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Convexity in Valued Fields
and Semi-equations

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by

Alex Jacob Mennen

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ABSTRACT OF THE DISSERTATION

Convexity in Valued Fields
and Semi-equations

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2022
Professor Artem Chernikov, Chair

This dissertation studies certain asymmetric (in the sense of not closed under complement) properties of families of sets, and how they relate to standard model-theoretic dividing lines and other combinatorial properties, particularly in the context of valued fields. In chapter 2, we investigate convex sets over valued fields, providing a classification result for them, and studying how the combinatorial properties satisfied by the family of convex sets over a valued field compares with the family of convex sets over $\mathbb{R}$. In chapter 3, we introduce and study two closely related concepts that we call semi-equationality, and weak semi-equationality, which are generalizations of equationality beyond the stable context, and also closely related to distality.

The dissertation of Alex Jacob Mennen is approved.

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2022

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CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

Much of contemporary model theory studies combinatorial properties of first-order formulas, or equivalently, of the families of sets they define. When a first-order theory implies that all formulas share some particular combinatorial regularity, this can enable the development of techniques to study that theory. The most influential of these combinatorial properties is stability, first studied by Shelah, allowed him to develop machinery that enabled him and others to classify the possible numbers of models (up to isomorphism) of each uncountable cardinality that a complete countable theory can have [She90]. Another important combinatorial property, called NIP (for "no independence property"), also introduced by Shelah, is useful for, among many other things, defining a notion of dimension of definable sets in NIP theories called dp-rank [She14] (one of several such notions in model theory), and for understanding the model theory of algebraically closed valued fields [HHM08] (which are NIP). Since then, these and many other combinatorial properties that first-order formulas and complete theories could have have been extensively studied.

There are other areas of mathematics in which combinatorial properties of families of sets are important, such as computational learning theory, and the study of convex sets, resulting in connections between model theory and these fields. For example, the same combinatorial property characterizing NIP turns out to also be behind the notion of PAC-learnability, resulting in connections between model theory and VC theory, which is motivated primarily by the latter.

In this thesis, we study combinatorial properties of families of sets that are not closed under complement, particularly as they relate to valued fields.

In model theory, the most commonly studied property of first-order formulas that is not closed under negation is equationality. Equations, and equational theories, were introduced by Srour [Sro88a, Sro88b, Sro90] as a way of characterizing formulas carrying positive information, in the sense of defining closed sets in a certain topology. Equations were designed to generalize algebraic sets in algebraically closed fields.

Another property not closed under complement, specifically for subsets of real vector spaces, is convexity. Convex sets are not closely studied in model theory because a real closed field expanded with a predicate for every convex set does not satisfy any of the tameness conditions of first-order theories commonly studied in model theory (in fact, there is a single convex subset of $\mathbb{R}^2$ that can be used to define $(\mathbb{Z}, *, +)$; see Example 2.5.6). Despite this, there are certain commonalities between the combinatorial properties of the family of convex sets over the reals and those of definable families of sets over the reals, such as satisfying the fractional Helly property, and having weak ε-nets of bounded size.

This chapter contains an overview of the main results in this thesis, and some background material.

In the second chapter, we study the family of convex subsets of finite-dimensional vector spaces over valued fields. In particular, we prove a classification theorem for convex sets over spherically complete valued fields (Theorem 2.3.6), and study the extent to which many known combinatorial properties of the family of convex subsets of a finite-dimensional real vector space hold in the valued fields setting. The Helly's theorem and the fractional Helly theorem are found to hold for convex sets over valued fields (Theorems 2.4.5 and 2.4.14, respectively). We find many results that hold for convex sets over valued fields that are significantly stronger versions of classic results for convex sets over the reals, including

Radon's theorem (Proposition 2.2.8), Tverberg's theorem (Theorem 2.4.15, which can also be seen as a strengthening of Carathéodory's theorem), and a theorem of Bárány that Matoušek calls the first selection lemma (Theorem 2.4.16). An analog of the (p, q)-theorem of Alon and Kleitman [AK92] also holds (Corollary 2.5.1).

Convex sets over valued fields are also found to satisfy certain properties that are not similar to any properties that hold for convex sets over the reals. In particular, the family of convex sets over a given finite-dimensional vector space over a valued field has finite VC-dimension (Theorems 2.4.8 and 2.4.10) and breadth (Theorem 2.4.3). As a result, expansions with predicates for convex sets are much more model-theoretically well-behaved for valued fields than they are for the reals. In particular, they are externally definable if the valued field is spherically complete (Remark 2.5.5).

Section 2.1 contains a more detailed overview of the second chapter. Section 2.2 covers background material on basic properties of convex sets over valued fields, including their connections to submodules over the valuation ring. Section 2.3 presents classification results for convex sets. Section 2.4 contains the main combinatorial results on the family of convex sets. Section 2.5 discusses applications and open questions.

In the third chapter, we introduce the notions of semi-equationality and weak semi-equationality (Definitions 3.2.3 and 3.2.1), which are closely related notions that can be seen either as ways to complete the analogy "stability is to equations as NIP is to what?", or as one-sided generalizations of distality. (Weak) semi-equationality is a property both of formulas and of theories, related by a theory being defined to be (weakly) semi-equational if every formula is a Boolean combination of semi-equations. As implied by the names, semi-equations are weak semi-equations (Proposition 3.2.6).

Some basic results about (weak) semi-equations are established that support the analogy with equations (Proposition 3.2.13), including that weak semi-equations are NIP, a formula is an equation if and only if it is a stable semi-equation, and in a stable theory, all weak semi-equations are equations.

We characterize weak semi-equations in terms of having a one-sided version of the strong honest definitions that characterize distal theories (Theorem 3.4.5). A formula being a semi-equation is equivalent to the family of instances of the formula having finite breadth (Proposition 3.2.20), and this can be interpreted as the formula having a one-sided strong honest definition that is a conjunction of instances of itself (Corollary 3.4.8), highlighting a similarity between equations and formulas in distal theories.

Numerical parameters associated to semi-equations (Definition 3.2.17) and introduced and studied. We show that 1-semi-equations relate to weakly normal formulas in a similar way that semi-equations relate to equations (Proposition 3.2.22).

We find some examples of (weakly) semi-equational theories. For instance, linear o-minimal expansions of groups (Proposition 3.3.1), arbitrary unary expansions of linear orders (Fact 3.3.5), and infinitely-branching dense trees (Theorem 3.3.16) are semi-equational. All distal structures are weakly semi-equational (Proposition 3.2.13(2)), and expansions of distal structures by a predicate are weakly semi-equational under certain conditions (Theorem 3.6.4). An example is found showing that semi-equationality and 1-semi-equationality are not preserved under expansions by constants: dense cyclic orders are not semi-equational, but dense cyclic orders with one named constant are (Proposition 3.3.8).

Techniques that can be used to show that a theory is not (weakly) semi-equational are identified (most notably Lemma 3.5.2, but also Proposition 3.7.6). These are used to show that algebraically closed valued fields (and several closely related structures) are not weakly semi-equational (Theorem 3.5.10), contrary to what might be predicted from their relationship with algebraically closed fields, which are equational, and fields of p-adic numbers, which are distal.

Section 3.1 contains a more detailed overview of the third chapter. Section 3.2 introduces the main definitions and goes over some basic properties. Section 3.3 goes over some examples of semi-equational theories. Section 3.4 discusses the relationship between (weak) semi-equationality and distality in terms of one-sided strong honest definitions. Section

3.5 introduces a technique for showing that structures are not weakly semi-equational, and examples of its use. Section 3.6 discusses conditions under which expansions of a distal structure by a predicate can be shown to be weakly semi-equational. Section 3.7 adapts the techniques of section 3.5 to get a criterion that could be used to show that a structure is not semi-equational.

1.2 VC theory

Vapnik-Chervonenkis (VC) theory is the study of a certain way of quantifying the combinatorial complexity of families of sets. It turns out to be useful in computational learning theory, since, if samples are given $\{0,1\}$-valued labels in some unknown way, a family of subsets of the sample space represents a space of hypotheses for how samples get labeled, each set corresponding to the hypothesis that samples are labeled 1 iff they are in that set. As we will see later, VC theory also has close connections to model theory.

Definition 1.2.1. 1. A set system on a set X is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of X.

2. Given a set system $\mathcal{F}$ on a set X, and $Y \subseteq X$, $\mathcal{F} \restriction_Y$ is the set system $\{S \cap Y \mid S \in \mathcal{F}\}$ on Y.

3. The shatter function $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ of a set system $\mathcal{F}$ is defined as

$$\pi_{\mathcal{F}}(n) := \max\{|\mathcal{F} \restriction_A| \mid A \subseteq X, |A| = n\}.$$

   This is the maximal number of subsets of an n-element set that can be cut out by sets in $\mathcal{F}$.

4. A subset $A \subseteq X$ is shattered by $\mathcal{F}$ if $\mathcal{F} \restriction_A = \mathcal{P}(A)$; that is, if every subset of A is its intersection with some set in $\mathcal{F}$.

5. The VC-dimension of $\mathcal{F}$, denoted $\mathrm{VC}(\mathcal{F})$, is the maximum size of finite sets shattered by $\mathcal{F}$, if such a maximum exists, and ∞ if $\mathcal{F}$ shatters arbitrarily large finite sets.

Equivalently, $\mathrm{VC}(\mathcal{F}) := \max\{n \mid \pi_{\mathcal{F}}(n) = 2^n\}$ (and ∞ if $\pi_{\mathcal{F}}(n) = 2^n$ for all n).

6. The VC-density of $\mathcal{F}$ is $\mathrm{vc}(\mathcal{F}) := \limsup_{n\to\infty} \frac{\log(\pi_{\mathcal{F}}(n))}{\log(n)}$, or equivalently, $\inf\left\{d \mid \pi_{\mathcal{F}}(n) = O\left(n^d\right)\right\}$. This can be thought of as the polynomial growth rate of $\pi_{\mathcal{F}}$.

Lemma 1.2.2 (Sauer-Shelah Lemma)**.** *[Sau72] If $VC(\mathcal{F}) = d$, then $\pi_{\mathcal{F}}(n) \leq \sum_{k=0}^{d} \binom{n}{k}$. Consequently, $vc(\mathcal{F}) \leq VC(\mathcal{F})$, and $VC(\mathcal{F}) = \infty \iff vc(\mathcal{F}) = \infty$.*

Example 1.2.3. Let X be an infinite set, and $\mathcal{F} := \{A \subseteq X \mid |A| = d\}$. Then $\mathrm{VC}(\mathcal{F}) = d$, and $F \restriction_Y = \{A \subseteq Y \mid |A| \leq d\}$, so $\pi_{\mathcal{F}}(n) = \sum_{k=0}^{d} \binom{n}{k}$, showing that the bound in lemma 1.2.2 is tight.

Definition 1.2.4.

1. Let $\mathcal{F}$ be a set system on a set X. Its dual set system is the set system on $\mathcal{F}$ given by $\mathcal{F}^* := \{\{S \in \mathcal{F} \mid x \in S\} \mid x \in X\} \subseteq \mathcal{P}(\mathcal{F})$. This can be thought of as keeping the same incidence relation between X and $\mathcal{F}$, but switching which is which.

2. The dual VC-dimension and dual VC-density of $\mathcal{F}$ are the VC-dimension and VC-density, respectively, of $\mathcal{F}^*$. These are denoted $\mathrm{VC}^*(\mathcal{F}) := \mathrm{VC}(\mathcal{F}^*)$, $\mathrm{vc}^*(\mathcal{F}) := \mathrm{vc}(\mathcal{F})$. Dual VC-dimension is also sometimes called independence dimension.

Lemma 1.2.5. *[Ass83, 2.13(b)] If $VC(\mathcal{F}) = d$, then $VC^*(\mathcal{F}) < 2^{d+1}$.*

Definition 1.2.6. The breadth of a set system is the smallest number d (if there is any) such that any intersection of finitely many sets in the set system is the intersection of at most d of them.

Lemma 1.2.7. *[ADH+16, lemma 2.9] $breadth(\mathcal{F}) \geq VC^*(\mathcal{F})$.*

1.3 Model theory

1.3.1 Notational conventions

Letters near the end of the alphabet will be used to denote variables, and letters near the beginning of the alphabet for constants. Each of these can be used to refer to tuples, and will not be assumed to refer to singletons unless it is specified. The length of a tuple x will be denoted $|x|$.

Variables in formulas can be separated by either semi-colons or commas, like $\varphi(x, y; z)$, with semi-colons serving to divide variables into groups (so the formula $\varphi(x, y; z)$ has its variables partitioned into two groups: (x, y), and (z)). When the variables are partitioned into two groups, the second group is typically to be thought of as variables to plug parameters into to define a relation on the first group (i.e. a formula $\varphi(x; y)$ defines the definable family of sets $\{\varphi(\mathcal{M}; b) \mid b \in \mathcal{M}\}$, where $\varphi(\mathcal{M}; b)$ is used as shorthand for the definable set $\{a \in \mathcal{M} \mid \mathcal{M} \models \varphi(a; b)\}$).

1.3.2 Types, monster models, and indiscernibility

Definition 1.3.1. 1. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq \mathcal{M}$, then $\mathcal{L}_A$ is the expansion of $\mathcal{L}$ also containing a constant symbol for every element of A. $\mathcal{M}$ is also considered an $\mathcal{L}_A$-structure, with symbols in $\mathcal{L}$ interpreted in the same way as they are in the $\mathcal{L}$-structure $\mathcal{M}$, and symbols corresponding to elements of A interpreted as the corresponding elements.

2. A partial n-type over A is a set of $\mathcal{L}_A$-formulas in n variables.

3. A partial n-type $p(x_1, ..., x_n)$ is consistent if for every finite subset $p_0 \subseteq p$, there are $a_1, ..., a_n \in \mathcal{M}$ such that $\mathcal{M} \models p_0(a_1, ..., a_n)$.

4. A complete n-type (or simply, an n-type) $p(x_1, ..., x_n)$ is a partial n-type that is

consistent and such that for every $\mathcal{L}_A$-formula $\varphi(x_1, ..., x_n)$, p contains either φ or $\neg\varphi$.

5. If $a_1, ..., a_n \in \mathcal{M}$ and $A \subseteq \mathcal{M}$, the type of $(a_1, ..., a_n)$ over A, denoted $\operatorname{tp}(a_1, ..., a_n/A)$, is the complete type consisting of all $\mathcal{L}_A$-formulas $\varphi(x_1, ..., x_n)$ such that $\mathcal{M} \models \varphi(a_1, ..., a_n)$. The type of a over $\emptyset$ is simply called the type of a, and denoted $\operatorname{tp}(a)$.

The space of n-types over a set A is denoted $S_n(A)$. If x is a tuple of variables, In all multi-sorted languages considered, there will be a main sort, and $S_n(A)$ will refer to the space of n-types over A with variables in the main sort. $S_x(A)$ will also be used to denote $S_{|x|}(A)$, when x is the variables appearing in the formulas in these types.

There is also an analog of a type localized to a single formula.

Definition 1.3.2. 1. For a formula $\varphi(x; y)$ and set A, a φ-type over A is a consistent partial $|x|$-type over A only containing formulas of the form $\varphi(x; b)$ or $\neg\varphi(x; b)$ for $b \in A$, and which, for every $b \in A$, does contain either $\varphi(x; b)$ or $\neg\varphi(x; b)$.

2. If $\varphi(x; y)$ is a formula, $a \in \mathcal{M}^x$ and $A \subseteq \mathcal{M}$, the φ-type of a over A, denoted $\operatorname{tp}_\varphi(a/A)$, is the φ-type $\{\varphi(x; b) \mid \mathcal{M} \models \varphi(a; b)\} \cup \{\neg\varphi(x; b) \mid \mathcal{M} \models \neg\varphi(a; b)\}$.

The space of φ-types over a set A is denoted $S_\varphi(A)$.

Definition 1.3.3. For an infinite cardinal κ, a structure $\mathcal{M}$ is κ-saturated if, for every $A \subseteq \mathcal{M}$ with $|A| < \kappa$, $n \in \mathbb{N}$, and every type $p \in S_n(A)$, there is some $a \in \mathcal{M}$ such that $\mathcal{M} \models p(a)$.

Definition 1.3.4. For an infinite cardinal κ, a structure $\mathcal{M}$ is κ-homogeneous if, for every $A \subseteq \mathcal{M}$ with $|A| < \kappa$, and $\sigma : A \to \mathcal{M}$ such that for any $n \in \mathbb{N}$ and $a \in A^n$, $\operatorname{tp}(a) = \operatorname{tp}(\sigma(a))$, σ extends to an automorphism of $\mathcal{M}$.

It is frequently convenient to work in a κ-saturated model for some sufficiently large κ, so that arguments about it being consistent for there to exist a family of elements satisfying some property can be simplified to arguments that there is such a family of elements. It is occasionally also useful to work in a κ-homogeneous model for some sufficiently large κ, when using arguments involving automorphisms. A model that is κ-saturated and κ-homogeneous for some sufficiently large κ is often called a monster model, and subsets of cardinality less than κ are often called small. Working in such a model is always possible, because:

Fact 1.3.5. *[Che19, Fact 1.2.2] For every structure $\mathcal{M}$ and infinite cardinal κ, there is a κ-saturated and κ-homogeneous elementary extension $\mathbb{M} \succ \mathcal{M}$.*

Another useful model-theoretic tool is indiscernible sequences.

Definition 1.3.6. Given a set $A \subseteq \mathcal{M}$, a totally ordered index set I, a sequence $(a_i)_{i\in I}$ is A-indiscernible if, for every formula $\varphi(x_1, \dots, x_n; y)$, $b \in A^{|y|}$, and every $i_1, \dots, i_n, j_1, \dots, j_n \in I$ with $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$, $\models \varphi(a_{i_1}, \dots, a_{i_n}, b) \iff \models \varphi(a_{j_1}, \dots, a_{j_n}, b)$. An $\emptyset$-indiscernible sequence is also called simply an indiscernible sequence.

In a saturated model, it is easy to find indiscernible sequences.

Fact 1.3.7. *[Che19, Proposition 2.4.4] If $A \subseteq \mathcal{M}$, I and J are linear orders, $(a_j)_{j\in J}$ is any sequence, and $\mathcal{M}$ is $\max(|A|, |I|)^+$-saturated, then there is an indiscernible sequence $(a'_i)_{i\in I}$ such that, for every formula $\varphi(x_1, ..., x_n; y)$ and $b \in A^y$, if $\models \varphi(a_{j_1}, \dots, a_{j_n}; b)$ for every $j_1 < \dots < j_n \in J$, then $\models \varphi\left(a'_{i_1}, \dots, a'_{i_n}; b\right)$ for every $i_1 < \dots < i_n \in I$.*

1.3.3 Stability and NIP

Much of modern model theory deals with combinatorial properties of first-order formulas, the most important of these being stability.

Proposition 1.3.8. *Let T be a complete theory. Given a formula $\varphi(x; y)$, the following are equivalent:*

1. *There is some $k \in \mathbb{N}$ such that in some/every model, there are no $(a_i, b_i)_{i\in[k]}$ such that $\models \varphi(a_i, b_j) \iff i < j$.*

2. *For some/every infinite linear order I, in some/every $|I|^+$-saturated model, there are no $(a_i, b_i)_{i\in I}$ such that $\models \varphi(a_i; b_j) \iff i < j$.*

3. *For some/every model $\mathcal{M}$ and infinite cardinal $\kappa \leq |\mathcal{M}|$, for every $A \subseteq \mathcal{M}$ with $|A| = \kappa$, there are κ φ-types over A.*

4. *For some/every model $\mathcal{M}$ and every $A \subseteq \mathcal{M}$, φ-types over A are uniformly definable, meaning that there is some formula $\psi(y; z)$ such that for every φ-type $p(x)$ over A, there is some $b \in A$ such that $\varphi(x; a) \in p(x)$ iff $\models \psi(a; b)$ for $a \in A$.*

Definition 1.3.9. A formula satisfying these conditions is called stable, and a theory is called stable if all formulas are stable.

Proof of proposition 1.3.8. The equivalence of "some" and "every" in (1) follows from the fact that T is complete.

(1) $\implies$ (2): If (1), then given any infinite linear order I and any $|I|^+$-saturated model $\mathcal{M}$, any $(a_i, b_i)_{i\in I}$ such that $\models \varphi(a_i; b_j) \iff i < j$ would contain arbitrarily long finite subsequences satisfying the same condition, contradicting (1).

(2) $\implies$ (1): If not (1), then let $\mathcal{M}$ be ω_1-saturated. Since for every $k \in \mathbb{N}$ there are $(a_i, b_i)_{i\in[k]}$ such that $\models \varphi(a_i, b_j) \iff i < j$, it follows by saturation that there are $(a_i, b_i)_{i\in\mathbb{N}}$ such that $\models \varphi(a_i, b_j) \iff i < j$.

For (3) and (4), see [Che19, Propositions 2.2.8, 2.2.13, Theorem 2.3.8] □

Although stability is a simple combinatorial property of individual formulas, a theory being stable turns out to have many important implications, such as the existence of a well-behaved independence relation generalizing, for instance, algebraic independence in algebraically closed fields, or linear independence in vector spaces. It is also useful for determining how many complete types there can be over a set of given cardinality, or how many models there are of a given cardinality up to isomorphism.

Another important combinatorial property of first-order formulas generalizing stability, is called NIP.

Definition 1.3.10. The alternation rank of a formula $\varphi(x;y)$ is the largest $k \in \mathbb{N}$ for which there is some b and indiscernible sequence $(a_i)_{i\in[k]}$ such that for $i < k$, $\models \varphi(a_i;b) \leftrightarrow \varphi(a_{i+1};b)$. If there is such an indiscernible sequence for every k, say that $\varphi(x;y)$ has infinite alternation rank.

Proposition 1.3.11. *Let T be a complete theory. Given a formula $\varphi(x;y)$ (where x and y may be tuples of variables), the following are equivalent:*

1. *There is some $k \in \mathbb{N}$ such that in some/every model, there are no $(a_i)_{i\in[k]}, (b_X)_{X\subseteq[k]}$ such that $\models \varphi(a_i, b_X) \iff i \in X$.*

2. *There is some finite bipartite graph on vertex sets A, B such that in some/every model, there are no $(a_i)_{i\in A}, (b_j)_{j\in B}$ such that $\models \varphi(a_i, b_j)$ iff i and j are connected by an edge.*

3. *For some/every infinite set I and some/every $\left(2^{|I|}\right)^+$-saturated model, there are no $(a_i)_{i\in I}, (b_X)_{X\subseteq I}$ such that $\models \varphi(a_i, b_X) \iff i \in X$.*

4. *$\varphi(x;y)$ has finite alternation rank.*

Definition 1.3.12. A formula is said to have the independence property if it does not satisfy these conditions, and is said to be NIP (for "no independence property") if it does. A theory is called NIP if all formulas are NIP.

Note that criterion (1) says that $\varphi(x;y)$ is NIP iff the family of sets $\{\varphi(\mathcal{M};b) \mid b \in \mathcal{M}\}$ has finite VC-dimension, since its VC-dimension is the smallest k such that there are no $(a_i)_{i\in[k]}, (b_X)_{X\subseteq[k]}$ such that $\models \varphi(a_i, b_X) \iff i \in X$.

Proof of proposition 1.3.11. In all three equivalent conditions, "every" implies "some", and the converse is clear in (1) and (2) because T is complete, so it suffices to show (2) $\implies$ (1) $\implies$ (3, every), and (3, some) $\implies$ (2).

