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THE AXIAL CURRENT IN DIMENSIONAL REGULARIZATION*

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ABSTRACT

We show that a fully anti-commuting γ_5 is a correct and natural prescription for the dimensional regularization of one fermion loop graphs in spontaneously broken gauge theories. Other prescriptions introduce spurious anomalies into Ward identities which are actually anomaly free. Our prescription is correct even though no such γ_5 exists: it cannot exist precisely because of the familiar chiral anomaly.

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I. INTRODUCTION

Dimensional regularization[1]) is the most elegant and convenient method for computing higher order corrections in spontaneously broken gauge theories. It is probably the only practical method for all but the simplest calculations. In the absence of axial couplings the method is straightforward but there is some confusion in the case of fermion loops with one or more factors of γ_5. This is because the totally antisymmetric four-tensor $\varepsilon^{\mu\nu\alpha\beta}$ is a uniquely four dimensional object, with no natural continuation to n dimensions. Indeed we propose here that the correct and natural prescription for γ_5 in dimensional regularization is to choose an n-dimensional γ_5 which does not exist at all, in the sense that it is not mathematically well defined.

The preceding sentence is not meant to be a Zen koan: the central word is <u>prescription</u>. In constructing a regulator we want a method which (A) renders divergent Feynman integrals finite and (B) honors the Ward identities of the theory. Our prescription for γ_5 satisfies criteria (A) and (B) even though it is not well defined in n dimensions. The failure to exist is actually a virtue: it reflects the essential ambiguity of the Adler-Bell-Jackiw anomaly[2]), which is an unavoidable clash of Ward identities in certain Green's functions. Our prescription is ambiguous only in the context of the A-B-J anomaly and leaves us the freedom to choose, according to the physical circumstances, which of the clashing Ward identities will be anomalous.

Other prescriptions in the literature, which are well defined, are really stronger than we would like[3,4]). They force on us a particular resolution of the A-B-J ambiguity. And, more seriously, they introduce spurious anomalies into Ward identities which are really free of any essential anomalies. Consequently these prescriptions lead to errors in the calculation of physical quantities, unless the underlying Ward identities are checked at each stage of the calculation and the spurious anomalies are subtracted by hand. This is a tedious and unnecessary procedure. It is much simpler to use our prescription, which is unambiguous and correct except for the known essential ambiguity of the Adler-Bell-Jackiw anomaly. With our prescription, the latter ambiguity can be handled in the usual ways. In particular, in spontaneously broken gauge theories the ambiguity is constrained to cancel between the different fermion species[5]).

In the body of this paper we will concentrate on comparing our prescription for γ_5 to the prescription of 't Hooft and Veltman[3]). We defer to the concluding section a brief discussion of another prescription due to Akyeampong and Delbourgo[4]) which has properties similar to the 't Hooft-Veltman prescription.

In their original paper on dimensional regularization 't Hooft and Veltman[3]) proposed that in n dimensions γ_5 be defined[†] by

$$\overline{\gamma}_5 \equiv i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \,. \tag{1.1}$$

We denote their prescription by $\overline{\gamma}_5$. Our prescription, denoted $\tilde{\gamma}_5$, is defined by the properties that it anticommutes with all γ^μ in n dimensions

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† Equation (1.1) appears to be possible only for $n \geqslant 4$. But generalized S-matrix elements are also defined for $n < 4$ by analytic continuation -- see ref. (3).

$$\{\tilde{\gamma}_5, \gamma^\mu\} = 0 \qquad \mu = 0, 1, \cdots, n-1 \tag{1.2a}$$

and that it satisfies

$$\tilde{\gamma}_5^{\,2} = 1 \,. \tag{1.2b}$$

In contrast to eq. (1.2), the 't Hooft-Veltman prescription obeys

$$\{\overline{\gamma}_5, \gamma^\mu\} = 0 \qquad \mu = 0, 1, 2, 3 \tag{1.3a}$$

$$[\overline{\gamma}_5, \gamma^\mu] = 0 \qquad \mu = 4, \cdots, n-1 \,. \tag{1.3b}$$

Since it is defined explicitly by construction, eq. (1.1), $\overline{\gamma}_5$ is unique and well defined. Our prescription, $\tilde{\gamma}_5$, eqs. (1.2), is sufficient to define uniquely fermion loops with even numbers of γ_5's.

Comparing eq. (1.2a) with eqs. (1.3), it is not surprising that $\overline{\gamma}_5$ and $\tilde{\gamma}_5$ imply different definitions of divergent fermion loops with even numbers of γ_5's. What is more surprising is that $\overline{\gamma}_5$ and $\tilde{\gamma}_5$ seem to yield different predictions for experimentally measurable quantities. We will show this explicitly with an example from the $SU(2) \times U(1)$ weak interaction model.

This ambiguity is resolved by examining the relevant Ward identities. We will see that these Ward identities are not satisfied by $\overline{\gamma}_5$ because of the anomalous commutation relations, eqs. (1.3). Since loops with even numbers of γ_5's are well known to be free of essential anomalies, it is clear that these anomalies are spurious. On the other hand it is easy to see why our prescription, $\tilde{\gamma}_5$, correctly reproduces the canonical Ward identities for loops with even numbers of γ_5's. The canonical Ward identities are derived by formal manipulations which ignore divergences and of course assume the naive Dirac algebra, including $\{\gamma_5, \gamma^\mu\} = 0$. Since the divergences are removed by the continuation in n and since $\{\tilde{\gamma}_5, \gamma^\mu\} = 0$ for all μ, our prescription must yield the canonical identities.

To discuss loops with odd number of γ_5's we require in addition to eq. (1.2) that

$$\mathrm{Tr}\left(\tilde{\gamma}_5\, \gamma^\mu\, \gamma^\nu\, \gamma^\omega\, \gamma^\tau\right) = 4i\, \varepsilon^{\mu\nu\omega\tau} + O(n-4) \qquad (1.4)$$

for $\mu, \nu, \omega, \tau = 0, 1, 2, 3,$

that is, when μ, ν, ω, τ are in the four dimensional subspace. Using eqs. (1.2), (1.4) and the usual Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad (1.5a)$$

$$g_\mu{}^\mu = n \qquad (1.5b)$$

we can evaluate all one loop graphs with odd numbers of γ_5's. The result is unique up to the polynomial ambiguity of the A-B-J anomaly. When the anomaly is required to cancel in the sum over fermion species the resulting amplitudes are well defined and obey the canonical Ward identities.

When no regulator is used, as in Adler's discussion[2]), the VVA triangle is finite but has a polynomial ambiguity of arbitrary magnitude. Its magnitude may be varied by shifting the origin of momentum space in the finite but superficially divergent integral. With our prescription the dimensionally regulated integral is not even

superficially divergent but the polynomial ambiguity still occurs: it is now the algebraic manifestation of the fact that $\tilde{\gamma}_5$ is not well defined in n dimensions.

As in Adler's discussion, we choose the magnitude of the arbitrary polynomial according to which Ward identity we wish to be canonical. The essential feature of the A-B-J anomaly is that there is no choice which allows all of the Ward identities to be true. By contrast, the 't Hooft-Veltman prescription is unambiguous since $\bar{\gamma}_5$ is defined explicitly by construction. It automatically yields the magnitude which guarantees vector current conservation for the VVA triangle. For physical reasons this is the correct choice for the application to $\pi^0 \to \gamma\gamma$. But in other contexts, e.g., in a theory with an unbroken $SU(2)_L$ gauge symmetry, other choices might be appropriate. In this sense the ambiguity of our prescription is a virtue. In the spontaneously broken gauge theories the normalization of the ambiguous polynomial is irrelevant, since it is constrained to cancel in any case.

