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Free Probability, Planar Algebras, and the Multicolored Guionnet-Jones-Shlyakhtenko Construction

by

Michael Aaron Hartglass

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

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in the

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Committee in charge:

Professor Vaughan F.R. Jones, Chair
Professor Marc Rieffel
Professor Ori Ganor

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Abstract

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Doctor of Philosophy in Mathematics

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Professor Vaughan F.R. Jones, Chair

This dissertation consists of three papers I have written or helped write during my time at UC Berkeley. The three papers all fall under the common theme of exploring connections between free probability and planar algebras.

In Chapter 2, an amalgamated free product algebra, $\mathcal{N}(\Gamma)$, is constructed for any connected, weighted, loopless graph Γ, and its isomorphism class can be nicely read off based on the weighting data on the graph. This construction was heavily influenced by an algebra appearing in [GJS11], and it is used, along with some standard embedding arguments, to show that the factors of Guionnet, Jones, and Shlyakhtenko appearing in [GJS10] are isomorphic to $L(\mathbb{F}_\infty)$ when the planar algebra is infinite depth.

Chapter 3 is the paper "Rigid C^*-tensor categories of bimodules over interpolated free group factors" [BHP12] which was co-authored with Arnaud Brothier and David Penneys. In this paper, we establish an unshaded planar algebra structure (with multiple colors of strings) which can be used to model a countably generated rigid C^*-tensor category, $\mathcal{C}$. We use this to construct a category of bifinite bimodules over a II_1 factor M_0, and we show that this category is equivalent to $\mathcal{C}$. Finally, we use the work in Chapter 2 to show that the factor M_0 is isomorphic to $L(\mathbb{F}_\infty)$.

Chapter 4 is a note regarding multishaded planar algebras. The problem studied consists of placing the "Fuss-Catalan" potential on a specific kind of subfactor planar algebra $\mathcal{Q}$, which is the standard invariant for and inclusion $N \subset M$ with an intermediate subfactor P. This potential is best understood by augmenting $\mathcal{Q}$, forming a bigger multishaded planar algebra $\mathcal{P}$. The isomorphism classes of the algebras M_α associated to $\mathcal{P}$ will be computed explicitly. While the isomorphism class of the von Neumann algebras $N_k^\pm$ associated to $\mathcal{Q}$ are still not yet known, they will be shown to be contained in a free group factors and contain a free group factors. This potential is shown to yield a nice free product expression for the law of $\cup$, an element which plays a critical role in understanding algebras that arise from this construction. Many of the ideas in this chapter influenced the work in the Chapter 3.

This dissertation is dedicated to Douglas Knowles.

Contents

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Chapter 1

Introduction

1.1 Subfactors and planar algebras

In [Jon83], Jones initiated the study of subfactor theory. The objects of interest are inclusions $M_0 \subset M_1$ of finite index II_1 factors M_0 and M_1. From this, he iterated the basic construction to produce a tower of finite index II_1 factors

$$M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots ,$$

and studied subfactors by examining their standard invariant. The standard invariant consists of two sequences of finite dimensional vector spaces, $P_{n,+} = M_0' \cap M_n$ and $P_{n,-} = M_1' \cap M_n$ [Jon83]. By 1995, the standard invariant became axiomatized by Ocneanu's paragroups [Ocn88], and Popa's $\lambda-$lattices [Pop95]. Around the same time, Jones discovered that operations and objects in subfactor theory (such as the Jones projections, traces, and conditional expectations) possess a very natural planar structure. He used these observations to invent subfactor planar algebras. Before talking about subfactor planar algebras, we will give a definition of a shaded planar tangle:

Definition 1.1.1. A shaded planar tangle, T consists of the following data:

(1) a rectangle $D_0(T) \subset \mathbb{R}^2$,

(2) Finitely many disjoint rectangles $\{D_i : i \in I\}$ in the interior of $D_0(T)$ (I may be empty),

(3) Finitely many disjoint smooth arcs in $D_0(T)$ called the strings of T which do not meet the interior of any D_i. The boundary points of a string of T (if it has any) lie in the boundaries of the D_i, and they meet these boundaries transversally (if they meet the boundaries at all).

(4) A choice of shading of the regions of T. Each region can be either shaded or unshaded, but an unshaded region can only border a shaded region or vice versa. Notice that this means that the number of strings intersecting any box must be even.

The boundaries of the strings divide the boundaries of the rectangles into intervals. For each rectangle, there is a distinguished interval denoted $\star$. The intervals of $D_i(T)$ are divided by the marked points of $D_i(T)$, i.e., the points at which the strings meet the boundary of $D_i(T)$. Starting at $\star$, the marked points on each rectangle are number clockwise.

If we have two planar tangles S, T satisfying the following boundary conditions:

- some internal rectangle $D_i(S)$ agrees with $D_0(T)$,

- the marked points of $D_i(S)$ agree with the marked points of $D_0(T)$,

- the distinguished interval of $D_i(S)$ agrees with the distinguished interval of T, and

- the label from $\mathcal{L}$ of each marked point of $D_i(S)$ agrees with the label of each marked point from $D_0(T)$,

then we may compose S and T to get the planar tangle $S \circ_{D_i} T$ by taking S union the interior of $D_0(T)$, removing the boundary of $D_i(S)$, and smoothing the strings.

An example of such a tangle is:
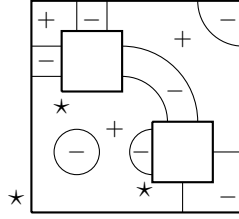
where a $+$ indicates an unshaded region and a $-$ indicates a shaded region. Loosely speaking, a shaded planar algebra consists of a family of vector spaces having which exhibit a natural action by planar diagrams. In most definitions of planar algebras (such as [Jon]), the rectangles are taken to be smoothly embedded subdisks in the plane, and the action of planar tangles is invariant under an orientation preserving diffeomorphism. Therefore, one should picture the rectangles as having their corners smoothed out. In future drawings, the $\star$ on a box will be placed at the bottom-left of a box unless otherwise indicated.

We now define a subfactor planar algebra.

Definition 1.1.2. A shaded planar algebra $\mathcal{P}$ comes equipped with the following data:

- a collection of vector spaces $(P_{2n,+})_{n\in\mathbb{N}}$ and $(P_{2n,-})_{n\in\mathbb{N}}$

- an action of planar tangles by multilinear maps, i.e., for each planar $(2n, \epsilon)$ tangle T, whose rectangles $D_i(T)$ are $(2n_i, \epsilon_i)$-rectangles, there is a multilinear map

$$Z_T \colon \prod_{i\in I} P_{2n_i,\epsilon_i} \to P_{2n,\epsilon}$$

satisfying the following axioms:

Isotopy: If θ is an orientation preserving diffeomorphism of $\mathbb{R}^2$, then $Z_{\theta(T)} = Z_T$. That is, let T^0 be the interior of T, and let $f \in \prod_{D\subset T^0} P_{2n_D,\epsilon_D}$. Then

$$Z_{\theta(T)}(f_\theta) = Z_T(f)$$

where $f_\theta(\theta(D)) = f(D)$.

Naturality: For S, T composable tangles, $Z(S \circ_D T) = Z(S) \circ_D Z(T)$, where the composition on the right hand side is the composition of multilinear maps.

Moreover, $\mathcal{P}$ is called a <u>subfactor planar algebra</u> if $\mathcal{P}$ is

- <u>evaluable</u>, i.e., $\dim(P_{2n,\pm}) < \infty$ for all n and $P_{0,\pm} \cong \mathbb{C}$ via the map that sends the empty diagram to $1 \in \mathbb{C}$. Hence, by naturality, there is a scalar $\delta \in \mathbb{C}$ such that any labeled diagram containing a closed loop is equal to the same diagram without the closed loop multiplied by δ.

- $\mathcal{P}$ is <u>unital</u> [Jon11]: Let S be a shaded planar $2n, \epsilon$ tangle with no input disks. Then, there is an element $Z(S) \in P_{2n,\epsilon}$ so that the following holds:

  Let T be a tangle with a nonempty set of internal disks such that S can be glued into the internal disk D^S of T. Then
  $$Z(T \circ S) = Z(T) \circ Z_S.$$
  Here $(Z(T) \circ Z_S)(f) = \tilde{f}$ where

  $$\tilde{f}(D) = \begin{cases} f(D) \text{ if } D \neq D^S \\ Z(S) \text{ if } D = D^S \end{cases}$$

- <u>involutive</u>, i.e. there is a map $* \colon P_{2n,\pm} \to P_{2n,\pm}$ with $* \circ * = \text{id}$ which is compatible with the reflection of tangles, i.e., if T is a planar tangle acting on $x_1, ..., x_n$, and φ is an orientation reversing diffeomorphism of $\mathbb{R}^2$, then

  $$Z(T)(f)^* = Z(\varphi(T))(\overline{f})$$

  where $\overline{f}(\varphi(D)) = f(D)^*$

- <u>spherical</u>, i.e., for all $\alpha \in \Lambda$ and all $x \in P_{2n,\pm}$, we have

  $$\text{tr}(x) = \boxed{\begin{array}{c} n \\ x \\ n \end{array}} \, n \;=\; n \, \boxed{\begin{array}{c} n \\ x \\ n \end{array}}.$$

- <u>positive</u>, i.e., the map $\langle \cdot, \cdot \rangle \colon P_{2n,\pm} \times P_{2n,\pm} \to P_{0,\pm} \cong \mathbb{C}$ given by

  $$\langle x, y \rangle = \boxed{x} \overset{2n}{-} \boxed{y^*}$$

  is a positive definite inner product. Hence $\delta > 0$. It further follows from the work of Jones [Jon83] that
  $$\delta \in \{2\cos(\pi/n) : n = 3, 4, 5, ...\} \cup [2, \infty)$$

Jones showed that the relative commutants $P_{2n,+} = M_0' \cap M_n$ and $P_{2n,-} = M_1' \cap M_{n+1}$ admit the structure of a subfactor planar algebra. In Chapters 3 and 4, we will define notions of planar algebras similar to the one presented above.

Remark 1.1.3. The spaces $P_{2n,\pm}$ as above are traditionally referred to as $P_{n,\pm}$ in the literature. We adopt this new numbering convention as we will be considering unshaded planar algebras in Chapter 3 which can admit actions by planar tangles with *odd* numbers of strings.

The positive definiteness assures that each $P_{2n,\pm}$ is a finite dimensional C^* algebra where the multiplication is given by

$$x \cdot y = \boxed{\begin{array}{c} n \\ \boxed{y} \\ n \\ \boxed{x} \\ n \end{array}}$$

and non-normalized trace Tr is given by

$$\mathrm{Tr}(x) = \boxed{x} \; n \; .$$

Of critical importance is the study of minimal projections in these algebras via a principal graph, Γ and dual principal graph Γ'. To begin, we consider the algebras $P_{2n,+}$. Let $p \in P_{2j,+}$ and $q \in P_{2k,+}$. We say that p and q are equivalent if there is a $u \in P_{j+k,+}$ satisfying

$$\begin{array}{c} j \\ \boxed{u} \\ k \\ \boxed{u^*} \\ j \end{array} = p \quad \text{and} \quad \begin{array}{c} k \\ \boxed{u^*} \\ j \\ \boxed{u} \\ k \end{array} = q.$$

Notice that for equivalence of p and q to make sense, we must have $k + j$ even. We then form Γ as follows:

- The vertices v of Γ are in a one to one correspondence with equivalence classes $[p_v]$ of minimal projections in the algebras $P_{2n,+}$.

- There are n edges connecting v and w if the following condition is met: Suppose $p \in P_{2j,+}$ is a minimal projection equivalent to p_v. Then the element

$$i(p) = \boxed{\begin{array}{c} j \\ p \\ j \end{array}}$$

  is a projection $P_{2(j+1),+}$. The number n will be the maximum k such that there exist orthogonal projections $q_1, ..., q_k \in P_{2(j+1),+}$ which are equivalent to p_w and satisfy $\sum_{i=1}^{k} q_i \leq i(p)$. This number is independent of the representative of p_v. Also, if there are n edges connecting v to w, then there are also n edges connecting w to v [Jon]. See Chapter 4 for an explanation of this fact for the multishaded case.

It follows from this construction that Γ is a connected, unoriented, bipartite graph. If we weight each vertex v by $\mathrm{Tr}(p_v)$, then it follows that the weights satisfy the Perron-Frobenius condition, i.e.

$$\delta \, \mathrm{Tr}(p_v) = \sum_{w \sim v} n_{v,w} \, \mathrm{Tr}(p_w).$$

We write $w \sim v$ if w is connected to v and we let $n_{v,w}$ be the number of edges connecting v and w. We will mark the vertex corresponding to the empty diagram with a *.

By looking at the algebras $P_{2n,-}$, one can similarly construct the dual principal graph of $\mathcal{P}$. It is straightforward to check that a $180°$ rotation is an anti $*-$isomorphism between $P_{2n,+}$ and $P_{2n,-}$ for n odd. It follows that the odd vertices of Γ and Γ' are in bijective correspondence.

We say that $\mathcal{P}$ is finite depth is Γ is finite and infinite depth if Γ is infinite. Note that if Γ is finite, then there are only finitely many odd vertices of Γ'. Since the algebras $P_{2n,-}$ are finite dimensional for each n, it follows from the construction of the edges that Γ' must be finite as well.

1.2   Free probability

Around the same time as the beginning of subfactor theory, Voiculescu developed an analogue of probability called free probability(see the book [VDN92]). At the heart of free probability is the concept of free independence, which we will now define

Definition 1.2.1. Let M be a von Neumann algebra with faithful normal state ϕ, and $(M_\iota)_{\iota \in I}$ a family of unital von Neumann subalgebras of M. Then we say that the algebras M_ι are freely independent (or free) if the following condition holds:

$$\phi(x_{\iota_1} \cdots x_{\iota_n}) = 0 \text{ whenever } \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n \text{ and } \phi(x_{\iota_j}) = 0 \text{ for all } j.$$

We say that elements $(x_\iota)_{\iota \in I}$ are free if the corresponding von Neumann algebras they generate are free.

In the rest of the thesis, ϕ will always be assumed to be tracial. Given any collection $(M_\iota)_{\iota \in I}$ of von Neumann algebras with faithful states ϕ_ι, one can form a von Neumann algebra M with faithful state ϕ which is generated by isomorphic images of the M_ι such that $\phi|_{M_\iota} = \phi_\iota$ and the family $(M_\iota)_{\iota \in I}$ is free. M is unique up to spatial isomorphism, and the state ϕ is uniquely determined by the states ϕ_ι. See [VDN92] for more details.

Voiculescu noticed strong parallels between commutative probability and free probability theory. By realizing that the semicircular law should play the same role in free probability as the gaussian law plays in commutative probability, he established a central limit theorem for additive free convolution which mimics the central limit theorem in probability [VDN92]. Building on the proof of Wigner's semicircular law [Wig55], Voiculescu proved that a family of independent self adjoint Gaussian random matrices converges in joint distribution to a free family of semicircular elements [Voi91]. Therefore, one can think of free probability as a random matrix limit of commutative probability.

1.3   Connection between subfactors and free probability

In the 1990's, evidence started to surface that the combinatorial structure of free probability is closely related to non-crossing diagrams. One of the key observations is that if x is a self adjoint random variable having the semicircular law of radius 2 with respect to some state ϕ, then $\phi(x^{2n}) = C_n$, the nth Catalan number. The Catalan numbers count many things including the numbers of noncrossing pair partitions of an ordered set of $2n$ elements.

It was shown by Speicher [Spe94] that additive free convolution of noncommutative random variables is easily described in terms of free cumulants which are naturally defined according to noncrossing partitions. A combinatorial proof of the free central limit theorem boils down to counting noncrossing pair partitions in the large N limit (as opposed to all pair partitions for the commutative central theorem) [Spe90]. An explicit use of planar diagrams in free probability and random matrix theory with regards to second-order freeness was considered by Mingo, Speicher, and others [MS06, MŚS07, CMŚS07]. Furthermore, Voiculescu's theorem on the asymptotic freeness of independent gaussian random matrices can be recast as a combinatorial proof involving counting certain noncrossing partitions [NS06].

These observations helped lead Guionnet, Jones, and Shlyakhtenko (GJS) to find connections between subfactors, planar algebras, free probability, and random matrix theory [GJS10]. In [GJS10], they constructed graded algebras $\mathrm{Gr}_n(\mathcal{P}^{\pm})$ associated to a subfactor planar algebra $\mathcal{P}$ and endowed them with a trace tr emanating from all Temperley-Lieb (planar) diagrams. They were able to model tr by a random matrices associated to the principal graph, Γ of $\mathcal{P}$ and were able to construct a Jones tower of factors $M_n^+ = \mathrm{Gr}(\mathcal{P}_{2n,+})''$ whose standard invariant is $\mathcal{P}$, thus recovering Popa's reconstruction theorem [Pop95]. It was also shown that in the case where $\mathcal{P}$ is finite depth (has a finite principal graph), the factors M_n^+ are interpolated free group factors, whose parameter depends on data in the planar algebra.

In Chapter 2, it will be shown that for infinite depth, the factors M_n^+ are all isomorphic to $L(\mathbb{F}_\infty)$. In Chapter 3, we examine the GJS construction with regards to unshaded planar algebras, and obtain a universality result regarding countably generated rigid C^*-tensor categories and bimodules over $L(\mathbb{F}_\infty)$. In the final chapter, we will examine the situation when the trace involving all Temperley-Lieb diagrams is replaced by the trace involving all Fuss-Catalan diagrams.

Planarity and freeness

The algebras $\mathrm{Gr}_0(\mathcal{P})$ in the following chapters are presented as variations of the following idea: We set

$$\mathrm{Gr}_0(\mathcal{P}) = \bigoplus_{n=0}^{\infty} P_n$$

with multiplication given by

$$x \cdot y = \boxed{x}\ \boxed{y}$$

and trace, tr given by

$$\mathrm{tr}(x) = \frac{\boxed{\Sigma TL}}{\boxed{x}}$$

where $\sum TL$ is the sum of all planar string diagrams with no strings emanating out of the top of the box or the sides. Assuming for the moment that tr is positive definite, we have the following lemma:

Lemma 1.3.1. *Suppose $x_1, ..., x_n$ are elements in $\mathcal{P}$ with the following property: For every tangle, T, with at least one string connecting two distinct internal disks D_1 and D_2, we have $Z(T)(f) = 0$*

*whenever $f(D_1) = x_{i_1}^{\epsilon_1}$, $f(D_2) = x_{i_2}^{\epsilon_2}$ with $i_1 \neq i_2$ and ϵ_j is either nothing or a *. Then $\{x_1, ..., x_n\}$ are free in $M_0 = \mathrm{Gr}_0(\mathcal{P})''$.*

Loosely speaking, this lemma says that whenever an x_i is connected by at least one string to x_j for $i \neq j$, then x_i and x_j are free.

Proof. Suppose p_j is a (non-commutative) polynomial in x_{i_j} and $x_{i_j}^*$ with $\mathrm{tr}(p_j) = 0$, and consider the element $p_1 \cdots p_m$ for some m. Suppose further that $i_k \neq i_{k+1}$ for $1 \leq k \leq m$. We need to show that $\mathrm{tr}(p_1 \cdots p_m) = 0$. By definition, we have:

$$\mathrm{tr}(x) = \begin{array}{c} \boxed{\Sigma TL} \\ \boxed{p_1} \cdots \boxed{p_m} \end{array}.$$

By assumption on the variables x_i, any element of TL which connects a summand of p_i to a summand of p_{i+1} gives a zero contribution. Therefore, by planarity, every contributing TL element must pair some summand of a p_j with itself. Summing over all of these partitions, and using $\mathrm{tr}(p_j) = 0$ for all j gives the result. $\qquad \square$

The idea in the sections that follow is to find elements x_i satisfying this property which generate M_0 in an appropriate sense. In practice, we will pass to an infinite amplification of M_0 as it will be more natural to find such elements there.

Chapter 2

Free Product von Neumann Algebras Associated to Graphs and the Guionnet, Jones, Shlyakhtenko Construction in Infinite Depth

2.1 Introduction

In [Jon83], Jones initiated the study of modern subfactor theory. Given a finite index II_1 subfactor $M_0 \subset M_1$, one computes its standard invariant: two towers $(M_0' \cap M_j : j \geq 0)$ and $(M_1' \cap M_j : j \geq 1)$ of finite dimensional von Neumann algebras [Jon83]. The standard invariant has been axiomatized by Ocneanu's paragroups [Ocn88], Popa's $\lambda-$lattices [Pop95], and Jones' subfactor planar algebras [Jon03]. Popa showed that given a standard invariant $\mathcal{P}$, we can reconstruct a II_1 subfactor $M_0 \subset M_1$ whose standard invariant is $\mathcal{P}$ [Pop95]. Guionnet, Jones, and Shlyakhtenko [GJS10] give a planar-algebraic proof of the above result. Moreover, if $\mathcal{P}$ is finite depth with loop paramater $\delta > 1$, they showed that M_k, the kth factor in the Jones tower, is isomorphic to $L(\mathbb{F}(1 + 2\delta^{-2k}(\delta - 1)I))$ where I is the global index of $\mathcal{P}$ [GJS11]. Kodiyalam and Sunder also obtained this formula when $\mathcal{P}$ is depth 2 [KS09b, KS09a]. In this paper, we prove the following theorem:

Theorem. *If $\mathcal{P}$ is infinite depth, then every factor in the construction of [GJS10] is isomorphic to $L(\mathbb{F}_\infty)$.*

Using this theorem, we recover a diagrammatic proof of a result of Popa and Shlyakhtenko for $\mathcal{P}$ infinite depth [PS03]:

Corollary. *Every infinite depth subfactor planar algebra is the standard invariant of $\mathcal{N} \subset \mathcal{M}$ where $\mathcal{N}, \mathcal{M} \cong L(\mathbb{F}_\infty)$.*

Outline of the proof

To prove the above theorem, we will bootstrap the proof from [GJS11] of the finite-depth case to the infinite-depth case using standard embedding tricks.

The GJS Construction: We will recall the construction of Guionnet, Jones and Shlyakhtenko. For more details, see [GJS10] and [GJS11], and Chapters 3 and 4. The placement of the $\star$ below is how the construction first appeared in [GJS10] and [GJS11]. The author has since found it more convenient to place the $\star$ at the bottom of the box. Let $\mathcal{P} = (P_{n,\pm})_{n \geq 0}$ be a subfactor planar algebra with loop parameter $\delta > 1$. Set $\text{Gr}_k(\mathcal{P}^+) = \bigoplus_{n \geq 0} P_{k,n}^+$ where $P_{k,n}^+ = \mathcal{P}_{2k+2n,+}$ and an element of $P_{k,n}^+$ is represented as
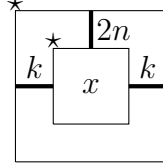

where the $\star$ is always in an unshaded region and a thick string with a j next to it denotes j parallel strings. If $x \in P_{k,n}^+$ and $y \in P_{k,m}^+$ then define a multiplication $\wedge_k$ by

$$x \wedge_k y = \quad$$
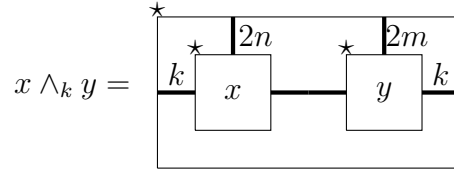
which is an element in $P_{k,m+n}^+$. One can endow $Gr_k(P^+)$ with the following trace: if $x \in P_{k,n}^+$ then

$$\text{tr}(x) = \delta^{-k} \cdot \quad$$
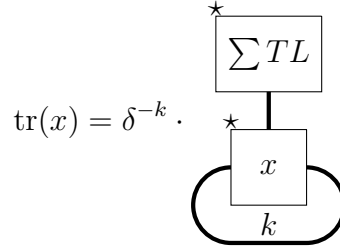
where $\sum TL$ denotes the sum of all Temperely-Lieb diagrams, i.e. all planar pairings of the $2n$ strings on top of x. This trace is positive definite, and one can form the von Neumann algebra M_k which is the strong closure of $Gr_k(P^+)$ acting on $L^2(Gr_k(P^+))$ by left multiplication (under $\wedge_k$). It is shown that M_k is a II_1 factor. Moreover one can view $x \in M_k$ as an element in M_{k+1} as follows:
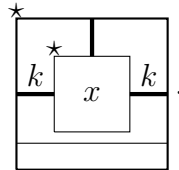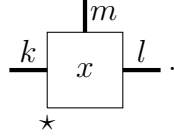

With this identification, M_k is a von Neumann subalgebra of M_{k+1} and $M_0 \subset M_1 \subset \cdots \subset M_k \subset \cdots$ is a Jones tower of II_1 factors with standard invariant $\mathcal{P}$.

To identify the isomorphism type of the M_k, we look at the semi-finite algebra

$$V_+ = \bigoplus_{k+l+m \text{ even}} P_{k,l,m}^+$$

where $P^+_{k,l,m} = P_{k+l+m,+}$ and is spanned by boxes of the form

$$\begin{array}{c} k\ \boxed{x}\ l \\ \star \end{array}$$

The element x above is identified with the following element of $P^+_{k+2p,l+2q,m}$:

$$\delta^{-(p+q)/2}\ \begin{array}{c} m \\ k\ \boxed{x}\ l \\ \star \\ :p \qquad q: \end{array} \tag{2.1}$$

where there are p cups on the left and q cups on the right. In the future chapters, it is convenient not to make such an identification, but this approach to the semifinite algebra was the original one that appeared in [GJS11]. Under these identifications, V_+ completes to a semifinite von Neumann algebra, $\mathcal{M}_+$ where the multiplication is given by

$$\left(\begin{array}{c} m \\ k\ \boxed{x}\ l \\ \star \end{array}\right) \cdot \left(\begin{array}{c} m' \\ k'\ \boxed{y}\ l' \\ \star \end{array}\right) = \delta_{l,k'}\ \begin{array}{c} n \qquad m' \\ k\ \boxed{x}\ l\ \boxed{y}\ l' \\ \star \qquad \star \end{array}$$

where we have assumed that we have added enough cups as in diagram (3.1) so that l and k' are either the same or differ by 1. The trace on $\mathcal{M}_+$ is given by

$$\mathrm{Tr}(x) = \begin{array}{c} \star \\ \boxed{\sum TL} \\ | \\ \boxed{x} \\ \star\ k \end{array}$$

provided that the number of strings on the left and right of x have the same parity and is zero otherwise. It is easy to check that the identification in diagram (3.1) respects both the trace and multiplication.

The algebras M_{2k} above are a compression of $\mathcal{M}_+$ by the projection p^+_{2k} where for general n,

$$p^+_n = \begin{array}{c} \boxed{n} \\ \hline \\ \star \end{array}$$

Similarly, we can consider a semifinite von Neumann algebra $\mathcal{M}_-$ generated by the $P_{n,-}$'s (where the $\star$ is in a *shaded* region and on the bottom of the box), and if we define projections p_n^-, then M_{2k+1} is the compression of $\mathcal{M}_-$ by p_{2k+1}^-.

A diagrammatic argument shows that $\mathcal{M}_+$ is generated by

$$\mathcal{A}_+ = \left(\bigcup_{k,\ell} P_{k,\ell,0}^+\right)'' \quad \text{and} \quad X = \text{s} - \lim_{k\to\infty} \boxed{ 2k } + \boxed{ 2k }$$

where the limit above is in the strong operator topology. This element is an $\mathcal{A}_+$-valued semicircular element in the sense of [Shl99] and is used in the calculation of the isomorphism class of the algebras M_k

The finite depth case: Let Γ denote the principal graph of $\mathcal{P}$ with edge set $E(\Gamma)$ and initial vertex $*$. Let $\ell^\infty(\Gamma)$ as the von Neumann algebra of bounded functions on the vertices of Γ and endow $\ell^\infty(\Gamma)$ with a trace tr such that $\text{tr}(p_v) = \mu_v$, where p_v is the delta function at v and μ_v is the entry corresponding to a fixed Perron-Frobenius eigenvector for Γ with $\mu_* = 1$. From [GJS11], $M_0 = p_* \mathcal{N}(\Gamma) p_*$ where $\mathcal{N}(\Gamma)$ is the von Neumann algebra generated by $(\ell^\infty(\Gamma), \text{tr})$ and $\ell^\infty(\Gamma)$-valued semicircular elements $\{X_e : e \in E(\Gamma)\}$ which are compressions of X by partial isometries in $\mathcal{A}_+$ and are free with amalgamation over $\ell^\infty(\Gamma)$. Each X_e is supported under $p_v + p_w$, where e connects v and w, and we have $X_e = p_v X_e p_w + p_w X_e p_v$. Assuming that $\mu_v \geq \mu_w$, the scalar-valued distribution of X_e^2 in $(p_v + p_w)\mathcal{N}(\Gamma)(p_v + p_w)$ is free-Poisson with an atom of size $\frac{\mu_v - \mu_w}{\mu_v + \mu_w}$ at 0. Therefore,

$$vN(\ell^\infty(\Gamma), X_e) = L(\mathbb{Z}) \otimes M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \ell^\infty(\Gamma \setminus \{v, w\})$$

with $p_w = (1 \otimes e_{1,1}) \oplus 0 \oplus 0$ and $p_v = (1 \otimes e_{2,2}) \oplus 1 \oplus 0$, where $\{e_{i,j} : 1 \leq i, j \leq 2\}$ is a system of matrix units for $M_2(\mathbb{C})$. If Γ is finite, Dykema's formulas for computing certain amalgamated free products [Dyk93, Dyk11] show that $\mathcal{N}(\Gamma)$ is an interpolated free group factor and the compression formula gives the result for M_0. Since M_{2n} is a δ^{2n}-amplification of M_0, the result holds for M_{2n}. The factor M_1 is a compression of $\mathcal{N}(\Gamma^*)$ with Γ^* the dual principal graph of $\mathcal{P}$. Applying the same analysis to Γ^* gives the formula for the M_{2n+1}'s.

The infinite depth case: We similarly define $\mathcal{N}(\Gamma)$ for an arbitrary connected, loopless (not necessarily bipartite) graph Γ. If Γ is finite, we show that $\mathcal{N}(\Gamma) \cong L(\mathbb{F}_t) \oplus A$ where A is finite-dimensional and abelian (A can possibly be $\{0\}$). Furthermore, if p_Γ is the identity of $L(\mathbb{F}_t)$ and Γ' is a finite graph containing Γ, then the inclusion $p_\Gamma \mathcal{N}(\Gamma) p_\Gamma \to p_\Gamma \mathcal{N}(\Gamma') p_\Gamma$ is a standard embedding of interpolated free group factors (see Definition 2.2.2 and Remark 2.2.3 below). Therefore, if $\mathcal{P}$ is infinite depth with principal graph Γ, we write Γ as an increasing union of finite graphs Γ_k where Γ_k is Γ truncated at depth k. Since standard embeddings are preserved by cut-downs, the inclusion $p_* \mathcal{N}(\Gamma_k) p_* \to p_* \mathcal{N}(\Gamma_{k+1}) p_*$ is a standard embedding. As M_0 is the inductive limit of the $p_* \mathcal{N}(\Gamma_k) p_*$'s, it is an interpolated free group factor where the parameter is the limit of the parameters for the $p_* \mathcal{N}(\Gamma_k) p_*$'s, which is ∞. Since the factors M_{2k} are amplifications of M_0, $M_{2k} \cong L(\mathbb{F}_\infty)$. Applying the same analysis to Γ^* (the dual principal graph of $\mathcal{P}$) shows that $M_{2k+1} \cong L(\mathbb{F}_\infty)$.

Organization: Section 2.2 covers some preliminary material on interpolated free group factors, free dimension, and standard embeddings. Section 2.3 introduces $\mathcal{N}(\Gamma)$ and establishes both its structure and how it includes into $\mathcal{N}(\Gamma')$ for Γ a subgraph of Γ'. Section 2.4 provides the proof that the factors M_k above are all isomorphic to $L(\mathbb{F}_\infty)$.

2.2 Preliminaries

Dykema [Dyk94] and Rădulescu [Răd94] independently developed interpolated free group factors $L(\mathbb{F}_t)$ for $1 < t \leq \infty$. These coincide with the usual free group factors when $t \in \mathbb{N} \cup \{\infty\}$ and they satisfy

$$L(\mathbb{F}_t) * L(\mathbb{F}_s) = L(\mathbb{F}_{s+t}) \text{ and } L(\mathbb{F}_t)_\gamma = L(\mathbb{F}(1 + \gamma^{-2}(t-1))),$$

where M_γ is the $\gamma-$amplification of the II_1 factor M. It is known that either the interpolated free group factors are all isomorphic or they are pairwise non-isomorphic [Dyk94, Răd94].

Notation 2.2.1. Throughout this paper, we will be concerned with finite von Neumann algebras $(\mathcal{M}, tr)$ which can be written in the form

$$\mathcal{M} = \overset{p_0}{\underset{\gamma_0}{\mathcal{M}_0}} \oplus \bigoplus_{j \in J} \overset{p_j}{\underset{\gamma_j}{L(\mathbb{F}_{t_j})}} \oplus \bigoplus_{k \in K} \overset{q_k}{\underset{\alpha_k}{M_{n_k}(\mathbb{C})}}$$

where $\mathcal{M}_0$ is a diffuse hyperfinite von Neumann algebra, $L(\mathbb{F}_{t_j})$ is an interpolated free group factor with parameter t_j, $M_{n_k}(\mathbb{C})$ is the algebra of $n_k \times n_k$ matrices over the scalars, and the sets J and K are at most finite and countably infinite respectively. We use p_j to denote the projection in $L(\mathbb{F}_{t_j})$ corresponding to the identity of $L(\mathbb{F}_{t_j})$ and q_k to denote a minimal projection in $M_{n_k}(\mathbb{C})$. The projections p_j and q_k have traces γ_j and α_k respectively. Let p_0 be the identity in $\mathcal{M}_0$ with trace γ_0. We write $\overset{p,q}{M_2}(\mathbb{C})$ to mean $M_2(\mathbb{C})$ with a choice of minimal orthogonal projections p and q.

If the interpolated free group factors turn out to be non-isomorphic, it is desirable to be able to calculate which interpolated free group factors appear in amalgamated free products. To help facilitate this calculation, Dykema defined the notion of free dimension. In general, one has

$$\text{fdim}(\mathcal{M}_1 \underset{D}{*} \mathcal{M}_2) = \text{fdim}(\mathcal{M}_1) + \text{fdim}(\mathcal{M}_2) - \text{fdim}(D)$$

whenever $\mathcal{M}_1$ and $\mathcal{M}_2$ are of the form of Notation 3.4.18 and D is finite dimensional [Dyk93, Dyk95, Dyk11, DR11]. In general, for the algebra $\mathcal{M}$ in Notation 3.4.18, we have

$$\text{fdim}(\mathcal{M}) = 1 + \sum_{j \in J} \gamma_j^2 (t_j - 1) - \sum_{k \in K} \alpha_k^2.$$

Notice that this includes the special case $\text{fdim}(L(\mathbb{F}_t)) = t$.

Of course if the interpolated free group factors are isomorphic, then the free dimension is not well defined; however, the only purpose of the free dimension is to determine the parameter of interpolated free group factors which show up in amalgamated free products. Therefore all the lemmas below will remain valid if all references to free dimension are removed.

Of critical importance will be the notion of a *standard embedding* of interpolated free group factors [Dyk93]. This is a generalization of a mapping $\mathbb{F}_n \to \mathbb{F}_m$ for $m > n$ sending the n generators of $\mathbb{F}_n$ onto n of the m generators for $\mathbb{F}_m$.

Definition 2.2.2. Let $1 < r < s$ and $\phi : L(\mathbb{F}_r) \to L(\mathbb{F}_s)$ be a von Neumann algebra homomorphism. We say that ϕ is a standard embedding if there exist nonempty sets $S \subset S'$, a family of projections $\{p_{s'} : s' \in S'\}$ with $p_{s'} \in R$ (the hyperfinite II_1 factor), a free family $\{X^{s'} : s' \in S'\}$ of semicircular elements which are also free from R, and isomorphisms

$$\alpha : L(\mathbb{F}_r) \to (R \cup \{p_s X^s p_s\}_{s \in S})'' \text{ and } \beta : L(\mathbb{F}_s) \to (R \cup \{p_{s'} X^{s'} p_{s'}\}_{s' \in S'})''$$

such that $\phi = \beta^{-1} \circ \iota \circ \alpha$ where $\iota : (R \cup \{p_s X^s p_s\}_{s \in S})'' \to (R \cup \{p_{s'} X^{s'} p_{s'}\}_{s' \in S'})''$ is the canonical inclusion. We will write $A \overset{s.e.}{\hookrightarrow} B$ to mean that the inclusion of A into B is a standard embedding.

Remark 2.2.3. Dykema in [Dyk93] and [Dyk95] shows the following useful properties of standard embeddings which we will use extensively in this paper.

(1) If A is an interpolated free group factor, the canonical inclusion $A \to A * \mathcal{M}$ is a standard embedding whenever $\mathcal{M}$ is of the form in Notation 3.4.18.

(2) A composite of standard embeddings is a standard embedding.

(3) If $A_n = L(\mathbb{F}_{s_n})$ with $s_n < s_{n+1}$ for all n and $\phi_n : A_n \overset{s.e.}{\hookrightarrow} A_{n+1}$, then the inductive limit of the A_n with respect to ϕ_n is $L(\mathbb{F}_s)$ where $s = \lim_{n \to \infty} s_n$.

(4) If $t > s$ then $\phi : L(\mathbb{F}_s) \overset{s.e.}{\hookrightarrow} L(\mathbb{F}_t)$ if and only if for any nonzero projection $p \in L(\mathbb{F}_s)$, $\phi|_{pL(\mathbb{F}_s)p} : pL(\mathbb{F}_s)p \overset{s.e.}{\hookrightarrow} \phi(p)L(\mathbb{F}_t)\phi(p)$.

Our work will rely heavily on the following two lemmas.

Lemma 2.2.4 ([DR11]). *Let* $\mathcal{N} = (M_n(\mathbb{C}) \overset{p}{\oplus} B) \underset{D}{*} C$ *and* $\mathcal{M} = (M_n(\mathbb{C}) \otimes A \oplus B) \underset{D}{*} C$ *where* A, B *and* C *are finite von Neumann algebras and* D *is a finite dimensional abelian von Neumann algebra. Let* E *be the trace-preserving conditional expectation of* $\mathcal{M}$ *onto* D. *Assume* p *lies under a minimal projection in* D *and* $E|_{M_n(\mathbb{C}) \otimes A} = E|_{M_n(\mathbb{C})} \otimes \mathrm{tr}_A$. *Then* $p\mathcal{M}p = p\mathcal{N}p * A$ *and the central support of* p *in* $\mathcal{M}$ *is the same as that in* $\mathcal{N}$.

Lemma 2.2.5 ([DR11]). *Let* $\mathcal{N} = (M_m(\mathbb{C}) \overset{p}{\oplus} M_{n-m}(\mathbb{C}) \overset{q}{\oplus} B) \underset{D}{*} C$ *and* $\mathcal{M} = (M_n(\mathbb{C}) \oplus B) \underset{D}{*} C$ *with* B, C, D *as in Lemma 2.2.4. Assume* p *and* q *sit under minimal projections in* D *and* p *is equivalent to* q *in* $\mathcal{N}$. *Then* $p\mathcal{M}p = p\mathcal{N}p * L(\mathbb{Z})$ *and the central support of* p *in* $\mathcal{M}$ *is the same as that in* $\mathcal{N}$.

Note that if A, B and C are in the form in Notation 3.4.18, and if $\mathcal{N}$ is an interpolated free group factor, then the proofs of the above lemmas in [DR11] show that $p\mathcal{N}p \to p\mathcal{M}p$ of Lemmas 2.2.4 and 2.2.5 are standard embeddings. This implies $\mathcal{N} \overset{s.e.}{\hookrightarrow} \mathcal{M}$ by Remark 2.2.3.

Also, we will use Dykema's results for free products of finite-dimensional, abelian, von Neumann algebras. More precisely, we will use the following theorem, which appears in [Dyk93]

Theorem 2.2.6. *We have the following formulas for free products:*

- *Assume*
$$1 \geq \alpha_1 \geq \beta_1 \geq \beta_2 \geq \alpha_2 \geq 0 \ \text{with} \ \alpha_1 + \alpha_2 = 1 = \beta_1 + \beta_2.$$
  Then
$$\left(\underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \underset{\alpha_2}{\overset{p_2}{\mathbb{C}}} \right) * \left(\underset{\beta_1}{\overset{q_1}{\mathbb{C}}} \oplus \underset{\beta_2}{\overset{q_2}{\mathbb{C}}} \right) = \left(\underset{2\alpha_2}{(M_2(\mathbb{C}) \otimes L(\mathbb{Z}))} \oplus \underset{\alpha_1 - \beta_1}{\overset{p_1 \wedge q_1}{\mathbb{C}}} \oplus \underset{\alpha_1 - \beta_2}{\overset{p_1 \wedge q_2}{\mathbb{C}}} \right)$$

- *Assume*
$$1 \geq \alpha_1 \geq \cdots \geq \alpha_n \geq 0 \ \text{and} \ 1 \geq \beta_1 \geq \cdots \geq \beta_m \geq 0$$
  with $n, m \geq 2$, $n + m \geq 5$, and $\sum_{i=1}^n \alpha_i = 1 = \sum_{j=1}^m \beta_j$. Set $B = \{(i,j) : \alpha_i + \beta_j > 1\}$. Then
$$\left(\underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \cdots \oplus \underset{\alpha_n}{\overset{p_n}{\mathbb{C}}} \right) * \left(\underset{\alpha_1}{\overset{q_1}{\mathbb{C}}} \oplus \cdots \oplus \underset{\beta_m}{\overset{q_m}{\mathbb{C}}} \right) = L(\mathbb{F}_t) \oplus \left(\bigoplus_{(i,j) \in B} \underset{\alpha_i + \beta_j - 1}{\overset{p_i \wedge q_j}{\mathbb{C}}} \right)$$
  where t is calculated using free dimension.

- *Assume that A is either a diffuse hyperfinite finite von Neumann algebra or an interpolated free group factor. Let α_i and β_j be as above, except that $i + j$ need not exceed 4 and $\sum_{i=1}^n \alpha_i < 1$. Then*
$$\left(A \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \cdots \oplus \underset{\alpha_n}{\overset{p_n}{\mathbb{C}}} \right) * \left(\underset{\alpha_1}{\overset{q_1}{\mathbb{C}}} \oplus \cdots \oplus \underset{\beta_m}{\overset{q_m}{\mathbb{C}}} \right) = L(\mathbb{F}_t) \oplus \left(\bigoplus_{(i,j) \in B} \underset{\alpha_i + \beta_j - 1}{\overset{p_i \wedge q_j}{\mathbb{C}}} \right)$$
  where t is calculated using free dimension.