(2) $\implies$ (1): Given such a bipartite graph on A, B, let $k = |A|$. Given $(a_i)_{i\in[k]}, (b_X)_{X\subseteq[k]}$ such that $\models \varphi(a_i, b_X) \iff i \in X$, use a bijection to identify A with $[k]$, and for $j \in B$, let $b_j := b_{i\in[k]|iEj}$, where E is the edge relation. Then $\models \varphi(a_i, b_j)$ iff iEj.

(1) $\implies$ (3): Given infinite I and $(a_i)_{i\in I}, (b_X)_{X\subseteq I}$ such that $\models \varphi(a_i, b_X) \iff i \in X$, for any $k \in \mathbb{N}$, identify $[k]$ with a k-element subset of I using an injection $[k] \to I$, and consider $(a_i)_{i\in[k]}, (b_X)_{X\subseteq[k]}$.

(3) $\implies$ (2): Given infinite I and $\left(2^{|I|}\right)^+$-saturated $\mathcal{M}$, if for every finite subsets $I_0 \subset I$ and $\mathcal{F} \subset 2^I$, there are $(a_i)_{i\in I_0}, (b_X)_{X\in\mathcal{F}}$ such that $\models \varphi(a_i, b_X) \iff i \in X$, then, by saturation, there are $(a_i)_{i\in I}, (b_X)_{X\subseteq I}$ such that $\models \varphi(a_i, b_X) \iff i \in X$. Otherwise, use $A = I_0, = \mathcal{F}$, and the finite bipartite graph given by the membership relation between them.

For (4), see [Che19, Proposition 2.44] □

Proposition 1.3.13. *Boolean combinations of stable/NIP formulas are stable/NIP, respectively.*

Proof. If $\neg\varphi(x; y)$ is unstable, then for each k, there are $(a_i, b_i)_{i\in[k+1]}$ such that $\models \varphi(a_i; b_j) \iff i \geq j$. We can turn $\geq$ into $<$ by reversing the order and shifting over by one step: let $a_i' := k + 2 - i$ and $b_i' := k + 1 - i$ for $i \in [k]$, so $\models \varphi\left(a_i'; b_j'\right) \iff i < j$. So $\varphi(x; y)$ is unstable.

If $\neg\varphi(x; y)$ has the independence property, then for each k, there are $(a_i,)_{i\in[k+1]}, (b_X)_{X\subseteq[k]}$ such that $\models \varphi(a_i; b_X) \iff i \notin X$. Let $b_X' := b_{[k]\setminus X}$. Then $\models \varphi\left(a_i; b_X'\right) \iff i \in X$, so $\varphi(x; y)$ has the independence property.

If $(\varphi \vee \psi)(x; y)$ is unstable, then, in an ω_1-saturated model, there are $(a_i, b_i)_{i\in\mathbb{N}}$ such that $\models (\varphi \vee \psi)(x; y) \iff i < j$. By Ramsey's theorem, there is an infinite set $X \subseteq \mathbb{N}$ such that either $\models \varphi(a_i; b_j)$ for all $i, j \in X$ with $i < j$, or $\not\models \varphi(a_i; b_j)$ for all $i, j \in X$ with $i < j$. In the latter case, we have $\models \psi(a_i; b_j)$ for all $i, j \in X$ with $i < j$. Thus at least one of $\varphi(x; y)$ or $\psi(x; y)$ must be unstable.

If $(\varphi \vee \psi)(x; y)$ has the independence property, then, for every $k \in \mathbb{N}$, there is b and $(a_i)_{i \in [2k]}$ such that $\models (\varphi \vee \psi)(a_i; b) \iff i$ is even. Let $X := \{i \in [k] \mid \models \varphi(a_{2i}; b)\}$. If $|X| \geq \frac{k}{2}$, then $(a_{2i}, a_{2i-1})_{i \in X})$ shows that $\varphi(x; y)$ has alternation rank as least k. Otherwise, $(a_{2i}, a_{2i-1})_{i \in [k] \setminus X})$ shows that $\psi(x; y)$ has alternation rank at least k. Thus at least one of these must have infinite alternation rank. $\square$

Proposition 1.3.14. *Stable formulas are NIP.*

Proof. If $\varphi(x; y)$ has the independence property, then, for $k \in \mathbb{N}$, let $(a_i)_{i \in [k]}, (b_X)_{X \subseteq [k]}$ such that $\models \varphi(a_i; b_X) \iff i \in X$. Then $\models \varphi\left(a_i; b_{[j-1]}\right) \iff i < j$. $\square$

1.3.4 Equations

Often we wish to distinguish some formulas as carrying "positive" information. For example, in algebraically closed fields, while all Boolean combinations of algebraic varieties are definable, it is the algebraic varieties themselves that we wish to single out as carrying positive information. The notion of an "equation" is one way to formalize this notion in the context of stable theories.

Definition 1.3.15. A set system $\mathcal{F}$ has the descending intersection condition if for every collection of sets in $\mathcal{F}$, their intersection is the intersection of some finite subcollection. Equivalently, for any sequence $(X_i)_{i \in \mathbb{N}}$ of sets in $\mathcal{F}$, its sequence of partial intersections $(\cap_{i<n} X_i)_{n \in \mathbb{N}}$ is eventually constant.

Definition 1.3.16. In a complete theory T, a formula $\varphi(x, y)$ is an equation if, in some/every ω_1-saturated model $\mathcal{M} \models T$, the set system $\{\varphi(\mathcal{M}, b) \mid b \in \mathcal{M}\}$ has the descending intersection condition. A theory is called equational if all formulas are equivalent to Boolean combinations of equations.

Example 1.3.17. In any theory, the formula $x = y$ is an equation, but $x \neq y$ is not an equation (except in theories with finite models). This shows that equations are not closed under Boolean combinations.

Proposition 1.3.18. *If $\varphi(x; y)$ and $\psi(x; y)$ are equations, then so are $(\varphi \wedge \psi)(x; y)$ and $(\varphi \vee \psi)(x; y)$. That is, equations are closed under positive Boolean combinations.*

Proof. If $\varphi(x; y)$ and $\psi(x; y)$ are equations, then for any sequence $(b_i)_{i \in \mathbb{N}}$, there are $n, m \in \mathbb{N}$ such that for any a, if $\models \varphi(a; b_i)$ for $i \leq n$, then $\models (a; b_i)$ for all i, and if $\models \psi(a; b_i)$ for $i \leq m$, then $\models \psi(a; b_i)$ for all i. Then, if $\models (\varphi \wedge \psi)(a; b_i)$ for $i \leq \max(n, m)$, then $\models (\varphi \wedge \psi)(a; b_i)$ for all i, and likewise for $(\varphi \vee \psi)(x; y)$, so $(\varphi \wedge \psi)(x; y)$ and $(\varphi \vee \psi)(x; y)$ are equations. $\square$

Example 1.3.19.

1. In any completion of the theory of fields, if $f(x; y)$ is any polynomial, the formula $f(x; y) = 0$ is an equation. This is the Hilbert basis theorem, and is the motivation for the name "equation".

2. By Chevalley's theorem, ACF_0 and ACF_p eliminate quantifiers. Since, in fields, all atomic formulas are equivalent to formulas of the form $f = 0$ for some polynomial f, it follows that in ACF_0 and ACF_p, all formulas are equivalent to Boolean combinations of the equations $f(x; y) = 0$, so these theories are equational.

Theorem 1.3.20. *Equations are stable. Since stable formulas are closed under Boolean combinations, it follows that Boolean combinations of equations are stable, and hence equational theories are stable.*

Proof. If $\varphi(x; y)$ is unstable, let $(a_i, b_i)_{i \in \mathbb{N}}$ be such that $\models \varphi(a_i; b_j) \iff i < j$. Then $\bigcap_{i \in \mathbb{N}} \varphi(\mathcal{M}; b_i)$ is not equal to any of its finite subintersections. $\square$

For an introductory resource on equations, see [O'H11a].

1.3.5 Distality

Proposition 1.3.21. *[CS15, Theorem 21] Let T be a complete theory. Given a formula $\varphi(x;y)$, the following are equivalent:*

1. *For some/every infinite linear orders I_L, I_R, in some/every $(I_L+I_R)^+$-saturated model $\mathcal{M} \models T$, for every $b \in \mathcal{M}^{|y|}$ and indiscernible sequence $(a_i)_{i\in I_L+(0)+I_R}$ such that $(a_i)_{i\in I_L+I_R}$ is b-indiscernible, $\mathcal{M} \models \varphi(a_0,b) \iff \mathcal{M} \models \varphi(a_i,b)$ for $i \in I_L+I_R$.*
2. *There is a formula $\theta(x;z)$ such that, in some/every model $\mathcal{M} \models T$, for every $a \in \mathcal{M}^{|x|}$ and finite $C \subset \mathcal{M}$, there is some $b \in \mathcal{C}^{|z|}$ such that $\mathcal{M} \models \theta(a;b)$, and $\theta(x;b) \vdash \mathrm{tp}_{\varphi}(a/C)$.*

Definition 1.3.22. A formula meeting these conditions is called distal. A theory is called distal if all formulas are distal.

Proposition 1.3.23. *Distal formulas are NIP.*

Proof. If $\varphi(x;y)$ has the independence property, then it has infinite alternation rank. That is, in an ω^+-saturated model, there is b and an indiscernible sequence $(a_i)_{i\in\mathbb{Z}}$ such that $\models \varphi(a_i;b) \iff i \in 2\mathbb{Z}$. Then b, together with the indiscernible sequence $(a_i)_{i \text{is odd or } 0}$ is a counterexample to distality of $\varphi(x;y)$. $\square$

Proposition 1.3.24. *[Sim13] No theory with infinite models can be both stable and distal.*

Because of proposition 1.3.24, among other reasons, distality can be seen, within NIP, as the opposite extreme from stability.

Example 1.3.25. [Sim13] $\mathbb{R}$ and $\mathbb{Q}_p$ are distal.

1.4 Valued fields

1.4.1 Basics

Definition 1.4.1. A valued field is a field K equipped with a surjective map $\nu : K^\times \to \Gamma$ for some ordered abelian group Γ called the value group, such that $\nu(xy) = \nu(x) + \nu(y)$ and $\nu(x+y) \geq \min(\nu(x), \nu(y))$. ν is extended to a map $K \to \Gamma_\infty$, where $\Gamma_\infty := \Gamma \sqcup \{\infty\}$ is an ordered commutative monoid with $\infty > \Gamma$ and $\infty + \gamma = \infty$ for $\gamma \in \Gamma_\infty$, by $\nu(0) = \infty$. The valuation ring of K is the subring $\{a \in K \mid \nu(a) \geq 0\}$, and is typically denoted $\mathcal{O}_K$, or just $\mathcal{O}$ if the valued field is unambiguous. $\mathcal{O}$ has a unique maximal ideal $\{a \in K \mid \nu(a) > 0\}$, typically denoted $\mathfrak{m}_K$, or just $\mathfrak{m}$ is the valued field is unambiguous. The field $\mathcal{O}/\mathfrak{m}$ is called the residue field, and typically denoted k. ν is called trivial if $\Gamma = \{0\}$, or equivalently, if $k = K$. The residue map $\mathcal{O} \to \mathcal{O}/\mathfrak{m} = k$ is typically denoted with an overline; i.e. $x \mapsto \bar{x}$.

Definition 1.4.2.

1. In a valued field K, the open ball of radius $r \in \Gamma$ around center $c \in K$ is $\{a \in K \mid \nu(a-c) > r\}$.
2. The closed ball of radius $r \in \Gamma$ around center $c \in K$ is $\{a \in K \mid \nu(a-c) \geq r\}$.
3. If $\Delta \subseteq \Gamma_\infty$ is nonempty and upward-closed, the quasi-ball of quasi-radius Δ around center $c \in K$ is $\{a \in K \mid \nu(a-c) \in \Delta\}$.

Note that open and closed balls are conventional notions, but the notion of a quasi-ball is ideosyncratic. The quasi-balls are exactly the translates of $\mathcal{O}$-submodules of K. Open balls of radius r are the same as quasi-balls of quasi-radius $\{\gamma \in \Gamma_\infty \mid \gamma > r\}$. Closed balls of radius r are the same as quasi-balls of quasi-radius $\{\gamma \in \Gamma_\infty \mid \gamma \geq r\}$. And any two quasi-balls are either nested or disjoint. Any two quasi-balls of the same quasi-radius are either equal or disjoint.

Valued fields are topological fields, with any of the open balls, the closed balls, or the quasi-balls with quasi-radius larger than $\{\infty\}$ as a basis of open sets; these generate the same topology if the valuation is nontrivial. All quasi-balls of quasi-radius larger than $\{\infty\}$ are clopen (hence so are open balls and closed balls, despite the names). Quasi-balls of quasi-radius $\{\infty\}$ are singleton sets, and are closed but not open.

Definition 1.4.3.

1. A nest of quasi-balls is a set of quasi-balls that is linearly ordered under inclusion.

2. A valued field K is spherically complete if the intersection of any nest of quasi-balls is nonempty; or equivalently, if the intersection of any nest of (open/closed) balls is nonempty.

Proposition 1.4.4. *If a valued field is complete, then it is spherically complete.*

Definition 1.4.5. A spherical completion of a field K with valuation $\nu : K \to \Gamma_\infty$ is an extension field $\tilde{K}/K$ with valuation $\tilde{\nu} : \tilde{K} \to \Gamma_\infty$ such that $\tilde{\nu} \restriction_K = \nu$, $\tilde{K}$ is spherically complete, and no proper intermediate extension $K \subseteq L \subsetneq \tilde{K}$ is spherically complete (with the valuation $\tilde{\nu} \restriction_L$).

Proposition 1.4.6. *[Sch50, section 2.3 Theorem 5] Every valued field has a spherical completion.*

Note that spherical completions are not always unique.

Definition 1.4.7. A valued field K is called henselian if for every polynomial $f \in \mathcal{O}[x]$ and $a \in \mathcal{O}$ such that $f(a) \in \mathfrak{m}$ and $f'(a) \notin \mathfrak{m}$, there is $\tilde{a} \in \mathcal{O}$ such that $\tilde{a} - a \in \mathfrak{m}$ and $f(\tilde{a}) = 0$. Or equivalently, if for every polynomial $f \in \mathcal{O}[x]$ and $a \in k$ such that $\bar{f}(a) = 0$ and $\bar{f}'(a) \neq 0$ (where the overline represents reduction mod $\mathfrak{m}[x]$), there is $\tilde{a} \in \mathcal{O}$ such that $\bar{\tilde{a}} = a$ and $f(\tilde{a}) = 0$; that is, every simple root of the residue of f lifts to a root of f.

Proposition 1.4.8 (Hensel's lemma)**.** *[Mar18, theorem 2.2] Spherically complete valued fields are henselian.*

Hensel's lemma essentially follows from Newton's method adapted to the valued fields setting. Note that algebraically closed valued fields are also henselian.

Definition 1.4.9. The characteristic of a valued field K is the ordered pair $(\mathrm{char}(K), \mathrm{char}(k))$. A valued field of characteristic $(0,0)$ is called equicharacteristic 0, a valued field of characteristic (p,p) for a prime p is called equicharacteristic p, and a valued field of characteristic $(0,p)$ for a prime p is called mixed characteristic. These are all the possible characteristics of a valued field.

1.4.2 First-order languages of valued fields

Definition 1.4.10.

1. The two-sorted language of valued fields is the language $\mathcal{L}_{2vf}$ with sorts K and Γ_∞, with the structure of a field on K, the structure of an ordered monoid on Γ_∞, and a valuation function $\nu : K \to \Gamma_\infty$.

2. The three-sorted language of valued fields is the language $\mathcal{L}_{3vf}$ with sorts K, Γ_∞, and k, with all the structure of $\mathcal{L}_{2vf}$ on K and Γ, and in addition, the structure of a field on k, and a residue function $\mathrm{res} : K \to k$, which is intended to be interpreted as the residue map on $\mathcal{O}$, and send everything outside of $\mathcal{O}$ to 0.

3. The one-sorted language of valued fields is the language $\mathcal{L}_{vf}$ with sort K, the structure of a field on K, and a unary relation $\mathcal{O}$ on K.

The unary relation $\mathcal{O}(x)$ is definable in $\mathcal{L}_{2vf}$ as $\nu(x) \geq 0$. Γ_∞ is interpretable in $\mathcal{L}_{1vf}$ as K, with $x =_{\Gamma_\infty} y$ defined by $(\exists z\, \mathcal{O}(z) \wedge xz = y) \wedge (\exists w\, \mathcal{O}(w) \wedge yw = x)$. k is interpretable in $\mathcal{L}_{2vf}$ as $\{x \in K \mid \nu(x) \geq 0\}$, with $x =_k y$ defined by $\nu(x-y) > 0$. Thus a valued field in each of these three languages are mutually biinterpretable, and thus the theories of valued fields in each of these languages are mutually biinterpretable. A few other minor variations of these languages, also biinterpretable with them, have also been considered in the literature.

Definition 1.4.11.

1. An ac-valued field is a valued field K equipped with an angular component map $\mathrm{ac} : K^\times \to k^\times$, a group homomorphism that agrees with the residue map on $\mathcal{O}^\times$.

2. The Denef-Pas language $\mathcal{L}_{dp}$ is the language with sorts K, Γ_∞, and k, the structure of $\mathcal{L}_{3vf}$, and a function $\mathrm{ac} : K \to k$. An ac-valued field is a structure in the Denef-Pas language by extending the angular component map so that $\mathrm{ac}(0) = 0$.

1.4.3 Quantifier elimination results

There are many quantifier elimination results for valued fields in various languages and under various conditions. For example, two that are relevent for us are:

Theorem 1.4.12. *[Pas89] Henselian ac-valued fields of equicharacteristic* 0 *eliminate quantifiers of sort* K *in the language* $\mathcal{L}_{3vf}$.

Theorem 1.4.13. *[Mar18, theorem 4.4] Algebraically closed valued fields (of any characteristic) eliminate quantifiers of sort* K *in the language* $\mathcal{L}_{2vf}$.

See [Mar18] for a more comprehensive survey of these types of results.

1.5 Convexity

A number of combinatorial properties of the class of convex sets in $\mathbb{R}^d$ have been discovered. Let $\mathrm{Conv}_{\mathbb{R}^d}$ denote the class of convex sets in $\mathbb{R}^d$. For $X \subseteq \mathbb{R}^d$, let $\mathrm{conv}(X)$ denote its convex hull.

Definition 1.5.1. $\mathcal{F} \subseteq \mathcal{P}(X)$ has Helly number k if $\forall n \, \forall S_1, ..., S_n \in \mathcal{F}$ if every k-subset of $\{S_1, ..., S_n\}$ has nonempty intersection, then $\bigcap_{1 \leq i \leq n} S_i \neq \emptyset$. The Helly number of $\mathcal{F}$ refers to the minimal k with this property. Say that $\mathcal{F}$ has the Helly property if it has a Helly number.

Theorem 1.5.2 (Helly's theorem)**.** *[Mat02, theorem 1.3.2]* $\mathrm{Conv}_{\mathbb{R}^d}$ *has the Helly property, with Helly number* $d+1$.

Definition 1.5.3. A family of sets $\mathcal{F}$ has fractional Helly number k if $\forall \alpha > 0 \; \exists \beta > 0 \; \forall n$ $\forall S_1, ..., S_n \in \mathcal{F}$ if there are $\geq \alpha \binom{n}{k}$ k-subsets of $\{S_1, ..., S_n\}$ with an intersection point, then there are $\geq \beta n$ sets from $\{S_1, ..., S_n\}$ that all intersect in a common point. The fractional Helly number of $\mathcal{F}$ refers to the minimal k with this property. Say that $\mathcal{F}$ has the fractional Helly property if it has a fractional Helly number.

Theorem 1.5.4 (Fractional Helly theorem)**.** *[Mat02, theorem 8.1.1] $Conv_{\mathbb{R}^d}$ has the fractional Helly property, with fractional Helly number $d+1$.*

Theorem 1.5.5 (Tverberg's theorem)**.** *[Mat02, theorem 8.3.1] Any set X of $\geq (d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into subsets $X_1, ..., X_r$ such that $\bigcap_{i\in[r]} conv(X_i) \neq \emptyset$.*

Theorem 1.5.6 (Colored Tverberg theorem)**.** *[Mat02, theorem 8.3.3] For any positive integers d and r, there is some $t \geq r$ such that for any $X \subseteq \mathbb{R}^d$ with $|X| = t(d+1)$, partitioned into $d+1$ color classes $C_1, ..., C_{d+1}$ of size t, there are disjoint $X_1, ..., X_r \subseteq X$ with $|X_i \cap C_j| = 1$ for $i \in [r]$ and $j \in [d+1]$, and $\bigcap_{i\in[r]} conv(X_i) \neq \emptyset$.*

Theorem 1.5.7 (First selection lemma)**.** *[Mat02, theorem 9.1.1] For each d, there is a constant $c > 0$ such that for any finite $X \subseteq \mathbb{R}^d$ (say $n := |X|$), there is some $a \in \mathbb{R}^d$ contained in the convex hulls of at least $c\binom{n}{d+1}$ of the $\binom{n}{d+1}$ $(d+1)$-subsets of X.*

Theorem 1.5.8 (Second selection lemma)**.** *[Mat02, theorem 9.2.1] For each d, there are $c, s > 0$ such that for all $\alpha \in (0,1]$ and any n, for every $X \subseteq \mathbb{R}^d$ with $|X| = n$, and every family $\mathcal{F}$ of $d+1$-subsets of X with $|\mathcal{F}| \geq \alpha\binom{n}{d+1}$, there is a point contained in the convex hulls of at least $c\alpha^s\binom{n}{d+1}$ of the elements of $\mathcal{F}$.*

CHAPTER 2

Combinatorial properties of non-archimedean convex sets

2.1 Introduction

Convexity in the context of non-archimedean valued fields was introduced in a series of papers by Monna in 1940's [Mon46], and has been extensively studied since then in non-archimedean functional analysis (see e.g. the monographs [PGS10, Sch13] on the subject). Convexity here is defined analogously to the real case, with the role of the unit interval played instead by a valuational unit ball (see Definition 2.2.1). Convex subsets of $\mathbb{R}^d$ admit rich combinatorial structure, including many classical results around the theorems of Helly, Radon, Carathéodory, Tverberg, etc. — we refer to e.g. [DLGMM19] for a recent survey of the subject. In the case of $\mathbb{R}$, or more generally a real closed field, there is a remarkable parallel between the combinatorial properties of convex and semi-algebraic sets (which correspond to definable sets from the point of view of model theory). They share many (but not all) properties in the form of various restrictions on the possible intersection patterns, including the fractional Helly theorem and existence of (weak) ε-nets. A well-studied phenomenon in model theory establishes strong parallels between definable sets in $\mathbb{R}$ and in many non-archimedean valued fields such as the p-adics $\mathbb{Q}_p$ or various fields of power series (see e.g. [vdD14]). In this paper we focus on the combinatorial study of convex sets over general valued fields, trying to understand if there is similarly a parallel theory. On the one hand, we demonstrate valued field analogs of some classical results for convex sets over the reals (e.g. the fractional Helly theorem and Bárány's theorem on points in many simplices). On the other, we establish some additional properties not satisfied by convex sets over the reals, including finite breadth and VC-dimension. This suggests that in a sense convex sets over valued fields are the best of both worlds combinatorially, and satisfy various properties enjoyed either by convex or by semialgebraic sets over the reals.

