degenerate, metastable ground states. What happens to these states in the limit of large soft breakings, where the discrete symmetry is lost and QCD is recovered? A priori, one possibility is that these states, and the associated branch structure, disappears. The possibility of phase transitions as parameters are varied is already realized in supersymmetric QCD in the controlled approximation of small soft breakings. Against this possibility is the usual large N scaling of perturbative correlation functions, suggesting that the branches should remain. As we briefly review, a possible microscopic realization of the branches in real QCD is provided by 't Hooft's proposal of *oblique confinement* [30] (particularly as realized in deformed $N = 2$ theories).³ On the other hand, the fact that instantons are not suppressed as e^{-N} in controlled situations raises questions about these arguments. Whether the states disappear or survive cannot be conclusively established without non-perturbative computations.

In Sec. 4.7, we summarize and conclude. We argue that while the traditional large N branched picture of [11, 14, 12, 13] remains likely, only lattice calculations can ultimately settle the issues.

2.3 Large N Scaling of the Gaugino Condensate

Much is understood about the dynamics of supersymmetric gauge theories. For a pure supersymmetric gauge theory, for example, the value of the gaugino condensate is known, from arguments which resemble neither perturbation theory nor a straightforward instanton computa-

³We thank Ed Witten and Davide Gaiotto for stressing this possibility to us.

tion [31, 32, 22, 33, 24, 34]. It is interesting that, as we now show, the N dependence agrees with that expected from the usual diagrammatic counting.

Let us recall the Coleman-Witten argument [35] for the N -scaling of the chiral condensate in QCD and apply it to supersymmetric QCD. By ordinary N counting, an effective potential for $\mathcal{M} = \langle \bar{\psi}\psi \rangle$ (with $\psi, \bar{\psi}$ two-component fermions) would take the form

$$V(\mathcal{M}) = Nf \left(\frac{\mathcal{M}^\dagger \mathcal{M}}{N^2 \Lambda_{QCD}^6} \right), \quad (2.8)$$

in the fermion normalization where $1/g^2$ sits in front of the whole action. Thus $\mathcal{M} \propto N\Lambda^3$. For supersymmetric gauge theories, the corresponding analysis for the $\langle \lambda\lambda \rangle$ effective potential gives

$$V(\langle \lambda\lambda \rangle) = N^2 f \left(\frac{\langle \lambda\lambda \rangle \langle \lambda\lambda \rangle^*}{N^2 \Lambda^6} \right) \quad (2.9)$$

again in the gaugino normalization where $1/g^2$ sits in front of the whole action. So, we expect $\langle \lambda\lambda \rangle = N\Lambda^3$.

The exact result in pure gauge theory is

$$\langle \lambda\lambda \rangle = 32\pi^2 \Lambda_{hol}^3 e^{\frac{2\pi i k}{N}}. \quad (2.10)$$

(For a review, see [36].) Here Λ_{hol} is the holomorphic Λ parameter, proportional to $e^{\frac{i\theta}{3N}}$. In general, as discussed in [37], the holomorphic Λ parameter differs from the more conventional Λ parameter, as defined in [38], by an N -dependent factor:

$$\Lambda_{hol} = \Lambda \left(\frac{b_0}{16\pi^2} \right)^{b_1/b_0^2}. \quad (2.11)$$

We review this connection in Appendix A. Eq. (2.11) reflects the fact that Λ is fixed as $N \rightarrow \infty$ with $g^2 N$ fixed, while $\Lambda_{hol}^3 \propto N\Lambda^3$. It is striking that the N scaling of $\langle \lambda\lambda \rangle$ agrees with the diagrammatic expectation, although the physics leading to the exact computation appears quite different.

2.4 θ and the η' Potential in SQCD

In this section, we will see that with small soft breakings, both without matter and with $N_f \ll N$, supersymmetric theories exhibit precisely the branched behavior anticipated by Witten, with the branches being associated with the breaking of an approximate discrete symmetry.

2.4.1 Supersymmetric $SU(N)$ Gauge Theory Without Matter

For vanishing gaugino mass, the gaugino condensate is given by Eq. (2.10). In the presence of a small holomorphic soft-breaking mass, m_λ , the vacuum energy is

$$V(\theta, k) \simeq m_\lambda |\Lambda_{hol}|^3 \cos\left(\frac{\theta + 2\pi k}{N}\right). \quad (2.12)$$

In terms of physical quantities,

$$m_\lambda \Lambda_{hol}^3 = N^2 m_{phys} \Lambda^3, \quad (2.13)$$

where $m_{phys} = g^2 m_\lambda$. Therefore, for very large N with θ and k fixed

$$V(\theta, k) \simeq N^2 m_{phys} |\Lambda|^3 \left(\frac{\theta + 2\pi k}{N}\right)^2. \quad (2.14)$$

This is compatible with the N -scaling and θ dependence of [13].

For small m_λ , the separate branches are long-lived. As m_λ increases, approaching real QCD, the fate of the branches is not clear; we will comment on this further in Sec. 2.8.

2.4.2 $N_f \ll N$ in supersymmetric QCD: A model for the η'

Supersymmetric QCD with $N_f < N$ flavors possesses an $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ symmetry. Dynamically, a non-perturbative superpotential is generated [22],

$$W_{np} = (N - N_f) \frac{\Lambda_{hol}^{\frac{3N-N_f}{N-N_f}}}{(\det \bar{Q}Q)^{\frac{1}{N-N_f}}}. \quad (2.15)$$

Including supersymmetric mass terms for the quarks, the system has N supersymmetric vacua.

Turning on general soft breakings gives a set of theories which, in certain limits, should reduce to $SU(N)$ QCD with N_f flavors of fermionic quarks. For small values of the supersymmetric mass terms and the soft breaking terms, the system can be studied in a systematic perturbative/semiclassical approximation [26, 29]. Consider first adding only soft squark and gaugino masses:

$$\delta V = \tilde{m}^2 \sum_f (|Q_f|^2 + |\bar{Q}_f|^2) + m_\lambda \lambda \lambda. \quad (2.16)$$

With universal soft scalar mass terms, the first terms respect the full $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ symmetry of the supersymmetric theory. The gaugino mass term breaks the $U(1)_R$.

Ignoring the gaugino mass, the potential

$$V = \sum_f \left(\left| \frac{\partial W}{\partial Q_f} \right|^2 + \left| \frac{\partial W}{\partial \bar{Q}_f} \right|^2 \right) + \delta V \quad (2.17)$$

(along with the $\sum (D^a)^2$ terms) yields a minimum at

$$Q_f^a = v \delta_f^a \quad \bar{Q}_f^a = Q_{f'}^a U_{f'f}, \quad (2.18)$$

where U is a unitary matrix describing the Goldstone fields. If $U = 1$, the symmetry is broken to the diagonal subgroup. v is given, in the large N limit, by:

$$v = \Lambda_{hol} \left(\frac{\Lambda_{hol}^2}{\tilde{m}^2} \right)^{1/4}. \quad (2.19)$$

(If we take $\tilde{m}^2 \sim \Lambda^2$, and recall that $\Lambda_{hol}^3 \sim N\Lambda^3$, then $v = f_{\eta'} \sim \sqrt{N}$, as expected by standard large N arguments. The same result is obtained if the moduli are stabilized by a small quark mass, $v^2 \sim \Lambda_{hol}^3/m \Rightarrow v \sim \sqrt{N}$.)

The gaugino bilinear $\lambda\lambda$ has an expectation value in this theory, which is essentially the derivative with respect to τ of the expectation value of the non-perturbative superpotential [24, 36],

$$\langle \lambda\lambda \rangle = 32\pi^2 \left\langle \frac{\Lambda_{hol}^{\frac{3N-N_f}{N-N_f}}}{(\det \bar{Q}Q)^{\frac{1}{N-N_f}}} \right\rangle. \quad (2.20)$$

To leading order, the expectation value is obtained simply using the value of v in Eq. (2.19). For large N , the condensate behaves as

$$\langle \lambda\lambda \rangle = \Lambda_{hol}^3 e^{\frac{2\pi ik}{N} + i \arg \det U^{1/N}}, \quad (2.21)$$

where U is the unitary matrix in Eq. (2.18).

Now consider turning on a small m_λ . The gaugino mass breaks the classical, anomalous $U(1)_R$ as well as the quantum, non-anomalous $U(1)_R$. It also breaks the quantum Z_N symmetry. Through a field redefinition, we can take $m_\lambda = |m_\lambda| e^{i\theta/N}$. Gaugino condensation then generates a potential for the fields U , which at large N takes the form:

$$V(\theta, \eta') = |m_\lambda| \Lambda_{hol}^3 \cos \left(\frac{\theta + 2\pi k + \frac{\eta'}{v}}{N} \right), \quad (2.22)$$

where we have written $\arg \det U = \frac{\eta'}{v}$. Recall that in conventional large N scaling, $m_\lambda \propto N m_\lambda^{phys}$, where m_λ^{phys} is the physical gaugino mass. Therefore, expanding for very large N and taking $m_\lambda^{phys} \sim \Lambda$ gives the potential for the η' proposed in [12]. The scaling with N is exactly as predicted.

For zero supersymmetric quark mass, θ and k can be removed by a redefinition of the η' field. In the presence of a quark mass term, this is no longer the case. The η' potential contains an

additional term, which at large N takes the form

$$V(\theta, \eta') = |m_\lambda| \Lambda_{hol}^3 \cos\left(\frac{\theta + 2\pi k + \frac{\eta'}{v}}{N}\right) + |m_q| \Lambda_{hol}^3 \cos\left(\frac{\eta'}{v} + \beta\right), \quad (2.23)$$

where β is the phase of the quark mass. We comment on the properties of this potential in Sec. 2.5.

2.5 Phases with General $N_f < N$

In QCD, the realization of branched structure is thought to vary with m_q [12]. At zero m_q , a field redefinition can eliminate θ -dependence. At large N , this corresponds to the fact that θ can be eliminated by a shift of the η' . On the other hand, at sufficiently large m_q , the quarks can be integrated out and θ -dependence should reappear, along with any branched structure.

In SBQCD, already in the limit of soft breakings, an intricate phase structure arises by varying the soft breaking parameters and the quark masses. This can be anticipated because in the theory of Eq. (2.16), before including the quark masses m_q , the discrete symmetry is Z_{N_f} , a preserved subgroup of the anomalous $U(1)_A$ axial symmetry acting on Q, \bar{Q} . If we set m_λ to zero, with non-zero m_q , the discrete symmetry is Z_N , a preserved subgroup of the anomalous $U(1)_R$ symmetry acting only on λ . It is easy to check that varying the parameter

$$x = \frac{m_\lambda}{m_q}, \quad (2.24)$$

the number of local minima of the potential changes from N at small x to N_f at large x .

To see this explicitly, take the simplified case $|m_q|^2, |m_\lambda|^2 \ll \tilde{m}^2$, and \tilde{m}^2, m_q proportional to the unit matrix in flavor space. We can then take $\bar{Q}Q = v_0^2 e^{i\eta'}$ (note here we are working

with a dimensionless η'). The potential for the η' then has the form:

$$V(\eta') = m_q \Lambda_{hol}^{\frac{3N-N_f}{N-N_f}} v_0^{-\frac{2N_f}{N-N_f}} \cos\left(\frac{N}{N-N_f} \eta'\right) + N m_\lambda \frac{\Lambda_{hol}^{\frac{3N-N_f}{N-N_f}}}{v_0^{\frac{2N_f}{N-N_f}}} \cos\left(\eta' \frac{N_f}{N-N_f}\right), \quad (2.25)$$

or, for $N \gg N_f$,

$$V(\eta') = m_q \Lambda_{hol}^3 v_0^{-\frac{2N_f}{N}} \cos(\eta') + N m_\lambda \Lambda_{hol}^3 \cos\left(\eta' \frac{N_f}{N}\right). \quad (2.26)$$

This potential is similar in structure to that for the ordinary η' proposed in [12]. It exhibits N vacua in the limit of small x , and N_f in the limit of large x . Analogously, in ordinary QCD, the large- N η' potential has N_f vacua in the limit $m_q \ll \Lambda/N$, and N vacua in the opposite limit.

In SQCD, the transitions between these phases occur for x of order one. As the vacua disappear, they become increasingly unstable. In the limit of large x , correlation functions with successively more insertions of $\int d^4x F \tilde{F}$ are suppressed by N_f , not N . The potential can also be analyzed in the case of $N_f = N - 1$, where a reliable instanton computation is possible. In this case, there are of order N branches in either limit, but one can still observe transitions between different phases, increasing confidence in the small N_f analysis.

The phase structure also offers some insight into the lifetimes of states of a system as one approaches the critical values x_0 where they disappear. The bounce action vanishes as a power of $x - x_0$ (of course, the semiclassical analysis breaks down once the lifetime becomes short).

2.6 Matter as a Perturbation

In the large N limit, we might expect that small changes in the number of flavors only affect the properties of the theory at order $1/N$: in this sense, matter is a perturbation.

There are two classes of quantities we might study. In actual QCD, we might ask about the N_f dependence of the glueball mass or $F\tilde{F}$ correlation functions, expecting weak sensitivity of these quantities to $\mathcal{O}(1)$ changes in N_f at large N . Alternatively, we can consider the structure of the quark sector. Here we expect the features of the effective action for the η' , for example, to be determined by the large N pure gauge theory.

In the supersymmetric theories, the gluino condensate is in the first class, and we expect small changes in the number of flavors to yield only small changes in the condensate. To test this idea, we must be precise about what is perturbed. As we vary N_f , we hold the ultraviolet cutoff M and the gauge coupling $g^2(M)$ fixed. For simplicity, we take all quarks to have mass m_q , with $m_q \gg \Lambda$, and we study the Wilsonian effective action at a scale μ such that $m_q \gg \mu \gg \Lambda$. Integrating out the quarks generates a term

$$\mathcal{L} = -\frac{1}{32\pi^2} \int d^2\theta \left(\frac{8\pi^2}{g^2} + 3N \log(\mu/M) - N_f \log(m_q/M) \right) W_\alpha^2. \quad (2.27)$$

From it, we can compute the holomorphic low energy scale, Λ_{LE} , which in turn determines $\langle \lambda\lambda \rangle$,

$$\langle \lambda\lambda \rangle = \Lambda_{LE}^3 = \Lambda^3 \left(\frac{m_q}{\Lambda} \right)^{\frac{N_f}{N}}. \quad (2.28)$$

This expression is clearly smooth with respect to changes in N_f . Indeed, we can treat an additional flavor as a perturbation, computing first the change in the effective action, and from that the change in Λ_{LE} .

Alternatively, we can consider a quantity involving the quark superfields, for small number of flavors. As before, we can think of a fixed cutoff scale and coupling, and take universal quark masses $m_q \gg \Lambda$. Then integrating out the heavy fermions yields

$$\langle \bar{Q}Q \rangle = \frac{1}{16\pi^2 m_q} \langle \lambda\lambda \rangle. \quad (2.29)$$

(this is an example of the Konishi anomaly [39]). This agrees with the exact result, and by holomorphy, it holds for all m_q . Thus for small N_f the quark condensate is determined in large N by the pure gauge theory. One can provide a heuristic derivation of this result at small m_q as well.

For larger values of N_f , small changes $\Delta N_f \ll N$ should also produce only small changes in the theory, for appropriate choices of ground states. This is particularly interesting for $N_f = N - 2, N - 1, N, N + 1, N + 2$, where the dynamics, when the quarks are light, is substantially different in each case (described via gaugino condensation, instantons, the deformed moduli space, s-confinement, and Seiberg duality, respectively [22, 23, 25].) Yet, in large N , all descriptions must in some sense converge, up to $1/N$ corrections!

Let us understand a few simple reflections of this fact, again taking $m_{f\bar{f}} \rightarrow m\delta_{f\bar{f}}$ and $Q\bar{Q}_{f\bar{f}} \rightarrow v^2\delta_{f\bar{f}}$. For $N_f = N - 1$, there is a Wilsonian effective superpotential [22],

$$W_{\text{Wilsonian}} = \frac{\Lambda^{2N+1}}{v^{N-1}} + N_f m v^2 \quad (2.30)$$

and the vacuum is

$$v = \Lambda \left(\frac{\Lambda}{m} \right)^{\frac{1}{2N}} \quad (2.31)$$

which approaches $v \rightarrow \Lambda$ in the large N limit, losing its m -dependence. In contrast, the case $N_f = N - 1$ has a deformed moduli space [23], described by a 1PI effective superpotential with Lagrange multiplier X ,

$$W_{1PI} = X (v^{2N} - B\bar{B} - \Lambda^{2N}) + N_f m v^2 . \quad (2.32)$$

Since there are no baryonic operators in $N_f = N - 1$, vacua on baryonic branches are not connected to vacua in $N_f = N - 1$. The meson vacuum, however, is: in the large N limit, the $N_f = N - 1$

vacuum becomes the $B = \bar{B} = 0$ vacuum $v = \Lambda$ of $N_f = N$. The gaugino condensates likewise match in large N , and vanish in the massless limit.

A similar result is obtained for $N_f = N + 1$ with small quark mass: the meson vev takes the form $v^{2N+1} = m\Lambda^{2N}$, so $v \rightarrow \Lambda$ in large N . The new feature of the $N_f = N + 1$ theory, the chiral preserving vacuum, is obtained in the limit $m \rightarrow 0$, which does not commute with $N \rightarrow \infty$.

2.7 Instantons at Large N

We see that approximately supersymmetric theories exhibit many of the features anticipated for real QCD, within controlled approximations. Much of our understanding of supersymmetric dynamics, on the other hand, involves instantons in an essential way. This suggests that instanton effects *are not necessarily* suppressed at large N , and can have controlled large N limits, at least in SQCD.

2.7.1 Heuristic treatment of instantons: the infrared cutoff

In the introduction, we discussed two potential behaviors for large N QCD as a function of θ , referred to as branched and instanton behaviors, respectively. We have seen that supersymmetric SQCD with small gaugino mass exhibits the former behavior. Ref. [11] offered a simple argument against the latter, suggesting that instanton effects are exponentially suppressed in large N . Let us recapitulate the argument.

Consider QCD without flavors. The one-instanton contribution to $V(\theta)$ has the structure:

$$V(\theta) = \int d\rho \rho^{-5 + \frac{11N}{3}} M^{\frac{11N}{3}} N e^{-\frac{8\pi^2}{g(M)^2}} \cos(\theta) \quad (2.33)$$

where M is a renormalization scale. Since $g^2(M) \sim 1/N$, this is formally exponentially sup-

pressed, but the expression is also infrared divergent. Suppose that the integral is cut off at $\rho \approx \Lambda^{-1}$.