In addition to the VVA anomaly we have also examined the related triangle, box, and pentagon anomalies that occur in non-Abelian theories. Our prescription affords a straightforward derivation of the full anomaly, which is much easier than the original derivations[6]).

The paper is organized as follows: In Section II we show that a measurable one loop correction to the Higgs-Z-Z coupling in the $SU(2) \times U(1)$ model[7]) seems to depend on whether we use $\tilde{\gamma}_5$ or $\bar{\gamma}_5$. In Section III we resolve this apparent ambiguity by showing that $\tilde{\gamma}_5$ but not $\bar{\gamma}_5$ obeys the relevant Ward identity. Examining the high energy behavior of $e^+e^- \to ZZ$ we show explicitly how this Ward identity is essential for renormalizability. In Section IV we discuss the VVA anomaly and show how our prescription may be used to compute the minimal set of chiral anomalies in a non-Abelian theory. In Section V we summarize our results, discuss briefly another prescription due to Akyeampong and Delbourgo[4], and comment on the extension to multi-loop diagrams.

II. CALCULATION OF PHYSICAL QUANTITIES

In this section we compute a measurable quantity which seems to depend on the γ_5 prescription. We study one fermion loop corrections in the standard $SU(2) \times U(1)$ model[7]. To simplify the calculations and because it is the context in which we originally encountered the issue[8], we consider a fermion doublet (F_1, F_2) with masses M_1, M_2 much heavier than the W and Z bosons, $M_1, M_2 \gg M_W, M_Z$. We compute the leading correction to the tree approximation relations

$$M_W = M_Z \cos\theta \tag{2.1}$$

$$\lambda_{HZZ} = g\, M_Z/\cos\theta \tag{2.2}$$

where λ_{HZZ} is the Higgs-Z-Z coupling, θ is the weak interaction mixing angle, and g is the $SU(2)$ gauge coupling constant.

Although these corrections must be finite (because of renormalizability), they are computed from the sums of Feynman diagrams which are individually divergent. Since $\tilde{\gamma}_5$ and $\bar{\gamma}_5$ differ by terms of order $(n-4)$, they could imply results which differ by finite amounts for the divergent diagrams. We will see that this is

indeed the case. When the divergent contributions are summed to obtain the finite physical corrections, it turns out that the corrections to (2.2), but not those to (2.1), seem to depend on whether we use $\bar{\gamma}_5$ or $\tilde{\gamma}_5$.

The proper relationship between the infinite diagrams is assured by the Ward identities, which we will see in the next section are not honored by the $\bar{\gamma}_5$ prescription. The crucial difference between (2.1) and (2.2) is that the W and Z self-energy diagrams both contain the same number of γ_5 vertices so that either γ_5 prescription guarantees the proper relationship between them and therefore the same correction to (2.1). But (2.2) has contributions with different numbers of γ_5 vertices: the Z self energy and the H-Z-Z proper vertex with two γ_5's and the Higgs wave function renormalization with none. In this case $\bar{\gamma}_5$ does not give the correct relationship.

Since the calculations are straightforward we will present them only in outline. The contribution to the W and Z self energies is determined by the general vacuum polarization tensor

$$\Pi^{\mu\nu}(0) = \int \frac{d^nk}{(2\pi)^4} \frac{\mathrm{Tr}\, \gamma^\mu(C_V + C_A\gamma_5)(\not{k} + M)\, \gamma^\nu(C_V + C_A\gamma_5)(\not{k} + M')}{(k^2 - M^2)(k^2 - M'^2)} . \tag{2.3}$$

Using our $\tilde{\gamma}_5$ prescription the tensor is given to leading order in fermion masses by

$$\Pi^{\mu\nu}(0)\Big|_{\tilde{\gamma}_5} = \frac{ig^2}{4\pi^2} g^{\mu\nu} \Gamma(2 - \frac{n}{2})$$

$$\cdot \int_0^1 \frac{dx}{X^{2-\frac{n}{2}}} \left[(C_V{}^2 + C_A{}^2) X - (C_V{}^2 - C_A{}^2) MM' \right] \tag{2.4}$$

where

$$X \equiv xM^2 + (1 - x)M'^2 \; .$$

When $\bar{\gamma}_5$ is used the result differs from (2.4) by a finite quantity

$$\Pi^{\mu\nu}(0)\Big|_{\bar{\gamma}_5} - \Pi^{\mu\nu}(0)\Big|_{\tilde{\gamma}_5} = - \frac{ig^2}{4\pi^2} g^{\mu\nu} C_A{}^2 (M^2 + M'^2) \; . \tag{2.5}$$

But using either $\bar{\gamma}_5$ or $\tilde{\gamma}_5$ we find the same contribution[8,9]) to the ratio M_W/M_Z, that is

$$\frac{M_W}{M_Z \cos\theta} = 1 + \frac{g^2}{64\pi^2 M_W{}^2} \left[\frac{M_1{}^2 M_2{}^2}{M_1{}^2 - M_2{}^2} \ln \frac{M_2{}^2}{M_1{}^2} + \frac{M_1{}^2 + M_1{}^2}{2} \right] . \tag{2.6}$$

The contribution of the difference between the two prescriptions, given by eq. (2.5), cancels in the ratio (2.6).

Consider next the Higgs-Z-Z coupling constant given in tree approximation by (2.2). To leading order in the fermion masses the

corrections come from three sources: the Z self-energy correction determined by the tensor (2.3), the HZZ proper vertex

$$\Gamma^{\mu\nu}_{HZZ}(0) = \sum_{i=1}^{2} \frac{-g M_i}{2M_W} \int \frac{d^n k}{(2\pi)^4} \frac{1}{(k^2 - M_i{}^2)^3}$$

$$\cdot \; \mathrm{Tr} \left\{ (\not{k} + M_i)\gamma^\nu (C_V + C_A \gamma_5)_i (\not{k} + M_i)\gamma^\mu (C_V + C_A \gamma_5)_i (\not{k} + M_i) \right\} \tag{2.7}$$

and the Higgs boson wave function renormalization, $\Pi_H'(0)$, given by

$$\Pi_H(p^2) = - \sum_{i=1,2} \left(\frac{g M_i}{2M_W}\right)^2 \int \frac{d^n k}{(2\pi)^4} \frac{\mathrm{Tr}(\not{k} + M_i)(\not{k} + \not{p} + M_i)}{(k^2 - M_i{}^2)((k+p)^2 - M_i{}^2)} \; . \tag{2.8}$$

The leading correction is given by

$$\lambda_{HZZ} = \frac{g M_Z}{\cos\theta} \left\{ 1 + \frac{i}{2} \Pi_H'(0) - \frac{\delta M_Z{}^2}{2 M_Z{}^2} - \frac{i \cos\theta}{g M_Z} \Gamma_{HZZ}(0) \right\} \tag{2.9}$$

where $\delta M_Z{}^2$ is the Z self-energy contribution determined from (2.3) and $\Gamma_{HZZ}(0)$ is the coefficient of $g^{\mu\nu}$ in (2.7).