We give a quick outline of the paper. Section 2.2 covers some basics concerning convexity for subsets of K^d over an arbitrary valued field K, in particular discussing the connection to modules over the valuation ring. These results are mostly standard (or small variations of standard results), and can be found e.g. in [PGS10, Sch13] under the unnecessary assumption that K is spherically complete and $(\Gamma, +) \subseteq (\mathbb{R}_{>0}, \times)$; we provide some proofs for completeness. In Section 2.3 we give a simple combinatorial description of the submodules of K^d over the valuation ring $\mathcal{O}_K$ in the case of a spherically complete field K (Theorem 2.3.6 and Corollary 2.3.14), and an analog for finitely generated modules over arbitrary valued fields (Corollary 2.3.16). We also give an example of a convex set over the field of Puiseux series demonstrating that the assumption of spherical completeness is necessary for our presentation in the non-finitely generated case (Example 2.3.11). In Section 2.4 we use this description of modules to deduce various combinatorial properties of the family of convex subsets Conv_{K^d} of K^d over an arbitrary valued field K. First we show that Conv_{K^d} has breadth d (Theorem 2.4.3), VC-dimension $d+1$ (Theorem 2.4.8), dual VC-dimension d (Theorem 2.4.10) — in stark contrast, all of these are infinite for the family of convex subsets of $\mathbb{R}^d$ for $d \geq 2$. On the other hand, we obtain valued field analogs of the following classical results: the family Conv_{K^d} has Helly number $d+1$ (Theorem 2.4.5), fractional Helly number $d+1$ (Theorem 2.4.14), satisfies a strong form of Tverberg's theorem (Theorem 2.4.15) and Boros-Füredi/Bárány theorem on the existence of a common point in a positive fraction of all geometric simplices generated by an arbitrary finite set of points in K^d (Theorem 2.4.16). Some of the proofs here are adaptations of the classical arguments, and some rely crucially on the finite breadth property specific to the valued field context. Finally, in Section 2.5 we point out some further applications, e.g. a valued field analogue of the celebrated (p, q)-theorem of Alon and Kleitman [AK92] (Corollary 2.5.1), and that all convex sets over a spherically complete field are externally definable in the sense of model theory (Remark 2.5.5); as well as pose some questions and conjectures.

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2.2 Preliminaries on convexity over valued fields

Notation. For $n \in \mathbb{N}_{\geq 1}$, we write $[n] = \{1, \dots, n\}$ and $\langle\rangle$ denotes the span in vector spaces. Throughout the paper, K will denote a valued field, with value group $\Gamma = \Gamma_K$, and valuation $\nu = \nu_K : K \to \Gamma_\infty := \Gamma \sqcup \{\infty\}$, valuation ring $\mathcal{O} = \mathcal{O}_K = \nu^{-1}([0, \infty])$, maximal ideal $\mathfrak{m} = \mathfrak{m}_K = \nu^{-1}((0, \infty])$, and residue field $k = \mathcal{O}/\mathfrak{m}$. The residue map $\mathcal{O} \to k$ will be denoted $\alpha \mapsto \bar{\alpha}$.

The following definition of convexity is analogous to the usual one over $\mathbb{R}$, with the unit interval replaced by the (valuational) unit ball.

Definition 2.2.1.
1. For $d \in \mathbb{N}_{\geq 1}$, a set $X \subseteq K^d$ is *convex* if, for any $n \in \mathbb{N}_{\geq 1}$, $x_1, \dots, x_n \in X$, and $\alpha_1, \dots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \ldots + \alpha_n = 1$ we have $\alpha_1 x_1 + \ldots + \alpha_n x_n \in X$ (in the vector space K^d).
2. The family of convex subsets of K^d will be denoted Conv_{K^d}.

It is immediate from the definition that the intersection of any collection of convex subsets of K^d is convex.

Definition 2.2.2. Given an arbitrary set $X \subseteq K^d$, its *convex hull* $\mathrm{conv}(X)$ is the convex set given by the intersection of all convex sets containing X, equivalently

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}, \alpha_i \in \mathcal{O}, x_i \in X, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Definition 2.2.3. A (valuational) *quasi-ball* is a set $B = \left\{x \in K^d : \nu(x-c) \in \Delta\right\}$ for some $c \in K$ and an upwards closed subset Δ of Γ_∞. In this case we say that B is *around* c, and refer to Δ as the quasi-radius of B. We say that B is a *closed* (respectively, *open*) *ball* if additionally $\Delta = \{\gamma \in \Gamma : \gamma \geq r\}$ (respectively, $\Delta = \{\gamma \in \Gamma : \gamma > r\}$) for some $r \in \Gamma$, and just *ball* if B is either an open or a closed ball (in which case we refer to r as its *radius*).

Remark 2.2.4.

1. If the value group Γ is Dedekind complete, then every quasi-ball is a ball (except for K itself, which is a quasi-ball of quasi-radius Γ_∞).
2. Note also that if B is a quasi-ball of quasi-radius Δ around c and $c' \in B$ is arbitrary, then B is also a quasi-ball of quasi-radius Δ around c'.
3. In particular, any two quasi-balls are either disjoint, or one of them contains the other.

Example 2.2.5.

1. The convex subsets of $K = K^1$ are exactly $\emptyset$ and the quasi-balls (see Proposition 2.2.9 and Example 2.2.10).
2. If $e_1, \ldots, e_d$ is the standard basis of the vector space K^d, then
$$\operatorname{conv}\left(\{0, e_1, \ldots, e_d\}\right) = \mathcal{O}^d.$$
3. The image and the preimage of a convex set under an affine map is convex. In particular, a translate of a convex set is convex, and a projection of a convex set is convex. (Recall that given two vector spaces V, W over the same field K, a map $f : V \to W$ is *affine* if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in V, \alpha, \beta \in K, \alpha + \beta = 1$.)

One might expect, by analogy with real convexity, that the definition of a convex set could be simplified to: if $x, y \in X$, $\alpha, \beta \in \mathcal{O}$ such that $\alpha + \beta = 1$, then $\alpha x + \beta y \in X$. The following two propositions show that this is the case if and only if the residue field is not isomorphic to $\mathbb{F}_2$, and that in general we have to require closure under 3-element convex combinations.

Proposition 2.2.6. *Let K be a valued field and $X \subseteq K^d$. If X is closed under 3-element convex combinations (in the sense that if $x, y, z \in X$ and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha+\beta+\gamma = 1$, then $\alpha x + \beta y + \gamma z \in X$), then X is convex.*

Proof. Suppose X is closed under 3-element convex combinations. We will show by induction on n that then X is closed under n-element convex combinations. Let $n \geq 3$, $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \ldots + \alpha_n = 1$ be given. Then one of the following two cases holds.

Case 1: $\alpha_1 + \alpha_2 \in \mathcal{O}^\times$.

Then $\frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\frac{\alpha_2}{\alpha_1+\alpha_2}$ are elements of $\mathcal{O}$ that sum to 1, so

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2 \in X$$

by assumption. But then

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = (\alpha_1 + \alpha_2) \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2 \right) + \alpha_3 x_3 + \ldots + \alpha_n x_n \in X$$

by the induction hypothesis, as it is a convex combination of $n-1$ elements of X.

Case 2: $\alpha_1 + \alpha_2 \in \mathfrak{m}$.

Then, as $\nu\left(\sum_{i=1}^n \alpha_i\right) = 0$, there must exist some i with $3 \leq i \leq n$ such that $\alpha_i \in \mathcal{O}^\times$. Hence $\alpha_1 + \alpha_2 + \alpha_i \in \mathcal{O}^\times$, so $\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_i}$, $\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_i}$, and $\frac{\alpha_i}{\alpha_1+\alpha_2+\alpha_i}$ are elements of $\mathcal{O}$ that sum to 1. Thus

$$\left(\frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_i}\right) x_1 + \left(\frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_i}\right) x_2 + \left(\frac{\alpha_i}{\alpha_1 + \alpha_2 + \alpha_i}\right) x_i \in X$$

by assumption, and so

$$\begin{gathered} \alpha_1 x_1 + \ldots + \alpha_n x_n = \\ (\alpha_1 + \alpha_2 + \alpha_i) \left(\frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_i} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_i} x_2 + \frac{\alpha_i}{\alpha_1 + \alpha_2 + \alpha_i} x_i \right) \\ + \alpha_3 x_3 + \ldots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \ldots + \alpha_n x_n \in X \end{gathered}$$

by the induction hypothesis, as it is a convex combination of $n-2$ elements of X.

$\square$

Proposition 2.2.7. *For any valued field K, the following are equivalent:*

1. *for every $d \geq 1$, every set in K^d that is closed under 2-element convex combinations is convex;*

2. *the residue field k is not isomorphic to $\mathbb{F}_2$.*

Proof. (1) implies (2). If $k = \mathbb{F}_2$, consider the set

$$X := \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathcal{O},\ \exists i\, a_i \in \mathfrak{m}\} \subseteq K^3.$$

We claim that X is closed under 2-element convex combinations. That is, given arbitrary $(a_1, a_2, a_3), (b_1, b_2, b_3) \in X$ and $\alpha, \beta \in \mathcal{O}$ with $\alpha + \beta = 1$, we must show that $\alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3) \in X$. We have $\bar{\alpha} + \bar{\beta} = 1$ in $k = \mathbb{F}_2$, so necessarily one of $\bar{\alpha}$ and $\bar{\beta}$ is 1 and the other is 0. Without loss of generality $\bar{\alpha} = 1$ and $\bar{\beta} = 0$. Then $\beta \in \mathfrak{m}$. By definition of X, $a_i \in \mathfrak{m}$ for some i. Then $\alpha a_i \in \mathfrak{m}$, and $\beta b_i \in \mathfrak{m}$ as $b_i \in \mathcal{O}$, so $\alpha a_i + \beta b_i \in \mathfrak{m}$. Thus $(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3) \in X$. However X is not convex: for an arbitrary $a \in \mathfrak{m}$ we have $(0,0,0), (1,0,0), (0,1,1) \in X$, $1, -1 \in \mathcal{O}$, but $(-1)(0,0,0) + 1(1,0,0) + 1(0,1,1) = (1,1,1) \notin X$. (This example can be modified to work in K^2.)

(2) implies (1). If $k \ncong \mathbb{F}_2$, suppose X is closed under 2-element convex combinations. By Proposition 2.2.6, we only need to check that it is then closed under 3-element convex combinations. Let $x, y, z \in X$, and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$. Then one of the following two cases holds.

Case 1: At least one of $\alpha + \beta, \beta + \gamma, \alpha + \gamma$ is an element of $\mathcal{O}^\times$.

Without loss of generality, $\alpha + \beta \in \mathcal{O}^\times$. Then $\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y \in X$ by assumption, and thus

$$\alpha x + \beta y + \gamma z = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta}x + \frac{\beta}{\alpha + \beta}y\right) + \gamma z \in X.$$

Case 2: $\alpha+\beta, \beta+\gamma, \alpha+\gamma \in \mathfrak{m}$.

In the residue field, $\bar{\alpha}+\bar{\beta}=\bar{\beta}+\bar{\gamma}=\bar{\alpha}+\bar{\gamma}=0$, and $\bar{\alpha}+\bar{\beta}+\bar{\gamma}=1$, hence necessarily $\bar{\alpha}=\bar{\beta}=\bar{\gamma}=1$, and $\operatorname{char}(k)=2$. Since $k \ncong \mathbb{F}_2$, there is $\delta \in \mathcal{O}$ such that $\bar{\delta} \notin \{0,1\}$. Then $\bar{\alpha}+\bar{\delta}=1+\bar{\delta} \neq 0$ and $\bar{\beta}-\bar{\delta}+\bar{\gamma}=\bar{\delta} \neq 0$, so

$$\alpha x+\beta y+\gamma z=$$
$$(\alpha+\delta)\left(\frac{\alpha}{\alpha+\delta}x+\frac{\delta}{\alpha+\delta}y\right)+(\beta-\delta+\gamma)\left(\frac{\beta-\delta}{\beta-\delta+\gamma}y+\frac{\gamma}{\beta-\delta+\gamma}z\right) \in X.$$

□

The following proposition gives a very strong form of Radon's theorem (not only we obtain a partition into two sets with intersecting convex hulls, but moreover one of the points is in the convex hull of the other ones).

Proposition 2.2.8. *Let K be a valued field. For any $d+2$ points $x_1,\ldots,x_{d+2} \in K^d$, one of them is in the convex hull of the others.*

Proof. There exist $a_1,\ldots,a_{d+2} \in K$, *not all* 0, such that $\sum_{i=1}^{d+2} a_i x_i = 0$ and $\sum_{i=1}^{d+2} a_i = 0$ (because those are $d+1$ linear equations on $d+2$ variables, as we are working in K^d). Let $i \in [d+2]$ be such that $\nu(a_i)$ is minimal among $\nu(a_1),\ldots,\nu(a_{d+2})$, in particular $a_i \neq 0$. Then $x_i = \sum_{j\neq i} \frac{-a_j}{a_i} x_j$, and this is a convex combination: for $i \neq j$ we have $\frac{-a_j}{a_i} \in \mathcal{O}$ (as $\nu\left(\frac{-a_j}{a_i}\right) = \nu(a_j)-\nu(a_i) \geq 0$ by the choice of i) and $\sum_{j\neq i} \frac{-a_j}{a_i} = \frac{-\sum_{j\neq i} a_j}{a_i} = \frac{a_i}{a_i} = 1$. □

Convex sets over valued fields have a natural algebraic characterization.

Proposition 2.2.9. *1. A subset $C \subseteq K^d$ is an $\mathcal{O}$-submodule of K^d if and only if it is convex and contains* 0.

2. Nonempty convex subsets of K^d are precisely the translates of $\mathcal{O}$-submodules of K^d.

Proof. (1) First, $\mathcal{O}$-submodules of K^d are clearly convex and contain 0. Now suppose $C \subseteq K^d$ is convex and $0 \in C$. Then for any $\alpha \in \mathcal{O}$ and $x \in C$, $\alpha x = \alpha x + (1-\alpha)0 \in C$. And for any $x, y \in C$, $x + y = 1 \cdot x + 1 \cdot y - 1 \cdot 0 \in C$. Therefore C is an $\mathcal{O}$-submodule. (2) Given a non-empty convex $C \subseteq K^d$, we can choose $a \in K^d$ such that the translate $C + a$ contains 0, and it is still convex, hence $C + a$ is an $\mathcal{O}$-submodule of K^d by (1). $\square$

Example 2.2.10. Let C be an $\mathcal{O}$-submodule of K, and take $\Delta := \nu(C)$. Then Δ is non-empty because it contains $\infty = \nu(0)$, and upward-closed because for $\gamma \in \Delta$ and $\delta > \gamma$, there is $x \in C$ with $\nu(x) = \gamma$, and $\alpha \in K$ with $\nu(\alpha) = \delta - \gamma$; then $\alpha x \in C$ and $\nu(\alpha x) = \delta$. Clearly $C \subseteq \{x \in K \mid \nu(x) \in \Delta\}$ by definition of Δ. To show $C \supseteq \{x \in K \mid \nu(x) \in \Delta\}$, given any $x \in K$ with $\nu(x) \in \Delta$, there is $y \neq 0 \in C$ with $\nu(y) = \nu(x)$, and $\frac{x}{y} \in \mathcal{O}$, so $x = \frac{x}{y} y \in C$. Thus $C = \{x \in K \mid \nu(x) \in \Delta\}$ is a quasi-ball around 0.

Corollary 2.2.11. *The convex hull of any finite set in K^d is the image of $\mathcal{O}^d$ under an affine map.*

Proof. By a repeated application of Proposition 2.2.8, the convex hull of a finite subset of K^d is the convex hull of some $d+1$ points $x_0, \dots, x_d$ from it (possibly with $x_i = x_j$ for some i, j). Let $e_1, \dots, e_d$ be the standard basis for K^d, and let f be an affine map $f : K^d \to K^d$ such that $f(0) = x_0$ and $f(e_i) = x_i$ for $1 \leq i \leq d$ (can take f to be the composition of two affine maps: the linear map sending e_i to $x_i - x_0$ for $1 \leq i \leq d$, and translation by x_0). Then we have $\operatorname{conv}(\{x_0, \dots, x_d\}) = f(\operatorname{conv}\{0, e_1, \dots, e_d\}) = f\left(\mathcal{O}^d\right)$ (by Example 2.2.5(2)). $\square$

Proposition 2.2.12. *For any convex $C \subseteq K^d$ and $a \in K^d$, the translate $C + a := \{x + a \mid x \in X\}$ is either equal to or disjoint from C.*

Proof. If $x \in C \cap (C + a)$, then $\forall y \in C$ $y + a = y + x - (x - a) \in C$, since that is a convex combination, and conversely, if $y + a \in C$ then $y = (y + a) - x + (x - a) \in C$. $\square$

Definition 2.2.13. Given a valued field K, by a *valued K-vector space* we mean a K-vector space V equipped with a surjective map $\nu = \nu_V : V \to \Gamma_\infty = \Gamma \cup \{\infty\}$ such that $\nu(x) = \infty$ if and only if $x = 0$, $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ and $\nu(\alpha x) = \nu_K(\alpha) + \nu(x)$ for all $x, y \in V$ and $\alpha \in K$.

Remark 2.2.14. Here we restrict to the case when V has the same value group as K, and refer to [Fuc75] for a more general treatment (see also [Joh16, Section 6.1.3], [Hru14, Section 2.5] or [AvdDvdH17, Section 2.3]).

By a morphism of valued K-vector spaces we mean a morphism of vector spaces preserving valuation. If V and W are valued K-vector spaces, their direct sum $V \oplus W$ is the direct sum of the underlying vector spaces equipped with the valuation $\nu(x,y) := \min\{\nu_V(x), \nu_W(y)\}$. In particular, the vector space K^d is a valued K-vector space with respect to the valuation $\nu_{K^d} : K^d \to \Gamma_\infty$ given by

$$\nu_{K^d}(x_1, \ldots, x_d) := \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\}.$$

Note that for any scalar $\alpha \in K$ and vector $v \in K^d$ we have $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v)$. By a *(valuational) ball* in K^d we mean a set of the form $\{x \in K^d : \nu_{K^2}(x - c)\Box r\}$ for some center $c \in K^d$, radius $r \in \Gamma \cup \{\infty\}$ and $\Box \in \{>, \geq\}$ (corresponding to open or closed ball, respectively). The collection of all open balls forms a basis for the *valuation topology* on K^d turning it into a topological vector space. Note that due to the "ultra-metric" property of valuations, every open ball is also a closed ball, and vice versa. Equivalently, this topology on K^d is just the product topology induced from the valuation topology on K.

Recall that the *affine span* $\mathrm{aff}(X)$ of a set $X \subseteq K^d$ is the intersection of all affine sets (i.e. translates of vector subspaces of K^d) containing X, equivalently

$$\mathrm{aff}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}_{\geq 1}, \alpha_i \in K, x_i \in X, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

We have $\mathrm{conv}(X) \subseteq \mathrm{aff}(X)$ for any X.

Proposition 2.2.15. *Any convex set in K^d is open in its affine span.*

Proof. For $x \in C \subseteq K^d$, C convex, let $d' \leq d$ be the dimension of the affine span of C, and let $y_1, \ldots, y_{d'} \in C$ be such that $x, y_1, \ldots, y_{d'}$ are affinely independent, and thus have the same affine span as C. Then the map $(\alpha_1, \ldots, \alpha_{d'}) \mapsto x + \alpha_1 (y_1 - x) + \ldots + \alpha_{d'} (y_{d'} - x)$ is a homeomorphism from $K^{d'}$ to the affine span of C, and sends $\mathcal{O}^{d'}$ (which is open in $K^{d'}$) to a neighborhood of x contained in C. $\square$

Corollary 2.2.16. *Convex sets in K^d are closed.*

Proof. For convex $C \subseteq K^d$ and $x \in \mathrm{aff}\,(C) \setminus C$, $C + x$ is an open subset of $\mathrm{aff}\,(C)$ that is disjoint from C, so C is a closed subset of its affine span, and hence closed in K^d, since affine subspaces are closed. $\square$

2.3 Classification of $\mathcal{O}$-submodules of K^d

In this section we provide a simple description for the $\mathcal{O}$-submodules of K^d over a spherically complete valued field K (and over an arbitrary valued field K in the finitely generated case). Combined with the description of convex sets in terms of $\mathcal{O}$-submodules from Section 2.2, this will allow us to establish various combinatorial properties of convex sets over valued fields in the next section.

Lemma 2.3.1. *Let K be a valued field, and $V \subseteq K^d$ a subspace. Then the quotient vector space K^d/V is a valued K-vector space equipped with the valuation*

$$\nu\,(u) := \max\left\{\nu_{K^d}\,(v) \mid \pi\,(v) = u, v \in V\right\},$$

where $\pi : K^d \to K^d/V$ is the projection map.

If $\dim(V) = n$, then $K^d/V \cong K^{d-n}$ as valued K-vector spaces, and there is a valuation preserving embedding of K-vector spaces $f : K^d/V \hookrightarrow K^d$ so that $\pi \circ f = \mathrm{id}_{K^d/V}$.

Proof. First we prove the lemma for $n = 1$. Let $V \subseteq K^d$ be one-dimensional. There exists $i \in [d]$ such that $\nu_{K^d}\,((x_1, \ldots, x_d)) = \nu_K\,(x_i)$ for all $(x_1, \ldots, x_d) \in V$ (indeed, if $\nu_K(x_i) = \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\}$ for some $(x_1, \ldots, x_d) \in V$, then we also have $\nu_K(\alpha x_i) = \nu_K(\alpha) + \nu_K(x_i) = \nu_K(\alpha) + \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\} = \min\{\nu_K(\alpha x_1), \ldots, \nu_K(\alpha x_d)\}$ for any

$\alpha \in K$). Given any $(x_1, \ldots, x_d) \in K^d$ with $x_i = 0$ and $(y_1, \ldots, y_d) \in V$, we have

$$\nu_{K^d}(x_1 + y_1, \ldots, x_d + y_d) = \min_{j \in [d]} \{\nu_K(x_j + y_j)\} =$$
$$\min\left(\nu_K(y_i), \min_{j \neq i} \{\nu_K(x_j + y_j)\}\right) \leq \nu_K(y_i) = \nu_{K^d}(y_1, \ldots, y_d).$$

Thus the maximum of the valuations of elements of any given affine translate of V is achieved by an element of that translate with zero ith coordinate, in particular the valuation ν on K^d/V is well-defined.

Let $K' := \{(x_1, \ldots, x_d) \in K^d \mid x_i = 0\}$, then we have $K^d = V \oplus K'$ as vector spaces, hence the projection of K^d onto K' along V induces an isomorphism between K^d/V and K', which in turn is naturally isomorphic to K^{d-1}, and these isomorphisms preserve the valuation and give the desired embedding $f : K^d/V \to K^d$. The general case follows by induction on n using the vector space isomorphism theorems. $\square$

We recall an appropriate notion of completeness for valued fields. Recall that a family $\{C_i : i \in I\}$ of subsets of a set X is *nested* if for any $i, j \in I$, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$.

Definition 2.3.2. A valued field K is *spherically complete* if every nested family of (closed or open) valuational balls has non-empty intersection.

For the following standard fact, see for example [Sch50, Theorem 5 in Section II.3 + Theorem 8 in section II.6].