The result would then be simply

$$V(\theta) = C\Lambda^4 \cos(\theta). \quad (2.34)$$

which is of order one in large N . Of course, this argument is handwaving at best. If the cutoff is $c\Lambda$, with c an order one constant, then the result can be exponentially suppressed or enhanced by c^N . Ref. [11] suggested that the most likely smooth limit for instanton effects in large N is zero.

Imagine, however, that c approaches 1 as $e^{1/N}$: in this case, the limit of the single instanton term would be smooth and finite. In QCD, such a picture could only be qualitative; perturbative corrections and instanton-antiinstanton corrections are all be nominally of the same order, and a reliable semiclassical calculation is not possible. The only statement one could make, in general, is that θ dependence would be described by a series of the form of Eq. (2.7). One could speculate on the convergence of the series, for example whether cusps arise in the potential. This appears to occur in the CP^N models, where finite temperature provides an infrared cut-off on instanton size [40, 41, 42], and the series (2.7) exhibits cusps in the limit $T \rightarrow 0$ (the Fourier expansion for $\frac{dE}{d\theta}$ does not converge). This will be discussed more fully in a subsequent publication.

2.7.2 Scaling of Reliable Instanton Computations with N

In SQCD with $N_f = N - 1$, the role of instantons in large N can be assessed sharply, exploiting the existence of a pseudomoduli space. The effective superpotential can be computed systematically, and infrared divergences are cut off by $Q\bar{Q}_{f\bar{f}} \equiv v^2 \delta_{f\bar{f}}$. The ρ integrals take the form

$$W \sim \int d\rho (\Lambda\rho)^{2N+1} (v^*)^{2N-2} \rho^{4N-5} e^{-c^2 \rho^2 |v|^2} \sim \frac{\Lambda^{2N+1}}{v^{2N-2}}. \quad (2.35)$$

A careful analysis yields [24]

$$W = \frac{\Lambda_{hol}^{2N+1}}{\det \bar{Q}Q}, \quad (2.36)$$

which is naïvely of order e^{-N} .

However, v^2 also depends on Λ . For simplicity, taking all of the quarks to have equal mass,

$$v^N = \Lambda_{hol}^N \left(\frac{\Lambda_{hol}}{m_q} \right)^{\frac{1}{N}}. \quad (2.37)$$

At the stationary point,

$$\langle W \rangle = a \Lambda_{hol}^2 m_q \left[\frac{\Lambda_{hol}}{m_q} \right]^{1/N}. \quad (2.38)$$

This structure is dictated by symmetries and holomorphy. In particular, there is a non-anomalous, spurious R symmetry under which

$$m_q \rightarrow e^{2i\alpha \frac{N}{N_f}} m_q. \quad (2.39)$$

Similarly, there is a non-anomalous R symmetry under which m_q (and Q, \bar{Q}) are neutral, and $\Lambda \rightarrow e^{i\alpha 2N/(2N+1)} \Lambda$.

Eq. (2.38) is notable. First, there is no exponential suppression with N :

$$\Lambda_{hol}^{2+\frac{1}{N}} = M^{2+\frac{1}{N}} e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}. \quad (2.40)$$

Not only do the $e^{-\frac{8\pi^2}{g^2}}$ factors appear with a suitable power to avoid e^{-N} suppressions, but there are no factors like π^N or 2^N which might have obstructed a suitable large N limit. At the same time, the result exhibits monodromy, arising from the N roots of Eq. (2.37).

It is also important to stress that, unless m_q is exponentially small, the stationary point lies in a region of strong coupling. So a reliable calculation is possible taking $m_q = \epsilon^N \Lambda$, for small

ϵ , and then using holomorphy and symmetries to extend the result to $m_q = \Lambda$. For $m_q \sim \Lambda$, the instanton result is not reliable in the sense that non-holomorphic quantities like the scalar potential are not properly computed. But the result for $\langle W \rangle$ qualitatively has the instanton structure, and it is equivalent to say that it is saturated by the single instanton.

We also note that in presence of a gaugino mass, we again find the usual formula for the vacuum energy,

$$E(\theta) = m_\lambda \langle W \rangle = m_\lambda \Lambda_{hol}^3 \cos\left(\frac{\theta + 2\pi k}{N}\right). \quad (2.41)$$

So in this case, we have *complete agreement* with expectations based on N counting of perturbative Feynman diagrams, yet the result arises entirely from an instanton! In particular, correlators of n $F\tilde{F}$ operators at zero momentum behave as N^{2-n} , precisely as expected. We have already noted how a cutoff might approach Λ in large N so that instanton amplitudes are unsuppressed. Here we see that, in the nearly supersymmetric case, the Λ which appears in the argument is the *holomorphic* Λ , yielding $\cos(\theta/N)$.

To summarize, on the one hand, we see evidence for a branched structure, a structure originally suggested by a presumed suppression of instanton effects. On the other hand, we see that instantons are not suppressed, and the branches are associated with an approximate discrete symmetry. We cannot draw conclusions about the fate of the branched structure as SUSY breaking is increased, but the instanton argument for the branched structure, by itself, is at least misleading in the nearly-SUSY limit.

2.7.3 Further circumstantial evidence for the role of instantons

Also instructive are instanton computations in the pure supersymmetric gauge theory. This subject was pioneered in [43, 44]. In pure $SU(N)$ supersymmetric QCD, one can attempt to calculate the correlation function

$$G^{2N} = \langle \lambda\lambda(x_1) \dots \lambda\lambda(x_N) \rangle . \quad (2.42)$$

A single instanton makes an infrared finite contribution to this correlator, $G^{2N} \sim \Lambda^{3N}$, which is formally of order e^{-N} . This paper argued that this correlation function, as the correlator of the lowest component of a set of chiral fields, was independent of coordinates, and in addition advanced arguments that it was not renormalized. The authors of [45] argued, invoking cluster decomposition, that the N^{th} root of this expression is $G = \langle \lambda\lambda \rangle$.

It is known that the single instanton computation makes an order one error in these quantities. The corrections can be understood as dilute gas corrections (in the sense that they can be shown to arise from the sector with topological number one [46]). If the naive reasoning were correct, these effects would be suppressed by further powers of e^{-N} , but this is not the case. This is consistent with the infrared cutoff computations suggested in [47].

2.8 Speculations on Real QCD

We have seen that instantons and large N behavior are not necessarily incompatible, and emphasized that the appearance of branches in supersymmetric QCD is associated with the spontaneous breaking of a discrete symmetry. As we take the soft breakings large, most of the N vacua might disappear, leading to what we have called “instanton” behavior. On the other hand, given

that the lifetimes of the states scale as e^{-N^4} (in the region over which we have control), they might survive.

2.8.1 Spontaneous breaking of an explicitly broken discrete symmetry

In this brief section, we describe the possible behaviors in terms of the realization of a spurious symmetry. At the level of the classical action, the softly broken supersymmetric theory exhibits a symmetry with m_λ viewed as a spurion:

$$\lambda\lambda \rightarrow e^{\frac{2\pi ik}{N}} \lambda\lambda \quad m_\lambda \rightarrow e^{-\frac{2\pi ik}{N}} m_\lambda. \quad (2.43)$$

If $E(m_\lambda) = E(|m_\lambda|, m_\lambda^N)$, this spurious symmetry is not spontaneously broken. If $E(m_\lambda)$ is not invariant under $m_\lambda \rightarrow e^{-\frac{2\pi ik}{N}} m_\lambda$, however, spontaneous symmetry breaking has occurred. This is the option realized in SBQCD, and is associated with N stationary points of the vacuum energy. E has an imaginary part outside a finite range of $\alpha = \arg m_\lambda$.

The existence of branches in real QCD can be mapped to the question of whether the spurious symmetry is broken or unbroken as m_λ becomes much larger than Λ . As $m_\lambda \rightarrow \infty$ and λ is integrated out, we generate $\theta = \arg(m_\lambda)N$. The question is: does E behave (in the pure gauge theory) as a function of $\arg(m_\lambda)$ or $\arg(m_\lambda)N$? Needless to say, analytic tools to address this question are not available, but we can look to toy models to gain some understanding of the possibilities.

We can illustrate these possible behaviors of the pure gauge theory in a field theory of scalars, treating the system classically and including certain non-renormalizable couplings. With a complex field, ϕ , the potential

$$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4 - \Gamma(\phi^N + \phi^{*N}) \quad (2.44)$$

respects a Z_N symmetry. If Γ is small, we can write:

$$\phi = f e^{ia/f}, \quad f = \sqrt{\frac{\mu^2}{\lambda}} \quad (2.45)$$

The field a acquires a potential

$$V(a) = -\Gamma f^N \cos\left(N \frac{a}{f}\right). \quad (2.46)$$

The system has N degenerate minima, at $\frac{a}{f} = \frac{2\pi k}{N}$, reflecting the spontaneous breaking of the discrete symmetry.

Adding a coupling

$$\delta V = m_\lambda \Lambda^2 \phi + \text{c.c.} \quad (2.47)$$

breaks the Z_N symmetry explicitly, and the parameter m_λ is a spurion analogous to m_λ in SUSY QCD. For small $m_\lambda = |m_\lambda| e^{i\alpha}$, ϕ does not shift significantly, and the classical vacuum energy has a contribution

$$E(\alpha, k) = |m_\lambda| \Lambda^2 f \cos\left(\alpha + \frac{2\pi k}{N}\right). \quad (2.48)$$

The potential reflects the spontaneous breaking of the spurious symmetry. Quantum mechanically, E has a small imaginary part except for k such that $|\alpha + \frac{2\pi k}{N}| < \pi$.

Elsewhere in the parameter space, however, the branches disappear. For example, for μ^2 negative, the potential has a unique minimum, and this is not altered by the addition of the m_λ term. Instead,

$$\langle \phi \rangle = \frac{m_\lambda^* \Lambda^2}{\mu^2}, \quad (2.49)$$

and

$$E(\alpha, k) = \frac{|m_\lambda|^2 \Lambda^4}{\mu^2}. \quad (2.50)$$

Thinking of this as a toy model of supersymmetric QCD, the parameters $\mu^2 \rightarrow \mu^2(m_\lambda)$, $\Gamma \rightarrow \Gamma(m_\lambda)$. If, for example, $\mu^2(m_\lambda)$ becomes negative and Γ does not grow too rapidly for large m_λ , the branched structure disappears. Alternatively, if for large m_λ , $\mu^2 > 0$ and if Γ grows rapidly with m_λ , then the branched structure survives. In this toy model, the N vacua reflect an approximate Z_N symmetry which survives in the limit.

2.8.2 Stability of Branches

In SBQCD, both with and without matter, we can ask about the stability of different branches. Take $k = 0$ and $0 < \theta < 2\pi$ and consider what happens as η' increases. At some point, the state with $k = -1$ has lower energy, and the system can tunnel. For small m_λ , the tunneling rate is highly suppressed, roughly as⁴,

$$\Gamma = C e^{-aN^4 \frac{\Lambda^3}{m_\lambda^3}}. \quad (2.51)$$

We can repeat this for larger k , producing a large set of metastable states. Increasing m_λ , eventually we can no longer perform a reliable computation, but based on (2.51) it is possible that tunneling rates remain exponentially suppressed with N . The presence of metastable states in QCD is an interesting target for the lattice [16], and could conceivably have implications for physics in the early universe.

⁴This estimate appears also in [48], which notes that due to numerical factors, even for $m_\lambda \Lambda_{QCD}$, the states may be short-lived unless $N > 100$ or so. If true, it would be challenging to understand how the large N limit could be valid for $N \approx 3$.

2.8.3 't Hooft's Picture of Confinement: A Candidate Setting for Branched Structure

Nambu, Mandelstam, and 't Hooft suggested that condensation of magnetically charged objects in a non-abelian theory could account for confinement of color charge [49, 50, 51]. Subsequently, 't Hooft studied the adjoint-valued composite field $\Phi = F_{\mu\nu}\tilde{F}^{\mu\nu}$, choosing a gauge in which Φ is everywhere diagonal and leaving unfixd a $U(1)^{N-1}$ symmetry [30]. He speculated that singular points with respect to the gauge choice correspond to massless, condensing monopoles of the $U(1)^{N-1}$ theory, and noted that in the presence of θ , the monopoles acquire a charge through the Witten effect. When $\theta \rightarrow \theta + 2\pi$, the spectrum is the same, but “rearranged”: what were monopoles with one charge at $\theta = 0$ become monopoles of a different charge at $\theta = 2\pi$. This picture of confinement thus gives rise to an explicit realization of branched structure with θ . The details, including whether there are N vacua of a spurious Z_N symmetry, depend on unknown features of the monopole/dyon spectrum. Such dynamical features are also suggested by consideration of the algebra of Wilson and 't Hooft lines [52, 53].

$N = 2$ supersymmetric Yang-Mills, with a small mass m_A for the adjoint chiral multiplet, exhibits many of these features explicitly, including a $U(1)^{N-1}$ symmetry in the small m_A limit. Seiberg and Witten showed that the theory possesses massless monopoles at points in the moduli space [54]. In the case of $SU(N)$, there are N such points, related by a discrete Z_N symmetry. Turning on m_A , the massless monopoles condense, and the theory confines. The condensate is proportional to m_A , and, for small m_A , the monopole and $U(1)$ gauge field masses are also suppressed by m_A . The theory possesses precisely the sort of branched structure anticipated by 't Hooft, with $\tau = \frac{8\pi^2}{g^2} + ia$, and the branches are associated with the Z_N symmetry of the theory. As m_A becomes

larger than Λ , it is not clear what becomes of the monopole picture; the $U(1)$ gauge bosons are no longer light relative to other states in the spectrum, nor are the monopoles. But we know that the $N = 1$ theory exhibits a branched structure.

If 't Hooft's picture for confinement is qualitatively correct for real QCD, it can account for a branched structure. However, the applicability of the monopole condensation picture to real QCD remains unclear. For example, one does not expect that the theory exhibits light states corresponding to $U(1)^{N-1}$ gauge bosons. Starting from $N = 2$, it is also not clear that the monopole picture is instructive for large m_A , let alone after adding a soft breaking gaugino mass.

2.9 Summary

We have studied the large N θ dependence of supersymmetric QCD, using small soft breakings as a probe of the nonsupersymmetric limit. We have seen that certain aspects of the usual large N picture, including the presence of branches and the behavior in theories with matter (both with $N_f \ll N$ and $N_f \sim N$), are reflected in SBQCD. However, there are also striking departures from ordinary QCD and the conventional large N description. First, in supersymmetric theories, instanton effects are sometimes calculable and do not fall off exponentially with N . Second, branched structure in SBQCD is always associated with approximate discrete symmetries, which are badly broken in the nonsupersymmetric limit.

In light of these differences, and to advance our understanding of nonperturbative phenomena in QCD, it would be of great interest to have additional lattice probes of the branched structure of large N QCD. In future work we will explore aspects of lattice tests, particularly the possibility of searching directly for the tower of metastable states at $\theta = 0$.

Appendix: Λ and Λ_{hol}

Quantities in supersymmetric gauge theories are readily derived in terms of an object referred to as the *holomorphic scale*, Λ_{hol} . In the case of $SU(N)$ SUSY QCD without chiral fields, we can make this notion precise in a very simple way, embedding the theory in an $\mathcal{N} = 4$ theory, with masses for the adjoint fields providing a cutoff for the SQCD theory [55, 47]. In a presentation in which the $SU(4)$ symmetry is (almost) manifest, the action is

$$\mathcal{L} = -\frac{1}{32\pi^2} \int d^2\theta \tau W_\alpha^2 + \frac{1}{g^2} \int d^4\theta \Phi_i^\dagger e^V \Phi^i + \int d^2\theta \frac{1}{g^2} f_{abc} \epsilon^{ijk} \Phi_i^a \Phi_j^b \Phi_k^c. \quad (2.52)$$

Here τ is

$$\tau = \frac{8\pi^2}{g^2} + i\theta. \quad (2.53)$$

In order that the superpotential be a holomorphic function of τ , we rescale the Φ^a fields. We can also add holomorphic mass terms:

$$\mathcal{L} = -\frac{1}{32\pi^2} \int d^2\theta \tau W_\alpha^2 + \frac{1}{g^{2/3}} \int d^4\theta \Phi_i^\dagger e^V \Phi^i + \int d^2\theta (f_{abc} \epsilon^{ijk} \Phi_i^a \Phi_j^b \Phi_k^c + M \Phi_i^a \Phi_i^a). \quad (2.54)$$

Holomorphy of the gauge coupling function gives, for the renormalized coupling,

$$\frac{8\pi^2}{g^2(m)} = \frac{8\pi^2}{g^2(M)} + b_0 \log(m/M). \quad (2.55)$$

Here m and M are holomorphic parameters (this is discussed further in [47]). The physical masses are related to these by a factor of $g^{2/3}(m)$, $g^{2/3}(M)$; substituting yields the standard β function through two loops (issues involving the exact β function are discussed, again, in [47]). Λ_{hol} is then defined through:

$$\Lambda_{hol} = M e^{-\tau/b_0} = g^{-2/3} M_{phys} e^{-\tau/b_0}. \quad (2.56)$$

This is *almost* the conventionally defined Λ parameter, but in large N it differs by a power of N , as noted in [37] and we now review.

The Particle Data Group presents the strong coupling as (with slight redefinition of b_0 and b_1 to agree with our conventions above):

$$\alpha_s(\mu) = \frac{4\pi}{b_0 t} \left(1 - \frac{b_1 \log t}{b_0^2 t} \right), \quad t = \log \left(\frac{\mu^2}{\Lambda^2} \right). \quad (2.57)$$

Comparing with the solution of the RGE,

$$\frac{8\pi^2}{g^2(\mu)} = \frac{8\pi^2}{g^2(M_{phys})} + b_0 \log(\mu/M_{phys}) - \frac{b_1}{b_0} \log(g(\mu)/g(M_{phys})), \quad (2.58)$$

we see that inserting

$$\Lambda = M_{phys} e^{-\frac{8\pi^2}{b_0 g^2(M_{phys})}} \left(\sqrt{\frac{b_0}{8\pi^2}} g(M_{phys}) \right)^{-b_1/b_0^2}, \quad (2.59)$$

and

$$\log t \approx \log \left(\frac{8\pi^2}{b_0 g^2(\mu)} \right) \quad (2.60)$$

into Eq. (2.57), we recover Eq. (2.58). Using $b_1^2/b_0 = 2/3$ in pure SYM, one obtains $(\Lambda_{hol}/\Lambda)^3 \sim N$ in large N .

Chapter 3

Axion Cosmology

3.1 Introduction

The axion remains a promising candidate for dark matter [56, 57, 58], perhaps more so as the window for conventional WIMPs shrinks. Searches for axion dark matter are underway [59], and there are proposals for future experiments which could conceivably widen the search window substantially [60].