2.3   A von Neumann algebra associated to a finite connected graph

Let Γ be a connected, loopless, finite graph with edge set $E(\Gamma)$ and vertex set $V(\Gamma)$. Assume further that each vertex $v \in V(\Gamma)$ is weighted by a real constant $\gamma_v > 0$ with $\sum_{v \in \Gamma} \gamma_v = 1$ (the weighting does *not* have to be the Perron-Frobenius weighting. Consider the abelian von Neumann algebra $(\ell^\infty(\Gamma), \text{tr})$ where tr is defined as follows: Let p_v be the indicator function on the vertex v. Then $\text{tr}(p_v)$ will be γ_v. We construct a finite von Neumann algebra associated to Γ (also see [KS11]).

Definition 2.3.1. Let Γ be as above, e be an edge in Γ connecting the vertices v and w, and assume $\gamma_v \geq \gamma_w$. Define
$$\mathcal{A}_e = \underset{2\gamma_w}{M_2(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \underset{\gamma_v - \gamma_w}{\overset{p_v^e}{\mathbb{C}}} \oplus \ell^\infty(\Gamma \setminus \{v, w\})$$

where the trace on $M_2(\mathbb{C}) \otimes L(\mathbb{Z})$ is $\operatorname{tr}_{M_2(\mathbb{C})} \otimes \operatorname{tr}_{L(\mathbb{Z})}$. $\mathcal{A}_e$ includes $\ell^\infty(\Gamma)$ by setting

$$p_w = 1 \otimes e_{1,1} \oplus 0 \oplus 0 \text{ and}$$
$$p_v = 1 \otimes e_{2,2} \oplus 1 \oplus 0.$$

Therefore, the trace preserving conditional expectation $E_e : \mathcal{A}_e \to \ell^\infty(\Gamma)$ has the property $E_e|_{M_2 \otimes L(Z)} = E_e|_{M_2} \otimes \operatorname{tr}|_{L(\mathbb{Z})}$. We define $\mathcal{N}(\Gamma)$, the von Neumann algebra associated to Γ, by

$$\mathcal{N}(\Gamma) = \underset{\ell^\infty(\Gamma)}{*} (\mathcal{A}_e, E_e)_{e \in E(\Gamma)}.$$

Remark 2.3.2. If Γ is an infinite graph with a weighting that is not ℓ^1, then we can still define $\mathcal{N}(\Gamma)$ as in 2.3.1 although it will be a semifinite algebra. Given $e \in E(\Gamma)$ connecting vertices v and w, the compressed algebra $(p_v + p_w)\mathcal{A}_e(p_v + p_w)$ is still finite, and if $E_e : \mathcal{A}_e \to \ell^\infty(\Gamma)$ is the (tracial-weight) preserving conditional expectation, then E_e is clearly normal on $(p_v + p_w)\mathcal{A}_e(p_v + p_w)$ and is the identity on $(1 - p_v - p_w)\mathcal{A}_e(1 - p_v - p_w)$. Therefore one can take the algebraic free product Q of $(\mathcal{A}_e)_{e \in E(\Gamma)}$ with amalgamation over $\ell^\infty(\Gamma)$ and represent it on $L^2(Q, Tr \circ \underset{\ell^\infty(\Gamma)}{*} E_e)$ to obtain $\mathcal{N}(\Gamma)$.

Definition 2.3.3. Let $v, w \in V(\Gamma)$ We write $v \sim w$ if v and w are connected by at least 1 edge in Γ and denote $n_{v,w}$ be the number of edges joining v and w. We set $\alpha_v^\Gamma = \sum_{w \sim v} n_{v,w} \gamma_w$, and define $B(\Gamma) = \{v \in V(\Gamma) : \gamma_v > \alpha_v^\Gamma\}$.

For the rest of this section, we assume Γ is finite. We show that $\mathcal{N}(\Gamma)$ is the direct sum of an interpolated free group factor and a finite dimensional abelian algebra. More precisely, we prove the following theorem, which has a direct analogy with Theorem 2.2.6:

Theorem 2.3.4. *Let Γ and Γ' be connected, finite, loopless, and weighted graphs with at least 2 edges. Then $\mathcal{N}(\Gamma) \cong L(\mathbb{F}_{t_\Gamma})^{p^\Gamma} \oplus \bigoplus_{v \in B(\Gamma)^{\gamma_v - \alpha_v^\Gamma}} \overset{r_v^\Gamma}{\mathbb{C}}$ where $r_v^\Gamma \leq p_v$ and t_Γ is such that this algebra has the appropriate free dimension. Furthermore, if Γ is a subgraph of Γ', then $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$.*

Notice that since we are assuming that all vertices have positive weight, it follows that $p_v p^\Gamma \neq 0$ for all $v \in \Gamma$. We will prove Theorem 2.3.4 in a series of lemmas.

Lemma 2.3.5. *Let Γ be a finite, connected, weighted, loopless graph with 2 edges. Then $\mathcal{N}(\Gamma)$ is of the form in Theorem 2.3.4.*

Proof. Set $D = \ell^\infty(\Gamma)$. There are two overlying cases to consider. One where Γ has 2 vertices and the other where Γ has 3 vertices.

 Case 1: Assume that Γ has 2 vertices v, w and 2 edges e_1 and e_2 connecting v and w and without loss of generality assume $\gamma_v \geq \gamma_w$. We obtain the desired formula for $\mathcal{N}(\Gamma)$ by examining the following sequence of inclusions:

$$\mathcal{N}_0 = \left(\overset{p_w}{\underset{\gamma_w}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\gamma_w}{\mathbb{C}}} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_w}{\underset{\gamma_w}{\mathbb{C}}} \oplus \overset{q_v^I}{\underset{\gamma_w}{\mathbb{C}}} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right)$$

$$\cap$$

$$\mathcal{N}_1 = \left(\overset{p_w, p_v^I}{\underset{\gamma_w}{M_2(\mathbb{C})}} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_w}{\underset{\gamma_w}{\mathbb{C}}} \oplus \overset{q_v^I}{\underset{\gamma_w}{\mathbb{C}}} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right)$$

$$\cap$$

$$\mathcal{N}_2 = \left(\overset{p_w, p_v^I}{\underset{\gamma_w}{M_2(\mathbb{C})}} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_w, q_v^I}{\underset{\gamma_w}{M_2(\mathbb{C})}} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right)$$

$$\cap$$

$$\mathcal{N}_3 = \left(\underset{2\gamma_w}{M_2(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_w}{\underset{\gamma_w}{M_2(\mathbb{C})}} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right)$$

$$\cap$$

$$\mathcal{N}(\Gamma) = \left(\underset{2\gamma_w}{M_2(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \underset{D}{*} \left(\underset{2\gamma_w}{M_2(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right),$$

where p_v decomposes as $(1 \otimes e_{2,2}) \oplus 1$ in $\mathcal{A}_{e_1}$ and $\mathcal{A}_{e_2}$ with $p_v^I = 1 \otimes e_{2,2}$ in $\mathcal{A}_{e_1}$ and $q_v^I = 1 \otimes e_{2,2}$ in $\mathcal{A}_{e_2}$. From Lemma 2.2.4 and [Dyk93], we see that

$$p_v \mathcal{N}_0 p_v = \overset{p_v^I}{\underset{\frac{\gamma_w}{\gamma_v}}{\mathbb{C}}} \oplus \underset{\frac{\gamma_v - \gamma_w}{\gamma_v}}{\mathbb{C}} * \overset{q_v^I}{\underset{\frac{\gamma_w}{\gamma_v}}{\mathbb{C}}} \oplus \underset{\frac{\gamma_v - \gamma_w}{\gamma_v}}{\mathbb{C}} = \begin{cases} \underset{2\frac{\gamma_v - \gamma_w}{\gamma_v}}{M_2(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \overset{p_v^I \wedge q_v^I}{\underset{\frac{2\gamma_w - \gamma_v}{\gamma_v}}{\mathbb{C}}} & \text{if } 2\gamma_w \geq \gamma_v \\ \underset{\frac{2\gamma_w}{\gamma_v}}{M_2(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \overset{(p_v - p_v^I) \wedge (p_v - q_v^I)}{\underset{\frac{\gamma_v - 2\gamma_w}{\gamma_v}}{\mathbb{C}}} & \text{if } \gamma_v > 2\gamma_w \end{cases}$$

where in the first algebra, the identity element copy of $\mathbb{C}$ is $p_v^I \wedge q_v^I$ and and in the second algebra, the identity of the copy of $\mathbb{C}$ is orthogonal to both p_v^I and q_v^I.

<u>*Case 1a*</u>: Assume $2\gamma_w \geq \gamma_v$. As $p_v \wedge q_v$ is minimal and central in $\mathcal{N}_0$, we see that

$$\mathcal{N}_1 = \underset{3(\gamma_v - \gamma_w)}{M_3(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \overset{p_v^I \wedge q_v^I}{\underset{2\gamma_w - \gamma_v}{M_2(\mathbb{C})}}.$$

By [Dyk93], the projections p_v^I and q_v^I are equivalent in $\mathcal{N}_0$, so it follows that p_w is equivalent to q_v^I in $\mathcal{N}_1$. Therefore by Lemma 2.2.5,

$$p_w \mathcal{N}_2 p_w = p_w \mathcal{N}_1 p_w * L(\mathbb{Z}) = (L(\mathbb{Z}) \oplus \underset{\frac{\gamma_v - \gamma_w}{\gamma_w}}{\mathbb{C}}) * L(\mathbb{Z}),$$

$$\underset{\frac{2\gamma_w - \gamma_v}{\gamma_w}}{}$$

which is an interpolated free group factor $L(\mathbb{F}_t)$ by Theorem 2.2.6. As the central support of p_w in $\mathcal{N}_2$ is 1, it follows that $\mathcal{N}_2$ is also an interpolated free group factor $L(\mathbb{F}_{t_1})$. To finish up this case,

we write

$$\mathcal{N}_2 \subset \mathcal{N}_3 = \left(M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{2\gamma_w}{\oplus} \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_w}{M_2(\mathbb{C})} \underset{\gamma_w}{\oplus} \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \text{ and}$$

$$\mathcal{N}_3 \subset \mathcal{N}(\Gamma) = \left(M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{2\gamma_w}{\oplus} \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right) \underset{D}{*} \left(M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{2\gamma_w}{\oplus} \underset{\gamma_v - \gamma_w}{\mathbb{C}} \right),$$

and use Lemma 2.2.4 twice, as well as the amplification formula for interpolated free group factors to obtain that $\mathcal{N}(\Gamma)$ is an interpolated free group factor.

Case 1b: The case $\gamma_v > 2\gamma_w$ for $\mathcal{N}_0$ is treated exactly the same as the first with only the caveat that the central support of p_w in $\mathcal{N}_1$ is a projection of trace $3\gamma_w$, so $\mathcal{N}_1$, and thus $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}(\Gamma)$, have a minimal central projection of trace $\gamma_v - 2\gamma_w$.

Case 2: Next we consider the case where Γ has 3 vertices v_1, v_2, and v_3 with weights γ_1, γ_2, and γ_3 respectively, where v_2 is connected to v_1 by e_1 and to v_3 by e_2. There are two sub-cases to consider. The first is when $\gamma_2 \geq \gamma_1 \geq \gamma_3$, and the second is when $\gamma_1 > \gamma_2$ and $\gamma_1 \geq \gamma_3$.

Case 2a: We examine the following sequence of inclusions:

$$\mathcal{N}_0 = \left(\overset{p_{v_1}}{\mathbb{C}} \underset{\gamma_1}{\oplus} \overset{p_2^I}{\mathbb{C}} \underset{\gamma_1}{\oplus} \overset{p_2^{II}}{\mathbb{C}} \underset{\gamma_2-\gamma_1}{\oplus} \overset{p_{v_3}}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_{v_1}}{\mathbb{C}} \underset{\gamma_1}{\oplus} \overset{q_2^I}{\mathbb{C}} \underset{\gamma_2-\gamma_3}{\oplus} \overset{q_2^{II}}{\mathbb{C}} \underset{\gamma_3}{\oplus} \overset{p_{v_3}}{\mathbb{C}} \right)$$

$$\cap$$

$$\mathcal{N}_1 = \left(\overset{p_{v_1},p_2^I}{M_2(\mathbb{C})} \underset{\gamma_1}{\oplus} \overset{p_2^{II}}{\mathbb{C}} \underset{\gamma_2-\gamma_1}{\oplus} \overset{p_{v_3}}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_{v_1}}{\mathbb{C}} \underset{\gamma_1}{\oplus} \overset{q_2^I}{\mathbb{C}} \underset{\gamma_2-\gamma_3}{\oplus} \overset{q_2^{II},p_{v_3}}{M_2(\mathbb{C})} \right)$$

$$\cap$$

$$\mathcal{N}_2 = \left(M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{2\gamma_1}{\oplus} \overset{p_2^{II}}{\mathbb{C}} \underset{\gamma_2-\gamma_1}{\oplus} \overset{p_{v_3}}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_{v_1}}{\mathbb{C}} \underset{\gamma_1}{\oplus} \overset{q_2^I}{\mathbb{C}} \underset{\gamma_2-\gamma_3}{\oplus} \overset{q_2^{II},p_{v_3}}{M_2(\mathbb{C})} \right)$$

$$\cap$$

$$\mathcal{N}(\Gamma) = \left(M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{\gamma_1}{\oplus} \overset{p_2^{II}}{\mathbb{C}} \underset{\gamma_2-\gamma_1}{\oplus} \overset{p_{v_3}}{\mathbb{C}} \right) \underset{D}{*} \left(\overset{p_{v_1}}{\mathbb{C}} \underset{\gamma_1}{\oplus} \overset{q_2^I}{\mathbb{C}} \underset{\gamma_2-\gamma_3}{\oplus} M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \right),$$

where p_{v_2} decomposes as $1 \otimes e_{22} \oplus 1 \oplus 0$ in $\mathcal{A}_{e_1}$ and $0 \oplus 1 \oplus 1 \otimes e_{1,1}$ in $\mathcal{A}_{e_2}$. We set p_2^I and p_2^{II} as the summands of p_{v_2} supported in the diffuse and atomic parts of $\mathcal{A}_{e_1}$ respectively and q_2^I and q_2^{II} as the summands of p_{v_2} supported in the atomic and diffuse parts of $\mathcal{A}_{e_2}$ respectively. As above,

$$p_{v_2}\mathcal{N}_0 p_{v_2} = \overset{p_2^I}{\underset{\frac{\gamma_1}{\gamma_2}}{\mathbb{C}}} \oplus \overset{p_2^{II}}{\underset{\frac{\gamma_2-\gamma_1}{\gamma_2}}{\mathbb{C}}} * \overset{q_2^I}{\underset{\frac{\gamma_2-\gamma_3}{\gamma_2}}{\mathbb{C}}} \oplus \overset{q_2^{II}}{\underset{\frac{\gamma_3}{\gamma_2}}{\mathbb{C}}} = \begin{cases} M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{2\frac{\gamma_2-\gamma_1}{\gamma_2}}{\oplus} \overset{p_2^I \wedge q_2^I}{\underset{\frac{\gamma_1-\gamma_3}{\gamma_2}}{\mathbb{C}}} \oplus \overset{p_2^I \wedge q_2^{II}}{\underset{\frac{\gamma_1-\gamma_2+\gamma_3}{\gamma_2}}{\mathbb{C}}} & \text{if } \gamma_2 \leq \gamma_1 + \gamma_3 \\[2em] M_2(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{2\frac{\gamma_3}{\gamma_2}}{\oplus} \overset{p_2^I \wedge q_2^I}{\underset{\frac{\gamma_1-\gamma_3}{\gamma_2}}{\mathbb{C}}} \oplus \overset{p_2^{II} \wedge q_2^I}{\underset{\frac{\gamma_2-\gamma_1-\gamma_3}{\gamma_2}}{\mathbb{C}}} & \text{if } \gamma_2 > \gamma_1 + \gamma_3 \end{cases}.$$

Case 2a(i): Assume $\gamma_2 \leq \gamma_1 + \gamma_3$. Since the two new matrix units in $\mathcal{N}_1$ introduce equivalences between p_{v_1} and p_2^I and between q_2^{II} and p_{v_3} respectively, we see that $\mathcal{N}_1$ has the same number of

summands as $p_{v_2}\mathcal{N}_0 p_{v_2}$, but with suitable amplifications. Explicitly, we find that

$$\mathcal{N}_1 = M_4(\mathbb{C}) \otimes L(\mathbb{Z}) \underset{4(\gamma_2-\gamma_1)}{\oplus} \overset{p_2^I \wedge q_2^I}{M_2(\mathbb{C})} \underset{\gamma_1-\gamma_3}{\oplus} \overset{p_2^I \wedge q_2^{II}}{M_3(\mathbb{C})}$$

where the central support of p_{v_1} is the identity. By applying Lemma 2.2.4 and applying the same reasoning as case 1, we see that $\mathcal{N}_2$ is an interpolated free group factor. Applying Lemma 2.2.4 again shows that $\mathcal{N}(\Gamma)$ is an interpolated free group factor.

Case 2a(ii): Assume $\gamma_2 > \gamma_1 + \gamma_3$. This case is treated in the same way as above except that in $\mathcal{N}_1$, $q_2^I \wedge p_2^{II}$ with trace $\gamma_2 - \gamma_3 - \gamma_1$ is minimal and central, so it is minimal and central in $\mathcal{N}(\Gamma)$.

Case 2b: Now let γ_1 be the largest weight. First assume $\gamma_3 \geq \gamma_2$. We consider the algebra

$$\mathcal{N}_1 = \left(\overset{p_1^I}{\underset{\gamma_1-\gamma_2}{\mathbb{C}}} \oplus \overset{p_{v_2}}{\underset{\gamma_2}{M_2(\mathbb{C})}} \oplus \overset{p_{v_3}}{\underset{\gamma_3}{\mathbb{C}}} \right) \underset{D}{*} \left(\overset{p_{v_1}}{\underset{\gamma_1}{\mathbb{C}}} \oplus \overset{p_{v_2}}{\underset{\gamma_2}{M_2(\mathbb{C})}} \oplus \overset{p_3^I}{\underset{\gamma_3-\gamma_2}{\mathbb{C}}} \right),$$

where the projections orthogonal to p_{v_2} in each copy of $M_2(\mathbb{C})$ sit under p_i and $p_i^I \leq p_{v_i}$ for $i = 1$ or 3. It follows that $\mathcal{N}_1 = \overset{p_1^I}{\underset{\gamma_1-\gamma_2}{\mathbb{C}}} \oplus \overset{p_{v_2}}{\underset{\gamma_2}{M_3(\mathbb{C})}} \oplus \overset{p_3^I}{\underset{\gamma_3-\gamma_2}{\mathbb{C}}}$, so tensoring each copy of $M_2(\mathbb{C})$ with $L(\mathbb{Z})$ and using the standard arguments as above show that

$$\mathcal{N}(\Gamma) = \overset{p_1^I}{\underset{\gamma_1-\gamma_2}{\mathbb{C}}} \oplus \overset{}{\underset{3\gamma_2}{L(\mathbb{F}_t)}} \oplus \overset{p_3^I}{\underset{\gamma_3-\gamma_2}{\mathbb{C}}}.$$

Finally, if $\gamma_2 > \gamma_3$ then we consider

$$\mathcal{N}_1 = \left(\overset{p_1^I}{\underset{\gamma_1-\gamma_2}{\mathbb{C}}} \oplus \overset{p_{v_2}}{\underset{\gamma_2}{M_2(\mathbb{C})}} \oplus \overset{p_{v_3}}{\underset{\gamma_3}{\mathbb{C}}} \right) \underset{D}{*} \left(\overset{p_{v_1}}{\underset{\gamma_1}{\mathbb{C}}} \oplus \overset{p_2^I}{\underset{\gamma_2-\gamma_3}{\mathbb{C}}} \oplus \overset{p_{v_3}}{\underset{\gamma_3}{M_2(\mathbb{C})}} \right) = \overset{p_1^I}{\underset{\gamma_1-\gamma_2}{\mathbb{C}}} \oplus \overset{}{\underset{\gamma_3}{M_3(\mathbb{C})}} \oplus \overset{}{\underset{\gamma_2-\gamma_3}{M_2(\mathbb{C})}},$$

where the central support of p_{v_2} is $1 - p_1^I$. Therefore, tensoring each copy of $M_2(\mathbb{C})$ with $L(\mathbb{Z})$ gives $\mathcal{N}(\Gamma) = \overset{p_1^I}{\underset{\gamma_1-\gamma_2}{\mathbb{C}}} \oplus \overset{}{\underset{2\gamma_2+\gamma_3}{L(\mathbb{F}_t)}}$ as desired. $\qquad\square$

We now inductively assume that for some Γ, $\mathcal{N}(\Gamma)$ has the form as described in Theorem 2.3.4.

Lemma 2.3.6. *Suppose Γ' is a graph obtained from Γ by adding an edge e connecting two vertices v and w of Γ (so that in particular Γ and Γ' have the same underlying set of vertices with the same weighting). Assume that*

$$\mathcal{N}(\Gamma) = \overset{p^\Gamma}{L(\mathbb{F}_{t_\Gamma})} \oplus \bigoplus_{v \in B(\Gamma)} \overset{r_v^\Gamma}{\underset{\gamma_v - \alpha_v^\Gamma}{\mathbb{C}}}$$

as in Theorem 2.3.4. Then

$$\mathcal{N}(\Gamma') = \overset{p^{\Gamma'}}{L(\mathbb{F}_{t_{\Gamma'}})} \oplus \bigoplus_{v \in B(\Gamma')} \overset{r_v^{\Gamma'}}{\underset{\gamma_v - \alpha_v^{\Gamma'}}{\mathbb{C}}}$$

where $p^\Gamma \leq p^{\Gamma'}$, $r_v^{\Gamma'} \leq r_v^\Gamma$, and $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \overset{s.e.}{\hookrightarrow} p_\Gamma \mathcal{N}(\Gamma') p_\Gamma$.

Proof. We use the convention that if the term $\overset{p}{\underset{\alpha}{\mathbb{C}}}$ appears where $\alpha \leq 0$ then this term is identically zero. All parts of the proof below are valid if this modification is made.

Set $D = \ell^\infty(\Gamma') = \ell^\infty(\Gamma)$ and let the new edge e connect v to w with $\gamma_v \geq \gamma_w$. We examine the following sequence of inclusions:

$$\mathcal{N}(\Gamma) \subset \mathcal{N}_1 = \mathcal{N}(\Gamma) \underset{D}{*} \left(\overset{p_w}{\underset{\gamma_w}{\mathbb{C}}} \oplus \left(\bigoplus_{k=1}^{n} \overset{p_{v,k}}{\underset{\gamma_w/n}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \right) \oplus \ell^\infty(\Gamma \setminus \{v,w\}) \right)$$

$$\cap$$

$$\mathcal{N}_2 = \mathcal{N}(\Gamma) \underset{D}{*} \left(\bigoplus_{k=1}^{n} \overset{p_{w,k}}{\underset{\gamma_w/n}{\mathbb{C}}} \oplus \left(\bigoplus_{k=1}^{n} \overset{p_{v,k}}{\underset{\gamma_w/n}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \right) \oplus \ell^\infty(\Gamma \setminus \{v,w\}) \right)$$

$$\cap$$

$$\mathcal{N}_3 = \mathcal{N}(\Gamma) \underset{D}{*} \left(\bigoplus_{k=1}^{n} \overset{p_{w,k},p_{v,k}}{\underset{\gamma_w/n}{M_2(\mathbb{C})}} \oplus \overset{p_v^I}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \oplus \ell^\infty(\Gamma \setminus \{v,w\}) \right)$$

$$\cap$$

$$\mathcal{N}(\Gamma') = \mathcal{N}(\Gamma) \underset{D}{*} \left(L(\mathbb{Z}) \otimes M_2(\mathbb{C}) \oplus \overset{p_v^I}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \oplus \ell^\infty(\Gamma \setminus \{v,w\}) \right).$$

The projections $p_{w,k}$ are an orthogonal family with trace γ_w/n in $\mathcal{A}_e$ whose sum is p_w. The projection p_v decomposes as $\sum_{k=1}^{n} p_{v,k} + p_v^I$ with p_v^I supported in the atomic part of $\mathcal{A}_e$ and the $p_{v,k}$ are an orthogonal family of projections with trace γ_w/n supported in the diffuse part of $\mathcal{A}_e$. The positive integer n is chosen large enough such that $\frac{1}{n} + \frac{\gamma_w - \alpha_w^\Gamma}{\gamma_w} < 1$ and $\frac{\gamma_w}{n\gamma_v} + \frac{\gamma_v - \alpha_v^\Gamma}{\gamma_v} < 1$. From the induction hypothesis,

$$p_v \mathcal{N}(\Gamma) p_v = \overset{p_v^\Gamma}{L(\mathbb{F}_{t_v})} \oplus \overset{r_v^\Gamma}{\underset{\frac{\gamma_v - \alpha_v}{\gamma_v}}{\mathbb{C}}}, \quad \text{and} \quad p_w \mathcal{N}(\Gamma) p_w = \overset{p_w^\Gamma}{L(\mathbb{F}_{t_w})} \oplus \overset{r_w^\Gamma}{\underset{\frac{\gamma_w - \alpha_w}{\gamma_w}}{\mathbb{C}}},$$

with $p_u^\Gamma = p^\Gamma p_u$ for any vertex u. From Lemma 2.2.4,

$$p_v \mathcal{N}_1 p_v = \left(\overset{p_v^\Gamma}{L(\mathbb{F}_{t_v})} \oplus \overset{r_v^\Gamma}{\underset{\frac{\gamma_v - \alpha_v^\Gamma}{\gamma_v}}{\mathbb{C}}} \right) * \left(\bigoplus_{k=1}^{n} \overset{p_{v,k}}{\underset{\frac{\gamma_w}{n\gamma_v}}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\frac{\gamma_v - \gamma_w}{\gamma_v}}{\mathbb{C}}} \right) = L(\mathbb{F}_{t_{v,1}}) \oplus \overset{p_v^I \wedge r_v^\Gamma}{\underset{\frac{\gamma_v - \alpha_v^{\Gamma'}}{\gamma_v}}{\mathbb{C}}}.$$

Lemma 2.2.4 applied to the inclusion

$$\left(\overset{p_v^\Gamma}{\mathbb{C}} \oplus \overset{r_v^\Gamma}{\underset{\frac{\gamma_v - \alpha_v}{\gamma_v}}{\mathbb{C}}} \right) * \left(\bigoplus_{k=1}^{n} \overset{p_{v,k}}{\underset{\frac{\gamma_w}{n\gamma_v}}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\frac{\gamma_v - \gamma_w}{\gamma_v}}{\mathbb{C}}} \right) \to \left(\overset{p_v^\Gamma}{L(\mathbb{F}_{t_v})} \oplus \overset{r_v^\Gamma}{\underset{\frac{\gamma_v - \alpha_v}{\gamma_v}}{\mathbb{C}}} \right) * \left(\bigoplus_{k=1}^{n} \overset{p_{v,k}}{\underset{\frac{\gamma_w}{n\gamma_v}}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\frac{\gamma_v - \gamma_w}{\gamma_v}}{\mathbb{C}}} \right),$$

shows that the inclusion $L(\mathbb{F}_{t_v}) = p_v^\Gamma \mathcal{N}(\Gamma) p_v^\Gamma \to p_v^\Gamma \mathcal{N}_1 p_v^\Gamma$ is equivalent to the canonical inclusion

$$L(\mathbb{F}_{t_v}) \to L(\mathbb{F}_{t_v}) * p_v^\Gamma \left[\left(\overset{p_v'}{\mathbb{C}} \oplus \underset{\frac{\gamma_v - \alpha_v}{\gamma_v}}{\mathbb{C}} \right) * \left(\bigoplus_{k=1}^{n} \overset{p_{v,k}}{\underset{\frac{\gamma_w}{n\gamma_v}}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\frac{\gamma_v - \gamma_w}{\gamma_v}}{\mathbb{C}}} \right) \right] p_v^\Gamma$$

so $p_v^\Gamma \mathcal{N}(\Gamma) p_v^\Gamma \overset{s.e.}{\hookrightarrow} p_v^\Gamma \mathcal{N}_1 p_v^\Gamma$. From Remark 2.2.3, $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}_1 p^\Gamma$ as well.

By Lemma 2.2.4 we have

$$p_w \mathcal{N}_2 p_w = L(\mathbb{Z}/n\mathbb{Z}) * p_w \mathcal{N}_1 p_w = L(\mathbb{Z}/n\mathbb{Z}) * \left(\overset{p_w^{\mathcal{N}_1}}{L(\mathbb{F}_{t_w})} \oplus \underset{\gamma_w}{\overset{r_w^{\mathcal{N}_1}}{\mathbb{C}}} \right),$$

where $p_w^{\mathcal{N}_1} = p_w p^{\mathcal{N}_1}$ with $p^{\mathcal{N}_1}$ the central support of p_Γ in $\mathcal{N}_1$ (note $p_w^{\mathcal{N}_1} \geq p_w^\Gamma$ so $r_w^{\mathcal{N}_1} \leq r_w^\Gamma$ which implies $\delta_w \leq \gamma_w - \alpha_w^\Gamma$). From these observations, it follows that $p_w \mathcal{N}_2 p_w$ is an interpolated free group factor (since n was chosen such that $\frac{\gamma_w}{n\gamma_v} + \frac{\gamma_v - \alpha_v}{\gamma_v} < 1$) and the arguments used in the inclusion $\mathcal{N}(\Gamma) \to \mathcal{N}_1$ imply $p_w^{\mathcal{N}_1} \mathcal{N}_1 p_w^{\mathcal{N}_1} \overset{s.e.}{\hookrightarrow} p_w^{\mathcal{N}_1} \mathcal{N}_2 p_w^{\mathcal{N}_1}$. Therefore $p_w^\Gamma \mathcal{N}_1 p_w^\Gamma \overset{s.e.}{\hookrightarrow} p_w^\Gamma \mathcal{N}_2 p_w^\Gamma$ so $p^\Gamma \mathcal{N}_1 p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}_2 p^\Gamma$. Also, observe that since the projections $p_{v,k}$ and $p_{w,k}$ lie in the interpolated free group factor summand of $\mathcal{N}_2$, they are equivalent in $\mathcal{N}_2$. We now define algebras $\mathcal{N}_{2,j}$ for $j = 0, ..., n$ so that

$$\mathcal{N}_2 = \mathcal{N}_{2,0} \subset \mathcal{N}_{2,1} \subset \mathcal{N}_{2,2} \subset \cdots \subset \mathcal{N}_{2,n} = \mathcal{N}_3 \text{ where}$$

$$\mathcal{N}_{2,j} = \left(\bigoplus_{k=j+1}^{n} \underset{\gamma_w/n}{\overset{p_{w,k}}{\mathbb{C}}} \oplus \bigoplus_{k=1}^{j} \underset{\gamma_w/n}{\overset{p_{w,k}, p_{v,k}}{M_2(\mathbb{C})}} \oplus \bigoplus_{k=j+1}^{n} \underset{\gamma_w/n}{\overset{p_{v,k}}{\mathbb{C}}} \oplus \underset{\gamma_v - \gamma_w}{\overset{p_v^I}{\mathbb{C}}} \oplus \ell^\infty(\Gamma \setminus \{v, w\}) \right) \underset{D}{*} \mathcal{N}(\Gamma).$$

Let $p^{\mathcal{N}_2}$ be the central support of p_Γ in $\mathcal{N}_2$. Applying Lemma 2.2.5 to the inclusion

$$p_{w,j+1} \mathcal{N}_{2,j} p_{w,j+1} \to p_{w,j+1} \mathcal{N}_{2,j+1} p_{w,j+1} = p_{w,j+1} \mathcal{N}_{2,j} p_{w,j+1} * L(\mathbb{Z})$$

shows that this inclusion is a standard embedding, so it follows from Remark 2.2.3 that $p^{\mathcal{N}_2} \mathcal{N}_{2,j} p^{\mathcal{N}_2} \overset{s.e.}{\hookrightarrow} p^{\mathcal{N}_2} \mathcal{N}_{2,j+1} p^{\mathcal{N}_2}$, implying $p^\Gamma \mathcal{N}_{2,j} p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}_{2,j+1} p^\Gamma$ for all j. Inductively,

$$\mathcal{N}_3 = \left(\overset{p^{\mathcal{N}_2}}{L(\mathbb{F}_{t_3})} \oplus \underset{\gamma_v - \alpha_v^{\Gamma'}}{\overset{p_v^I \wedge r_v^\Gamma}{\mathbb{C}}} \bigoplus_{u \in L(\Gamma) \setminus \{v,w\}} \underset{\gamma_u - \alpha_u^{\Gamma'}}{\overset{r_u^\Gamma}{\mathbb{C}}} \right)$$

and $p^\Gamma \mathcal{N}_2 p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}_3 p^\Gamma$. To finish, we look at the sequence of algebras

$$\mathcal{N}_3 = \mathcal{N}_{3,0} \subset \mathcal{N}_{3,1} \subset ... \subset \mathcal{N}_{3,n} = \mathcal{N}(\Gamma') \text{ where}$$

$$\mathcal{N}_{3,j} = \left(\bigoplus_{k=1}^{j} \overset{p_{w,k} + q_{w,k}}{M_2(\mathbb{C}) \otimes L(\mathbb{Z})} \oplus \bigoplus_{k=j+1}^{n} \overset{p_{w,k}, p_{v,k}}{M_2(\mathbb{C})} \oplus \underset{\gamma_v - \gamma_w}{\overset{p_v^I}{\mathbb{C}}} \bigoplus \ell^\infty(\Gamma \setminus \{v, w\}) \right) \underset{D}{*} \mathcal{N}(\Gamma).$$

Lemma 2.2.4 implies that the inclusion

$$p_{w,j+1} \mathcal{N}_{3,j} p_{w,j+1} \to p_{w,j+1} \mathcal{N}_{3,j+1} p_{w,j+1} = p_{w,j+1} \mathcal{N}_{3,j} p_{w,j+1} * L(\mathbb{Z})$$

is a standard embedding, so by Remark 2.2.3, $p^{\mathcal{N}_2} \mathcal{N}_{3,j} p^{\mathcal{N}_2} \overset{s.e.}{\hookrightarrow} p^{\mathcal{N}_2} \mathcal{N}_{3,j+1} p^{\mathcal{N}_2}$ and thus $p^\Gamma \mathcal{N}_{3,j} p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}_{3,j+1} p^\Gamma$. Therefore the inclusion $p^\Gamma \mathcal{N}_3 p^\Gamma \to p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$ is standard since it is a composite of standard embeddings. This implies $\mathcal{N}(\Gamma')$ has the desired formula and $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$. $\square$

We again assume that $\mathcal{N}(\Gamma)$ is in the form of Theorem 2.3.4.

Lemma 2.3.7. *Let Γ' be a weighted graph obtained from Γ by adding a vertex v and an edge e connecting v to $w \in V(\Gamma)$ woth weighting γ_v, and assume $\mathcal{N}(\Gamma) = L(\mathbb{F}_{t_\Gamma}) \overset{p^\Gamma}{\oplus} \underset{v\in B(\Gamma)^{\gamma_v - \alpha_v^\Gamma}}{\bigoplus} \overset{r_v^\Gamma}{\mathbb{C}}$ with notation as in Theorem 2.3.4. Then*

$$\mathcal{N}(\Gamma') = L(\mathbb{F}_{t_{\Gamma'}}) \overset{p^{\Gamma'}}{\oplus} \underset{u\in B(\Gamma')^{\gamma_u - \alpha_u^{\Gamma'}}}{\bigoplus} \overset{r_u^{\Gamma'}}{\mathbb{C}}$$

where $p^\Gamma \leq p^{\Gamma'}$, $r_u^{\Gamma'} \leq r_u^\Gamma$ for all u, and $p_\Gamma \mathcal{N}(\Gamma) p_\Gamma \overset{s.e.}{\hookrightarrow} p_\Gamma \mathcal{N}(\Gamma') p_\Gamma$.

Notice that the natural inclusion $\mathcal{N}(\Gamma) \to \mathcal{N}(\Gamma')$ is not unital, but the compressed inclusion $p_\Gamma \mathcal{N}(\Gamma) p_\Gamma \to p_\Gamma \mathcal{N}(\Gamma') p_\Gamma$ is.

Proof. Just as in the proof of Lemma 2.3.6, if the term $\overset{p}{\underset{\alpha}{\mathbb{C}}}$ appears where $\alpha \leq 0$ then this term is identically zero.

Set $D = \ell^\infty(\Gamma')$. We rescale all of the weights on Γ such that all of the weights on Γ' sum to 1. We have 2 cases: when $\gamma_v > \gamma_w$ and when $\gamma_w \geq \gamma_v$.

<u>*Case 1, $\gamma_v > \gamma_w$*</u>: We look at the following sequence of inclusions:

$$\mathcal{N}(\Gamma) \overset{p_v}{\underset{\gamma_v}{\oplus}} \mathbb{C} \subset \mathcal{N}_1 = \left(\mathcal{N}(\Gamma) \overset{p_v}{\underset{\gamma_v}{\oplus}} \mathbb{C} \right) \underset{D}{*} \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^{n} \overset{p_{w,k}}{\underset{\gamma_w/n}{\mathbb{C}}} \oplus \left(\bigoplus_{k=1}^{n} \overset{p_{v,k}}{\underset{\gamma_w/n}{\mathbb{C}}} \oplus \overset{p_v^I}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \right) \right)$$

$$\cap$$

$$\mathcal{N}_2 = \left(\mathcal{N}(\Gamma) \overset{p_v}{\underset{\gamma_v}{\oplus}} \mathbb{C} \right) \underset{D}{*} \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^{n} \overset{p_{w,k}, p_{v,k}}{\underset{\gamma_w/n}{M_2(\mathbb{C})}} \oplus \overset{p_v^I}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \right)$$

$$\cap$$

$$\mathcal{N}(\Gamma') = \left(\mathcal{N}(\Gamma) \overset{p_v}{\underset{\gamma_v}{\oplus}} \mathbb{C} \right) \underset{D}{*} \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus L(\mathbb{Z}) \underset{2\gamma_w}{\otimes} M_2(\mathbb{C}) \oplus \overset{p_v^I}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \right).$$

The projections $p_{w,k}$ are an orthogonal family with trace γ_w/n in $\mathcal{A}_e$ whose sum is p_w. In $\mathcal{A}_e$, p_v decomposes as $\sum_{k=1}^n p_{v,k} + p_v^I$ with p_v^I supported in the atomic part of $\mathcal{A}_e$, and the $p_{v,k}$ are an orthogonal family of projections with trace γ_w/n supported in the diffuse part of $\mathcal{A}_e$. By the inductive hypothesis,

$$p_w \mathcal{N}(\Gamma) p_w = L(\mathbb{F}_{t_w}) \overset{p_w^\Gamma}{\underset{\frac{\alpha_w^\Gamma}{\gamma_w}}{\oplus}} \overset{r_w^\Gamma}{\underset{\frac{\gamma_w - \alpha_w^\Gamma}{\gamma_w}}{\mathbb{C}}},$$

with $p_w^\Gamma = p_w p^\Gamma$. We choose n large enough such that $\frac{1}{n} + \frac{\gamma_w - \alpha_w}{\gamma_w} < 1$, i.e., so that $p_w \mathcal{N}(\Gamma) p_w * L(\mathbb{Z}/n\mathbb{Z})$ is an interpolated free group factor. From Lemma 2.2.4,

$$p_w \mathcal{N}_1 p_w = p_w \mathcal{N}(\Gamma) p_w * \left(\bigoplus_{k=1}^{n} \overset{p_{w,k}}{\underset{1/n}{\mathbb{C}}} \right) = \left(L(\mathbb{F}_{t_w}) \overset{p'_w}{\underset{\frac{\alpha_w}{\gamma_w}}{\oplus}} \underset{\frac{\gamma_w - \alpha_w}{\gamma_w}}{\mathbb{C}} \right) * \left(\bigoplus_{k=1}^{n} \overset{p_{w,k}}{\underset{1/n}{\mathbb{C}}} \right),$$

so it is an interpolated free group factor, and applying Lemma 2.2.4 again, we see that

$$p_w^\Gamma \mathcal{N}_1 p_w^\Gamma = p_w^\Gamma \mathcal{N}(\Gamma) p_w^\Gamma * p_w^\Gamma \left[\left(\underset{\frac{\alpha_w}{\gamma_w}}{\overset{p_w^\Gamma}{\mathbb{C}}} \oplus \underset{\frac{\gamma_w - \alpha_w}{\gamma_w}}{\overset{r_w^\Gamma}{\mathbb{C}}} \right) * \left(\bigoplus_{k=1}^n \underset{1/n}{\overset{p_{w,k}}{\mathbb{C}}} \right) \right] p_w^\Gamma$$

with the inclusion $p_w^\Gamma \mathcal{N}(\Gamma) p_w^\Gamma \to p_w' \mathcal{N}_1 p_w'$ the canonical one. Therefore $p_w^\Gamma \mathcal{N}(\Gamma) p_w^\Gamma \overset{s.e.}{\hookrightarrow} p_w^\Gamma \mathcal{N}_1 p_w^\Gamma$, so it follows that $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}_1 p^\Gamma$ as well. It is clear that p_v^I will be a minimal central projection in $\mathcal{N}_2$, and since the projections $p_{v,k}$ lie under the minimal projection $p_v \in \mathcal{N}(\Gamma) \oplus \underset{\gamma_v}{\overset{p_v}{\mathbb{C}}}$, it follows that

$$\mathcal{N}_2 = L(\mathbb{F}_{t_2}) \oplus \underset{\gamma_v - \gamma_w}{\overset{p_v^I}{\mathbb{C}}} \oplus \bigoplus_{u \in B(\Gamma) \setminus \{w\}} \underset{\gamma_u - \alpha_u^\Gamma}{\overset{r_u^\Gamma}{\mathbb{C}}},$$

where $L(\mathbb{F}_{t_2})$ is an amplification of $p^\Gamma \mathcal{N}_1 p^\Gamma$. Hence $p^\Gamma \mathcal{N}_1 p^\Gamma = p^\Gamma \mathcal{N}_2 p^\Gamma$. As a final step, we tensor each copy of $M_2(\mathbb{C})$ with $L(\mathbb{Z})$ to obtain $\mathcal{N}(\Gamma')$ and apply Lemma 2.2.4 and Remark 2.2.3 n times as in the proof of Lemma 2.3.6 to conclude that

$$\mathcal{N}(\Gamma') = L(\mathbb{F}_{t_3}) \oplus \underset{\gamma_v - \gamma_w}{\overset{p_v^I}{\mathbb{C}}} \oplus \bigoplus_{u \in B(\Gamma) \setminus \{w\}} \underset{\gamma_u - \alpha_u^{\Gamma'}}{\overset{r_u^\Gamma}{\mathbb{C}}}$$

and $p^\Gamma \mathcal{N}_2 p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$. Therefore $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$ and $\mathcal{N}(\Gamma')$ has the desired form.