Fact 2.3.3. *Every valued field K (with valuation ν_K, value group Γ_K and residue field k_K) admits a* spherical completion*, i.e. a valued field $\widetilde{K}$ (with valuation $\nu_{\widetilde{K}}$, value group $\Gamma_{\widetilde{K}}$ and residue field $k_{\widetilde{K}}$) so that:*

1. *$\widetilde{K}$ is an* immediate extension *of K, i.e. $\widetilde{K}$ is a field extension of K, $\nu_{\widetilde{K}} \restriction_K = \nu_K$, $\Gamma_{\widetilde{K}} = \Gamma_K$ and $k_{\widetilde{K}} = k_K$;*
2. *$\widetilde{K}$ is spherically complete.*

We remark that in general a valued field might have multiple non-isomorphic spherical completions.

Lemma 2.3.4. *If K is spherically complete, then every nested family of non-empty convex subsets of K^d has a non-empty intersection.*

Proof. By induction on d. For $d = 1$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, so each C_i is a quasi-ball (see Example 2.2.5(1)). If there exists some $i \in I$ so that C_i is the smallest of these under inclusion then any element of C_i is in the intersection of the whole family. Hence we may assume that for each $i \in I$ there exists some $i' \in I$ such that $C_{i'} \subsetneq C_i$. Let Δ_i and $\Delta_{i'}$ be the quasi-radii of C_i and $C_{i'}$, respectively. We may assume that both quasi-balls are around the same point $x_i \in C_{i'}$ (by Remark 2.2.4), hence necessarily $\Delta_{i'} \subsetneq \Delta_i$. Let $r_i \in \Delta_i \setminus \Delta_{i'}$, and let C'_i be a (open or closed) ball of radius r_i around x_i. We have $C'_i \subseteq C_i$, so if $\bigcap_{i\in I} C'_i$ is nonempty, then so is $\bigcap_{i\in I} C_i$. Hence it is sufficient to show that $\{C'_i\}_{i\in I}$ is nested, and then the intersection is non-empty by spherical completeness of K. By construction for any $i, j \in I$ there exists some $\ell \in I$ such that $C_\ell \subseteq C'_i \cap C'_j$, so C'_i and C'_j have non-empty intersection, and are thus nested as they are balls.

For $d \geq 2$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, and let $\pi_1 : K^d \to K$ be the projection onto the first coordinate. Then $\{\pi_1 (C_i)\}_{i\in I}$ is a nested family of nonempty convex sets in K, hence has an intersection point x. Then $\left\{\pi_1^{-1} (x) \cap C_i\right\}_{i\in I}$ is a nested family of nonempty convex sets in $\pi_1^{-1} (x) \cong K^{d-1}$, which is nonempty by the induction hypothesis. $\square$

Lemma 2.3.5. *If $C \subseteq K^d$ is an $\mathcal{O}$-module, and $\gamma \in \Gamma_\infty$, then the set*

$$X_\gamma = \left\{(x_1, \ldots, x_{d-1}) \in \mathcal{O}^{d-1} \mid \exists\alpha \in K \ \nu (\alpha) = \gamma,\ (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C\right\}$$

is convex.

Proof. Let $x = (x_1, \ldots, x_{d-1}), y = (y_1, \ldots, y_{d-1}), z = (z_1, \ldots, z_{d-1}) \in X_\gamma$ and $\beta_1, \beta_2, \beta_3 \in \mathcal{O}$ with $\beta_1 + \beta_2 + \beta_3 = 1$ be arbitrary. Then there exist some $\alpha_1, \alpha_2, \alpha_3 \in K$ with $\nu (\alpha_i) = \gamma$ so that

$$(\alpha_1, \alpha_1 x_1, \ldots, \alpha_1 x_{d-1}), (\alpha_2, \alpha_2 y_1, \ldots, \alpha_2 y_{d-1}), (\alpha_3, \alpha_3 z_1, \ldots, \alpha_3 z_{d-1}) \in C.$$

Taking $\alpha := \alpha_1$, we have

$$x' := (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}), y' := (\alpha, \alpha y_1, \ldots, \alpha y_{d-1}), z' := (\alpha, \alpha z_1, \ldots, \alpha z_{d-1}) \in C,$$

as for every $i \in [3]$, $\frac{\alpha}{\alpha_i} \in \mathcal{O}$, and hence $\frac{\alpha}{\alpha_i} v \in C$ for any $v \in C$ as C is an $\mathcal{O}$-module. Using this and convexity of C we thus have

$$\Big(\alpha, \alpha(\beta_1 x_1 + \beta_2 y_1 + \beta_3 z_1), \ldots, \alpha(\beta_1 x_{d-1} + \beta_2 y_{d-1} + \beta_3 z_{d-1})\Big) =$$

$$\beta_1 (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) + \beta_2 (\alpha, \alpha y_1, \ldots, \alpha y_{d-1}) + \beta_3 (\alpha, \alpha z_1, \ldots, \alpha z_{d-1}) =$$

$$\beta_1 x' + \beta_2 y' + \beta_3 z' \in C.$$

This shows that $\beta_1 x + \beta_2 y + \beta_3 z \in X_\gamma$, and hence that X_γ is convex by Proposition 2.2.6. $\square$

Combining the lemmas, we obtain the following description of the $\mathcal{O}_K$-submodules of K^d for spherically complete K.

Theorem 2.3.6. *Suppose K is a spherically complete valued field, $d \in \mathbb{N}_{\geq 1}$, and let $C \subseteq K^d$ be an $\mathcal{O}$-submodule. Then there exists a complete flag of vector subspaces $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and a decreasing sequence of nonempty, upwards-closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_d$ of Γ_∞ such that $C = \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$.*

Remark 2.3.7. If F_i, Δ_i satisfy the conclusion of Theorem 2.3.6 for C, then $\nu_{K^d}(C \cap F_1) = \nu_{K^d}(C) = \Delta_1$.

Indeed, any $v \in C$ is of the form $v = v_1 + \ldots + v_d$ with $v_i \in F_i$, $\nu(v_i) \in \Delta_i$ and $\Delta_1 \supseteq \Delta_i$ for all $i \in [d]$, hence $\nu(v) \geq \min\{\nu(v_i) : i \in [d]\} \in \Delta_1$, hence $\nu(v) \in \Delta_1$ as Δ_1 is upwards closed, so $\nu(C) \subseteq \Delta_1$. Conversely, assume $\gamma \in \Delta_1$. If $\gamma = \infty$, then $\nu(0) = \infty$ and $0 \in F_1$. So assume $\gamma \in \Gamma$ and let v be any non-zero vector in F_1, in particular $\delta := \nu(v) \in \Gamma$. Taking $\alpha \in K$ so that $\nu_K(\alpha) = \gamma - \delta$, we have $\alpha v \in F_1$ and $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v) = \gamma$. Note also that $\alpha v = v_1 + \ldots + v_d$ with $v_1 := \alpha v, v_i := 0$ for $2 \leq i \leq d$, in particular $v_i \in F_i$ and $\nu(v_i) \in \Delta_i$, so $\alpha v \in C$, hence $\Delta_1 \subseteq \nu(F_1 \cap C)$.

Proof of Theorem 2.3.6. By induction on d. For $d = 1$, every $\mathcal{O}$-submodule of K is a quasi-ball $C = \{x \in K : \nu(x) \in \Delta\}$ for some upwards-closed $\Delta \subseteq \Gamma \cup \{\infty\}$ (see Example 2.2.10), hence we take $F_1 := K$ and $\Delta_1 := \Delta$.

For $d > 1$, let $\Delta_1 := \{\gamma \in \Gamma_\infty \mid \exists v \in C\ \nu_{K^d}(v) = \gamma\}$. Note that Δ_1 is nonempty because it contains $\infty = \nu(0)$. Then there is some $i \in [d]$ such that every $\gamma \in \Delta_1$ is the valuation of the ith coordinate of some element of C. To see this, note that for each $i \in [d]$, the set

$$S_i := \{\gamma \in \Gamma_\infty \mid \exists v = (v_1, \ldots, v_d) \in C\ \nu_{K^d}(v) = \nu(v_i) = \gamma\}$$

is upwards closed in Γ_∞. Indeed, assume $v = (v_1, \ldots, v_d) \in C$, $\gamma = \nu(v_i) = \min\{\nu(v_j) : j \in [d]\}$ and $\delta \geq \gamma$ in Γ_∞. Let $\alpha \in K$ be arbitrary with $\nu(\alpha) = \delta - \gamma$, then $\alpha \in \mathcal{O}$, hence $\alpha v \in C$, and so $\nu_{K^d}(\alpha v) = \min\{\nu(\alpha v_j) : j \in [d]\} = \nu(\alpha v_j) = \delta$. As we also have $\Delta_1 = \bigcup_{i \in [d]} S_i$, it follows that $\Delta_1 = S_i$ for some $i \in [d]$ as wanted (and in particular Δ_1 is upwards closed in Γ_∞).

Without loss of generality we may assume $i = 1$. Then, given any $\gamma \in \Delta_1$, there is some $(\alpha, y_1, \ldots, y_{d-1}) \in C$ such that $\gamma = \nu(\alpha) \leq \min\{\nu(y_j) : j \in [d-1]\}$. Taking $x_j := \frac{y_j}{\alpha} \in \mathcal{O}$, we thus have $(\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C$. Hence for any $\gamma \in \Delta_1$, the set

$$X_\gamma := \left\{(x_1, \ldots, x_{d-1}) \in \mathcal{O}^{d-1} \mid \exists \alpha \in K\ \nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C\right\}$$

is nonempty, and convex (by Lemma 2.3.5). Note that for $\gamma < \delta \in \Gamma_\infty$ we have $X_\gamma \subseteq X_\delta$, hence $\bigcap_{\gamma \in \Delta_1} X_\gamma \neq \emptyset$ by Lemma 2.3.4. That is, there exists $(x_1, \ldots, x_{d-1}) \in \mathcal{O}^{d-1}$ such that $\forall \gamma \in \Delta_1\ \exists \alpha \in K\ \ (\nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C)$. Hence

$$\forall \alpha \in K,\ \nu(\alpha) \in \Delta_1 \implies (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C \tag{2.3.1}$$

(since we have $\exists \beta \in K\ \nu(\beta) = \nu(\alpha) \wedge (\beta, \beta x_1, \ldots, \beta x_{d-1}) \in C$, so $\frac{\alpha}{\beta} \in \mathcal{O}$ and multiplying by it we get $(\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C$).

Let $F_1 := \langle (1, x_1, \ldots, x_{d-1}) \rangle$. Let $\pi : K^d \twoheadrightarrow K^d / F_1$ be the projection map, $f : K^d / F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 2.3.1, and $\pi' := f \circ \pi : K^d \to K^d$.

Note that $K^d/F_1 \cong K^{d-1}$ as a valued K-vector space by Lemma 2.3.1, and that $\widetilde{C} := \pi(C)$ is still an $\mathcal{O}$-submodule of K^d/F_1. By induction hypothesis there is a full flag $\{0\} \subsetneq \widetilde{F}_2 \subsetneq \ldots \subsetneq \widetilde{F}_d = K^d/F_1$ and upwards-closed subsets $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2 \supseteq \ldots \supseteq \Delta_d$ of Γ_∞ satisfying the conclusion of the theorem with respect to $\widetilde{C}$ (the equality $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2$ is by Remark 2.3.7). Note that

$$\forall v \in K^d,\ \nu_{K^d}(\pi'(v)) = \nu_{K^d/F_1}(\pi(v)) \geq \nu_{K^d}(v). \tag{2.3.2}$$

In particular we have $\Delta_1 \supseteq \Delta_2$.

Let the subspaces $F_2, \ldots, F_d$ be the preimages of $\widetilde{F}_2, \ldots, \widetilde{F}_d$ in K^d. We let $W := f(K^d/F_1) \subseteq K^d$ be the image of the valuation preserving embedding $f : K^d/F_1 \hookrightarrow K^d$. Then we have

$$C = \{v_1 + w \mid v_1 \in F_1,\ \nu_{K^d}(v_1) \in \Delta_1,\ w \in C \cap W\}. \tag{2.3.3}$$

To see this, given an arbitrary $v \in C$, let $w := \pi'(v)$ and $v_1 := v - w$. As $\pi \circ f = \mathrm{id}_{K^d/F_1}$ by assumption, we have $\pi(w) = \pi(\pi'(v)) = \pi(f(\pi(v))) = \pi(v)$, hence $v_1 \in F_1$. By (2.3.2) we have $\nu_{K^d}(w) \geq \nu_{K^d}(v)$, and thus $\nu_{K^d}(v_1) \geq \min\{\nu_{K^d}(v), \nu_{K^d}(w)\} \geq \nu_{K^d}(v)$ as well. Thus $\nu_{K^d}(v_1) \in \Delta_1$, and $v_1 \in F_1$, which together with (2.3.1) and the definition of F_1 implies $v_1 \in C$; hence $w = v - v_1 \in C$ as well. The opposite inclusion is obvious.

Furthermore, applying the isomorphism $f : K^d/F_1 \to W$ to

$$\widetilde{C} = C/F_1 = \left\{v_2 + \ldots + v_d \mid v_i \in \widetilde{F}_i, \nu_{K^d/F_1}(v_i) \in \Delta_i\right\}$$

we get

$$C \cap W = \{v_2 + \ldots + v_d \mid v_i \in F_i \cap W,\ \nu_{K^d}(v_i) \in \Delta_i\},$$

which together with (2.3.3) implies

$$C = \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i,\ v_i \in W \text{ for } i \geq 2\}.$$

Now $C = \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$ follows because for any such $v_1, \ldots, v_d$, v_i (for $i \geq 2$) can be moved into W by subtracting an element of F_1 with valuation in Δ_1, and collecting the differences in with v_1. That is, given arbitrary $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$, let $w_i := \pi'(v_i) \in W$ for $i \geq 2$, and let $w_1 := v_1 + (v_2 - \pi'(v_2)) + \ldots + (v_d - \pi'(v_d))$. As above, using (2.3.2), for each $i \geq 2$ we have $\nu_{K^d}(v_i - \pi'(v_i)) \geq \min\{\nu_{K^d}(v_i), \nu_{K^d}(\pi'(v_i))\} \geq \nu_{K^d}(v_i) \in \Delta_i \subseteq \Delta_1$. Hence $\nu_{K^d}(w_1) \geq \min\{v_1, v_2 - \pi'(v_2), \ldots, v_d - \pi'(v_d)\} \in \Delta_1$. We also have $\nu_{K^d}(w_i) \geq \nu_{K^d}(v_i) \in \Delta_i$ for $i \geq 2$ by (2.3.2). Using that f is a one-sided inverse of π as above, we also have $v_i - \pi'(v_i) \in F_1 \subseteq F_i$ for $i \geq 2$. It follows that $w_i \in F_i$ for all $i \in [d]$. Putting all of this together, we get $w_1 + \ldots + w_d = v_1 + \ldots + v_d$, $w_i \in F_i$, $\nu(w_i) \in \Delta_i$, and $w_i \in W$ for $i \geq 2$. □

Remark 2.3.8. Note that as $F_d = K^d$ in Theorem 2.3.6, we have

$$\Delta_d = \left\{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C\right\}.$$

That is, Δ_d is the quasi-radius of the largest quasi-ball around 0 contained in C.

Remark 2.3.9. Given a convex set $0 \in C \subseteq K^d$ and any $F_i, \Delta_i, i \in [d]$ satisfying the conclusion of Theorem 2.3.6 with respect to it, for every $j \in [d]$ we have

$$C \cap F_j = \{v_1 + \ldots + v_j \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i \text{ for all } j \in [i]\}.$$

Indeed, if $x \in C \cap F_j$, then $x = v_1 + \ldots + v_d \in F_j$ for some $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$ for $i \in [d]$. Then, using that the F_i are increasing under inclusion and Δ_i are increasing under inclusion and upwards closed, $v_{j+1} + \ldots + v_d \in F_j$ and taking $v'_j := v_j + \ldots + v_d$ we have $v'_j \in F_j, \nu(v'_j) \geq \min\{\nu(v_i) : j \leq i \leq d\} \in \Delta_j$ and $x = v_1 + \ldots + v_{j-1} + v'_j$. Conversely, any element $x = v_1 + \ldots + v_j$ with $v_i \in F_i, \nu(v_i) \in \Delta_i$ for $i \in [j]$ can be written as $x = v_1 + \ldots + v_d$ with $v_i := 0 \in F_i$ and $\nu(v_i) = \infty \in \Delta_i$ for $j + 1 \leq i \leq d$. So $x \in C \cap F_j$.

Remark 2.3.10.

1. It follows from the conclusion of Theorem 2.3.6 that the subspace F_{d-1} is a linear hyperplane in K^d, and every element of C differs from an element of F_{d-1} (and hence of $F_{d-1} \cap C$ in view of Remark 2.3.9) by a vector in K^d with valuation in Δ_d (with Δ_d as in Remark 2.3.8).

2. Conversely, F_{d-1} can be chosen to be *any* linear hyperplane H in K^d such that every element of C differs from an element of H by a vector in K^d with valuation in Δ_d. To see this, let H be such a hyperplane in K^d. Then $C \cap H$ is a convex subset of $H \cong K^{d-1}$ containing 0, hence an $\mathcal{O}$-submodule of H by Proposition 2.2.9. Applying Theorem 2.3.6 to $C \cap H$ in H (with the induced valuation on H), there are $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_{d-1}$ and a full flag $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_{d-1} = H$, such that $C \cap H = \{v_1 + \ldots + v_{d-1} \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$. Then

$$\{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} = \{w + v_d \mid w \in C \cap H,\ \nu(v_d) \in \Delta_d\} = C.$$

Example 2.3.11. The assumption of spherical completeness of K is necessary in Theorem 2.3.6. For example, let $K := \bigcup_{n \geq 1} k\left(\left(t^{\frac{1}{n}}\right)\right)$ be the field of Puiseux series over a field k, and let $\widetilde{K} := k\left[\left[t^{\mathbb{Q}}\right]\right]$ be the field of Hahn series over k with rational exponents, it is the spherical completion of K (both fields have value group $\mathbb{Q}$ and valuation $\nu(x) = q$ where x has leading term t^q; see e.g. [AvdDvdH17, Example 3.3.23]). In particular $\sum_{n \geq 1} t^{1-\frac{1}{n}} \in \widetilde{K} \setminus K$, and let

$$\widetilde{C} := \left\{\alpha\left(1, \sum_{n \geq 1} t^{1-\frac{1}{n}}\right) + v \mid \alpha \in \widetilde{K}, v \in \widetilde{K}^2, \nu_{\widetilde{K}}(\alpha) \geq 0,\ \nu_{\widetilde{K}^2}(v) \geq 1\right\} \subseteq \widetilde{K}^2,$$

and let $C := \widetilde{C} \cap K^2$. Then $\widetilde{C}$ is convex in $\widetilde{K}^2$, and hence C is also convex as a subset of K^2. The basic idea behind why C is not of the form described in Theorem 2.3.6 is that C is close enough to $\widetilde{C}$, and the subspace F_1 appearing in the conclusion of Theorem 2.3.6 for $\widetilde{C}$ must be close to $\left\langle\left(1, \sum_{n \geq 1} t^{1-\frac{1}{n}}\right)\right\rangle$; specifically, it must be $\left\langle\left(1, x + \sum_{n \geq 1} t^{1-\frac{1}{n}}\right)\right\rangle$ for some $x \in K^2$ with $\nu(x) \geq 1$, but K^2 contains no such subspaces.

Indeed, by Remark 2.3.7, given any F_i, Δ_i satisfying the conclusion of Theorem 2.3.6 with respect to C, the valuation of every element of C must also be the valuation of some element of $F_1 \cap C$. So, to show that C is not of the form described in Theorem 2.3.6, it suffices to show that C contains elements of valuation arbitrarily close to 0, but that for every 1-dimensional subspace $F_1 \subset K^2$, there is some $q > 0$ in Γ such that every element of $F_1 \cap C$ has valuation at least q (and note that from the definition of C, every element in it has positive valuation).

Claim 2.3.12. *For every $n \in \mathbb{N}_{\geq 1}$, there is some $v \in C$ with $\nu_{K^2}(v) = \frac{1}{n}$.*

Proof. To see this, note that

$$t^{\frac{1}{n}}\left(1, \sum_{m=1}^{n-1} t^{1-\frac{1}{m}}\right) = t^{\frac{1}{n}}\left(1, \sum_{m\geq 1} t^{1-\frac{1}{m}}\right) - t^{\frac{1}{n}}\left(0, \sum_{m\geq n} t^{1-\frac{1}{m}}\right) \in C$$

as $\nu_K\left(t^{\frac{1}{n}}\right) = \frac{1}{n} \geq 0$ and $\nu_{K^2}\left(t^{\frac{1}{n}}\left(0, \sum_{m\geq n} t^{1-\frac{1}{m}}\right)\right) = \frac{1}{n} + \left(1 - \frac{1}{n}\right) \geq 1$. □

Claim 2.3.13. *For every 1-dimensional subspace $F_1 \subset K^2$, there is some $n \in \mathbb{N}_{n\geq 1}$ such that every element of $F_1 \cap C$ has valuation at least $\frac{1}{n}$.*

Proof. We prove this by breaking into two cases.

Case 1. $F_1 = \langle(0,1)\rangle$.

Assume $x \in F_1 \cap C$, then $x = (x_1, x_2) = \alpha\left(1, \sum_{n\geq 1} t^{1-\frac{1}{n}}\right) + v$ for some $\alpha \in K, v = (v_1, v_2) \in \widetilde{K}^2$ with $\nu_{\widetilde{K}}(\alpha) \geq 0, \nu_{\widetilde{K}^2}(v) \geq 1$, and $x_1 = 0$, so $\alpha = -v_1$. But $1 \leq \nu_{\widetilde{K}^2}(v) = \min\{\nu_{\widetilde{K}}(v_1), \nu_{\widetilde{K}}(v_2)\}$, hence $\nu_{\widetilde{K}}(\alpha) \geq 1$ as well. Since $\nu_{\widetilde{K}}\left(\sum_{n\geq 1} t^{1-\frac{1}{n}}\right) = 0$, it follows that $\nu_{\widetilde{K}^2}(x) = \min\left\{\nu_{\widetilde{K}}(0), \nu_{\widetilde{K}}\left(\alpha\left(\sum_{n\geq 1} t^{1-\frac{1}{n}}\right)\right)\right\} \geq 1$. Thus every element of $F_1 \cap C$ has valuation at least 1.

Case 2. $F_1 = \langle(1,x)\rangle$ for some $x \in K$.

Given any $x \in K$, there must exist some $n \in \mathbb{N}$ such that $\nu_{\widetilde{K}}\left(x - \sum_{m\geq 1} t^{1-\frac{1}{m}}\right) \leq 1 - \frac{1}{n}$. Given any $v \in F_1 \cap C$, we have

$$v = \alpha(1, x) = \beta\left(1, \sum_{m\geq 1} t^{1-\frac{1}{m}}\right) + w$$

for some $\alpha \in K$, some $\beta \in \widetilde{K}$ with $\nu_{\widetilde{K}}(\beta) \geq 0$ and $w = (w_1, w_2) \in \widetilde{K}^2$ with $\nu_{\widetilde{K}^2}(w) \geq 1$. Without loss of generality $\alpha \neq 0$, so we have

$$x = \frac{\alpha x}{\alpha} = \left(w_2 + \beta \sum_{m\geq 1} t^{1-\frac{1}{m}}\right)(w_1 + \beta)^{-1} = \left(\frac{w_2}{\beta} + \sum_{m\geq 1} t^{1-\frac{1}{m}}\right)\left(1 + \frac{w_1}{\beta}\right)^{-1}.$$

If $\nu_{\widetilde{K}}(\beta) < \frac{1}{n}$, then

$$\nu_{\widetilde{K}}\left(\frac{w_1}{\beta}\right) > 1 - \frac{1}{n},\ \nu_{\widetilde{K}}\left(\frac{w_2}{\beta}\right) > 1 - \frac{1}{n},\ \nu_{\widetilde{K}}\left(\left(1 + \frac{w_1}{\beta}\right)^{-1}\right) = 0, \text{ and}$$

$$\nu_{\widetilde{K}}\left(\left(1 + \frac{w_1}{\beta}\right)^{-1} - 1\right) > 1 - \frac{1}{n}, \text{ so}$$

$$\nu\left(x - \sum_{m\geq 1} t^{1-\frac{1}{m}}\right) = \nu\left(\frac{w_2}{\beta}(w_1+\beta)^{-1} + \left(\sum_{m\geq 1} t^{1-\frac{1}{m}}\right)\left(\left(1 + \frac{w_1}{\beta}\right)^{-1} - 1\right)\right) > 1 - \frac{1}{n},$$

a contradiction to the choice of n. Thus $\nu(\beta) \geq \frac{1}{n}$, and hence $\nu(v) \geq \frac{1}{n}$. $\square$

Thus no 1-dimensional subspace F_1 of K^2 can fill its desired role in the presentation for C.