In the early universe, the axion begins to oscillate coherently when its thermal mass, $m_a(T)$, becomes comparable to the Hubble scale. The axion mass is related to the QCD topological susceptibility by

$$m_a^2(T) f_a^2 = \chi(T), \quad \chi = \int d^4x \langle F \tilde{F}(x) F \tilde{F}(0) \rangle_T = \partial_\theta^2 F(\theta, T), \quad (3.1)$$

where, in the last expression, $F(\theta, T)$ is the θ -dependent free energy.

At high temperatures, $F(\theta, T)$ can be calculated by standard instanton methods [61]. At low temperatures, $F(\theta, T)$ is known from chiral perturbation theory, and in fact converges rapidly

to its $T = 0$ limit below the confining phase transition [62, 63]. However, for plausible cosmologies and a range of axion parameters, it is the case that the axion starts to oscillate at intermediate temperatures, $T \sim \text{GeV}$, where α_s is approaching strong coupling and neither calculation applies. Instead, one can hope to extract $\chi(T)$ from lattice QCD.

Recently there have been a number of papers reporting lattice calculations of $\chi(T)$ at temperatures above the critical temperature, both in pure gauge theory/quenched approximations [64, 65, 63] and in QCD [66, 67, 1, 62, 68, 69, 70]. In some cases, discrepancies are found at high temperatures compared with the dilute instanton gas prediction. The free energies found in [67, 1] differ by about an order of magnitude from the leading-order semiclassical result above the GeV scale, while the computation and extrapolation obtained in [62, 68] (see also [71]) differs by many orders of magnitude (although systematic effects are not under control in the extrapolation, and thus the level of compatibility with the controlled high-temperature computations in [67, 1] is unclear.) These results suggest a level of uncertainty in the microscopic parameters (the zero temperature axion mass m_a and possibly the misalignment angle θ_0) required to achieve a given relic density Ω .

Here we will assess the theoretical uncertainty on $\Omega(m_a, \theta_0)$ in analytic computations and compare with lattice results. Our analysis has two prongs. First, in Sec. 3.2, we reexamine the uncertainty on the leading-order semiclassical result for the free energy above the GeV scale. We consider a number of possible sources of error, the most important of which are likely to be corrections to the effective action that shift the IR cutoff on instanton sizes. However, in contrast to suggestions in the literature, we argue that infrared divergences that plague ordinary finite-temperature perturbation theory are not numerically relevant in the instanton effective action, and that the size of higher-order corrections can be reasonably estimated. As a result, the topological susceptibility has known

low and high temperature asymptotics which appear compatible with [67, 1]. Then, in Sec. 3.3, we introduce a family of models for the topological susceptibility that interpolate through the region where neither analysis is reliable. In Sec. 3.4, we compute the relic density over this range of models, including the instanton uncertainty at the high temperature boundary and uncertainties in QCD parameters. In this way we determine the sensitivity of $\Omega(m_a, \theta_0)$ to theoretical uncertainties. We find the sensitivity is limited, and overall the axion mass prediction from analytical methods appears robust at the level of a factor of 2 – 3. We comment on the implications of these results in Sec. 4.7 and conclude.

3.2 Theoretical Uncertainties on the Instanton Contribution to the Free Energy

3.2.1 The Standard Computation

At high temperatures, the θ dependence of the free energy is controlled by instantons ([61], henceforth GPY). Classically, even at finite temperature, there are instantons of all scale sizes. But at one loop, there are two sources of scale invariance violation: the usual ultraviolet divergences familiar in the zero temperature theory, and the finite temperature itself. Both correct the effective action, rendering finite the scale size integral both at small and large ρ .

Heuristically, the latter effect is associated with the effective mass of the A_4 field,

$$m_D^2 = \frac{1}{3}(g^2 T^2)(N + \frac{N_f}{2}). \quad (3.2)$$

GPY note that a term in the effective action $\frac{1}{2g^2} m_D^2 A_4^2$ gives rise to a correction to the instanton

action, for $\rho \gg T^{-1}$ (and $\alpha_s(\rho^{-1}) \ll 1$), proportional to ρ^2 :

$$\int d^4x \frac{1}{2g^2} m_D^2 A_4^2 = \frac{\pi^2}{2g^2} m_D^2 \rho^2. \quad (3.3)$$

Note the g^{-2} in front of m_D^2 , reflecting the $1/g^2$ in front of the whole action, and the fact that the actual screening length is of order $\frac{1}{gT}$. If this were the complete result for the correction to the effective action, the ρ integration for the free energy would take the form, in the case of three flavors,

$$F(T) \propto m_u m_D m_s \int \frac{d\rho}{\rho^2} (\Lambda\rho)^9 e^{-3\pi^2 \rho^2 T^2}. \quad (3.4)$$

The integral is finite, and dominated by $\rho \sim (\pi T)^{-1}$.

Since the dominant scale is of order T^{-1} , the effective action cannot be expanded in powers of ρ ; in a derivative expansion of the background field effective action, terms of the form

$$\frac{g^2}{T^{n-2}} A_4(\vec{x}) \partial_{i_1} \dots \partial_{i_n} A_4(\vec{x}) \quad (3.5)$$

are all of the same order, $g^2 T^2$, in the instanton background.

GPY indeed computed the full one-loop determinant [61]. At small ρ , in particular, the above expression for the action is modified:

$$\delta S = \frac{1}{3} \pi^2 \rho^2 T^2 (2N + N_f) - \frac{1}{18} \pi^2 \rho^2 T^2 (N - N_f). \quad (3.6)$$

For $N_f = 0$, for example, this is not parametrically smaller than the Debye screening term, though it is numerically smaller. At one loop, the complete expression for the free energy in the presence of a single instanton is given by [61]

$$F(\theta, T) = - \int \frac{d\rho}{\rho^5} \left(\frac{4\pi^2}{g^2} \right)^{2N} e^{-\frac{8\pi^2}{g^2(\rho)}} C_N \prod_{i=1}^{N_f} (\xi \rho m_i) e^{-1/3\lambda^2(2N+N_f) - 12A(\lambda)[1 + \frac{1}{6}(N-N_f)] + i\theta} \quad (3.7)$$

where

$$A(\lambda) = -\frac{1}{12} \ln(1 + \lambda^2/3) + \alpha(1 + \gamma\lambda^{-2/3})^{-8} \quad (3.8)$$

$$\lambda = \pi\rho T \quad C_N = 0.097163; \quad \xi = 1.3388 \quad \alpha = .01290 \quad \gamma = 0.1586 \quad (3.9)$$

and $N_F = 3$ in temperature regimes where three quarks are excited. At a temperature of $T = 1.5$ GeV and using a renormalization scale $\mu = T$, we obtain

$$F_0(1.5) = -3.7 \times 10^{-14} \text{ GeV}^{-4} \quad (3.10)$$

where the subscript indicates $\theta = 0$. Here we have used the program RunDec [72, 73, 74] to obtain $\alpha_s(1.5 \text{ GeV}) \simeq 0.345$ with three active flavors.

This computation of the free energy is subject to certain theoretical uncertainties, including higher-order corrections sensitive to the UV cutoff, parametric uncertainties on α_s , effects of heavier quarks, and higher-order corrections that modify the infrared cutoff on ρ . In the next subsection, we estimate the uncertainties from the first three of these sources. In our view there has been some confusion in the literature about the uncertainty associated with corrections to the ρ cutoff, which is plausibly the dominant source of uncertainty. We therefore devote a separate subsection to this source.

3.2.2 UV-Sensitive Corrections, Heavy Quarks, and Parametric Uncertainties

In the previous section, we evaluated the one-loop expression for F at $T = 1.5$ GeV with $\mu = T$ and three flavors. Let us comment on a few of the knobs we can turn in this calculation to obtain estimates of theoretical uncertainty.

- Because the dominant instanton size is of order $(\pi T)^{-1}$, $\mu = \pi T$ is another natural choice for the renormalization scale.

- At $T = 1.5$ GeV, πT is substantially above the charm threshold and near the bottom quark mass. We might therefore include at least the charm quark in the free energy.
- A complete two-loop calculation of F is not available at present. In some places in the literature, UV-divergent two-loop corrections to the free energy are incorporated using renormalization group considerations, as discussed in [75]. These corrections are generally written as powers of $\alpha_s(\rho^{-1})/\alpha_s(\mu)$ in the ρ integrand. However, for μ of order T (or πT), there is no justification for including some two-loop corrections and not others. Without performing an actual two-loop computation of the θ -dependent part of the free energy, the only principled approach is to use the complete one-loop expression with μ of order T (or πT). However, the “UV two-loop” computation might instead be useful as an uncertainty estimator.

We therefore recompute the free energy using $\mu = \{1, \sqrt{\pi}, \pi\} \times T$, three or four active flavors, and including or not including the UV-divergent two-loop corrections. For the change in renormalization scale, we again use RunDec to accurately determine $\alpha_s(\mu)$ with different numbers of active flavors. The two-loop corrections are incorporated by running α_s in the exponent to $\mu = \rho^{-1}$ at two-loop order and running the quark masses and the coupling in the prefactor to $\mu = \rho^{-1}$ at one-loop order. For example,

$$e^{-\frac{2\pi}{\alpha_s(\mu)}} \rightarrow e^{-\frac{2\pi}{\alpha_s(\mu)}} (\mu\rho)^{b_0} \left(\frac{\alpha_s(\rho^{-1})}{\alpha_s(\mu)} \right)^{2b_1/b_0} \quad (3.11)$$

where $b_0 = 9$ and $2b_1/b_0 = 32/9$ for $N_F = 3$, and $\alpha_s(\rho^{-1})$ is determined from $\alpha_s(\mu)$ at one-loop order.

Results are reported in Table 3.1. The largest value for the free energy is obtained in the three-flavor scheme with $\mu = T$, adding the partial two-loop terms. This is not a surprise, since the

| | |
|-----------------------|-----|
| 3F, 1L, T | 3.6 |
| 3F, 2L, T | 10 |
| 3F, 1L, $\sqrt{\pi}T$ | 4.9 |
| 3F, 2L, $\sqrt{\pi}T$ | 7.2 |
| 4F, 1L, $\sqrt{\pi}T$ | 3.2 |
| 4F, 2L, $\sqrt{\pi}T$ | 5.2 |
| 3F, 1L, πT | 6.0 |
| 3F, 2L, πT | 5.5 |
| 4F, 1L, πT | 4.0 |
| 4F, 2L, πT | 3.8 |

Table 3.1: The instanton-induced free energy in units of $-10^{-14} \text{ GeV}^{-4}$ at $\theta = 0$ and $T = 1.5$. Rows correspond to a variety of computations: (3F,4F) = three or four light flavors; (1L,2L) = one-loop complete or partial two-loop; (T, $\sqrt{\pi}T$, πT) = renormalization scale.

included two-loop terms correspond entirely to running from T to $\rho^{-1} \sim \pi T$. However, it is likely an overestimate of the correction; if we use $\mu = \pi T$ in the same computation, the partial two-loop result is smaller and much closer to the complete one-loop result. In reality, the complete two-loop result is likely to involve a mixture of scales, motivating the choice $\mu = \sqrt{\pi}T$. We observe that (excluding the $3F, 2L, \mu = T$ result), the envelope of the values is contained within the $\mu = \sqrt{\pi}T$ calculations, corresponding to an $\mathcal{O}(1)$ uncertainty the free energy,

$$\frac{\Delta F_0(1.5)}{F_0(1.5)} \simeq 1. \quad (3.12)$$

We can also estimate a ‘‘parametric’’ uncertainty stemming from experimental uncertainty in α_s . Using the 1-sigma error bar on $\alpha_s(m_Z)$, running down to $\mu = \pi \times 1.5 \text{ GeV}$ and converting

to the three-flavor scheme with RunDec [72, 73, 74], we obtain less than 2% uncertainty in α_s .¹ This results in an uncertainty in $F_0(1.5)$ of about a factor of 2, similar to the uncertainty from UV-sensitive corrections.

The lattice result for the topological susceptibility obtained in Ref. [1] corresponds to $F_0(1.5) \approx -4 \times 10^{-13} \text{ GeV}^{-4}$, which lies outside the uncertainty range that we estimate from these sources. Similar conclusions were drawn in the lattice studies [67, 70]. In [71, 67], it was suggested that the uncertainty in the 1-loop instanton computation arising from higher order terms could actually be much larger, associated with infrared divergences in QCD perturbation theory at finite temperatures and with large shifts in the Debye screening length. We now turn to corrections of this type.

3.2.3 Corrections to the IR Cutoff on ρ and Infrared Sensitivity

As discussed previously, the IR cutoff on the instanton size can be *qualitatively* associated with the Debye mass term in the effective action. In the perturbative vacuum, the Debye mass does receive large corrections beyond leading order [76, 77]. A simple, heuristic understanding of these corrections can be obtained by considering Π_{44} as a function of (spatial) momentum, \vec{q} , for small \vec{q} . There are a variety of effects, but already at one loop, for example, there is a contribution to $\partial\Pi/\partial q^2$ that diverges linearly as $m_D \rightarrow 0$ at $q = 0$. The linear IR divergence is cut off by the leading-order m_D , leaving a weaker logarithmic IR divergence cut off by the nonperturbative magnetic mass. The NLO Debye mass has the form [76, 77]:

$$m_D^2 = (m_D)_0^2 + \frac{2Ng^2}{4\pi} T(m_D)_0 \ln(m_D/g^2T) + \dots \quad (3.13)$$

¹Uncertainty from higher order corrections to the running of α_s are extremely subdominant.

The NLO correction is of order g^3 , signaling a breakdown of the perturbation expansion. It has been suggested [71, 67] that the uncertainty on $\chi(\theta, T)$ might be much larger than estimated in the previous section, due to the presence of such IR divergences.²

In the instanton computation, there is both a question of principle and a question of numerics. We have seen that it is not low spatial momenta that are relevant in the instanton background, but momenta of order $k \sim 1/\rho \sim T$. Consequently, for the dominant semiclassical configurations with $\rho \sim 1/T \ll \Lambda$, the IR divergences in Π_{44} are cut off at T in the instanton effective action S_{eff} . The corrections to S_{eff} from individual diagrams are then well-behaved and proportional to $g(T)^2 T^2$. Thus, as a matter of principle, IR divergent corrections to m_D in the perturbative vacuum do not indicate a loss of perturbative control or a significant source of uncertainty in the instanton computation of $F(\theta, T)$.

However, until T is extremely large, $g(T)$ is $\mathcal{O}(1)$ in QCD, and there is no parametric separation between m_D and T . Therefore Eq. (3.13), valid in the perturbative vacuum, might still be used as an estimate for the typical size of corrections to the effective action in the instanton background. Numerically, it gives rise to

$$\frac{(m_D)_1}{(m_D)_0} \simeq 0.6 \quad (3.14)$$

at $T = 1.5$ GeV. The instanton-induced free energy scales approximately as the 7th – 8th power of the infrared cutoff on ρ , so from Eq. (3.14) we are led to associate an uncertainty in the free energy due to two-loop finite temperature corrections,

$$\frac{\Delta F_0(1.5)}{F_0(1.5)} \simeq 20. \quad (3.15)$$

²It should be noted that the existence of such corrections is not connected with the presence of light fermions. In particular, fermions do not introduce infrared divergences at high temperature. Therefore, even lattice studies focusing on the size of corrections in the pure gauge theory are of interest.

We emphasize that a correction to the free energy of this size does *not* reflect a breakdown of the semiclassical analysis at this order. Organizing the instanton effective action as

$$S_{inst} = S_0 + S_1 + S_2 + \dots, \quad (3.16)$$

at leading order, the action is

$$S_0 = \frac{8\pi^2}{g^2} \simeq 17 \quad (3.17)$$

at $T = 1.5$ GeV. A shift in the free energy of order Eq. (3.15) corresponds to

$$\frac{S_2^{\text{Debye}}}{S_0} \lesssim 0.2, \quad (3.18)$$

a controlled correction to the effective action. More generally, we could estimate terms in the series by the three-dimensional loop factor, which is of order

$$\lambda = \frac{Ng^2(T)}{(4\pi)^{3/2}}. \quad (3.19)$$

At $T = 1.5$ GeV, $\lambda = 0.3$. Two-loop corrections to the effective action would then be expected to be of order

$$\frac{S_2}{S_0} \simeq \lambda^2 = 0.1 \quad (3.20)$$

consistent with the Debye estimate. In other words, there is no reason to expect *arbitrarily* large corrections. The action is exponentiated in the free energy, leading to the order-of-magnitude uncertainty estimate in Eq. (3.15).

There is also the question of actual infrared divergent contributions to the instanton action. These are associated with low momentum \vec{A} fields and corrections to the effective action involving no background fields. In the zero-instanton sector, such infrared divergences arise in the free energy

first at four-loop order. It is believed that they are cut off at a scale of order g^2T , the presumed mass gap of the three dimensional gauge theory. The typical diagram involves six vertices connected by propagators, and the divergence arises when all vertices are well-separated. A computation at high order in the instanton background is complex, but for ρ of order T^{-1} , the infrared divergence should be similar. At zero temperature, the propagators are known [78, 79], and at distances large compared to ρ , they are close to free-field propagators. At finite temperatures, when all coordinates except x_4 are large compared to T^{-1} and ρ , we expect something similar, leading to an infrared divergent correction at the same order as at zero temperature. At 1.5 GeV, this suggests a perturbatively incalculable correction to the instanton action at the 1% level.

In summary, IR divergences do not appear relevant to the instanton computation, and semiclassical analysis is under sufficient theoretical control to admit uncertainty estimates. Absent a complete 2-loop computation, we will take the perturbative Debye mass correction, Eq. (3.15), as a *conservative* estimate of the uncertainty in the θ -dependent free energy. UV cutoff-sensitive corrections and uncertainties in α_s are expected to be subdominant to the finite- T corrections to the effective action, and Eq. (3.17) indicates that dilute gas corrections are expected to be negligible.

We therefore know with some confidence the range of possible behaviors for the axion potential both at temperatures below the critical temperature (~ 150 MeV for $N_f = 3$) and at temperatures a few GeV and above. These boundary properties constrain the behavior in the intermediate range of temperatures, which happen to lie where the axion begins to oscillate in conventional scenarios. For this reason lattice computations (that successfully reproduce the high temperature behavior) can be of value. On the other hand, as we will describe below, if we simply assume a smooth interpolation between the two regimes, the axion relic density is not very sensitive either to

the form of the interpolation or the uncertainty in the high energy semiclassical computation.

In closing this section, we note that there have been arguments that the behavior of χ is drastically different at high temperatures than the semiclassical result, even turning off exponentially rapidly with temperature [80, 81, 82, 83]. We will not address this possibility further here, but it is certainly true that in such a circumstance substantially different axion relic densities can be obtained [65, 82, 83].

3.3 $\chi(T)$ At Intermediate Temperatures

We have argued that we know the high-temperature behavior of χ to about an order of magnitude. At scales below 1 GeV, the coupling rapidly becomes strong, and other methods are needed to determine the axion mass.