Case 2, $\gamma_w \geq \gamma_v$: We look at a sequence of inclusions similar to those in the previous case:

$$\mathcal{N}(\Gamma) \oplus \underset{\gamma_v}{\overset{p_v}{\mathbb{C}}} \subset \mathcal{N}_1 = \left(\mathcal{N}(\Gamma) \oplus \underset{\gamma_v}{\overset{p_v}{\mathbb{C}}} \right) \underset{D}{*} \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \left(\bigoplus_{k=1}^n \underset{\gamma_v/n}{\overset{p_{w,k}}{\mathbb{C}}} \oplus \underset{\gamma_w - \gamma_v}{\overset{p_w^I}{\mathbb{C}}} \right) \oplus \bigoplus_{k=1}^n \underset{\gamma_v/n}{\overset{p_{v,k}}{\mathbb{C}}} \right)$$

$$\cap$$

$$\mathcal{N}_2 = \left(\mathcal{N}(\Gamma) \oplus \underset{\gamma_v}{\overset{p_v}{\mathbb{C}}} \right) \underset{D}{*} \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^n \underset{\gamma_v/n}{\overset{p_{w,k}, p_{v,k}}{M_2(\mathbb{C})}} \oplus \underset{\gamma_w - \gamma_v}{\overset{p_w^I}{\mathbb{C}}} \right)$$

$$\cap$$

$$\mathcal{N}(\Gamma') = \left(\mathcal{N}(\Gamma) \oplus \underset{\gamma_v}{\overset{p_v}{\mathbb{C}}} \right) \underset{D}{*} \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \underset{2\gamma_v}{\overset{}{L(\mathbb{Z})}} \otimes M_2(\mathbb{C}) \oplus \underset{\gamma_w - \gamma_v}{\overset{p_w^I}{\mathbb{C}}} \right).$$

The projections $p_{v,k}$ are an orthogonal family with trace γ_v/n in $\mathcal{A}_e$ whose sum is p_v. In $\mathcal{A}_e$, p_w decomposes as $\sum_{k=1}^n p_{w,k} + p_w^I$ where p_w^I is supported in the atomic part of $\mathcal{A}_e$, and the $p_{w,k}$ are an orthogonal family of projections with trace γ_v/n supported in the diffuse part of $\mathcal{A}_e$. We choose n large enough so that $\frac{\gamma_w - \alpha_w}{\gamma_w} + \frac{\gamma_v}{n\gamma_w} < 1$. Observe by the condition on n that $p_w \mathcal{N}_1 p_w = L(\mathbb{F}_{t_1'}) \oplus \underset{\gamma_w - \alpha_w - \gamma_v}{\overset{p_w^{\mathcal{N}_1}}{\mathbb{C}}}$ where the copy of $\mathbb{C}$ is orthogonal to each $p_{w,k}$. Therefore as in the proof of Lemma 2.3.6 $p_\Gamma \mathcal{N}(\Gamma) p_\Gamma \overset{s.e.}{\hookrightarrow} p_\Gamma \mathcal{N}_1 p_\Gamma$. We next look at

$$\mathcal{N}_1 \subset \mathcal{N}_2 = \left(\mathcal{N}(\Gamma) \oplus \underset{\gamma_v}{\overset{p_v}{\mathbb{C}}} \right) \underset{D}{*} \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^n \underset{\gamma_v/n}{\overset{p_{w,k}, p_{v,k}}{M_2(\mathbb{C})}} \oplus \underset{\gamma_w - \gamma_v}{\overset{p_v^I}{\mathbb{C}}} \right).$$

Since the $p_{v,k}$ lie under the minimal central projection $p_v \in \mathcal{N}(\Gamma) \oplus \underset{\gamma_v}{\overset{p_v}{\mathbb{C}}}$, the arguments above imply

$$\mathcal{N}_2 = L(\mathbb{F}_{t_2}) \oplus \underset{\gamma_w - \gamma_v - \alpha_w}{\mathbb{C}} \oplus \bigoplus_{u \in L(\Gamma) \setminus \{w\}} \underset{\gamma_u - \alpha_u}{\overset{r_u^\Gamma}{\mathbb{C}}}$$

and $p^\Gamma \mathcal{N}_1 p^\Gamma = p^\Gamma \mathcal{N}_2 p^\Gamma$. To finish, we tensor each copy of $M_2(\mathbb{C})$ with $L(\mathbb{Z})$ and apply Lemma 2.2.4 and Remark 2.2.3 n times as in the end of the proof of Lemma 2.3.6 to obtain

$$\mathcal{N}(\Gamma') = L(\mathbb{F}_{t_3}) \oplus \underset{\gamma_w - \gamma_v - \alpha_w}{\mathbb{C}} \oplus \bigoplus_{u \in L(\Gamma) \setminus \{w\}} \underset{\gamma_u - \alpha_u}{\overset{r_u^\Gamma}{\mathbb{C}}}$$

with the inclusion $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \to p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$ standard. $\qquad\square$

Proof of Theorem 2.3.4. Note that if Γ' and Γ are connected, loopless, finite graphs, then Γ' can be constructed form Γ by considering the steps in Lemmas 2.3.6 and 2.3.7. Therefore, we can deduce Theorem 2.3.4 by observing that the composite of standard embeddings is a standard embedding and that standard embeddings are preserved by cut-downs by projections. $\qquad\square$

2.4   The GJS construction in infinite depth

Recall that the vertices on a principal graph for $M_0 \subset M_1$ represent isomophism classes of irreducible $M_0 - M_0$ and $M_0 - M_1$ subbimodules of tensor products of $X =_{M_0} L^2(M_1)_{M_1}$ and its dual, $X^* =_{M_1} L^2(M_1)_{M_0}$. Assume Γ is the principal graph for an infinite-depth subfactor. If $*$ is the depth-0 vertex of Γ, then the factor M_0 as in the introduction is isomorphic to $p_* \mathcal{N}(\Gamma) p_*$. $\mathcal{N}(\Gamma)$ is now a semifinite algebra where the weighting γ on $\ell^\infty(\Gamma)$ corresponds to the bimodule dimension obtained by identifying each vertex with an irreducible bimodule as above. Under this identification, $\gamma_* = 1$ and $\delta \cdot \gamma_v = \sum_{w \sim v} n_{v,w} \gamma_w$ where $\delta = [M_1 : M_0]^{1/2}$. To circumvent the difficulty of dealing with a semifinite algebra, we realize that M_0 is an inductive limit of the algebras $p_* \mathcal{N}(\Gamma_k) p_*$ where Γ_k is Γ truncated at depth k. To aid our computation of the isomorphism class of M_0, we have the following lemma, whose proof is a routine calculation and is identical to that in [GJS11].

Lemma 2.4.1. *The free dimension of $\mathcal{N}(\Gamma_k)$ is*

$$1 + \frac{1}{\operatorname{Tr}(F_k)^2} \left(-\sum_{v \in \Gamma_k} \gamma_v^2 + \sum_{v \in \Gamma_k} \sum_{w \sim v} n_{v,w} \gamma_v \gamma_w \right)$$

where $w \sim v$ means w is connected to v in Γ_k, $F_k = \sum_{u \in \Gamma_k} p_u$, and Tr is the trace on the semifinite algebra $\mathcal{N}(\Gamma)$.

Theorem 2.4.2. *Let $\mathcal{P}$ be an infinite depth subfactor planar algebra. Then the factor M_0 in the construction of [GJS10] is isomorphic to $L(\mathbb{F}_\infty)$.*

Proof. For a given k, we write

$$\mathcal{N}(\Gamma_k) = \underset{\underset{w\notin B(\Gamma_k)}{\sum \gamma_w} + \underset{v\in B(\Gamma_k)}{\sum \alpha_v^{\Gamma_k}}}{L(\mathbb{F}_{t_k})}^{p_k} \oplus \bigoplus_{v\in B(\Gamma_k)} \overset{p_v^{\Gamma_k}}{\underset{\gamma_v - \alpha_v^{\Gamma_k}}{\mathbb{C}}}.$$

The free dimension of this algebra is

$$1 + (t_k - 1)\left(\frac{\sum_{w\notin B(\Gamma_k)}\gamma_w + \sum_{v\in B(\Gamma_k)}\gamma_v}{\mathrm{Tr}(F_k)}\right)^2 - \frac{\sum_{v\in B(\Gamma_k)}(\gamma_v - \alpha_v^{\Gamma_k})^2}{\mathrm{Tr}(F_k)^2},$$

so by Lemma 2.4.1, we have the equation

$$(t_k - 1)\left(\sum_{w\notin B(\Gamma_k)}\gamma_w + \sum_{v\in B(\Gamma_k)}\alpha_v^{\Gamma_k}\right)^2 = \sum_{u\in\Gamma_k}\sum_{w\sim u} n_{u,w}\gamma_u\gamma_w - \sum_{u\in\Gamma_k}\gamma_u^2 + \sum_{v\in B(\Gamma_k)}(\gamma_v - \alpha_v^{\Gamma_k})^2.$$

Observe that in Γ_k, the vertices up to depth $k-1$ are connected to all of their neighbors in Γ, so by the Perron-Frobenius condition and the fact that $\delta > 1$, none of these vertices are in $B(\Gamma_k)$. If we let $B'(\Gamma_k)$ be the vertices v at depth k with $\gamma_v \le \sum_{w\sim v} n_{v,w}\gamma_w$, then the right hand side of the above equality becomes

$$(\delta - 1)\sum_{v\in\Gamma_{k-2}}\gamma_v^2 + \sum_{v\in\Gamma_{k-1}\setminus\Gamma_{k-2}}\gamma_v\left(-\gamma_v + \sum_{\substack{w\notin B(\Gamma_k)\\w\sim v}} n_{v,w}\gamma_w + \sum_{\substack{w\in B(\Gamma_k)\\w\sim v}}\alpha_w^{\Gamma_k}\right)$$

$$+ \sum_{v\in B'(\Gamma_k)}\gamma_v\left(-\gamma_v + \sum_{w\sim v} n_{v,w}\gamma_w\right)$$

where we have used $\alpha_v^{\Gamma_k} = \sum_{w\sim v} n_{v,w}\gamma_w$. This quantity majorizes $(\delta - 1)\sum_{v\in\Gamma_{k-2}}\gamma_v^2$. Since the bimodule dimensions of any irreducible sumbimodule of $(X\otimes_{M_1} X^*)^{\otimes_{M_0}^n}$ and $(X\otimes_{M_1} X^*)^{\otimes_{M_0}^n}\otimes_{M_0} X$ are bounded below by 1, $\gamma_v \ge 1$ for all $v\in\Gamma$ so we conclude that

$$(t_k - 1)\left(\sum_{w\notin L(\Gamma_k)}\gamma_w + \sum_{v\in L(\Gamma_k)}\alpha_v^{\Gamma_k}\right)^2 \to \infty$$

as $k\to\infty$. From the amplification formula, $p_*\mathcal{N}(\Gamma_k)p_* = L(\mathbb{F}_{t'_k})$ where

$$t'_k = 1 + (t_k - 1)\left(\sum_{w\notin L(\Gamma_k)}\gamma_w + \sum_{v\in L(\Gamma_k)}\alpha_v^{\Gamma_k}\right)^2.$$

Hence $p_{\Gamma_k}\mathcal{N}(\Gamma_k)p_{\Gamma_k} \overset{s.e.}{\hookrightarrow} p_{\Gamma_k}\mathcal{N}(\Gamma_{k+1})p_{\Gamma_k}$ so by Remark 2.2.3, $p_*\mathcal{N}(\Gamma_k)p_* \overset{s.e.}{\hookrightarrow} p_*\mathcal{N}(\Gamma_{k+1})p_*$. As $p_*\mathcal{N}(\Gamma)p_*$ is the inductive limit of the $p_*\mathcal{N}(\Gamma_k)p_*$, it follows that $p_*\mathcal{N}(\Gamma)p_* = L(\mathbb{F}_t)$ where $t = \lim t'_k = \infty$. $\qquad\square$

Corollary 2.4.3. *The factors M_k are isomorphic to $L(\mathbb{F}_\infty)$.*

Proof. If k is even, then M_k is an amplification of M_0 so it follows for M_k. If k is odd, then M_k are cut-downs of $\mathcal{N}(\Gamma')$ with Γ' the dual principal graph of $\mathcal{P}$. Applying the same analysis as in Theorem 2.4.2 shows that $M_k \cong L(F_\infty)$. $\square$

Chapter 3

Universality of $L(\mathbb{F}_\infty)$ for Rigid C^*-Tensor Categories

3.1   Introduction

Jones initiated the modern theory of subfactors in his breakthrough paper [Jon83] in which he classified the possible values for the index of a II_1 subfactor to the range $\{4\cos^2(\pi/n)|n \geq 3\}\cup[4,\infty]$, and he found a subfactor of the hyperfinite II_1-factor R for each allowed index.

A finite index subfactor $N \subset M$ is studied by analyzing its standard invariant, i.e., two rigid C^*-tensor categories of $N - N$ and $M - M$ bimodules and the module categories of $N - M$ and $M - N$ bimodules which arise from the Jones tower. The standard invariant has been axiomatized in three similar ways, each emphasizing slightly different structure: Ocneanu's paragroups [Ocn88, EK98], Popa's λ-lattices [Pop95], and Jones' planar algebras [Jon].

In [Pop93, Pop95, Pop02], Popa starts with a λ-lattice $A_{\bullet,\bullet} = (A_{i,j})$ and constructs a II_1-subfactor whose standard invariant is $A_{\bullet,\bullet}$. Hence for each subfactor planar algebra $P_\bullet$, there is some subfactor whose planar algebra is $P_\bullet$. However, the following question remains unanswered:

Question 3.1.1. *For which subfactor planar algebras $\mathcal{P}$ is there a subfactor of R whose planar algebra is $\mathcal{P}$?*

Using his reconstruction theorems, Popa gave a positive answer to Question 3.1.1 for (strongly) amenable subfactor planar algebras [Pop94].

In [PS03], Popa and Shlyakhtenko were able to identify the factors in certain cases of Popa's reconstruction theorems. Using this, they gave a positive answer to Question 3.1.1 for $L(\mathbb{F}_\infty)$, i.e., every subfactor planar algebra arises as the standard invariant of some subfactor $N \subset M$ such that N, M are both isomorphic to $L(\mathbb{F}_\infty)$. This theorem was reproduced by Hartglass [Har12] using the reconstruction results of Guionnet-Jones-Shlyakhtenko-Walker (GJSW) [GJS10, JSW10, GJS11] which produce subfactors of interpolated free group factors.

It is natural to extend these questions to rigid C^*-tensor categories, i.e.,

Question 3.1.2. *For which rigid C^*-tensor categories $\mathcal{C}$ is there a category $\mathcal{C}_{bim}$ of bifnite bimodules over R such that $\mathcal{C}_{bim}$ is equivalent to $\mathcal{C}$?*

As in the subfactor case, Hayashi and Yamagami gave a positive result for amenable rigid C^*-tensor categories [HY00] (amenability for C^*-tensor categories was first studied by Hiai and

Izumi [HI98]). Moreover, given a rigid C^*-tensor category $\mathcal{C}$, Yamagami constructed a category of bifite bimodules $\mathcal{C}_{bim}$ over an amalgamated free product II_1-factor such that $\mathcal{C}_{bim}$ is equivalent to $\mathcal{C}$ [Yam03]. However, one can show these factors have property Γ, so they are not interpolated free group factors (we briefly sketch this in Appendix 3.5).

In this paper, we give a result analogous to Popa and Shlyakhtenko's results for $L(\mathbb{F}_\infty)$ for countably generated rigid C^*-tensor categories, which answers part of Question 9 in [Müg10, Section 6]. Recall that a rigid C^*-tensor category $\mathcal{C}$ is generated by a set of objects $\mathcal{S}$ if for every $Y \in \mathcal{C}$, there are $X_1, \ldots, X_n \in \mathcal{S}$ such that

$$\mathcal{C}(X_1 \otimes \cdots \otimes X_n, Y) \neq (0),$$

i.e., Y is (isomorphic to) a sub-object of $X_1 \otimes \cdots \otimes X_n$.

Theorem 3.1.3. *Every countably generated rigid C^*-tensor category can be realized as a category of bifinite bimodules over $L(\mathbb{F}_\infty)$.*

Remark 3.1.4. Note that when $\mathcal{C}$ is finitely generated, we can prove Theorem 3.1.3 using [PS03] by adapting the technique in [FR12, Theorem 4.1]. We provide a sketch of the proof in Appendix 3.5, where we also point out some difficulties of using the results of [PS03] when $\mathcal{C}$ is not finitely generated (see also [Yam03, Section 4]).

Hence we choose to use planar algebra technology to prove Theorem 3.1.3 since it offers the following advantages. First, the same construction works for both the finitely and infinitely generated cases. Second, planar diagrams arise naturally in the study of tensor categories, and a reader familiar with the diagrams may benefit from a planar algebraic approach. Third, we get an elegant description of the bimodules over $L(\mathbb{F}_\infty)$ directly from the planar algebra (see Section 3.3).

There are three steps to the proof of Theorem 3.1.3.

(1) Given a countably generated C^*-tensor category $\mathcal{C}$, we get a factor planar algebra $\mathcal{P}$ such that the C^*-tensor category Pro of projections of $\mathcal{P}$ is equivalent to $\mathcal{C}$.

 A <u>factor planar algebra</u> (called a fantastic planar algebra in [MP12]) is an unshaded, spherical, evaluable C^*-planar algebra.

 This step is well known to experts; we give most of the details in Section 3.2.

(2) Given a factor planar algebra $\mathcal{P}$, we construct a II_1-factor M and two rigid C^*-tensor categories of bifinite bimodules over M:

  • Bim, built entirely from $\mathcal{P}$ and obviously equivalent to Pro, and

  • $\mathcal{CF}$, formed using Connes' fusion and linear operators.

 These categories are defined in Definitions 3.3.23 and 3.3.25. We then show Bim $\simeq \mathcal{CF}$ in Theorem 3.3.26.

 We use results of GJSW to accomplish this step in Section 3.3. Along the way, we adapt Brothier's treatment [Bro12] of GJSW results [GJS10, JSW10] for unshaded planar algebras.

(3) We show $M \cong L(\mathbb{F}_\infty)$.

 This last step is similar to results of GJS [GJS11] and Chapter 2.

One can use similar analysis as in the proof of Theorem 3.1.3 to prove the following theorem:

Theorem 3.1.5. *Suppose that in addition, $\mathcal{C}$ has finitely many isomorphism classes of simple objects, i.e., $\mathcal{C}$ is a unitary fusion category. Picking an object $X \in \mathcal{C}$ which generates $\mathcal{C}$, then $\mathcal{C}$ can be realized as a category of bifinite bimodules over $L(\mathbb{F}_t)$ with*

$$t = 1 + \dim(\mathcal{C})(\dim(X \oplus \overline{X}) - 1) = 1 + \dim(\mathcal{C})(2\dim(X) - 1),$$

where $\dim(\mathcal{C})$ is the Frobenius-Perron dimension of $\mathcal{C}$.

Remark 3.1.6. Note that a version of Theorem 3.1.5 can be obtained by using [FR12, Theorem 4.1] together with [GJS11]. If $Z_1, \ldots, Z_n$ are representatives for the simple objects in $\mathcal{C}$ and $Y = \bigoplus_{k=1}^{n} Z_k$, then one obtains the factor $L(\mathbb{F}_s)$ for

$$s = 1 + \dim(\mathcal{C})(\dim(Y) - 1),$$

which will be a different parameter than what we obtained in Theorem 3.1.5.

On the other end of the spectrum, one should also note that there has been interesting work on rigid C^*-tensor categories of bimodules over II_1-factors by Vaes, Falguières, and Raum [FV11, FR12]. Given a rigid C^*-tensor category $\mathcal{C}$ which is either $\text{Rep}(G)$ for G a compact quantum group [FV11] or a unitary fusion category [FR12], they construct a II_1-factor M whose category $\text{Bim}(M)$ of bifinite bimodules is exactly $\mathcal{C}$ (up to equivalence). Their results can be interpreted as rigidity results in contrast to the universality of $L(\mathbb{F}_\infty)$ to rigid C^*-tensor categories.

3.2   Tensor categories and planar algebras

We briefly recall how to go back and forth between rigid C^*-tensor categories and factor planar algebras. The contents of this subsection are well known to experts. Our treatment follows [MPS10, Jon11, Yam12, Gho11, Yam03, Müg10].

Notation 3.2.1. Categories will be denoted with the letter $\mathcal{C}$ unless otherwise specified. We write $X \in \mathcal{C}$ to mean X is an object in $\mathcal{C}$, and we write $\mathcal{C}(X \to Y)$ or $\mathcal{C}(X, Y)$ for the set of morphisms from X to Y in $\mathcal{C}$.

Factor planar algebras and principal graphs

We briefly recall the definition of an unshaded factor planar algebra where the strings are labelled. For more details, see [Jon11].

Definition 3.2.2. Given a set $\mathcal{L}$, the planar operad with string labels $\mathcal{L}$, denoted $\mathbb{P}_\mathcal{L}$, is the set of all planar tangles whose strings are labelled by elements of $\mathcal{L}$, e.g.,

 where $a, b, c, d \in \mathcal{L}$.
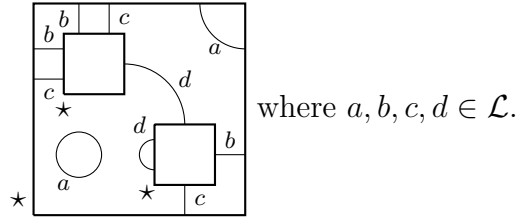
A planar tangle T consists of the following data:

(1) a rectangle $D_0(T) \subset \mathbb{R}^2$,

(2) Finitely many disjoint rectangles $D_1(T), \ldots, D_s(T)$ in the interior of $D_0(T)$ (s may be zero),

(3) Finitely many disjoint smooth arcs in $D_0(T)$ called the strings of T which do not meet the interior of any $D_1(T), \ldots, D_s(T)$. The boundary points of a string of T (if it has any) lie in the boundaries of the $D_i(T)$, and they meet these boundaries transversally (if they meet the boundaries at all).

The boundaries of the strings divide the boundaries of the rectangles into intervals. For each rectangle, there is a distinguished interval denoted $\star$. The intervals of $D_i(T)$ are divided by the marked points of $D_i(T)$, i.e., the points at which the strings meet the boundary of $D_i(T)$. Starting at $\star$, the marked points on each rectangle are number clockwise.

Each string is labelled by an element from $\mathcal{L}$, which induces a labeling on the marked points of the $D_i(T)$. Reading clockwise around the boundary of $D_i(T)$ starting at $\star$, we get a word $\alpha_i \in \Lambda$, the set of all finite words on $\mathcal{L}$. We call $D_i(T)$ an α_i-rectangle, and we call such a planar tangle T a planar α_0-tangle.

If we have two planar tangles S, T satisfying the following boundary conditions:

- some internal rectangle $D_i(S)$ agrees with $D_0(T)$,

- the marked points of $D_i(S)$ agree with the marked points of $D_0(T)$,

- the distinguished interval of $D_i(S)$ agrees with the distinguished interval of T, and

- the label from $\mathcal{L}$ of each marked point of $D_i(S)$ agrees with the label of each marked point from $D_0(T)$,

then we may compose S and T to get the planar tangle $S \circ_i T$ by taking S union the interior of $D_0(T)$, removing the boundary of $D_i(S)$, and smoothing the strings.

Remark 3.2.3. When we draw a planar tangle, we will often suppress the external rectangle, which is assumed to be large. If we omit the $\star$, it is always assumed the $\star$ is in the <u>lower left corner</u>. Finally, if $a_1, \ldots, a_k \in \mathcal{L}$ and $\alpha = a_1 \ldots a_k \in \Lambda$, we draw one string labelled α rather than k parallel strings labelled $a_1, \ldots, a_k$ where we always read the strands from <u>left to right</u> and <u>top to bottom</u>.

For each word $\alpha = a_1 \cdots, a_n \in \Lambda$, we write $\bar{\alpha} = a_n \cdots a_1$ for the word in the reverse order.

Definition 3.2.4. A <u>planar algebra</u> $\mathcal{P}$ with string labels $\mathcal{L}$ is

- a collection of vector spaces $(P_\alpha)_{\alpha \in \Lambda}$ (recall Λ is the set of finite words on $\mathcal{L}$)

- an action of planar tangles by multilinear maps, i.e., for each planar α-tangle T, whose rectangles $D(T)$ are α_D-rectangles, there is a multilinear map

$$Z_T \colon \prod_{D \subset T_0} P_{\alpha_D} \to P_\alpha$$

  satisfying the following axioms:

<u>Isotopy:</u> If θ is an orientation preserving diffeomorphism of $\mathbb{R}^2$, then $Z_{\theta(T)} = Z_T$. This means that

$$Z_{\theta(T)}(f_\theta) = Z_T(f)$$

  where $f_\theta(\theta(D)) = f(D)$.

<u>Naturality:</u> For S, T composable tangles, $Z(S \circ_D T) = Z(S) \circ_D Z(T)$, where the composition on the right hand side is the composition of multilinear maps.

Moreover, $\mathcal{P}$ is called a <u>factor planar algebra</u> if $\mathcal{P}$ is

- <u>evaluable</u>, i.e., $\dim(P_\alpha) < \infty$ for all $\alpha \in \Lambda$ and $P_\emptyset \cong \mathbb{C}$ via the map that sends the empty diagram to $1 \in \mathbb{C}$. Hence, by naturally, to each $a \in \mathcal{L}$, there is a scalar $\delta_a \in \mathbb{C}$ such that any labelled diagram containing a closed loop labelled a is equal to the same diagram without the closed loop multiplied by δ_a. We use the notation $\delta_\alpha = \delta_{a_1} \cdots \delta_{a_n}$ if $\alpha = a_1 \cdots a_n$.

- <u>involutive</u>, i.e., for each $\alpha \in \Lambda$, there is a map $* \colon P_\alpha \to P_{\bar{\alpha}}$ with $* \circ * = \mathrm{id}$ which is compatible with the reflection of tangles, i.e., if T is a planar tangle labelled by $\alpha_1, \ldots, \alpha_s$, then

$$T(\alpha_1, \ldots, \alpha_n)^* = T^*(\alpha_1^*, \ldots, \alpha_n^*)$$

  where T^* is the reflection of T.

- <u>unital</u> [Jon11]: Let S be a shaded planar α_0 tangle with no input disks. Then, there is an element $Z(S) \in P_{\alpha_0}$ so that the following holds:

  Let T be a tangle with a nonempty set of internal disks such that S can be glued into the internal disk D^S of T. Then

$$Z(T \circ S) = Z(T) \circ Z_S.$$

  Here $(Z(T) \circ Z_S)(f) = \tilde{f}$ where

$$\tilde{f}(D) = \begin{cases} f(D) \text{ if } D \neq D^S \\ Z(S) \text{ if } D = D^S \end{cases}$$

- <u>spherical</u>, i.e., for all $\alpha \in \Lambda$ and all $x \in P_{\alpha\bar\alpha}$, we have

$$\mathrm{tr}(x) = \overset{\alpha}{\boxed{x}}\ \bar\alpha\ =\ \bar\alpha\ \overset{\alpha}{\boxed{x}}\ .$$

- <u>positive</u>, i.e., for every $\alpha \in \Lambda$, the map $\langle\cdot,\cdot\rangle\colon P_\alpha \times P_\alpha \to P_\emptyset \cong \mathbb{C}$ given by

$$\langle x, y\rangle = \boxed{x}\overset{\alpha}{\ }\boxed{y^*}$$

  is a positive definite inner product. Hence for all $a \in \mathcal{L}$, $\delta_a > 0$.

Notation 3.2.5. For each $\alpha, \beta \in \Lambda$, we write $P_{\alpha\to\beta}$ to denote the box space $P_{\alpha\bar\beta}$ of elements of the form

$$\overset{|\alpha}{\underset{|\beta}{\boxed{x}}}\ .$$

Remark 3.2.6. Note that for each $\alpha \in \Lambda$, the multiplication tangle

$$\overset{|\alpha}{\underset{\underset{|\alpha}{\boxed{}}}{\overset{\boxed{}}{|\alpha}}}$$

makes $P_{\alpha\to\alpha}$ into an associative algebra. If $\mathcal{P}$ is a factor planar algebra, then the multiplication tangle makes $P_{\alpha\to\alpha}$ a finite dimensional C^*-algebra.

Definition 3.2.7. Suppose $\mathcal{P}$ is a factor planar algebra. A <u>projection</u> in $\mathcal{P}$ is an element $p \in P_{\alpha\to\alpha}$ satisfying $p = p^2 = p^*$ where the multiplication is as in Remark 3.2.6. A projection $p \in P_{\alpha\to\alpha}$ is called <u>simple</u> if it is a minimal projection in $P_{\alpha\to\alpha}$. Since $P_{\alpha\to\alpha}$ is a finite dimensional C^*-algebra, every projection is the sum of finitely many simple projections. This property is called <u>semi-simpicity</u>.

Given a projection $p \in P_{\alpha\to\alpha}$, the dual projection $\bar{p} \in P_{\bar\alpha\to\bar\alpha}$ is obtained by

$$\bar{p} = \overset{\bar\alpha}{\Big|}\ \overset{\alpha}{\boxed{p}}\ \underset{\alpha}{\Big|}_{\bar\alpha} = \ _{\bar\alpha}\Big|\ \overset{\alpha}{\boxed{p}}\ \overset{\bar\alpha}{\Big|}\ .$$

Projections $p \in P_{\alpha\to\alpha}$ and $q \in P_{\beta\to\beta}$ are <u>isomorphic</u> or <u>equivalent</u>, denoted $p \simeq q$, if there is a $u \in P_{\alpha\to\beta}$ such that

$$u^*u = \overset{|\alpha}{\underset{\underset{|\alpha}{\boxed{u^*}}}{\overset{\boxed{u}}{|\beta}}} = p \text{ and } uu^* = \overset{|\beta}{\underset{\underset{|\beta}{\boxed{u}}}{\overset{\boxed{u^*}}{|\alpha}}} = q.$$

Given a projection $p \in P_{\alpha \to \alpha}$ and a $\beta \in \Lambda$, we can form the projection

$$p \otimes \mathrm{id}_\beta = \boxed{\;p\;}\,\Big|_\alpha^\alpha\,\Big|\beta \;\; \in P_{\alpha\beta \to \alpha\beta}.$$

The principal graph of $\mathcal{P}$ with respect to $\beta \in \Lambda$, denoted Γ_β, is the graph whose vertices are the isomorphism classes of simple projections in $\mathcal{P}$, and if $p \in P_{\alpha \to \alpha}$ and $q \in P_{\alpha\beta \to \alpha\beta}$ are simple projections, then the vertices $[p]$ and $[q]$ are connected by $\dim(q P_{\alpha\beta \to \alpha\beta}(p \otimes \mathrm{id}_\beta))$ edges.

The principal graph of $\mathcal{P}$, denoted Γ, is the push out of the Γ_b for $b \in \mathcal{L}$ over the isomorphism classes of simple projections, i.e., the vertices are the same as before, and the edge set is the union of the edge sets of the Γ_b for $b \in \mathcal{L}$.

Since $\mathcal{P}$ is factor and $\mathcal{L}$ is countable, Γ has countably many vertices, although it may not be locally finite. However, Γ_b is always locally finite for $b \in \mathcal{L}$.

Given a vertex $[p]$ of Γ, the number $\mathrm{tr}(p)$ is independent of the choice of representative of $[p]$. The vector $(\mathrm{tr}(p))_{[p] \in V(\Gamma)}$ defines a Frobenius-Perron weight vector on the vertices of Γ satisfying the following equation for each $b \in \mathcal{L}$:

$$\delta_b \, \mathrm{tr}(p) = \sum_{[q] \in V(\Gamma_b)} n^b_{[p],[q]} \, \mathrm{tr}(q)$$

where $n^b_{[p],[q]}$ is the number of edges connecting $[p]$ and $[q]$ in Γ_b.

Rigid C^*-tensor categories and fusion graphs

We briefly recall the definition of a rigid C^*-tensor category.

Definition 3.2.8. A rigid C^*-tensor category is a pivotal, spherical, positive/unitary, rigid, semisimple, linear (Vect-enriched) monoidal category such that $\mathrm{End}(1) \cong \mathbb{C}$.

We now unravel this definition and state many properties that follow. The interested reader should see [Yam03, Müg10] for more details. As we go through the properties, we will also go through the well-known graphical calculus used for strict tensor categories. We will immediately see that we get a factor planar algebra from a rigid C^*-tensor category.

We start with an abelian category $\mathcal{C}$ together with

- a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative up to a natural isomorphism (the pentagon axiom is satisfied), and

- a unit object $1 \in \mathcal{C}$ which is a left and right identity for $\otimes$ up to natural isomorphism (the triangle identity is satisfied).

Remark 3.2.9. Recall that a tensor category is called strict if the above natural isomorphisms are identities, i.e., for each $X, Y, Z \in \mathcal{C}$, we have

$$(X \otimes Y) \otimes Y = X(\otimes(Y \otimes Z) \text{ and}$$
$$1 \otimes X = X = X \otimes 1.$$

Since each tensor category is equivalent to a strict tensor category by Theorem 7.2.1 in [ML98], our tensor categories will be assumed to be strict unless otherwise stated. Note that even with all the properties we want, we can still restrict our attention to strict categories.

First, since $\mathcal{C}$ is Vect-enriched, for each $X, Y \in \mathcal{C}$, $\mathcal{C}(X \to Y)$ is a finite dimensional complex vector space. The morphisms in $\mathcal{C}$ are drawn as boxes with strings emanating from the top and bottom. The strings are labelled by the objects, and the diagram is read from top to bottom. For example,

$$f = \boxed{f} \in \mathcal{C}(X \to Y),$$

and the identity morphism id_X is denoted by the horizontal strand labelled X. We compose morphisms by vertical concatenation

$$\boxed{f \circ g} = \boxed{\substack{g \\ f}} \in \mathcal{C}(X \to Z),$$

and we tensor morphisms by horizontal concatenation:

$$\boxed{f_1 \otimes f_2} = \boxed{f_1}\ \boxed{f_2} \in \mathcal{C}(X_1 \otimes X_2 \to Y_1 \otimes Y_2).$$

Since $\mathcal{C}$ is rigid, for each $X \in \mathcal{C}$, there is a *dual* or *conjugate* $\overline{X} \in \mathcal{C}$, and there is a natural isomorphism $\overline{\overline{X}} \cong X$. Along with the dual object, we have an *evaluation map* $\mathrm{ev}_X \colon \overline{X} \otimes X \to 1$ and a *coevaluation* map $\mathrm{coev}_X \colon 1 \to X \otimes \overline{X}$ such that the diagram

$$
\begin{array}{ccc}
 & X \otimes \overline{X} \otimes X & \\
\nearrow^{\mathrm{coev}_X \otimes 1} & & \searrow^{1 \otimes \mathrm{ev}_X} \\
X \xrightarrow{\quad \mathrm{id}_X \quad} & & X \\
\searrow_{1 \otimes \mathrm{coev}_{\overline{X}}} & & \nearrow_{\mathrm{ev}_{\overline{X}} \otimes 1} \\
 & X \otimes \overline{X} \otimes X &
\end{array}
$$

commutes. The evaluation is denoted by a cap, and we a draw a cup for the coevaluation:

$$\mathrm{ev}_X = \underset{1}{\overset{\overline{X} \quad X}{\smile}} \quad \text{and} \quad \mathrm{coev}_X = \overset{1}{\underset{X \quad \overline{X}}{\frown}} .$$

The diagram above commuting is sometimes referred to as the *zig-zag relation*, since it is the straightening of the kinked string:

In general, we don't draw a string connected to the trivial object $1 \in \mathcal{C}$. For each $X, Y \in \mathcal{C}$, $\overline{X \otimes Y}$ is naturally isomorphic to $\overline{Y} \otimes \overline{X}$, and the diagram

$$
\begin{array}{ccc}
\overline{X \otimes Y} \otimes (X \otimes Y) & \xrightarrow{\ \mathrm{ev}_{X \otimes Y}\ } & 1 \\
\downarrow & & \uparrow {\scriptstyle \mathrm{ev}_Y} \\
\overline{Y} \otimes \overline{X} \otimes X \otimes Y & \xrightarrow{\ 1 \otimes \mathrm{ev}_X \otimes 1\ } & \overline{Y} \otimes Y
\end{array}
$$

commutes, and similarly for the coev's. This diagram just means that we can write one cap labelled $X \otimes Y$ and its dual instead of two separate caps labelled X and Y and their duals:

and similarly for the cups.

The *pivotality* axiom in $\mathcal{C}$ requires that for all $f \in \mathcal{C}(X \to Y)$,

$$(\mathrm{ev}_Y \otimes \mathrm{id}_{\overline{X}}) \circ (\mathrm{id}_{\overline{Y}} \otimes f \otimes \mathrm{id}_{\overline{X}}) \circ (\mathrm{id}_{\overline{Y}} \otimes \mathrm{coev}_X) = (\mathrm{id}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_{\overline{Y}} \otimes f \otimes \mathrm{id}_{\overline{X}}) \circ (\mathrm{coev}_{\overline{X}} \otimes \mathrm{id}_{\overline{Y}}).$$

The equation above has an elegant representation in diagrams:

For $f \in \mathcal{C}(X \to Y)$, the above diagram defines a dual map $\overline{f} \in \mathcal{C}(\overline{Y} \to \overline{X})$, and $\overline{\overline{f}} = f$.

The evaluations and coevaluations together with pivotality allow us to define a *left* and *right* trace on $\mathrm{End}_{\mathcal{C}}(X)$:

$$\mathrm{tr}_L(f) = \mathrm{ev}_X \circ (\mathrm{id}_{\overline{X}} \otimes f) \circ \mathrm{coev}_{\overline{X}} = \ \ \ \in \mathrm{End}(1) \cong \mathbb{C} \text{ and}$$

$$\mathrm{tr}_R(f) = \mathrm{ev}_{\overline{X}} \circ (f \otimes \mathrm{id}_{\overline{X}}) \circ \mathrm{coev}_X = \ \ \ \in \mathrm{End}(1) \cong \mathbb{C}.$$

Similarly, for each $X \in \mathcal{C}$, there are numbers d_X^L and d_X^R which are the left and right traces of the identity morphism respectively, and $d_X^L = d_{\overline{X}}^R$ and $d_X^R = d_{\overline{X}}^L$

Sphericality means that these two traces are equal, and we denote the common number by $\mathrm{tr}(f)$. The sphericality allows us to perform isotopy on closed diagrams as if they were drawn on a sphere. Hence $\dim(X) := d_X^L = d_X^R$ and $\dim(X) = \dim(\overline{X})$ for all $X \in \mathcal{C}$.

The *positivity* or *unitarity* of $\mathcal{C}$ means there is a contravariant functor $*\colon \mathcal{C} \to \mathcal{C}$ which is the identity on all objects, and on morphisms, it is anti-linear, involutive ($* \circ * = \mathrm{id}_\mathcal{C}$), monoidal ($(f \circ g)^* = g^* \circ f^*$ for composable f, g), and positive ($f^* \circ f = 0$ implies $f = 0$). We require $*$ to be compatible with the duality ($\overline{f}^* = \overline{f^*}$) and with the evaluations and coevaluations (for all $X \in \mathcal{C}$, $\mathrm{coev}_X = \mathrm{ev}_{\overline{X}}^*$). On diagrams, we perform $*$ by reflecting the diagram, keeping the labels on the strings, and placing a $*$ on all morphisms.

For all $X, Y \in \mathcal{C}$, we now have that $\mathcal{C}(X \to Y)$ is a Banach space with positive definite inner product

$$\langle f, g \rangle = \mathrm{tr}(g^* f) = \begin{array}{c} X \\ \boxed{f} \\ Y \\ \boxed{g^*} \\ X \end{array} \overline{X} \ .$$

The inner product makes $\mathrm{End}_\mathcal{C}(X)$ a finite dimensional C^*-algebra, so in particular, all projections are sums of finitely many simple projections, and $\mathcal{C}$ is *semi-simple*, i.e., every exact sequence in $\mathcal{C}$ splits. This also means that any object in $\mathcal{C}$ can be written as a finite direct sum of simple objects. Recall that $X \in \mathcal{C}$ is *simple* if $\dim(\mathrm{End}_\mathcal{C}(X)) = 1$. Thus if X, Y are non-isomorphic simple objects, $\mathcal{C}(X, Y) = (0)$. This means that for each simple $X, Y, Z \in \mathcal{C}$, there are non-negative integers $N_{X,Y}^Z$ such that $X \otimes Y = \bigoplus_{Z \in \mathcal{C}} N_{X,Y}^Z Z$, i.e.,

$$N_{X,Y}^Z = \dim(\mathcal{C}(X \otimes Y \to Z)).$$

Moreover, we have *Frobenius reciprocity*, i.e., for each $X, Y, Z \in \mathcal{C}$, there are natural isomorphisms

$$\mathcal{C}(X \otimes Y \to Z) \cong \mathcal{C}(X \to Z \otimes \overline{Y}) \cong \mathcal{C}(Y \to \overline{X} \otimes Z)$$

which are implemented by the evaluation and coevaluation maps:

$$\begin{array}{ccccc} \begin{array}{c} X \quad Y \\ \boxed{f} \\ Z \end{array} & \leftrightarrow & \begin{array}{c} X \quad Y \\ \boxed{f} \\ Z \quad \overline{Y} \end{array} & \leftrightarrow & \begin{array}{c} X \quad Y \\ \boxed{f} \\ \overline{X} \quad Z \end{array} \end{array} \ .$$

Hence, for all simple $X, Y, Z \in \mathcal{C}$, we have $N_{X,Y}^Z = N_{Z,\overline{Y}}^X = N_{\overline{X},Z}^Y$.