Using the $\tilde{\gamma}_5$ prescription the proper vertex is given to leading order in the fermion masses by

$$\Gamma_{HZZ}(0)\Big|_{\tilde{\gamma}_5} = \frac{i g^3}{32\pi^2 \cos^2\theta} \frac{1}{M_W} \sum_{i=1}^{2} M_i{}^2 \left(\frac{\Gamma(2 - \frac{n}{2})}{M_i{}^{4-n}} - 1 \right) . \tag{2.10}$$

With $\bar{\gamma}_5$ the vertex differs by a finite amount

$$\Gamma_{HZZ}(0)\Big|_{\bar{\gamma}_5} = \Gamma_{HZZ}(0)\Big|_{\tilde{\gamma}_5} - \frac{i\, g^3 (M_1{}^2 + M_2{}^2)}{32\pi^2\, M_W \cos^2\theta} . \qquad (2.11)$$

The Higgs wave function renormalization, since it does not involve γ_5, is given in either case by

$$\Pi_H'(0) = \frac{i\, g^2}{32\pi^2\, M_W{}^2} \sum_{i=1}^{2} M_i{}^2 \left(\frac{\Gamma(2-\frac{n}{2})}{M_i{}^{4-n}} - \frac{2}{3} \right) . \qquad (2.12)$$

Substituting these results into eq. (2.9) we find for the $\tilde{\gamma}_5$ prescription the finite correction

$$\lambda_{HZZ}\Big|_{\tilde{\gamma}_5} = \frac{g\, M_Z}{\cos\theta} \left(1 - \frac{g^2 (M_1{}^2 + M_2{}^2)}{48\pi^2\, M_W{}^2} \right) . \qquad (2.13)$$

If we use the $\bar{\gamma}_5$ prescription we find a different answer:

$$\lambda_{HZZ}\Big|_{\bar{\gamma}_5} = \lambda_{HZZ}\Big|_{\tilde{\gamma}_5} - \frac{g\, M_Z}{\cos\theta} \cdot \frac{3\, g^2 (M_1{}^2 + M_2{}^2)}{64\pi^2\, M_W{}^2} . \qquad (2.14)$$

The differences (2.5) and (2.11) do not cancel in their contribution to the correction (2.9).

Equations (2.13) and (2.14) seem to imply an ambiguity in the relationship between experimentally measurable quantities. We will see in the next section that the correct relationship is given by the

$\tilde{\gamma}_5$ prescription (2.13).†

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† Another apparent physical manifestation of different γ_5 prescriptions was found by Nachtmann and Wetzel[10]). They computed $\langle V^\alpha V^\beta \rangle_0$ and $\langle A^\alpha A^\beta \rangle_0$ in the limit of zero fermion mass using the prescription of Akyeampong and Delbourgo (see Appendix). They obtained different answers and concluded that chiral symmetry was in this way broken in Q.C.D. But this difference is just an example of the spurious anomalies generated by the prescription of Akyeampong and Delbourgo. A correct evaluation will yield no difference between the two terms in the limit of zero fermion mass.

III. CANONICAL WARD IDENTITIES

By virtue of the canonical Ward identities the spontaneously broken gauge theory maintains the good ultra-violet behavior of the unbroken theory. In this section we show that chiral Ward identities which do not contain essential (Adler-Bell-Jackiw) anomalies are obeyed by our $\tilde{\gamma}_5$ prescription but not by the 't Hooft-Veltman $\overline{\gamma}_5$ construction. This resolves the ambiguous prediction for λ_{HZZ} discussed in Section II: the Ward identities imply relationships between δM_Z, Π_H, and Γ_{HZZ} in eq. (2.9) which are satisfied by $\tilde{\gamma}_5$ but not by $\overline{\gamma}_5$. We will show explicitly that one of these Ward identities is essential to maintain acceptable high energy behavior in the process $e^+e^- \to ZZ$ and therefore to maintain the renormalizability of the theory.

Before discussing the Ward identities relevant to eq. (2.18) we first consider the simplest possible example--the chiral Ward identity relating the chiral vacuum polarization tensor to the vacuum expectation value of the scalar operator $\overline{\psi}\psi$. We define operators bilinear in a free fermion field $\psi(x)$ of mass M:

$$A^\mu(x) \equiv \overline{\psi}(x)\gamma^\mu \gamma_5 \psi(x)$$

$$P(x) \equiv \overline{\psi}(x)\gamma_5 \psi(x)$$

$$S(x) \equiv \overline{\psi}(x) \psi(x) \; . \tag{3.1}$$

The equal time commutation relation of the chiral charge

$$Q_5(t) \equiv \int d^3x \, A^0(\vec{x},t)$$

with the pseudoscalar density P is

$$\left[Q^5(t), P(\vec{x}, t)\right] = -2 \, S(\vec{x}, t) \; . \tag{3.2}$$

The Green's functions are defined as

$$\Pi_5{}^\mu(p) \equiv \int dx \, e^{ipx} \left\langle T \, A^\mu(x) \, P(0)\right\rangle_0$$

$$\Pi_5(p) \equiv \int dx \, e^{ipx} \left\langle T \, P(x) \, P(0)\right\rangle_0 \; . \tag{3.3}$$

Then from the equal-time commutator (3.2) we obtain the canonical Ward identity

$$p_\mu \, \Pi_5{}^\mu(p) = -2 \, M \, \Pi_5(p) - 2i \left\langle S(0)\right\rangle_0 \; . \tag{3.4}$$

To make the example completely transparent we consider the value $p = 0$, for which

$$M \, \Pi_5(0) = -i \left\langle S(0)\right\rangle_0 \; . \tag{3.5}$$

We now compute the one fermion loop contribution to eq. (3.5). The right-hand side is given by the tadpole diagram

$$-i\left\langle S\right\rangle_0 = -\int \frac{d^nq}{(2\pi)^4} \, \frac{\mathrm{Tr}(\not{q} + M)}{q^2 - M^2}$$

$$= -4 \, M \int \frac{d^nq}{(2\pi)^4} \, \frac{1}{q^2 - M^2} \tag{3.6}$$

and the left-hand side is

$$M \, \Pi_5(0) = M \int \frac{d^nq}{(2\pi)^4} \, \frac{\mathrm{Tr}\left[\gamma_5(\not{q} + M)\gamma_5 \, (\not{q} + M)\right]}{(q^2 - M^2)^2} \; . \tag{3.7}$$

Evaluating the trace using $\tilde{\gamma}_5$ it is easy to see that the Ward identity (3.5) is satisfied,

$$M\, \Pi_5(0)\Big|_{\tilde{\gamma}_5} = -i \langle S \rangle_0 \ . \qquad (3.8)$$

But using $\overline{\gamma}_5$, $\Pi_5(0)$ acquires an extra term. This occurs just as it did in the examples discussed in Section II: $\{q, \overline{\gamma}_5\}$ is nonvanishing of order $O(n - 4)$ and leaves a finite contribution since the divergent integral has a simple pole at $n = 4$. Instead of eq. (3.5) we obtain the anomalous Ward identity

$$M\, \Pi_5(0)\Big|_{\overline{\gamma}_5} = -i \langle S \rangle_0 - \frac{i}{2\pi^2} M^3 \ . \qquad (3.9)$$

This simple example illustrates the basic principle: the $\tilde{\gamma}_5$ prescription reproduces the canonical Ward identities because it mimics the naive manipulations used in the formal derivations in four dimensions. But because of eq. (1.3) the $\overline{\gamma}_5$ construction induces algebraic "anomalies" in the evaluation of Dirac traces which have no counterparts in the naive four dimensional calculations. As a result the $\overline{\gamma}_5$ construction introduces spurious anomalies into chiral Ward identities which are actually free of any essential anomalies. This occurs when, as in (3.4) and (3.5), there are divergent contributions from terms with different numbers of γ_5's.