Theorem 2.3.6 implies the following simple description of convex sets over spherically complete valued fields.

Corollary 2.3.14. *If K is a spherically complete valued field and $d \in \mathbb{N}_{\geq 1}$, then the non-empty convex subsets of K^d are precisely the affine images of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$.*

Proof. Let $C \subseteq K^d$ be an affine image of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$. Note that $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ is convex, and an image of a convex set under an affine map is convex (Example 2.2.5), hence C is convex.

Conversely, let $\emptyset \neq C \subseteq K^d$ be convex. Since the affine images of $\mathcal{O}$-submodules of K^d give us all non-empty convex sets by Proposition 2.2.9, without loss of generality $0 \in C$ and C is an $\mathcal{O}$-submodule of K^d. Let $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and $\nu_{K^d}(C) = \Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_d$ be as given by Theorem 2.3.6 for C. Using Lemma 2.3.1 we can choose $v_1, \ldots, v_d \in K^d$ such that for every $i \in [d]$ we have:

1. $v_1, \ldots, v_i$ is a basis for F_i,

2. $\nu(v_i) = 0$,

3. $\nu(v_i + x) \leq 0$ for all $x \in F_{i-1}$.

Then C is the image of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ under the linear map $f : K^d \to K^d$ such that $f(e_i) = v_i$, where e_i is the ith standard basis vector. Indeed, if $x \in f(\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d))$ then $x = \sum_{i=1}^d c_i v_i$ for some c_i with $\nu(c_i) \in \Delta_i$. Using (2) this implies $\nu(c_i v_i) = \nu(c_i) \in \Delta_i$, and $c_i v_i \in F_i$, hence $x \in C$. Conversely, let x be an arbitrary element of C, then $x = w_1 + \ldots + w_d$ for some $w_i \in F_i$ with $\nu(w_i) \in \Delta_i$. Each w_i is a linear combination of $v_1, \ldots, v_i$, say $w_i = \sum_{j=1}^i c_{i,j} v_j$.

Now we claim that for any $i \in [d]$, $\alpha \in K$ and $v \in F_{i-1}$ we have $\nu(\alpha v_i + v) = \min\{\nu(\alpha v_i), \nu(v)\}$. Indeed, replacing v and α by $\alpha^{-1} v \in F_{i-1}$ and $\alpha^{-1}\alpha \in K$, respectively, changes both sides of the claimed equality by the same amount, hence we may assume that $\alpha = 0$ or $\alpha = 1$. The first case holds trivially, in the second case we need to show that $\nu(v_i + v) = \min\{\nu(v_i), \nu(v)\}$. If $\nu(v_i) \neq \nu(v)$ this holds by the ultrametric inequality, so we assume $\nu(v_i) = \nu(v) = 0$ (using (2)). Then, using (3), $0 \geq \nu(v_i + v) \geq \min\{\nu(v_i), \nu(v)\} = 0$, so $\nu(v_i + v) = 0$ as well.

Applying this claim by induction on $i \in [d]$, we get

$$\nu\left(\sum_{j=1}^{i} c_{i,j} v_j\right) = \min_j \{\nu(c_{i,j} v_j)\},$$

which using (2) implies $\nu(w_i) = \nu\left(\sum_{j=1}^i c_{i,j} v_j\right) = \min_j \{\nu(c_{i,j})\}$ for each $i \in [d]$. As for each $i \in [d]$ we have $\nu(w_i) \in \Delta_i$ and Δ_i is upwards closed, it follows that $\nu(c_{i,j}) \in \Delta_i$ for all $i \in [d], j \in [i]$. Regrouping the summands $c_{i,j} v_i$, it follows that $x = w_1 + \ldots + w_d$ is a linear combination of $v_1, \ldots, v_d$ where the coefficient of v_i has valuation in Δ_i, hence x belongs to $f(\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d))$. $\square$

We can eliminate the assumption of spherical completeness of the field when only considering convex hulls of finite sets. We will say that a convex set is *finitely generated* if it is the convex hull of a finite set of points.

Lemma 2.3.15. *A subset $C \subseteq K^d$ is a finitely generated $\mathcal{O}$-module if and only if it is a finitely generated convex set and contains* 0.

Proof. If an $\mathcal{O}$-module $C \subseteq K^d$ is generated as an $\mathcal{O}$-module by some finite set X, then it is the convex hull of $X \cup \{0\}$. If a set C is the convex hull of some finite set X and contains 0, then it is an $\mathcal{O}$-module by Proposition 2.2.9, clearly generated as an $\mathcal{O}$-module by X. □

We have the following analog of Theorem 2.3.6 in the finitely generated case over an arbitrary valued field.

Corollary 2.3.16. *Let K be an* arbitrary *valued field and C a* finitely generated *convex set containing* 0. *Then there is a full flag $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and an increasing sequence $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_d \in \Gamma_\infty$ such that*

$$C = \{v_1 + \ldots + v_d \mid v_i \in F_i, \ \nu(v_i) \geq \gamma_i\}.$$

Proof. Let $C \ni 0$ be the convex hull of some finite set $X \subseteq K^d$. By a repeated application of Proposition 2.2.8, C is the convex hull of some $d+1$ elements $v_0, \ldots, v_d$ from X (possibly with $x_i = x_j$ for some i, j). As $0 \in C$, we have $0 = \sum_{i=0}^{d} \alpha_i v_i$ for some $\alpha_i \in \mathcal{O}$ with $\sum_{i=0}^{d} \alpha_i = 1$. Let j be such that $\nu(\alpha_j)$ is minimal among $\{\nu(\alpha_i) : 0 \leq i \leq d\}$. In particular $\alpha_j \neq 0$, hence $v_j = \left(1 - \sum_{i \neq j} \frac{\alpha_i}{\alpha_j}\right) 0 + \sum_{i \neq j} \frac{\alpha_i}{\alpha_j} v_i$. By the choice of j we have $\frac{\alpha_i}{\alpha_j} \in \mathcal{O}$ for all $i \neq j$, hence also $1 - \sum_{i \neq j} \frac{\alpha_i}{\alpha_j} \in \mathcal{O}$, thus $v_j \in \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$, and so also $C = \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$. Reordering if necessary, we can thus assume that C is the convex hull of some $\{0, v_1, \ldots, v_d\} \subseteq C$ with $\nu(v_1) \leq \nu(v_i)$ for each $i \in [d]$.

Let $F_1 := \langle v_1 \rangle$ and $\gamma_1 := \nu(v_1)$. Let $\pi_1 : K^d \twoheadrightarrow K^d/F_1$ be the projection map, $f_1 : K^d/F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 2.3.1, $V_1 := f_1\left(K^d/F_1\right)$ and $\pi_1' := f_1 \circ \pi_1 : K^d \to K^d$.

For $i \geq 2$, as explained after (2.3.3) in the proof of Theorem 2.3.6 we have $v_i - \pi_1'(v_i) \in F_1$; and by (2.3.2) there and assumption we have $\nu(\pi_1'(v_i)) \geq \nu(v_i) \geq \nu(v_1)$. So $v_i - \pi_1'(v_i) \in \mathcal{O}v_1$

for all $i \geq 2$, which implies

$$\operatorname{conv}\left(\{0, v_1, \pi_1'(v_2), \ldots, \pi_1'(v_d)\}\right) = \operatorname{conv}\left(\{0, v_1, \ldots, v_d\}\right) = C.$$

Without loss of generality we suppose $\nu\left(\pi_1'(v_2)\right) \leq \nu\left(\pi_1'(v_i)\right)$ for $i \geq 3$, and let $F_2 := \langle v_1, \pi_1'(v_2)\rangle$ and $\gamma_2 := \nu\left(\pi_1'(v_2)\right) \geq \nu(v_1) = \gamma_1$ by assumption. By definition of the valuation on the quotient space, using the properties of f, we have

$$\nu_K(\pi_1'(v_i)) = \nu_{K^d/F_1}(\pi_1(v_i)) = \nu_{K^d/F_1}(\pi_1(\pi_1'(v_i))) \geq \nu_{K^d}(\pi_1'(v_i) + \alpha v_1)$$

for all $\alpha \in K$. As in the proof of Corollary 2.3.14, this implies $\nu(\beta\pi_1'(v_i) + \alpha v_1) = \min\{\beta\nu(\pi_1'(v_i)), \nu(\alpha v_1))\}$ for all $i \geq 2$ and $\alpha, \beta \in K$. It follows that

$$\{nv_1 + m\pi_1'(v_2) \mid n, m \in \mathcal{O}\} = \{w_1 + w_2 \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}.$$

To see that the set on the right is contained in the set on the left, assume $x = w_1 + w_2$ for some $w_i \in F_i, \nu(w_i) \geq \gamma_i$. Then $w_1 = \alpha_1 v_1$ and $w_2 = \alpha_2 v_1 + \beta\pi_1'(v_2)$ for some $\alpha_1, \alpha_2, \beta \in K$, and by the observation above $\gamma_2 \leq \nu(w_2) = \min\{\nu(\alpha_2 v_1), \nu(\beta\pi_1'(v_2))\}$. So $x = (\alpha_1 + \alpha_2)v_1 + \beta\pi_1'(v_2)$, $\nu((\alpha_1 + \alpha_2)v_1) \geq \gamma_1 = \nu(v_1)$, so $(\alpha_1 + \alpha_2) \in \mathcal{O}$, and $\nu(\beta) \geq \gamma_2$, as wanted.

Now we replace v_i by $\pi_1'(v_i)$ for $i \geq 2$, and let $\pi_2 : K^d \twoheadrightarrow K^d/F_2$ be the projection map, $f_2 : K^d/F_2 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 2.3.1, $V_2 := f_2\left(K^d/F_2\right)$ and $\pi_2' := f_2 \circ \pi_2 : K^d \to K^d$. For $i \geq 3$, $v_i - \pi_2'(v_i) \in F_2$ and $v_i - \pi_2'(v_i) \in \mathcal{O}v_1 + \mathcal{O}v_2$, so again replacing v_i with $\pi_2'(v_i)$ for $i \geq 3$ does not change the convex hull. Again we may assume $\nu\left(\pi_2'(v_3)\right) \leq \nu\left(\pi_2'(v_i)\right)$ for $i \geq 4$, and let $F_3 := \langle v_1, v_2, v_3\rangle$ and $\gamma_3 := \nu\left(\pi_2'(v_3)\right)$. Repeating this argument as above d times, we have chosen vectors v_i, increasing spaces $F_i = \langle v_1, \ldots, v_i\rangle$ and increasing $\gamma_i = \nu(v_i) \in \Gamma$ for $i \in [d]$ so that

$$C = \operatorname{conv}\left(\{0, v_1, \ldots, v_d\}\right) =$$
$$\{n_1 v_1 + \ldots + n_d v_d \mid n_i \in \mathcal{O}\} = \{w_1 + \ldots + w_d \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}.$$

□

2.4 Combinatorial properties of convex sets

The following definition is from [ADH$^+$16, Section 2.4].

Definition 2.4.1. Given a set X and $d \in \mathbb{N}_{\geq 1}$, a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *breadth* d if any nonempty intersection of finitely many sets in $\mathcal{F}$ is the intersection of at most d of them, and d is minimal with this property.

Lemma 2.4.2. *Let K be a valued field and S a convex subset of K^d.*

1. *If $0 \in S$ and S is finitely generated, then it is generated as an $\mathcal{O}$-module by a finite linearly independent set of vectors.*
2. *Let $\widetilde{K}$ be a valued field extension of K and $\widetilde{S} := \operatorname{conv}_{\widetilde{K}^d}(S) \subseteq \widetilde{K}^d$. Then $\widetilde{S} \cap K^d = S$.*

Proof. (1) By Lemma 2.3.15, S is generated as an $\mathcal{O}$-module by some finite set $v_1, \ldots, v_n \in S$. Assume these vectors are not linearly independent, then $0 = \sum_{i \in [n]} \alpha_i v_i$ for some $\alpha_i \in K$ not all 0. Let $i \in [n]$ be such that $\nu(\alpha_i) \leq \nu(\alpha_j)$ for all $j \in [n]$, in particular $\alpha_i \neq 0$. Then $v_i = \sum_{j \neq i} \frac{\alpha_j}{-\alpha_i} v_j$ and $\nu\left(\frac{\alpha_j}{-\alpha_i}\right) = \nu(\alpha_j) - \nu(\alpha_i) \geq 0$, hence $\frac{\alpha_j}{-\alpha_i} \in \mathcal{O}$ for all $j \neq i$, and S is still generated as an $\mathcal{O}$-module by the set $\{v_j : j \neq i\}$. Repeating this finitely many times, we arrive at a linearly independent set of generators.

(2) Since convexity is invariant under translates, we may assume $0 \in S$. Since every element in the convex hull of a set is in the convex hull of some finite subset, we may also assume that S is finitely generated as an $\mathcal{O}$-module, and by (1) let $v_1, \ldots, v_n \in S$ be a linearly independent (in the vector space K^d, so $n \leq d$) set of its generators. Let $v_{n+1}, \ldots, v_d \in K^d$ be so that $\{v_i : i \in [d]\}$ is a basis of K^d, and say $v_i = (v_{i,j} : j \in [d])$ with $v_{i,j} \in K$. Then the square matrix $A := (v_{i,j} : i, j \in [d]) \in M_{d \times d}(K)$ is invertible, so $A^{-1} \in M_{d \times d}(K) \subseteq M_{d \times d}(\widetilde{K})$, so A is also invertible in $M_{d \times d}(\widetilde{K})$, hence $\{v_i : i \in [d]\}$ are linearly independent vectors in $\widetilde{K}^d$ as well. But now if $\sum_{i \in [n]} \alpha_i v_i = u$ for some $\alpha_i \in \widetilde{K}$ and $u \in K^d$, then necessarily $\alpha_i \in K$ for all i (otherwise we would get a non-trivial linear combination of $v_1, \ldots, v_d$ in $\widetilde{K}^d$). In

particular, any element of the $\mathcal{O}_{\widetilde{K}}$-module generated by $v_1, \ldots, v_n$ which is in K^d already belongs to the $\mathcal{O}_K$-module generated by $v_1, \ldots, v_n$, hence $\widetilde{S} \cap K^d = S$. $\square$

We can now demonstrate an (optimal) finite bound on the breadth of the family of convex sets over valued fields. In sharp contrast, over the reals there is no finite bound on the breadth already for convex subsets of $\mathbb{R}^2$ (for any n, a convex n-gon in $\mathbb{R}^2$ is the intersection of n half-planes, but not the intersection of any fewer of them).

Theorem 2.4.3. *Let K be a valued field and $d \geq 1$. Then the family Conv_{K^d} has breadth d. That is, any nonempty intersection of finitely many convex subsets of K^d is the intersection of at most d of them.*

Proof. The family Conv_{K^d} cannot have breadth less than d because the d coordinate-aligned hyperplanes are convex, have common intersection $\{0\}$, but any $d-1$ of them intersect in a line.

We now show that Conv_{K^d} has breadth at most d, by induction on d. Then case $d = 1$ is clear. For $d > 1$, assume $C_1, \ldots, C_n \in \mathrm{Conv}_{K^d}$ with $n \geq d$ are convex and satisfy $\bigcap_{i\in[n]} C_i \neq \emptyset$. Translating, we may assume $0 \in \bigcap_{i\in[n]} C_i$.

We may also assume that K is spherically complete. Indeed, let $\widetilde{K}$ be a spherical completion of K as in Fact 2.3.3, and let $\widetilde{C}_i := \mathrm{conv}_{\widetilde{K}^d}(C_i) \in \mathrm{Conv}_{\widetilde{K}^d}$. By Lemma 2.4.2(2), $\widetilde{C}_i \cap K^d = C_i$ for each $i \in [n]$. Hence $\bigcap_{i\in[n]} \widetilde{C}_i \neq \emptyset$, and if $\bigcap_{i\in[n]} \widetilde{C}_i = \bigcap_{i\in S} \widetilde{C}_i$ for some $S \subseteq [n]$ with $|S| \leq d$, then also $\bigcap_{i\in[n]} C_i = \bigcap_{i\in S} C_i$.

Then let the vector subspaces $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and the upwards closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_d$ of Γ_∞ be as given by Theorem 2.3.6 for the convex set $C := C_1 \cap \ldots \cap C_n$. By Remark 2.3.8 we have

$$\Delta_d = \left\{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C_1 \cap \ldots \cap C_n\right\}.$$

It follows that there is some $i_d \in [n]$ such that in fact

$$\Delta_d = \left\{\gamma \in \Gamma_\infty \mid\ \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C_{i_d}\right\} \tag{2.4.1}$$

(since these are finitely many upwards closed sets in Γ, their intersection is already given by one of them).

Let $\{0\} \subsetneq F'_1 \subsetneq \ldots \subsetneq F'_d = K^d$ and $\Delta'_1 \supseteq \Delta'_2 \supseteq \ldots \supseteq \Delta'_d$ be as given by Theorem 2.3.6 for C_{i_d}. By Remark 2.3.10(1), F'_{d-1} is a linear hyperplane so that every element of C_{i_d} differs from an element of $F'_{d-1} \cap C_{i_d}$ by a vector with valuation in Δ'_d. As $\Delta_d = \Delta'_d$ by (2.4.1) and $C \subseteq C_{i_d}$, by Remark 2.3.10(1) we may assume that $F_{d-1} = F'_{d-1}$, hence every element in C_{i_d} differs from an element of $F_{d-1} \cap C_{i_d}$ by a vector with valuation in Δ_d.

Consider $C \cap F_{d-1} = C_1 \cap \ldots \cap C_n \cap F_{d-1} = (C_1 \cap F_{d-1}) \cap \ldots \cap (C_n \cap F_{d-1})$. Note that each $C_i \cap F_{d-1}$ is a convex subset of $F_{d-1} \cong K^{d-1}$, so by induction hypothesis there exist $i_1, \ldots, i_{d-1} \in [n]$ such that

$$C_{i_1} \cap \ldots \cap C_{i_{d-1}} \cap F_{d-1} = C_1 \cap \ldots \cap C_n \cap F_{d-1} = C \cap F_{d-1}. \tag{2.4.2}$$

Let $x \in C_{i_1} \cap \ldots \cap C_{i_d}$ be arbitrary. As $x \in C_{i_d}$, by the choice of F_{d-1}, $x = w + v_d$ for some $w \in F_{d-1}$ and $v_d \in K^d$ with $\nu(v_d) \in \Delta_d$. By the choice of Δ_d we have in particular $v_d \in C_{i_1} \cap \ldots \cap C_{i_d}$. And as each C_i is a module, it follows that also $w \in C_{i_1} \cap \ldots \cap C_{i_d}$. Combining this with (2.4.2) and using Remark 2.3.9 (for $j = d-1$) we thus have

$$\begin{gathered} C_{i_1} \cap \ldots \cap C_{i_d} = \{w + v_d \mid w \in C_{i_1} \cap \ldots \cap C_{i_d} \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} = \\ \{w + v_d \mid w \in C \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} = \\ \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} = \\ C_1 \cap \ldots \cap C_n. \end{gathered}$$

□

Definition 2.4.4. 1. A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *Helly number* $k \in \mathbb{N}_{\geq 1}$ if given any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$, if every k-subset of $\{S_1, \ldots, S_n\}$ has nonempty intersection, then $\bigcap_{i \in [n]} S_i \neq \emptyset$.

2. The *Helly number of* $\mathcal{F}$ refers to the minimal k with this property (or ∞ if it does not exist).

3. We say that $\mathcal{F}$ has the *Helly property* if it has a finite Helly number.

Theorem 2.4.5. *Let K be a valued field and $d \geq 1$. Then the Helly number of* Conv_{K^d} *is* $d+1$.

Proof. Let n be arbitrary, and let $S_1, \ldots, S_n \subseteq K^d$ be convex sets so that any $d+1$ of them have a non-empty intersection. We will show by induction on n that $S_1 \cap \ldots \cap S_n \neq \emptyset$.

Base case: $n = d+2$.

By assumption for each $i \in [d+2]$ there exists some $x_i \in K^d$ so that $x_i \in \bigcap_{j \in [d+2] \setminus \{i\}} S_j$. By Proposition 2.2.8 there exists some $i^* \in [d+2]$ so that $x_{i^*} \in \mathrm{conv}\left(\{x_i \mid i \neq i^*\}\right)$. By the choice of the x_i's we have $x_{i^*} \in S_i$ for all $i \neq i^*$. We also have $x_i \in S_{i^*}$ for all $i \neq i^*$, S_{i^*} is convex and $x_{i^*} \in \mathrm{conv}\left(\{x_i \mid i \neq i^*\}\right)$, hence $x_{i^*} \in S_{i^*}$. Thus $x_{i^*} \in \bigcap_{i \in [d+2]} S_i$, as wanted.

Inductive step: $n > d+2$.

Let $\widetilde{S}_{n-1} := S_{n-1} \cap S_n$, in particular $\widetilde{S}_{n-1}$ is convex. By induction hypothesis, any $n-1$ sets from $\{S_1, \ldots, S_n\}$ have a non-empty intersection. Hence any $n-2$ sets from $\left\{S_1, \ldots, S_{n-2}, \widetilde{S}_{n-1}\right\}$ have a non-empty intersection. As $n-2 \geq d+1$ by assumption, applying the induction hypothesis again we get

$$S_1 \cap \ldots \cap S_n = S_1 \cap \ldots \cap S_{n-2} \cap \widetilde{S}_{n-1} \neq \emptyset.$$

This completes the induction, and shows that Conv_{K^d} has Helly number $d+1$.

It remains to show that Conv_{K^d} does not have Helly number d. Let $e_i \in K^d$ be the ith standard basis vector. In particular the set $E := \{0, e_1, \ldots, e_d\}$ is affinely independent, hence the intersection of the affine spans of its $d+1$ maximal proper subsets is empty. The convex hull of a subset of K^d is contained in its affine hull, hence the intersection of the $d+1$ convex hulls of its maximal proper subsets is also empty. But for any d among the $(d+1)$ maximal proper subsets of E, some element of E belongs to their intersection, and hence in particular the intersection of their convex hulls is non-empty. $\square$

We recall some terminology around the *Vapnik-Chervonenkis dimension* (and refer to [ADH$^+$16, Sections 1 and 2] for further details).

Definition 2.4.6. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X.