Definition 3.2.10. An object $X \in \mathcal{C}$ has a <u>self-duality</u> if there is an invertible $\varphi \in \mathcal{C}(X, \overline{X})$, which must satisfy certain compatibility axioms. We would like this φ to allow us to define evaluation and coevaluation maps $X \otimes X \to 1$ and $1 \to X \otimes X$, i.e, they are adjoint to each other, satisfy the zig-zag relation, and give a positive scalar for $\dim(X)$ when composed in the natural way. We define these maps by

$$\begin{array}{c} X \\ \boxed{\varphi} \\ \overline{X} \end{array} X \quad \text{and} \quad \begin{array}{c} \overline{X} \\ \boxed{\varphi^*} \\ X \end{array} X$$

respectively. Since X is naturally isomorphic to $\overline{\overline{X}}$, $\overline{\varphi}$ is naturally in $\mathcal{C}(X, \overline{X})$. Therefore, the compatibility requirements are that φ must satisfy $\varphi\overline{\varphi}^* = \mathrm{id}_X$ and $\mathrm{tr}(\varphi\varphi^*) = \dim(X)$. However, to be able to draw these diagrams naively by just a cup and a cap without the label φ, we must have that each of these maps is preserved by rotation:

i.e., the <u>Frobenius-Schur indicator</u> [NS07] of the evaluation must be equal to $+1$. This tells us that $\varphi = \overline{\varphi}$, and φ is unitary ($\varphi^*\varphi = \mathrm{id}_X$ and $\varphi\varphi^* = \mathrm{id}_{\overline{X}}$). A self-duality satisfying this extra axiom is called a <u>symmetric self-duality</u>.

Assumption 3.2.11. We assume that our rigid C^*-tensor category $\mathcal{C}$ is countably generated, i.e., there is a countable set $\mathcal{S}$ of objects in $\mathcal{C}$ such that for each $Y \in \mathcal{C}$, there are $X_1, \ldots, X_n \in \mathcal{S}$ such that

$$\mathcal{C}(X_1 \otimes \cdots \otimes X_n, Y) \neq (0),$$

i.e., Y is (isomorphic to) a sub-object of $X_1 \otimes \cdots \otimes X_n$.

To perform the calculations needed to prove Theorem 3.1.3, we want the planar algebra $\mathcal{P}$ associated to $\mathcal{C}$ in Definition 3.2.13 to be non-oriented, have a non-oriented fusion graph, and have all loop parameters greater than 1.

Hence given a countable generating set $\mathcal{S}$, we work with the generating set $\mathcal{L} = \left\{ X \oplus \overline{X} \,\middle|\, X \in \mathcal{S} \right\}$. Note that the objects in $\mathcal{L}$ are <u>not</u> simple, but they are symmetrically self-dual and have dimension greater than 1.

Definition 3.2.12. The <u>fusion graph of $\mathcal{C}$ with respect to $Y \in \mathcal{C}$</u>, denoted $\mathcal{F}_\mathcal{C}(Y)$, is the oriented graph whose vertices are the isomorphism classes of simple objects of $\mathcal{C}$, and between simple objects $X, Z \in \mathcal{C}$, there are $N^Z_{X,Y} = \dim(\mathcal{C}(X \otimes Y \to Z))$ oriented edges pointing from X to Z. Note that if Y is self-dual, then by semi-simplicity, we have $N^Z_{X,Y} = N^X_{Z,Y}$, and we may ignore the orientation of the edges.

The <u>fusion graph of $\mathcal{C}$ with respect to $\mathcal{L}$</u> (with $\mathcal{L}$ as in Assumption 3.2.11), denoted $\mathcal{F}_\mathcal{C}(\mathcal{L})$, is the push out of the $\mathcal{F}_\mathcal{C}(Y)$ over the isomorphism classes of simple objects $Y \in \mathcal{L}$, i.e., the vertices are the same as before, and the edge set is the union of the edge sets of the $\mathcal{F}_\mathcal{C}(Y)$ for $Y \in \mathcal{L}$. If e is an edge in $\mathcal{F}_\mathcal{C}(\mathcal{L})$ which comes from an edge in $\mathcal{F}_\mathcal{C}(Y)$, then we color e by Y.

Since $\mathcal{L}$ is countable, $\mathcal{F}_\mathcal{C}(\mathcal{L})$ has countably many vertices, although it may not be locally finite. However, $\mathcal{F}_\mathcal{C}(X)$ is always locally finite for $X \in \mathcal{C}$.

Given a vertex $[X]$ of $\mathcal{F}_\mathcal{C}(\mathcal{L})$, the number $\dim(X)$ is independent of the choice of representative of $[X]$. Again, we get a <u>Frobenius-Perron weight vector</u> on the vertices of $\mathcal{F}_\mathcal{C}(\mathcal{L})$, given by $(\dim(X))_{[X] \in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))}$, which satisfies the following equation for each $Y \in \mathcal{L}$:

$$\dim(X)\dim(Y) = \sum_{[Z] \in V(\mathcal{F}_\mathcal{C}(Y))} N^Z_{X,Y} \dim(Z).$$

For convenience, we will identify words on $\mathcal{L}$ with their products, i.e., the word $\alpha = X_1 X_2 \ldots X_n$ is identified with $X_1 \otimes X_2 \otimes \cdots \otimes X_n$.

Definition 3.2.13. To get a factor planar algebra $\mathcal{PA}(\mathcal{C})_\bullet$, for each word α on $\mathcal{L}$, let $\mathcal{PA}(\mathcal{C})_\alpha = \mathcal{C}(\alpha \to 1)$, whose elements are represented diagrammatically as

$$
\boxed{f} \, \big|{\alpha}
$$

Frobenius reciprocity allows us to identify $\mathcal{PA}(\mathcal{C})_\alpha$ with $\mathcal{C}(\beta \to \gamma)$ where $\alpha = \beta\overline{\gamma}$:

$$
\boxed{f} \,\Big|{\alpha} \;=\; \boxed{f} \,\Big|{\gamma}^{\beta}.
$$

We may now interpret any planar tangle in $\mathbb{P}_\mathcal{L}$ labelled by morphisms of $\mathcal{C}$ as one morphism in $\mathcal{C}$ in the usual way. First, isotope the tangle so that each string travels transversally to each horizontal line, except at finitely many critical points. Then isotope the tangle so that each labelled rectangle and each critical point occurs at a different vertical height, and read the diagram from bottom to top to see what the morphism is. The zig-zag relation, Frobenius-reciprocity, pivotality, and symmetric self-dualities of the objects in $\mathcal{L}$ ensure that the answer is well-defined.

From planar algebras to tensor categories

Given a factor planar algebra $\mathcal{P}$, we obtain its C^*-tensor category $\mathrm{Pro}(\mathcal{P})$ of projections as described in [MPS10]. We briefly recall the construction here.

Definition 3.2.14. Let $\mathrm{Pro}(\mathcal{P})$ (abbreviated Pro when $\mathcal{P}$ is understood) be the rigid C^*-tensor category given as follows.

Objects: The objects of Pro are formal finite direct sums of projections in $\mathcal{P}$, i.e., all $p \in P_{\alpha\to\alpha}$ satisfying $p = p^2 = p^*$ for all words α on $\mathcal{L}$. The trivial object is the empty diagram.

Tensor: We tensor objects in Pro by horizontal concatenation; e.g., if $p \in P_{\alpha\to\alpha}$ and $q \in P_{\beta\to\beta}$, then $p \otimes q \in P_{\alpha\beta\to\alpha\beta}$ is given by

$$
\boxed{p \otimes q} \,{}_{\alpha\beta}^{\alpha\beta} \;=\; \boxed{p}\,{}_{\alpha}^{\alpha} \;\; \boxed{q}\,{}_{\beta}^{\beta} \;\in P_{\alpha\beta\to\alpha\beta}.
$$

Note that the simple objects in Pro are the simple projections in $\mathcal{P}$.

We extend the tensor product to direct sums of projections linearly.

<u>Morphisms:</u> The morphisms in Pro are matrices of intertwiners between the projections. If $p \in P_{\alpha \to \alpha}$ and $q \in P_{\beta \to \beta}$, then elements in $\mathrm{Pro}(p, q) = qP_{\alpha \to \beta}p$ are all $x \in P_{\alpha \to \beta}$ such that $x = qxp$, i.e.,

$$
\boxed{x}\,\Big|_{\alpha}^{\alpha} \quad = \quad \boxed{x}\,\Big|_{\beta}^{\alpha} \;\; .
$$

We compose morphisms by vertical concatenation of elements in the planar algebra. If we have $x \in \mathrm{Pro}(p \to q)$ and $y \in \mathrm{Pro}(q \to r)$, then the composite xy is given by

$$
\boxed{xy} \;=\; \frac{\boxed{x}}{\boxed{y}} \;\in\; \mathrm{Pro}(p \to r).
$$

Composition of matrices of morphisms occurs in the usual way.

<u>Tensoring:</u> We tensor morphisms by horizontal concatenation. If $x \in \mathrm{Pro}(p_1 \to q_1)$ and $y \in \mathrm{Pro}(p_2 \to q_2)$, then the tensor product $x \otimes y$ is given by

$$
\boxed{x \otimes y} \;=\; \boxed{x}\;\boxed{y}\;\; .
$$

The tensor product of matrices of intertwiners is the tensor product of matrices followed by tensoring of morphisms.

<u>Duality:</u> The *duality* operation on objects and morphisms is rotation by π

$$
\boxed{\overline{p}} \;=\; \boxed{p} \;=\; \boxed{p} \;\; .
$$

The evaluation and coevaluation maps are given by the caps and cups between the projections in the obvious way.

<u>Adjoint:</u> The *adjoint* operation in Pro is the identity on objects. The adjoint of a 1-morphism is the same as the adjoint operation in the planar algebra $\mathcal{P}$. If $x \in \mathrm{Pro}(p \to q)$ where $p \in P_{\alpha \to \alpha}$ and $q \in P_{\beta \to \beta}$, then consider $x \in P_{\alpha \to \beta}$, take the adjoint, which is an element in $P_{\beta \to \alpha}$, and consider the result x^* as an element in $\mathrm{Pro}(q \to p)$.

For matrices of intertwiners, the adjoint is the $*$-transpose.

Example 3.2.15. We copy the example from [MPS10] as it is highly instructional. If $p, q \in P_{\alpha \to \alpha}$ are orthogonal, then if we define the matrix

$$u = \begin{pmatrix} p & q \end{pmatrix} \in \mathrm{Pro}(\mathrm{id}_\alpha \oplus \mathrm{id}_\alpha \to \mathrm{id}_\alpha),$$

we get an isomorphism $p \oplus q = u^*u \simeq uu^* = p + q$.

Remark 3.2.16. Note that Pro is strict. For any projection $p \in P_{\alpha \to \alpha}$, $p \otimes 1_{\mathrm{Pro}} = 1_{\mathrm{Pro}} \otimes p = p$ since 1_{Pro} is the empty diagram. For all projections $p, q, r \in \mathcal{P}$,

$$(p \otimes q) \otimes r = \boxed{p}\ \boxed{q}\ \boxed{r} = p \otimes (q \otimes r).$$

The following theorem is well-known to experts, and one can easily work it out from the definitions. See part (ii) of the remark on page 10 of [Yam12] for more details.

Theorem 3.2.17.

(1) Let $\mathcal{C}$ be a strict rigid C^*-tensor category. Then $\mathrm{Pro}(\mathcal{PA}(\mathcal{C})_\bullet)$ is equivalent to $\mathcal{C}$.

(2) Let $\mathcal{P}$ be a factor planar algebra. Then $\mathcal{PA}(\mathrm{Pro}(\mathcal{P}))_\bullet = \mathcal{P}$.

Corollary 3.2.18. *Suppose that*

- *$\mathcal{C} = \mathrm{Pro}(\mathcal{P})$ and $\mathcal{P}$ has a countable set of string labels $\mathcal{L}$, or*

- *$\mathcal{P} = \mathcal{PA}(\mathcal{C})_\bullet$ and $\mathcal{C}$ has countable generating set $\mathcal{L}$ of symmetrically self-dual objects.*

Then we may identify the fusion graph Γ of $\mathcal{P}$ with the fusion graph $\mathcal{F}_\mathcal{C}(\mathcal{L})$ of $\mathcal{C}$.

3.3   GJS results for factor planar algebras

Given a subfactor planar algebra $\mathcal{P}$, GJSW constructed a subfactor $N \subset M$ whose planar algebra is $\mathcal{P}$ [GJS10, JSW10]. Moreover, they identified the factors as interpolated free group factors [GJS11].

Suppose we have a factor planar algebra $\mathcal{P}$ with a countable set of string labels $\mathcal{L}$ such that for each $c \in \mathcal{L}$, $\delta_c > 1$. (One can assume $\mathcal{P}$ is the factor planar algebra associated to a rigid C^*-tensor category $\mathcal{C}$ with generating set $\mathcal{L}$ as in Assumption 3.2.11.) We mimic the construction of GJSW to obtain a factor M_0 and rigid C^*-tensor categories Bim and $\mathcal{CF}$ of bifinite bimodules over M_0 such that Pro is equivalent to Bim and $\mathcal{CF}$.

Remark 3.3.1. Recall that when we suppress the $\star$ of an input rectangle, it is assumed that $\star$ is in the <u>lower-left</u> corner. Recall that if a string is labelled by the word $\alpha \in \Lambda$, it is read either <u>top to bottom</u> or <u>left to right</u>.

The graded algebras and their orthogonalized pictures

To start, we set $\mathrm{Gr}_0(P) = \bigoplus_{\alpha \in \Lambda} P_\alpha$ where Λ denotes the set of all finite sequences of colorings for strings and endow $\mathrm{Gr}_0(\mathcal{P})$ with a multiplication $\wedge$ which satisfies

$$x \wedge y = \boxed{x}^{\alpha} \boxed{y}^{\beta}$$

where $x \in P_\alpha$ and $y \in P_\beta$. We endow $\mathrm{Gr}_0(P)$ with the following trace:

$$\mathrm{tr}(x) = \frac{\boxed{\Sigma CTL}}{\boxed{x}^{\alpha}} \tag{3.1}$$

where $x \in P_\alpha$ and $\sum CTL$ denotes the sum of all colored Temperely-Lieb diagrams, i.e. all planar ways of pairing the colors on top of x in a way which respects the word α.

Lemma 3.3.2. *The inner product on $\mathrm{Gr}_0(\mathcal{P})$ given by $\langle x, y \rangle = \mathrm{tr}(y^*x)$ is positive definite. Furthermore, left and right multiplication by elements in $\mathrm{Gr}_0(\mathcal{P})$ is bounded with respect to this inner product*

The proof of the above lemma will closely follow the orthogonalization approach in [JSW10]. To begin, we define a new algebraic structure $\star$ on $\mathrm{Gr}_0(\mathcal{P})$ defined as follows. Suppose $x \in P_\alpha$ and $y \in P_\beta$. Then by letting $|\alpha|$ denote the length of α, we have

$$x \star y = \sum_{\substack{\gamma \text{ s.t.} \\ \alpha = \alpha'\gamma \\ \beta = \bar{\gamma}\beta'}} \boxed{x}^{\alpha'} \overset{\gamma}{\frown} \boxed{y}^{\beta'}$$

where it is understood that if a string connects two different colors, then that term in the sum is zero. We let $F_0(\mathcal{P})$ be the vector space $\mathrm{Gr}_0(\mathcal{P})$ endowed with the multiplication $\star$. Given x in $F_0(\mathcal{P})$, let $x_\emptyset$ denote the component of x in $P_\emptyset \cong \mathbb{C}$. We define a trace tr_F on $F_0(\mathcal{P})$ by $\mathrm{tr}_F(x) = x_\emptyset$. Since $\mathcal{P}$ is a C^*-planar algebra, the sesquilinear form

$$\langle x, y \rangle = \mathrm{tr}_{F_0(\mathcal{P})}(x \star y^*) = \boxed{x} \!-\! \boxed{y^*}$$

is a positive definite inner product.

Set $\mathrm{Epi}(CTL)$ to be the set of colored Temperely-Lieb boxes with strings at the top and bottom where any string touching the top of the box must be through. One can argue exactly as in section 5 of [JSW10] that the map $\Phi : \mathrm{Gr}_0(\mathcal{P}) \to F_0(\mathcal{P})$ given by

$$\Phi(x) = \sum_{E \in \mathrm{Epi}(CTL)} \frac{\boxed{E}}{\boxed{x}}$$

is a bijection with the property that $\Phi(x \wedge y) = \Phi(x) \star \Phi(y)$, $\Phi(x^*) = \Phi(x)^*$ and $\mathrm{tr}(x) = \mathrm{tr}_F(\Phi(x))$. Hence $\star$ is an associative multiplication, $F_0(\mathcal{P})$ and $\mathrm{Gr}_0(\mathcal{P})$ are isomorphic as $*$-algebras, and the inner product on $\mathrm{Gr}_0(\mathcal{P})$ is positive definite.

We now prove that left multiplication by $x \in F_0(\mathcal{P})$ is bounded (this will closely follow arguments in [Bro12]). We may assume $x \in P_\alpha$ for a fixed word α. For fixed words β and γ such that $\alpha = \overline{\beta}\gamma$, the element

is positive in the finite dimensional C^* algebra $P_{\overline{\gamma} \to \overline{\gamma}}$, since for any $y \in P_{\overline{\gamma} \to \overline{\gamma}}$,

Given x and w with $x \in P_\alpha$ with $\alpha = \overline{\beta}\gamma$, $x \star w$ is a sum of terms of the form

and we see that the 2-norm of the above diagram is

                                                     (3.2)

where $\|\cdot\|_{P_{\overline{\gamma} \to \overline{\gamma}}}$ is the operator norm in the C^*-algebra $P_{\overline{\gamma} \to \overline{\gamma}}$. Hence using Equation (3.2) repeatedly, we have

$$\|x \star w\|_{L^2(F_0(P))} \leq \left(\sum_{\alpha = \overline{\beta}\gamma} \|x\|_{P_{\overline{\gamma} \to \overline{\gamma}}} \right) \cdot \|w\|_{L^2(F_0(\mathcal{P}))},$$

and thus left multiplication is bounded on $F_0(\mathcal{P})$. The boundedness of right multiplication is similar.

Since the multiplication is bounded, we can represent $F_0(\mathcal{P})$ on $L^2(F_0(\mathcal{P}))$ acting by left multiplication. We denote $M_0 = F_0(\mathcal{P})''$. We also use M_0 to denote $\mathrm{Gr}_0(\mathcal{P})''$ acting on $L^2(\mathrm{Gr}_0(\mathcal{P}))$, but it will be clear from context which picture we are using. Of course, from the discussion above, both von Neumann algebras are isomorphic.

Given $\alpha \in \Lambda$, we draw a blue string for a string labelled α. We will provide the α label only when it is possible to confuse α and $\overline{\alpha}$.

We define the graded algebra $\mathrm{Gr}_\alpha(\mathcal{P}) = \bigoplus_{\beta \in \Lambda} P_{\overline{\alpha}\beta\alpha}$ with multiplication $\wedge_\alpha$ by

for $x \in P_{\overline{\alpha}\beta\alpha}$ and $x \in P_{\overline{\alpha}\gamma\alpha}$, and trace

$$\text{tr}(x) = \frac{1}{\delta^\alpha} \; {}_\alpha\boxed{\begin{array}{c} \framebox{ΣCTL} \\ \beta \\ \boxed{x} \\ \overline{\alpha} \end{array}}{}_\alpha \; .$$

Note 3.3.3. Be warned that the multiplication $\wedge_\alpha$ in the GJSW diagrams when restricted to $P_{\overline{\alpha}\to\overline{\alpha}}$ is in the opposite order with the multiplication in the introduction!

The inner product

$$\langle \cdot, \cdot \rangle_{F_\alpha(\mathcal{P})} = {}_\alpha\boxed{\quad}\!\!-\!\!\boxed{\quad}$$

makes $P_{\overline{\alpha}\beta\alpha} \perp P_{\overline{\alpha}\gamma\alpha}$ for $\beta \neq \gamma$. We get a trace on $F_\alpha(\mathcal{P})$ by $\text{tr}_{F_\alpha(\mathcal{P})}(x) = \langle x, 1_\alpha \rangle_{F_\alpha(\mathcal{P})}$ where 1_α the the horizontal strand labelled α.

The multiplication $\star_\alpha$ given by

$$x \star_\alpha y = \sum_{\substack{\kappa \text{ s.t.} \\ \beta = \beta'\kappa \\ \gamma = \overline{\kappa}\gamma'}} {}_{\beta'}\boxed{x}\!-\!\boxed{y}\,{}_{\gamma'}$$

for $x \in P_{\overline{\alpha}\beta\alpha}$ and $x \in P_{\overline{\alpha}\gamma\alpha}$ makes $F_\alpha(P)$ isomorphic as a $*$-algebra to $\text{Gr}_\alpha(\mathcal{P})$, preserving the inner product. The same techniques as above with heavier notation show that the inner product on $F_\alpha(\mathcal{P})$ (hence $\text{Gr}_\alpha(\mathcal{P})$) is positive definite and that left and right multiplication in $F_\alpha(\mathcal{P})$ (hence $\text{Gr}_\alpha(\mathcal{P})$) is bounded. We can therefore form the von Neumann algebra $M_\alpha = (F_\alpha(\mathcal{P}))''$ acting by left multiplication on $L^2(F_\alpha)(\mathcal{P})$ (or $(\text{Gr}_\alpha(\mathcal{P}))''$ acting by left multiplication on $L^2(\text{Gr}_\alpha)(\mathcal{P})$). Again, it will be clear from context which picture we are considering.

Factorality of M_α

In this section, we aim to prove the following theorem:

Theorem 3.3.4. *The algebra M_α is a II_1 factor. We have an embedding $\iota_\alpha : M_0 \hookrightarrow M_\alpha$ which is the extension of the map $F_0(\mathcal{P}) \to F_\alpha(\mathcal{P})$ given by*

$$\iota_\alpha \left(\boxed{x} \right) = \boxed{x} \; ,$$

and $\iota_\alpha(M_0)' \cap M_\alpha = P_{\overline{\alpha}\to\overline{\alpha}}^{\text{op}} \cong P_{\alpha\to\alpha}$.

Throughout this subsection, we use the orthogonal picture $M_\alpha = (F_\alpha(\mathcal{P}))''$. Pick a specific color, $c \in \mathcal{L}$, which we will denote by the color green, and recall $\delta_c > 1$. Let A be the abelian von Neumann subalgebra of M_α generated by the cup element

$$\cup_\alpha = \boxed{\cup} \; .$$

We will obtain the factorality of M_α be first examining $L^2(M_\alpha)$ as an $A - A$ bimodule.

To begin, assuming $|\gamma| \geq 2$, we set

$$V_\gamma = \left\{ x \in P_{\overline{\alpha}\gamma\alpha} : \begin{array}{c} c \\ \boxed{x} \end{array} = \begin{array}{c} c \\ \boxed{x} \end{array} = 0 \right\},$$

$V = \oplus_{\gamma \in \Lambda} V_\gamma$ and $\mathcal{V}$ the $A - A$ bimodule generated by V. For $b \in \mathcal{L}$, let $\mathcal{V}_b$ be the $A - A$ bimodule generated by elements of the form

$$\begin{array}{c} b \\ \boxed{x} \end{array},$$

and let W be the $A - A$ bimodule generated by $P_{\alpha \to \alpha}$. We claim that we have the following decomposition:

Lemma 3.3.5. *As $A - A$ bimodules, $L^2(M_\alpha) = W \oplus \mathcal{V} \oplus \bigoplus_{b \in \mathcal{L}} \mathcal{V}_b$.*

Proof. The proof is exactly the same as [Bro12] Proposition 2.1 except that we induct on the length of a word in Λ as opposed to the number of strands of a single color. $\square$

Notice that we can decompose V further as $V = V_{cc} \oplus V_{co} \oplus V_{oc} \oplus V_{oo}$. Here, V_{cc} consists of elements in V whose leftmost and rightmost strings are colored c, V_{co} consists of elements of V whose leftmost string is colored c and rightmost string is colored differently than c, V_{oc} consists of elements of V whose leftmost string is colored differently than c and whose rightmost string is colored c, and V_{oo} consists of the elements in V whose leftmost and rightmost strings are colored other than c. We set $\mathcal{V}_{cc}$ the $A - A$ bimodule generated by V_{cc} and we define $\mathcal{V}_{co}$, $\mathcal{V}_{oc}$, and $\mathcal{V}_{oo}$ analogously.

Let $(V_{cc})_n$ be the subspace of V_{cc} spanned by boxes with a word of length n on top and let $\{\zeta_{n,i}\}$ be an orthonormal basis for $(V_{cc})_n$. It straightforward to see that the set

$$\left\{ \zeta_{n,i}^{l,r} = \delta_c^{-(l+r)/2} \begin{array}{c} \overbrace{}^{l} \quad \overbrace{}^{r} \\ \boxed{\zeta_{n,i}} \end{array} : n, l, r \in \mathbb{N} \right\}$$

is an orthonormal basis for $\mathcal{V}_{cc}$. There are similar orthonormal bases for $\mathcal{V}_{co}$, $\mathcal{V}_{oc}$ and $\mathcal{V}_{oo}$. Let B denote $V_{cc}, V_{co}, V_{oc}, V_{oo}$ or $P_{\overline{\alpha}b\alpha}$ for $b \neq c$ and let $\mathcal{B}$ denote $\mathcal{V}_{cc}, \mathcal{V}_{co}, \mathcal{V}_{oc}, \mathcal{V}_{oo}$, or $\mathcal{V}_b$. Let π, ρ denote the left, right representation of M_0 on $L^2(M_0)$ respectively. We have the following lemma whose proof is straightforward:

Lemma 3.3.6. *Let $\phi : \mathcal{B} \to \ell^2(\mathbb{N}) \otimes B \otimes \ell^2(\mathbb{N})$ be defined on the orthonormal basis of $\mathcal{B}$ by*

$$\phi \left(\delta_c^{-(l+r)/2} \begin{array}{c} \overbrace{}^{l} \quad \overbrace{}^{r} \\ \boxed{v} \end{array} \right) = \xi_l \otimes v \otimes \xi_r$$

with $v \in B_n$, where $\{\xi_i | i \in \mathbb{N}\}$ is the usual orthonormal basis of $\ell^2(\mathbb{N})$. Then ϕ extends to a unitary operator and we have the following representations of $\cup_\alpha$:

- If $\mathcal{B} = \mathcal{V}_{cc}$, $\mathcal{V}_{co}$, $\mathcal{V}_{oc}$ or $\mathcal{V}_{oo}$ then

$$\phi \cdot \pi \left(\frac{\cup_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = x \otimes 1 \otimes 1 \ where$$

  x is $s + s^*$ if the top leftmost color in for boxes in B is c and $s + s^* - \delta_c^{-1/2} e_{\xi_0}$ if the this color differs from c. Similarly,

$$\phi \cdot \rho \left(\frac{\cup_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = 1 \otimes 1 \otimes x \ where$$

  x is $s + s^*$ if the top rightmost color in for boxes in B is c and $s + s^* - \delta_c^{-1/2} e_{\xi_0}$ if the this color differs from c.

- If $\mathcal{B} = \mathcal{V}_b$ and $b \neq c$ then

$$\phi \cdot \pi \left(\frac{\cup_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = (s + s^* - \delta_c^{-1/2} e_{\xi_0}) \otimes 1 \otimes 1 \ and$$

$$\phi \cdot \rho \left(\frac{\cup_\alpha - 1}{\delta_c^{1/2}} \right) \cdot \phi^* = 1 \otimes 1 \otimes (s + s^* - \delta_c^{-1/2} e_{\xi_0}).$$

- (See [Bro12]) If $\mathcal{B} = \mathcal{V}_c$ then there is a unitary (which is <u>not</u> ϕ) $v : \mathcal{V}_c \to \ell^2(\mathcal{N}) \otimes L^2(P_{\bar{\alpha}c\alpha}) \otimes \ell^2(\mathbb{N})$ such that

$$v \cdot \pi \left(\frac{\cup_\alpha - 1}{\delta_c^{1/2}} \right) \cdot v^* = (s + s^*) \otimes 1 \otimes 1 \ and$$

$$v \cdot \rho \left(\frac{\cup_\alpha - 1}{\delta_c^{1/2}} \right) \cdot v^* = 1 \otimes 1 \otimes (s + s^*).$$

Here, s is the unilateral shift operator on $\ell^2(\mathbb{N})$ and e_{ξ_0} is the orthogonal projection in $\mathcal{B}(\ell^2(\mathbb{N}))$ onto the one-dimensional space spanned by ξ_0.

We now show that the operators $s + s^*$ and $s + s^* - \delta_c^{-1/2} e_{\xi_0}$ are unitary equivalent in $\mathcal{B}(\ell^2(\mathbb{N}))$. We begin with the following lemma

Lemma 3.3.7. *The spectra of $s + s^*$ and $y = s + s^* - \delta_c^{-1/2} e_{\xi_0}$ are the same.*

Proof. Since the operators differ by a finite rank operator, by the Weyl-von Neumann Theorem (see [Kat95] p.523), they have the same essential spectrum. The operator $s + s^*$ has essential spectrum $[-2, 2]$ and since the complement of the essential spectrum in the spectrum is an isolated set of eigenvalues, we just need to show that y has no eigenvalues outside of $[-2, 2]$ since y is self adjoint and must have real spectrum.

To this end, let $\lambda > 2$ and $\xi = \sum_j x_j \xi_j$. If ξ is an eigenvector of y with eigenvalue λ then we have the equations

$$x_1 = \left(\lambda + \delta_c^{-1/2} \right) x_0 \ and \ \lambda x_{n+1} = x_n + x_{n+2}$$

for $n \geq 0$. The characteristic equation for this linear recurrence is $x^2 - \lambda x + 1 = 0$ which has roots

$$l = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \text{ and } r = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}.$$

This implies the existence of constants C and D with $x_n = C \cdot l^n + D \cdot r^n$ for all n. Since the sequence x_n is ℓ^2, $C = 0$ meaning $x_n = x_0 \cdot r^n$ for all n. However, this means $x_1 = (\lambda + 1/\sqrt{\delta_c})x_0$ and $x_1 = rx_0$ must both be satisfied. This can only happen if $x_0 = 0$ since $\lambda + 1/\sqrt{\delta_c} > 1$ and $r < 1$. This implies $\xi = 0$.

Now let $\lambda < -2$. With l and r as above we must have $x_n = C \cdot l^n + D \cdot r^n$. This time we must have $D = 0$ and we obtain the equation $x_n = x_0 l^n$. This gives us the two equations

$$x_1 = \left(\lambda + \delta_c^{-1/2}\right) x_0 \text{ and } x_1 = lx_0$$

which implies $(\lambda - \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}) + \delta_c^{-1/2})x_0 = 0$. This forces x_0 to be 0 since by the choice of λ, $\lambda - \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}) < -1$ and $|\delta_c^{-1/2}| < 1$. Therefore $\xi = 0$ and y has no eigenvectors. $\qquad\square$

For any self-adjoint operator $a \in B(\mathcal{H})$, we set $\sigma(a)$ to be the spectrum of a. Given $\xi \in \mathcal{H}$, a Radon measure μ_ξ on the real line is induced by the formula $\mu_\xi(f) = \langle f(a)\xi, \xi \rangle$ for any bounded continuous f. We set $\mathcal{H}_{ac}$ the Hilbert space of vectors ξ where μ_ξ is absolutely continuous with respect to the Lebesgue measure, $\mathcal{H}_{sc}$ the Hilbert space of vectors ξ where μ_ξ is singular with respect to the Lebesgue measure, and $\mathcal{H}_{pp}$ the Hilbert space of vectors ξ where μ_ξ has purely atomic measure. We define the absolutely continuous spectrum of a as the spectrum of a on $\mathcal{H}_{ac}$ and denote it as $\sigma_{ac}(a)$. We define $\sigma_{cc}(a)$ and $\sigma_{pp}(a)$ in a similar manner.

We say a has uniform multiplicity 1 if there is a measure μ on the spectrum of a and a unitary $w : \mathcal{H} \to L^2(\sigma(a), \mu)$ such that for any $f \in L^2(\sigma(a), \mu)$, $waw^*(f)(x) = xf(x)$. In this case $\sigma_{ac}(a)$, $\sigma_{cc}(a)$ and $\sigma_{pp}(a)$ form a partition of $\sigma(a)$. See [Kat95] for more details.

Lemma 3.3.8. *y is unitary equivalent to $s + s^*$.*

Proof. We first show that y has uniform multiplicity 1. To this end, consider the following sequence of polynomials:

$$
\begin{aligned}
p_0(x) &= 1 \\
p_1(x) &= x + \delta_c^{-1/2} \\
p_{n+2}(x) &= xp_{n+1} - p_n(x) \text{ for } n \geq 0
\end{aligned}
$$

By induction, it is straightforward to check that $p_n(y)(\xi_0) = \xi_n$ and hence the map $w : \mathcal{H} \to L^2(\sigma(a), \mu_{\xi_0})$ given by $w(\xi_n) = p_n$ is a unitary satisfying $wyw^*(f)(x) = xf(x)$ for all $f \in L^2(\sigma(a), \mu_{\xi_0})$. By the Kato-Rosenblum theorem (see [Kat95] p.540), if a and b are self adjoint operators and b is trace class then the absolutely continuous parts of b and $a + b$ are unitary equivalent. This implies that the absolutely continuous parts of y and $s + s^*$ are unitary equivalent. The absolutely continuous spectrum of $s + s^*$ is $[-2, 2] = \sigma(s + s^*)$ so the same must hold for y. As was discussed in the previous lemma, y has no spectral values outside $[-2, 2]$ and from above, the spectrum of y is partitioned into σ_{ac}, σ_{cc} and σ_{pp}. Therefore $\sigma_{pp} = \sigma_{cc} = \emptyset$ and $\sigma_{ac} = \sigma$ showing the unitary equivalence. $\qquad\square$

Therefore by conjugating by a unitary, we may assume that $L^2(M_\alpha) \cong L^2(A) \oplus \ell^2(\mathbb{N}) \otimes \mathcal{H} \otimes \ell^2(\mathbb{N})$ for some Hilbert space $\mathcal{H}$ where on the second summand, $\pi\left(\frac{\cup -1}{\sqrt{\delta_c}}\right)$ acts as $(s+s^*) \otimes 1 \otimes 1$ and $\rho\left(\frac{\cup -1}{\sqrt{\delta_c}}\right)$ acts as $1 \otimes 1 \otimes (s + s^*)$. The lemmas below follow [JSW10] but we supply proofs for the readers' convenience.

Lemma 3.3.9. $A' \cap M_\alpha = AP^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}}$.

Proof. First, note that $AP^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}} \subseteq A' \cap M_\alpha$ in the obvious way

where we need the "op" since multiplication in the GJSW picture happens in the opposite order (see Note 3.3.3).

If $A' \cap M_\alpha$ were larger than $AP^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}}$ then by looking at the orthogonal compliment of $AP^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}}$ in $L^2(M_\alpha)$, there is a nonzero vector $\xi \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ with $((s+s^*) \otimes 1)\xi = (1 \otimes s + s^*)\xi$. Viewing $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ as the Hilbert Schmidt operators on $\ell^2(\mathbb{N})$ this means $\xi(s+s^*) = (s+s^*)\xi$ which is impossible since $s + s^*$ has no eigenvalues. $\square$

We will realize M_0 as a unital subalgebra of M_α via the map ι_α.

Lemma 3.3.10. $M_0' \cap M_\alpha = P^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}} \cong P_{\alpha \to \alpha}$ as an algebra.

Proof. If $x \in M_0' \cap M_\alpha$ then by Lemma 3.3.9, $x \in AP^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}}$, so we write x as an ℓ^2 sum

$$x = \sum_{n=0}^\infty \frac{1}{\sqrt{\delta_c^n}} \cdot x_n \star \boxed{ \overset{\cdots}{n} },$$

where $x_n \in P^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}}$ for all n. Consider the following elements of M_0:

$$z = \boxed{}, l_n = \frac{1}{\sqrt{\delta_c^n}} \boxed{ \overset{\cdots}{n} }, \text{ and } r_n = \frac{1}{\sqrt{\delta_c^n}} \boxed{ \overset{\cdots}{n} }.$$

By a direct diagrammatic computation, $z \star x - x \star z$ is the orthogonal sum

$$\sum_{n=1}^\infty (x_n + \frac{1}{\sqrt{\delta_c}} \cdot x_{n+1}) \star (l_n - r_n).$$

For this to be zero, we must have $x_{n+1} = -\sqrt{\delta_c} \cdot x_n$ for $n \geq 1$. Since the sum for x must be ℓ^2, this implies $x_n = 0$ for $n \geq 1$, i.e., $x \in P^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}}$.

Obviously the rotation by 180 degrees gives the isomorphism $P^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}} \cong P_{\alpha \to \alpha}$. $\square$

Proof of Theorem 3.3.4. If x were in the center of M_α then we know that $x \in P^{\mathrm{op}}_{\overline{\alpha} \to \overline{\alpha}}$. The element x must commute with

giving the equation

Joining the leftmost strings to the top implies that x is a scalar multiple of the identity. $\square$

Jones' Towers Associated to M_0

If α and β are in Λ such that $\alpha = \beta\gamma$ with $\gamma \in \Lambda$ then we have a unital, trace-preserving inclusion $\iota : M_\beta \to M_\alpha$ given by

$$\iota(x) = \begin{array}{c} \beta \boxed{x} \beta \\ \hline \gamma \end{array}.$$

When we write $M_\beta \subset M_\alpha$ we mean that M_β is included into M_α in the manner described above. We have the following theorem:

Theorem 3.3.11. *The following is a Jones' tower of factors:*

$$M_0 \subset M_\alpha \subset M_{\alpha\bar\alpha} \subset \cdots \subset M_{(\alpha\bar\alpha)^n \alpha} \subset M_{(\alpha\bar\alpha)^{n+1}} \subset \cdots .$$

Moreover $[M_\alpha : M_0] = \delta_\alpha^2$.

Proof. This proof closely follows Section 4 in [JSW10]. We will show that $M_{\alpha\bar\alpha}$ is the basic construction of $M_0 \subset M_\alpha$. The proof for higher steps in the tower is the same but with heavier notation. To begin, set

$$e_0 = \frac{1}{\delta_\alpha} \begin{array}{c}\supset\ \subset\end{array} \in P^{op}_{\alpha\bar\alpha \to \alpha\bar\alpha} \subset M_{\alpha\bar\alpha}$$

then e_0 is a projection. It is a straightforward diagrammatic computation to show that if $x \in M_\alpha$ then $e_0 x e_0 = E_{M_0}(x)$ with E_{M_0} the trace preserving conditional expectation.

We now claim that $(M_\alpha, e_0)''$ is a II_1 factor. Indeed, if y were in the center of $(M_\alpha, e_0)''$ it would have to commute with M_0, implying that $y \in P_{\alpha\bar\alpha \to \alpha\bar\alpha}$. The element y also has to commute with M_α, in particular it has to commute with the element

$$\begin{array}{c}\boxed{}\\ \bar\alpha \end{array}.$$

This implies y must be of the form

$$\begin{array}{c} \overline{} \\ \bar\alpha \boxed{y} \bar\alpha \end{array},$$

but commuting with e_0 forces this diagram to be a scalar multiple of the identity.

One then observes that if $z \in (M_\alpha, e_0)''$, then $z e_0 = (\delta^\alpha)^2 E_{M_\alpha}(z e_0) e_0$. This is done by realizing that as a von Neumann algebra, $(M_\alpha, e_0)''$ is generated by M_α and $M_\alpha e_0 M_\alpha$ and hence one can assume z is in either of these spaces. The equality then becomes a straightforward diagrammatic check. From this, one can deduce that the map $x \mapsto \delta^u x e_0$ from M_α to $(M_\alpha, e_0)''$ is a surjective isometry intertwining E_{M_0} on $L^2(M_\alpha)$ and left multiplication by e_0. From this, we deduce that $(M_\alpha, e_0)''$ is the basic construction of $M_0 \subset M_\alpha$ and evaluating $[M_\alpha : M_0]$ is a matter of calculating the trace of e_0.

The same arguments applied to $M_\alpha \subset M_{\alpha\bar\alpha}$ implies that $[M_{\alpha\bar\alpha} : M_\alpha] = (\delta^\alpha)^2$ hence $[M_{\alpha\bar\alpha} : (M_\alpha, e_0)''] = 1$ i.e. $M_{\alpha\bar\alpha} = (M_\alpha, e_0)''$. $\square$

As an aside, since $M_0' \cap M_\beta = P_{\beta \to \beta}$ for any β, it follows that the sequence of vector spaces $P_0, P_{\alpha \to \alpha}, P_{\alpha\bar\alpha \to \alpha\bar\alpha}, \cdots$ forms a subfactor planar algebra.

Some bifinite bimodules over M

Notation 3.3.12. We now use the notation M instead of M_0 as it makes the rest of this section easier on the eyes.