Next we consider the Ward identities which must be satisfied by the Green's functions that contribute to λ_{HZZ} in eq. (2.9). The relevant operators are

$$A_3^{\ \mu}(x) \equiv \overline{\psi}(x)\gamma^\mu \gamma_5 \frac{\tau_3}{2} \psi(x)$$

$$P_3(x) \equiv \overline{\psi}\, \gamma_5 \frac{\tau_3}{2} \psi(x)$$

$$S(x) \equiv \overline{\psi}(x)\, \psi(x) \ . \qquad (3.10)$$

They satisfy the equal time commutation relations

$$\left[Q_3^{\ 5}(t),\, A_3^{\ \mu}(\vec{x}, t)\right] = 0$$

$$\left[Q_3^{\ 5}(t),\, P_3(\vec{x}, t)\right] = -\frac{1}{2} S(x)$$

$$\left[Q_3^{\ 5}(t),\, S(\vec{x}, t)\right] = -2 P_3(x) \ . \qquad (3.11)$$

We define the Green's functions

$$\Pi^{\mu\nu S}_{33}(p_1, p_2) \equiv \int_{x,y} e^{i(p_1 x + p_2 y)} \langle T\, A_3^{\ \mu}(x) A_3^{\ \nu}(y)\, S(0) \rangle_0$$

$$\Pi^{PPS}_{33}(p_1, p_2) \equiv \int_{x,y} e^{i(p_1 x + p_2 y)} \langle T\, P_3(x)\, P_3(y)\, S(0) \rangle_0$$

$$\Pi^{PP}_{33}(p) \equiv \int_x e^{ipx} \langle T\, P_3(x)\, P_3(0) \rangle_0$$

$$\Pi^{SS}(p) \equiv \int_x e^{ipx} \langle T\, S(x)\, S(0) \rangle_0 \ . \qquad (3.12)$$

$\Pi^{\mu\nu S}_{33}$ is proportional to the $C_A^{\ 2}$ term in the HZZ proper vertex,

$\Gamma^{\mu\nu}_{HZZ}$, eq. (2.7), and Π^{SS} is proportional to the Higgs self energy, Π_H , eq. (2.8). They are related by the canonical Ward identity which follows from the commutation relations (3.11),

$$p_{1\mu}\, p_{2\nu}\, \Pi^{\mu\nu S}_{33}(p_1, p_2) = 4\, M^2\, \Pi^{PPS}_{33}(p_1, p_2) + i\, M\, \Pi^{SS}(p_1 + p_2)$$

$$+ 4iM\left(\Pi^{PP}_{33}(p_1) + \Pi^{PP}_{33}(p_2)\right) - \left\langle S \right\rangle_0 . \qquad (3.13)$$

For simplicity we have assumed that the two partners in the fermion doublet $\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ have a common mass, $M_1 = M_2 = M$.

We have studied eq. (3.13) in two cases--the low energy limit, $p_1 = p_2 = 0$, and a high energy limit, $(p_1 + p_2)^2 \to \infty$ with $p_1{}^2$ and $p_2{}^2$ fixed. In both cases, the Ward identity is obeyed if the Green's functions are computed with $\tilde{\gamma}_5$ but not with $\overline{\gamma}_5$. As in the simpler example of eq. (3.4), the salient feature of eq. (3.13) is that it relates divergent fermion loops with different numbers of γ_5's. Here we will present only the results for the high energy limit, since it involves just the Green's functions in eq. (3.13) that contribute to λ_{HZZ} and since it illustrates very clearly the relationship of the Ward identity to the renormalizability of the theory.

In the limit $(p_1 + p_2)^2 \to \infty$ with $p_1{}^2$ and $p_2{}^2$ fixed the terms $p_{1\mu}\, p_{2\nu}\, \Pi^{\mu\nu S}$ and Π^{SS} are of order $(p_1 + p_2)^2$ while the other terms in (3.13) are $O(1)$. Therefore in this limit the Ward identity becomes

$$p_{1\mu}\, p_{2\nu}\, \Pi^{\mu\nu S}_{33}(p_1, p_2) \cong i\, M\, \Pi^{SS}(p_1 + p_2) . \qquad (3.14)$$

The quantity $\Pi^{SS}(p_1 + p_2)$ is given by $\Pi_H(p_1 + p_2)$, eq. (2.8), except for the factor $(-igM/2M_W)^2$. The leading term is

$$\Pi^{SS}(p_1 + p_2) \cong \frac{-i}{4\pi^2}\, (p_1 + p_2)^2 \left\{\Gamma(2 - \tfrac{n}{2}) - \ell n\left[-(p_1 + p_2)^2\right] + 2\right\} . \qquad (3.15)$$

The three point function evaluated using $\tilde{\gamma}_5$ is to leading order

$$\Pi^{\mu\nu S}_{33}(p_1, p_2)\Big|_{\tilde{\gamma}_5} \cong \frac{M}{2\pi^2}\, g^{\mu\nu} \left\{\Gamma(2 - \tfrac{n}{2}) - \ell n\left[-(p_1 + p_2)^2\right] + 2\right\}$$

$$- \frac{M}{4\pi^2}\left(g^{\mu\nu} - \frac{p_1{}^{\nu}\, p_2{}^{\mu}}{p_1 \cdot p_2}\right) \int_0^1 dx \int_0^{1-x} dy\, \frac{2xy - x - y}{xy - \left[M^2/(2p_1 \cdot p_2)\right]} . \qquad (3.16)$$

Multiplying by $p_{1\mu}\, p_{2\nu}$ the second term in (3.16) vanishes and the first term gives $iM\, \Pi^{SS}(p_1 + p_2)$ as required by the Ward identity,

$$p_{1\mu}\, p_{2\nu}\, \Pi^{\mu\nu S}(p_1, p_2)\Big|_{\tilde{\gamma}_5} \cong i\, M\, \Pi^{SS}(p_1 + p_2) . \qquad (3.17)$$

Computing with $\overline{\gamma}_5$ we find that the <u>exact</u> difference with the $\tilde{\gamma}_5$ result, eq. (3.16), is

$$\Pi^{\mu\nu S}_{33}(p_1, p_2)\Big|_{\overline{\gamma}_5} - \Pi^{\mu\nu S}_{33}(p_1, p_2)\Big|_{\tilde{\gamma}_5} = -\frac{i\, M}{2\pi^2}\, g^{\mu\nu} . \qquad (3.18)$$

In the high energy limit, $\bar{\gamma}_5$ then yields the anomalous result

$$p_{1\mu}\, p_{2\nu}\, \Pi^{\mu\nu S}_{33}(p_1, p_2)\Big|_{\bar{\gamma}_5} = i\, M\, \Pi^{SS}(p_1, p_2) - \frac{M}{2\pi^2}\, p_1 \cdot p_2 \,. \tag{3.19}$$

In the calculation of the order $O(g^4 M^2)$ corrections to λ_{HZZ}, eq. (2.9), we computed contributions from $\Pi'_H(0) \propto \Pi^{SS'}(0)$, from $\Gamma_{HZZ}(0) \propto \Pi^{\mu\nu S}_{33}(0, 0)$, and from the axial current part of the vector boson self energy $\Pi^{\mu\nu}_Z(0) \propto \Pi^{\mu\nu}_{33}(0)$. Π_H and Γ_{HZZ} are related by the Ward identity (3.13) and because of eq. (3.18), which is exact for all values of p_1 and p_2, this relationship is not obeyed if the $\bar{\gamma}_5$ construction is used. The third $O(g^4 M^2)$ contribution to λ_{HZZ}, $\Pi^{\mu\nu}_W(0) \propto \Pi^{\mu\nu}_{33}(0)$, is related to $\Pi^{\mu\nu S}_{33}(0, 0)$ by the trace identity[11]). Because the trace identity is a relationship between Green's functions with the same number of γ_5's, it is obeyed for both $\bar{\gamma}_5$ and $\tilde{\gamma}_5$. Therefore $\Pi^{\mu\nu}_W\big|_{\bar{\gamma}_5}$ does not combine with $\Gamma_{HZZ}\big|_{\bar{\gamma}_5}$ to restore the proper relationship with Π_H.