At very low temperatures, the θ dependence of the vacuum energy is known reliably from current algebra,

$$F(\theta, 0) = -3.6 \times 10^{-5} \text{ GeV}^4 \cos(\theta). \quad (3.21)$$

Finite temperature lattice computations indicate that the topological susceptibility ,

$$\chi(T) = \left. \frac{\partial^2 V(T)}{\partial \theta^2} \right|_{\theta=0}, \quad (3.22)$$

is near its zero-temperature ChPT value at temperatures of order 100 MeV, and remains approximately equal to it until at least the chiral phase transition near 150 MeV [63]. Beyond this scale, only lattice computations can accurately determine $\chi(T)$, and at present there are varied results in the literature.

However, for the purposes of computing the axion relic density, it turns out to be sufficient

to consider simple models that interpolate between the ChPT and instanton regimes. We will adopt the following class of models for $F(\theta, T)$:

$$F(\theta, T) = \begin{cases} -\chi(0) \cos \theta, & 0 < T < T_2 \\ -\chi(T_0) \left(\frac{T_0}{T}\right)^n \cos \theta, & T_2 < T < T_0 \\ -\chi(T_0) \left(\frac{T_0}{T}\right)^8 \cos \theta, & T > T_0 \end{cases} \quad (3.23)$$

Here T_0 is the ‘‘anchor point’’ for the instanton regime. The results of [63] suggest that the slope of χ is instanton-like down to temperatures a few times T_c ; however, to maintain a minimal uncertainty in the semiclassical computation, we fix $T_0 = 1.5$ GeV. As discussed below, our modeling still includes the possibility of instanton-like slopes at lower T . We will vary $\chi(T_0)$ within the uncertainty on the instanton computation. T_2 , the anchor point for the ChPT regime, is related to T_0 and the slope of the power law in the model by

$$T_2^n = T_0^n \times \frac{\chi(T_0)}{\chi(0)}. \quad (3.24)$$

We vary n such that T_2 varies between 100 and 500 MeV. Given $T_0 = 1.5$ GeV, values of T_2 of order 150 in fact correspond to $n \simeq 8$, equivalent to assuming instanton-like behavior persists significantly below T_0 . Larger values for T_2 above the critical temperature T_c are not based on physical considerations, but instead are included to partially accommodate the lattice results of [62, 68], which found very *shallow* falloff of $\chi(T)$ above the chiral phase transition. This behavior is then approximated in the models of Eq. (3.23) for larger T_2 by zero falloff until T_2 . However, Ref. [62, 68, 71] extrapolated the shallow power law behavior up to high temperatures, leading to values for F_0 many orders of magnitude different from the semiclassical result. Our insistence on reaching instanton behavior by 1.5 GeV (within the uncertainty (3.15)) requires an even *steeper* power law to set in above T_2 when $T_2 \gg T_c$. Numerically, n will fall in the range 7 – 20, with the

lower values corresponding to lower values of T_2 .

3.4 Axion Relic Density from Misalignment

We can now assess the sensitivity of the axion relic density to uncertainties in $\chi(T)$, including both the uncertainties in the instanton computation and the range of models for the behavior at intermediate temperatures. In the figures below we will numerically integrate the equation of motion,

$$\ddot{a} + 3H\dot{a} + V'(a) = 0. \quad (3.25)$$

However, for qualitative purposes, a good approximation is obtained by treating the axion as frozen until a temperature T_{osc} [84, 85]:

$$m_a(T_{osc}) = 3H(T_{osc}). \quad (3.26)$$

At this point, the axion begins to oscillate with a time (temperature) dependent mass. Approximating the energy density by

$$\rho(t) = \frac{1}{2}\dot{a}^2 + \frac{1}{2}m_a^2(T)a^2, \quad (3.27)$$

one can show that it evolves with temperature as

$$\rho(T) = \rho(T_{osc}) \left(\frac{R^3(T_{osc})}{R^3(T)} \right) \frac{m_a(T)}{m_a(T_{osc})}. \quad (3.28)$$

Within the range of $m_a(T)$ that we consider, T_{osc} is always less than the instanton anchor point T_0 . Therefore, Eq. (3.26) can be solved by substituting the intermediate power-law behaviors for $m_a(T)$; the instanton asymptotics constrain the range of intermediate power laws considered.

Ω

| n | $\chi_0 = 1/10$ | $\chi_0 = 1$ | $\chi_0 = 10$ |
|-----|-----------------|--------------|---------------|
| 8 | 0.22 | 0.18 | 0.15 |
| 14 | 0.18 | 0.16 | 0.14 |
| 20 | 0.17 | 0.15 | 0.14 |

Table 3.2: Axion relic density as a function of model parameters as computed with the approximate formula (3.29). Here $m_a(0) = 30 \mu\text{eV}$, χ_0 is given in units of $3.7 \cdot 10^{-14} \text{ GeV}^4$, and the misalignment angle is set to the value appropriate for post-inflationary breaking of the Peccei-Quinn symmetry, $\theta_0 = 2.16$.

The relic density Ω_a can then be expressed as a function of the parameters $\chi(T_0)$ and n (or T_2). The result is:

$$\Omega_{axion} = 0.13 \times (7.3)^{\frac{2}{4+n}} \left(\frac{m_a}{30 \mu\text{eV}} \right)^{-\frac{6+n}{4+n}} \left(\frac{\chi_0(1.5)}{3.7 \cdot 10^{-14} \text{ GeV}^4} \right)^{-\frac{1}{4+n}} \left(\frac{\theta_0}{2.155} \right)^2 \quad (3.29)$$

where θ_0 is the initial misalignment angle and m_a is the zero-temperature axion mass. Taking, for example, $m_a = 30 \mu\text{eV}$ and a few values for $\chi_0(1.5)$ and n gives the results in Table 3.2.

Alternatively, for fixed $\Omega_{axion} = \Omega_{DM}$, we obtain

$$\left(\frac{m_a}{30 \mu\text{eV}} \right) = 0.51 \times (2.4)^{\frac{6}{6+n}} \left(\frac{\chi_0(1.5)}{3.7 \cdot 10^{-14} \text{ GeV}^4} \right)^{-\frac{1}{6+n}} \left(\frac{\theta_0}{2.155} \right)^{\frac{8+2n}{6+n}} \left(\frac{\Omega_{DM}}{0.25} \right)^{-\frac{4+n}{6+n}}. \quad (3.30)$$

Eq. (3.29) indicates that the relic density is substantially insensitive to the magnitude of the free energy at high temperatures; for $T_2 = 150 \text{ MeV}$ ($n \sim 8$), χ_0 enters to the $-\frac{1}{12}$ power. Therefore, sizable uncertainties in χ_0 translate into modest uncertainties in m_a , also observed in [63]. Similarly the dependence on n (T_2) is mild. The lattice results of [63, 1], for example, differ from our estimate of $\chi_0(1.5)$ by a factor of about 10, and exhibit power-law behavior corresponding to

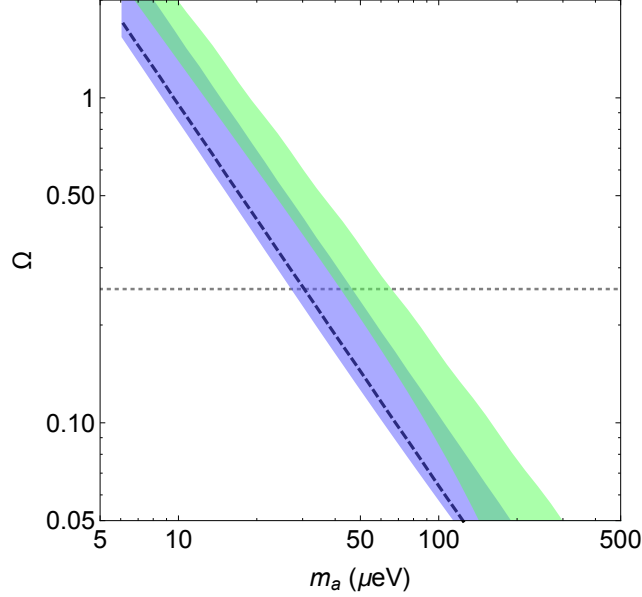


Figure 3.1: Axion relic density from misalignment in the post-inflationary scenario. Colors correspond to different models for the temperature-dependent free energy between the dilute gas at high temperatures and chiral perturbation theory at low temperatures. Specifically, the blue (green) band sets the anchor point for ChPT at $T_2 = 100$ (500) MeV. The width of each band reflects the uncertainty in the instanton computation of the free energy used as an anchor at $T = 1.5$ GeV, $F_0 \rightarrow (1/20, 20) \times F_0$, c.f. Eqs. (3.10),(3.15). The dashed line corresponds to the value of $F_0(1.5)$ obtained in the lattice calculation of Ref. [1].

$n \simeq 8$. If the Peccei-Quinn phase transition occurs after inflation, this factor of 10 leads to about a 15% decrease in the value of the axion mass required to account for the observed dark matter density.

To obtain a more accurate result for the late-time relic density, we solve the full axion equation of motion numerically through the time where it starts to oscillate. Fig. 3.1 shows the relic density obtained in this way for two values of T_2 and a range of χ_0 , in the post-inflationary PQ-breaking scenario ($\theta_0 = 2.155$). Compared to the analytic estimate (3.29), the full numerical solution yields marginally higher Ω for fixed m_a . Even with the factor of 5 variation in T_2 and the factor of 20^2 variation in χ_0 , we find that the axion mass required to account for all of dark matter

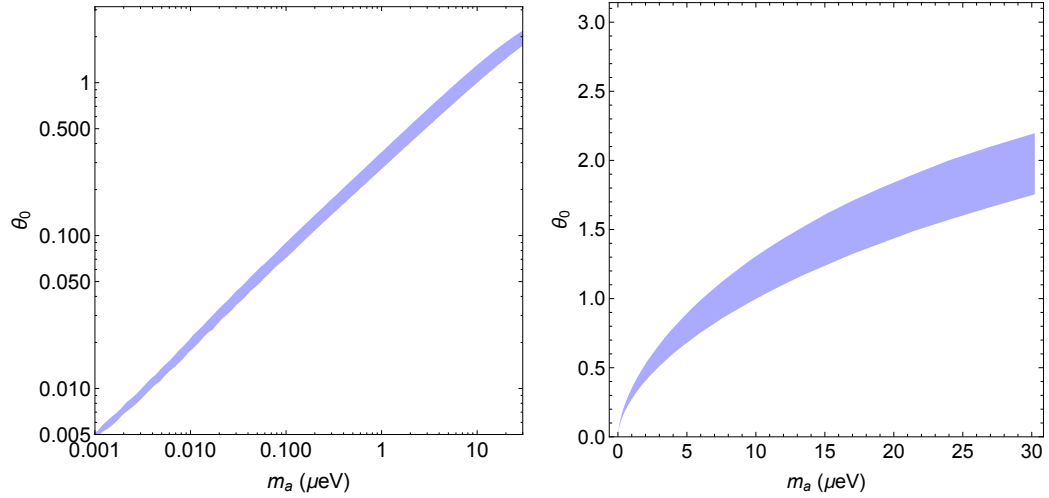


Figure 3.2: Axion relic density from misalignment in the pre-inflationary scenario. The curve shows the misalignment angle needed to obtain $\Omega = 0.258$. The band reflects the uncertainty in the instanton computation of the free energy, Eq. (3.15), used as an anchor at $T = 1.5$ GeV, and the anchor point for ChPT has been fixed to $T_2 = 140$ MeV. Left panel: log-log axes over a broad range of axion masses. Right panel: linear axes over a range of axion masses in reach of current and next generation ADMX [2].

varies by only a factor of 2-3.

Additional sources of axion production (cosmic strings) can force a larger axion mass. These masses are, indeed, at the edge of capability of cavity experiments like ADMX, and are the focus of much future planning. However, these sources of energy density, as well as the constraint on θ_0 , are not necessarily present in the early universe.

In fact, there is not necessarily a Peccei-Quinn transition at all [86]. The approximate Peccei-Quinn symmetry, if it exists, is almost certainly an accident. This accident may not occur at the high temperatures or high curvatures that characterize the early universe. In this case, the initial value of θ is a fixed number, or possibly one of a set of discrete numbers. This number might well be small, or might be $\mathcal{O}(1)$. Either has significant implications for the final dark matter density.

Alternatively, there may be an approximate symmetry both for low and high tempera-

ture (or curvature). The question of whether the symmetry is broken during or after inflation then depends, for example, on the coupling of the inflaton to the field responsible for PQ symmetry breaking. For example, there might be an effective mass term for this field, of either sign. There seems to be no particular reason to believe that one or the other outcome is favored.

These different possibilities have been extensively studied in the literature. If the symmetry breaking occurs after inflation, one has to average over random initial misalignment angles, which fixes the parameter θ_0 as above. In the case of symmetry breaking before inflation, θ_0 is a free parameter. In Fig. 3.2 we show the sensitivity of the misalignment angle required to saturate the relic density to the uncertainty in χ_0 . As in the post-inflationary case studied above, we find that the theoretical uncertainties have essentially no qualitative impact on the required parameters. Furthermore, for a wide range of $\mathcal{O}(1)$ values for θ_0 , the relevant axion masses are compatible with current and next-generation cavity experiments [59, 2].

3.5 Conclusions

Within the conventional picture of axion cosmology, we have found that the standard computation of the axion relic density is relatively robust against theoretical uncertainties stemming from the dilute gas computation of the QCD free energy at high temperatures and the behavior of the free energy at strong coupling. In particular, we have argued that the instanton computation is under sufficient control at temperatures of 1-2 GeV to allow a reasonable assessment of uncertainties due to higher-order corrections. These corrections cannot amount to much more than an order of magnitude in the free energy without an unexplained breakdown in the semiclassical analysis. In particular, we have argued that infrared divergences in the Debye mass in the perturbative vac-

uum are not relevant in the instanton background and cannot inject arbitrarily large corrections to $F(\theta, T)$. Thus, while an improved determination of the finite-temperature topological susceptibility would lead to improvement in the precision of the (relic density, axion mass) relation, it is not expected to lead to qualitative (order-of-magnitude) changes, and modern cavity experiments retain significant discovery potential.

However, in closing, we note that it has long been recognized that the underlying cosmological assumptions of the standard calculation may not hold, and that there is good theoretical motivation to consider lighter axions with larger decay constants. Within conventional effective field theory, for example, it is hard to account for the requisite quality of the Peccei-Quinn symmetry without invoking large discrete symmetries. String theory points to a different picture, in which the Peccei-Quinn symmetry appears more natural [87]. The assumption that the underlying mass scale is of order the Planck or unification scale is suggestive of larger decay constants. It could also be that four-dimensional effective field theory is not useful at scales orders of magnitude below the Planck scale, as in large or warped extra dimension scenarios, and early universe cosmology might be substantially modified.

Chapter 4

DebyeMass

4.1 Introduction

At high temperatures, non-abelian gauge theories undergo a phase transition to an unconfined phase. The high temperature theory exhibits two mass or length scales. The first of these is the Debye mass, $m_D \sim gT$, loosely speaking a scale beyond which static electric charges are screened. The second arises because the theory at high temperatures and long distances becomes a three (Euclidean) dimensional Yang-Mills theory with coupling $g_3^2 = g^2T$ without matter fields. The second scale is the mass gap of this theory, on dimensional grounds, $m_{mag} = g_3^2T$.

But the high temperature theory is not exactly a weakly coupled theory. If one attempts to formulate a perturbation theory, quantities like the free energy, and even short distance Green's functions, suffer severe infrared divergences, and can be calculated, at best at low orders. This is usually described by saying that these quantities are logarithmically divergent at some order,

diverging with an additional power of momentum each further order. Assuming an infrared cutoff

$$m_{mag}^2 = ag^4T^2 = ag^4T^2, \quad (4.1)$$

each additional order makes a comparable contribution. For the free energy, divergences first arise at order g^6 (four loop order). Gauge invariant Greens functions, like $\langle F^2(x)F^2(0) \rangle$ similarly exhibit such divergences at high enough order.

All of this arises because the theory, at high temperatures, is a three dimensional theory, with a dimensionful coupling $g_3^2 = g^2T$. At best, one can hope for a perturbation expansion valid for short distances or high momenta, $g_3^2r \ll 1$; $g_3^2/|p| \ll 1$. But loop corrections, even in these limits, are dominated, at sufficiently high order, by low momenta, leading to a breakdown of weak coupling.

On the other hand, if one computes a *Wilsonian* effective action for the three dimensional theory, integrating out momenta between scales

$$\frac{1}{\epsilon}g_3^2 < k < \Lambda \sim T \quad (4.2)$$

one should have a valid expansion in powers of ϵ .¹ The remaining contributions to physical quantities, Greens functions, and the like must be obtained from a fully non-perturbative analysis of the strongly coupled three dimensional theory. This suggests that quantities such as the free energy can be calculated as a sum of two parts: the perturbative, Wilsonian contribution, which can be obtained reliably, and the non-perturbative contribution. This latter is typically, on dimensional grounds, a power of g_3^2 times an unknown, dimensionless number. Assuming that the dimensionless number is of order one, this means that, with a straightforward (if possibly challenging) perturbative computa-

¹Our discussion can be viewed as a reframing of an analysis of Braaten and Pisarski[88].

tion, one can obtain an estimate of such quantities, accompanied by an error estimate, of irreducible size.

Applied to the free energy, as we will explain in section 4.2, this means that one can reliably compute through order g^4 . At order g^6 , there is a contribution which, again, can be reliably extracted proportional to the log of the ultraviolet cutoff, and a contribution without a log which cannot be obtained perturbatively. This non-perturbative contribution represents the irreducible uncertainty.

For the Debye length, we will see that there is a similar story. The existence of a mass – or correlation length – for A_4 is well known. At finite temperature, one does not have the full $O(4)$ (in Euclidean space) symmetry. At one loop, if one calculates the vacuum polarization tensor, gauge invariance and the remaining $O(3)$ symmetry are enough to insure vanishing of Π_{ij} as $\vec{q} \rightarrow 0$. However, this is not the case for Π_{44} at one loop. If q_0 is the discrete frequency of the finite temperature theory, one finds that for $q_0 = 0$, as $\vec{q} \rightarrow 0$,

$$\Pi^{44}(0, \vec{q}) \rightarrow m_D^2 \equiv g^2 T^2 (N + 3N_f) \quad (4.3)$$

In coordinate space, this mass for the A_4 field translates into a characteristic length scale. The A_4 Greens function, in leading order and at large distances, is given by

$$D^{44}(0, \vec{y}) = \frac{1}{4\pi|\vec{y}|} e^{-m_D y}. \quad (4.4)$$

As a result of these considerations, there is a scale, of order $\sqrt{g_3 T} \ll T$ (for small g_3), at which one has a three dimensional gauge theory with an adjoint scalar, ϕ , of mass $\mu = m_D$. Corrections to the Debye length have been considered in the literature[76, 77]. We will consider them from two points of view. We'll first examine the direct computation of $D^{44}(\vec{y})$ in perturbation

theory. We'll see that one can obtain a reliable estimate of the Greens function at distances parametrically large compared to μ^{-1} , where $\mu = m_D$, by a factor $(g_3^2 \log(\mu/g_3^2))^{-1}$. Beyond this scale, the computation of the Greens function, order by order in the perturbation expansion, is not under control.