1. For a subset $Y \subseteq X$, we let $\mathcal{F} \cap Y := \{S \cap Y : S \in Y\} \subseteq \mathcal{P}(Y)$.

2. We say that $\mathcal{F}$ *shatters* a subset $Y \subseteq X$ if $\mathcal{F} \cap Y = \mathcal{P}(Y)$.

3. The *VC-dimension* of $\mathcal{F}$, or $\mathrm{VC}(\mathcal{F})$, is the largest $k \in \mathbb{N}$ (if one exists) such that $\mathcal{F}$ shatters some subset of X size k. If $\mathcal{F}$ shatters arbitrarily large finite subsets of X, then it is said to have infinite VC-dimension.

4. The *dual family* $\mathcal{F}^* \subseteq \mathcal{P}(\mathcal{F})$ is given by
   $\mathcal{F}^* = \{S_x \mid x \in X\}$, where $S_x = \{A \in \mathcal{F} \mid x \in A\}$.

5. The *dual VC-dimension* of $\mathcal{F}$, or $\mathrm{VC}^*(\mathcal{F})$, is the VC-dimension of $\mathcal{F}^*$. Equivalently, it is the largest $k \in \mathbb{N}$ (or ∞ if no such k exists) such that there are sets $S_1, \ldots, S_k \in \mathcal{F}$ that generate a Boolean algebra with 2^k atoms (i.e. for any distinct $I, J \subseteq [k]$, $\bigcap_{i \in I} S_i \cap \bigcap_{i \in [k] \setminus I} (X \setminus S_i) \neq \bigcap_{i \in J} S_i \cap \bigcap_{i \in [k] \setminus J} (X \setminus S_i)$).

6. The *shatter function* $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{F}$ is
   $$\pi_{\mathcal{F}}(n) := \max \{|\mathcal{F} \cap Y| : Y \subseteq X, |Y| = n\}.$$

7. By the Sauer-Shelah lemma (see e.g. [ADH$^+$16, Lemma 2.1], if $\mathrm{VC}(\mathcal{F}) \leq d$, then $\pi_{\mathcal{F}}(n) \leq \left(\frac{e}{d}\right)^d n^d$ for all $n \geq d$ (and $\pi_{\mathcal{F}}(n) = 2^n$ for all n if $\mathrm{VC}(\mathcal{F}) = \infty$).

8. The *VC-density* of $\mathcal{F}$, or $\mathrm{vc}(\mathcal{F})$, is the infimum of all $r \in \mathbb{R}_{>0}$ so that $\pi_{\mathcal{F}}(n) = O(n^r)$, and ∞ if there is no such r. (In particular $\mathrm{vc}(\mathcal{F}) \leq \mathrm{VC}(\mathcal{F})$.)

9. Finally, we define the *dual shatter function* $\pi^*_{\mathcal{F}} := \pi_{\mathcal{F}^*}$ and the *dual* VC-*density* $\mathrm{vc}^*(\mathcal{F}) := \mathrm{vc}(\mathcal{F}^*)$ of the family $\mathcal{F}$.

Remark 2.4.7. Note that if $\mathcal{F} \subseteq \mathcal{P}(X)$ and $Y \subseteq X$, then $\mathrm{VC}(\mathcal{F} \cap Y) \leq \mathrm{VC}(\mathcal{F})$ and $\mathrm{VC}^*(\mathcal{F} \cap Y) \leq \mathrm{VC}^*(\mathcal{F})$.

The following results is in stark contrast with the situation for the family of convex sets over the reals, where already the family of convex subsets of $\mathbb{R}^2$ has infinite VC-dimension (e.g., any set of points on a circle is shattered by the family of convex hulls of its subsets).

Theorem 2.4.8. *Let K be a valued field and $d \geq 1$. Then the family Conv_{K^d} has VC-dimension $d+1$.*

Proof. We have $\mathrm{VC}\left(\mathrm{Conv}_{K^d}\right) \geq d+1$ since the set $E := \{0, e_1, \ldots, e_d\} \subseteq K^d$, with e_i the ith vector of the standard basis, is shattered by Conv_{K^d}. Indeed, the convex hull of any subset is contained in its affine span, and for any $S \subseteq E$, $\mathrm{aff}(S)$ does not contain any of the points in $E \setminus S$.

On the other hand, $\mathrm{VC}\left(\mathrm{Conv}_{K^d}\right) \leq d+1$ as no subset Y of K^d with $|Y| \geq d+2$ can be shattered by Conv_{K^d}. Indeed, by Proposition 2.2.8, at least one of the points of Y belongs to every convex set containing all the other points of Y. □

The dual VC-dimension of a family of sets is bounded by its breadth.

Fact 2.4.9. *[ADH+16, Lemma 2.9] Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X of breadth at most d. Then also $\mathrm{VC}^*(\mathcal{F}) \leq d$.*

Using it, we get the following:

Theorem 2.4.10. *For any valued field K and $d \geq 1$, the family Conv_{K^d} has dual VC-dimension d.*

Proof. The dual VC-dimension of Conv_{K^d} is at least d because the d coordinate-aligned (convex) hyperplanes in K^d generate a Boolean algebra with 2^d atoms.

Conversely, the breadth of Conv_{K^d} is d by Theorem 2.4.3, hence by Fact 2.4.9 its dual VC-dimension is also at most d. □

Definition 2.4.11. 1. A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *fractional Helly number* $k \in \mathbb{N}_{\geq 1}$ if for every $\alpha \in \mathbb{R}_{>0}$ there exists $\beta \in \mathbb{R}_{>0}$ so that: for any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$ (possibly with repetitions), if there are $\geq \alpha \binom{n}{k}$ k-element subsets of the multiset $\{S_1, \ldots, S_n\}$ with a non-empty intersection, then there are $\geq \beta n$ sets from $\{S_1, \ldots, S_n\}$ with a non-empty intersection.

2. The *fractional Helly number* of $\mathcal{F}$ refers to the minimal k with this property. Say that $\mathcal{F}$ has the *fractional Helly property* if it has a fractional Helly number.

Note that any finite family of sets trivially has fractional Helly number 1 by choosing β sufficiently small with respect to the size of $\mathcal{F}$. We will use the following theorem of Matoušek.

Fact 2.4.12. *[Mat04, Theorem 2] Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a set system whose dual shatter function satisfies $\pi^*_{\mathcal{F}}(n) = o(n^k)$, i.e. $\lim_{n\to\infty} \pi^*_{\mathcal{F}}(n)/n^k = 0$, where k is a fixed integer. Then $\mathcal{F}$ has fractional Helly number k.*

Remark 2.4.13. Moreover, if $\mathrm{VC}^*(\mathcal{F}) = d < \infty$, then the fractional Helly number is $\leq d+1$, and the β witnessing this can be chosen depending only on d and α (and not on the family $\mathcal{F}$).

Indeed, by Definition 2.4.6, if $\mathrm{VC}^*(\mathcal{F}) \leq d$, then $\pi^*_{\mathcal{F}}(n) \leq \left(\frac{e}{d}\right)^d n^d$ for all $n \geq d$, hence $\pi^*_{\mathcal{F}}(n) \leq cn^d$ for all $n \in \mathbb{N}$, where $c = c(d) := \left(\frac{e}{d}\right)^d + 2^d$. In particular we can choose $m = m(d, \alpha)$ so that $\pi^*_{\mathcal{F}}(m) < \frac{1}{4}\alpha\binom{m}{d+1}$. Then it follows from the proof of [Mat04, Theorem 2] that $\beta = \frac{1}{2m}$ works for all $n \geq \frac{m}{\beta} = 2m^2$, and trivially $\beta = \frac{1}{2m^2}$ works for all $n \leq 2m^2$, hence $\beta := \beta(\alpha, d) := \frac{1}{2m^2}$ works for all $n \in \mathbb{N}$.

Using this, we get the following:

Theorem 2.4.14. *If K is a valued field, $d \geq 1$, and $X \subseteq K^d$ is an arbitrary subset, then the fractional Helly number of the family*

$$\mathrm{Conv}_{K^d} \cap X = \{C \cap X : X \in \mathrm{Conv}_{K^d}\} \subseteq \mathcal{P}(X)$$

is at most $d+1$. Moreover, β in Definition 2.4.11 can be chosen depending only on d and α (and not on the field K or set X). And if K is infinite, then the fractional Helly number of the family Conv_{K^d} *is exactly $d+1$.*

Proof. By Fact 2.4.12 we have that the fractional Helly number of a set system is at most the smallest integer larger than its dual VC-density. Dual VC-density is, in turn, at most its dual VC-dimension. Also for any set $X \subseteq K^d$ we have $\mathrm{VC}^*\left(\mathrm{Conv}_{K^d} \cap X\right) \leq \mathrm{VC}^*\left(\mathrm{Conv}_{K^d}\right)$ by Remark 2.4.7. So $\mathrm{Conv}_{K^d} \cap X$ has dual VC-density at most d by Theorem 2.4.10, hence its fractional Helly number is at most $d+1$ by Fact 2.4.12. And an appropriate β can be chosen depending only on d and α by Remark 2.4.13.

To show that the fractional Helly number of Conv_{K^d} is at least $d+1$ when K is infinite, we can use the standard example with affine hyperplanes in general position. We include the details for completeness. First note that as the field K is infinite, for any K-vector space V of dimension k and $v \in V \setminus \{0\}$ there exists an infinite set $S \subseteq V$ so that $v \in S$ and any k vectors from S are linearly independent. This is clear for $k=1$ by taking any infinite set of non-zero vectors, so assume that $k > 1$. By induction on $i \in \mathbb{N}_{\geq k}$ we can find sets S_i such that $v \in S_i, |S_i| \geq i$ and every k vectors from S_i are linearly independent, for all i. Let S_k be any basis of V containing v. Assume $i > k$ and S_i satisfies the assumption. Since K is infinite, V is not a union of finitely many proper subspaces, in particular there exists some

$$w \in V \setminus \bigcup_{s \subseteq S_i, |s|=k-1} \langle s \rangle.$$

Let $S_{i+1} := S_i \cup \{w\}$. Since in particular any $s \subseteq S_i$ with $|s| = k-1$ is linearly independent by the inductive assumption, it follows that $s \cup \{w\}$ is also linearly independent, hence S_{i+1} satisfies the assumption. Finally, $S := \bigcup_{i \in \mathbb{N}_{\geq k}} S_i$ is as wanted.

In particular, we can find an infinite set of vectors S in $K^d \times K$ so that any $d+1$ of them are linearly independent and the standard basis vector $e_{d+1} \in S$. Then

$$X := \{\langle v, - \rangle : v \in S\} \subseteq \left(K^d \times K\right)^*$$

is an infinite set of dual vectors such that any $d+1$ of them are linearly independent, and it contains the projection map onto the last coordinate $\pi_{d+1} := \langle e_{d+1}, -\rangle : (x_1, \ldots, x_{d+1}) \mapsto x_{d+1}$. Consider the family

$$\mathcal{H} := \{\ker(f) \mid f \in X \setminus \{\pi_{d+1}\}\} \subseteq \mathcal{P}\left(K^d \times K\right)$$

of kernels of these dual vectors (excluding the projection map onto the last coordinate), and let

$$\mathcal{H}' := \left\{\left\{v \in K^d \mid (v,1) \in H\right\} \mid H \in \mathcal{H}\right\} \subseteq \mathcal{P}\left(K^d\right).$$

Then $\mathcal{H}'$ is an infinite family of affine hyperplanes in K^d, and we wish to show that any d element of $\mathcal{H}'$ intersect in a point, and any $d+1$ elements of $\mathcal{H}'$ have empty intersection. For any pairwise distinct $f_1, \ldots, f_d \in X \setminus \{\pi_{d+1}\}$, by linear independence

$$\dim(\ker(f_1) \cap \ldots \cap \ker(f_d)) = d + 1 - \dim(\langle f_1, \ldots, f_d\rangle) = 1.$$

And by their linear independence with π_{d+1},

$$\dim(\ker(f_1) \cap \ldots \cap \ker(f_d) \cap \ker(\pi_{d+1})) = 0.$$

That is, $\ker(f_1) \cap \ldots \cap \ker(f_d)$ is a line in $K^d \times K$ that intersects $\ker(\pi_{d+1}) = K^d \times \{0\}$ only at the origin, and thus must also intersect $K^d \times \{1\}$ in a single point; this shows that every d elements of $\mathcal{H}'$ intersect in a point. And any pairwise distinct $f_1, \ldots, f_{d+1} \in X \setminus \{\pi_{d+1}\}$ span $\left(K^d \times K\right)^*$ by linear independence, so $\ker(f_1) \cap \ldots \cap \ker(f_{d+1}) = \{0\}$, and thus has empty intersection with $K^d \times \{1\}$. This shows that every $d+1$ elements of $\mathcal{H}'$ have empty intersection.

Using $\alpha = 1$, for any $\beta > 0$, take an arbitrary $n \geq \frac{d+1}{\beta}$. Let $H_1, \ldots, H_n \in \mathcal{H}'$ be any distinct hyperplanes from this collection. All d-subsets (so, $\alpha\binom{n}{d}$ of them) of $\{H_1, \ldots, H_n\}$ have an intersection point, but there are no $\beta n \geq d+1$ of them with a common intersection point. Therefore Conv_{K^d} does not have fractional Helly number d. $\square$

Note that Theorems 2.4.5 and 2.4.14 replicate results for real convex sets, while Theorems 2.4.3, 2.4.8, and 2.4.10 do not: as we have already remarked, $\mathrm{Conv}_{\mathbb{R}^2}$ has infinite breadth, VC-dimension, and dual VC-dimension. The following result is somewhere in between: it is a much stronger version of the *Tverberg theorem* for real convex sets (note that any element of the non-empty set X_r in the statement of the theorem belongs to the convex hulls of each of the sets $X_i, i \in [r]$ — which gives the usual conclusion of Tverberg's theorem over the reals).

Theorem 2.4.15. *Let K be a valued field and $d, r \in \mathbb{N}_{\geq 1}$. Then any set $X \subseteq K^d$ with*

$$|X| \geq (d+1)(r-1)+1$$

points in K^d can be partitioned into subsets $X_1, \ldots, X_r$ such that $|X_i| = d+1$ for $i < r$, $|X_r| = |X| - (d+1)(r-1)$, and $\operatorname{conv}(X_i) \supseteq \operatorname{conv}(X_j)$ for all $i \leq j \in [r]$.

Proof. Since any finitely generated convex set is the convex hull of some $d+1$ points from it by Proposition 2.2.8, we can find $X_1 \subseteq X$ with $|X_1| = d+1$ and $\operatorname{conv}(X_1) = \operatorname{conv}(X)$, $X_2 \subseteq X \setminus X_1$ with $|X_2| = d+1$ and $\operatorname{conv}(X_2) = \operatorname{conv}(X \setminus X_1)$, and so on: once $X_1, \ldots, X_{i-1}$ have been chosen, pick $X_i \subseteq X \setminus \left(\bigcup_{j=1}^{i-1} X_j\right)$ such that $|X_i| = d+1$, $\operatorname{conv}(X_i) = \operatorname{conv}\left(X \setminus \bigcup_{j=1}^{i-1} X_j\right)$, and then let X_r consist of everything left over at the end. ☐

From this strong Tverberg theorem and the fractional Helly property, we finally get an analog of the result due to Boros-Füredi [BF84] and Bárány [Bár82] on the common points in the intersections of many "simplices" over valued fields (note that the conclusion is actually stronger than over the reals: the common point comes from the set X itself). This answers a question asked by Kobi Peterzil and Itay Kaplan. Our argument is an adaptation of the second proof in [Mat02, Theorem 9.1.1].

Theorem 2.4.16. *For each $d \geq 1$ there is a constant $c = c(d) > 0$ such that: for any valued field K and any finite $X \subseteq K^d$ (say $n := |X|$), there is some $a \in X$ contained in the convex hulls of at least $c\binom{n}{d+1}$ of the $\binom{n}{d+1}$ subsets of X of size $d+1$.*

Proof. Let $X \subseteq K^d$ with $|X| = n$ be given, and let

$$\mathcal{F} := \operatorname{Conv}_{K^d} \cap X = \{C \cap X : C \in \operatorname{Conv}_{K^d}\}$$

be the family of all subsets of X cut out by the convex subsets of K^d. Let $(S_i)_{i\in[N]}$ with $S_i \in \mathrm{Conv}_{K^d}$ be the sequence listing all convex hulls of subsets of X of size $d+1$ in an arbitrary order (possibly with repetitions). Then $N = \binom{n}{d+1}$, and for a $(d+1)$-element subset $Y \subseteq X$ we let $g(Y) \in [N]$ be the index at which $\mathrm{conv}(Y)$ appears in this sequence. For each $i \in [N]$ let $S'_i := S_i \cap X \in \mathcal{F}$. It is thus sufficient to show that there exists some $\alpha > 0$, depending only on d, such that at least $\alpha\binom{N}{d+1}$ of the $(d+1)$-element subsets $I \subseteq [N]$ satisfy $\bigcap_{i\in I} S'_i \neq \emptyset$ — as then Theorem 2.4.14 applied to $\mathcal{F} \subseteq \mathcal{P}(X)$ shows the existence of $c > 0$ depending only on α, d, and hence only on d, so that for some $I \subseteq [N]$ with $|I| \geq cN = c\binom{n}{d+1}$ there exists some $a \in \bigcap_{i\in I} S'_i \subseteq \bigcap_{i\in I} S_i$ (in particular $a \in X$).

Now we find an appropriate α. For any $(d+1)^2$-element subset $Y \subseteq X$, by Theorem 2.4.15 (with $r := d+1$), we can fix a partition of Y into $d+1$ disjoint parts $Y_1, \ldots, Y_{d+1}$, each of which has $d+1$ elements, and so that $\mathrm{conv}(Y_i) \supseteq \mathrm{conv}(Y_j)$ for all $i \leq j \in [d+1]$. In particular any element of the non-empty set $Y_{[d+1]} \subseteq X$ belongs to $\bigcap_{i\in[d+1]} (\mathrm{conv}(Y_i) \cap X) = \bigcap_{i\in[d+1]} \left(S'_{g(Y_i)}\right)$. As g is a bijection, $Y \mapsto \{g(Y_i) : i \in [d+1]\}$ gives a function f from $(d+1)^2$-element subsets of X to $(d+1)$-element subsets $I \subseteq [N]$ so that $\bigcap_{i\in I} S'_i \neq \emptyset$. Moreover, f is an injection. Indeed, given a set $\{j_i : i \in [d+1]\}$ in the image of f, as g is a bijection, there is a unique set $\{Y_1, \ldots, Y_{d+1}\}$ with $Y_i \subseteq X$ disjoint of size $d+1$ so that $g(Y_i) = j_i$ for all $i \in [d+1]$, and there can be only one set $Y \subseteq X$ of size $(d+1)^2$ for which it is a partition. If follows that the number of sets $I \subseteq [N]$ with $\bigcap_{i\in I} S'_i \neq \emptyset$ is at least

$$\binom{n}{(d+1)^2} = \Omega\left(n^{(d+1)^2}\right) \geq \alpha\binom{N}{d+1}$$

for some sufficiently small α depending only on d. □

2.5 Final remarks and questions

The results of Section 2.4 imply the following analog of the celebrated (p, q)-*theorem* of Alon and Kleitman [AK92] for convex sets over valued fields.

Corollary 2.5.1. *For any $d, p, q \in \mathbb{N}_{\geq 1}$ with $p \geq q \geq d+1$ there exists $T \in \mathbb{N}$ such that: if K is a valued field and $\mathcal{F}$ is a family of convex subsets of K^d such that among every p sets of $\mathcal{F}$, some q have a non-empty intersection, then there exists a T-element set $Y \subseteq K^d$ intersecting all sets of $\mathcal{F}$.*

Corollary 2.5.1 follows formally by applying [AKMM02, Theorem 8] since the family Conv_{K^d} has fractional Helly property (Theorem 2.4.14) and is closed under intersections. Alternatively, it follows with a slightly better bound on T by combining the fractional Helly property with the existence of ε-nets for families of bounded VC-dimension (Theorem 2.4.8), as outlined at the end of [Mat04, Section 1]. The problem of determining the optimal bound on $T(p, q, d)$ is widely open over the reals (see [BK21, Section 2.6]), and we expect that it might be easier in the valued fields setting.

Kalai [Kal84] and Eckhoff [Eck85] proved that in the fractional Helly property for convex sets over the reals, one can take $\beta(d, \alpha) = 1 - (1-\alpha)^{\frac{1}{d+1}}$ (and this bound is sharp).

Problem 2.5.2. What is the optimal dependence of β on d, α in Theorem 2.4.14?

We expect that the *colorful* Tverberg theorem also holds over valued fields, however the proofs for convex sets over the reals rely on topological arguments not readily available in the valued field context:

Conjecture 2.5.3. For any integers $r, d \geq 2$ there exists $t \geq r$ such that: for any valued field K and $X \subseteq K^d$ with $|X| = t\,(d+1)$, partitioned into $d+1$ color classes $C_1, \ldots, C_{d+1}$ each of size t, there exist pairwise disjoint $X_1, \ldots, X_r \subseteq X$ with $|X_i \cap C_j| = 1$ for $i \in [r]$ and $j \in [d+1]$, and $\bigcap_{i \in [r]} \mathrm{conv}\,(X_i) \neq \emptyset$.

It would formally imply (see e.g. [Mat02, Section 9.2]) the "second selection lemma" over valued fields generalizing Theorem 2.4.16:

Conjecture 2.5.4. For each $d \in \mathbb{N}_{\geq 1}$ there exist $c, s > 0$ such that: for any valued field K, $\alpha \in (0, 1]$ and $n \in \mathbb{N}$, for every $X \subseteq K^d$ with $|X| = n$, and every family $\mathcal{F}$ of $(d+1)$-element subsets of X with $|\mathcal{F}| \geq \alpha\binom{n}{d+1}$, there is a point contained in the convex hulls of at least $c\alpha^s\binom{n}{d+1}$ of the elements of $\mathcal{F}$.

Finally, Corollary 2.3.14 has the following immediate model-theoretic application.

Remark 2.5.5. If K is a spherically complete valued field, then every convex subset of K^d is definable in the expansion of the field K by a predicate for each Dedekind cut of the value group (so in particular definable in *Shelah expansion of* K by all externally definable sets [She09, CS13]). And conversely, every Dedekind cut of the value group is definable in the expansion of K by a predicate for each $\mathcal{O}$-submodule of K. In particular, if K has value group $\mathbb{Z}$, then all convex subsets of K^d form a definable family.

Example 2.5.6. In contrast, naming a single (bounded) convex subset of $\mathbb{R}^2$ in the field of reals allows to define the set of integers. Indeed, we can define a continuous and piecewise linear function $f : [0,1] \to [0,1]$ such that

$$C := \{(x,y) : x \in [0,1], 0 \leq y \leq f(x)\}$$

is convex but the set of points where f is not differentiable is exactly $\left\{\frac{1}{n} : n \in \mathbb{N}_{\geq 2}\right\}$. Now in the field of reals with a predicate for C we can define f and the set of points where it is not differentiable, hence $\mathbb{N}$ is also definable.

CHAPTER 3

Semi-equational theories

3.1 Introduction

Equations and equational theories were introduced by Srour [Sro88a, Sro88b, Sro90] in order to distinguish "positive" information in an arbitrary first order theory, i.e. to find a well behaved class of "closed" sets among the definable sets, by analogy to the algebraic sets among the constructible ones in algebraically closed fields. We recall the definition:

Definition 3.1.1. 1. A partitioned formula $\varphi(x, y)$, with x, y tuples of variables, is an *equation* (with respect to a first-order theory T) if there do not exist $\mathcal{M} \models T$ and tuples $(a_i, b_i : i \in \omega)$ in $\mathcal{M}$ such that $\mathcal{M} \models \varphi(a_i, b_j)$ for all $j < i$, and $\mathcal{M} \models \neg\varphi(a_i, b_i)$ for all i.