Finally we will show explicitly how the high energy limit (3.14) of the Ward identity (3.13) is required to maintain the renormalizability of the theory at the two loop level. Consider $e^+e^- \to ZZ$ where the Z bosons are longitudinally polarized. In Born approximation there are two contributions--t-channel electron exchange and s-channel Higgs boson exchange. For equal $e^\pm$ helicities, $h_+ = h_- \equiv h$, and to leading order in the center-of-mass energy $E \gg M_Z$ they are given respectively by

$$\mathcal{M}_t \cong hg^2 \frac{m_e E}{2M_W^2} + O\left(\frac{M_W^2}{E^2}, \frac{m_e^2}{E^2}\right) \tag{3.20}$$

$$\mathcal{M}_s \cong - hg^2 \frac{m_e E}{2M_W^2} \frac{4E^2}{4E^2 - m_H^2} + O\left(\frac{M_W^2}{E^2}, \frac{m_e^2}{E^2}\right) . \tag{3.21}$$

The two diagrams individually have "bad" high energy behavior, growing like E, but the terms linear in E cancel, as they must, to keep the theory renormalizable at the one loop level.

Next we compute the one loop radiative corrections to $e^+e^- \to ZZ$ due to a fermion doublet (F_1, F_2) of common mass $M \gg M_W$. The only order $O(g^2 M^2/M_W^2)$ corrections are to the s-channel Higgs exchange: the fermion loop contributions to the H propagator and to the HZZ proper vertex. In the notation of Section II, the Higgs propagator is modified by

$$\frac{i}{4E^2 - m_H^2} \longrightarrow \frac{i}{4E^2 - m_H^2}\left(1 + \frac{i\,\Pi_H(4E^2)}{4E^2 - m_H^2}\right) \tag{3.22}$$

and the HZZ vertex by

$$\frac{ig\, M_Z}{\cos\theta}\, g_{\mu\nu} \longrightarrow \frac{ig\, M_Z}{\cos\theta}\left(g_{\mu\nu} - \frac{i\cos\theta}{g\, M_Z}\, \Gamma^{HZZ}_{\mu\nu}(p_1 + p_2, p_1, p_2)\right) \tag{3.23}$$

where p_i are the Z four-momenta and $4E^2 = (p_1 + p_2)^2$. In the high energy limit the longitudinal Z boson polarization vectors are approximately proportional to the Z momenta, $\varepsilon_i^\mu \cong p_i^\mu / M_Z$, so that the sum of the two corrections is proportional to

$$p_1^{\ \mu}\, p_2^{\ \nu} \left\{ \frac{i\, \Pi_H(4E^2)}{4E^2}\, g_{\mu\nu} - \frac{i \cos\theta}{g\, M_Z}\, \Gamma^{HZZ}_{\mu\nu}(p_1 + p_2, p_1, p_2) \right\} . \tag{3.24}$$

Using $p_1 \cdot p_2 \cong 2E^2$, $\Pi_H = -\left(\frac{g\, M}{2M_W}\right)^2 \Pi^{SS}$, and

$p_1^{\ \mu}\, p_2^{\ \nu}\, \Gamma^{HZZ}_{\mu\nu} = +\, i\, \frac{g^3\, M}{8M_W \cos^2\theta}\, p_1^{\ \mu}\, p_2^{\ \nu}\, \Pi^{33}_{\mu\nu S}$, we see that (3.24) vanishes to leading order in E just because of the high energy limit (3.14) of the Ward identity (3.13). If (3.24) did not vanish to leading order then the cancellation between the linear terms (3.20) and (3.21) would be undone at the one loop level, which would in turn render the theory nonrenormalizable at the two loop level.

IV. ANOMALOUS WARD IDENTITIES

Our prescription, $\{\tilde{\gamma}_5, \gamma^\mu\} = 0$ and $\tilde{\gamma}_5^{\ 2} = 1$, uniquely defines fermion loops with even numbers of γ_5's and honors the canonical Ward identities for such loops. This is not true of other prescriptions for γ_5 in n dimensions. Now we consider fermion loops with odd numbers of γ_5's. At the one loop level we will show it is sufficient to require that

$$\mathrm{Tr}\left[\tilde{\gamma}_5\, \gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\phi\right] \equiv 4i\, \varepsilon^{\alpha\beta\delta\phi} + O(n-4)$$

$$\{\alpha, \beta, \delta, \phi\} = \{0, 1, 2, 3\}$$

That is, we require only that $\mathrm{Tr}\left[\tilde{\gamma}_5\, \gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\phi\right]$ reduce to the usual answer as $n \to 4$ when $\alpha, \beta, \delta, \phi$ are in the four dimensional subspace.

For $\{\alpha, \beta, \delta, \phi\} = \{0, 1, 2, 3\}$ it is clear from (1.2) that $\mathrm{Tr}\left[\tilde{\gamma}_5\, \gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\phi\right]$ must be proportional to $\varepsilon^{\alpha\beta\delta\phi}$ if it is not identically zero. Indeed, the most "natural" dimensional continuation of γ_5 (cf. Section V) would give $\mathrm{Tr}\left[\gamma_5\, \gamma^\alpha \gamma^\beta \gamma^\delta \delta^\phi\right]$ an essential zero at $n = 4$. This would be a mathematically consistent but physically useless prescription. The only remaining freedom is that the coefficient of $\varepsilon^{\alpha\beta\delta\phi}$ could be "$4 + b(4 - n)$" with "b" an arbitrary parameter. That is, we could take

$$\mathrm{Tr}\left[\tilde{\gamma}_5\, \gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\phi\right] = \left[4 + b(4-n)\right] i\varepsilon^{\alpha\beta\delta\phi} \tag{4.1'}$$

$$\{\alpha, \beta, \delta, \phi\} = \{0, 1, 2, 3\}$$

and still obtain the correct results for finite loops since the term proportional to $n - 4$ would not then contribute. In fact we will show below that no results depend on the choice of "b" and for convenience we set it equal to zero, taking

$$\mathrm{Tr}\left[\tilde{\gamma}_5\, \gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\phi\right] = 4i\varepsilon^{\alpha\beta\delta\phi} , \tag{4.1}$$

$$\{\alpha, \beta, \delta, \phi\} = \{0, 1, 2, 3\} .$$

We also use the property, which follows from the anti-commutativity of $\tilde{\gamma}_5$, that $\mathrm{Tr}\left[\tilde{\gamma}_5\, \gamma^\alpha \gamma^\beta\right]$ and $\mathrm{Tr}\left[\tilde{\gamma}_5\, \gamma^\alpha \gamma^\beta \gamma^\delta\right]$ have essential zeros at $n = 4$.