But as explained in [77] and we elaborate further here, it is possible to define a gauge-invariant, non-perturbative Debye mass which controls the very large r behavior of the Greens function. As we explain, viewing the three dimensional theory as a *Minkowski* theory with an adjoint scalar, the theory has a Z_2 symmetry.² The mass of the lightest Z_2 odd state controls the Euclidean large distance behavior of Green's functions of Z_2 -odd operators, and is naturally defined as the Debye mass. One can obtain an estimate of this mass by studying the divergence structure of the perturbation theory, noting that there is logarithmic infrared sensitivity at one loop in the computation of this mass. But the structure of the perturbation expansion is complicated, with a variety of infrared singularities at higher orders. Instead, one can also use the Wilsonian language, as for the free energy, applied to a suitable non-relativistic effective action. Here there is, again, an ultraviolet divergence (the cutoff for the low energy theory is now $\mu = m_D$). Again, one obtains an *estimate* of the Debye length, as well as an irreducible, perturbative uncertainty.

But while providing, perhaps, a different language, for the free energy and the Debye length, this serves simply to confirm longstanding results. But our particular interest is in the nature of the semiclassical estimate of θ -dependent effects at finite temperature, interesting in themselves and relevant to the problem of axion cosmology. Here the object of interest is the free energy as a

²These are statements about QCD, in the absence of weak interactions. As noted in [77], in theories which are not vector-like, the low energy theory may not respect the Z_2 . In this case, strictly speaking, there is no sharp definition of the Debye mass. However the Z_2 breaking is often highly suppressed. For example, in non-vector-like theories without scalars, there is still a Z_2 , related to CP in the four dimensional theory. As a result, there is often a range of (large) distances where correlators invariant under the approximate Z_2 exhibit a rapid exponential falloff, even if, at extremely large distances, the falloff is power law.

function of θ and T , $F(\theta, T)$. Much of the literature, particularly the lattice literature, focuses on the topological susceptibility

$$\chi(T) \equiv \frac{\partial^2 F}{\partial \theta^2}. \quad (4.5)$$

The leading term as a function of g^2 can be computed in the dilute gas approximation, and is known[61]. At some order, one expects infrared divergences to arise as in the computation of the free energy in ordinary perturbation theory. One of the goals of this paper is to determine the order of the corrections to the leading semiclassical approximation at which these divergences arise. We will seek to determine the nature of these divergences, in order to assess the uncertainties in the standard computation of the finite temperature. We will examine the large distance behavior of Feynman diagrams in an instanton background, determining the order of the expansion in g_3^2 about the classical solution at which first logarithmic divergences and then power law divergences arise. We will interpret the log divergence, as for the perturbative free energy, as a term in a Wilsonian action involving the log of the cutoff ($\Lambda = T$). This will, again, permit us to make an estimate of χ and to determine an irreducible uncertainty in the semiclassical computation.

Apart from the intrinsic interest of finite temperature gauge theory, our work has been motivated, in part, by arguments in the literature for large corrections to the semiclassical computation of $\chi(F(\theta, T))$. In particular, it has been asserted that at one loop there is a correction to the instanton action which is (fractionally) of order $g \log g$ rather than g^2 and that at temperatures of interest it is numerically of order one. This effect is exponentiated in χ , and as such could lead to an uncertainty of orders of magnitude. This claim has been used, in turn, to argue that lattice computations are essential to determine the behavior of hypothetical axions in the early universe. Such computations have yielded values of χ varying by orders of magnitude at relevant temperatures,

both from each other and from the leading semiclassical result. [62, 68, 63, 1, 71].

The basis for these concerns is the assertion that the corrections to the Debye length which we have described above are large [76, 77] and that the Debye length acts as an infrared cutoff on the instanton size in the dilute gas approximation. As explained in [61], the Debye *mass term in the effective lagrangian* does provide an infrared cutoff, but as noted in [89], the actual cutoff in the instanton scale size in the computation of χ is T^{-1} . As a result, as we will explain further in this paper, uncertainties associated with the cutoff on the ρ integration are small.

Still, the θ -dependence of the free energy will, at some point, exhibit infrared sensitivity. One of the points of this paper will be to argue that the expansion for the topological susceptibility, χ exhibits infrared divergences at lower order than that for the perturbative free energy:

$$\chi(T) = aT^4(1 + bg^2 + cg^4(\log(g^2) + C)). \quad (4.6)$$

This follows from the explicit form of the instanton, and in particular its large distance behavior.

This paper is organized as follows. In section 4.2, we review the high temperature behavior of the free energy in QCD. In particular, we note that the “infrared” log is also an ultraviolet logarithm from the point of view of the three dimensional effective theory, and that the coefficient of this logarithm can be reliably calculated. As a result, there is a well-justified computation of the free energy, whose accuracy has an intrinsic limitation and whose error can be reliably estimated.

In section 4.3, we consider the perturbative calculation of the $\phi(A_4)$ Green’s function, with a focus on the large distance, coordinate space behavior. After reviewing the leading computation, we consider broad classes of Feynman diagrams. These become progressively more singular at large distances, and large, infinite sets are actually infrared divergent. Assuming a cutoff m_{mag} , one can establish that the leading correction dominates, up to some maximum distance.

In section 4.4, we explain that, with the assumption the three dimensional theory is gapped, there is a well-defined notion of the Debye mass, and we provide a definition. Then, in section 4.5, we turn to the question of to what extent we can estimate this mass, i.e. to what order in the perturbation expansion, and with what level of uncertainty. We will argue that this computation is robust.

In section 4.6, we turn to the question of the calculation of the topological susceptibility. We review some features of the finite temperature instanton computation, explaining that, at low orders in the semiclassical computation, the dominant instanton scales, ρ , are of order T^{-1} . Then we ask about the appearance of infrared divergences in this computation. We work in coordinate space; noting that at large distances, the instanton solution falls off rapidly, so that for purposes of isolating the infrared divergence, the modifications of the relevant Greens functions from their tree level forms are small. Essentially, we are able to treat the background instanton as a perturbation. We note that individual Feynman diagrams are actually divergent already at two loops, but gauge invariance requires that these divergences cancel, and the leading infrared divergence occurs at three loops (order g^4). Again, we argue that the computation of the logarithmic term at three loops is robust, and gives us both an estimate of the size of χ and of irreducible uncertainties.

Finally, in section 4.7, we consider the implications of these observations for some physical problems. We focus on the calculation of the finite temperature axion mass, which is proportional to χ . We note, again, that the cutoff on the instanton scale size integration is of order T^{-1} , but stress that this distance is not only parameterically small compared to the Debye length, but it is *even smaller* than the scale at which the large corrections to the Debye come into play. We note that the infrared divergence at order g^4 implies an error in the computation of the susceptibility at

the 1% level or better, which is not significant for the calculation of the axion energy density.

4.2 g^2 Expansion of the Free Energy

In this section, we consider the expansion of the free energy in perturbation theory in powers of g^2 . For the perturbative free energy, ignoring, at first, the adjoint scalar (i.e. A_4), there is formally an expansion of the form:

$$F(T) = T^4 \sum_{n=0}^{\infty} a_n g^{2n} \quad (4.7)$$

but this expansion breaks down due to infrared divergences at a certain order. This can be understood in terms of the behavior of the three dimensional theory, with effective coupling $g_3^2 = g^2 T$, and an ultraviolet cutoff $\Lambda \sim T$. Let's first ignore the adjoint scalar ϕ and consider the computation of a Wilsonian effective action in the pure gauge theory, integrating out physics between scales Λ and $\epsilon\Lambda$, where $\frac{g_3^2}{\epsilon\Lambda} \ll 1$. Then in the Wilsonian action, one obtains for the vacuum energy (coefficient of the unit operator) a series:

$$E_0 = a\Lambda^3(1 + \mathcal{O}(\epsilon)) + bg_3^2\Lambda^2(1 + \mathcal{O}(\epsilon)) + cg_3^4\Lambda(1 + \mathcal{O}(\epsilon)) + dg_3^6 \log(\epsilon). \quad (4.8)$$

In individual Feynman diagrams, the logarithmic behavior is readily identified by power counting. Higher order terms in the expansion are suppressed by powers of g_3^2/Λ . This is characteristic of a superrenormalizable theory. The computation of the Wilsonian action terminates at some power of the coupling.

From the requirement $\frac{g_3^2}{\epsilon\Lambda} \ll 1$, we have $1 \gg \epsilon \gg g^2(T)$. So thinking of ϵ as several times g^2 , say $\epsilon = Ag^2$, where $1 \ll A \ll \frac{1}{g^2}$, we can reliably say that the Wilsonian action includes a contribution to the coefficient of the unit operator (ground state energy) of the form of equation

4.8. In the low energy, cutoff theory, there will be a contribution to the energy of size $g_3^6 \log(A)$, which requires non-perturbative evaluation.

Including the adjoint field in the analysis, we might expect an expansion in μ^2 and g_3^2 . However, already at one loop, the expansion is actually an expansion in μ ; at one loop order, there is a contribution behaving as $\mu^3 T \sim g^3 T^3$ [90]. Higher orders yield more complicated dependence on μ .

Returning to the infrared perspective we discussed in the introduction, we saw that, beyond the $g^6 \log g$ term, there are an infinite number of perturbative contributions which are nominally of order g^6 . From the perspective of the three dimensional effective theory, assuming that the vacuum energy is well-defined, after integrating out modes well above g_3^2 , any further contribution necessarily scales as g_3^6 . Moreover, these contributions are not perturbatively accessible. Formally, we can attempt to tame the infrared divergences by various strategies. Apart from introducing a magnetic mass or simply cutting off momentum integrals at that scale, we can resum the contributions to the propagator, yielding $1/(g_3^2 k)$ behavior at small k . But it is easy to check that all diagrams beyond four loop order are of order g_3^6 .

4.3 Perturbative Computation of the Greens function at Large Distances and its Limitations

In the next section we will see that from knowledge of the three dimensional *Minkowski* theory we can obtain information about the large distance behavior of the Euclidean theory. We first consider the direct computation of the Green's function in coordinate space, and then move on to non-perturbative considerations.

At tree level, the Euclidean Greens function in coordinate space can be evaluated by Fourier transforming the momentum space expression. Performing the angular integrals, the remaining momentum integral (integral over p) can be treated as an integral in the complex plane. Deforming into, say, the upper half plane one picks up the pole at $p = i\mu$. This corresponds to the on shell point in the Minkowski description.

As one works to higher order in perturbation theory, the self energy, Σ , as we will shortly see, has a branch cut starting at $p = i\mu$. So now for the Greens function, deforming the contour, one encircles the branch cut. Calling the new integration variable δ , the integral involves a factor $e^{-(m+\delta)r}$. For large r , it is dominated by $\delta \sim r^{-1}$. In the Minkowski language, this corresponds to ϕ being off shell by an amount of order $\mu\delta \sim \frac{\mu}{r}$. So in *momentum* space, we are interested in $\Sigma(p)$ for $p^2 = \mu^2 - 2\mu\delta$, with δ small. We want to ask: how small can δ be and still yield a reliable estimate.

Consider, first, the one-loop contribution to the self-energy, $\Sigma(p)$ (figure 4.1). Work slightly off shell:

$$p^2 = \mu^2 + 2\mu\delta. \quad (4.9)$$

Then

$$\begin{aligned} \Sigma &= Ng^2 \int \frac{d^3k}{(2\pi)^3} \frac{4\mu^2}{k^2 + i\epsilon} \frac{1}{2\mu(\delta + k^0) + i\epsilon} \\ &= \frac{Ng^2}{\pi} \mu \log(\mu/\delta) \end{aligned} \quad (4.10)$$

Adding $\alpha \frac{k^\mu k^\nu}{k^2}$ to the propagator, it is easy to check that the logarithmic term in this expression is gauge invariant.

As we will now show, successive orders in the expansion of Σ are progressively more

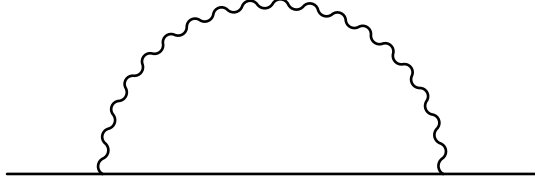


Figure 4.1: The leading contribution to the ϕ self-energy, $\Sigma(p)$.

singular in δ . Moreover, for fixed δ , one encounters actual infrared divergences at two loop order and beyond.

We can consider several classes of higher order perturbative corrections to $\Sigma(p)$ to illustrate the behavior of the perturbation expansion for small δ and to determine where it breaks down.

There are three issues:

1. Singular behavior for small δ
2. Actual infrared divergences
3. Divergent series for some value of δ and plausible infrared cutoff.

One class of diagrams involves “rainbows” of gluons emitted by ϕ (figure 4.2). For these, a typical contribution is of the form:

$$g_3^{2n} \int \frac{d^3k_1 \dots d^3k_n}{(2\pi)^{3n} k_1^2 \dots k_n^2} \frac{1}{2\mu(k_1 + \delta)} \frac{1}{2\mu(k_1 + k_2 + \delta)} \dots \frac{1}{2\mu(k_1 + k_2 + \delta)} \frac{1}{2\mu(k_1 + \delta)} \quad (4.11)$$

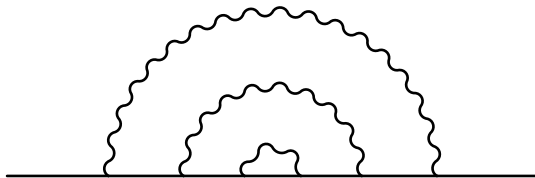


Figure 4.2: One class of diagrams singular in the limit $\delta \rightarrow 0$.

$$\sim g^{2n} \delta^{-(n-1)}.$$

If we restrict

$$\delta \geq \frac{Ng^2}{\pi} \mu \log(\mu/\delta), \quad (4.12)$$

then the perturbation series would appear to be an expansion in powers of $1/\log(\delta)$.

But other classes of diagrams leads to a stricter requirement. The first consists of “rainbows” as above, but each with a one loop vacuum polarization correction on the gluon line (figure 4.3). Such diagrams, by power counting, have an actual infrared divergences, and are also more singular as g_3^4/δ^2 for each additional loop. In other words, assuming an infrared cutoff of order g_3^2 ,

$$\frac{g_3^4 \mu \log(m_{mag}/\mu)}{\delta^2}. \quad (4.13)$$

So, for δ satisfying the condition 4.12, these contributions are all nominally of similar size, each suppressed by $\log(\delta) \sim \log(\mu/g_3^2)$ relative to the leading contribution.

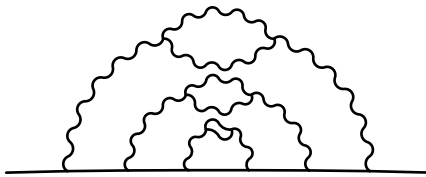


Figure 4.3: A second class of diagrams singular in the limit $\delta \rightarrow 0$, with actual infrared divergences.

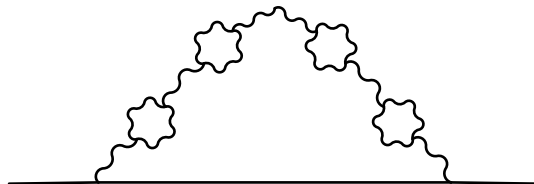


Figure 4.4: One class of diagrams singular in the limit $\delta \rightarrow 0$.

A third class of diagrams involve propagator corrections to the gluon line of the one loop contribution (figure 4.4). With n one-loop corrections to the gluon line, these are infrared divergent,



Figure 4.5: One class of diagrams singular in the limit $\delta \rightarrow 0$.

behaving as

$$\delta\Sigma^{(n)} \sim (g_3^2)^{n+1} \int \frac{d^3k}{k^2 k^n} \sim \frac{g_3^2 \mu}{n\delta} \frac{(g_3^2)^n}{m_{mag}^{(n)}} \quad (4.14)$$

These diagrams are individually suppressed at large δ , only by a single power of δ ; the series $\sum_n \frac{1}{n}$ is log divergent. Assuming that this sum is of order $\log(\mu/m_{mag})$, this requires, again, that $\delta \gg g_3^2 \log(\mu/m_{mag})$.

A similar requirement arises from the class of diagrams in figure 4.5. These behave, with n one-loop corrections to the ϕ propagator, as

$$\delta\Sigma^{(n)} \sim g_3^2 \int \frac{d^3k}{k^2(k+\delta)^n} (g_3^2 \log(\delta/\mu))^n \sim \frac{g_3^2}{n} (g_3^2 \log(\delta/\mu))^n. \quad (4.15)$$

So at each order in n , we have a comparable contribution, suppressed by a logarithm relative to our leading contribution. The sum would appear poorly behaved. But if we sum before integration, we obtain a correction to the leading contribution down by a logarithm, in other words, of order g_3^2 .

Of course, there are many other diagrams – mixtures of these various types and other topological classes altogether. The main lesson is that, *at best*, one can calculate for $\delta \gg \delta_0 = \frac{g_3^2 N}{2\pi} \log(\mu/g_3^2)$. With this understanding of the behavior of Σ , let's consider the fourier transform of the propagator in Euclidean space,

$$G(r) = \int \frac{d^p k}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \frac{1}{k^2 + \mu^2 - \Sigma(p)} \quad (4.16)$$

Performing the angular integrals leaves:

$$G(r) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} \frac{1}{ir} \frac{e^{ipr}}{p^2 + \mu^2 - \Sigma(p)}. \quad (4.17)$$

The integrand has branch cuts starting at $p = \pm i\mu^2$. Treating as a contour integral and deforming so as to encircle the branch cut in the upper half plane, the integral becomes:

$$G(r) = \int_{\mu}^{\infty} \frac{d\delta}{(2\pi)^2} \frac{1}{r} e^{-\delta r} \text{Disc} \frac{1}{2\delta\mu + \Sigma(\delta)}. \quad (4.18)$$

The main noteworthy feature here is that for large r , $\delta \sim \frac{1}{r}$. So for sufficiently large r ,

$$r \approx (\log(\frac{\mu}{g_3^2}) g_3)^{-1} \quad (4.19)$$

one has lost control of the expansion of $\Sigma(p)$, and the computation of the propagator has broken down. Still, parameterically in g_3 , there is a large range of distance where the propagator can be computed.