2. A theory T is *equational* if every formula $\varphi(x, y)$, with x, y arbitrary finite tuples of variables, is equivalent in T to a Boolean combination of finitely many equations $\varphi_1(x, y), \ldots, \varphi_n(x, y)$.

It is immediate from the definition that every equational theory is stable. Structural properties of equational theories in relation to forking and stability theory are studied in [PS84, HS89, Jun00, JK02, JL01]. Many natural stable theories are equational; [HS89] provided the first example of a stable non-equational theory. More recently it was demonstrated that the stable theory of non-abelian free groups is not equational [Sel12, MS17], and further examples are constructed in [MPZ21]. It is demonstrated in [MPZ20] that all theories of separably closed fields are equational (generalizing earlier work of Srour [Sro86]). See also [O'H11b] for an accessible introduction to equationality.

In this paper we propose a generalization of equations and equational theories to the larger class of NIP theories (see Section 3.2.1 for a more detailed discussion):

Definition 3.1.2. 1. A partitioned formula $\varphi(x, y)$ is a *semi-equation* if there is no infinite sequence $(a_i, b_i : i \in \omega)$ such that for all $i, j \in \omega$, $\models \varphi(a_i, b_j) \iff i \neq j$.

2. A partitioned formula $\varphi(x, y)$ is a *weak semi-equation* if there does not exist an ($\emptyset$-) indiscernible sequence $(a_i : i \in \mathbb{Z})$ and tuple b such that the subsequence $(a_i : i \in \mathbb{Z} \setminus \{0\})$ is indiscernible over b, $\models \varphi(a_i, b)$ for all $i \in \mathbb{Z} \setminus \{0\}$, but $\models \neg\varphi(a_0, b)$.

3. A theory is *(weakly) semi-equational* if every formula is a Boolean combination of (weak) semi-equations.

Semi-equations are in particular weak semi-equations, every weakly semi-equational theory is NIP, and in a stable theory all three notions coincide (see Proposition 3.2.13). Some parts of the basic theory of equations naturally generalize to (weak) semi-equations, but there are also some new phenomena and complications appearing outside of stability. In particular, weak semi-equationality provides a simultaneous generalization of equationality and distality, bringing out some curious parallels between those two notions (see Section 3.4). In this paper we develop the basic theory of (weak) semi-equations, and investigate (weak) semi-equationality in some examples. We view this as a first step, and a large number of rather basic questions remain open and can be found throughout the paper.

In Section 3.2.1 we define weak semi-equations (Definition 3.2.1), semi-equations (Definition 3.2.3), (weakly) semi-equational theories (Definition 3.2.4), and provide some equivalent characterizations in terms of indiscernibles. We discuss closure of (weak) semi-equations under Boolean combinations (Proposition 3.2.6), reducts and expansions (Proposition 3.2.9). In Section 3.2.2 we discuss how (weak) semi-equationality relates to the more familiar notions: all weakly semi-equational theories are NIP, distal theories are weakly semi-equational, and in a stable theory a formula is an equation if and only if it is a (weak) semi-equation (Proposition 3.2.13). In Section 3.2.3 we introduce some quantitive parameters associated to semi-equations. This parameter is related to breadth (Definition 3.2.19) of the family defined by the instances of a formula, and we observe that a formula is a semi-equation if and only if the family of its instances has finite breadth (Proposition 3.2.20). The case when this parameter is minimal, i.e. 1-semi-equations, provide a generalization of weakly normal formulas characterizing 1-based stable theories (Proposition 3.2.22). Hence 1-semi-equationality can be viewed as a form of "linearity", or "1-basedeness" for NIP theories. We discuss its connections to a different form of "linearity" considered in [BCS$^+$21], namely basic relations and almost linear Zarankiewicz bounds (see Proposition 3.2.26 and Remark 3.2.27), observing in particular that $(2, 1)$-semi-equational theories do not define infinite fields.

In Section 3.3 we consider some examples of semi-equational theories. In Section 3.3.1 we show that an o-minimal expansion of a group is linear if and only if it is $(2, 1)$-semi-equational. It remains open if the field of reals is semi-equational (Problem 3.3.3). We demonstrate that arbitrary unary expansions of linear orders (Section 3.3.2) and many ordered abelian groups (Section 3.3.4) are 1-semi-equational. In Section 3.3.5 we demonstrate that the theory of infinitely-branching dense trees is semi-equational (Theorem 3.3.16), but not 1-semi-equational (even after naming parameters, see Theorem 3.3.17 and Corollary 3.3.18). Semi-equationality of arbitrary trees remains open (Problem 3.3.21). In Section 3.3.3 we observe that dense circular orders are not semi-equational, but become 1-semi-equational after naming a single constant (in contrast to equationality being preserved under naming and forgetting constants).

In Section 3.4 we consider the relation of weak semi-equationality and distality in more detail. We show that in an NIP theory, weak semi-equationality of a formula is equivalent to the existence of a *one-sided* strong honest definition for it (Theorem 3.4.5). This is a simultaneous generalization of the existence of strong honest definitions in distal theories from [CS15] and the isolation property for the positive part of φ-types for equations (replacing a conjunction of finitely many instances of φ by some formula θ, see Fact 3.4.3). We also make some remarks about forking for weak semi-equations.

In Section 3.5 we provide some examples of NIP theories that are not weakly semi-equational. First, in Section 3.5.1 we provide a sufficient criterion for when a formula is not a Boolean combination of weak semi-equations (generalizing the criterion for equations from [MS17]). We then apply it to show that the theory of dense valued trees (Section 3.5.2) and many theories of valued fields with an infinite stable residue field, e.g. ACVF (Section 3.5.3), are not weakly semi-equational. In both cases the proof relies on a detailed analysis of the behavior of indiscernible sequences. It remains open if the field $\mathbb{Q}_p$ is semi-equational (Problem 3.5.19).

In Section 3.6 we consider preservation of weak semi-equationality in expansions by naming a new predicate, partially adapting a result for NIP from [CS13]. Namely, we demonstrate in Theorem 3.6.4 that if $\mathcal{M} \models T$ is distal, A is a subset of $\mathcal{M}$ with a distal induced structure and the pair (M, A) is almost model complete (i.e. every formula in the pair is equivalent to a Boolean combination of formulas which only quantify existentially over the predicate, see Definition 3.6.3), then the pair $(\mathcal{M}, A)$ is weakly semi-equational. This implies in particular that dense pairs of o-minimal structures are weakly semi-equational (but not distal by [HN17]).

Finally, in Section 3.7 we establish a sufficient criterion for when a formula is not a Boolean combination of semi-equations, which we hope will find applications in the future. In comparison to the case of weak semi-equations in Section 3.5.1, the situation is complicated by the fact that semi-equations are not closed under disjunction. This requires working with indiscernible arrays of higher dimension, see Proposition 3.7.6 for the details.

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3.2 Semi-equations and their basic properties

Let T be a complete theory in a language $\mathcal{L}$, we work inside a sufficiently saturated and homogeneous monster model $\mathbb{M} \models T$. All sequences of elements are assumed to be small relative to the saturation of $\mathbb{M}$, and we write $x, y, \ldots$ to denote finite tuples of variables. Given two linear orders I, J, $I + J$ denotes the linear order given by their sum (i.e. $I < J$); and (0) denotes a linear order with a single element. We write $\mathbb{N} = \{0, 1, \ldots\}$ and for $k \in \mathbb{N}$, $[k] = \{1, \ldots, k\}$. Given a partitioned formula $\varphi(x, y)$, we let $\varphi^*(y, x) := \varphi(x, y)$.

3.2.1 (Weak) semi-equations

Definition 3.2.1. A (partitioned) formula $\varphi(x, y)$ is a *weak semi-equation* (in T) if there do not exist infinite linear orders I_L and I_R, $b \in \mathbb{M}^y$ and an ($\emptyset$-)indiscernible sequence

$$(a_i : i \in I_L + (0) + I_R)$$

with $a_i \in \mathbb{M}^x$ such that the subsequence $(a_i : i \in I_L + I_R)$ is indiscernible over b, $\models \varphi(a_i, b)$ for all $i \in I_L + I_R$, but $\models \neg\varphi(a_0, b)$.

Proposition 3.2.2. *The following are equivalent for a formula $\varphi(x, y)$.*

1. *There is no b, infinite linear orders I_L, I_R and indiscernible sequence $(a_i)_{i \in I_L + (0) + I_R}$ such that $\models \varphi(a_i, b)$ for $i \in I_L + I_R$, but $\not\models \varphi(a_0, b)$.*
2. *There is no infinite order I and sequence $(a_i, b_i)_{i \in I}$ such that for all $i, j \in I$, $\models \varphi(a_i, b_j) \iff i \neq j$.*

Proof. (2)⇒(1). Assume that there exist $b \in \mathbb{M}^y$ and indiscernible sequence $(a_i)_{i \in I_L+(0)+I_R}$ such that $\models \varphi(a_i, b)$ for all $i \in I_L + I_R$, but $\models \neg\varphi(a_0, b)$. Given $n \in \mathbb{N}$, choose arbitrary $j_{-2n} < \ldots < j_{2n} \in I_L + (0) + I_R$ with $j_0 = 0$ (so $j_{-i} \in I_L$ and $j_i \in I_R$ for $0 < i \leq 2n$). By indiscernibility of the sequence $(a_i)_{i \in I_L+(0)+I_R}$ there exists an automorphism σ of $\mathbb{M}$ such that $\sigma(a_{j_i}) = a_{j_{i+1}}$ for $-2n \leq i < 2n$. Let $a'_i := a_{j_i}$ and $b_i := \sigma^i(b)$ for $-n \leq i \leq n$. Since $\not\models \varphi(a_0, b)$ and σ^i is an automorphism, $\not\models \varphi(a_{j_i}, \sigma^i(b))$, that is $\not\models \varphi(a'_i, b_i)$. And for distinct $i, \ell \in \{-n, \ldots, n\}$, $\models \varphi\left(a_{j_{i-\ell}}, b\right)$, so $\models \varphi\left(a_{j_i}, \sigma^\ell(b)\right)$, that is $\models \varphi(a'_i, b_\ell)$. Thus we have $(a'_i, b_i)_{-n \leq i \leq n}$ such that $\models \varphi(a'_i, b_j) \iff i \neq j$. Since this can be done for every $n \in \mathbb{N}$, by compactness there is $(a'_i, b_i)_{i \in I}$ with the same property for any infinite I.

(1)⇒(2). Assume that there exists a sequence $(a_i, b_i)_{i \in I}$ such that $\models \varphi(a_i, b_j) \iff i \neq j$. By compactness it is sufficient to show, for every $n \in \mathbb{N}$ and every finite set Δ of formulas, that there is a tuple b and sequence $(a_i)_{-n \leq i \leq n}$ which is Δ-indiscernible (i.e. for each formula $\psi(x_1, \ldots, x_k) \in \Delta$, the truth value of $\psi(a_{i_1}, \ldots, a_{i_k})$ is the same for every increasing sequence $-n \leq i_1 < \ldots < i_k \leq n$) and $\models \varphi(a_i, b) \iff i \neq 0$. By Ramsey's theorem, $(a_i)_{i \in I}$ has a Δ-indiscernible subsequence $\left(a_{i_j}\right)_{-n \leq j \leq n}$ of length $2n+1$ ($i_{-n} < \ldots < i_n$). Let $b := b_{i_0}$. Then b together with $\left(a_{i_j}\right)_{-n \leq j \leq n}$ has the desired property. □

Definition 3.2.3. A formula $\varphi(x, y)$ satisfying the equivalent conditions in Proposition 3.2.2 is called a *semi-equation*.

Definition 3.2.4. A theory T is (*weakly*) *semi-equational* if every formula $\varphi(x, y) \in \mathcal{L}$, with x, y arbitrary finite tuples of variables, is a Boolean combination of finitely many (weak) semi-equations $\psi_1(x, y), \ldots, \psi_n(x, y) \in \mathcal{L}$.

Remark 3.2.5. Definitions 3.2.1 and 3.2.3 do not depend on the choice of the infinite linear orders I_L and I_R. For example, for weak semi-equations: if there exist some $b \in \mathbb{M}_y$ and $(a_i : i \in I_L + (0) + I_R)$ satisfying the given conditions, and J_L and J_R are arbitrary infinite linear orders, then by saturation of $\mathbb{M}$ there exists an indiscernible sequence $(a'_i : i \in J_L + (0) + J_R)$ satisfying the same EM-type as $(a_i : i \in I_L + (0) + I_R)$, with $a'_0 = a_0$, and with $(a'_i : i \in J_L + J_R)$ satisfying the same EM-type over b as $(a_i : i \in I_L + I_R)$.

Proposition 3.2.6. *1. If $\varphi(x,y)$ is a semi-equation, then $\varphi(x,y)$ is a weak semi-equation. Hence every semi-equational theory is weakly semi-equational.*

2. Semi-equations are closed under conjunctions and exchanging the roles of the variables.

3. Weak semi-equations are closed under conjunctions and disjunctions.

Proof. (1) Clear from definitions using condition (1) in Proposition 3.2.2 as the definition of a semi-equation.

(2) Suppose $\varphi(x,y) \wedge \psi(x,y)$ is not a semi-equation. By Proposition 3.2.2, there is b and an indiscernible sequence $(a_i)_{i\in\mathbb{Z}}$ such that $\models \varphi(a_i,b) \wedge \psi(a_i,b) \iff i \neq 0$. Either $\not\models \varphi(a_0,b)$, in which case $\varphi(x,y)$ is not a semi-equation, or $\not\models \psi(a_0,b)$, in which case $\psi(x,y)$ is not a semi-equation. And $\varphi(x,y)$ is a semi-equation if and only if $\varphi^*(y,x) := \varphi(x,y)$ is a semi-equation by the symmetry of the property in Proposition 3.2.2(2).

(3) For conjunctions, same as the proof of (2), but with the stipulation that $(a_i)_{i\neq 0}$ is b-indiscernible added. Now suppose $\varphi(x,y) \vee \psi(x,y)$ is not a weak semi-equation. Then there is b and an indiscernible sequence $(a_i)_{i\in\mathbb{Z}}$ such that $(a_i)_{i\neq 0}$ is b-indiscernible, and $\models \varphi(a_i,b) \vee \psi(a_i,b) \iff i \neq 0$. Either $\models \varphi(a_1,b)$ or $\models \psi(a_1,b)$, and then, by b-indiscernibility, either $\models \varphi(a_i,b)$ for all $i \neq 0$ or $\models \psi(a_i,b)$ for all $i \neq 0$. In the first case, $\varphi(x,y)$ is not a weak semi-equation, and in the second case, $\psi(x,y)$ is not a weak semi-equation. $\square$

Remark 3.2.7. 1. To see that neither property is closed under negation, note that $x = y$ is a semi-equation (hence also a weak semi-equation), but $x \neq y$ is not a semi-equation, and in a stable theory is not even a weak semi-equation (by Proposition 3.2.13). In fact, if $\varphi(x,y)$ is stable with infinitely many distinct instances $\varphi(\mathbb{M},b), b \in \mathbb{M}^y$, then either $\varphi(x,y)$ is not a semi-equation, or $\neg\varphi(x,y)$ is not a semi-equation (combining [ADH+16, Proposition 2.20] and Proposition 3.2.20).

2. To see that semi-equations are not closed under disjunction, note that in a linear order, $x < y$ and $y < x$ are both semi-equations, but their disjunction is equivalent to $x \neq y$,

which is not.

Problem 3.2.8. Are weak semi-equations closed under exchanging the roles of the variables, at least in NIP theories? Fact 3.6.6 can be viewed as establishing this for the definition of distality, however the proof does not seem to be sufficiently local with respect to a formula witnessing failure of distality.

We observe some basic properties of (weak) semi-equations with respect to reducts and expansions of theories.

Proposition 3.2.9. *Assume we are given languages $\mathcal{L} \subseteq \mathcal{L}'$, a complete $\mathcal{L}$-theory T and an $\mathcal{L}'$-theory T' with $T \subseteq T'$.*

1. *A formula $\varphi(x, y) \in \mathcal{L}$ is a semi-equation in T if and only if it is a semi-equation in T'.*
2. *If $\varphi(x, y) \in \mathcal{L}$ is a weak semi-equation in T, then it is also a weak semi-equation in T'.*

Proof. (1) Left to right is immediate from the definition (Proposition 3.2.2). For the converse, assume that in some model of T we can find an infinite sequence $(a_i, b_i)_{i \in I}$ such that for all $i, j \in I$, $\models \varphi(a_i, b_j) \iff i \neq j$. By completeness of T, we can find arbitrarily long finite sequences with the same property in *every* model of T, in particular in some model of T'. By compactness we can thus find an infinite sequence with the same property in a model of T', demonstrating that $\varphi(x, y)$ is not a semi-equation in T'.

(2) If $\varphi(x, y) \in \mathcal{L}$ is not a weak semi-equation in T', then (in a monster model of T', and hence of T) there is b and an $\mathcal{L}'$-indiscernible $(a_i)_{i \in I_L + (0) + I_R}$ such that $(a_i)_{i \in I_L + I_R}$ is $\mathcal{L}'$-indiscernible over b and $\models \varphi(a_i, b)$ for $i \in I_L + I_R$, but $\not\models \varphi(a_0, b)$, for infinite linear orders I_L, I_R. Then, in particular, $(a_i)_{i \in I_L + (0) + I_R}$ is $\mathcal{L}$-indiscernible, and $(a_i)_{i \in I_L + I_R}$ is $\mathcal{L}$-indiscernible over b, so $\varphi(x, y)$ is a not a weak semi-equation in T. $\square$

Remark 3.2.10. The converse to Proposition 3.2.9(2) does not hold. Let $T' :=$ DLO be the theory of dense linear orders, and T its reduct to $\mathcal{L} := \{=\}$. Then the $\mathcal{L}$-formula $x \neq y$ is not a weak semi-equation in T (by Proposition 3.2.13(5)), but it is a weak semi-equation in T' (by Proposition 3.2.13(2), as DLO is distal).

Problem 3.2.11. Is weak semi-equationality of a theory preserved under reducts? This appear to be open already for equationality (see [Jun00, Question 3.10]), and fails for semi-equationality (see Section 3.3.3).

Problem 3.2.12. Is (weak) semi-equationality of theories invariant under bi-interpretability without parameters? Equivalently, if T is (weakly) semi-equational, does it follow that so is T^{eq}?

3.2.2 Relationship to equations and NIP

Comparing the definitions, (weak) semi-equations can be seen as a way to complete the analogy "stable : equation" to "NIP : ?". One way to see this is that the definition of semi-equationality can be obtained by modifying the definition of NIP in the same way as modifying the definition of stability gives you the definition of an equation. The descending chain condition on instances of a formula $\varphi(x, y)$, characterizing equationality, can be rephrased as follows: there is no sequence $(a_i, b_i)_{i \in \mathbb{N}}$ such that $\models \varphi(a_i, b_j)$ for all $j < i$, and $\not\models \varphi(a_i, b_i)$ (so that $\bigwedge_{j<i} \varphi(x, b_j)$ is consistent, and does not imply $\varphi(x, b_i)$). This is similar to the order property charactering stability, except with the stipulation that $\not\models \varphi(a_i, b_j)$ for $j > i$ dropped. The definition of NIP also involves a (partial) order: the inclusion order on a power set. If we keep all stipulations of positive instances of φ from the independence property, but only the extremal negative instances, we get: $(a_i)_{i \in I}, (b_X)_{X \in \mathcal{P}(I)}$ such that $\models \varphi(a_i, b_X)$ for $i \notin X$, and $\not\models \varphi\left(a_i, b_{I \setminus \{i\}}\right)$. The nonexistence of such a pattern is equivalent to semi-equationality (in the sense of Definition 3.2.3), using $b_i = b_{I \setminus \{i\}}$ (b_X for $|I \setminus X| \neq 1$ can easily be chosen to satisfy the given conditions if $b_{I \setminus \{i\}}$ satisfying the given conditions are given). We make some observations providing further evidence that semi-equationality can be naturally viewed as a generalization of equationality (in the sense of Srour) in stable theories to the NIP context.

Proposition 3.2.13. *1. Weak semi-equations are NIP formulas, and weakly semi-equational theories are NIP.*

2. Every formula in a distal theory is a weak semi-equation.

3. Equations are semi-equations.

4. A formula is an equation if and only if it is both stable and a semi-equation.

5. In a stable theory, all weak semi-equations are equations. In particular, a stable theory is equational if and only if it is weakly semi-equational, if and only if it is semi-equational.

Proof. (1) If $\varphi(x, y)$ is not NIP, then there is an indiscernible sequence $(a_i)_{i \in \mathbb{N}}$ and b such that $\models \varphi(a_i, b) \iff i$ is even. For any finite set of formulas $\Delta(x_1, \ldots, x_n, y)$, by Ramsey's theorem, there is an infinite $I \subseteq 2\mathbb{N}$ on which the truth value of all formulas in $\Delta(a_{i_1}, \ldots, a_{i_n}, b)$ is constant for all $i_1 < \ldots < i_n \in I$. Thus, by letting $a'_0 := a_i$ for some sufficiently large odd i, we can find an indiscernible sequence $(a'_i)_{i \in I_L + (0) + I_R}$ (using $I_L \sqcup I_R = I$, and $a'_i = a_i$ for $i \in I$) for some infinite I_R and arbitrarily large finite I_L, such that $(a'_i)_{i \in I_L + I_R}$ is Δ-indiscernible over b. By compactness, it follows that $\varphi(x, y)$ is not a weak semi-equation.

(2) By a standard characterization of distality (see e.g. [ACGZ22, Corollary 1.11]; see also Section 3.4 for an extended discussion on connections to distality).

(3) If $\varphi(x, y)$ is not a semi-equation, then there is a sequence $(a_i, b_i)_{i \in \mathbb{N}}$ such that $\models \varphi(a_i, b_j) \iff i \neq j$. In particular, $\models \varphi(a_i, b_j)$ for all $j < i$, and $\not\models \varphi(a_i, b_i)$, so this is a counterexample to the descending chain condition, and $\varphi(x, y)$ is not an equation.

(4) Suppose $\varphi(x, y)$ is not an equation, so there exist $(a_i, b_i)_{i \in \mathbb{N}}$ such that $\models \varphi(a_i, b_j)$ for all $j < i$, and $\not\models \varphi(a_i, b_i)$. Color pairs (i, j) with $i < j$ by whether or not $\varphi(a_i, b_j)$ holds. By Ramsey's theorem, there's an infinite homogeneous subset of $\mathbb{N}$. If $\varphi(a_i, b_j)$ holds for $i < j$ in this homogeneous subset, then $\varphi(x, y)$ is not a semi-equation. Otherwise, $\varphi(x, y)$ is not stable.