Having made the choice (4.1) it now necessarily follows that other traces are <u>not</u> well defined. Therefore our prescription $\tilde{\gamma}_5$ does not correspond to a well-defined Dirac matrix. But the ambiguities which render it ill defined correspond precisely to the inescapable ambiguities of the Adler-Bell-Jackiw anomaly.

Consider the trace $\mathrm{Tr}(\tilde{\gamma}_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi\,\gamma_\mu\,\gamma^\mu)$, where again $\alpha, \beta, \delta, \phi$ are in the four-dimensional subspace. Using the standard n-dimensional Dirac algebra, eg. (1.5), we find immediately

$$\mathrm{Tr}(\tilde{\gamma}_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi\,\gamma_\mu\,\gamma^\mu) = 4i\,n\,\varepsilon^{\alpha\beta\delta\phi} \,. \tag{4.2}$$

If instead we rewrite the trace as $\mathrm{Tr}(\gamma^\mu\,\tilde{\gamma}_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi\,\gamma_\mu)$ and then anticommute γ^μ to the right we find

$$\mathrm{Tr}(\tilde{\gamma}_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi\,\gamma_\mu\,\gamma^\mu) = 4i(8-n)\varepsilon^{\alpha\beta\delta\phi} \,. \tag{4.3}$$

The two results only agree for $n = 4$. We simply accept this ambiguity as a necessary consequence of our prescription. We introduce an arbitrary parameter "a" and define the trace to be $a \times$ eq(4.3) + $(1-a) \times$ eq.(4.2), which is

$$\mathrm{Tr}(\tilde{\gamma}_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi\,\gamma_\mu\,\gamma^\mu) = 4i\,\varepsilon^{\alpha\beta\delta\phi}\Big[n + 2a(4-n)\Big] \,. \tag{4.4}$$

The ambiguity is of course proportional to $n - 4$.

It is now easy to see why it doesn't matter whether we defined $\mathrm{Tr}(\tilde{\gamma}_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi)$ by (4.1) or (4.1'). If we had used (4.1') then in place of (4.4) we would have had the same result except that the arbitrary parameter "a" would have been replaced by the equally arbitrary parameter $a + \frac{b}{2}$.

If we were to insist that our prescription be well defined, then by the manipulations that led to eqs. (4.2) and (4.3) we would obtain $(n-4)\mathrm{Tr}(\gamma_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi) = 0$, which would imply that the trace has an essential zero at $n = 4$. This would be a consistent but useless prescription, which would imply that the VVA triangle diagram has an essential zero at $n = 4$.

It is important to stress that the ambiguity in (4.4) is not a defect of our prescription but is rather a necessary consequence of the Adler-Bell-Jackiw anomaly. If it were possible to define uniquely an n-dimensional γ_5 satisfying our prescription, eqs. (1.2) and (4.1), then it would follow that there is no A-B-J anomaly. For the VVA Green's function defined by such a γ_5 construction would obey all canonical Ward identities and be finite for $n \neq 4$. Then the four dimensional Green's function defined by the limit $n \to 4$ would also obey canonical Ward identities. But the essence of the A-B-J anomaly is that is impossible to satisfy simultaneously the canonical vector and chiral Ward identities for the VVA triangle. The existence of the anomaly therefore proves that no such γ_5 can actually be constructed.

Since the $\overline{\gamma}_5$ of 't Hooft and Veltman is defined by the explicit construction eq. (1.1), it yields the unambiguous result

$$\mathrm{Tr}(\overline{\gamma}_5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\delta\,\gamma^\phi\,\gamma_\mu\,\gamma^\mu) = 4i\,n\,\varepsilon^{\alpha\beta\delta\phi} \tag{4.6}$$

for $\{\alpha, \beta, \delta, \phi\} = \{0, 1, 2, 3\}$.

This correspond to our result, eq. (4.4), when the arbitrary parameter "a" is set to zero.

With eqs. (1.2) and (4.1) we have specified our $\overline{\gamma}_5$ prescription sufficiently to discuss the VVA amplitude and the related anomaly. We define the pseudoscalar and pseudovector amplitudes respectively by

$$\Pi_5^{\mu\nu}(p_1, p_2) \equiv \int_{x,y} e^{i(p_1 x + p_2 y)} \left\langle T\, V^\mu(x)\, V^\nu(y)\, P(0) \right\rangle_0$$

$$\Pi_5^{\mu\nu\tau}(p_1, p_2) = \int_{x,y} e^{i(p_1 x + p_2 y)} \left\langle T\, V^\mu(x)\, V^\nu(y)\, A^\tau(0) \right\rangle_0 . \qquad (4.7)$$

Where $\psi(x)$ is a fermion field of mass M, the current densities are

$$P(x) \equiv \bar\psi(x)\, \gamma_5 \psi(x)$$

$$A^\mu(x) \equiv \bar\psi(x)\, \gamma^\mu\, \gamma_5\, \psi(x)$$

$$V^\mu(x) \equiv \bar\psi(x)\, \gamma^\mu\, \psi(x) . \qquad (4.8)$$

The canonical Ward identities for the vector currents are

$$p_{1\mu}\, \Pi_5^{\mu\nu\tau}(p_1, p_2) = p_{2\nu}\, \Pi_5^{\mu\nu\tau}(p_1, p_2) = 0 \qquad (4.9)$$

and the canonical chiral Ward identity is

$$(p_1 + p_2)_\tau\, \Pi_5^{\mu\nu\tau}(p_1, p_2) = 2M\, \Pi_5^{\mu\nu}(p_1, p_2) . \qquad (4.10)$$

Adler, Bell, and Jackiw showed that it is impossible to satisfy eqs. (4.9) and (4.10) simultaneously.

To understand the anomaly it suffices to evaluate $\Pi_5^{\mu\nu\tau}$ to first order and $\Pi_5^{\mu\nu}$ to second order in the external momenta p_1 and p_2. $\Pi_5^{\mu\nu}$ is finite and unambiguous and can be evaluated without a regulator. It is given by

$$\Pi_5^{\mu\nu}(p_1, p_2) = \frac{-i}{4M\,\pi^2}\, \varepsilon^{\mu\nu\alpha\beta}\, p_{1\alpha}\, p_{2\beta} + O(p_i^{\,4}) . \qquad (4.11)$$