In this subsection and the next, we define a category $\mathcal{C}_{bim}$ of bifinite bimodules over M. To begin, we use $\mathcal{P}$ to construct another von Neumann algebra $\mathcal{M}_{\alpha,\beta}$ which contains the factors M_α and M_β as corners/cut-downs. Let $\mathrm{Gr}_{\alpha,\beta}(P) = \bigoplus_{\gamma \in \Lambda} P_{\overline{\alpha}\gamma\beta}$. If $x \in P_{\overline{\alpha}\gamma\beta}$, we view x as

$$-\boxed{x}-\,.$$

where we now draw red strings for strings labelled β. There is a sesquilinear form $\langle \cdot, \cdot \rangle$ on $Gr_{\alpha,\beta}(P)$ given by

$$\langle x, y \rangle = \boxed{\sum CTL \atop \boxed{x}-\boxed{y^*}}\,.$$

As a vector space, $\mathrm{Gr}_{\alpha,\alpha}(\mathcal{P}) = \mathrm{Gr}_\alpha(\mathcal{P})$. For $\alpha \neq \beta$, we form the $*$-algebra $\mathcal{G}_{\alpha,\beta}$ which is generated by the vector spaces $\mathrm{Gr}_\alpha(\mathcal{P})$, $\mathrm{Gr}_\beta(\mathcal{P})$, $\mathrm{Gr}_{\alpha,\beta}(\mathcal{P})$ and $\mathrm{Gr}_{\beta,\alpha}(\mathcal{P})$, under the multiplication

$$\boxed{\kappa\,x\,\theta}^{\gamma} \wedge \boxed{\omega\,x\,\chi}^{\gamma'} = \delta_{\omega,\theta} \boxed{\kappa\,x\,\theta}^{\gamma}\boxed{y^*\,\chi}^{\gamma'}\,,$$

where $\kappa, \theta, \omega, \chi \in \{\alpha, \beta\}$. There is also a (non-normalized) trace on $\mathcal{G}_{\alpha,\beta}$ given by

$$\mathrm{tr}(x) = \boxed{\Sigma CTL \atop \boxed{x}}$$

if $x \in \mathrm{Gr}_\alpha(\mathcal{P})$ or $\mathrm{Gr}_\beta(\mathcal{P})$, and is zero otherwise. Just as in the case for the algebras $\mathrm{Gr}_\alpha(\mathcal{P})$ one can show (by orthogonalizing) that the trace is positive definite and that multiplication is bounded. Therefore, one can form the von Neumann algebra $\mathcal{M}_{\alpha,\beta} = \mathcal{G}_{\alpha,\beta}''$ acting on $L^2(\mathcal{G}_{\alpha,\beta})$ by left and right multiplication. Set

$$p_\alpha = \boxed{\alpha} \text{ and } p_\beta = \boxed{\beta}\,.$$

We see that $p_\gamma \mathcal{M}_{\alpha,\beta} p_\gamma = M_\gamma$ (with the non-normalized trace) for $\gamma = \{\alpha, \beta\}$ so $L^2(\mathcal{M}_{\alpha,\beta})$ is naturally an $M_\alpha \oplus M_\beta$ bimodule. Hence we may consider $L^2(\mathcal{M}_{\alpha,\beta})$ as an $M - M$ bimodule via the embedding $x \mapsto \iota_\alpha(x) \oplus \iota_\beta(x) \in M_\alpha \oplus M_\beta$. Under this identification, we define $\mathcal{H}_{\alpha,\beta}$ to be the the $M - M$ bimodule

$$\mathcal{H}_{\alpha,\beta} = p_\alpha \wedge L^2(\mathcal{M}_{\alpha,\beta}) \wedge p_\beta.$$

As mentioned above, we can give $\mathcal{G}_{\alpha,\beta}$ an orthogonalized inner product and multiplication exactly as in Section 3.3. We use the notation $\mathcal{F}_{\alpha,\beta}$ to denote the vector space $\mathcal{G}_{\alpha,\beta}$ with the orthogonalized

inner product and multiplication $\star$. The vector space $p_\alpha \star L^2(\mathcal{F}_{\alpha,\beta}) \star p_\beta$ is naturally an $M - M$ bimodule which is isomorphic to $p_\alpha \wedge L^2(\mathcal{M}_{\alpha,\beta}) \wedge p_\beta$. Our first lemma is proven by making use of the orthogonal picture:

Lemma 3.3.13. *The vector space of $M - M$ central vectors of $p_\alpha \star L^2(\mathcal{F}_{\alpha,\beta}) \star p_\beta$ is the vector space $P_{\overline{\alpha} \to \overline{\beta}}$ (rotating the GJS diagrams 90 degrees clockwise).*

Proof. As in Section 3.3, we let A be the von Neumann subalgebra of M generated by $\cup$ and let W be the $A - A$ bimodule generated by $P_{\overline{\alpha} \to \overline{\beta}}$. With the same approach as 3.3, we see that as an $A - A$ bimodule, we have

$$\mathcal{H}_{\alpha,\beta} \cong W \bigoplus \left(\ell^2(\mathbb{N}) \otimes \mathcal{H} \otimes \ell^2(\mathbb{N}) \right)$$

for $\mathcal{H}$ an auxiliary Hilbert space. The operator $\cup$ acts on the left of the second factor by $(s+s^*) \otimes 1 \otimes 1$ and on the right by $1 \otimes 1 \otimes (s+s^*)$. Therefore the proof of Lemma 3.3.9 applies here and we see that the $A - A$ central vectors of $L^2(\mathcal{F}_{\alpha,\beta})$ are exactly W. To finish the proof, one repeats the argument in Lemma 3.3.10. $\qquad\square$

Corollary 3.3.14. *The vector space of $M - M$ central vectors of $p_\alpha \wedge L^2(\mathcal{F}_{\alpha,\beta}) \wedge p_\beta$ is the vector space $P_{\overline{\alpha} \to \overline{\beta}}$ (rotating the GJS diagrams 90 degrees clockwise).*

For the rest of this section, we use the non-orthogonalized picture for $\mathcal{H}_{\alpha,\beta}$. We first write down some straightforward isomorphisms between these bimodules:

Lemma 3.3.15. *Let $\alpha, \beta, \gamma \in \Lambda$. Then as $M - M$ bimodules, $\mathcal{H}_{\alpha\beta,\gamma} \cong \mathcal{H}_{\alpha,\overline{\beta}\gamma}$.*

Proof. The map
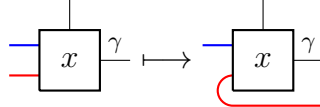
is a unitary operator intertwining the left and right M-actions. $\qquad\square$

Definition 3.3.16. By Lemma 3.3.15, we now define the bimodule $H_\alpha := \mathcal{H}_{\alpha,\emptyset} \cong \mathcal{H}_{\emptyset,\overline{\alpha}}$. We draw elements of H_α with strings emanating from the bottom instead of the sides
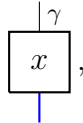
and the left and right M-actions are given by the obvious diagrams. For $\xi \in p_\alpha \wedge \mathcal{M}_{\alpha,\emptyset} \wedge p_\emptyset \cong p_\emptyset \wedge \mathcal{M}_{\emptyset,\overline{\alpha}} \wedge p_{\overline{\alpha}}$ and $x, y \in M$,
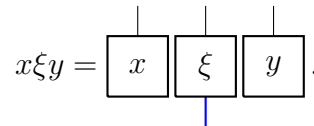
The inner product on H_α is given by

$$\langle x, y \rangle_{H_\alpha} = \qquad\qquad\qquad\qquad (3.3)$$

where $\overline{y^*}$ is the rotation of y^* by 180 degrees.

Our goal will now be to show that $H_\alpha \otimes_M H_\beta \cong H_{\alpha\beta}$. In diagrams:
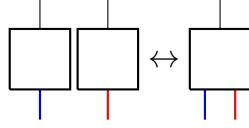
We will need the following lemma, whose purpose is to "close up" the space between the boxes.

Lemma 3.3.17. *The $\alpha-$cup element* *is positive and invertible in M.*

Proof. We induct on the size of α. If $|\alpha| = 1$ then it follows from [GJS10] that the operator is a free-poisson element whose spectrum is supported away from 0 hence the lemma holds for this case. Suppose that the lemma holds for some color pattern β and suppose $\alpha = \beta c$ for a fixed color c, denoted by a green string. It follows that the element

$$x = \;\;\boxed{\;\;}$$

is positive invertible in M_c. Let $z \in \mathcal{M}_{\emptyset,c}$ be the element

$$z = \;\;\boxed{\;\;}$$

Then the $\alpha-$cup element has the form $z^* \wedge x \wedge z \in \mathcal{M}_{\emptyset,c}$ so it immediately follows that it is positive. Note that $z^* \wedge z$ is invertible so there is a positive constant k_z so that $\langle z^* \wedge z \wedge \xi, \xi \rangle \geq k_z \|\xi\|_2^2$ for all $\xi \in L^2(M)$. Letting r be the positive square root of x, there is a strictly positive constant k_r so that for all $\eta \in L^2(M_c)$, $\langle r \wedge \eta, r \wedge \eta \rangle \geq k_r \|\eta\|_2^2$. Therefore, for all $\xi \in L^2(M)$,

$$\langle z^* \wedge x \wedge z \wedge \xi, \xi \rangle = \langle r \wedge z \wedge \xi, r \wedge z \wedge \xi \rangle \geq k_r \langle z \wedge \xi, z \wedge \xi \rangle$$
$$= k_r \langle z^* \wedge z \wedge \xi, \xi \rangle \geq k_r k_z \|\xi\|_2^2$$

implying invertibility. □

Definition 3.3.18. Recall from [Con80, Bis97, Jon08] that given a bimodule $_M K_M$, a vector $\xi \in K$ is called right M-bounded if the map $L(\xi)^0 \colon M \to K$ given by $m \mapsto \xi m$ extends to a bounded linear operator $L(\xi) \colon L^2(M) \to K$. There is a similar definition of a left M-bounded vector, and if K is bifinite, then the sets of left and right M-bounded vectors agree. For such a bimodule K, we denote the set of left/right M-bounded vectors by $D(K)$.

Example 3.3.19. For all $\alpha, \beta \in \Lambda$, we have $D(H_{\alpha\beta}) = D(\mathcal{H}_{\alpha,\overline{\beta}}) = p_\alpha \wedge \mathcal{M}_{\alpha,\overline{\beta}} \wedge p_{\overline{\beta}}$.

Lemma 3.3.20. *As $M-M$ bimodules, the Connes' fusion $H_\alpha \otimes_M H_\beta \cong H_{\alpha\beta}$.*

Proof. By [Bis97], $H_\alpha \otimes_M H_\beta$ is the completion of the algebraic tensor product $D(H_\alpha) \odot D(H_\beta)$ (first modding out by vectors of length zero) under the semi-definite inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{H_\alpha \otimes_M H_\beta} = \langle \langle \xi_2 | \xi_1 \rangle_M \cdot \eta_1, \eta_2 \rangle_{H_\beta}$$

where $\langle \xi_2 | \xi_1 \rangle_M$ is the unique element in M satisfying

$$\langle \xi_1 x, \xi_2 \rangle_{H_\alpha} = \text{tr}(x \langle \xi_2 | \xi_1 \rangle_M) \text{ for all } x \in M.$$

We define a map u by the linear extension of

$$D(H_\alpha) \odot D(H_\beta) \longrightarrow \mathcal{H}_{\alpha,\bar{\beta}} \cong H_{\alpha\beta} \text{ by}$$

Obviously the map u intertwines the left and right M-actions and is M-middle linear. We will show it is isometric with dense range, and thus u has a unique extension to a $M - M$ bilinear unitary giving the desired isomorphism.

First, it is not hard to check directly that the M-valued inner product of the M-bounded vectors $\xi_1, \xi_2 \in D(H_\alpha)$ is given by

Hence if

$$\zeta = \sum_{i=1}^{n} \xi_i \otimes \eta_i \in D(H_\alpha) \odot D(H_\beta),$$

we immediately have

$$\|\zeta\|_2^2 = \sum_{i,j=1}^{n} \langle \langle \xi_j | \xi_i \rangle_M \cdot \eta_i, \eta_j \rangle_{H_\beta} = \quad \tag{3.4}$$

Since

$$u(\zeta) = \sum_{i=1}^{n} \boxed{\xi_i}\,\boxed{\eta_i} ,$$

we see that $\|u(\zeta)\|_2^2$ is the same diagram as in (3.4). Hence u can be extended to be an isometry on $H_\alpha \otimes_M H_\beta$.

For the rest of this proof we will only use the $\wedge$ multiplication, so to clean up notation we will omit the $\wedge$. We must show the image of u is dense in $H_{\alpha\beta}$. We may assume $\delta_\alpha \leq \delta_\beta$, and the proof is similar if $\delta_\beta < \delta_\alpha$. Set

$$w = \boxed{\;\diagup\;} .$$

Then w^*w is positive and invertible in M, so if v is the polar part of w, then $v \in D(H_\alpha)$ with right support $p_\emptyset$ and left support e under p_α. Choose partial isometries $x_1, ..., x_n$ in M_α such that

$x_1, ..., x_{n-1}$ have right support e and orthogonal left supports. Choose the partial isometry x_n so that its left support is $1 - \sum_{i=1}^{n-1} x_i x_i^*$ and its right support is under e.

Similarly, there is a partial isometry $\tilde{v} \in D(H_\beta)$ whose right support is $f \leq p_{\overline{\beta}}$ and whose left support is $p_\emptyset$. Choose partial isometries $y_1, ..., y_n \in M_{\overline{\beta}}$ such that $y_1, ..., y_{n-1}$ have left support f and orthogonal right supports. Choose the partial isometry y_n so its right support is orthogonal to $\sum_{i=1}^{n-1} y_i^* y_i$ and its left support is the right support of $x_n v \tilde{v}$ (we can do this since $\delta_\alpha \leq \delta_{\overline{\beta}}$).

It now follows that $z = \sum_{i=1}^{n} x_i v \tilde{v} y_i$ is a partial isometry in $D(H_{\alpha\beta})$ with full left support p_α. Note that every element, r in $D(H_{\alpha\beta})$ is of the form zx for $x \in M_{\overline{\beta}}$ since we simply choose $x = z^* r$. Since $zx = \sum_{i=1}^{n}(x_i v) \cdot (\tilde{v} y_i x)$ is in the image of u, we are finished. $\qquad\square$

Lemma 3.3.21. *As an $M - M$ bimodule, H_α is bifinite. Thus by Lemma 3.3.15, so is $\mathcal{H}_{\alpha,\beta}$.*

Proof. By the proof of Lemma 3.3.20, there is a partial isometry $v \in p_\alpha \wedge \mathcal{M}_{\alpha,\emptyset} \wedge p_\emptyset$ with right support $p_\emptyset$. The argument at the end of Lemma 3.3.20 shows that every element of $p_\alpha \wedge \mathcal{M}_{\alpha,\emptyset} \wedge p_\emptyset$ is of the form $x \wedge v$ for $x \in M_\alpha$. Therefore v is cyclic for the left action of M_α on $\mathcal{H}_{\alpha,\emptyset} \cong H_\alpha$, and since $[M_\alpha : M] < \infty$, $\dim_{M-}(H_\alpha) < \infty$. Similarly $\dim_{-M}(H_\alpha) < \infty$. $\qquad\square$

The categories $\mathrm{Bim}(\mathcal{P})$ and $\mathcal{CF}(\mathcal{P})$

We now define two rigid C^*-tensor categories $\mathrm{Bim}(\mathcal{P})$ and $\mathcal{CF}(\mathcal{P})$ whose objects are bifinite $M - M$ bimodules. We show given a factor planar algebra $\mathcal{P}$, we have equivalences $\mathrm{Pro}(\mathcal{P}) \cong \mathrm{Bim}(\mathcal{P}) \cong \mathcal{CF}(\mathcal{P})$.

Definition 3.3.22. If p is a projection in $P_{\alpha \to \alpha}$, we define $H_p = \mathcal{H}_{\emptyset,\overline{\alpha}} \wedge p$, which is a bifinite $M - M$ bimodule. Note that elements of H_p are obtained from elements in H_α by putting a p on the bottom. Thus the linear span of elements of the form

where $x \in p_\emptyset \wedge \mathcal{M}_{\emptyset,\overline{\alpha}} \wedge p$ forms a dense subset of H_p. Recall that elements of M act on the left and right as in the non-orthogonal picture. These actions clearly do not affect the p on the bottom.

Definition 3.3.23. Let $\mathrm{Bim}(\mathcal{P})$ (abbreviated Bim) be the strict rigid C^*-tensor category defined as follows.

Objects: The objects of Bim are finite direct sums of the bimodules H_p for the projections $p \in P_{\alpha \to \alpha}$ for $\alpha \in \Lambda$. Note that the unit object $1_{\mathrm{Bim}} = H_\emptyset \cong L^2(M)$.

Tensor: For $p \in P_{\alpha \to \alpha}$ and $q \in P_{\beta \to \beta}$, we define $H_p \otimes_{\mathrm{Bim}} H_q = H_{p \otimes q}$ where $p \otimes q$ is the tensor product in $\mathcal{P}$. The tensor product is extended to direct sums linearly.

Note that since the tensor product of projections in $\mathcal{P}$ is strict, so is the tensor product in Bim.

Morphisms: For $p \in P_{\alpha \to \alpha}$ and $q \in P_{\beta \to \beta}$, we define $\mathrm{Bim}(H_p \to H_q) = qP_{\alpha \to \beta}p$ and composition is the usual composition in $\mathcal{P}$. Morphisms between direct sums are matrices of such maps.

Tensoring: For $x_i \in \mathrm{Bim}(H_{p_i} \to H_{q_i}) = q_i P_{\alpha_i \to \beta_i} p_i$ for $i = 1, 2$, we define $x_1 \otimes_{\mathrm{Bim}} x_2$ as the tensor product of morphisms in $\mathcal{P}$. Similarly for matrices of such maps.

Duality: The dual of H_p is $H_{\bar{p}}$. The evaluation map $\mathrm{ev}_{H_p} \colon H_{\bar{p}} \otimes_{\mathrm{Bim}} H_p \to H_\emptyset = 1_{\mathrm{Bim}}$ is given by the α-cup with projections $\bar{p}, p$ on top:

$$\boxed{\bar{p}} \quad \boxed{p}$$
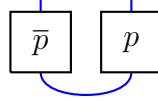
Of course, the coevalutation map $1_{\mathrm{Bim}} \to H_p \otimes H_{\bar{p}}$ is given by the adjoint of $\mathrm{ev}_{H_{\bar{p}}}$, which is the α-cap with projections underneath:

$$\boxed{p} \quad \boxed{\bar{p}} \, .$$
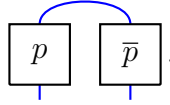
One easily checks that the necessary relations hold.

The dual map is extended to direct sums linearly.

Adjoint: The adjoint map $*$ is the identity on all objects, and the adjoint of a morphism $x \in \mathrm{Bim}(H_p \to H_q) = qP_{\alpha \to \beta}p$ is the adjoint in the planar algebra $x^* \in pP_{\beta \to \alpha}q = \mathrm{Bim}(H_q \to H_p)$. The adjoint of a matrix of maps is the $*$-transpose of the matrix.

Theorem 3.3.24. *The map* $\mathrm{Pro} \to \mathrm{Bim}$ *by* $p \mapsto H_p$ *and the identity on morphisms is an equivalence of categories.*

Proof. Note that $\mathrm{Pro}(p \to q) = qP_{\alpha \to \beta}p = \mathrm{Bim}(H_p \to H_q)$. One now checks that the described map is an additive, monoidal, dual-preserving, $*$-preserving, fully faithful, essentially surjective functor. $\qquad\square$

Definition 3.3.25. Let $\mathcal{CF}(\mathcal{P})$ (abbreviated $\mathcal{CF}$, which stands for *Connes' fusion*) be the rigid C^*-tensor category defined as follows.

Objects: The objects of $\mathcal{CF}$ are finite direct sums of the bimodules H_p as in Bim.

Tensor: For bimodules $K, L \in \mathcal{CF}$, we define $K \otimes_{\mathcal{CF}} L = K \otimes_M L$, the Connes' fusion of II_1-factor bimodules.

The associator $a_{\mathcal{CF}}$ is defined by restricting to M-bounded vectors as in [Bis97].

Morphisms: $\mathcal{CF}(K \to L)$ is the set of $M-M$ bilinear maps $K \to L$. Composition is the usual composition of linear maps.

Tensoring: For $M - M$ bilinear maps $\varphi_i \colon K_i \to L_i$ for $i = 1, 2$, we define $\varphi_1 \otimes_M \varphi_2$ by the Connes' fusion of intertwiners. If we have M-bounded vectors $\xi_i \in K_i$ for $i = 1, 2$, then the map $\xi_1 \otimes \xi_2 \mapsto \varphi_1(\xi_1) \otimes \varphi_2(\xi_2)$ is clearly M-middle linear and bounded, so it extends to a unique $M - M$ bilinear map.

<u>Duality:</u> The dual of K is the contragredient bimodule $\overline{K} = \{\overline{\xi} | \xi \in K\}$ where $\overline{\lambda\xi + \eta} = \overline{\lambda}\overline{\xi} + \overline{\eta}$ for all $\lambda \in \mathbb{C}$ and $\eta, \xi \in K$, and the action is given by $a\overline{\xi}b = \overline{b^*\xi a^*}$. The evaluation map

$$\mathrm{ev}_K \colon \overline{K} \otimes_M K \longrightarrow H_\emptyset = L^2(M) = 1_{\mathcal{CF}}$$

is the unique extension of the map $\overline{\xi} \otimes \eta \mapsto \langle \xi | \eta \rangle_M$ where ξ, η are M-bounded vectors in K. The coevaluation map coev_K is the unique map in $\mathcal{CF}(1 \to K \otimes_M \overline{K})$ corresponding to $\mathrm{id}_K \in \mathcal{CF}(K \to K)$ under the natural isomorphism given by Frobenius reciprocity. For an explicit formula, just pick an orthonormal left M-basis $\{\zeta\} \subset K$ (e.g., see [Con80, Pen12]), and we have that

$$\mathrm{coev}_K(1_M) = \sum_\zeta \zeta \otimes \overline{\zeta}$$

is M-central and independent of the choice of $\{\zeta\}$. The zig-zag relation is now given by

$$\sum_\zeta \zeta \langle \zeta | \xi \rangle_M = \xi$$

for all M-bounded $\xi \in K$.

<u>Adjoint:</u> The adjoint map $*$ is the identity on all objects and on a morphism $x \in \mathcal{CF}(K \to L)$ is the adjoint linear operator $x^* \in \mathcal{CF}(L \to K)$.

Note $\mathcal{CF}$ is a rigid C^*-tensor category by well known properties of Connes' fusion (e.g., see [Bis97, Pen12]). It is now our task to prove the following theorem:

Theorem 3.3.26. *Define a map $\Phi \colon \mathrm{Bim} \to \mathcal{CF}$ as follows. First, Φ is the identity on objects. Second, for a morphism $x \in qP_{\alpha \to \beta}p$, we get an $M - M$ bimodule map $\Phi_x \colon H_p \to H_q$ by*

$$(3.5)$$

Finally, Φ is applied entry-wise to matrices over such morphisms.
  The map Φ is an equivalence of categories $\mathrm{Bim} \simeq \mathcal{CF}$.

In the lemmas below, unless otherwise stated, p, q are projections in $P_{\alpha \to \alpha}, P_{\beta \to \beta}$ respectively.

Remark 3.3.27. Note that composition of the bimodule maps given by Equation (3.5) corresponds to the usual composition in $\mathcal{P}$, i.e., if we have $x \in qP_{\alpha \to \beta}p$ and $y \in rP_{\beta \to \gamma}q$, then $\Phi_y \circ \Phi_x = \Phi_{yx} \colon H_p \to H_q$, where

It is obvious that $\Phi_p = \mathrm{id}_{H_p}$, so Φ is a functor.

Lemma 3.3.28. *For p, q projections in $\mathcal{P}$, the maps*

$$\phi_{p,q} \colon H_p \otimes_M H_q \to H_p \otimes_{\mathrm{Bim}} H_q = H_{p \otimes q}$$

(where $p \otimes q$ is the tensor product in $\mathcal{P}$) given by the unique extension of
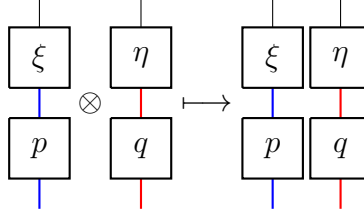
for $\xi \in D(H_p)$ and $\eta \in D(H_q)$, are $M - M$ bilinear isomorphisms which satisfy associativity, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
(H_p \otimes_M H_q) \otimes_M H_r & \xrightarrow{\ a_{\mathcal{CF}}\ } & H_p \otimes_M (H_q \otimes_M H_r) \\
\phi_{p,q} \otimes_M \mathrm{id}_{H_r} \downarrow & & \downarrow \mathrm{id}_{H_p} \otimes_M \phi_{q,r} \\
(H_p \otimes_{\mathrm{Bim}} H_q) \otimes_M H_r & & H_p \otimes_M (H_q \otimes_{\mathrm{Bim}} H_r) \\
\phi_{p \otimes q, r} \downarrow & & \downarrow \phi_{p, q \otimes r} \\
(H_p \otimes_{\mathrm{Bim}} H_q) \otimes_{\mathrm{Bim}} H_r & \xrightarrow{\quad = \quad} & H_p \otimes_{\mathrm{Bim}} (H_q \otimes_{\mathrm{Bim}} H_r).
\end{array}
$$

Proof. As in the proof of Lemma 3.3.20, the map for $\xi \in D(H_p), \eta \in D(H_q)$ is an isometry intertwining the left and right M actions. Since the linear span of elements of the form $x \wedge y$ with $x \in D(\mathcal{H}_{\alpha,\emptyset})$ and $y \in D(\mathcal{H}_{\emptyset,\overline{\beta}})$ is equal to $D(\mathcal{H}_{\alpha,\overline{\beta}}) \cong D(H_{\alpha\beta})$ by the proof of Lemma 3.3.20, the above map is also surjective.

Associativity follows from looking at M-bounded vectors (see [Bis97]). $\qquad\square$

Lemma 3.3.29. *As complex vector spaces,*

$$\mathcal{CF}(H_p \to H_q) \cong q(P_{\alpha \to \beta})p = \mathrm{Bim}(H_p \to H_q).$$

Moreover, the composition of maps $\varphi \in \mathcal{CF}(H_p \to H_q)$ and $\psi \in \mathcal{CF}(H_q \to H_r)$ corresponds to the composition in Bim.

Proof. Recall that an element $x \in q(P_{\alpha \to \beta})p = \mathrm{Bim}(H_p \to H_q)$ gives a map $\Phi_x \in \mathcal{CF}(H_p \to H_q)$ as in Equation (3.5), and composition of morphisms is exactly multiplication in $\mathcal{P}$ by Remark 3.3.27. Note further that $\mathrm{Bim}(H_p \to H_q)$ and $\mathcal{CF}(H_p \to H_q)$ are finite dimensional, so it remains to show they have the same dimension.

By Frobenius reciprocity, we have a natural isomorphism

$$\mathrm{Hom}_{M-M}(H_p \to H_q) \cong \mathrm{Hom}_{M-M}(1 \to \overline{H_p} \otimes H_q),$$

and the latter space is naturally identified with the $M - M$ central vectors in $\overline{H_p} \otimes H_q$. Note that

$$\overline{H_p} \otimes_M H_q \cong H_{\overline{p}} \otimes_M H_q \cong H_{\overline{p} \otimes q} \cong p \wedge \mathcal{H}_{\overline{\alpha},\overline{\beta}} \wedge q.$$

From Corollary 3.3.14, the set of central vectors in $p \wedge \mathcal{H}_{\overline{\alpha},\overline{\beta}} \wedge q$ is $q(P_{\alpha\to\beta})p$

$$p \wedge_{\overline{\alpha}} \mathcal{H}_{\overline{\alpha},\overline{\beta}} \wedge_{\overline{\beta}} q \ni \boxed{p} - \boxed{x} - \boxed{q} \longleftrightarrow \begin{array}{c} \boxed{p} \\ \boxed{x} \\ \boxed{q} \end{array} \in q(P_{\alpha\to\beta})p,$$

proving $\dim(\mathcal{CF}(H_p \to H_q)) = \dim(\mathrm{Bim}(H_p \to H_q))$. □

Lemma 3.3.30. *If $x \in \mathrm{Bim}(H_p \to H_q) = qP_{\alpha\to\beta}p$ and $\Phi_x \in \mathcal{CF}(H_p \to H_q)$ is as in Equation (3.5), then $\Phi_x^* = \Phi_{x^*}$.*

Proof. By Equation (3.3), if $\xi \in D(H_p)$ and $\eta \in D(H_q)$, then

$$\langle \Phi_x \xi, \eta \rangle_{H_q} = \begin{array}{c} \boxed{\sum CTL} \quad \boxed{\sum CTL} \\ \boxed{\overline{\eta^*}} \; \boxed{\xi} \\ \boxed{\overline{q}} \; \boxed{p} \\ \boxed{x} \end{array} = \begin{array}{c} \boxed{\sum CTL} \quad \boxed{\sum CTL} \\ \boxed{\overline{\eta^*}} \; \boxed{\xi} \\ \boxed{\overline{q}} \; \boxed{p} \\ \boxed{\overline{x}} \end{array} = \langle \xi, \Phi_{x^*} \eta \rangle_{H_p}$$

where $\overline{x}$ is the 180 degree rotation of x. □

For the next lemma, recall that the conjugate Hilbert space of K, is the set of formal symbols $\{\overline{\xi} \,|\, \xi \in H_p\}$ such that $\lambda\overline{\xi} + \overline{\eta} = \overline{\overline{\lambda}\xi + \eta}$ for all $\lambda \in \mathbb{C}$ and $\eta, \xi \in H_p$, together with left and right M-actions given by $x\overline{\xi}y = \overline{y^*\xi x^*}$ for $x, y \in M$.

Lemma 3.3.31. *For $p \in P_{\alpha\to\alpha}$, define $\psi_p \colon H_{\overline{p}} \to \overline{H_p}$ (where $\overline{p}$ is the dual projection of p in $\mathcal{P}$) by the unique extension of the map*

$$\begin{array}{c} \boxed{x} \\ \boxed{\overline{p}} \end{array} \longmapsto \begin{array}{c} \boxed{x^*} \\ \boxed{p} \end{array}$$

for $\xi \in D(H_{\overline{p}})$ (the blue strand on the left is labelled $\overline{\alpha}$). Then ψ_p is an $M-M$ bilinear isomorphism such that the following diagrams commute:

$$\begin{array}{ccccc} H_{\overline{p}} \otimes_M H_{\overline{q}} & \xrightarrow{\psi_p \otimes_M \psi_q} & \overline{H_p} \otimes_M \overline{H_q} & \xrightarrow{\cong} & \overline{H_q \otimes_M H_p} \\ {\scriptstyle \phi_{\overline{p},\overline{q}}} \downarrow & & & & \downarrow {\scriptstyle \phi_{q,p}} \\ H_{\overline{p}\otimes\overline{q}} & \xrightarrow{=} & H_{\overline{q\otimes p}} & \xrightarrow{\psi_{p\otimes q}} & \overline{H_{q\otimes p}} \end{array}$$

and

$$
\begin{array}{ccc}
H_{\overline{\overline{p}}} & \xrightarrow{\psi_p} & \overline{H_{\overline{p}}} \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \overline{\psi_p}} \\
H_p & \xrightarrow{\cong} & \overline{\overline{H_p}}.
\end{array}
$$

where we just write $\cong$ for the obvious isomorphism.

Proof. That ψ_p is a well-defined $M - M$ bilinear isomorphism is trivial. Commutativity of the diagrams follows by looking at M-bounded vectors. $\qquad \square$

Lemma 3.3.32. *For $p \in P_{\alpha \to \alpha}$, $\Phi_{\text{ev}^{\text{Bim}}_{H_p}} = \text{ev}^{\mathcal{CF}}_{H_p}$ and $\Phi_{\text{coev}^{\text{Bim}}_{H_p}} = \text{coev}^{\mathcal{CF}}_{H_p}$. Hence Φ preserves the rigid structure, and $\overline{\Phi_x} = \Phi_{\overline{x}} \in \mathcal{CF}(H_{\overline{q}} \to H_{\overline{p}})$ for all $x \in \text{Bim}(H_p \to H_q)$.*

Proof. In diagrams, $\text{ev}^{\mathcal{CF}}_{H_p}$ is given for M-bounded vectors by

Hence using the isomorphism

$$
\overline{H_p} \otimes_M H_p \xrightarrow{\psi_p \otimes_M \text{id}_{H_p}} H_{\overline{p}} \otimes_M H_p \xrightarrow{\phi_{\overline{p},p}} H_{\overline{p} \otimes p}
$$

from Lemmas 3.3.28 and 3.3.31, we have that $\text{ev}^{\mathcal{CF}}_{H_p} = \Phi_{\text{ev}^{\text{Bim}}_{H_p}}$. Since Φ is $*$-preserving by Lemma 3.3.30, we must have that $\Phi_{\text{coev}^{\text{Bim}}_{H_p}} = \text{coev}^{\mathcal{CF}}_{H_p}$. Thus

It is now easy to check that $\overline{\Phi_x} = \Phi_{\overline{x}} \in \mathcal{CF}(H_{\overline{q}} \to H_{\overline{p}})$ for all $x \in \text{Bim}(H_p \to H_q)$. $\qquad \square$

Proof of Theorem 3.3.26. We must show Φ is additive, monoidal, dual-preserving, $*$-preserving, fully faithful, and essentially surjective.

Additivity on objects and essentially surjective come for free, since the objects are the same. Additivity on morphisms and fully faithful follows from Lemma 3.3.29. Monoidal follows from Lemma 3.3.28, $*$-preserving follows from Lemmas 3.3.29 and 3.3.30, and dual-preserving follows from Lemmas 3.3.31 and 3.3.32. Note that the results of the previous five lemmas extend in the obvious ways to direct sums by looking at matrices over the maps $H_p \to H_q$. $\qquad \square$

3.4   The isomorphism class of M

We now determine the isomorphism class of the II_1-factor M from Section 3.3.

Assumption 3.4.1. Note that if our rigid C^*-tensor category $\mathcal{C}$ is countably generated, then the isomorphism classes of objects of $\mathcal{C}$ form a countable set. Let $\mathcal{S}$ consist of a set of representatives for the isomorphism classes of objects in $\mathcal{C}$, and let $\mathcal{L} = \left\{ X \oplus \overline{X} \,\middle|\, X \in \mathcal{S} \right\}$ as in Assumption 3.2.11. Again, the objects in $\mathcal{L}$ are not simple, and all have dimension greater than 1. In particular, $\mathcal{F}_{\mathcal{C}}(\mathcal{L})$ is not locally finite, and each vertex has self loops, since objects of the form $2 \oplus X \oplus \overline{X}$ are in $\mathcal{L}$.

With this assumption, we prove in Section 3.4 that $M \cong L(\mathbb{F}_\infty)$. In the case that $\mathcal{C}$ has finitely many isomorphism classes of simple objects, i.e., $\mathcal{C}$ is a unitary fusion category, we can find a single object X that generates $\mathcal{C}$. We explain in Remark 3.4.27 that if we choose $\mathcal{L} = \{X \oplus \overline{X}\}$, then $M \cong L(\mathbb{F}_t)$ with

$$t = 1 + \dim(\mathcal{C})(2\dim(X) - 1),$$

similar to the result in [GJS11].

To begin, we describe in the next two sections a semifinite algebra associated to $\mathcal{P}$. Many of the ideas in these two sections mirror [GJS11], except that the semifinite algebra makes all box-spaces orthogonal. One also must be more careful since the fusion graph $\mathcal{F}_{\mathcal{C}}(\mathcal{L})$ is not bipartite.

A semifinte algebra associated to $\mathcal{P}$

Set $\mathcal{G}_\infty = \bigoplus_{\alpha,\beta \in \Lambda} \mathrm{Gr}_{\alpha,\beta}(\mathcal{P})$. We endow $\mathcal{G}_\infty$ with the multiplication

$$\kappa \boxed{x}_\theta^{\,|\gamma} \wedge \omega \boxed{x}_{\,|\gamma'}^{\,\chi} = \delta_{\omega,\theta} \; \kappa \boxed{x}_\theta^{\,|\gamma} \boxed{y^*}_{\,|\gamma'}^{\,\chi} \,,$$

where $\kappa, \theta, \omega, \chi \in \Lambda$. There is a (semifinite) trace, Tr on $\mathcal{G}_\infty$ given by

$$\mathrm{Tr}(x) = \boxed{\Sigma CTL} \\ \boxed{x}$$

for $x \in \mathrm{Gr}_\alpha$, and $\mathrm{Tr}(x) = 0$ for $x \in \mathrm{Gr}_{\alpha,\beta}$ with $\alpha \neq \beta$.

Definition 3.4.2. As in Section 3.3, one argues (by once again orthogonalizing) that Tr is positive definite on $\mathcal{G}_\infty$ and multiplication is bounded on $L^2(\mathcal{G}_\infty)$. Therefore, we form the (semifinite) von Neumann algebra $\mathcal{M}_\infty = \mathcal{G}_\infty''$ acting on $L^2(\mathcal{G}_\infty, \mathrm{Tr})$.

We also use the von Neumann subalgebra $\mathcal{A}_\infty \subset \mathcal{M}_\infty$ which is generated by all boxes in $\mathcal{G}_\infty$ with <u>no</u> strings on top.

As in Section 3.3, let

$$p_\alpha = \boxed{}.$$

It is easy to see that $p_\alpha \wedge \mathcal{M}_\infty \wedge p_\alpha = M_\alpha$ so that all factors in the various Jones towers in Section 3.3 appear as cut-downs/corners of $\mathcal{M}_\infty$. We now record some lemmas about the structure of $\mathcal{M}_\infty$ and $\mathcal{A}_\infty$.

Lemma 3.4.3. $\mathcal{M}_\infty$ *is a II_∞ factor.*

Proof. Note that $p_\emptyset \wedge \mathcal{M}_\infty \wedge p_\emptyset = M$ is a II_1 factor and the trace of $1 = 1_{\mathcal{M}_\infty} = \sum_{\alpha \in \Lambda} p_\alpha$ (with convergence of the orthogonal sum in the strong operator topology) is infinite, so we only need to show that $\mathcal{M}_\infty$ is a factor. To do so, we show that the central support of $p_\emptyset$ in $\mathcal{M}_\infty$ is 1, which is enough since $p_\emptyset \wedge \mathcal{M}_\infty \wedge p_\emptyset$ is a factor. We let

$$w_\alpha = \boxed{\;\diagdown\;} ,$$

and we let v_α be the polar part of w_α. As was discussed in the proof of Lemma 3.3.20, v_α induces an equivalence of projections between $p_\emptyset$ and $e \le p_\alpha$. Let z be the central support of $p_\emptyset$ (and e). Since $p_\alpha \wedge \mathcal{M}_\infty \wedge p_\alpha$ is a factor, the central support of e must lie above p_α, so $z \ge p_\alpha$. Since this holds for all $\alpha \in \Lambda$, we have $z \ge 1$. Hence $z = 1$, and we are finished. $\qquad\square$

Corollary 3.4.4. *The factor M_α is a δ_α-amplification of M.*

Corollary 3.4.5. *The algebras $\mathcal{M}_{\alpha,\beta}$ are II_1 factors.*

Proof. $\mathcal{M}_{\alpha,\beta}$ is the compression of $\mathcal{M}_\infty$ by the finite projection $p_\alpha + p_\beta$. $\qquad\square$

Lemma 3.4.6. *We have a direct sum decomposition*

$$\mathcal{A}_\infty = \bigoplus_{v \in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))} \mathcal{A}_v$$

where the sum is over all vertices v in the fusion graph $\mathcal{F}_\mathcal{C}(\mathcal{L})$, and each $\mathcal{A}_v$ is a type I_∞ factor. If p is a minimal projection in $P_{\alpha \to \alpha}$ whose equivalence class represents the vertex v, then $p \le 1_{\mathcal{A}_v}$.

Proof. For each vertex v, choose a minimal projection $p_v \in P_{\alpha_v \to \alpha_v}$ whose equivalence class corresponds to the vertex v. We first see that

$$p_v \wedge \mathcal{A}_\infty \wedge p_v = p_{\alpha_v} P_{\alpha_v \to \alpha_v} p_{\alpha_v} \cong \mathbb{C}$$

so $\mathcal{A}_\infty$ has minimal projections. Letting $p \in P_{\alpha \to \alpha}$ and $q \in P_{\beta \to \beta}$, then the observation that $p \wedge \mathcal{A}_\infty \wedge q = q P_{\alpha \to \beta} p$ implies p is equivalent to q in the planar algebra sense if and only if p is equivalent to q in $\mathcal{A}_\infty$. Let 1_v be the central support of p_v. Then by construction, $\mathcal{A}_v = 1_v \wedge \mathcal{A}_\infty \wedge 1_v$ is a type I factor which must be type I_∞ since there are infinitely many mutually orthogonal projections equivalent to p_v in $\mathcal{P}$. It is easy to see that by construction, if $v \ne w$ then $1_v \wedge 1_w = 0$, which implies that

$$\mathcal{A}_\infty = \bigoplus_{v \in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))} \mathcal{A}_v \oplus \mathcal{B}$$

for some von Neumann algebra $\mathcal{B}$.

We claim that $\mathcal{B} = \{0\}$. Indeed we know that p_α can be written as an orthogonal sum of projections equivalent to some of the p_w's. Since $1 = \sum_{\alpha \in \Lambda} p_\alpha$, this implies $\sum_v 1_v = 1$, and thus $\mathcal{B} = \{0\}$. $\qquad\square$

Remark 3.4.7. For the rest of this section, all multiplication will be the $\wedge$ multiplication in the GJS picture. Hence for the rest of this section, we just write xy for $x \wedge y$ for convenience.

We now show that we can obtain $\mathcal{M}_\infty$ from a base "building block" $\mathcal{A}_\infty$ and various free "corner elements." Fixing a color c, which we again represent by the color green, we define

$$X_c = \sum_{\substack{\alpha \in \Lambda \\ |\alpha| \in 2\mathbb{N}}} \boxed{} + \boxed{}.$$

Remark 3.4.8. We note that this sum defines a bounded operator. Indeed, the individual terms in the sum are supported under the mutually orthogonal family of projections $\{p_{c\alpha} + p_\alpha : |\alpha| \in 2\mathbb{N}\}$ and each term in the sum has operator norm

$$\left\| \boxed{} + \boxed{} \right\|_\infty^{1/2} = \left\| \boxed{} + \boxed{} \right\|_\infty^{1/2}.$$

We also note the simple fact that for $|\alpha|$ even, $p_{c\alpha} X_c p_\alpha$ is the corner diagram

$$\boxed{},$$

so each term in the sum defining X_c appears in the von Neumann algebra $W^*(\mathcal{A}_\infty, X_c)$. We only sum over α with $|\alpha|$ even as it makes computations involving freeness in Section 3.4 much easier.

The X_c elements give us a very nice way of obtaining $\mathcal{M}_\infty$ from $\mathcal{A}_\infty$.

Lemma 3.4.9. $\mathcal{M}_\infty \cong W^*(\mathcal{A}_\infty, \{X_c : c \in \mathcal{L}\})$.