In the following sections, we will see that general arguments establish that the falloff of the propagator is that of a massive scalar field. The corrections to the mass can be estimated with a definite error. The results are consistent with our estimates above, in that the propagator is exponentially modified from its perturbative form at distances $r > \delta_0$, but the error on the exponent is controlled.

4.4 Defining the Debye Mass

At high temperatures, as is well known, four dimensional field theories with massless fields behave as three dimensional systems. In the case of Abelian gauge theories with light matter, and general non-abelian theories, the field A_4 of the four dimensional theory behaves like a massive field, with mass parameterically less than T by a single power of g , the gauge coupling. In this section, we'll consider a theory with a massive adjoint field, ϕ , with lagrangian mass parameter μ^2 , and attempt to determine the behavior of the charged field propagator at large distances.

In QCD, the resulting effective theory has a Z_2 symmetry, arising from the four dimensional symmetry of the Euclidean theory $A_4 \rightarrow -A_4$, $t \rightarrow -t$. Calling $A_4 \equiv \phi$ (with corresponding transformations on the fermions), the symmetry takes $\phi \rightarrow -\phi$ in the three dimensional theory.

To extract properties of the theory at large Euclidean distances, it is helpful to consider the theory continued to three dimensional *Minkowski* space (the argument which follows is familiar in lattice gauge theory and other contexts). We expect in the three dimensional system, there will be states odd under the Z_2 , which are bound states of ϕ and gluons. Suitable interpolating fields for such states would be:

$$\Phi = \text{Tr}(\phi F_{\mu\nu}^2), \quad (4.20)$$

and operators with more $F_{\mu\nu}$ and/or D_μ factors.

If we continue the three dimensional *Euclidean* theory to Minkowski space, we can write a spectral representation of the Green's function:

$$G_\Phi(x) = \langle \Phi(x)\Phi(0) \rangle = \int dM \rho(M) D_F(x, M). \quad (4.21)$$

In 4.21, $D_F(x, M)$ is the free propagator for a field of mass M . If the spectrum is gapped, we can write

$$\rho(M) = Z\delta(M - M_{phys}) + \theta(M - M_0)f(M) \quad (4.22)$$

where we will refer to M_{phys} as the *physical mass* of the lightest Z_2 -odd particle, and $M_0 > M_{phys}$. Then we can continue eqn. 4.21 back to Euclidean space, and (in three dimensions) show that the asymptotic behavior of the propagator is:

$$G_\Phi(x) = \frac{Z}{8\pi r} e^{-M_{phys}r} \quad (4.23)$$

for $r = |\vec{x}| \gg M_{phys}^{-1}$. So the spatial falloff of the propagator is determined in terms of M_{phys} . This quantity is gauge invariant. If we can estimate M_{phys} in some regime where the perturbation expansion is reliable, then, using unitarity, we can continue to the regime of arbitrarily large distance, with an error in the estimate which we can hope to control. M_{phys} we define to be the Debye mass.

4.5 Calculating the Debye Mass

In this section, we determine how well one can estimate the Debye mass, and compare with the analysis of section 4.3.

4.5.1 Non-Relativistic Effective Theory (NRET)

In our discussion of the perturbative free energy, we were able to argue that the *infrared* logarithm at four loop order was robust; it could be understood as a renormalization of the unit operator of the effective three dimensional theory at low energies. Given that our focus, for the

Debye mass, is also on infrared (long distance) issues, we can ask whether we can isolate a similar *ultraviolet* logarithm.

We can consider the problem from the point of view of non-relativistic effective field theory. For scalars (see, for example, [91]), one conventionally defines:

$$\phi = \frac{1}{\sqrt{2m}} e^{-imv \cdot x} \chi; \quad v^2 = 1. \quad (4.24)$$

The action for χ is then:

$$i\chi^* v_\mu D^\mu \chi. \quad (4.25)$$

At one loop, there is an ultraviolet and infrared divergent correction to the χ propagator, similar to the one loop correction to Σ which has been our focus:

$$\frac{g_3^2 N}{2\pi} \log(\Lambda/m_{mag}). \quad (4.26)$$

For the non-relativistic theory, the ultraviolet cutoff is the mass, μ . This divergence is cured by a counterterm for the operator $\chi^\dagger \chi$; it corresponds to a mass shift in the non-relativistic theory. Because the operator has dimension 2, and g_3^2 has dimension one, there is only an ultraviolet divergent correction at one loop. We can again think of the logarithm as a term arising in a Wilsonian effective action from integrating out high energy modes of the gauge field, in this case, between the ultraviolet cutoff, μ , and a scale

$$\lambda = \frac{1}{\epsilon} g_3^2. \quad (4.27)$$

In principle ϵ is a small number, $1 \gg \epsilon \gg \frac{g_3^2}{\mu}$.

This counterterm eliminates all μ -dependence in the perturbation expansion. The theory does contain the parameter $\log \epsilon$, which can be thought of as an order one number. Thus the effective

theory has only the dimensionful parameter g_3^2 . So necessarily any further correction to the mass, beyond the counterterm, is a constant times g_3^2 . In other words, this argument establishes that

$$m^2 = \mu^2 + \frac{g_3^2 N}{2\pi} (\log(\mu/g_3^2) + A). \quad (4.28)$$

Note that this is in accord with our discussion of the previous section. We can compute, in conventional perturbation theory, the Green's function out to distances of order $r \ll \frac{1}{g_3^2 \log(\mu/g_3^2)}$. Beyond this distance scale, the corrections quickly become large compared to one. The non-perturbative analysis gives us a reliable estimate of the Green's function to scales of order $r \sim g_3^{-2}$. Note that both of these scales are parameterically large compared to μ^{-1} . As we have remarked, for the question of θ dependence of the free energy, one is interested in much shorter distance scales.

4.5.2 Calculation of NRET Analysis to Conventional Perturbation Theory

The argument based on non relativistic effective theory gives a sharp, principled argument that one can calculate, perturbatively, a correction of order $g_3^2 \mu \log(\mu/g_3^2)$ to the mass of ϕ , while the remaining correction is of the form $A g_3^2$. We'd like to verify this structure in a more conventional perturbative analysis. In particular, for the order by order coordinate space computation of the Greens function, we identified several classes of Feynman diagrams. Here we would like to verify that, apart from one loop corrections with logarithmic sensitivity to μ , all corrections are proportional to g_3^2 .

The results of the NRET analysis of the previous section can be understood in a more conventional Feynman diagram analysis. Dividing up the integration over the gluon momentum into two regions, one with momentum $k > k_0 = \frac{1}{\epsilon m_{mag}}$, and one with $k < k_0$, we also take the

external momentum to satisfy

$$p^2 = \mu^2 + \delta_0 + \delta'; \quad \delta_0 = \frac{Ng^2}{2\pi} \log(\mu/k_0). \quad (4.29)$$

where δ_0 represents the lowest order mass shift. We take the propagator to be the resummed propagator, with the one loop contribution to Σ . With this choice, all higher loop contributions to Σ are dominated by the infrared, and are thus insensitive to μ , apart from an overall factor. They behave as $(g_3^2\mu)(g_3^2/\delta')^n$. The mass shell condition takes the form

$$\Sigma(\delta') = \mu\delta'. \quad (4.30)$$

So assuming $\Sigma(\delta') = g_3^2\mu A(\frac{g_3^2}{\delta'})$ we have that $\delta' = ag_3^2$ for some constant a .

4.5.3 Implications of the Debye Mass Calculation for the Topological Susceptibility

As we have remarked, one situation where it has been suggested that large corrections to the Debye mass might be important is in instanton computations of the free energy (topological susceptibility) at high temperatures[1, 71, 67]. It has been argued that the instanton computation is proportional to a large power of an infrared cutoff, and that this cutoff is the Debye length. But as noted in [89] the actual cutoff is T (or T^{-1} in coordinate space)[61]. As we have now seen, this is a regime where perturbation theory is valid, at least until one encounters magnetic divergences at high orders. These corrections are not likely to be numerically large, or terribly important for estimating, for example, the axion dark matter density[89].

4.6 Infrared Sensitivity in the Instanton Computation

While we have argued that the Debye mass is not the relevant cutoff for the instanton computation, we do expect actual infrared divergences to appear at some order; in other words, we do not expect to be able to perform a semiclassical computation of the topological susceptibility to arbitrary accuracy. In this section, after first considering the question of what does play the role of the cutoff on the ρ integration at large ρ , we turn to a determination of the order in perturbation theory infrared divergences actually arise in the computation of the topological susceptibility. In the spirit of our earlier Wilsonian analyses, we use this result to determine the irreducible uncertainty of the semiclassical computation.

4.6.1 Instantons as a Perturbation at Large Distances

At finite temperatures, for $r = |\vec{x}| \gg \beta$, the instanton takes the form:

$$A^{ia} = \frac{\epsilon_{aij}x^j}{(r^2 + r^3/(\rho^2 T))} \quad A^{4a} = -\frac{x^a}{(r^2 + r^3/(\rho^2 T))}$$

From a three dimensional perspective, the instanton solution is well behaved at large distances, with \vec{E} and \vec{B} fields falling off as $1/r^2$, but singular at short distances. The temperature and the scale size (which is of order T) act as short distance cutoffs, yielding a finite action. This is complicated to describe from our Wilsonian perspective. In addition to generating contributions to local operators, the short distance physics yields boundary conditions for the three dimensional classical solutions as well as an integration measure. Rather than describe the process of integrating out short distance physics in this way, we will content ourselves with an examination of corrections to the leading semiclassical approximation, isolating contributions which behave as $\log(T/m_{mag})$ and $\log(\rho m_{mag})$. From this analysis, we will infer the extent to which one can estimate χ , and the

irreducible uncertainty.

If we want to investigate actual infrared divergences, we need to study loop corrections to the instanton computation in this background. Because we are interested in effects at large distances, we are interested in integration regions where the fields of the instanton are small, and Greens functions of the fluctuating fields are close to their free field expressions. In particular, we can attempt to treat the instanton as a perturbation. By this we mean that we break up the fields as

$$A_4 = A_{inst}^4 + a_4; \quad A^i = A_{inst}^i + a^i. \quad (4.31)$$

Then there are interactions involving two fluctuating fields, a^i proportional to one or two powers of the background field, and three powers of the fluctuating fields and one power of the background field. For the instanton fields, we will take the large r limits. We will need to integrate over collective coordinates for translations, dilations and rotations. The integral over ρ will be controlled by the same exponential terms as in the leading approximation, up to small corrections. As a result, the dominant ρ is of order T^{-1} . The rotational collective coordinate is simple to deal with as χ is itself rotational and gauge invariant.

The translational collective coordinate, \vec{x}_0 , requires closer attention. In perturbation theory at order n , we will have vertices labeled by x_i , $i = 1, \dots, n$. At vertices with insertions of the instanton field, $A_{inst} = A_{inst}(\vec{x}_i - \vec{x}_0)$. We will also have products of free Greens functions (and derivatives), $\Delta(x_i - x_j)$. So if we shift $\vec{x}_i \rightarrow \vec{x}_i + \vec{x}_0$, the integral over \vec{x}_0 factors out, yielding the factor of volume appropriate to the three dimensional vacuum energy. The free propagators, in coordinate space, are simply

$$\Delta(\vec{x}_i - \vec{x}_j) = \frac{g_3^2}{4\pi} \frac{1}{|\vec{x}_i - \vec{x}_j|}. \quad (4.32)$$

The asymptotic behavior of the instanton is:

$$(A_4)_{inst}^a = \frac{\rho^2 T x^a}{x^3} \quad (A_{inst}^i)^a = \frac{\rho^2 T \epsilon_{aij} x^j}{x^3}. \quad (4.33)$$

So formally, in the large distance regime we have an expansion in powers of g_3^2 and $\rho' = \rho^2 T$. We can power count on diagrams at $n + 1$ loops. These will have n factors of g_3^2 . Then if there are m insertions of the background field we have m factors of $\rho' = \rho^2 T$. Schematically, the graph has the structure

$$(g_3^2)^n \rho'^m \prod \int d^3 x_i^{m+2n} \partial_i^{m+2n} \frac{1}{|\vec{x}_i - \vec{x}_j|^{3n+m}} \quad (4.34)$$

where the partial derivatives are meant to indicate vertices with derivatives and the factors of $\frac{1}{|\vec{x}_i - \vec{x}_j|}$ to indicate the number of propagators. Then we can assign a "superficial degree of divergence", $n - m$ to each graph. Then if

1. $n < m$, the graph has power law divergence in the ultraviolet, corresponding to domination of the contribution to the Wilsonian action by high momenta. Such diagrams will yield powers of T relative to the leading contribution.
2. $n = m$, the graph is logarithmically divergent in the ultraviolet and infrared, similar to the g^6 contributions to the perturbative free energy.
3. $n > m$, the diagram exhibits a power law divergence in the infrared, and should be thought of as a contribution from the low energy, three dimensional theory.

For $n = m = 1$, however, the relevant Feynman diagrams vanish. So the action indicates infrared sensitivity first at order $g_3^4 \rho'^2$ (figure 4.6).

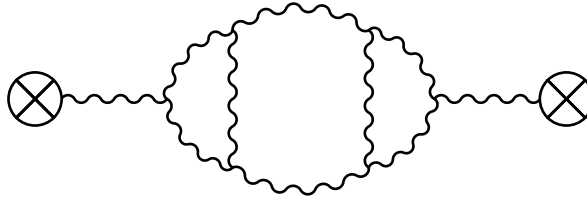


Figure 4.6: Infrared divergent contributions to χ at three loops.

4.6.2 Subtleties at Two Loops

At two loop order, there are diagrams which, individually, are ultraviolet and infrared divergent. These arise if, for example, we consider an insertion of two instanton fields at a point, and integrate over the instanton pair. Graphically, this (and similar contributions) correspond to examining the effects of the instanton on the large distance behavior of the A^i two-point function. This effect can be summarized in terms of the insertion of a local operator. Here it is necessary to consider

$$\langle A^i(\vec{z}_1) A^j(\vec{z}_2) \rangle \quad (4.35)$$

The corrections which arise from insertions of two powers of the background instanton field have the form::

$$\delta \langle A^i(\vec{z}_1) A^j(\vec{z}_2) \rangle = g_3^2 \int \frac{d\rho}{\rho^4} \int d^3 x_0 f(\rho) \int d^3 z_3 \frac{1}{|\vec{z}_1 - \vec{z}_3|} \frac{1}{|\vec{z}_2 - \vec{z}_3|} A_{inst}^2(\vec{z}_3 - \vec{x}_0). \quad (4.36)$$

Shifting $z_3 \rightarrow z_3 + x_0$, if one estimates the integral over z_3 by ignoring the z_3 dependence of the first two propagators, one has:

$$\delta\langle A^i(\vec{z}_1)A^j(\vec{z}_2)\rangle = g_3^2 \int \frac{d\rho}{\rho^4} \int d^3x_0 f(\rho) \frac{1}{|\vec{z}_1 - \vec{x}_0|} \frac{1}{|\vec{z}_2 - \vec{x}_0|} \int d^3z_3 A_{inst}^2(\vec{z}_3). \quad (4.37)$$

The z_3 integral is convergent and dominated by scales of order ρ . Since $\rho \ll |\vec{z}_1|, |\vec{z}_2|$, this is a short distance effect. This is the result one would obtain from a term in the effective lagrangian

$$\delta\mathcal{L} = \mu^2 (A^i)^2, \quad (4.38)$$

with μ^2 the result of the z_3 integral above. But such a term is not gauge invariant, so the leading short distance contributions must cancel. The lowest dimension gauge invariant operator is

$$\delta\mathcal{L} = F_{ij}^2. \quad (4.39)$$

This insertion of this operator at two loops does not lead to an expression which is infrared divergent.

So the leading divergence arises at three loops. The form of the susceptibility is:

$$\chi(T) = \chi_0(T)(1 + ag_3^2/T + bg_3^4 \log(T) + cg_3^4) \quad (4.40)$$

where a and b can be computed in the semiclassical approximation, but c requires a lattice computation. This last term represents, again, the irreducible uncertainty in the semiclassical analysis.

4.7 Conclusions

After reviewing the nature of the leading infrared effects in high temperature perturbation theory, we have studied two physical quantities in high temperature QCD: the Debye mass and the topological susceptibility. For χ , we have seen that uncontrolled infrared divergences first arise at

order g^4 . The leading semiclassical approximation would appear to be reliable at the fraction of a percent level.

As we have reviewed, the Debye length – which determines the exponential falloff of the A_4 Green's function (or the gauge invariant Green's function which we have discussed) at very large distances – is not critical to understanding the behavior of the susceptibility. But it is interesting in its own right. We have explained why, even though fundamentally a strong coupling problem, one can obtain a reliable estimate for this length. This involves carefully considering the fact that the system is gapped and the structure of the perturbation series. We have seen that, as a Minkowski theory, one can calculate the leading order contribution to the position of the pole, which is larger by a logarithm than the expected uncertainties due to confinement. Infrared divergences appear in this computation at precisely the order where one expects confinement effects to be important. We have stressed, as in [89], that these large corrections to the Debye mass are not important for the calculation of the susceptibility.

Chapter 5

Anthropic Theta

5.1 Introduction

At present, however frustrating it may be, the most compelling solution we have to the cosmological constant (c.c.) problem is provided by the anthropic landscape[92, 93, 94, 95]. The picture scored an enormous success with the *discovery* of the dark energy[96, 97], with a value only somewhat smaller than expected from the simplest version of Weinberg's argument. More refined versions of the argument may come closer. This success has raised the specter that anthropic considerations may play an important role in determining *all* of the laws of physics. But there are good reasons for skepticism. One might expect, in such a picture, that the parameters of the Standard Model should either be anthropically determined or should be random numbers. A principled objection to these ideas, then, emerges from the fact that some parameters of the Standard Model appear to be neither random nor anthropically constrained[98]. Possibly the most dramatic of these is the θ parameter[98, 99, 100]. If a landscape picture is ultimately to make sense, one needs to find correlations between the θ parameter and other quantities that are anthropically constrained. For

example, it is conceivable that dark matter is an anthropic requirement[101], and it might be that the most efficient way to obtain dark matter in a landscape is through an axion. This is a tall order. In particular, the requirement of a dark matter axion does not necessarily imply a Peccei-Quinn symmetry of sufficient quality to explain $\theta < 10^{-10}$ [102].