$\Pi_5^{\mu\nu\tau}$ is also finite but not unambiguous. It is given exactly by

$$\Pi_5^{\mu\nu\tau}(p_1, p_2) = i \int \frac{d^n\ell}{(2\pi)^4}\, \frac{\mathrm{Tr}\left[\gamma^\alpha \tilde\gamma_5 (\not\ell - \not p_2 + M)\gamma^\nu (\not\ell + M)\gamma^\mu (\not\ell + \not p_1 + M)\right]}{((\ell - p_2)^2 - M^2)(\ell^2 - M^2)((\ell + p_1)^2 - M^2)}$$

$$+ (\mu \leftrightarrow \nu,\ p_1 \leftrightarrow p_2) . \qquad (4.12)$$

We introduce Feynman parameters and evaluate $\Pi_5^{\mu\nu\tau}$ to linear order in the p_i, after which the integration over the Feynman parameters is trivially performed. We rewrite the trace, using symmetric integration but <u>without anticommuting $\not\ell$ and $\tilde\gamma_5$</u>. The result is

$$\Pi_5^{\mu\nu\tau}(p_1, p_2) = -2i \int \frac{d^n\ell}{(2\pi)^4}\, \frac{N^{\mu\nu\tau}}{(\ell^2 - M^2)^3} + O(p_i^{\,3}) \qquad (4.13)$$

where

$$N^{\mu\nu\tau} \equiv \frac{2}{3}\, \frac{\ell^2}{n}\, \mathrm{Tr}\left[\tilde\gamma_5\, \gamma^\mu\, \gamma^\nu\, \gamma^\tau (\not p_1 - \not p_2)\gamma_\omega\, \gamma^\omega\right]$$

$$- \frac{2}{3}\left(4\frac{\ell^2}{n} + M^2\right) \mathrm{Tr}\left[\tilde\gamma_5\, \gamma^\mu\, \gamma^\nu\, \gamma^\tau (\not p_1 - \not p_2)\right] + O(p_i^{\,3}) . \qquad (4.14)$$

Since μ, ν, τ, and $\not p_i$ are external and four-dimensional, the second term in (4.14) is given unambiguously by eq. (4.1) while for the first term we use the ambiguous "a" --dependent prescription eq. (4.4). The result is

$$N^{\mu\nu\tau} = \frac{8i}{3}\,\varepsilon^{\mu\nu\tau\rho}(p_1 - p_2)_\rho \left\{\ell^2(1-2a)(1-\frac{4}{n}) - M^2\right\} + O(p_i^{\,3}) \,. \tag{4.15}$$

Performing the integration over ℓ we obtain finally

$$\Pi_5^{\mu\nu\tau}(p_1\,,\,p_2) = a\,\frac{i}{3\pi^2}\,\varepsilon^{\mu\nu\tau\rho}\,(p_1 - p_2)_\rho + O(p_i^{\,3}) \,. \tag{4.16}$$

The integral $\int \ell^2/(\ell^2 - M^2)^3$ diverges but we obtain a finite result because of the factor $1 - \frac{4}{n}$ in eq. (4.15). The "a" dependence of this term gives rise to the ambiguity which in the original analyses[2]) reflected the fact that the four dimensional integration is not well defined. Here instead the ambiguity is algebraic in origin. For $a = 0$ the vector Ward identities, eq. (4.9), are obeyed but the chiral identity (4.10) is not. For $a = 3/4$ the chiral identity is satisfied but the vector identities are not. For other values of "a" none of the Ward identities are satisfied. And there is clearly no value of "a" for which all three identities are valid: this is the essential feature of the anomaly.

The usual designation, "chiral anomaly," reflects the choice $a = 0$, which happens to be appropriate in the application to $\pi^0 \to \gamma\gamma$. However one can certainly imagine other contexts in which other choices might be appropriate. As we have already noted, the result $a = 0$ is uniquely selected by the $\overline{\gamma}_5$ construction of 't Hooft and Veltman.

Next we show that it is straightforward to use our prescription to derive the minimal Bardeen set of chiral anomalies in theories with non-Abelian currents. The calculation is much easier than previous

calculations[6]) which used point splitting or Pauli-Villars regularization. In those calculations counter terms were needed to put the anomaly in its minimal form and, in the case of point splitting, to make the three and four point functions finite. In contrast, using our approach the anomalous Green's functions are finite though ambiguous and the minimal anomaly emerges immediately.

It is clear from our discussion of the Abelian case that the presence of anomalies is intimately related to the ambiguity in defining traces with an odd number of γ_5's. Except for these ambiguities our prescription is guaranteed to reproduce the canonical Ward identities. As in eq. (4.4) ambiguities are proportional to $n - 4$ so they can only survive if multiplied by a divergent loop integral. It is easy to verify at the one loop level that this only occurs in the Green's functions $\langle VVA \rangle$, $\langle AAA \rangle$, $\langle VVVA \rangle$, and $\langle VAAA \rangle$. From this list of ambiguous Green's functions it is in turn easy to obtain the minimal Bardeen anomaly.

Where λ_i denote the group generators in the representation of the fermions, the vector and axial currents are defined as

$$V_i^{\,\mu}(x) \equiv \overline{\psi}(x)\,\gamma^\mu\,\lambda_i\,\psi(x)$$

$$A_i^{\,\mu}(x) \equiv \overline{\psi}(x)\,\gamma^\mu\,\gamma_5\,\lambda_i\,\psi(x) \,.$$

Green's functions are written in momentum space, e.g.,

$$\langle V_i^{\,\mu}\,V_j^{\,\nu}\,A_k^{\,\tau}\rangle \equiv \int_{x,y} e^{\,i(p_i x + p_j y)} \langle T\,V_i^{\,\mu}(x)\,V_j^{\,\nu}(y)\,A_k^{\,\tau}(0)\rangle_0 \,.$$

Then, as in the Abelian case, the three-point functions are well defined except for finite polynomial ambiguities which are linear in the external momenta:

$$\left\langle V_i^{\mu}\ V_j^{\nu}\ A_k^{\tau}\right\rangle = K\left(p_i - p_j\right)_{\rho}\ \varepsilon^{\mu\nu\tau\rho}\ \mathrm{Tr}(\lambda_i\ \lambda_j\ \lambda_k)\ \cdots \tag{4.17}$$

$$\left\langle A_i^{\mu}\ A_j^{\nu}\ A_k^{\tau}\right\rangle = \left(L^{(1)}p_i + L^{(2)}p_j + L^{(3)}p_k\right)_{\rho} \times \varepsilon^{\mu\nu\tau\rho}\ \mathrm{Tr}(\lambda_i\ \lambda_j\ \lambda_k) + \cdots \tag{4.18}$$

K, $L^{(1)}$, $L^{(2)}$, and $L^{(3)}$ are arbitrary parameters, like "a" in eq. (4.16). We have exhibited only the ambiguous terms; the well defined terms are $O(p^3)$ and higher.

Notice that the ambiguous terms are independent of fermion masses. This reflects the well-known fact that the anomaly arises from the leading ultra-violet behavior of the one-loop graphs, which is scale-invariant. This means that fermion mass splitting will not effect our calculation, so we may for convenience take all fermions to have a common mass. Then canonically all vector currents would be conserved, $\partial_{\mu} V_i^{\mu} = 0$.

We now impose the conventional requirement that vector current Ward identities be canonical. Then $p_{i\mu}\left\langle V_i^{\mu}\ V_j^{\nu}\ A_k^{\tau}\right\rangle = 0$ (we work to one loop order) which implies that $K = 0$ in (4.17); the well-defined terms, which were not exhibited in (4.17), are guaranteed by our prescription to have the canonical behavior. In (4.18) we fix the ambiguous polynomial by requiring Bose symmetry (which again is automatically satisfied by the well-defined contribution). Now $\mathrm{Tr}(\lambda_i\ \lambda_j\ \lambda_k)$ is in general the sum of total symmetric and totally antisymmetric pieces, which we denote generically

$$\mathrm{Tr}(\lambda_i\ \lambda_j\ \lambda_k) \equiv d_{ijk} + f_{ijk}\ .$$

Then symmetry under $\mu, i \leftrightarrow \nu, j$ implies using $p_k = -p_i - p_j$ that

$$(L^{(1)} - L^{(3)})(f_{ijk} - d_{ijk}) = (L^{(2)} - L^{(3)})(f_{ijk} + d_{ijk})$$

$$(L^{(1)} - L^{(3)})(f_{ijk} + d_{ijk}) = (L^{(2)} - L^{(3)})(f_{ijk} - d_{ijk})$$

which implies $L^{(1)} = L^{(2)}$. Requiring Bose symmetry for all three currents we find that $L^{(1)} = L^{(2)} = L^{(3)}$ is required, and the ambiguous term in (4.18) is proportional to $p_i + p_j + p_k$ and vanishes by momentum conservation.