Proof. Every element in $\mathcal{G}_\infty$ is a linear combination of elements of the following form:

$$\boxed{x}.$$
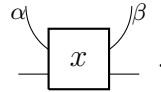
The above diagram is $x \in \mathcal{A}_\infty$ multiplied on the left and right by diagrams of the form

$$\boxed{}$$

and their adjoints. This diagram is a product of diagrams of the form

$$\boxed{},$$

so we only need to check that the above diagram is in $W^*(\mathcal{A}_\infty, \{X_c : c \in \mathcal{L}\})$ when $|\alpha|$ is odd. This is easy since it can be written as the product

$$\boxed{} \cdot \boxed{}.$$

$\square$

There is a Tr-preserving conditional expectation $E : \mathcal{M}_\infty \to \mathcal{A}_\infty$ given by

$$E(x) = \boxed{\begin{array}{c}\Sigma CTL\\ | \\ \boxed{x}\end{array}} \quad,$$

and E induces normal completely positive maps $(\eta_{c,c} = \eta_c)_{c \in \mathcal{L}}$ on $\mathcal{A}_\infty$ satisfying

$$\eta_c(y) = E(X_c y X_c) = \boxed{X_c} - \boxed{y} - \boxed{X_c} \quad.$$

For $b \neq a$, we have trivial "off-diagonal" maps $\eta_{a,b}$ on $\mathcal{A}_\infty$ satisfying $\eta_{a,b}(y) = E(X_a y X_b) = 0$. This gives a straightforward diagrammatic procedure for evaluating $E(y_0 X_{c_1} y_1 X_{c_2} \cdots X_{c_n} y_n)$ for $y_i \in \mathcal{A}_\infty$. First, write the word $y_0 X_{c_1} y_1 X_{c_2} \cdots X_{c_n} y_n$ as

$$\boxed{y_0} - \boxed{X_{c_1}} - \boxed{y_1} - \cdots - \boxed{y_{n-1}} - \boxed{X_{c_n}} - \boxed{y_n} \quad.$$

Then sum over all planar ways to connect the strings on top. Whenever we see a term of the form

$$\boxed{X_{c_i}} - \boxed{y} - \boxed{X_{c_j}} \quad,$$

we replace it with $\eta_{c_i,c_j}(y)$. It is straightforward to check that E and the η_{c_i,c_j} satisfy the following recurrence relation:

$$E(y_0 X_{c_1} y_1 X_{c_2} \cdots y_{n-1} X_{c_n} y_n)$$
$$= \sum_{k=2}^{n} y_0 \cdot \eta_{c_1,c_k}(E(y_1 X_{c_2} \cdots X_{c_{k-1}} y_{k-1})) \cdot E(y_k X_{c_{k+1}} ... X_{c_n} y_n). \qquad (3.6)$$

Also, by definition of η_{c_i,c_j}, it follows that the map on $\mathcal{A}_\infty \otimes B(\mathcal{H})$ given by $(y_{i,j}) \mapsto (\eta_{c_i,c_j}(y_{i,j}))$ is normal and completely positive. This, combined with Recurrence (3.6) implies that the elements $(X_c)_{c \in \mathcal{L}}$ form an $\mathcal{A}_\infty$-valued semicircular family with covariance (η_{c_i,c_j}) as in [Shl99]. Since $\eta_{c_i,c_j} = 0$ for $c_i \neq c_j$, the family $(X_c)_{c \in \mathcal{L}}$ is free with amalgamation over $\mathcal{A}_\infty$ with respect to E [Shl99]. We record what we have established above in the following lemma.

Lemma 3.4.10. $\mathcal{M}_\infty = W^*(\mathcal{A}_\infty, \{X_c : c \in \mathcal{L}\})$, and the elements $(X_c)_{c \in \mathcal{L}}$ form an $\mathcal{A}_\infty$-valued semicircular family with covariance (η_{c_i,c_j}) and are free with amalgamation over $\mathcal{A}_\infty$.

M as an amalgamated free product

By compressing the algebra $\mathcal{A}_\infty$ by the projection $1_e = \sum_{|\alpha| \in 2\mathbb{N}} p_\alpha$, Lemma 3.4.6 implies that if we set $\mathcal{A}_e = W^*((P_{\beta \to \alpha})_{\alpha,\beta \in 2\mathbb{N}})$ then

$$\mathcal{A}_e = \bigoplus_{v \in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))} \mathcal{B}_v.$$

$\mathcal{B}_v$ is a type I_∞ factor which is a cut-down of $\mathcal{A}_v$ by 1_e. Note that every vertex in $v \in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))$ appears in the direct sum because every vertex possesses at least one self-loop.

Similarly, if one sets $1_o = \sum_{|\alpha| \in (1+2\mathbb{N})} p_\alpha$ and $\mathcal{A}_o = 1_o \mathcal{A}_\infty 1_o$, then we have

$$\mathcal{A}_o = \bigoplus_{v \in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))} \mathcal{C}_v$$

where $\mathcal{C}_v$ is a type I_∞ factor which is a cut-down of $\mathcal{A}_v$ by 1_o.

For each vertex $v \in \mathcal{F}_\mathcal{C}(\mathcal{L})$, we choose a minimal projection $p_v \in \mathcal{A}_e$ whose equivalence class is represented by v with $p_\emptyset$ the empty diagram. If $x \in \mathcal{A}_e$, it follows that

$$\eta_c(x) = \boxed{x}$$

so that $\eta_c(p_v)$ is a finite projection in $\mathcal{A}_o$, and each η_c acts as a (non-unital) W^* algebra homomorphism from $\mathcal{A}_e$ into $\mathcal{A}_o$. Set $Q = \sum_v p_v$. Then there is a family of partial isometries $(V_i)_{i \in I}$ satisfying $V_i^* V_i = Q$ and $\sum_{i \in I} V_i V_i^* = 1_{\mathcal{A}_e}$. Note that $\sum_{c \in \mathcal{L}} \eta_c(Q)$ defines a projection in $\mathcal{A}_o$ since $\eta_a(Q) \perp \eta_b(Q)$ for $a \neq b$. We examine the compression of $\mathcal{M}_\infty$ by $T = Q + \sum_{c \in \mathcal{L}} \eta_c(Q)$.

Lemma 3.4.11. $T\mathcal{M}_\infty T = W^*(T\mathcal{A}_\infty T, (TX_c T)_{x \in \mathcal{L}})$

Proof. By the choices of the partial isometries V_i, we know that if we let $W_i = \sum_{c \in \mathcal{L}} \eta_c(V_i)$ then $\sum_{i \in I}(V_i + W_i)(V_i + W_i)^* = 1$, so

$$T\mathcal{M}_\infty T = W^* \left(T\mathcal{A}_\infty T, \{(V_i + W_i)^* X_c (V_j + W_j) : i, j \in I \; c \in \mathcal{L}\} \right)$$

The relation $y \cdot X_c = X_c \eta_c(y)$ for any $y \in \mathcal{A}_e$ is a straightforward diagrammatic check. Applying this relation to each $(V_i + W_i)^* X_c (V_j + W_j)$ gives $TX_c T$. $\qquad\square$

Since $\sum_{i \in I} W_i W_i^* = 1_{\mathcal{A}_o}$, it follows that if $v \in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))$ then there is a minimal projection $q_v \in \mathcal{A}_o$ whose equivalence class represents v and sits under $1_{\mathcal{A}_o} \cdot T$. Note that this means that for all v, q_v and p_v are orthogonal minimal projections that sit under 1_v. We will next examine the cut-down of $T\mathcal{M}_\infty T$ by $F = \sum_v p_v + q_v$.

Suppose the vertices v and w are distinct and connected in $\mathcal{F}_\mathcal{C}(\mathcal{L})$ and let $C(e)$ be the color of an edge connecting v and w. Let $e_1, ..., e_k$ be the (necessarily finite) collection of edges connecting v and w with $C(e_i) = c$. We let $u_{e_i}^v$ be a collection of partial isometries such that

$$(u_{e_i}^v)^* u_{e_j}^v = \delta_{ij} q_v \text{ and } \sum_{i=1}^k u_{e_i}^v (u_{e_i}^v)^* = \eta_c(p_w) \cdot 1_v$$

Let $r_v \in F\mathcal{A}_\infty F$ be a partial isometry satisfying $r_v r_v^* = p_v$ and $r_v^* r_v = q_v$ and $r_w \in F\mathcal{A}_\infty F$ be a partial isometry satisfying $r_w r_w^* = p_w$ and $r_w^* r_w = q_w$. The elements defined can be described diagrammatically as follows:

$$p_v = \;{}^{\alpha_v}\boxed{p_v}^{\alpha_v} \; , \; p_w = \;{}^{\alpha_w}\boxed{p_w}^{\alpha_w} \; , \; q_v = \;{}^{\beta_v}\boxed{q_v}^{\beta_v} \; , \; q_w = \;{}^{\beta_w}\boxed{q_w}^{\beta_w} \; ,$$

$$u_{e_i}^v = \;{}^{\alpha_w}\boxed{u_{e_i}^v}^{\beta_v} \; , \; r_v = \;{}^{\alpha_v}\boxed{r_v}^{\beta_v} \; , \; r_w = \;{}^{\alpha_w}\boxed{r_w}^{\beta_w} \; .$$

Set x_i to be the following diagram:

$$x_i = \quad \boxed{\alpha_w \;\; u^v_{e_i} \;\; \beta_v}$$

We have the following lemma about the left and right supports of x_i:

Lemma 3.4.12. *The left support of x_i is p_w and the right support of x_i lies under the projection $1_w \cdot i_c(q_v)$ where we have for $z \in P_{\overline{\gamma} \to \overline{\chi}}$*

$$i_c(z) = \quad \boxed{\gamma \;\; z \;\; \chi} \quad .$$

Furthermore if $i \neq j$ then the right supports of x_i and x_j are orthogonal.

Proof. Note that $x_i x_i^*$ is the diagram

$$\boxed{\alpha_w \;\; u^v_{e_i} \;\; \beta_v \;\; (u^v_{e_i})^* \;\; \alpha_w} \quad .$$

For any $\gamma \in \Lambda$, let $\Phi : P^{\mathrm{op}}_{\overline{\gamma} c \to \overline{\gamma} c} \to P^{\mathrm{op}}_{\overline{\gamma} \to \overline{\gamma}}$ be the trace preserving conditional expectation where $P^{\mathrm{op}}_{\overline{\gamma} \to \overline{\gamma}}$ includes unitally into $P^{\mathrm{op}}_{\overline{\gamma} c \to \overline{\gamma} c}$ via i_c. The above diagram is a scalar multiple of $\Phi(u^v_{e_i} \cdot (u^v_{e_i})^*)$. As $u^v_{e_i} \cdot (u^v_{e_i})^*$ lies under $\eta_c(p_w) = i_c(p_w)$ it follows that this diagram is a scalar multiple of p_w, proving the claim about the left support. As left and right supports are equivalent projections, it follows that the right support of x_i lies under 1_w. We note that $\Phi(x_i^* x_i) = k(u^v_{e_i})^* u^v_{e_i} = k q_v$ for some scalar k, implying that the right support of x_i must lie under $i_c(q_v)$. Finally, if $i \neq j$ then $x_i x_j^*$ is a constant multiple of $\Phi(u^v_{e_i} \cdot (u^v_{e_j})^*)$. We know this must be a scalar multiple of p_w, but that scalar must be zero as $u^v_{e_i} \cdot (u^v_{e_j})^*$ has trace zero. This proves the orthogonality of the right supports of the x_i. $\square$

Let f_i be the right support of x_i. As $i_c(p_v) \cdot 1_w$ can be written as a sum of k orthogonal projections equivalent to p_w, we conclude that $\sum_{i=1}^k f_i = i_c(p_v) \cdot 1_w$. We also recognize that $i_c(r_v)$ is a partial isometry with left support $i_c(q_v)$ and right support $i_c(p_v)$. We consider the elements $y_i = r_w^* \cdot x_i \cdot i_c(r_v^*)$ which diagrammatically look like

$$y_i = \quad \boxed{\beta_w \;\; r_w^* \;\; \alpha_w \;\; u^v_e \;\; \beta_v \;\; r_v^* \;\; \alpha_v}$$

note that the left support of y_i is q_w. From Lemma 3.4.12, the right supports of the y_i are orthogonal and sum up to $i_c(p_v) \cdot 1_w$. As q_w is minimal in $\mathcal{A}_\infty$, it follows that y_i is a scalar multiple of a partial isometry. We define the partial isometry u^w_e by the equation $(u^w_{e_i})^* = l \cdot y_i$ with l an appropriate constant. It follows that $(u^w_{e_i})^* u^w_{e_j} = \delta_{ij} q_w$ and $\sum_{i=1}^k u^w_{e_i}(u^w_{e_i})^* = \eta_c(p_v) \cdot 1_w$.

One can manipulate the diagram above to show that the relation

$$(u^v_e)^* = (\bar{l})^{-1} \cdot \quad \boxed{\beta_v \;\; r_v^* \;\; \alpha_v \;\; u^w_e \;\; \beta_w \;\; r_w^* \;\; \alpha_w}$$

holds. This discussion thus proves the following useful lemma:

Lemma 3.4.13. *Let $e_1, \ldots, e_k$ be all of the edges of color c in the fusion graph connecting distinct vertices v and w. Then one can find partial isometries $\{u_{e_i}^v : 1 \leq i \leq k\}$ and $\{u_{e_i}^w : 1 \leq i \leq k\}$ in $\mathcal{A}_o$ satisfying*

$$(u_{e_i}^v)^* u_{e_j}^v = \delta_{ij} q_v, \quad \sum_{i=1}^k u_{e_i}^v (u_{e_i}^v)^* = \eta_c(p_w) \cdot 1_v$$

$$(u_{e_i}^w)^* u_{e_j}^w = \delta_{ij} q_w, \quad \sum_{i=1}^k u_{e_i}^w (u_{e_i}^w)^* = \eta_c(p_v) \cdot 1_w$$

and the relations

$$(u_e^v)^* = (\bar{l})^{-1} \cdot \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}} \boxed{\begin{smallmatrix} u_e^w \end{smallmatrix}} \boxed{\begin{smallmatrix} \beta_w & r_w^* & \alpha_w \end{smallmatrix}} \quad and \quad (u_e^w)^* = l \cdot \boxed{\begin{smallmatrix} \beta_w & r_w^* & \alpha_w \end{smallmatrix}} \boxed{\begin{smallmatrix} u_e^v \end{smallmatrix}} \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}}$$

for some nonzero constant l.

We now turn our attention over to the edges e which are loops. Suppose v is a vertex and $e_i, \ldots, e_k$ represent all of the loops of color c connecting v to itself. As above, we can find partial isometries $u_{e_i}^v$ with right support q_v and orthogonal left supports under $1_v \cdot i_c(p_v)$, however for reasons that will become apparent later we desire a stronger property about these partial isometries.

Lemma 3.4.14. *There exists a set of partial isometries $\{u_{e_i}^v : 1 \leq i \leq k\}$ satisfying*

$$(u_{e_i}^v)^* u_{e_j}^v = \delta_{ij} q_v, \quad \sum_{i=1}^k u_{e_i}^v (u_{e_i}^v)^* = \eta_c(p_v) \cdot 1_v,$$

and the following relation:

$$(u_e^v)^* = k \cdot \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}} \boxed{\begin{smallmatrix} u_e^v \end{smallmatrix}} \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}}$$

for k a unimodular constant.

Proof. Let $u_1 \in \mathcal{A}_\infty$ be a partial isometry with right support q_v and left support under $1_v \cdot i_c(p_v)$. Consider the operator y_1 satisfying

$$y_1^* = \boxed{\begin{smallmatrix} \beta_v & u_1^* & \alpha_v \end{smallmatrix}} + \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}} \boxed{\begin{smallmatrix} u_1 \end{smallmatrix}} \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}} \cdot$$

If this is zero, then by setting $k = -1$, we have produced a partial isometry satisfying the appropriate diagrammatic relation. If not, then y_1^* has left support q_v and right support under $1_v \cdot i_c(p_v)$ so it is a scalar multiple of a partial isometry. The operator y_1^* is also fixed under the map

$$x \mapsto \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}} \boxed{\begin{smallmatrix} x^* \end{smallmatrix}} \boxed{\begin{smallmatrix} \beta_v & r_v^* & \alpha_v \end{smallmatrix}}$$

so a scalar multiple of y_1 satisfies the appropriate diagrammatic relation and is a partial isometry. In either case, we have produced a partial isometry $u_{e_1}^v$ satisfying the diagrammatic relation. To produce another partial isometry $u_{e_2}^v$ with right support q_v, left support orthogonal to that of $u_{e_1}^v$,

and satisfying the diagrammatic relation, we pick a partial isometry u_2 with right support q_v and left support orthogonal to that of $u_{e_1}^v$. By Lemma 3.4.12 and the discussion afterwards, the right supports of

$$\boxed{\beta_v}\ \boxed{r_v^*}\ \boxed{\alpha_v}\ \boxed{u_2}\ \boxed{\beta_v}\ \boxed{r_v^*}\ \boxed{\alpha_v} \quad \text{and}\ k\cdot\ \boxed{\beta_v}\ \boxed{r_v^*}\ \boxed{\alpha_v}\ \boxed{u_{e_1}^v}\ \boxed{\beta_v}\ \boxed{r_v^*}\ \boxed{\alpha_v} = (u_{e_1}^v)^*$$

are orthogonal so it follows that by considering the element

$$y_2^* = \ \boxed{\beta_v}\ \boxed{u_2^*}\ \boxed{\alpha_v}\ +\ \boxed{\beta_v}\ \boxed{r_v^*}\ \boxed{\alpha_v}\ \boxed{u_2}\ \boxed{\beta_v}\ \boxed{r_v^*}\ \boxed{\alpha_v}\ .$$

and arguing as in the beginning of the proof, we produce the desired element $u_{e_2}^v$. Iterating this procedure produces the set $\{u_{e_i}^v : 1 \le i \le k\}$. $\qquad\square$

We therefore assume in the rest of the section that our partial isometries satisfy the relations in either Lemma 3.4.13 or Lemma 3.4.14. The identity $\sum_v \sum_{e\sim v} u_e^v(u_e^v)^* = \sum_{c\in\mathcal{L}} \eta_c(Q)$ follows by the choices of the partial isometries u_e^v. We therefore have the following lemma:

Lemma 3.4.15. *For each edge e of the fusion graph $\mathcal{F}_\mathcal{C}(\mathcal{L}))$, we define operators Y_e as follows: If $C(e) = c$ and e connects distinct vertices v and w then $Y_e = p_v X_c u_e^w + (u_e^w)^* X_v p_v + p_w X_c u_e^v + (u_e^v)^* X_c p_w$ and if e connects the vertex v to itself then $Y_e = p_v X_c u_e^v + (u_e^v)^* X_c p_v$. Then we have $F\mathcal{M}_\infty F = W^*(F\mathcal{A}_\infty F, (Y_e)_{e\in E})$*

Proof. First note that by compressing by the appropriate elements, of $F\mathcal{A}_\infty F$, each term in the sums defining Y_e is in $W^*(F\mathcal{A}_\infty F, (Y_e)_{e\in E(\mathcal{F}_\mathcal{C}(\mathcal{L}))})$. The identity

$$\sum_{v\in V(\mathcal{F}_\mathcal{C}(\mathcal{L}))} \sum_{\substack{e:s(e)=v \\ C(e)=c}} = p_v X_c u_e^{t(e)}(u_e^{t(e)})^* = TX_c T$$

implies that words in $TX_c T$ (compressed on either end by F) can be approximated by words in $p_v X_c u_e^w$ and their adjoints so by Lemma 3.4.11 we are done. $\qquad\square$

Lemma 3.4.16. *If v and w are distinct vertices connected by an edge e with $C(e) = c$ then $r_v(u_e^v)^* X_c p_w r_w$ and $p_v X_c(u_e^w)$ are nonzero scalar multiples of each other.*
    If f is a loop at v then $r_v(u_f^v)^* X_c p_v r_v$ and $p_v X_c(u_f^v)$ are nonzero scalar multiples of each other.

Proof. First observe that

$$p_v X_c u_e^w = X_c \eta_c(p_v) u_e^w = X_c u_e^w.$$

and $r_v(u_e^v)^* X_c p_w r_w = r_v(u_v^e)^* X_c r_w$. The element $r_v(u_v^e)^* X_c r_w$ is represented diagrammatically as

$$\boxed{\alpha_v}\ \boxed{r_v}\ \boxed{\beta_v}\ \boxed{(u_e^v)^*}\ \boxed{\alpha_w}\ \boxed{r_w}\ \boxed{\beta_w}$$

which can be rewritten as

$$\boxed{\alpha_v}\ \boxed{r_v}\ \boxed{\beta_v}\ \boxed{(u_e^v)^*}\ \boxed{\alpha_w}\ \boxed{r_w}\ \boxed{\beta_w} = k\cdot X_c\cdot u_e^w$$

for an appropriate constant $k \neq 0$, proving the first statement. The proof of the second statement uses the exact same diagrammatic argument. $\qquad\square$

We now examine the algebras

$$\mathcal{M}_e = W^*(F\mathcal{A}_\infty F, Y_e) = W^*(F\mathcal{A}_\infty F, p_v X_c u_e^w + (u_e^w)^* X_c p_v)$$

where the last equality follows from Lemma 3.4.16. First assume v and w are distinct and $\mathrm{Tr}(p_v) \leq \mathrm{Tr}(p_w) = \mathrm{Tr}(q_w)$. One can show, using exactly the same techniques as in [GJS10] and [GJS11] (see the appendix) that $(u_e^w)^* X_c p_v X_c u_e^w$ is a free Poisson element with an atom of size $\frac{\mathrm{Tr}(p_w)-\mathrm{Tr}(p_v)}{\mathrm{Tr}(p_w)}$ at 0 in the compressed algebra $p_v \mathcal{M}_\infty p_v$. Also, $p_v X_c u_e^w (u_e^w)^* X_c p_v$ is a free poisson element with no atoms in the compressed algebra $q_v \mathcal{M}_\infty q_v$. Using the polar part of $p_v X_c u_e^w$ as well as the partial isometries r_v and r_w and Lemma 3.4.16, we see that

$$(1_v + 1_w) F\mathcal{M}_e F(1_v + 1_w) = (L(\mathbb{Z}) \otimes M_4(\mathbb{C})) \bigoplus_{\mathrm{Tr}(p_v)-\mathrm{Tr}(p_w)}^{p'_v, q'_v} M_2(\mathbb{C})$$

where the notation means that the copy of $M_2(\mathbb{C})$ has two orthogonal minimal projections p'_v and q'_v such that $p'_v \leq p_v$ and $q'_v \leq q_v$ and $\mathrm{Tr}(p'_v) = \mathrm{Tr}(p_v) - \mathrm{Tr}(p_w)$.

If e is a loop at v then using the partial isometry r_v and Lemma 3.4.16, we see that

$$1_v \cdot F\mathcal{M}_e F \cdot 1_v = L(\mathbb{Z}) \otimes M_2(\mathbb{C}).$$

For simplicity, we set $X_e = p_v X_c u_e^w + (u_e^w)^* X_c p_v$ so that $\mathcal{M}_e = W^*(F\mathcal{M}_\infty F, X_e)$.

Lemma 3.4.17. *The algebras $(\mathcal{M}_e)_{e \in E(\mathcal{F}_\mathcal{C}(\mathcal{L}))}$ are free with amalgamation over $F\mathcal{A}_\infty F$.*

Proof. From [Shl99] the elements X_e are $F\mathcal{A}_\infty F$ semicircular, so to show freeness we need only to show that if $e \neq e'$, $E(X_e y X_{e'}) = 0$ for all $y \in F\mathcal{A}_\infty F$. This can only be nonzero if e and e' have the came color, so we assume $C(e) = c = C(e')$. Also, this expectation can be nonzero only if e and e' share a common vertex, v.

We first assume that e and e' are not loops. Assume e connects v and w and e' connects v and w'. We see that $E(X_e p_v X_{e'}) = (u_e^w)^* i_c(p_v) u_{e'}^{w'} = (u_e^w)^* u_{e'}^{w'}$ which is zero since u_e^w and $u_{e'}^{w'}$ have orthogonal left supports if $w = w'$ and is clearly zero if $w \neq w'$. For $E(X_e q_w X_{e'})$ to be nonzero, we need $w = w'$. Assuming this, we get $p_v \Phi(u_e^w \cdot q_w \cdot (u_{e'}^{w'})^*) p_v$. As was discussed in the proof of Lemma 3.4.12,
$$\Phi(u_e^w \cdot q_w \cdot (u_{e'}^{w'})^*) = \Phi(u_e^w (u_{e'}^{w'})^*) = 0$$
since u_e^w and $u_{e'}^{w'}$ have orthogonal left supports. It is also straightforward to check that the only elements y in $F\mathcal{A}_\infty F$ where $X_e y X_{e'} \neq 0$ are of the form $y = a \cdot p_v + b \cdot q_w$.

We now assume that e is a loop at v and e' is an arbitrary edge connected to v. The discussion in the previous paragraph implies that $E(X_e p X_{e'}) = 0$ for any projection $p \in F\mathcal{A}_\infty F$; however, we also have to consider $E(X_e r_v^* X_{e'})$. This is the following diagram:

$$\boxed{\alpha_v}\,\boxed{p_v}\,\boxed{\alpha_v}\,\boxed{u_e^v}\,\boxed{\beta_v}\,\boxed{r_v^*}\,\boxed{\alpha_v}\,\boxed{p_v}\,\boxed{\alpha_v}\,\boxed{u_{e'}^v}\,\boxed{\beta'_v} \;=\; \boxed{\alpha_v}\,\boxed{r_v}\,\boxed{\beta_v}\,\boxed{r_v^*}\,\boxed{\alpha_v}\,\boxed{u_e^v}\,\boxed{\beta_v}\,\boxed{r_v^*}\,\boxed{\alpha_v}\,\boxed{u_{e'}^v}\,\boxed{\beta'_v} \;,$$

where $\beta'_v = \beta_v$ if e' is also a loop at v. By our choice of u_v^e, this diagram is a scalar multiple of $r_v \cdot (u_e^v)^* u_{e'}^v$ which is 0. Similarly, $E(X_e r_v^* X_{e'}) = 0$ and we are done. $\qquad\square$

Note that we have

$$FM_\infty F = \underset{F\mathcal{A}_\infty F}{*}(\mathcal{M}_e)_{e\in E(\mathcal{F}_\mathcal{C}(\mathcal{L}))}.$$

The algebra $F\mathcal{A}_\infty F$ has the decomposition

$$F\mathcal{A}_\infty F = \bigoplus_{v\in\mathcal{F}_\mathcal{C}(\mathcal{L})} \overset{p_v,q_v}{M_2(\mathbb{C})}.$$

Recall that $Q = \sum_v p_v$. Thus Q is an abelian projection with central support 1 in $F\mathcal{A}_\infty F$. Therefore (see for example [Dyk11] or [PS03])

$$Q\mathcal{M}_\infty Q = \underset{Q\mathcal{A}_\infty Q}{*}(Q\mathcal{M}_e Q)_{e\in E(\mathcal{F}_\mathcal{C}(\mathcal{L}))}.$$

The algebra $Q\mathcal{A}_\infty Q$ is simply $\ell^\infty(V(\mathcal{F}_\mathcal{C}(\mathcal{L})))$, the bounded functions on the vertices of $\mathcal{F}_\mathcal{C}(\mathcal{L})$. If e connects two distinct vertices v and w with $\mathrm{Tr}(p_v) \geq \mathrm{Tr}(p_w)$ then

$$Q\mathcal{M}_e Q = L(\mathbb{Z}) \otimes M_2(\mathbb{C}) \bigoplus \overset{p'_v}{\mathbb{C}} \bigoplus \ell^\infty(V(\mathcal{F}_\mathcal{C}(\mathcal{L})) \setminus \{v,w\})$$

and if e is a loop at v then

$$Q\mathcal{M}_e Q = L(\mathbb{Z}) \bigoplus \ell^\infty(V(\mathcal{F}_\mathcal{C}(\mathcal{L})) \setminus \{v\}).$$

In the next section, we will use the free product expression for $Q\mathcal{M}_\infty Q$ to determine the isomorphism class of $M = p_\emptyset(Q\mathcal{M}_\infty Q)p_\emptyset$.

von Neumann algebras associated to graphs

The following notation will be useful in this section:

Notation 3.4.18. Throughout this section, we will be concerned with finite von Neumann algebras $(\mathcal{N}, tr)$ which can be written in the form

$$\mathcal{N} = \overset{p_0}{\underset{\gamma_0}{\mathcal{N}_0}} \oplus \bigoplus_{j\in J} \overset{p_j}{\underset{\gamma_j}{L(\mathbb{F}_{t_j})}} \oplus \bigoplus_{k\in K} \overset{q_k}{\underset{\alpha_k}{M_{n_k}}}$$

where $\mathcal{N}_0$ is a diffuse hyperfinite von Neumann algebra, $L(\mathbb{F}_{t_j})$ is an interpolated free group factor with parameter t_j, M_{n_k} is the algebra of $n_k \times n_k$ matrices over the scalars, and the sets J and K are at most finite and countably infinite respectively. We use p_j to denote the projection in $L(\mathbb{F}_{t_j})$ corresponding to the identity of $L(\mathbb{F}_{t_j})$ and q_k to denote a minimal projection in M_{n_k}. The projections p_j and q_k have traces γ_j and α_k respectively. Let p_0 be the identity in $\mathcal{N}_0$ with trace γ_0. We write $\overset{p,q}{M_2}$ to mean M_2 with a choice of minimal orthogonal projections p and q.

We begin by letting Γ be a connected weighted graph with weighting γ. Let $\ell^\infty(\Gamma)$ be the bounded functions on the vertices of Γ and let p_v be the delta function at v. We endow $\ell^\infty(\Gamma)$ with a trace Tr satisfying $\mathrm{Tr}(p_v) = \gamma_v$. As in [Har12] we now describe how to use the edges to associate a free product von Neumann algebra $\mathcal{N}(\Gamma)$ to Γ.

Definition 3.4.19. Let e be an edge in Γ. We define algebras $\mathcal{N}_e$ as follows: If e connects two distinct edges v and w with $\gamma_v \geq \gamma_w$ then

$$\mathcal{N}_e = M_2(\mathbb{C}) \underset{2\gamma_w}{\otimes} L(\mathbb{Z}) \oplus \underset{\gamma_v - \gamma_w}{\overset{p_v^e}{\mathbb{C}}} \oplus \ell^\infty(\Gamma \setminus \{v, w\}).$$

If e is a loop at the vertex v then

$$\mathcal{N}_e = \underset{\gamma_v}{\overset{p_v}{L}}(\mathbb{Z}) \oplus \ell^\infty(\Gamma \setminus \{v\}).$$

If e connects two different vertices then the trace on $M_2(\mathbb{C}) \underset{2\gamma_w}{\otimes} L(\mathbb{Z}) \oplus \underset{\gamma_v - \gamma_w}{\overset{p_v^e}{\mathbb{C}}}$ is given by $\mathrm{tr}_{M_2} \otimes \mathrm{tr}_{L(\mathbb{Z})} + \mathrm{tr}_{\mathbb{C}}$. $\mathcal{N}_e$ includes $\ell^\infty \Gamma$ by letting $p_w = e_{1,1} \otimes 1$ and $p_v = e_{2,2} \otimes 1 + p_v^e$ so that $p_v^e \leq p_v$. If e is a loop that $\mathcal{N}_e$ then $\mathcal{N}_e$ includes $\ell^\infty(\Gamma)$ in the obvious way.

Let E_e be the $\mathrm{Tr}-$preserving conditional expectation from $\mathcal{N}_e$ to $\ell^\infty(\Gamma)$. We define $\mathcal{N}(\Gamma) = \underset{\ell^\infty(\Gamma)}{*} (\mathcal{N}_e, E_e)_{e \in E(\Gamma)}$.

Note that if Γ has no loops then this definition of $\mathcal{N}(\Gamma)$ is the same as $\mathcal{M}(\Gamma)$ in [Har12]. Also observe that $Q\mathcal{M}_\infty Q$ in section 3.4 is $\mathcal{N}(\mathcal{F}_{\mathcal{C}}(\mathcal{L}))$. We now define some notation which will be useful in determining the isomorphism class of $\mathcal{N}(\Gamma)$.

Definition 3.4.20. We write $v \sim w$ if v and w are connected by at least 1 edge in Γ and denote $n_{v,w}$ be the number of edges joining v and w. We set $\alpha_v^\Gamma = \sum_{w \sim v} n_{v,w} \gamma_w$, and define $B(\Gamma) = \{v \in V(\Gamma) : \gamma_v > \alpha_v^\Gamma\}$. Note that if there is a loop, e, at v then $v \notin B(\Gamma)$.

Assume for the moment that $\Gamma \subset \Gamma'$ are finite, connected, weighted graphs with at least two edges (so that $\mathcal{N}(\Gamma)$ and $\mathcal{N}(\Gamma')$ are finite von Neumann algebras). There is a natural inclusion $\mathcal{N}(\Gamma) \to \mathcal{N}(\Gamma')$ which will not be unital if Γ' has a larger vertex set. We will prove the following theorem, which is along the same lines as [Har12].

Theorem 3.4.21. $\mathcal{N}(\Gamma)$ *has the form*

$$\mathcal{N}(\Gamma) \cong \overset{p^\Gamma}{L}(\mathbb{F}_{t_\Gamma}) \oplus \bigoplus_{v \in B(\Gamma)} \underset{\gamma_v - \alpha_v^\Gamma}{\overset{r_v^\Gamma}{\mathbb{C}}}$$

where $r_v^\Gamma \leq p_v$ and t_Γ is such that this algebra has the appropriate free dimension. If Γ is a proper subgraph of Γ' then the unital inclusion $p^\Gamma \mathcal{N}(\Gamma) p^\Gamma \to p^\Gamma \mathcal{N}(\Gamma') p^\Gamma$ is a standard embedding of interpolated free group factors.

See [Dyk93] and [Dyk11] for a discussion of free dimension and rules for computing it. Also we refer the reader to [Dyk93] for the definition of a standard embedding of interpolated free group factors. Whenever A and B are interpolated free group factors and A is unitally included into B then we write $A \overset{s.e.}{\hookrightarrow} B$ to indicate that the inclusion of A into B is a standard embedding. In this section, we will extensively use the following properties of standard embeddings which can be found in [Dyk93] and [Dyk95].

(1) If A is an interpolated free group factor, the canonical inclusion $A \to A * \mathcal{N}$ is a standard embedding whenever $\mathcal{N}$ is of the form in Notation 3.4.18.

(2) A composite of standard embeddings is a standard embedding.

(3) If $A_n = L(\mathbb{F}_{s_n})$ with $s_n < s_{n+1}$ for all n and $\phi_n : A_n \overset{s.e.}{\hookrightarrow} A_{n+1}$, then the inductive limit of the A_n with respect to the ϕ_n is $L(\mathbb{F}_s)$ where $s = \lim_{n\to\infty} s_n$.

(4) If $t > s$ then $\phi : L(\mathbb{F}_s) \overset{s.e.}{\hookrightarrow} L(\mathbb{F}_t)$ if and only if for any nonzero projection $p \in L(\mathbb{F}_s)$, $\phi|_{pL(\mathbb{F}_s)p} : pL(\mathbb{F}_s)p \overset{s.e.}{\hookrightarrow} \phi(p)L(\mathbb{F}_t)\phi(p)$.

Theorem 3.4.21 was proved in [Har12] in the case that Γ contained no loops, so we only to need to modify the arguments there to incorporate what happens when Γ contains loops. We will prove Theorem 3.4.21 by inducting on the number of edges in the graph. We divide this into two lemmas.

Lemma 3.4.22. *Suppose Γ is a connected graph with 2 edges, one of which is a loop. Then $\mathcal{N}(\Gamma)$ is of the form in Theorem 3.4.21.*

Proof. There are two cases to consider. The first when Γ has one vertex with two loops and the other when Γ has two vertices, w and w with a loop at v and an edge connecting v to w.

<u>Case 1:</u> Assume Γ has one vertex with two loops. It follows immediately from the definition of $\mathcal{N}(\Gamma)$ that $\mathcal{N}(\Gamma) = L(\mathbb{F}_2)$ which is in agreement with Theorem 3.4.21.

<u>Case 2:</u> Assume Γ has two vertices w and w with a loop, e, at v and an edge, f, connecting v to w. There are two subcases to consider: when $\gamma_v \geq \gamma_w$ and $\gamma_v < \gamma_w$.

<u>Case 2a:</u> Assume $\gamma_v \geq \gamma_w$ and set $D = \ell^\infty(\Gamma)$. $\mathcal{N}(\Gamma)$ has the form

$$\mathcal{N}(\Gamma) = \left(\overset{p_v}{\underset{\gamma_v}{L(\mathbb{Z})}} \oplus \overset{p_w}{\underset{\gamma_w}{\mathbb{C}}} \right) \underset{D}{*} \left(\underset{2\gamma_w}{M_2 \otimes L(\mathbb{Z})} \oplus \overset{p_v^f}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \right).$$

Note that the central support of p_v is 1 in $\mathcal{N}_f$. By [DR11],

$$p_v \mathcal{N}(\Gamma) p_v = L(\mathbb{Z}) * p_v \mathcal{N}_f p_v = L(\mathbb{Z}) * \left(\underset{\gamma_w}{L(\mathbb{Z})} \oplus \overset{p_f^v}{\underset{\gamma_v - \gamma_w}{\mathbb{C}}} \right)$$

which is an interpolated free group factor $L(\mathbb{F}_t)$ for some appropriate t. By amplifying, it follows that $\mathcal{N}(\Gamma)$ is an interpolated free group factor, in agreement with Theorem 3.4.21.

<u>Case 2b:</u> Assume $\gamma_v < \gamma_w$. We note that the central support of p_v in $\mathcal{N}_f$ is $1 - p_w^f$ and following the same algorithm as in Case 2a gives

$$p_v \mathcal{N}(\Gamma) p_v = L(\mathbb{Z}) * p_v \mathcal{N}_f p_v = L(\mathbb{Z}) * L(\mathbb{Z}) = L(\mathbb{F}_2),$$

so by amplifying, $\mathcal{N}(\Gamma) = L(\mathbb{F}_t) \oplus \overset{p_w^f}{\underset{\gamma_w}{\mathbb{C}}}$ for appropriate t. This agrees with Theorem 3.4.21. $\square$

Lemma 3.4.23. *Suppose $\Gamma \subset \Gamma'$ are finite weighted graphs and assume $\mathcal{N}(\Gamma)$ is of the form in Theorem 3.4.21. In addition, suppose Γ' is obtained from Γ by one of the following three operations:*

(1) Γ and Γ' have the same edge set and Γ' is obtained from Γ be adding a loop e at a vertex v,

(2) Γ and Γ' have the same edge set and Γ' is obtained from Γ be adding an edge e between two distinct vertices v and w, or

(3) Γ' is obtained from Γ be adding a vertex v and an edge e connecting $w \in V(\Gamma)$ to v.

Then $\mathcal{N}(\Gamma')$ is also of the form in Theorem 3.4.21 and $p^\Gamma \mathcal{N}(\Gamma)p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}(\Gamma')p^\Gamma$.

Proof. The steps in proving the lemma assuming operations (2) or (3) are exactly the same as in [Har12], so we only need to assume that Γ' is obtained from Γ via operation (1). Set $D = \ell^\infty(\Gamma) = \ell^\infty(\Gamma')$. We have $\mathcal{N}(\Gamma') = \mathcal{N}_e \underset{D}{*} \mathcal{N}(\Gamma)$ and by assumption,

$$p_v \mathcal{N}(\Gamma)p_v = \overset{p_v^\Gamma}{L(\mathbb{F}_t)} \oplus \overset{r_v^\Gamma}{\mathbb{C}}$$

where r_v^Γ can possibly be zero but $p_v^\Gamma = p_v \cdot p^\Gamma \neq 0$. By [DR11],

$$p_v \mathcal{N}(\Gamma')p_v = L(\mathbb{Z}) * p_v \mathcal{N}(\Gamma)p_v = L(\mathbb{Z}) * \left(\overset{p_v^\Gamma}{L(\mathbb{F}_t)} \oplus \overset{r_v^\Gamma}{\mathbb{C}} \right)$$

which is an interpolated free group factor. By amplification, it follows that $\mathcal{N}(\Gamma')$ is of the form in Theorem 3.4.21. By [Dyk93], The inclusion $p_v^\Gamma \mathcal{N}(\Gamma)p_v^\Gamma \to p_v^\Gamma \mathcal{N}(\Gamma')p_v^\Gamma$ is equivalent to the inclusion

$$L(\mathbb{F}_t) \to L(\mathbb{F}_t) * p_v^\Gamma \left(L(\mathbb{Z}) * \left(\overset{p_v^\Gamma}{\mathbb{C}} \oplus \overset{r_v^\Gamma}{\mathbb{C}} \right) \right) p_v^\Gamma$$

which is a standard embedding. As $p_v^\Gamma \leq p^\Gamma$ it follows that $p^\Gamma \mathcal{N}(\Gamma)p^\Gamma \overset{s.e.}{\hookrightarrow} p^\Gamma \mathcal{N}(\Gamma')p^\Gamma$. $\qquad\square$

We notice that Theorem 3.4.21 follows from Lemmas 3.4.22 and 3.4.23. Indeed, if $\Gamma \subset \Gamma'$ are finite, connected, weighted graphs then Γ' can be obtained from Γ by applying operations (1), (2), and (3) above. Also, the composite of standard embeddings is a standard embedding and standard embeddings are preserved by cut-downs.

Before determining the isomorphism $M \cong L(\mathbb{F}_\infty)$ we note the following lemma whose proof is a straightforward induction exercise using the algorithms in [DR11].

Lemma 3.4.24. *Suppose Γ consists of a vertex v connected to vertices w_1 through w_k by a total of n edges. Assume further that $\gamma_v \leq \gamma_{w_i}$ for all i. Then $p_v \mathcal{N}(\Gamma)p_v = L(\mathbb{F}_t)$ where $t \geq n$.*

Theorem 3.4.25. $M \cong L(\mathbb{F}_\infty)$.