The notion of a landscape can hardly be considered well-developed; we don't have gravity theories in which we can reliably demonstrate the existence of even a small number of non-supersymmetric vacua, whereas we require a vast number of states. At best, we have toy models for the phenomenon, with which we can develop speculations about questions like statistical distributions of Standard Model parameters. Because we require both an understanding of the microphysics of the landscape and its cosmology, even the precise questions of greatest interest are not clear. But we might like to know, among the subset of states with Standard Model degrees of freedom at low energies, what is the distribution of Lagrangian parameters, $P(\Lambda, m_H^2, g_i, y_{f\bar{f}}, \theta)$, where Λ is the c.c., g_i are the gauge couplings, $y_{f\bar{f}}$ are the Yukawa couplings, m_H^2 is the Higgs mass, and θ is the QCD θ parameter. It seems plausible, as Weinberg assumed, that the cosmological constant is a uniformly distributed random variable near its observed value. The question of *naturalness* of the Higgs mass is the question of whether the same is true for the Higgs mass. If the distribution is uniform, then one would seem to require an anthropic explanation of the Higgs mass, or alternatively rely on an extraordinary piece of luck. On the other hand, supersymmetry or dynamics could enhance the probability that one finds the mass near zero, providing a realization of supersymmetry in a landscape framework.

The problem of θ in a landscape is that, absent a light axion, it is not clear why a distribution of θ should peak at $\theta = 0$. As we will explain below, in a model, say, like that of KKLT[95], one

would expect that θ is a discrete, uniformly distributed random variable. As we have noted, there is no obvious anthropic preference for extremely small θ . For critics of the landscape program, this is perhaps the most principled argument that the landscape idea may not be correct. For it to survive, one almost certainly has to find correlations between θ and other quantities which *are* anthropically constrained.

It is possible that small θ is selected by some other consideration, dark matter being one possible candidate. Recently, Kaloper and Terning (KT)[10] have put forward another proposal to account for small θ , which would correlate the value of θ with the problem of the cosmological constant. In this note, we will attempt to flesh out this proposal, determining what is required at a microscopic level to realize their picture, and what might be the parameters of such a model and their possible distributions. Then we ask in what range of parameters one would in fact account for a small value of θ .

KT assume that the cosmological constant has two contributions, one from a structure similar to that of Bousso and Polchinski (BP)[93], which we will denote by Λ_{BP} , and another, independent, one from QCD.

$$\Lambda = \Lambda_{BP} + \frac{1}{2}m_\pi^2 f_\pi^2 \theta^2. \quad (5.1)$$

The typical spacing of values of Λ_{BP} will be denoted as $\Delta\Lambda$; one can think of this as roughly some fundamental scale raised to the fourth power (say M_p^4) divided by the number of states. KT assume that θ is a continuous variable, and argue that, for a range of $\Delta\Lambda$, a small, negative value of Λ_{BP} will be compensated by a small θ , bringing the c.c. into the anthropically allowed range.

But this picture raises several puzzles. In our generic landscape picture, above, we would expect that θ is some combination of axion expectation values in the underlying theory. If θ can be

considered to be independent of the values of the fluxes, and if QCD is the dominant source of the θ potential, then we would seem to have a conventional axion.

On the other hand, we might expect the axion expectation values to be determined, in a scenario like that of BP, by values of fluxes or other quantities which label the different states. In this picture, the θ angles would take on random, discrete values, not obviously correlated with other quantities. The KKLTT[95] model provides a sharp implementation of this picture. In that model, the axion is not light. There, θ is fixed by the expectation values of Kähler moduli, which are themselves fixed by the (random) expectation value of the superpotential of complex structure moduli. This superpotential, $\langle W \rangle$, itself is complex. In order that θ vanish, one might require that the expectation value of the Kähler modulus be real. This would be the case among the tiny subset of fluxes (a fraction of order $(1/2)^N$, where N is the number of flux types) which conserve CP[103]. But this might not be enough once one has identified the origin of the CKM phase. One would require an additional layer of structure, similar to that of Nelson-Barr models[104, 105].

It is also possible that in the would-be string landscape there are typically one or more light axions, yielding a conventional Peccei-Quinn solution of the strong CP problem. Such a prospect leads to the notion of an *axiverse*[106]. The criteria for a solution of the strong CP problem in such a setting have been elaborated in [107].

All of this is to say that a “conventional” landscape picture suggests that θ should be a discrete variable, with no obvious correlation with questions like the value of the c.c..

An alternative is that the axions are heavy, with, for a fixed choice of the fluxes, a *very* large number of nearly degenerate minima, with the degeneracy lifted by (the *very* small) effects of QCD. For θ to play any role in determining the c.c., one needs, in this case, very small steps in θ ,

$\Delta\theta$. We call Λ_a the upper limit of the anthropic window, which we assume close to the observed value, i.e. $\Lambda_a \sim 10^{-47} \text{ GeV}^4$. Then a minimal requirement on $\Delta\theta$ is that it leads to steps in V , ΔV , small enough to bracket Λ_a :

$$|\Delta V| \approx \frac{1}{2} m_\pi^2 f_\pi^2 \Delta\theta^2 < \Lambda_a \quad (5.2)$$

yielding

$$\Delta\theta < 10^{-22}. \quad (5.3)$$

In this case, it would not be critical that fluxes conserve CP. The minimum of the θ potential, for any choice of fluxes, would lie at the CP conserving point for QCD; smaller θ contributions to the c.c. would be associated with smaller values of θ . We will develop several models of this type, in the spirit of outlining some of the ingredients required to achieve a correlation between θ and the c.c. Modest numbers of states can be accounted for within conventional field theory principles. The extension of these models to account for small enough $\Delta\theta$ requires features which are not particularly plausible. Conceivably there is some more plausible structure which could give rise to these features.

Allowing for such a structure, we then ask: does the requirement that the observed cosmological constant lie within the anthropic window, $0 < \Lambda < \Lambda_a$, favor any particular range of θ ? We will survey a two-parameter space: $\Delta\Lambda$, the typical spacing of cosmological constants in the flux vacua; and $\Delta\theta$. We find that throughout this parameter space, the anthropic constraint on the dark energy favors *large* θ .

The rest of this paper is organized as follows. In section 5.2, we review the problem of obtaining suitable axions assuming an anthropic requirement of dark matter[102]. We consider

models where a discrete Z_N symmetry accounts for an accidental Peccei-Quinn symmetry, and ask whether minimal requirements for dark matter yield an N large enough to account for $\theta < 10^{-10}$. We also mention the (possibly different) expectations for string theory.

In section 5.3, we consider models which yield a discretuum for θ . We first note that the *irrational axion*[108] has many of the desired features. It is not clear that such a structure ever arises in some more fundamental theory (string theory), and we will see that, in any case, it would tend to predict large θ . We then construct two models which implement at least some aspects of the KT program. One involves a single axion coupled to a *very* large additional gauge group; the other involves multiple axions and requires an intricate discrete symmetry. These models are useful in that they *do* allow us to address a subset of the questions we have raised:

1. In the theory θ is discrete, and the potential behaves as $V = -m_\pi^2 f_\pi^2 \cos \theta$.
2. The system is described in terms of two parameters, $\Delta\Lambda$, the typical spacing of the BP contribution to the c.c.¹, and $\Delta\theta$, the spacings in θ .

We will describe another model in the appendix. In section 5.4, we will ask, terms of these parameters, where are the bulk of the states which satisfy the anthropic condition. We will see that throughout the parameter space, cancellation of the c.c. (to within anthropic constraints) is most effective at $\theta \sim 1$.

¹We will use “BP” to refer more generally to features of the theory, other than θ , which allow for many possible values of the c.c.

5.2 Anthropic Axions

It is conceivable that dark matter, with something close to its observed density, is an anthropic requirement[101]. In that case, the question becomes: does the requirement of dark matter lead to a Peccei-Quinn symmetry of high enough quality to account for the smallness of θ [102]? One might imagine that in a landscape setting, an axion might be a favorable dark matter candidate. Models based on string constructions, for example, often have many axions, and there may be a significant fraction of the space of vacua in which one or more of these is very light. We might model this by a field, ϕ , subject to a Z_N symmetry,

$$\phi \rightarrow e^{\frac{2\pi i}{N}} \phi, \quad (5.4)$$

leading to an approximate $U(1)$ symmetry,

$$\phi \rightarrow e^{i\alpha} \phi. \quad (5.5)$$

We will assume suppression of higher dimension operators by the scale M_p . Suppose that the leading PQ symmetry-violating operator is:

$$\delta V = M_p^{4-N} (\gamma \phi^N + \text{c.c.}). \quad (5.6)$$

Writing $\phi = f_a e^{ia/f_a} \equiv f_a e^{i\theta}$, and assuming $\gamma \sim 1$, with an order one phase, the effective θ is roughly

$$\theta = \text{Im} \gamma \frac{M_p^4}{f_a^2 m_\pi^2} \left(\frac{f_a}{M_p} \right)^N. \quad (5.7)$$

If, say, $f_a = 10^{11}$ GeV, then $\theta < 10^{-10}$ requires $N \geq 12$.

Consider, instead, the requirement that the axion be the dark matter. In [102] a number of cosmological scenarios were considered, resulting in different constraints on N . But a minimal

requirement is that the lifetime of the axion should be longer than the age of the universe². Again with $f_a = 10^{11}$ GeV, the requirement is $m_a < 10^{-3}$ GeV, or $N > 9$. This is a weaker requirement on N than the demands of θ . More stringent requirements can arise from a detailed cosmological picture. For example, things may be different if there is some approximate supersymmetry and the universe is dominated by a saxion for some period. Depending on the details, the requirement of dark matter can sometimes account for a small enough θ . It is also possible, of course, that one has a bit of luck — that N is somewhat larger than it needs to be.

String theory suggests different possibilities[102]. The small parameter might be something like $A = e^{-\frac{2\pi}{\alpha}}$, with α some small coupling constant. A might be extremely small; if the dark matter requirement is a suppression of the mass by A^2 , for example, this might also be sufficient to account for the quality of the Peccei-Quinn symmetry.

One concludes from this that it is *plausible* that anthropic considerations could favor an axion suitable to solve the strong CP problem, but that it is by no means certain; many cosmological and microphysical details would need to be understood to settle the question.

5.3 Models Which Achieve a θ Discretuum

The basic structure of the potential of Kaloper and Terning is rather puzzling. In their picture, θ is continuous, and its potential, for any choice of the fluxes, has a minimum at $\theta = 0$. The QCD contribution to the potential dominates over other microscopic contributions. It is not clear how θ , as a continuous variable subject to a superselection rule, scans. (KT refer to an earlier paper

²Conceivably, the lifetime could be slightly shorter, being constrained only by the requirement of structure formation. Alternatively, effects of the radiation due to decaying axions could yield a stronger constraint. Observationally, the constraint is significantly stronger[109]. This range translates into about six orders of magnitude in m_a^2 , which is typically a change of 1 or 2 in the constraint on N , for a reasonable range of f_a .

of Linde's[110] which does not really provide a precise picture). But if somehow θ is selected from this distribution, it must be compatible with the anthropic bound for the c.c. Then, at least for some range of parameters, they argue, small cosmological constant implies small θ .

In this section, instead, we consider models which create a discretuum of values of θ , and ask about the distribution of ground state energies with θ . High energy dynamics give rise to a large number of degenerate vacua; the degeneracy is lifted by QCD.

We start by revisiting the idea of the *irrational axion*[108], noting that this is, in fact, a possible setting for these ideas. However, it is not clear whether the irrational axion is realized in any underlying theory, so we then consider two other possible models. These models are more concrete. On the other hand, while these theories generate a discretuum with a non-zero θ , to actually play a role in the cosmological constant problem, the discretuum must be extremely fine, and the theories then exhibit some rather implausible features.

5.3.1 Prelude: The Irrational Axion

The irrational axion is a hypothetical setting with many vacua with different θ [108]. If it were realized in an underlying theory, it might provide a setting for the ideas of [10]. In the irrational axion proposal, the θ potential receives contributions with different periodicities, which are not rational multiples of one another. This could arise from two groups, for example, with couplings to the same axion,

$$\sum_{i=1}^2 \frac{a}{16\pi^2 f_a} q_i F_i \tilde{F}_i, \quad (5.8)$$

where q_1 is not a rational multiple of q_2 .

Then, taking the group 2 to be the Standard Model $SU(3)$, and the group 1 another group

with scale $M^4 \gg m_\pi^2 f_\pi^2$,

$$V(\theta) = -M^4 \cos(q_1(\theta - \theta_0)) - m_\pi^2 f_\pi^2 \cos(q_2\theta). \quad (5.9)$$

For simplicity take $q_1 = x$, with x irrational, and $q_2 = 1$. Then the system has an infinity of nearly degenerate vacua with $\theta \approx \frac{2\pi n}{x} + \theta_0$, hence a true discretuum of values of θ_{QCD} . In this case, $\Delta\theta$ is zero.

One has, then, a picture where for each value of the fluxes, there is a distribution of states of different θ , with minimal energy at the point at which the strong interactions preserve CP. The crucial element here is the absence of corrections to the potential of eqn. 5.9, other than those from QCD, which lift the degeneracy. Lacking a model, it is difficult to address the question of what sorts of corrections might arise to eqn. 5.9.

One would have a similar picture if, say, q_2/q_1 were not irrational, but a ratio of two extremely large primes. In any case, as discussed in [108] and subsequently by others, it is not clear if such an axion actually emerges in string theory. We will see shortly, however, that in this model, selection for the cosmological constant favors large, rather than small, θ .

Ref. [111] does not directly address our questions here, but explores the interesting possibility that a theory with many axions might realize some features of the irrational axion. In this situation, the number of states may be exponentially large. One obtains bands of cosmological constant. However, small c.c. and small θ are not immediately correlated. We will discuss a variant with many axions in Section 5.3.3.

5.3.2 Models with a single axion

We consider in this section a more concrete model for the small $\Delta\theta$'s required to tune the cosmological constant. The model (if it is to yield extremely small $\Delta\theta$) is not particularly plausible, but illustrates the main ingredients required to implement the KT solution. We will see that for a limited range of parameters, one can account simultaneously for both the observed c.c. and the limits on θ . Conceivably there exists a more compelling structure with the features we describe below. We will offer another model in an appendix.

The model has two sectors, actual QCD and an $SU(N)$ gauge theory with a single adjoint fermion, λ (N will be *extremely* large), and a Z_P symmetry acting on λ . We include a complex scalar, ϕ , coupled to λ and to a (heavy) quark:

$$\phi\lambda\lambda + \phi\bar{Q}Q. \quad (5.10)$$

Under the Z_P symmetry, $\phi \rightarrow e^{\frac{2\pi i}{P}}\phi$. For general P , the discrete symmetry is anomalous with respect to both groups. We will comment on this in a moment. We assume that ϕ develops a vev, breaking the approximate Peccei-Quinn symmetry of the model,

$$\phi = fe^{i\theta}. \quad (5.11)$$

Integrating out the heavy fields yields $F\tilde{F}$ couplings to both groups:

$$\frac{N\theta}{16\pi^2}F'\tilde{F}' + \frac{\theta}{16\pi^2}F\tilde{F}. \quad (5.12)$$

This yields a potential

$$V = -\Lambda^4 \cos(N\theta) - m_\pi^2 f_\pi^2 \cos(\theta). \quad (5.13)$$

Both terms, for general P , violate the Z_P symmetry. The phenomenon of discrete symmetries with different apparent anomalies does occur in string theory[112] (but probably not with the enormous values of N we will shortly require).

Note here we do not have to assume an alignment of the CP conserving points in the two theories. Rather we need to assume that any additional contributions to the potential from other sources (such as other gauge groups) behaving as, say,

$$\delta V = \epsilon M^4 \cos(\theta - \theta_0), \quad (5.14)$$

where M indicates some fundamental scale and ϵ is a small parameter, are adequately suppressed. Then we require $\epsilon M^4 < 10^{-10} m_\pi^2 f_\pi^2$. This is similar to the requirement in theories with a light axion of a Peccei-Quinn symmetry of sufficient *quality* to solve the strong CP problem. It imposes a minimum on P .

Neglecting effects of QCD, the minima of θ lie at points: $\theta = \frac{2\pi k}{N}$. So steps in θ , are of size

$$\Delta\theta = \frac{2\pi}{N}. \quad (5.15)$$

In the absence of QCD, these states are degenerate. In the presence of QCD there is a contribution to the vacuum energy

$$V = m_\pi^2 f_\pi^2 [1 - \cos(k\Delta\theta)] , \quad (5.16)$$

with k an integer, which reduces to eqn. (5.2) for small $\Delta\theta$. To scan the c.c. finely enough, the requirement $\Delta\theta < 10^{-22}$ here corresponds to $N > 10^{22}$.

In the appendix, we provide an alternative model, based on the *clockwork axion* idea[113, 114], which does not require such huge gauge groups.

5.3.3 Models with Multiple Axions

Taking our clue from the irrational axion idea, we can proceed in another direction. If one is willing to pay the price of a large number of gauge groups (say 10), with large fermion representations (more generally with large anomaly coefficient), one can avoid the gigantic single gauge group. There are still stringent requirements regarding discrete symmetries. The idea is to have, say, M gauge groups (plus one more, QCD) and M approximate PQ symmetries (so M axions). The M approximate discrete symmetries arise as a consequence of M Z_P types symmetries. There are M axions. Take the fermions to be in the adjoint representation of the groups (or possibly larger representations — the point is to have big anomalies, so large $\cos(N\theta)$ type terms). More generally, one has something like

$$V = \sum \Lambda_i^4 \cos(N_i(a_j\theta_j)). \quad (5.17)$$

This has a large number of degenerate solutions,

$$\mathcal{N} = \prod_{i=1}^M N_i, \quad (5.18)$$

which can readily be huge.

An example that is simple to analyze contains M groups, $SU(N_i)$, $i = 1, \dots, M$; M discrete (and approximate continuous) symmetries; and M scalars. Under the symmetries,

$$\phi_i \rightarrow e^{\frac{2\pi ik}{n_i}} \phi_i; \quad \phi_i = f_i e^{i\theta_i}; \quad \theta_i = \frac{a_i}{f_i}. \quad (5.19)$$

Each scalar couples to adjoint fermions, $\phi_i \psi_i \psi_i$. The resulting potential is

$$V = - \sum \Lambda_i^4 \cos(\theta_i N_i). \quad (5.20)$$

The (degenerate) vacua have

$$\theta_i = \frac{2\pi k}{N_i}. \quad (5.21)$$

The discrete symmetries are anomaly free if $n_i = p N_i$, with integer p (with different conditions if the fermions are in representations other than the adjoint representation). Whether this is a condition we need to impose will be discussed later.

It is important that the vacua be degenerate to a high degree of approximation, with splittings smaller than $10^{-10} m_\pi^2 f_\pi^2$. As it stands, the symmetries allow couplings

$$\delta\mathcal{L} = \frac{\gamma}{M_p^{n_i-4}} \phi^{n_i} + \text{c.c.} \quad (5.22)$$

Depending on f_i , one obtains different conditions on n_i , but inevitably n_i must be rather large — typically 12 or more.