Having eliminated the linear polynomials, (4.17) and (4.18) are of $O(p^3)$ and well defined. Now just as in the Abelian case we find that the axial current Ward identity must be anomalous, since $p_{\tau k}\left\langle V_i^{\mu}\ V_j^{\nu}\ A_k^{\tau}\right\rangle$ and $p_{\tau k}\left\langle A_i^{\mu}\ A_j^{\nu}\ A_k^{\tau}\right\rangle$ are $O(p^4)$ while $\left\langle V_i^{\mu}\ V_j^{\nu}\ P_k\right\rangle$ and $\left\langle A_i^{\mu}\ A_j^{\nu}\ P_k\right\rangle$ are $O(p^2)$. The anomaly is given by the unambiguous $O(p^2)$ terms in $\langle VVP\rangle$ and $\langle AAP\rangle$. The $O(p^4)$ terms are all well defined and necessarily obey the canonical Ward identity.

In the four-point functions the ambiguous terms are of zero'th order in the external momenta:

$$\left\langle V_i^{\mu}\ V_j^{\nu}\ V_k^{\tau}\ A_{\ell}^{\rho}\right\rangle = R\ \varepsilon^{\mu\nu\tau\rho}\ \mathrm{Tr}(\lambda_i\ \lambda_j\ \lambda_k\ \lambda_{\ell}) + \cdots \tag{4.19}$$

$$\left\langle V_i^{\mu}\ A_j^{\nu}\ A_k^{\tau}\ A_{\ell}^{\rho}\right\rangle = S\ \varepsilon^{\mu\nu\tau\rho}\ \mathrm{Tr}(\lambda_i\ \lambda_j\ \lambda_k\ \lambda_{\ell}) + \cdots\ . \tag{4.20}$$

Again vector current conservation requires $R = S = 0$ and the axial anomaly is determined by the $0(p)$ terms in $\langle VVVP\rangle$ and $\langle VAAP\rangle$. The well-defined contributions, suppressed in (4.19) and (4.20), are $0(p^2)$ and canonical. In the chiral Ward identity they match the canonical $0(p^3)$ contributions from $\langle VVVP\rangle$ and $\langle VAAP\rangle$ and from the three-point functions which come from the equal time commutators.

Finally the five point functions $\langle VVVVA\rangle$, $\langle VVAAA\rangle$, $\langle AAAAA\rangle$ are finite and unambiguous. But their chiral Ward identities are anomalous because of the ambiguous four point functions generated by the equal-time commutator terms. Again power-counting determines the anomaly. $\langle VVVVA\rangle$ is $0(p)$ so the left-hand side of the chiral Ward identity is $0(p^2)$. By the previous considerations the four point functions $\langle VVVA\rangle$ on the right-hand side are also $0(p^2)$. But $\langle VVVVP\rangle$ is unambiguous and of $0(1)$ hence there is an $0(1)$ anomaly to cancel the zero momentum limit of $\langle VVVVP\rangle$.

For six- and higher-point functions, all Green's functions appearing in the Ward identities are finite and unambiguous so the Ward identities are necessarily canonical.

These simple considerations show that our prescription yields anomalies in precisely the same Ward identities as Bardeen's minimal set. It is also easy to see that the values of the anomalies agree with Bardeen's expressions. From our explicit calculation of the Abelian VVA anomaly, we find that our non-Abelian VVA anomalies are equal to Bardeen's. But then the integrability relations of Wess and Zumino[12]) imply that our calculation of all other anomalies must agree with Bardeen's. Our calculation is guaranteed to satisfy the integrability relations because our prescription respects the group symmetry from which they follow.

V. CONCLUDING REMARKS

The starting point for this work is the observation, illustrated by a specific example in Section II, that different prescriptions for γ_5 in n dimensions seem to imply different predictions for physical quantities. This apparent ambiguity was resolved in Section III by studying the underlying Ward identities. We showed that the prescription $\{\tilde{\gamma}_5, \gamma^\mu\} = 0$ and $\tilde{\gamma}_5^{\,2} = 1$ defines one fermion loop Green's functions with even numbers of γ_5's so that they obey the canonical Ward identities in n dimensions. Other prescriptions which differ from ours by terms of order $(n - 4)$ introduce spurious anomalies into these Ward identities. We exhibited these spurious anomalies in Section III for the prescription of 't Hooft and Veltman. We showed explicitly in Section III that these anomalies are not consistent with renormalizability. If our prescription is not used they must be subtracted by hand.

The confusion surrounding the meaning of γ_5 in n dimensions arises because there is no explicit n-dimensional γ_5 which obeys the prescription (1.2). In Section IV we showed that this need not be confusing: it is just the Adler-Bell-Jackiw anomaly in an unfamiliar guise. In particular we extended our prescription to diagrams with odd numbers of γ_5's by requiring that $Tr(\gamma_5\, \gamma^\mu\, \gamma^\nu\, \gamma^\omega\, \gamma^\tau) = 4i\, \varepsilon^{\mu\nu\omega\tau}$ when μ, ν, ω, τ are in the four-dimensional subspace. This sufficed to define one-loop diagrams except for a polynomial ambiguity, which A) renders our prescription ill defined but B) is nothing more than the usual A-B-J anomaly. If there were a well-defined γ_5 satisfying our prescripton, then there would be no A-B-J anomaly. Our method provides a computationally convenient method of recovering the known results for both the Abelian and non-Abelian chiral anomalies.

We will comment briefly on two prescriptions discussed by Akyeampong and Delbourgo[4]. The first is the most "natural" choice: in space-time of even dimension take γ_5 to be the product $\gamma^0 \gamma^1 \cdots \gamma^{n-1}$. This is an unambiguous realization of a fully anti-commuting γ_5. As Akyeampong and Delbourgo observed[4], it therefore suffers from the disease discussed in Section IV: it gives the VVA amplitude an essential zero at $n = 4$. The second prescription, which is the one advocated in ref. (4), is to use for the pseudoscalar vertex the quadruply antisymmetric product

$$\propto \sum_{\text{Perm.}} (-1)^P \gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\phi$$

where α, β, δ, ϕ vary over all n indices. This is covariant in n dimensions but otherwise is very similar to the prescription of 't Hooft and Veltman: it also introduces spurious anomalies and also forces the A-B-J anomaly into the chiral current. As the authors acknowledged in a subsequent paper[13], if it is used it is necessary to compute by hand counter terms to cancel the spurious anomalies.

The prescription defined in Sections III and IV provides a complete and correct description of all one-fermion loop diagrams. It is also clear that our prescription correctly describes graphs with arbitrary numbers of fermion loops if each loop contains an even number of γ_5's, since the validity of the canonical Ward identities is guaranteed for this case as noted in Section III. We are now considering multi-loop graphs in which there are loops with odd

numbers of γ_5's. If our method can be extended to that case it would constitute a complete prescription for dimensional regularization of spontaneously broken gauge theories.

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