Proof. Let Γ denote $\mathcal{F}_\mathcal{C}(\mathcal{L})$ so that $M \cong p_\emptyset \mathcal{N}(\Gamma)p_\emptyset$. We build up Γ by an increasing union of finite connected subgraphs Γ_k, each of which contain the vertex * so that M is the inductive limit of the algebras $p_\emptyset \mathcal{N}(\Gamma_k)p_\emptyset$. All vertices in Γ_k must have weight larger than 1 since for p an irreducible projection in $\mathcal{P}$, $p \otimes \overline{p}$ must have a subprojection equivalent to the trivial one. It follows that for k sufficiently large, $p_\emptyset \mathcal{N}(\Gamma_k)p_\emptyset$ is an interpolated free group factor and since $p_\emptyset \mathcal{N}(\Gamma_k)p_\emptyset \overset{s.e.}{\hookrightarrow} p_\emptyset \mathcal{N}(\Gamma_{k+1})p_\emptyset$ we know that $M = p_\emptyset \mathcal{N}(\Gamma)p_\emptyset$ must be an interpolated free group factor. Let $\tilde{\Gamma}_n$ be the subgraph of Γ_n whose vertices are * and v_1, ..., v_k which are connected to * in Γ_k and whose edges are exactly the edges connecting * to each v_i to * in Γ_n. Assuming there are k_n such vertices, we see from Lemma 3.4.24 that $p_\emptyset \mathcal{N}(\tilde{\Gamma}_n)p_\emptyset = L(\mathbb{F}_{r_n})$ for $r_n \geq k_n$. Since either $p_\emptyset \mathcal{N}(\tilde{\Gamma}_n)p_\emptyset = p_\emptyset \mathcal{N}(\Gamma_n)p_\emptyset$ or $p_\emptyset \mathcal{N}(\tilde{\Gamma}_n)p_\emptyset \overset{s.e.}{\hookrightarrow} p_\emptyset \mathcal{N}(\Gamma_n)p_\emptyset$ it follows that $p_\emptyset \mathcal{N}(\Gamma_n)p_\emptyset = L(\mathbb{F}_{t_n})$ for $t_n \geq r_n$. There are infinitely many edges emanating from *. Therefore k_n and hence t_n can be arbitrarily large so it follows that $M = L(\mathbb{F}_t)$ where $t = \lim_n t_n = \infty$. $\qquad \square$

Corollary 3.4.26. $M_\alpha \cong L(\mathbb{F}_\infty)$ *for all* $\alpha \in \Lambda$.

Proof. M_α is an amplification of M. $\qquad \square$

Remark 3.4.27. With a bit more careful analysis, using the techniques in [Har12], one can show that if $\mathcal{C}$ has infinitely many simple objects then the factor M is $L(\mathbb{F}_\infty)$ no matter the choice of the generating set $\mathcal{S}$ (using $\mathcal{L} = \mathcal{S} \oplus \overline{\mathcal{S}}$). When $\mathcal{C}$ has finitely many simple objects, then we can find t finite with $M \cong L(\mathbb{F}_t)$. To do this, we can find a single object X which generates $\mathcal{C}$, and we let $\mathcal{L} = \{X \oplus \overline{X}\}$. Applying the analysis in Section 3.4 shows that M is a cutdown of $\mathcal{N}(\mathcal{F}_\mathcal{C}(X \oplus \overline{X}))$. Keeping track of the free dimension yields the isomorphism $M \cong L(\mathbb{F}(1 + \dim(\mathcal{C})(2\dim(X) - 1)))$.

3.5 Appendix

The factors in [Yam03] have property Γ

First, we sketch the proof that the factors in [Yam03] have property Γ, and thus are not interpolated free group factors. We use some notation from [Yam03].

Sketch of proof. Let $\mathcal{C}$ be a countably generated rigid C^*-tensor category. Let $\mathcal{S}$ be the set of isomorphism classes of simple objects. Let $\{e_s | s \in \mathcal{S}\}$ be a family of non-zero, pairwise orthogonal projections in the hyperfinite II_1 factor R. Consider the von Neumann algebras

$$A = \bigoplus_{s \in \mathcal{S}} e_s R e_s \text{ and } B = \bigoplus_{s,t \in \mathcal{S}} \mathcal{C}(s,t)\overline{\otimes}e_s R e_t.$$

Note that we have a unital, connected inclusion $A \subset B$. The factors considered in [Yam03] are of the form

$$N = (Q\overline{\otimes}A) *_A B,$$

where Q is an arbitrary finite von Neumann algebra. To show that N admits a non-trivial central sequence, it suffices to find a sequence in A which commutes asymptotically with B. Let $x = (x_n)$ be a central sequence of R. One can check that the sequence (y_n) given by

$$y_n = \bigoplus_{s \in \mathcal{S}} e_s x_n e_s \in A$$

has the desired property. Hence N has property Γ and is not isomorphic to an interpolated free group factor. $\qquad \square$

Proofs using λ-lattices

We now sketch a proof of Theorem 3.1.3 using the techniques of [PS03] in the case where the rigid C^*-tensor category $\mathcal{C}$ is finitely generated. This sketch closely follows [FR12, Theorem 4.1].

Sketch of proof. Suppose $\mathcal{S} = \{X_1, \ldots, X_r\}$ generates $\mathcal{C}$. First, set

$$X = X_1 \oplus \overline{X_1} \oplus \cdots \oplus X_r \oplus \overline{X_r}$$

which is a symmetrically self-dual object in $\mathcal{C}$. Setting

$$P_{n,+} = \mathrm{End}_{\mathcal{C}}(X \otimes \overline{X} \otimes \cdots \otimes X^{\pm})$$
$$P_{n,-} = \mathrm{End}_{\mathcal{C}}(\overline{X} \otimes X \otimes \cdots \otimes X^{\pm})$$

where $X^{\pm} = X, \overline{X}$ depending on the parity of n, $\mathcal{P}$ is a subfactor planar algebra. (This follows by results similar to those in Section 3.2). A subfactor planar algebra is naturally a λ-lattice [Pop95, Jon], and thus [PS03] gives us a subfactor $N \subset M$ where N, M are both isomorphic to $L(\mathbb{F}_\infty)$ and whose subfactor planar algebra is isomorphic to $\mathcal{P}$. The result now follows by a modified version of Theorem 3.2.17. $\qquad \square$

Difficulties arise when trying to adapt an approach along these lines for $\mathcal{C}$ not finitely generated. One wants to define an inductive limit Popa system and use [PS03] to get an inclusion of factors isomorphic to $L(\mathbb{F}_\infty)$ which remembers $\mathcal{C}$.

One hope is to look at finitely generated subcategories of $\mathcal{C}$ as follows. Set

$$Y^r = X_1 \oplus \overline{X_1} \oplus \cdots \oplus X_r \oplus \overline{X_r},$$

and define Z^r inductively by setting $Z^1 = Y^1$ and $Z^r = Z^{r-1} \otimes Y^r$. Now for all r, we have subfactor planar algebras $\mathcal{P}^r$ given by

$$P^r_{n,+} = \mathrm{End}_{\mathcal{C}}(Z^r \otimes \overline{Z^r} \otimes \cdots \otimes Z^{r,\pm})$$
$$P^r_{n,-} = \mathrm{End}_{\mathcal{C}}(\overline{Z^r} \otimes Z^r \otimes \cdots \otimes Z^{r,\pm}).$$

Again, using [PS03], we get an inclusion $N^r \subset M^r$ where both factors are isomorphic to $L(\mathbb{F}_\infty)$, and the associated category of $N^r - N^r$ bimodules is equivalent to the subcategory of $\mathcal{C}$ generated by $X_1, \overline{X_1}, \ldots, X_r, \overline{X_r}$.

One problem with this approach is that while $P^r_{n,\pm}$ includes unitally into $P^{r+1}_{n,\pm}$, the inductive limit planar algebra $\mathcal{P}^\infty$ has infinite dimensional n-box spaces, i.e., $\dim(P^\infty_{n,\pm}) = \infty$ for all n. Hence we cannot directly use [PS03] to get a subfactor. Another problem is that the inclusion $P^r_{\bullet} \hookrightarrow \mathcal{P}^{r+1}$ does not induce an inclusion of N^r into N^{r+1} nor an inclusion M^r into M^{r+1}. Hence we do not get an inductive limit subfactor $N^\infty \subset M^\infty$.

Of course, one can try other approaches along these lines, but so far, the authors have not succeeded in finding an inductive limit inclusion.

The law of the element $p_v X_e u_e^w (u_e^w)^* X_e p_v$

We will adapt the technique in [GJS10] to compute the law of $R = p_v X_e u_e^w (u_e^w)^* X_e p_v$. A similar computation will compute the law of $(u_e^w)^* X_e p_v X_e u_e^w$. To begin, we set $r_w = u_e^w (u_e^w)^*$ and notice that this element diagrammatically looks like

$$-\boxed{p_v}\,\cup\!\!\cup\,\boxed{r_w}\,\cup\!\!\cup\,\boxed{p_v}-$$

Let ϕ be the moment generating function of R i.e. $\phi(z) = \sum_{z=0}^\infty \frac{\mathrm{Tr}(R^n)}{\mathrm{Tr}(p_v)} z^n$. Set $\alpha_e = \frac{\mathrm{Tr}(r_w)}{\mathrm{Tr}(p_v)} = \frac{\mathrm{Tr}(p_w)}{\mathrm{Tr}(p_v)} \geq 1$. Using the recurrence relation 3.6, and the fact that

$$E\left(-\boxed{r_w}\,\cup\!\!\cup\,\right) = \alpha_e p_v,$$

we see that for $n \geq 1$,

$$E(R^n) = \alpha_e E(R^{n-1}) + \left(\sum_{k=1}^{n-2} E(R^k) E(R^{n-1-k})\right) + E(R^{n-1}) = (\alpha_e - 1) E(R^{n-1}) + \sum_{k=0}^{n-1} E(R^k) E(R^{n-1-k})$$

Multiplying both sides by z^n and summing gives the equation

$$\phi - 1 = (\alpha_e - 1) z \phi + z \phi^2$$

which implies

$$\phi(z) = \frac{-((\alpha_e - 1)z - 1) + \sqrt{((\alpha_e - 1)z - 1)^2 - 4z}}{2z}$$

where the branch of the square root will be momentarily determined. Let μ be the measure supported on the spectrum of R in $p_v \mathcal{M}_\infty p_v$ satisfying

$$\int x^n d\mu(x) = \mathrm{Tr}(R^n)/\mathrm{Tr}(p_v).$$

ale let $G_\mu(z) = z^{-1} \phi(z^{-1})$ be its Cauchy transform. It follows that

$$G_\mu(z) = \frac{-((\alpha_e - 1) - z) + \sqrt{((\alpha_e - 1) - z)^2 - 4z}}{2z}.$$

The Cauchy transform of μ also has the form

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{z - x}$$

and it is straightforward to check that $\Im(G(z)) < 0$ whenever $\Im(z) > 0$, and $\lim_{z \to \infty} z G_\mu(z) = 1$ non-tangentially to the real axis. We therefore choose the branch of the square root so that the algebraic expression for $G_\mu(x)$ satisfies these two conditions. The measure μ is recovered by writing $z = x + iy$ and computing $\lim_{y \downarrow 0} -\frac{1}{\pi} \Im(G_\mu(z))$ [VDN92]. Doing so gives

$$d\mu(x) = \frac{\sqrt{-((\alpha_e - 1) - x)^2 + 4x}}{2\pi x} \cdot \chi_{[\sqrt{\alpha_e} - 1, \sqrt{\alpha_e} + 1]} dx$$

with χ_I the indicator function of the Lebesgue measurable set I. This is the distribution of a free Poisson element [NS06].

Chapter 4

The GJS Construction for Fuss Catalan and $N - P - M$ Planar Algebras

4.1   Introduction

The work of Chapter 3 was heavily influenced by a problem presented to the author by Vaughan Jones. The problem is as follows: Given a subfactor planar algebra, $\mathcal{Q}$, one can consider the algebras $\mathrm{Gr}_k^{\pm}(\mathcal{Q})$ as defined in the previous chapters and place the following "toy potential" on $\mathcal{Q}$:

$$\mathrm{tr}(x) = \frac{\boxed{\sum V}}{\boxed{x}}$$

where V is a rotationally invariant set of elements in $\mathcal{Q}$. If one is fortunate, tr is positive definite on $\mathcal{Q}$ and left multiplication is bounded on $L^2(\mathrm{Gr}(\mathcal{Q}))$. To this end, it is an interesting problem is to study the von Neumann algebras, $N_k^{\pm}(= \mathrm{Gr}_k^{\pm}(\mathcal{Q})'')$ associated to $\mathcal{Q}$ and V.

The case that will be considered here is the case where $\mathcal{Q}$ contains is the standard invariant for a subfactor $N \subset M$ that contains an intermediate subfactor, P. As such, it follows that $\mathcal{Q}$ contains the Fuss Catalan planar algebra as a sub planar algebra [BJ97]. Therefore, we can consider the following potential on $\mathrm{Gr}_0^+(\mathcal{Q})$:

$$\mathrm{tr}(x) = \frac{\boxed{\sum FC}}{\boxed{x}}$$

where $\sum FC$ represents the sum of all Fuss Catalan diagrams. Recall that Fuss Catalan diagrams are planar two-colored string diagrams satisfying the following condition: If the colors of the strings are a and b, then the colors of strings intersecting the boundary form the pattern, $aabbaabbaabb...$

To attack this problem, it is most natural to introduce a new planar algebra, $\mathcal{P}$ which will be called an $N - P - M$ planar algebra. A pleasing feature of this planar algebra is that the boundary conditions of the input disks can be taken to be any word in aa and bb. The planar algebra $\mathcal{Q}$ will be realized as a subalgebra of $\mathcal{P}$ by replacing each strand with an a strand next to a b strand so that the pattern formed by the input disks in the $\mathcal{Q}-$tangles is of the form $aabbaabbaabb....$ Whenever

$\mathcal{Q}$ is itself the Fuss Catalan planar algebra, we can take $\mathcal{P}$ to be generated the set of <u>all</u> string diagrams having boundary conditions any word in aa and bb.

The purpose of the $N - P - M$ planar algebra is that it gives one a natural way to decouple the strings a and b and treat them as free generators in an appropriate sense (see Section 4.5 for how this is done). We form algebras M_α for α a suitable word in a and b which are the $N - P - M$ analogues of the algebras $N_k^\pm$, and will use a semifinite algebra as in Chapter 3 to find the isomorphism class of the algebras M_α. More precisely, we will prove the following theorem:

Theorem 4.1.1. *M_α is a II_1 factor and is isomorphic to $L(\mathbb{F}(1 + 2I\delta_\alpha(\delta_a + \delta_b - 2)))$ for $\mathcal{P}$ finite depth. Here, δ_α is defined as in Chapter 3, and $I = \sum_{v \in \Gamma_N^N} \mu(v)^2$ with μ the Perron Frobenius weighting on the principal graphs on $\mathcal{P}$.*

This formula has some interest, because it contains information about the inclusions $N \subset P$ and $P \subset M$ (δ_a and δ_b respectively) as well as the larger inclusion $N \subset M$ (the global index I). Just as in the case for the GJS algebras, we will also prove the following theorem:

Theorem 4.1.2. *$M_\alpha \cong L(\mathbb{F}_\infty)$ when $\mathcal{P}$ is infinite depth.*

In addition to understanding the algebraic structure of the N_k, we will make use of the semifinite algebra to show that the law of $\cup \in \mathbb{N}_0^+$ has a nice expression in terms of known laws.

Unfortunately, the author has not yet been able to identify the isomorphism classes of the algebras $N_k^\pm$, however the semifinite algebra will show that there is evidence that the algebras $N_k^\pm$ are free group factors:

Theorem 4.1.3. *The von Neumann algebras, $N_k^\pm$, are each contained in a free group factor and contain a free group factor.*

4.2 $N - P - M$ **planar algebras**

The object that is at the heart of all of these computations is an $N - P - M$ planar algebra which can be thought of an augmentation of such a $\mathcal{Q}$ as above. Given parameters a and b, we denote $\mathcal{W}$ as the set of finite words on aa and bb. We define what it means for a planar tangle to be an $N - P - M$ tangle:

Definition 4.2.1. A planar tangle is said to be an $N - P - M$ tangle if its regions are shaded by three colors, N, P, and M such that the following conditions are met:

- A region colored N only borders a region colored P

- A region colored M only borders a region colored P

- A region colored P can border a region colored N or M but not P.

We will denote the set of $N - P - M$ tangles as the $N - P - M$ planar operad. As in the previous chapters, all tangles will be drawn so that the internal and external disks are rectangles. In order for smooth isotopy to make sense, the rectangles have their corners smoothed out.

Any string serving as the boundary string of a region colored N will be called an a string and any string bordering a region colored M will be called a b string. We note that the conditions on

the regions show that if α is a word in a and b, then α serves as a word of strings intersecting a disk if and only if $\alpha \in \Delta$ where Δ is the set

$$\Delta = \{awb : w \in \mathcal{W}\} \cup \{bwa : w \in \mathcal{W}\} \cup \{awa : w \in \mathcal{W}\} \cup \{bwb : w \in \mathcal{W}\} \cup \mathcal{W}$$

The following notation will be useful:

Notation 4.2.2. We denote Δ^Q to be the set of all words in Δ where the shading before the first letter is Q (for $Q = N$, P, or M). If $\alpha \in \Delta$, we define $s(\alpha)$ to be the first letter of the word α. Whenever a word is mentioned, part of the data is its initial region (hence the choice of region between every pair of letters), not just its letters. If a tangle, T, has boundary condition α on its outer disk, we will call T a planar α tangle. Any internal rectangle with boundary condition α will be called an α rectangle.

We remark that just as for shaded planar algebras, there is a natural gluing operation. Namely, if we have two planar tangles S, T satisfying the following boundary condition:

- Some internal rectangle D_S of S has boundary data which agrees with T, i.e. the shadings along the boundaries of T and D_S agree when counting clockwise from the marked point.

then we may compose S and T to get the planar tangle $S \circ_{D_S} T$ by taking S union the interior of T, removing the boundary of D_S, and smoothing the strings.

As in the previous chapter, given $\alpha \in \Delta$, we let $\overline{\alpha}$ be α read in the opposite order. When a string appears with a label α, then the string is meant to be a band of strings having colors ordered by the word α. As in the previous chapter, the strings are read in the order of top to bottom and left to right. Also, unless otherwise marked, all marked regions of rectangles will be assumed to be on the bottom-left corner of the box. In addition, whenever there is a box written without a tangle, it is assumed that the box is placed in a larger tangle whose boundary data agrees with the boundary data for the box and whose marked region is the same as the marked region of the box.

We now define an $N - P - M$ planar algebra:

Definition 4.2.3. An $N - P - M$ planar algebra consists of the following data:

- Given parameters a and b as above, there is a finite dimensional complex vector space $\mathcal{P}_\alpha$ for every nonempty word, $\alpha \in \Delta$. There are three vector spaces $\mathcal{P}_\emptyset^N$, $\mathcal{P}_\emptyset^P$, and $\mathcal{P}_\emptyset^M$ in the case when α is empty. These are one dimensional complex vector spaces.

- an action of planar tangles by multilinear maps, i.e., for each planar α tangle T, whose rectangles $D_i(T)$ are α_i rectangles, there is a multilinear map

$$Z_T \colon \prod_{i \in I} P_{\alpha_i} \to P_\alpha$$

satisfying the following axioms:

Isotopy: If θ is an orientation preserving diffeomorphism of $\mathbb{R}^2$, then $Z_{\theta(T)} = Z_T$. That is, let T^0 be the interior of T, and let $f \in \prod_{D \subset T^0} P_{\alpha_D}$. Then

$$Z_{\theta(T)}(f_\theta) = Z_T(f)$$

where $f_\theta(\theta(D)) = f(D)$.

<u>Naturality:</u> For S, T composable tangles, $Z(S \circ_D T) = Z(S) \circ_D Z(T)$, where the composition on the right hand side is the composition of multilinear maps.

- $\mathcal{P}$ is <u>unital</u> [Jon11]: Let S be an $N - P - M$ tangle with no input disks and boundary condition $\alpha \in \Delta$. Then, there is an element $Z(S) \in P_\alpha$ so that the following holds:

  Let S be a tangle a nonempty set of internal disks such that S can be glued into the internal disk D^S of T. Then
  $$Z(T \circ S) = Z(T) \circ Z_S.$$
  Here $(Z(T) \circ Z_S)(f) = \tilde{f}$ where

  $$\tilde{f}(D) = \begin{cases} f(D) \text{ if } D \neq D^S \\ Z(S) \text{ if } D = D^S \end{cases}$$

  This condition allows isotopy classes of of such an S to be elements of P_α. This action allows us to identify the empty diagrams (shaded N, P, and M) with the scalar $1 \in \mathbb{C}$. We make this assumption in the rest of this chapter. The naturality axiom, combined with this identification, forces closed strings with parameters a and b to be replaced by scalars δ_a and δ_b respectively.

- There is a conjugate linear involution, $* : \mathcal{P}_\alpha \to \mathcal{P}_{\bar{\alpha}}$. It is compatible with reflection of tangles i.e., if $\overline{T}$ is a tangle which is produced by an orientation <u>reversing</u> diffeomorphism, φ, of T, then we have
  $$(Z_T(f))^* = Z_{\overline{T}}(\bar{f})$$
  where $\overline{f}(\varphi(D)) = f(D)^*$.

- Each $\mathcal{P}_\alpha$ comes equipped with the positive definite sesquilinear form:

  $$\langle x, y \rangle = \boxed{x}\,{}^\alpha\!\boxed{y^*}$$

- $\mathcal{P}$ is <u>spherical</u>, i.e. for all $\alpha \in \Delta$ and all $x \in P_{\alpha\bar{\alpha}}$, we have

  $$\mathrm{tr}(x) = \boxed{x}\,\bar{\alpha} \quad = \quad \bar{\alpha}\,\boxed{x}.$$

  This says that we can think of our planar tangles as living in a sphere instead of a plane.

Remark 4.2.4. In viewing the action of a tangle, the letter Z will often be omitted.

A-priori, it is not clear that an $N - P - M$ planar algebra should exist. The following example shows that this is the case. For the rest of this paper, an a string will be colored blue and a b string will be colored red.

Example 4.2.5. Let $\delta_a, \delta_b \in \{2\cos(\pi/n) : n = 3, 4, 5, ...\} \cup [2, \infty)$ and define P_α by

$$P_\alpha = \text{span}\{\text{planar string diagrams with with boundary condition } \alpha\}$$

i.e., P_α is the $\mathbb{C}-$linear span of isotopy classes of α tangles with no input disks and no loops. For example,

$$P_{abba} = \text{span}\left\{ \boxed{}, \boxed{}, \boxed{} \right\}.$$

The action of $N - P - M$ tangles is as follows: All string diagrams are inserted into the necessary input disks. The result of this operation is a new string diagram with except with some loops. These loops are replaced with a parameter δ_a or δ_b, depending on the color of a loop. The adjoint operation is the conjugate linear extension of reflection of diagrams.

It is straightforward to check that $\mathcal{P}$ satisfies all of the axioms of an $N - P - M$ planar algebra except positive definiteness. Given $\alpha \in \Delta$, we form the word α' which is a word of colors that can appear in a Fuss Catalan diagram, and is obtained from α by inserting the right combination of $aa's$ or $bb's$ between letters in α. For example, if $\alpha = aaabbaabbbba$, then $\alpha' = a(bb)aabbaabb(aa)bba$. We then define a map $\phi : P_\alpha \to P_{\alpha'}$ which is given by inserting a cup of the appropriate color whenever that color has been inserted into α, and then dividing by $\delta_a^m \cdot \delta_b^n$. Here, aa was inserted m times and bb was inserted n times. For example,

$$\phi\left(\boxed{x} \right) = \frac{1}{\delta_a^2 \delta_b} \cdot \boxed{x}.$$

This map is easily seen to be preserve the desired sesquilinear form, and we know that this form is positive semidefinite on the Fuss Catalan algebras, with positive definiteness in the case $\delta_a, \delta_b \geq 2$, from [BJ97]. Therefore, after taking a quotient in the case that δ_a or δ_b is less than 2, this example produces an $N - P - M$ planar algebra.

By unitality, this planar algebra is represented in every $N - P - M$ planar algebra.

Principal graphs of $N - P - M$ planar algebras

We first remark that if $\bar{\alpha}\alpha \in \Delta$, then the axioms of an $N - P - M$ planar algebra show that $P_{\alpha\bar{\alpha}}$ is a finite dimensional C^* algebra with multiplication given by

$$x \cdot y = \boxed{x} \boxed{y}.$$

Let $p \in P_{\alpha\bar{\alpha}}$ and $q \in P_{\gamma\bar{\gamma}}$ be projections. Then we say p is <u>equivalent</u> to q if there is a $u \in P_{\alpha\bar{\gamma}}$ so that

$$\boxed{\begin{array}{c} u \\ u^* \end{array}} = p \quad \text{and} \quad \boxed{\begin{array}{c} u^* \\ u \end{array}} = q.$$

To an $N - P - M$ planar algebra $\mathcal{P}$, there are three <u>principal graphs</u> associated to $\mathcal{P}$, Γ_N, Γ_P, and Γ_M. We will call Γ_Q the $Q-$principal graph of $\mathcal{P}$. Each Γ_Q has three sets of vertices, Γ_Q^N, Γ_Q^P and Γ_Q^M. They are described by the following procedure:

The vertices $v \in \Gamma_{Q_1}^{Q_2}$ correspond to equivalence classes of minimal projections p_v in the finite-dimensional C^* algebra $P_{\alpha\overline{\alpha}}$ for some α depending on v where $\alpha\overline{\alpha} \in \Delta^{Q_1}$ and $\overline{\alpha}\alpha \in \Delta^{Q_2}$. There are <u>$a-$colored edges</u> connecting the vertices Γ_Q^N to the vertices Γ_Q^P as well as <u>$b-$colored edges</u> connecting the the vertices Γ_Q^P to Γ_Q^M. The $a-$colored edges are created as follows:

Suppose $v \in \Gamma_Q^N$ and $w \in \Gamma_Q^P$, let $p \in P_{\beta\overline{\beta}}$ be equivalent to p_v. It follows that the element

$$i_a(p) = \begin{array}{c} \star \\ \boxed{\beta \; \boxed{p} \; \beta} \end{array}$$

is a projection in $P_{\beta a a \overline{\beta}}$. We draw n $a-$colored edges between v and w if n is the maximal number such that there exist orthogonal projections $q_1, ..., q_n \in P_{\beta a a \overline{\beta}}$ which are each equivalent to p_w and satisfy $\sum_{i=1}^{n} q_i \leq i_a(p)$. We can also get edges from w to v in a similar manner. In principle, the construction of the $a-$edges leads to oriented edges, however, the presence of the Jones basic construction shows that the edges can be unoriented. More precisely, consider the projection

$$e = \frac{1}{\delta_a} \begin{array}{c} \star \boxed{\beta} \\ \boxed{} \end{array} \in P_{\beta a a a a \overline{\beta}},$$

and z be its central support. We note that $P_{\beta\overline{\beta}}$ unitally includes into $P_{\beta a a a a \overline{\beta}}$ by applying the map i_a twice. It is also a straightforward check to see that the mapping $P_{\beta\overline{\beta}} \to P_{\beta a a a a \overline{\beta}}$ given by $x \mapsto i_a(i_a(x))e$ is an isometry, and $e i_a(y) e = i_a(E_{P_{\beta\overline{\beta}}}(y))e$ for $y \in P_{\beta a a \overline{\beta}}$. Therefore, from [JS97] it follows $z P_{\beta a a a a \overline{\beta}} z$ is isomorphic to the basic construction of $P_{\beta\overline{\beta}}$ in $P_{\beta a a \overline{\beta}}$. If A, B, and C are finite dimensional C^* algebras with C the basic construction of A in B, then the Bratteli diagram of $B \subset C$ is the reflection of that of $A \subset B$ [JS97]. Therefore, if there are n $a-$colored edges from v to w, then there are n $a-$colored edges from w to v.

There is an analogous way to determine the $b-$colored edges that go between Γ_Q^P and Γ_Q^M. We also note that if $p \in P_{\alpha\overline{\alpha}}$ is a minimal projection corresponding to a vertex $v \in \Gamma_{Q_1}^{Q_2}$, then the mapping in Example 4.2.5 shows that p is equivalent to a minimal projection in P_w where $w = abbaabbaa..., aabbaabb..., bbaabbaa...$ or $baabbaa....$ In particular, if $\mathcal{Q}$ is as in the introduction, then we have established the following proposition once it is shown that we can consider $\mathcal{Q}$ inside an augmentation $\mathcal{P}$.

Lemma 4.2.6. *Let Γ and Γ' be the principal and dual principal graphs of $\mathcal{Q}$ respectively. Then there are one-to-one correspondences between the following sets of vertices:*

$$\Gamma_+ \leftrightarrow \Gamma_N^N, \; \Gamma_- \leftrightarrow \Gamma_N^M, \; \Gamma_+' \leftrightarrow \Gamma_M^M, \; and \; \Gamma_-' \leftrightarrow \Gamma_M^N$$

Also observe that rotation by $180°$ is an anti-isomorphism of each P_γ. This induces a one-to-one correspondence $\Gamma_{Q_1}^{Q_2} \leftrightarrow \Gamma_{Q_2}^{Q_1}$. Finally, if the vertices of the principal graphs Γ_Q are weighted according to the traces of their corresponding projections, then it follows by the definition of principal graph that the graph with vertices Γ_Q^N and Γ_Q^P is bipartite with Perron Frobenius eigenvalue δ_a. Also, the graph with vertices Γ_Q^P and Γ_Q^M is bipartite with Perron Frobenius eigenvalue δ_b.

4.3 $N - P - M$ planar algebras from intermediate subfactors

The goal of this section is to see that such a $\mathcal{Q}$ as above can be faithfully realized inside an $N-P-M$ planar algebra $\mathcal{P}$. Much of this section was influenced from discussions with David Penneys and Noah Snyder, and many of the proofs of the following theorems are taken from them. We will first describe how such an algebra arises from an inclusion $N \subset P \subset M$ of finite index II_1 factors. To start, we consider the following bifinite bimodules:

$$_N L^2(P)_P \text{ and } _P L^2(M)_M$$

and their duals (contragredients)

$$_P L^2(P)_N \text{ and } _M L^2(M)_P.$$

Let $\alpha \in \Delta$. Since part of the prescribed data for α is a choice of initial shading, we note that the shading of α, i.e. the shading between any two letters on α is uniquely determined. Assume that the shading of α is the sequence $Q_1 \cdots Q_k$ for $Q_i = N$, P or M. We define Z_α to be the following:

$$Z_\alpha = {}_{Q_1}L^2(Q_1)_{Q_1 \cap Q_2} \underset{Q_1 \cap Q_2}{\otimes} {}_{Q_1 \cap Q_2}L^2(Q_2)_{Q_2 \cap Q_3} \cdots \underset{Q_{k-1} \cap Q_k}{\otimes} {}_{Q_{k-1} \cap Q_k}L^2(Q_k)_{Q_k \cap Q_1} \underset{Q_k \cap Q_1}{\otimes} L^2(Q_1)_{Q_k \cap Q_1}.$$

and we set $P_\alpha = \text{Hom}_{Q_1 - Q_1}(L^2(Q_1), Z_\alpha)$ (Notice that $Q_i \cap Q_{i+1}$ is necessarily N, P, or M). We note from [Bis97, Con80] that this can be identified with the $Q_1 - Q_1$ central vectors of Z_α.

To help understand the planar structure, we let

$$N(= M_0) \subset M(= M_1) \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

be the Jones tower for $N \subset M$, where M_n is generated by M_{n-1} and e_{n-1}. Here, e_{n-1} is the orthogonal projection from $L^2(M_{n-1})$ onto $L^2(M_{n-2})$. We will define e_P to be the orthogonal projection from $L^2(M)$ onto $L^2(P)$. We will also let $B = \{b_i\}_{i=1}^n$ be an orthonormal Pimsner Popa basis for M over N where $n-1$ is the largest integer which is bounded above by the index $[M : N]$. The b_i are elements in M satisfying the following equivalent conditions:

$$x = \sum_{i=1}^n E_N(xb_i)b_i^* \; \forall x \in M$$

$$x = \sum_{i=1}^n b_i E_N(b_i^* x) \; \forall x \in M$$

$$1 = \sum_{i=1}^n b_i e_1 b_i^*,$$

as well as $E_N(b_i b_j^*) = \delta_{i,j}$ if $i \leq n-1$ and $E_N(b_n b_n^*)$ is a projection of trace $[M : N] - (n-1)$ in M. If we let e_P be the orthogonal projection from $L^2(M)$ onto $L^2(P)$ and $\{c_i\}_{i=1}^m$ be an orthonormal Pimsner-Popa basis of P over N. Then we have the following lemma.

Lemma 4.3.1. $e_P = \sum_{i=1}^m c_i e_1 c_i^*$.

Proof. We compute the 2-norm of $e_P - \sum_{i=1}^m c_i e_1 c_i^*$. Doing so gives:

$$\|e_p - \sum_{i=1}^m c_i e_1 c_i^*\|_2^2 = \operatorname{tr}(e_P) - 2\sum_{i=1}^m \operatorname{tr}(c_i e_1 c_i^* e_P) + \sum_{i,j=1}^m \operatorname{tr}(c_i e_1 c_i^* c_j e_1 c_j^*).$$

Since e_P commutes with the elements c_i, the term in the middle becomes $2\sum_{i=1}^m \operatorname{tr}(c_i e_1 c_i^*)$. Using $e_1 c_i^* c_j e_1 = E_N(c_i^* c_j e_1)$, and orthonormality of the basis, the last term becomes $\sum_{i=1}^m \operatorname{tr}(c_i e_1 c_i^*)$. Therefore, we get:

$$\|e_p - \sum_{i=1}^m c_i e_1 c_i^*\|_2^2 = \operatorname{tr}(e_P) - \sum_{i=1}^m \operatorname{tr}(c_i e_1 c_i^*)$$
$$= \operatorname{tr}(e_P) - [M:N]^{-1}\sum_{i=1}^m \operatorname{tr}(c_i c_i^*)$$
$$= [M:P]^{-1} - [M:N]^{-1}[P:N] = 0$$

as desired. □

We will now show the bimodules Z_α can be isometrically embedded in $L^2(M_n)$ for some n. As some notation, we will let $\delta_Q = [M:Q]^{1/2}$ for $Q = N$, P, or M. We also set $E_1^Q = \delta_Q e_Q$, and $v_n^Q = E_n E_{n-1} \cdots E_2 E_1^Q$.

Theorem 4.3.2. *The map $\phi : Z_\alpha \to M_k$ given by*

$$\phi(x_1 \underset{Q_1 \cap Q_2}{\otimes} x_2 \underset{Q_2 \cap Q_3}{\otimes} \cdots \underset{Q_{k-1} \cap Q_k}{\otimes} x_k) = x_1 v_1^{Q_1 \cap Q_2} x_2 v_2^{Q_2 \cap Q_3} \cdots v_{k-1}^{Q_{k-1} \cap Q_k} x_k$$

is an isometry.

Proof. Note that the map is well defined as v_r^Q commutes with Q. We proceed by induction on k. The result is clearly true for $k = 1$, so assume that it holds for $k - 1$. Using the previous lemma as

well as the relation $E_i E_j = E_j E_i$ for $|i - j| \geq 2$, we have:

$$\left\langle x_1 v_1^{Q_1 \cap Q_2} x_2 v_2^{Q_2 \cap Q_3} \cdots v_{k-1}^{Q_{k-1} \cap Q_k} x_k, \ y_1 v_1^{Q_1 \cap Q_2} y_2 v_2^{Q_2 \cap Q_3} \cdots v_{k-1}^{Q_{k-1} \cap Q_k} y_k \right\rangle_{M_k}$$

$$= \mathrm{tr}_{M_k} \left(y_k^* (v_{k-1}^{Q_{k-1} \cap Q_k})^* \cdots (v_2^{Q_2 \cap Q_3})^* y_2^* (v_1^{Q_1 \cap Q_2})^* y_1^* x_1 v_1^{Q_1 \cap Q_2} x_2 v_2^{Q_2 \cap Q_3} \cdots v_{k-1}^{Q_{k-1} \cap Q_k} x_k \right)$$

$$= \delta_Q^2 \, \mathrm{tr}_{M_k} \left(y_k^* (v_{k-1}^{Q_{k-1} \cap Q_k})^* \cdots y_2^* E_{Q_1 \cap Q_2} (y_1^* x_1) e_{Q_1 \cap Q_2} x_2 \cdots v_{k-1}^{Q_{k-1} \cap Q_k} x_k \right)$$

$$= \delta_Q^2 \, \mathrm{tr}_{M_k} \left(y_k^* (v_{k-2}^{Q_{k-1} \cap Q_k})^* E_{k-1} \cdots y_2^* E_{Q_1 \cap Q_2} (y_1^* x_1) e_{Q_1 \cap Q_2} x_2 \cdots E_{k-1} v_{k-2}^{Q_{k-1} \cap Q_k} x_k \right)$$

$$= \frac{\delta_Q^2}{[M : N]^{1/2}} \cdot$$

$$\mathrm{tr}_{M_k} \left(y_k^* (v_{k-2}^{Q_{k-1} \cap Q_k})^* \cdots y_2^* E_{Q_1 \cap Q_2} (y_1^* x_1) E_{k-1} \cdots E_2 \sum_{i=1}^{m} (b_i E_1 b_1^*) E_2 \cdots E_{k-1} x_2 \cdots v_{k-2}^{Q_{k-1} \cap Q_k} x_k \right)$$

$$= \frac{\delta_Q^2 [Q_1 \cap Q_2 : N]}{[M : N]^{1/2}} \cdot \mathrm{tr}_{M_k} \left(y_k^* (v_{k-2}^{Q_{k-1} \cap Q_k})^* \cdots y_2^* E_{Q_1 \cap Q_2} (y_1^* x_1) E_{k-1} x_2 \cdots v_{k-2}^{Q_{k-1} \cap Q_k} x_k \right)$$

$$= \mathrm{tr}_{M_{n-1}} \left(y_k^* (v_{k-2}^{Q_{k-1} \cap Q_k})^* \cdots y_2^* E_{Q_1 \cap Q_2} (y_1^* x_1) x_2 \cdots v_{k-2}^{Q_{k-1} \cap Q_k} x_k \right)$$

$$= \left\langle E_{Q_1 \cap Q_2} (y_1^* x_1) x_2 \underset{Q_2 \cap Q_3}{\otimes} \cdots \underset{Q_{k-1} \cap Q_k}{\otimes} x_k, \ y_2 \underset{Q_2 \cap Q_3}{\otimes} \cdots \underset{Q_{k-1} \cap Q_k}{\otimes} y_k \right\rangle$$

$$= \left\langle x_1 \underset{Q_1 \cap Q_2}{\otimes} x_2 \underset{Q_2 \cap Q_3}{\otimes} \cdots \underset{Q_{k-1} \cap Q_k}{\otimes} x_k, \ y_1 \underset{Q_1 \cap Q_2}{\otimes} y_2 \underset{Q_2 \cap Q_3}{\otimes} \cdots \underset{Q_{k-1} \cap Q_k}{\otimes} y_k \right\rangle$$

as desired. □

The map ϕ above is clearly a bimodule map, so central vectors get mapped into $N' \cap L^2(M_k)$. Since $N' \cap L^2(M_k) = N' \cap M_k$ is finite dimensional, it follows that each P_α is finite dimensional.

Action of $N - P - M$ tangles on P_α

We now describe how the $N - P - M$ planar operad acts on the various P_α. Given an $N - P - M$ tangle T, we isotope it so that it is in standard form. This means:

1. All of the input and output disks are rectangles and all strings (that are not closed loops) emanate from the top of the rectangles.

2. All the input disks are in different horizontal bands and all maxima and minima of strings are at different vertical levels, and not in the horizontal bands defined by the input disks.

3. The starred intervals of the input disks are all at the bottom-left corner. When we have a diagram of this form, the $\star$ is omitted.

One then positions an imaginary horizontal line at the bottom of the tangle, T, and then slides it to the top. One starts with the central vector $1_Q \in L^2(Q)$ whenever the bottom of the box is shaded Q. The central vector gets altered as the line crosses either an input box, a maximum on

a string, or a minimum on a string. When the line reaches the top, you get the central vector produced by the action of the tangle.

Suppose the horizontal line passes through the ith rectangle (with respect to the isotopy) in the m_i^{th} region which is shaded Q_{m_i} (reading left to right along the line), and suppose that the vector v_i has been assigned to the box. We simply insert v_i into the m_i^{th} slot, i.e.

$$\sum_j x_1^j \otimes \cdots \otimes x_{m_i}^j \otimes \cdots \otimes x_n \mapsto \sum_j x_1^j \otimes \cdots \otimes x_{m_i}^j v_i \otimes \cdots \otimes x_n$$

$$= \sum_j x_1^j \otimes \cdots \otimes v_i x_{m_i}^j \otimes \cdots \otimes x_n$$

Now suppose the horizontal line passes through a minimum, and suppose $Y \subset X$ with $X, Y \in \{N, P, M\}$ and X and Y resemble the regions on either side of the minimum. Let $B_{X,Y}$ be a Pimsner-Popa basis for X over Y. Then we have the diagrammatic rules:

$$\boxed{\begin{array}{c} \fbox{Y} \\ \hline X \end{array}} \to \boxed{\begin{array}{c} \fbox{Y} \\ X \end{array}} \quad x \mapsto \frac{1}{[X:Y]^{1/2}} \sum_{b \in B_{X,Y}} xb \underset{Y}{\otimes} 1_Y \underset{Y}{\otimes} b^* = \sum_{b \in B_{X,Y}} b \underset{Y}{\otimes} 1 \underset{Y}{\otimes} b^* x$$

$$\boxed{\begin{array}{c} \fbox{X} \\ \hline Y \end{array}} \to \boxed{\begin{array}{c} \fbox{X} \\ Y \end{array}} \quad x \mapsto x \underset{Y}{\otimes} 1_X \underset{Y}{\otimes} 1_Y = 1_Y \underset{Y}{\otimes} 1_X \underset{Y}{\otimes} x$$

Whenever a dotted line passes over a maximum, the following rules apply:

$$\boxed{\begin{array}{c} Y \\ \fbox{X} \end{array}} \to \boxed{\begin{array}{c} Y \\ \fbox{X} \end{array}} \quad y_1 \underset{Y}{\otimes} x \underset{Y}{\otimes} y_2 \mapsto [X:Y]^{1/2} y_1 E_Y(x) y_2$$

$$\boxed{\begin{array}{c} X \\ \fbox{Y} \end{array}} \to \boxed{\begin{array}{c} X \\ \fbox{Y} \end{array}} \quad x_1 \underset{Y}{\otimes} y \underset{Y}{\otimes} x_2 \mapsto x_1 y x_2$$

Here is an example of a tangle acting on $y_1 \underset{Y}{\otimes} x \underset{Y}{\otimes} y_2$:

$$\boxed{\begin{array}{c} Y \\ X \end{array}} \quad \left(y_1 \underset{Y}{\otimes} x \underset{Y}{\otimes} y_2 \right) = \frac{1}{[X:Y]^{1/2}} \cdot \sum_{b \in B_{X,Y}} b y_1 x \underset{Y}{\otimes} y_2 \underset{Y}{\otimes} b^*$$

Note also that our rules dictate that a loop with an X on one side and Y on the other counts for a factor $[X:Y]^{1/2}$. As $[M:P]$ and $[P:N]$ are the only two such indices that will appear, we will let $\delta_a = [P:N]^{1/2}$ and $\delta_b = [M:P]^{1/2}$.