Now suppose the QCD axion arises from a field ϕ_{QCD} with Z_N charges (q_1, \dots, q_m) , with couplings to a heavy quark $\phi \bar{Q} Q$. Also suppose $\Lambda_{QCD} \ll \Lambda_i$. Then the QCD contribution is a small perturbation, of the desired type, lifting the degeneracy among the vacua. Note

$$\phi_{QCD} = e^{i\theta_{QCD}}, \quad (5.23)$$

where

$$\theta_{QCD} = \sum q_i \theta_i. \quad (5.24)$$

This construction is also quite complicated. An elaborate set of fields and couplings will be required to generate a vev for ϕ_{QCD} , and to avoid additional approximate global symmetries. But at least there is no group $SU(\text{Avogadro's number})$.

5.3.4 A Stringy Variant

Rather than postulate a large set of gauge groups, we might consider a string theory with a large number of axions, and suppose that some non-perturbative effect (e.g. instantons in the string theory) gives rise to a potential for the axions. Such a possibility has been considered as an alternative to fluxes to obtain a large discretuum of states in [111]. Here we are essentially considering both fluxes and multiple axions as sources of the c.c. (this will be quite explicit in the next few sections). This potential might be a sum of terms:

$$\sum_{a=1}^N \cos \left(\sum_{i=1}^M \theta_i r_i^a \right) M_p^4 e^{-s_a}, \quad (5.25)$$

where s_a might be the (assumed large) expectation value of some modulus, and r_i^a are some integers. We have rather arbitrarily chosen M_p as the fundamental scale. The sum is over those terms which are large compared to QCD. If $M = N$, there are a large (discrete) number of degenerate minima. If $M < N$, there are not a large number of degenerate solutions (this is essentially the idea in [111] for generating a discretuum of cosmological constants). If $M > N$, there are one or more light axions.

5.4 Canceling the Cosmological Constant in Different Parameter Ranges

In this section, we assume that the underlying theory has a discretuum of states of different θ , and we ask whether anthropic selection for the c.c. leads to small θ . We start from eqn. (5.1), and we treat both θ and Λ_{BP} as discrete parameters. We write $\theta = k\Delta\theta$, with k an integer, and $\Lambda_{BP} = -q\Delta\Lambda$, with q taking discrete values, not necessarily integer, ranging from order 1 to large values, in order to stress that the spacing of states in the Bousso-Polchinski landscape is not uniform.

The cosmological constant,

$$\Lambda = -q\Delta\Lambda - \cos(k\Delta\theta)m_\pi^2 f_\pi^2 \approx -q\Delta\Lambda + \frac{1}{2}k^2\Delta\theta^2 m_\pi^2 f_\pi^2, \quad (5.26)$$

must satisfy the anthropic bound

$$0 < \Lambda < \Lambda_a, \quad (5.27)$$

with $\Lambda_a = 3 \times 10^{-47} \text{ GeV}^4$. For given step sizes $\Delta\Lambda$ and $\Delta\theta$, there is a definite number of states [labeled by (k, q)] that satisfy the bound of eqn. 5.27. The question we wish to ask is: are these states characterized by large or small θ ?

At a simple-minded level, there are a few constraints on the parameters that appear in eqn. (5.26). There is a maximum possible k , k_{max} , given by $k_{max}\Delta\theta = 2\pi$ (there are also degeneracies in k , which will not be particularly important in what follows). Also, we require $\Delta\Lambda < m_\pi^2 f_\pi^2$, or we won't, in general, have any possibility of canceling off the c.c. as we scan in k .

We organize the discussion according to whether we are scanning finely or coarsely in the Λ_{BP} direction:

$$\text{Fine scanning} \quad \Delta\Lambda < \Lambda_a, \quad (5.28)$$

$$\text{Coarse scanning} \quad \Lambda_a < \Delta\Lambda < m_\pi^2 f_\pi^2. \quad (5.29)$$

5.4.1 Fine $\Delta\Lambda$

For $\Delta\Lambda < \Lambda_a$, we write $y = -q\Delta\Lambda$ and treat y as a continuous parameter. Considering first $\Delta\theta \ll 10^{-22}$, we can approximate θ as continuous as well. The c.c. is

$$\Lambda = y + \frac{1}{2}m_\pi^2 f_\pi^2 \theta^2. \quad (5.30)$$

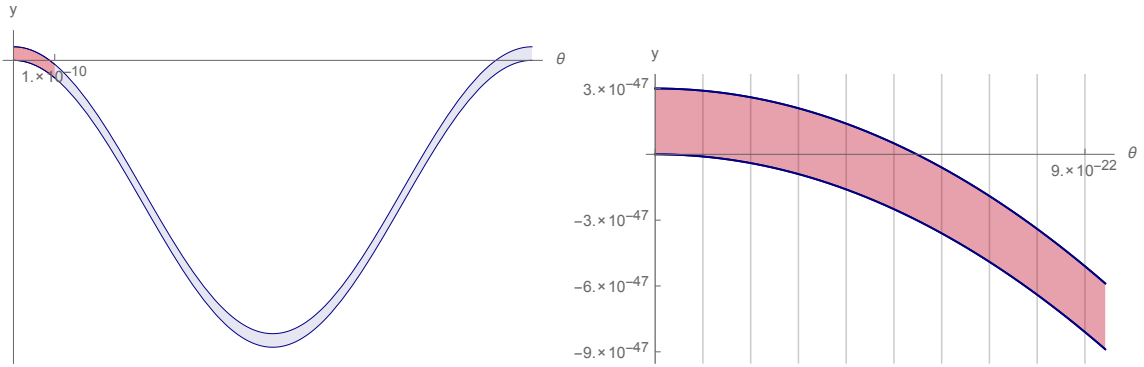


Figure 5.1: Left: The shaded region is the portion of the region in (θ, y) space that produces an anthropically allowed value for the c.c., as determined by eqn. (5.30). The red shading is the portion of this region for which $\theta < 10^{-10}$ [not drawn to scale]. This shows that values of θ of order one are favored by the c.c. anthropic selection. Right: zoomed in to very small θ , drawn to scale.

Because in this case we scan finely in $\Delta\Lambda$, it makes sense to define:

$$y_{max} = \Lambda_a - \frac{1}{2}m_\pi^2 f_\pi^2 \theta^2; \quad y_{min} = -\frac{1}{2}m_\pi^2 f_\pi^2 \theta^2. \quad (5.31)$$

We expect a uniform distribution in y and θ , so the fraction of states satisfying the anthropic condition is:

$$\int d\theta dy \Theta(y_{max} - y)\Theta(y - y_{min}) = \int d\theta (y_{max} - y_{min}), \quad (5.32)$$

which is independent of θ . So large θ is favored.

The same conclusion holds even if $\Delta\theta > 10^{-22}$ and we treat it as a discrete variable. Also in this case there is a larger number of states with large θ , and the energy density of any such state can be canceled to the desired accuracy by y [see Fig. 5.1].

5.4.2 Coarse $\Delta\Lambda$

In this subsection, we consider the case $\Lambda_a < \Delta\Lambda < m_\pi^2 f_\pi^2$. Here, because we do not scan finely in $\Delta\Lambda$, we can't treat q as a continuous variable as in our previous analysis. It is convenient

to define

$$a \equiv \frac{m_\pi^2 f_\pi^2 \Delta\theta^2}{2\Lambda_a}, \quad b \equiv \frac{\Delta\Lambda}{\Lambda_a}, \quad 1 < b < \frac{m_\pi^2 f_\pi^2}{\Lambda_a}. \quad (5.33)$$

As the theta potential goes as k^2 , there can be a value, k_0 , above which also the scanning in the theta direction becomes coarse (larger than Λ_a). k_0 is obtained from:

$$\Delta V(k_0) = m_\pi^2 f_\pi^2 k_0 \Delta\theta^2 = \Lambda_a, \quad \Rightarrow \quad k_0 = \frac{1}{2a} \quad (5.34)$$

We distinguish 2 cases. In the first

$$0 < a < \frac{1}{2} \frac{\Lambda_a}{m_\pi^2 f_\pi^2}, \quad (5.35)$$

we scan finely for all values of k , $k \leq k_{max}$. In the second

$$\frac{1}{2} \frac{\Lambda_a}{m_\pi^2 f_\pi^2} < a < \frac{1}{2}, \quad (5.36)$$

$1 < k_0 < k_{max}$, so the scanning becomes coarse for $k > k_0$.

Eq. (5.27) in terms of a and b becomes

$$\frac{qb}{a} < k^2 < \frac{qb}{a} + \frac{1}{a}. \quad (5.37)$$

Now we want to ask: are there more (k, q) states satisfying the above inequalities at small or large k ?

We can write eq. (5.37) as

$$k_- < k < k_+, \quad (5.38)$$

with

$$k_- = \sqrt{\frac{qb}{a}}, \quad k_+ = \sqrt{\frac{qb}{a} + \frac{1}{a}} \approx \sqrt{\frac{qb}{a}} + \frac{1}{2\sqrt{a}qb}. \quad (5.39)$$

Consider, first, the case of eqn. 5.35. Then the number of states, for fixed q , satisfying the anthropic constraint is

$$N(q) \sim k_+ - k_- \approx \frac{1}{2\sqrt{a}qb} < 1, \quad (5.40)$$

Treating q and θ as approximately continuous, $q \sim \theta^2$, and we see that the number of states satisfying the anthropic condition is larger at large θ (q) by $\sqrt{q} \propto \theta$.

Now take the case of eqn. 5.36. Let's start with fixing large q , and correspondingly large θ , where, for general q there is not a choice of k satisfying the anthropic condition. In other words, for

$$\sqrt{\frac{qb}{a}} > k_0 \quad \Rightarrow \quad \frac{1}{2\sqrt{aqb}} < 1 \quad (5.41)$$

the window $k_+ - k_-$ in this case is smaller than 1. Still, there is a small chance, for any given q , to find an integer k in such a window. The probability is

$$P(q) \sim k_+ - k_- \approx \frac{1}{2\sqrt{aqb}} < 1, \quad (5.42)$$

which decreases with increasing q (θ). The point, however, is that at large q the number of possible states increases. We can estimate the number of large q states which satisfy the anthropic condition as

$$N_{\text{large}} = \int_{\frac{a}{b}k_0^2}^{\frac{a}{b}k_{\text{max}}^2} P(q) dq = \frac{1}{\sqrt{2ab}} \sqrt{\frac{m_\pi^2 f_\pi^2}{\Lambda_a} - \frac{1}{2ab}}. \quad (5.43)$$

Note that when a saturates the upper bound of (5.35), $N_{\text{large}} = 0$, and remains zero for smaller a .

Next, consider small q :

$$\sqrt{\frac{qb}{a}} < k_0 \quad \Rightarrow \quad \frac{1}{2\sqrt{aqb}} > 1. \quad (5.44)$$

The expansion in (5.39) still holds, because $qb > 1$. Now, however, we have

$$k_+ - k_- \approx \frac{1}{2\sqrt{aqb}} > 1, \quad (5.45)$$

implying there is at least one integer k which satisfies (5.38), for any q which satisfies (5.44). Thus, we can estimate the number of states which satisfy the anthropic condition in this case as

$$N_{\text{small}} \approx \int_1^{\frac{a}{b}k_0^2} dq = \frac{1}{4ab} - 1. \quad (5.46)$$

Now we can compare N_{small} to N_{large} .

In the first case of eq. (5.35) we have $N_{\text{small}} > N_{\text{large}} = 0$, and the favored states have $\theta < k_0 \Delta\theta$. However, $k_0 \Delta\theta > 1$ in this case, so large theta is favored. In the second case of eq. (5.36), N_{small} and N_{large} can be comparable as long as a is close to the lower bound $\frac{1}{2} \frac{\Lambda_a}{m_\pi^2 f_\pi^2}$. This, again, favors $\theta \sim 1$. For larger values of a we quickly obtain $N_{\text{large}} \gg N_{\text{small}}$, which also favors large theta.

When $a > \frac{1}{2}$ the chance of canceling the c.c. is always smaller than one, and decreases going to larger values of q . However, as in the analysis above, the bulk of the states which satisfy the anthropic condition are at large q , so large θ is favored.

5.5 Conclusions

Arguably if one cannot find a suitable solution to the strong CP problem in a landscape framework, the landscape idea may be unsupportable. As a result, it is important to study any proposal to understand how the value of θ might be correlated with other physical quantities which might be anthropically determined. One possibility is that axions are selected by an anthropic requirement for dark matter; the main issue is whether the Peccei-Quinn symmetry is of sufficient quality to account for the small value of θ . We have reviewed the challenges to such a possibility, and concluded that such an anthropic explanation of θ is plausible, but that whether it is realized depends on questions about the microphysical theory and cosmology.

Kaloper and Terning propose to correlate θ with the value of the cosmological constant. We have attempted to flesh out this proposal. Rather than a continuous range of θ , which, as we have explained, is likely to correspond to a conventional axion, we have argued that one should

consider the possibility that, absent QCD, there is a *massive* axion with a vast number of nearly degenerate ground states. QCD then lifts this degeneracy. We have put forward models in which θ takes on a discretuum of discrete values, with a potential on this discretuum of the desired type. Having reduced the system to a discrete system, we were able to assess the probability of finding larger or smaller θ , in the sense of asking: are most of the states with anthropically favored c.c. at large or small θ ? The system has two parameters; on all of the space, $\theta \sim 1$ is favored.

It is conceivable that some other consideration might favor small θ within a θ discretuum. The models we have proposed to obtain such a discretuum are not particularly attractive; indeed they are hardly plausible. The irrational axion[108] has some of the desired features of such a system, but it is not clear that such axions actually arise in any theory of quantum gravity, and, in any case, this would predict $\theta \sim 1$. Another class of models involves a huge gauge group and a large discrete symmetry; others a very large number of fields. For the moment, we conclude, however, that in a landscape framework, an anthropic axion is a more plausible solution of the strong CP problem.

Appendix: A Clockwork Construction

The construction relies on the potential [114]

$$V(\Phi) = \sum_{j=1}^{N+1} \left(-\mu_{\Phi}^2 \Phi_j^{\dagger} \Phi_j + \frac{\lambda_{\Phi}}{4} |\Phi_j^{\dagger} \Phi_j|^2 \right) + \sum_{j=1}^N \left(\epsilon_{\Phi} \Phi_j^{\dagger} \Phi_{j+1}^3 + \text{h.c.} \right), \quad (5.47)$$

where Φ_j 's are complex scalar fields. The terms in the first sum respect a global $U(1)^{N+1}$ symmetry, while the second sum explicitly breaks it to a $U(1)$. The fields Φ_j have charges $Q = 1, \frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{3^N}$ under the unbroken $U(1)$. We take $\mu_{\Phi}^2 > 0$, so all the $U(1)$'s are spontaneously broken at a scale $f = \sqrt{(2\mu_{\Phi}^2)/\lambda_{\Phi}}$. All the radial modes then have a mass of order f . Neglecting the second sum in

the potential, we have $N + 1$ massless Nambu-Goldstone bosons (NGBs). Taking into account the second sum, with the explicit breaking parameter $\epsilon_\Phi \ll 1$, we find that N of these NGBs get a mass of order $\sqrt{\epsilon_\Phi} f$, while one, ϕ , remains massless. The latter corresponds to the linear combination $\phi = \mathcal{N} \left(1 \frac{1}{3} \frac{1}{9} \dots \frac{1}{3^N} \right)$. Here \mathcal{N} is a normalization factor.

Now, we can write the Yukawa coupling $\Phi_{N+1} \bar{\psi}_{N+1} \psi_{N+1}$, where ψ_{N+1} is a fermion that lives at the site $N + 1$ and is in the fundamental representation of $SU(3)_c$, the QCD gauge group. This is a KSVZ axion model, and the QCD anomaly leads to the coupling

$$\frac{\alpha_s}{8\pi} \frac{\phi}{F} G\tilde{G}, \quad (5.48)$$

which in turn gives us the cosine potential

$$V_{\text{QCD}} = m_\pi^2 f_\pi^2 \left(1 - \cos \frac{\phi}{F} \right) = m_\pi^2 f_\pi^2 (1 - \cos \theta). \quad (5.49)$$

Here we have defined $\theta \equiv \frac{\phi}{F}$, and we have $F = 3^N f$ from the clockwork construction.

Next, we introduce fermions ψ_1 in the fundamental of a new gauge group $SU(N_1)$, with confinement scale Λ , and we write the Yukawa coupling $\Phi_1 \bar{\psi}_1 \psi_1$. Again there is an anomaly with respect to $SU(N_1)$, from which we get the potential

$$V_1 = \Lambda^4 \left(1 - \cos \frac{\phi}{f} \right) = \Lambda^4 (1 - \cos(3^N \theta)). \quad (5.50)$$

Note that, as this potential arises from a coupling at the first clockwork site where the axion ϕ is mostly localized, the periodicity is given by f .

The sum $V_{\text{QCD}} + V_1$ gives the desired structure of eqn. (5.13).

Chapter 6

Conclusions

Starting with QCD and the strong CP problem, we explored different subjects in each Chapter. Studies of these problems could help understand Standard Model in a deeper way and also provide new ideas and tools for various kinds of questions beyond the Standard Model.

We began, in Chapter 2, by studying the large N θ dependence and the η' potential in supersymmetric QCD with SUSY-breaking terms as a probe of the non-SUSY limit. We found the presence of branched behavior, which quite agrees with the conjectured large N behavior of QCD, but we also found some disagreement. For example, in certain range of parameters in SQCD, instanton effects are calculable and don't fall off exponentially with N . And we also found that the discrete symmetry associated with the branched structure, could be badly broken when taking the soft breaking large.

In Chapter 3, we looked into the axion in early cosmology. We computed the axion relic density using standard dilute gas methods and considered the uncertainty on the instanton contribution, arguing that it amounts to less than 20% in the effective action, or a factor of 20 in χ

at $T = 1.5$ GeV. We then combined the instanton uncertainty with a range of models for $\chi(T)$ at intermediate temperatures and determined the impact on the axion relic density. We found that for a given relic density and initial misalignment angle, the combined uncertainty amounts to a factor of 2-3 in the zero-temperature axion mass.

In Chapter 4, we studied the Debye mass, m_D , and the topological susceptibility, χ , at high temperatures in non-abelian gauge theory. Both exhibit, at some order in the perturbation expansion, infrared sensitivity. For χ , the infrared divergence arises at order g^4 . For Debye mass, one can obtain a reliable estimate for this scale by calculating the leading order contribution to the position of the pole, and infrared divergences appear in this computation. We stressed that these large corrections to the Debye mass are not important for the calculation of the susceptibility.

In Chapter 5, We considered models where θ might be a discrete parameter and correlate it with the cosmological constant. We then asked whether large or small θ is favored with the anthropic constraint on c.c. Given such a discretuum, we found no circumstances where small θ might be selected by anthropic requirements on the cosmological constant.

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