It is a straightforward check to see that each of these maps preserves central vectors. Each map is also locally a bimodule map, hence the action of T will also preserve invariant elements.

Checking that T is well defined up to isotopy involves checking the same (finite number of) relations as in [Jon]. For example, checking

$$\boxed{\begin{array}{c} Y \\ X \end{array}} = \boxed{Y \, | \, X}$$

boils down to checking the relation $x = \sum_{b \in B_{X,Y}} E_Y(xb)b^*$, which always holds. The key to checking that the action of T is defined up to isotopy is to show that rotation by 2π is the identity.

Let $x = \sum_j x_1^j \underset{Q_1 \cap Q_2}{\otimes} x_2^j \cdots \underset{Q_n \cap Q_1}{\otimes} x_{n+1}^j \in P_\alpha$, and let T be the tangle which is rotation by one-click clockwise, namely

$$T = \boxed{}.$$
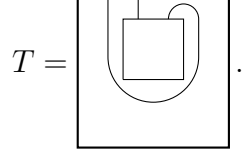
where there are $n-1$ strings that are not bent. By definition, we have:

$$Z(T)(x) = \begin{cases} \frac{1}{[Q_n:Q_1]^{1/2}} \sum_{b \in B(Q_n,Q_1)} \sum_j b \underset{Q_1}{\otimes} x_1 \underset{Q_1 \cap Q_2}{\otimes} x_2 \cdots \underset{Q_{n-1} \cap Q_n}{\otimes} x_n x_{n+1} b^* & \text{if } Q_1 \subset Q_n \\[2mm] [Q_1 : Q_n]^{1/2} \sum_j 1 \underset{Q_n}{\otimes} x_1 \underset{Q_1 \cap Q_2}{\otimes} x_2 \cdots \underset{Q_{n-1} \cap Q_n}{\otimes} x_n E_{Q_n}(x_{n+1}) & \text{if } Q_n \subset Q_1 \end{cases}$$

To help our computations, we define the following left and right creation operators, L_x and R_x for $x \in Q$. These are given by:

$$L_x : Z_\alpha \to L^2(Q) \underset{Q \cap Q_1}{\otimes} Z_\alpha \text{ such that } L_x(x_1 \otimes \cdots \otimes x_{n+1}) = x \otimes x_1 \otimes \cdots \otimes x_{n+1}$$

$$R_x : Z_\alpha \to Z_\alpha \underset{Q \cap Q_1}{\otimes} L^2(Q) \text{ such that } R_x(x_1 \otimes \cdots \otimes x_{n+1}) = x_1 \otimes \cdots \otimes x_{n+1} \otimes x$$

It follows from the definition of the bimodule tensor product that

$$L_x^*(x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1}) = E_{Q \cap Q_1}(x^* x_0) x_1 \otimes \cdots \otimes x_{n+1} \text{ and}$$
$$R_x^*(\otimes x_1 \otimes \cdots \otimes x_{n+1} \otimes y) = x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} E_{Q_1 \cap Q_0}(yx^*)$$

Therefore, we have the following formulae the rotation tangle, T:

$$Z(T)(x) = \begin{cases} \frac{1}{[Q_n:Q_1]^{1/2}} \sum_{b \in B} L_b R_b^*(x) & \text{if } Q_1 \subset Q_n \\[2mm] [Q_1 : Q_n]^{1/2} L_1 R_1^*(x) & \text{if } Q_n \subset Q_1 \end{cases}$$

From Burns' rotation trick [Bur03] we have the following lemma which is similar to lemmas that appear in [JP11]:

Lemma 4.3.3. *Let $\rho(\alpha)$ be the word formed when the words in α are cyclically permuted clockwise by one, and let $y = y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \in Z_{\rho(\alpha)}$. Then*

- $\langle T(x), y \rangle = \dfrac{1}{[Q_n : Q_1]^{1/2}} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle$ *if $Q_1 \subset Q_n$*

- $\langle T(x), y \rangle = [Q_1 : Q_n]^{1/2} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle$ *if $Q_n \subset Q_1$*

Proof. For the first case, using that x is central, we have

$$\langle T(x), y \rangle = \frac{1}{[Q_n : Q_1]^{1/2}} \langle \sum_{b \in B} L_b R_b^*(x), y \rangle = \sum_{b \in B} \frac{1}{[Q_n : Q_1]^{1/2}} \langle x, R_b L_b^*(y) \rangle$$

$$= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle x, E_{Q_1}(b^* y_1) y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b \rangle$$

$$= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle (E_{Q_1}(b^* y_1))^* x, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b \rangle$$

$$= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle x(E_{Q_1}(b^* y_1))^*, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b \rangle$$

$$= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle x, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b E_{Q_1}(b^* y_1) \rangle$$

$$= \frac{1}{[Q_n : Q_1]^{1/2}} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle.$$

For the second case, we have

$$\langle T(x), y \rangle = [Q_1 : Q_n]^{1/2} \langle L_1 R_1^*(x), y \rangle = [Q_1 : Q_n]^{1/2} \langle x, R_1 L_1^*(y) \rangle$$

$$= [Q_1 : Q_n]^{1/2} \langle x, E_{Q_1}(y_1) y_2 \otimes \cdots \otimes y_{n+1} \otimes 1 \rangle$$

$$= [Q_1 : Q_n]^{1/2} \langle x, y_1 y_2 \otimes \cdots \otimes y_{n+1} \otimes 1 \rangle$$

$$= [Q_1 : Q_n]^{1/2} \langle y_1^* x, y_2 \otimes \cdots \otimes y_{n+1} \otimes 1 \rangle$$

$$= [Q_1 : Q_n]^{1/2} \langle x y_1^*, y_2 \otimes \cdots \otimes y_{n+1} \otimes 1 \rangle$$

$$= [Q_1 : Q_n]^{1/2} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle$$

as desired. $\qquad \square$

Corollary 4.3.4. *Rotation by 2π is the identity.*

Proof. The nature of the shading dictates that all index factors cancel when applying the 2π rotation. The rest follows from the previous lemma. $\qquad \square$

One can now continue as in [Jon].

Realizing $\mathcal{Q}$ inside $\mathcal{P}$

Suppose $\mathcal{Q}$ is a planar algebra containing the Fuss-Catalan algebra, so that $\mathcal{Q}$ is the planar algebra for a finite index inclusion $N \subset M$ with intermediate subfactor P. We note that $Q_{2n,+}$ is the space of $N - N$ central vectors of

$$({}_N L^2(M)_N)^{\otimes^n_N} = ({}_N L^2(M)_M \underset{M}{\otimes} {}_M L^2(M)_N)^{\otimes^n_N}.$$

and $Q_{2n,-}$ is the space of $M - M$ central vectors of $({}_M L^2(M)_N \underset{N}{\otimes} {}_N L^2(M)_M)^{\otimes^n_M}$. Since

$$_N L^2(M)_M =_N L^2(P)_P \underset{P}{\otimes} {}_P L^2(M)_M,$$

it follows that

$$Q_{2n,+} = P_{(abba)^{n/2}} \text{ and } Q_{2n,-} = P_{(baab)^{n/2}}.$$

Furthermore, a tangle S that acts on $\mathcal{Q}$ can be made into an $N - P - M$ tangle S' by replacing each string with an a string cabled to a b string. This is done so that the shaded regions in S become the $M-$regions in S' and the unshaded regions in S become the $N-$regions in S'. Notice that this implies that the strings along any disk in S' read (clockwise from the marked region) as *abbaabbaa...* or *baabbaabb....* It directly follows from the definitions that if $x_1, ..., x_n$ are in $\mathcal{Q}$ and S is as above, then $Z(S)(f) = Z(S')(f)$ where the left hand side denotes the action of a shaded tangle and the right hand side denotes the action of an $N - P - M$ tangle.

Definition 4.3.5. If $\mathcal{Q}$ is the planar algebra of an inclusion $N \subset M$ of finite index II_1 factors with an intermediate subfactor, P, then we define the $\mathcal{P}$ constructed as above as the $\underline{P-\text{augmentation of } \mathcal{Q}}$.

4.4 The GJS construction for the $\sum FC$ potential

Suppose $\mathcal{Q}$ is a subfactor planar algebra containing a copy of the Fuss Catalan planar algebra. For each $k \geq 0$, we study the graded algebra $\text{Gr}_k^{\pm}(\mathcal{Q})$ as above, and place the following trace on $\text{Gr}_k^{\pm}(\mathcal{Q})$:

$$\text{tr}(x) = \frac{1}{(\delta_a \delta_b)^k} \quad \boxed{\begin{array}{c} \boxed{\sum FC} \\ | \\ \boxed{x} \\ \underset{k}{\smile} \end{array}}.$$

where the shading on the *upper left* corner is $\pm$ (therefore, the shading in the starred region depends on k).

As in Section 4.3, we realize $\mathcal{Q}$ inside an augmentation, $\mathcal{P}$, and we consider the algebras $\text{Gr}_\alpha(\mathcal{P})$ where $\overline{\alpha}\alpha \in \Delta$. If $\alpha \neq \emptyset$ then the shading after the last letter of α is uniquely determined and hence we can write,

$$\text{Gr}_\alpha(\mathcal{P}) = \oplus_{(\beta : \overline{\alpha}\beta\alpha \in \Delta)} \mathcal{P}_{\overline{\alpha}\beta\alpha}.$$

$\text{Gr}_\alpha(\mathcal{P})$ is endowed with a multiplication $\wedge$ given by

$$x \wedge y = \overset{\beta}{\underset{\alpha}{\boxed{x}}} \overset{\gamma}{\underset{\alpha}{\boxed{y}}} \alpha$$

and normalized trace

$$\text{tr}(x) = \frac{1}{\delta^\alpha} \cdot \quad \boxed{\begin{array}{c} \boxed{\Sigma CTL} \\ \beta | \\ \alpha \boxed{x} \alpha \\ \underset{\overline{\alpha}}{\smile} \end{array}}$$

where, as in Chapter 3, $\sum CTL$ is the sum of all colored Temperley-Lieb diagrams. In the case where $\alpha = \emptyset$, then we have three such algebras, one for each shading N, P, and M. We therefore form graded algebras Gr_0^N, Gr_0^P and Gr_0^M where

$$\text{Gr}_0^Q = \bigoplus_{(\beta : \beta \in \Delta^Q)} \mathcal{P}_\beta.$$

Using the graded algebras associated to $\mathcal{P}$, we see that we have the following trace preserving inclusions:

$$\mathrm{Gr}_{2k}^{+}(\mathcal{Q}) \subset \mathrm{Gr}_{abba\cdots abba}(\mathcal{P})$$
$$\mathrm{Gr}_{2k+1}^{+}(\mathcal{Q}) \subset \mathrm{Gr}_{abba\cdots baab}(\mathcal{P})$$
$$\mathrm{Gr}_{2k}^{-}(\mathcal{Q}) \subset \mathrm{Gr}_{baab\cdots baab}(\mathcal{P})$$
$$\mathrm{Gr}_{2k+1}^{-}(\mathcal{Q}) \subset \mathrm{Gr}_{baab\cdots abba}(\mathcal{P}).$$

One advantage to working in the $N - P - M$ planar algebra $\mathcal{P}$ is that the map $\Phi : \mathrm{Gr}_{\alpha}(\mathcal{P}) \to \mathrm{Gr}_{\alpha}(\mathcal{P})$ given by

$$\Phi(x) = \sum_{E \in \mathrm{Epi}(CTL)} \boxed{\begin{array}{c} E \\ \hline x \end{array}}$$

as in Section 3.3 of the previous chapter is a well defined trace preserving isomorphism between $\mathrm{Gr}_{\alpha}(\mathcal{P})$ with the $\sum CTL$ trace and $\mathrm{Gr}_{\alpha}(\mathcal{P})$ with the orthogonalized trace. Therefore, we have proven the following lemma:

Lemma 4.4.1. *The potential $\sum CTL$ gives a positive definite trace on $\mathrm{Gr}_{\alpha}(\mathcal{P})$.*

Furthermore, by considering either the a or $b \cup$ element, the same analysis as in Section 3.3 of the previous chapter proves the following theorem:

Theorem 4.4.2. *Left (and right) multiplication of elements of $Gr_{\alpha}(\mathcal{P})$ on $L^2(Gr_{\alpha}(\mathcal{P}))$ is bounded and the associated von Neumann algebra, $M_k = Gr_{\alpha}(\mathcal{P})''$ is a II_1 factor.*

We note that if γ and β are words in a and b such that $\overline{\beta}\beta \in \Delta$ and $\overline{\beta\gamma}\beta\gamma \in \Delta$ then we have a unital inclusion of M_{γ} into $M_{\beta\gamma}$ given the extension of

$$x \mapsto \begin{array}{c} \beta \boxed{x} \beta \\ \hline \gamma \end{array} .$$

We therefore have the following theorem, whose proof is exactly the same as in Section 3.3 of the previous chapter.

Theorem 4.4.3. *The following is a Jones' tower of II_1 factors:*

$$M_0^{Q} \subset M_{\alpha} \subset M_{\alpha\overline{\alpha}} \subset \cdots \subset M_{(\alpha\overline{\alpha})^n} \subset M_{\overline{\alpha}(\alpha\overline{\alpha})^n}.$$

Furthermore, $[M_{\alpha} : M_0^{Q}] = \delta_{\alpha}$, and the Jones projection for $M_0^{Q} \subset M_{\alpha}$ is

$$e_0 = \frac{1}{\delta_{\alpha}} \boxed{a} \; \boxed{a}$$

4.5 A semifinite algebra associated to $\mathcal{P}$

As in section 3.4 of the previous chapter, we will realize the isomorphism class of all of the M_α by examining realizing them as "corners" of a semifinite algebra. To begin, we consider the set semifinite algebra Gr^Q_∞ for $Q = N, P, M$. As a vector space,

$$\mathrm{Gr}^Q_\infty = \bigoplus_{\substack{\alpha,\gamma,\beta \\ \overline{\alpha}\gamma\beta \in \Delta,\, \alpha \in \Delta^Q}} \mathcal{P}_{\overline{\alpha}\gamma\beta}.$$

Pictorially, we realize elements in Gr^Q_∞ as linear combinations of boxes of the form

where the *bottom* (starred) region is shaded Q (thus the top left corner varies in shading). Gr^Q_∞ comes endowed with the following multiplication:

and semifinite trace, Tr, which is given by:

$$\mathrm{Tr}(x) = $$

if $x \in \mathrm{Gr}(\mathcal{P}_\alpha)$ and is zero otherwise. Just as in the analysis of Section 3.4 from the previous chapter, we see that Gr^Q_∞ completes to a II_∞ factor, $\mathcal{M}^Q_\infty$ when being represented on $L^2(\mathrm{Gr}^Q_\infty)$.

Also of importance will be the von Neumann subalgebra $\mathcal{A}^Q_\infty \subset \mathcal{M}^Q_\infty$ which is generated by all boxes in $\mathcal{G}^Q_\infty$ with <u>no</u> strings on top. Notice that there is a normal, faithful, Tr-preserving conditional expectation $E : \mathcal{M}^Q_\infty \to \mathcal{A}^Q_\infty$ given by

$$E(x) = $$

Furthermore, we have the following lemma, whose proof is identical to that of Lemma 3.4.6 of the previous chapter:

Lemma 4.5.1. $\mathcal{A}^Q_\infty = \bigoplus_{v \in \Gamma_Q} \mathcal{A}_v$ *where* Γ_Q *is the* $Q-$*principal graph of* $\mathcal{P}$ *and each* $\mathcal{A}_v$ *is a type* I_∞ *factor.*

We now aim to figure out the isomorphism class of the algebras $\mathcal{M}_\alpha$. For the remainder of the section, we will assume for simplicity that $Q = N$ and hence we will be finding the isomorphism class of M_α such that $s(\overline{\alpha}) = N$. The other two cases will follow a similar analysis. We first define elements X_a and X_b as follows:

$$X_a = \sum_{\substack{\alpha \\ \alpha, \overline{\alpha} \in \Delta^N}} \boxed{\overset{\fbox{a}}{}\; \alpha} + \boxed{\alpha \;\overset{\fbox{a}}{}} \quad \text{and} \quad X_b = \sum_{\substack{\beta \\ \beta \in \Delta^P\, \overline{\beta} \in N, s(\overline{\beta}) = a}} \boxed{\overset{\fbox{b}}{}\; \beta} + \boxed{\beta \;\overset{\fbox{b}}{}} .$$

As in Section 3.4 of the previous chapter, X_a and X_b are sums of orthogonally supported operators, and each summand has uniformly bounded operator norm. Therefore, $X_a, X_b \in \mathcal{M}_\infty^N$. Furthermore, we have the following lemma:

Lemma 4.5.2. $\mathcal{M}_\infty^N$ *is generated as a von Neumann algebra by* $(\mathcal{A}_\infty^N, X_a, X_b)$.

Proof. As in the proof of Lemma 3.4.9 from Chapter 3, all that needs to be shown is that the following diagrams lie in the von Neumann algebra generated by $(\mathcal{A}_\infty^N, X_a, X_b)$:

$$\boxed{\overset{\fbox{N}}{}\alpha} \quad \text{and} \quad \boxed{\overset{\fbox{P}}{}\beta}$$

If $s(\alpha) = a$, then the exact same method as in the proof of Lemma 3.4.9 from Chapter 3 shows that this element is in the algebra. if $s(\alpha) = b$, the following multiplication produces the diagram:

$$\frac{1}{\delta_a} \cdot \boxed{\alpha \;\cdot} \cdot \boxed{} \cdot \boxed{\cdot\; \alpha}.$$

A similar argument works for the element

$$\boxed{\overset{\fbox{P}}{}\beta}.$$

$\square$

Much as in the previous chapter, we also have the following lemma:

Lemma 4.5.3. X_a *and* X_b *are free with amalgamation over* $\mathcal{A}_\infty^N$ *with respect to the conditional expectation,* E.

We now define maps η_a and η_b on $\mathcal{A}_\infty^N$ as follows:

$$\eta_c(y) = E(X_c y X_c)$$

for $c = a$ or b. Notice that by definition of E, η_c is a completely positive map of $\mathcal{A}_I^N$ into itself. Furthermore it is a straightforward inductive check to note that the formula

$$E(y_0 X_c y_1 X_c \cdots y_{n-1} X_c y_n)$$
$$= \sum_{k=2}^n y_0 \cdot \eta_c(E(y_1 X_c \cdots X_c y_{k-1})) \cdot E(y_k X_c ... X_c y_n),$$

as in Section 3.4 of the previous chapter holds. Pictorially, there are nice expressions for $\eta_c(w)$ for various choices of w. To begin, for $i = 1, 2$, let α_i be a word with $\alpha_i \in \Delta^N$, β_i be a word with $s(\beta_i) = a$ and $\beta_i \in \Delta^P$, and γ_i be a word such that $\gamma_i \in \Delta^M$. Furthermore, suppose $x \in \mathcal{P}_{\overline{\alpha_1}\alpha_2}$, $y \in \mathcal{P}_{\overline{\beta_1}\beta_2}$ and $z \in \mathcal{P}_{\overline{\gamma_1}\gamma_2}$. We then have the following easily verifiable formulae:

$$\eta_a(x) = \overline{\boxed{{}^{\alpha_1}\,x\,^{\alpha_2}}} \qquad \eta_a(y) = {}^{\beta_1}\overparen{\boxed{y}}{}^{\beta_2} \qquad \eta_a(z) = 0$$

$$\eta_b(x) = 0 \qquad \eta_b(y) = \overline{{}^{\beta_1}\boxed{y}\,^{\beta_2}} \qquad \eta_b(z) = {}^{\gamma_1}\overparen{\boxed{z}}{}^{\gamma_2}$$

With the pictures above, the following useful lemma is easily verified:

Lemma 4.5.4. *Let α_i be a word with $s(\alpha_i) = N$ and β_i be a word with $s(\beta_i) = P$ for $i = 1$ or 2. In addition, suppose $x \in \mathcal{P}_{\overline{\alpha_1}\alpha_2}$ and $y \in \mathcal{P}_{\overline{\beta_1}\beta_2}$. We have the following formulae:*

$$x \cdot X_a = X_a \cdot \eta_a(x) \text{ and } y \cdot X_b = X_b \cdot \eta_b(y).$$

This lemma will be used to help describe certain compressions of $\mathcal{M}^N_\infty$.

A suitable compression of $\mathcal{M}^N_\infty$

To begin, it will be useful to define three projections in $\mathcal{A}^N_\infty$

$$1_{\mathcal{A}^N_N} = \sum_{\alpha \in \Delta^N} \boxed{{}^{\alpha}} \qquad 1_{\mathcal{A}^N_{P_a}} = \sum_{\alpha \in \Delta^N} \boxed{\begin{matrix}a\\\alpha\end{matrix}} \qquad 1_{\mathcal{A}^N_{M_a}} = \sum_{\alpha \in \Delta^N} \boxed{{}^{\alpha}}.$$

Note that $1_{\mathcal{A}^N_N} + 1_{\mathcal{A}^N_{P_a}} + 1_{\mathcal{A}^N_{M_a}}$ is the smallest projection dominating the support projections of X_a and X_b.

Our goal, much as in Section 3.4 of the previous chapter, is to better understand what happens when $\mathcal{M}^N_\infty$ is compressed by certain projections. To begin our study, we consider Γ_N, the $N-$principal graph of $\mathcal{P}$. For each vertex, v, at the $N-N$ level of the graph, we choose a minimal projection $p_v \in \mathcal{A}_\infty$, and for the vertex, $*$, we choose the empty $N-$shaded diagram. Notice that for each v we can choose $p_v \in \mathcal{P}_{\overline{\alpha}\alpha}$ for $\alpha \in \Delta^N$.

By the definition of the principal graph, we know that there exists a countable index set, I and partial isometries $(V_i)_{i \in I} \subset \mathcal{A}^N_\infty$ such that

$$V_i V_i^* = \sum_{v \in \Gamma^N_N} p_v \ \forall i \text{ and } \sum_{i \in I} V_i^* V_i = 1_{\mathcal{A}^N_N}.$$

This necessarily implies that

$$\sum_{i \in I} \eta_a(V_i)^* \eta_a(V_i) = 1_{\mathcal{A}^N_{P_a}} \text{ and } \sum_{i \in I} \eta_b(\eta_a(V_i))^* \eta_b(\eta_a(V_i)) = 1_{\mathcal{A}^N_{M_a}}$$

We define R^1 by the following formula:

$$R^1 = \sum_{v \in \Gamma^N_N} (p_v + \eta_a(p_v) + \eta_b(\eta_a(p_v))).$$

If we set $Z_i = V_i + \eta_a(V_i) + \eta_b(\eta_a(V_i))$ then

$$Z_i Z_i^* = R^1 \text{ and } \sum_{i \in I} Z_i^* Z_i = 1_{\mathcal{A}_N^N} + 1_{\mathcal{A}_{P_a}^N} + 1_{\mathcal{A}_{M_a}^N}.$$

We have the following lemma regarding compression of $\mathcal{M}_\infty^N$ by R^1.

Lemma 4.5.5. *As a von Neumann algebra, $R^1 \mathcal{M}_\infty^N R^1$ is generated by $R^1 \mathcal{A}_I^N R^1$, $R^1 X_a R^1$, and $R^1 X_b R^1$.*

Proof. Note that $\sum_{i,j \in I} Z_i^* X_c Z_j = X_c$ for $c = a$ or b by repeated applications of Lemma 4.5.4. Therefore, every word involving X_a or X_b and elements $x \in \mathcal{A}_\infty$ (whose ending letters are supported under R^1) can be replaced by sums of words involving terms of the form $R^1 X_c R^1$ and $R^1 x R^1$ by inserting the relation

$$\sum_{i \in I} Z_i^* Z_i = 1_{\mathcal{A}_N^N} + 1_{\mathcal{A}_{P_a}^N} + 1_{\mathcal{A}_{M_a}^N}.$$

between every letter of the word. $\qquad \square$

We now investigate the action of compressing $R^1 \mathcal{M}_\infty^N R^1$ by subprojections of R^1. To begin, for each vertex $w \in \Gamma_N^P$, let p_w be a minimal projection in $\mathcal{A}_I^N$ corresponding to the vertex w such that $p_w \leq \sum_{v \in V} \eta_a(p_v)$. For each edge e connecting a vertex in Γ_N^N to a vertex in Γ_N^P, we let $s(e)$ and $t(e)$ be the vertices in Γ_N^N and Γ_N^P respectively which e connects. We define partial isometries $\omega_e \in R^1 \mathcal{A}_\infty^N R^1$ such that

$$\omega_e^* \omega_{e'} = \delta_{e,e'} p_{t(e)} \text{ and } \sum_{s(e)=v} \omega_e \omega_e^* = \eta_a(p_v).$$

Once the p_w have been chosen, for each vertex $u \in \Gamma_N^M$, choose a minimal projection p_u corresponding to u such that $p_u \leq \sum_{w \in \Gamma_N^P} \eta_b(p_w)$.

For each edge, f, connecting the $N - P$ vertices to the $N - M$ vertices, $s(f)$ and $t(f)$ be the vertices in Γ_N^P and Γ_N^M respectively which f connects. We define partial isometries ν_f satisfying:

$$\nu_f^* \nu_{f'} = \delta_{f,f'} p_{t(f)} \text{ and } \sum_{s(f)=w} \nu_f \nu_f^* = \eta_a(p_w)$$

We now define operators $X_a^e \, X_b^f$ by the formulae

$$X_a^e = p_{s(e)} X_a \omega_e + \omega_e^* X_a p_{s(e)} \text{ and } X_b^f = p_{s(f)} X_b \nu_f + \nu_e^* X_b.$$

We have the following lemma, whose proof is the same as the arguments of Section 3.4 of Chapter 3, except easier as there are no loops on the principal graph and only one minimal projection for each vertex.

Lemma 4.5.6. *Set $R = \sum_{v \in \Gamma} p_v$. Then $R \mathcal{M}_\infty^N R$ is generated by $R \mathcal{A}_\infty^N R$ and the elements $X_a^e \, X_b^f$ for all e and f, and each of the elements are free with amalgamation over $R \mathcal{A}_\infty^N R$ with respect to E.*

Note that the algebra $R\mathcal{A}_\infty^N R$ is simply the bounded functions on the vertices of Γ, and the element $p_{s(e)} X_a \omega_e$ has left support under $p_{s(e)}$ and right support under $p_{t(e)}$. Furthermore, if $\mathrm{Tr}(p_{s(e)}) \geq \mathrm{Tr}(p_{t(e)})$ then the analysis in the appendix of Chapter 3 shows that $(p_{s(e)} X_a \omega_e)^* p_{s(e)} X_a \omega_e$ is a free poisson element with absolutely continuous spectrum in $p_{t(e)} \mathcal{M}_\infty^N p_{t(e)}$. If $\mathrm{Tr}(p_{s(e)}) \leq \mathrm{Tr}(p_{t(e)})$, then $p_{s(e)} X_a \omega_e (p_{s(e)} X_a \omega_e)^*$ is a free poisson element with absolutely continuous spectrum in $p_{s(e)} \mathcal{M}_N^\infty p_{s(e)}$. Analogous statements hold for the elements $p_{s(f)} X_b \nu_f$.

An amalgamated free product representation for $R\mathcal{M}_N^\infty R$

The work of the previous section shows that

$$R\mathcal{M}_\infty^N R = \mathcal{N}(\Gamma_N)$$

with $\mathcal{N}(\Gamma)$ as in the previous two chapters. We use this to obtain a formula for M_0 when Γ_N is finite. Let g be an edge in Γ_N connecting v and w with $\mathrm{Tr}(p_v) \geq \mathrm{Tr}(p_w)$. The basic rules for computing free dimension show that

$$\mathrm{fdim}(R\mathcal{M}_e R) = 1 - \frac{(\mathrm{Tr}(p_v) - \mathrm{Tr}(p_w))^2 - \sum_{u \neq v,w} \mathrm{Tr}(p_u)^2}{\mathrm{Tr}(R)^2} = 1 - \frac{\sum_{u \in \Gamma} \mathrm{Tr}(p_v)^2 - 2\,\mathrm{Tr}(p_v)\,\mathrm{Tr}(p_w)}{\mathrm{Tr}(R)^2}.$$

Using the additivity of free dimension, as well as

$$\mathrm{fdim}(\ell^\infty(\Gamma)) = 1 - \frac{\sum_{u \in \Gamma} \mathrm{Tr}(p_v)^2}{\mathrm{Tr}(R)^2},$$

we obtain

$$\mathrm{fdim}(R\mathcal{M}_\infty^N R) = 1 + \frac{-\sum_{u \in \Gamma} \mathrm{Tr}(p_v)^2 + 2 \sum_{g \in E(\Gamma_N)} \mathrm{Tr}(p_{s(g)})\,\mathrm{Tr}(p_{t(g)})}{\mathrm{Tr}(R)^2}$$

$$= 1 + \frac{\sum_{u \in \gamma} \mathrm{Tr}(p_u) \sum_{v \sim u}(\mathrm{Tr}(p_v) - \mathrm{Tr}(p_u))}{\mathrm{Tr}(R)^2}.$$

Using the Perron-Frobenius condition, this becomes

$$\mathrm{fdim}(R\mathcal{M}_\infty^N R) = 1 + \frac{2I((\delta_a - 1) + (\delta_b - 1))}{\mathrm{Tr}(R)^2}$$

Where $I = \sum_{v \in \Gamma_N^N} \mathrm{tr}(p_v)^2 (= \sum_{w \in \Gamma_N^P} \mathrm{tr}(p_w)^2 = \sum_{u \in \Gamma_N^M} \mathrm{tr}(p_u)^2)$. Therefore, $R\mathcal{M}_\infty^N R$ is an interpolated free group factor with the above parameter. The compression formula for free group factors proves the following lemma

Lemma 4.5.7. $M_0^N \cong L(\mathbb{F}_t)$ where $t = 1 + 2I(\delta_a + \delta_b - 2)$.

This gives us the following corollary:

Corollary 4.5.8. *The factors M_α have the formula*

$$M_\alpha \cong L(\mathbb{F}(1 + 2I\delta_\alpha^{-2}(\delta_a + \delta_b - 2)))$$

Proof. If $s(\overline{\alpha}) = N$, then it follows from the semifinte algebra $\mathcal{M}_\infty^N$ that M_α is a δ_α amplification of M_0^N. If the shading is different, apply similar analysis to the semifinite algebras $\mathcal{M}_\infty^P$ and $\mathcal{M}_\infty^M$. $\square$

We now handle the case where $\mathcal{P}$ is infinite depth:

Lemma 4.5.9. *If $\mathcal{P}$ is infinite-depth, then $\mathcal{M}_0^N \cong L(\mathbb{F}_\infty)$, and hence $\mathcal{M}_\alpha \cong L(\mathbb{F}_\infty)$ for all α*

Proof. Let Γ_k be the graph of Γ_N truncated up to depth k as in the proof of Theorem 2.4.2 in Chapter 2. As in that proof, we let

$$B(\Gamma_k) = \{v \in \Gamma_k; \operatorname{Tr}(p_v) > \sum_{w \sim v \in \Gamma_k} n_{v,w} \operatorname{Tr}(p_w)\}$$

where $n_{v,w}$ is the number of edges that connect v and w. We note that by the Perron Frobenius condition, no vertices in Γ_{k-1} are in $B(\Gamma_k)$. Following the proof of Theorem 2.4.2 in Chapter 2 step-by-step, we arrive at the formula

$$\operatorname{fdim}(M_0^N) \geq 1 + (\delta_a - 1) \sum_{\substack{v \in \Gamma_{k-2} \text{ and} \\ v \in \Gamma_N^N \cup \Gamma_N^P}} \operatorname{Tr}(p_v)^2 + (\delta_b - 1) \sum \sum_{\substack{v \in \Gamma_{k-2} \text{ and} \\ v \in \Gamma_N^P \cup \Gamma_N^M}} \operatorname{Tr}(p_v)^2$$

which gets arbitrarily large as k does. The standard embedding arguments of Chapter 2 show that $M_0^N \cong L(\mathbb{F}_\infty)$. We arrive at the result for the other M_α's either by amplification or by examining $\mathcal{M}(\Gamma_P)$ or $\mathcal{M}(\Gamma_M)$. $\square$

We can use this result to give a complete diagrammatic reproof of the universality result of Popa and Shlyakhtenko regarding the universality of $L(\mathbb{F}_\infty)$ in subfactor theory.

Lemma 4.5.10 ([PS03]). *Every subfactor planar algebra $\mathcal{P}'$ is the standard invariant for a finite-index inclusion $\mathcal{N} \subset \mathcal{M}$ with $\mathcal{N} \cong L(\mathbb{F}_\infty) \cong \mathcal{M}$.*

Proof. A diagrammatic proof of this fact for $\mathcal{P}'$ infinite depth was done is Chapter 2. If $\mathcal{P}'$ is finite depth, let $\mathcal{P}'$ be the planar algebra for a finite index inclusion $N' \subset P'$ of II_1 factors. Let $B \subset C$ be a finite index inclusion of II_1 factors with principal graph A_∞. We consider the inclusions

$$N' \otimes B(= N) \subset P' \otimes B(= P) \subset P' \otimes C(= M).$$

Let $\mathcal{Q}$ be the planar algebra for $N \subset M$, and $\mathcal{P}$ the augmentation of $\mathcal{Q}$ for $N \subset P \subset M$. We note that $\mathcal{P}'$ is also the planar algebra for $N \subset P$, and the planar subalgebra of $\mathcal{P}$ generated by words whose only color is a is $\mathcal{P}'$. From Theorem 4.4.3, it follows that the standard invariant of $M_\emptyset^N \subset M_a$ is $\mathcal{P}'$ and from the above calculation, $M_\emptyset^N \cong L(\mathbb{F}_\infty) \cong M_a$ $\square$

Note that Rădulescu provided the original construction of $\mathcal{N} \subset \mathcal{M}$ both isomorphic to $L(\mathbb{F}_\infty)$ having standard invariant $\mathcal{P}'$ for $\mathcal{P}'$ finite depth. [Răd94].

The law of $\cup \in N_0^+$

One pleasing feature of the semifinite algebra constructed above is that is gives one a transparent way to find equations of the spectrum of the element $\cup \in N_0$. This corresponds to the double-cup element

$$\boxed{} \in M_\emptyset^N$$

Picturing $\cup$ as living in $\mathcal{M}_I^N$, we note that $\mathrm{Tr}(\cup^n) = \mathrm{Tr}(x^n)$ where x is the element

$$x = \boxed{} = \boxed{} \cdot \boxed{}$$

which is supported by

$$1_a = \boxed{}$$

Note that x is expressed as a product of free elements $1_a \mathcal{M}_\infty^N 1_a$. Let y be the element

$$y = \boxed{}$$

and r be the element

$$r = \boxed{}.$$

We know that $\cup_a$ is distributed as a free-poisson element, and its moment generating function is

$$M_{\cup_a}(z) = \frac{-((\delta_a - 1)z - 1) + \sqrt{((\delta_a - 1)z - 1)^2 - 4z}}{2z}.$$

In the algebra $1_a \mathcal{M}_\infty^N 1_a$ with *normalized* trace, τ, we must have $\tau(y^n) = \mathrm{Tr}(\cup^n)/\delta_a$ for $n \geq 1$ and $\tau(y^0) = 1$. Therefore, the moment generating function of y is

$$M_y(z) = M_{\cup_a}(z)\delta_a + \frac{\delta_a - 1}{\delta_a}.$$

The tool we will use to calculate the moments of x is Voiculescu's $S-$transform [VDN92]. From [VDN92], it is known that is S_s and S_t are the $S-$ transforms for free elements s and t in a tracial von Neumann algebra, then

$$S_{st}(z) = S_{s^{1/2}ts^{1/2}}(z) = S_s(z)S_t(z).$$

Furthermore, to compute the $S-$transform of an element s, one finds formal power series ψ_s, χ_s and S_s satisfying:

$$\psi_s(z) = M_s(z) - 1 \quad \chi_s(\psi_s(z)) = z = \psi_s(\chi_s(z)) \quad \text{and} \quad S_s(z) = \frac{(z+1)\chi_s(z)}{z}$$

These formulas produce the following expression for the $S-$transform of x:

$$S_x(z) = \frac{(z+1)^2(z-1)(\delta_a z - 1)}{((\delta_b - 1)z + \delta_b)((\delta_a - 1)z + 1)},$$

which can be inverted to give the Cauchy transform of x. We know that the law of a single-colored $\cup$ (as in [GJS10]) is absolutely continuous with respect to Lebesgue measure and is supported away from the origin. Therefore, the law of y contains an atom of at the origin of measure $\frac{\delta_a - 1}{\delta_a}$ at the origin and is absolutely continuous away from the origin. It follows that the law of $y^{1/2} r y^{1/2}$ has an atom of measure $\frac{\delta_a - 1}{\delta_a}$ at the origin and is absolutely continuous away from the origin. Furthermore, the spectral projection corresponding to $\{0\}$ for $y^{1/2} r y^{1/2}$ must be the same as the spectral projection corresponding to $\{0\}$ for y. From this, we use the polar part of


to conclude that $\cup \in \mathcal{N}_0^+$ has law absolutely continuous to Lebesgue measure and supported away from the origin.

N_k contains a free group factor

Recall that $N_k^{\pm} = \mathrm{Gr}_k^{\pm}(\mathcal{Q})''$. We will use the moment calculation of $\cup$ above as well as similar elements arising in the semifinite algebra to find a free group factor contained in $N_k^{\pm}$. To simplify matters, we note that we need only consider the case where $\mathcal{Q}$ is Fuss Catalan, as any such planar algebra will contain the Fuss Catalan planar algebra.

By the usual amplification tricks, we need need only show that N_0^+ contains a free group factor. We embed $\mathcal{Q}$ into its augmentation $\mathcal{P}$ as in Example 4.2.5, which produces an embedding $\mathrm{Gr}_\infty^+(\mathcal{Q}) \hookrightarrow \mathrm{Gr}_\infty^N(\mathcal{P})$ where $\mathrm{Gr}_\infty^+(\mathcal{Q})$ is realized as the subalgebra of $\mathrm{Gr}_\infty^N(\mathcal{P})$ generated by the Fuss Catalan diagrams. Let $\mathcal{N}_\infty = \mathrm{Gr}_\infty^+(\mathcal{Q})$. We have the following lemma:

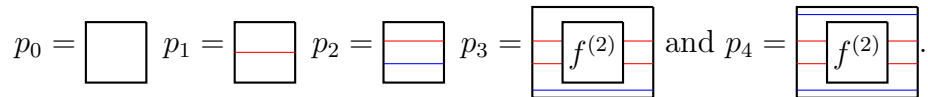**Lemma 4.5.11.** *Let p_{ab} be the following diagram*

$$p_{ab} = \;\; \text{}$$
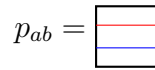
and set
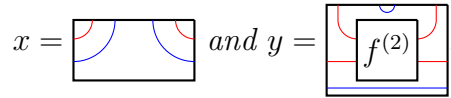
$$x = \;\; \text{} \quad and \quad y = \;\; \text{} \;.$$

with $f^{(2)}$ the second Jones-Wenzl idempotent in Temperely Lieb. Then x and y are free in $p_{ab} \mathcal{M}_\infty^N p_{ab}$.

Proof. From [BJ97], the projections

$$p_0 = \text{} \quad p_1 = \text{} \quad p_2 = \text{} \quad p_3 = \text{} \quad and \;\; p_4 = \text{} \;.$$

are inequivalent minimal projections in $\mathcal{P}$, and there exists exactly one edge e_i which goes between the vertices representing p_{i-1} and p_i. Therefore, by choosing p_0, p_2, and p_4, to line up with our choices of minimal projections lying under Q, It follows that

$$x = p_2 X_b^{e_2} X_a^{e_1} X_a^{e_1} X_b^{e_2} p_2 \text{ and } y = p_2 X_b^{e_3} X_a^{e_4} X_a^{e_4} X_b^{e_3} p_2$$

so x and y are free with amalgamation over $\mathcal{A}_\infty^N$. Since $p_{ab} = p_2$ is minimal in $\mathcal{A}_\infty^N$, the result follows. $\qquad \square$

Proof of Theorem 4.1.3. Clearly, $N_0^+ \subset M_\emptyset^N$ so N_0 is contained in an interpolated free group factor. Conversely, we know that $p_{ab}\mathcal{N}_\infty^N p_{ab}$ contains a copy of $W^*(x) * W^*(y)$. Since the law of $\cup \in \mathcal{N}_k$ has no atoms, it follows that

$$W^*(x) = L(\overset{q}{\underset{1}{\mathbb{Z}}}) \oplus \underset{\delta_a \delta_b - 1}{\mathbb{C}}$$

where q is equivalent to p_0 via the polar part of

.

It follows from the arguments in Chapter 2 that

$$q(W^*(x) * W^*(y))q = L(\mathbb{Z}) * q\left(\left(\overset{q}{\underset{1}{\mathbb{C}}} \oplus \underset{\delta_a \delta_b - 1}{\mathbb{C}}\right) * W^*(y)\right)q$$

which is an interpolated free group factor. By the equivalence of q and p_0 in $\mathcal{N}_\infty^N$, and the identity $N_0^+ = p_0 \mathcal{N}_\infty^N p_0$, it follows that N_0^+ contains an interpolated free group factor. $\square$

Unfortunately, at this point, the author is unable to determine the isomorphism class of the $N_k^\pm$. While it is straightforward to show that a suitable compression of the algebra $\mathcal{N}_\infty^N$ is generated by products of the form $X_a^e X_b^f$, terms of the form

$$X_a^{e_1} X_b^f \text{ and } X_a^{e_2} X_b^f$$

appear with $e_1 \neq e_2$. The very nature of $\mathcal{N}_\infty^N$ makes it difficult to "decouple" this into a free family which still lies in $\mathcal{N}_\infty^N$.

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