(5) If $\varphi(x, y)$ is not an equation, let $(a_i, b_i)_{i \in \mathbb{N}}$ be such that $\models \varphi(a_i, b_j)$ for all $j < i$, and $\not\models \varphi(a_i, b_i)$. By Ramsey and compactness we may assume that the sequence of pairs $(a_i, b_i)_{i \in \mathbb{N}}$ is indiscernible. Then $(a_i, b_i)_{i \geq 1}$ can be made indiscernible over $a_0 b_0$ by extracting an indiscernible sequence with the same EM type over $a_0 b_0$. Now $(a_i)_{i \in \mathbb{N}}$ is indiscernible, with $(a_i)_{i \geq 1}$ indiscernible over b_0, and $\models \varphi(a_i, b_0) \iff i \neq 0$. As the theory is stable, every infinite indiscernible sequence is totally indiscernible. Fix an arbitrary bijection $f : \mathbb{N} \to \mathbb{Z}$ so that $f(0) = 0$, by total indiscernibility there exists an automorphism σ of $\mathbb{M}$ so that $\sigma(a_i) = a_{f(i)}, \sigma(b_i) = b_{f(i)}$ for all $i \in \mathbb{Z}$. For $i \in \mathbb{Z}$, let $a'_i := a_{f^{-1}(i)}$ and $b := b_0$. Using that σ is an automorphism, we still have that $(a'_i)_{i \in \mathbb{Z}}$ is indiscernible, $(a'_i)_{i \neq 0}$ is indiscernible over b, $\models \varphi(a'_i, b) \iff i \neq 0$, so $\varphi(x, y)$ is not a weak semi-equation. $\square$

Problem 3.2.14. Is there an NIP theory without a (weakly) semi-equational expansion? We note that while the theory ACF_p for $p > 0$ is known not to have a distal expansion [CS18], it is equational, and hence semi-equational.

3.2.3 Weakly normal formulas, (k, n)-semi-equations and breadth.

In this section we discuss some refinements of semi-equationality.

Definition 3.2.15. [Pil96, Chapter 4, Definition 1.1]

1. A formula $\varphi(x, y)$ is *k-weakly normal* if for every $b_1, \ldots, b_k \in \mathbb{M}^y$ such that $\models \exists x\, \varphi(x, b_1) \wedge \ldots \wedge \varphi(x, b_k)$, there are some $i \neq j \in [k]$ such that $\models \forall x\, \varphi(x, b_i) \leftrightarrow \varphi(x, b_j)$.

2. A formula $\varphi(x, y)$ is *weakly normal* if it is k-weakly normal for some k.

Weakly normal formulas are a special kind of equations characterizing "linearity" of forking in stable theories (see [Pil96, Chapter 4, Proposition 1.5 + Remark 1.8.4 + Lemma 1.9]):

Fact 3.2.16. *A stable theory T is 1-based if and only if in T, every formula $\varphi(x, y) \in \mathcal{L}$, with x, y arbitrary finite tuples of variables, is equivalent to a Boolean combination of some weakly normal formulas $\psi_1(x, y), \ldots, \psi_n(x, y) \in \mathcal{L}$.*

We introduce some numeric parameters characterizing semi-equations, minimal values of which give rise to a generalization of weak normality.

Definition 3.2.17.

1. For $k, n \in \mathbb{N}$, a formula $\varphi(x, y)$ is a (k, n)-*semi-equation* if, for every $b_1, \ldots, b_k \in \mathbb{M}^y$, if $\models \exists x\ \varphi(x, b_1) \wedge \ldots \wedge \varphi(x, b_k)$, then for some pairwise distinct $i_1, \ldots, i_n, j \in [k]$, $\models \forall x\ (\varphi(x, b_{i_1}) \wedge \ldots \wedge \varphi(x, b_{i_n})) \to \varphi(x, b_j)$.
2. And $\varphi(x, y)$ is an *n-semi-equation* if it is a (k, n)-semi-equation for some k.
3. A theory T is *n-semi-equational* (respectively, (k, n)-*semi-equational*) if every formula $\varphi(x, y) \in \mathcal{L}$, with x, y arbitrary finite tuples of variables, is equivalent in T to a Boolean combination of n-semi-equations (respectively, (k, n)-semi-equations) $\psi_1(x, y), \ldots, \psi_n(x, y) \in \mathcal{L}$.

Proposition 3.2.18.

1. *If $\varphi(x, y)$ is a (k, n)-semi-equation, then $n < k$, and $\varphi(x, y)$ is also an (ℓ, m)-semi-equation for any $\ell \geq k$ and $n \leq m < \ell$. If $\varphi(x, y)$ is an n-semi-equation, then it is also an m-semi-equation for every $m \geq n$.*
2. *A formula is a semi-equation if and only if it is an n-semi-equation for some n, if and only if it is an $(n, n-1)$-semi-equation for some n.*

Proof. (1) Clear from the definitions.

(2) If $\varphi(x, y)$ is not a semi-equation, let $(a_i, b_i)_{i \in \mathbb{N}}$ be such that $\models \varphi(a_i, b_j) \iff i \neq j$. Then for any (k, n) we have $\models \varphi(a_0, b_1) \wedge \ldots \wedge \varphi(a_0, b_k)$, but for any pairwise distinct

$i_1, \ldots, i_n, j \in [k]$, $\models \varphi(a_j, b_{i_1}) \wedge \ldots \wedge \varphi(a_j, b_{i_n}) \wedge \neg\varphi(a_j, b_j)$, hence $\varphi(x, y)$ is not a (k, n)-semi-equation.

Conversely, for any $k \in \mathbb{N}$, if $\varphi(x, y)$ is not a $(k, k-1)$-semi-equation, then there exist $b_1, \ldots, b_k$ such that for each $j \in [k]$, there is a_j such that $\models \varphi(a_j, b_i)$ for $i \neq j$, but $\not\models \varphi(a_j, b_j)$. Hence if $\varphi(x, y)$ is not a $(k, k-1)$-semi-equation for any k, then by compactness $\varphi(x, y)$ is not a semi-equation.

And if $\varphi(x, y)$ is not an n-semi-equation, then it is not an $(n+1, n)$-semi-equation by definition, so a formula that is not an n-semi-equation for any n is also not a $(k, k-1)$-semi-equation for any k. $\square$

We recall the notion of breadth from lattice theory.

Definition 3.2.19. [ADH+16, Section 2.4] Given a set X and $d \in \mathbb{N}_{\geq 1}$, a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *breadth* d if any nonempty intersection of finitely many sets in $\mathcal{F}$ is the intersection of at most d of them, and d is minimal with this property.

Proposition 3.2.20. *A formula $\varphi(x, y)$ is a $(k+1, k)$-semi-equation if and only if the family of sets $\mathcal{F}_\varphi := \{\varphi(\mathbb{M}, b) \mid b \in \mathbb{M}^y\}$ has breadth at most k. In particular, $\varphi(x, y)$ is a semi-equation if and only if the family of sets $\mathcal{F}_\varphi$ has finite breadth.*

Proof. The family of sets $\{\varphi(\mathbb{M}, b) \mid b \in \mathbb{M}^y\}$ has breadth at most k if and only if every finite consistent conjunction of instances of φ is implied by the conjunction of at most k of those instances. In particular this applies to consistent conjunctions of $(k+1)$ instances of φ, showing that if the breadth of $\mathcal{F}_\varphi$ is $\leq k$, then it is a $(k+1, k)$-semi-equation. Conversely, assume $\varphi(x, y)$ is a $(k+1, k)$-semi-equation. Given any consistent conjunction of $n > k$ instances of φ, any $(k+1)$ of them contain an instance implied by the other k instances. Removing this implied instance, we reduce to the case of a consistent conjunction of $(n-1)$ instances, and after $(n-k)$ steps to a conjunction of k instances of φ implying all the other ones. The "in particular" part follows by Proposition 3.2.18(2). $\square$

Example 3.2.21. 1. Let T be an NIP theory expanding a group, and let a formula $\varphi(x,y)$ be such that for every $b \in \mathbb{M}^y$, $\varphi(\mathbb{M},b)$ is a subgroup. Then, by Baldwin-Saxl [BS76], there exists $n \in \omega$ such that for all finite $B \subseteq \mathbb{M}^y$, there is $B_0 \subseteq B$ with $|B_0| \leq n$ such that $\bigcap_{b \in B_0} \varphi(\mathbb{M},b) = \bigcap_{b \in B} \varphi(\mathbb{M},b)$. Hence $\varphi(x,y)$ is a semi-equation by Proposition 3.2.20.

2. Let K be a valued field (viewed as a structure in the language of rings with a predicate for the valuation ring $\mathcal{O}$), $d \in \omega$ and let $\mathcal{F}$ be the family of all convex subsets of K^d in the sense of Monna (equivalently, the family of all translates of $\mathcal{O}$-submodules of K^d). Then $\mathcal{F}$ is a definable family, and a formula defining it is a semi-equation by [CM21, Theorem 4.3].

The following suggests that 1-semi-equationality might be viewed as a generalization of being 1-based to the NIP context.

Proposition 3.2.22. *A formula $\varphi(x,y)$ is weakly normal if and only if it is both stable and a* 1*-semi-equation. In particular, a theory is* 1*-based if and only if it is stable (or just NSOP) and* 1*-semi-equational.*

Proof. Clearly every k-weakly normal formula is a $(k,1)$-semi-equation. If $\varphi(x,y)$ is unstable, then for every $k \in \mathbb{N}$ it has the *k-order property*, meaning that there exist $(a_i,b_i)_{i \in [k]}$ such that $\models \varphi(a_i,b_j) \iff i \leq j$. Then $(b_1,\ldots,b_k)$ is a counter-example to $\varphi(x,y)$ being k-weakly normal, as $a_1 \models \bigwedge_{i \in [k]} \varphi(x,b_i)$, but for any $i < j \in [k]$ we have $\models \varphi(a_j,b_j) \wedge \neg\varphi(a_j,b_i)$.

Conversely, suppose that $\varphi(x,y)$ is a $(k,1)$-semi-equation and does not have the $(\ell+1)$-order property for some $\ell \in \mathbb{N}$. We will show that then $\varphi(x,y)$ is k^{ℓ}-weakly normal (this bound is not optimal). Let $(b_\eta)_{\eta \in [k]^\ell}$ be such that $\models \exists x \bigwedge_{\eta \in [k]^\ell} \varphi(x,b_\eta)$. For $\sigma \in [k]^{\leq \ell}$, we will show by induction on $m := \ell - |\sigma|$ that there are pairwise distinct $\eta_0,\ldots,\eta_m \in [k]^\ell$ extending σ (as sequences) such that $\varphi(\mathbb{M},b_{\eta_0}) \subseteq \varphi(\mathbb{M},b_{\eta_1}) \subseteq \ldots \subseteq \varphi(\mathbb{M},b_{\eta_m})$. With $m = \ell$, so that $\sigma = \langle\rangle$ is the empty sequence, this implies (using that $\varphi(x,y)$ does not have the $(\ell+1)$-order property) that there are $\eta \neq \eta' \in [k]^{\ell}$ such that $\varphi(\mathbb{M}, b_{\eta}) = \varphi(\mathbb{M}, b_{\eta'})$, as desired. The base case $(m = 0)$ is trivial, with $\eta_0 = \sigma$.

Now assume the claim holds for m, and let $\sigma \in [k]^{\ell-(m+1)}$. For each $i \in [k]$, there exist pairwise distinct $\eta_{i,0}, \ldots, \eta_{i,m} \in [k]^{\ell}$ extending $\sigma^\frown i$ such that $\varphi\left(\mathbb{M}, b_{\eta_{i,0}}\right) \subseteq \ldots \subseteq \varphi\left(\mathbb{M}, b_{\eta_{i,m}}\right)$. Among the sets $\left\{\varphi\left(\mathbb{M}, b_{\eta_{i,0}}\right) \mid i \in [k]\right\}$, one must be contained in another by $(k,1)$-semi-equationality. Say $\varphi\left(\mathbb{M}, b_{\eta_{j,0}}\right) \subseteq \varphi\left(\mathbb{M}, b_{\eta_{i,0}}\right)$ for some $i \neq j$. Then $\varphi\left(\mathbb{M}, b_{\eta_{j,0}}\right) \subseteq \varphi\left(\mathbb{M}, b_{\eta_{i,0}}\right) \subseteq \varphi\left(\mathbb{M}, b_{\eta_{i,1}}\right) \subseteq \ldots \subseteq \varphi\left(\mathbb{M}, b_{\eta_{i,m}}\right)$, and $\eta_{j,0}, \eta_{i,0}, \eta_{i,1}, \ldots, \eta_{i,m}$ are pairwise distinct and extend σ, as desired.

The "in particular" part follows by Fact 3.2.16. $\square$

Remark 3.2.23.

1. The family of weakly normal formulas is closed under conjunctions.

   Indeed, let $\varphi_1(x,y), \varphi_2(x,y)$ be both k-weakly normal, and let $\varphi(x,y) := \varphi_1(x,y) \wedge \varphi_2(x,y)$. Then $\varphi(x,y)$ is weakly normal (potentially, for some $K \gg k$). Assume $\varphi(x,y)$ is not weakly normal, then by compactness we have an infinite sequence $(b_i : i \in \omega)$ such that $\bigcap_{i \in \omega} \varphi(\mathbb{M}, b_i) \neq \emptyset$, yet all of the sets $\varphi(\mathbb{M}, b_i)$ are pairwise different. By the choice of k, for every $i_1 < \ldots < i_k \in \omega$ there are some $j_1 < j_1' < k$ and $j_2 < j_2' < k$ such that $\varphi_i\left(\mathbb{M}, b_{i_{j_t}}\right) = \varphi_i\left(\mathbb{M}, b_{i_{j_t'}}\right)$ for $t \in \{1,2\}$. By Ramsey's theorem, we can find an infinite subsequence $I \subseteq \omega$ such that j_1, j_1', j_2, j_2' are fixed for all increasing tuples $i_1 < \ldots < i_k$ from I. But then, choosing such increasing tuples in I appropriately, we find some $j < j' \in I$ such that $\varphi_i(\mathbb{M}, b_j) = \varphi_i(\mathbb{M}, b_{j'})$ for both $i \in \{1,2\}$ simultaneously, hence $\varphi(\mathbb{M}, b_j) = \varphi(\mathbb{M}, b_{j'})$, a contradiction.

2. While semi-equations are closed under conjunctions by Proposition 3.2.6(2), this is not the case for the family of 1-semi-equations.

   Indeed, in a dense linear order, the formulas $x < y_1$ and $x > y_2$ are 1-semi-equations, but the formula $\varphi(x; y_1, y_2) := y_2 < x < y_1$ is not a 1-semi-equation since we can have any number of intervals with a non-empty intersection, so that none of them is contained in the other.

3. The definition of $(2,1)$-semi-equationality is analogous to VC-minimality [Adl08], but for formulas $\varphi(x,y)$ with x an arbitrary tuple of variables, as opposed to a singleton in the latter case.

We also observe a connection to another notion of "linearity", or "1-basedeness", for NIP theories considered in [BCS+21], where various combinatorial results are proved for relations that are Boolean combinations of *basic* relations. The following is [BCS+21, Definition 2], in the case of binary relations (using the equivalence in [BCS+21, Proposition 2.8 + Remark 2.9]).

Definition 3.2.24. A binary relation $R \subseteq X \times Y$ is *basic* if there exist a linear order $(S,<)$ and functions $f : X \to S, g : Y \to S$ for such that for any $a \in X, b \in Y$, $(a,b) \in R \iff f(a) < g(b)$.

Fact 3.2.25. *[GL13, Claim 1 in the proof of Proposition 2.5] Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ a family of subsets of X such that there no $A, B \in \mathcal{F}$ satisfying $A \cap B \neq \emptyset, B \setminus A \neq \emptyset$ and $B \setminus A \neq \emptyset$ simultaneously. Then there exists a linear order $<$ on X so that every $A \in \mathcal{F}$ is a $<$-convex subset of X.*

Proposition 3.2.26. *1. Given a formula $\varphi(x,y) \in \mathcal{L}$, if the relation $R_\varphi := \{(a,b) \in \mathbb{M}^x \times \mathbb{M}^y\}$ is basic, then $\varphi(x,y)$ is a $(2,1)$-semi-equation.*

2. If $\varphi(x,y)$ is $(2,1)$-semi-equation, then $R_\varphi = R_1 \cap R_2$ for some (not necessarily definable) basic relations $R_1, R_2 \subseteq \mathbb{M}^x \times \mathbb{M}^y$.

Proof. (1) Let $(S,<), f, g$ be as in Definition 3.2.24 for R_φ. Given any $b_1, b_2 \in \mathbb{M}^y$, the sets $\{x \in S : x < g(b_i)\}$ for $i \in \{1,2\}$ are initial segments of S. Say $g(b_1) \leq g(b_2)$. Then for any $a \in \mathbb{M}^x$, $f(a) < g(b_1) \Rightarrow f(a) < g(b_2)$, so $\varphi(\mathbb{M}, b_1) \subseteq \varphi(\mathbb{M}, b_2)$, and the other case is symmetric.

(2) If $\varphi(x,y)$ is a $(2,1)$-semi-equation, then the family $\mathcal{F}_\varphi$ of subsets of $\mathbb{M}^x$ satisfies the assumption in Fact 3.2.25. Hence there exists a (not necessarily definable) linear ordering $<'$

of $\mathbb{M}^x$ so that for every $b \in \mathbb{M}^y$, $\varphi(\mathbb{M}, b)$ is $<'$-convex. Let $(S, <)$ be the Dedekind completion of $(\mathbb{M}^x, <')$. Consider the functions $g_1, g_2 : \mathbb{M}^y \to S$ so that $g_1(b)$ is the infimum of $\varphi(\mathbb{M}, b)$ in S, and $g_2(b)$ is the supremum of $\varphi(\mathbb{M}, b)$ in S. Then $R_\varphi = \{(a, b) \in \mathbb{M}^x \times \mathbb{M}^y : g_1(b) \leq a\} \cap \{(a, b) \in \mathbb{M}^x \times \mathbb{M}^y : a \leq g_2(b)\}$, and both of this relations are basic (see [BCS$^+$21, Remark 2.7]). $\square$

Remark 3.2.27.

1. In view of Proposition 3.2.26(2), if $\varphi(x, y)$ is a Boolean combination of $(2, 1)$-semi-equations, then by [BCS$^+$21, Theorem 2.17 + Remark 2.20] the relation R_φ satisfies an almost linear Zarankiewicz bound. In particular, no infinite field can be defined in a $(2, 1)$-semi-equational theory (see [BCS$^+$21, Corollary 5.11] or [Wal21, Proposition 6.3] for a detailed explanation).

2. Note that if $\varphi(x, y)$ is a $(2, 1)$-semi-equation, then R_φ need not be basic. Indeed, the family of cosets of a subgroup is $(2, 1)$-semi-equational. If it was basic, then its complement is also basic, hence $(2, 1)$-semi-equational by the lemma above. But if the index of the subgroup is ≥ 3, the family of complements of cosets is clearly not $(2, 1)$-semi-equational.

Problem 3.2.28. If $\varphi(x, y)$ is a $(k, 1)$-semi-equation for $k \geq 3$, is it still a Boolean combination of basic relations?

Problem 3.2.29. Show that no infinite field is definable in a 1-semi-equational theory.

Problem 3.2.30. Is every 1-semi-equational theory rosy? (Note that dense trees are not 1-semi-equational by Theorem 3.3.17.)

Analogously, we could also define weak 1-semi-equationality as follows:

Definition 3.2.31. A partitioned formula $\varphi(x; y)$ is a *weak* 1-*semi-equation* if for every indiscernible sequence $(b_i)_{i \in \omega}$ and a such that $\models \varphi(a, b_i)$ for every $i \in \omega$, there exist $i \neq j \in \omega$ such that $\operatorname{tp}(a/b_i) \vdash \varphi(x, b_j)$.

In this definition, we could equivalently replace ω by any infinite linear order, and require that the sequence $(b_i)_{i\in\omega}$ is indiscernible over a.

Proposition 3.2.32. *1. If $\varphi(x, y)$ is a 1-semi-equation, the it is also a weak 1-semi-equation.*

2. If $\varphi(x, y)$ is a weak 1-semi-equation, then $\varphi^*(y, x) = \varphi(x, y)$ is a weak semi-equation.

Proof. 1. If $\varphi(x, y)$ is not a weak 1-semi-equation, let $(b_i)_{i\in\mathbb{Z}}$ be indiscernible, and a such that $\models \varphi(a, b_i)$ for every $i \in \mathbb{Z}$, but $\operatorname{tp}(a/b_i) \nvdash \varphi(x, b_j)$ for every distinct $i, j \in \mathbb{Z}$. Since $\operatorname{tp}(a/b_i) \vdash \varphi(a, b_i)$, in particular $\varphi(x, b_i) \nvdash \varphi(x, b_j)$ for any distinct i, j, so $\varphi(x, y)$ is not a 1-semi-equation.

2. If $\varphi^*(y, x)$ is not a weak semi-equation, there is a_0 and indiscernible $(b_i)_{i\in\mathbb{Z}}$ such that $(b_i)_{i\neq 0}$ is a_0-indiscernible, and $\models \varphi(a_0, b_1)$ but $\not\models \varphi(a_0, b_0)$. Let σ be an automorphism such that $\sigma(b_i) = b_{i+1}$ for all $i \in \mathbb{Z}$, and let $a_j := \sigma^j(a_0)$. By compactness, there is a such that $\operatorname{tp}(a/b_i) = \operatorname{tp}(a_j/b_i)$ for all distinct $i, j \in \mathbb{Z}$. For any distinct $i, j \in \mathbb{Z}$, $\models \varphi(a, b_i)$, $a_j \models \operatorname{tp}(a/b_i)$, and $\not\models \varphi(a_j, b_j)$. Thus $\varphi(x, y)$ is not a weak 1-semi-equation. $\square$

Problem 3.2.33. In a stable theory, are all weak 1-semi-equations weakly normal? More generally, if a formula is both a semi-equation and a weak 1-semi-equation, must it be a 1-semi-equation?

3.3 Examples of semi-equational theories

In this section we consider some (unstable) examples of (weakly) semi-equational theories.

3.3.1 O-minimal structures

All o-minimal theories (and more generally, ordered dp-minimal theories) are distal (see [Sim13]), hence they are weakly semi-equational by Proposition 3.2.13(2). Semi-equationality appears more subtle. We will say that an o-minimal structure is *linear* if it has the *CF property* in the sense of [LP93] (a weakening of local modularity of the pregeometry induced by the algebraic closure), so e.g. an ordered vector space over an ordered division ring.

Proposition 3.3.1. *Let* $T = \mathrm{Th}(\mathcal{M})$*, with* $\mathcal{M} = (M; <, +, \ldots)$ *an o-minimal structure.*

1. *If* T *is an expansion of an ordered group and linear, then* T *is* $(2,1)$*-semi-equational.*

2. *Conversely, if* T *is* $(2,1)$*-semi-equational, then* T *is linear.*

Proof. (1) Let $\mathcal{L} = (<, +, \ldots)$ be the language of T. A *partial endomorphism* of $\mathcal{M}$ is a map $f : (-c, c) \to M$, for c an element of M or ∞, such that if $a, b, a+b$ are all in the domain, then $f(a+b) = f(a) + f(b)$. Let $\mathcal{M}'$ be the reduct of $\mathcal{M}$ to the language $\mathcal{L}'$ consisting of:

- $+, <$,
- constant symbols naming $\mathrm{acl}_{\mathcal{L}}(\emptyset)$,
- for each $\mathcal{L}(\emptyset)$-definable partial endomorphism $f : (-c, c) \to M$ with $c \in \mathrm{acl}_{\mathcal{L}}(\emptyset)$ or $c = \infty$, a unary function symbol interpreted as f on $(-c, c)$ and as 0 outside of the domain of f.

By [LP93, Proposition 4.2], a subset of M^n is $\emptyset$-definable in $\mathcal{M}$ if and only if it is $\emptyset$-definable in $\mathcal{M}'$. Hence it suffices to show that $T' := \mathrm{Th}_{\mathcal{L}'}(\mathcal{M}')$ is $(2,1)$-semi-equational. By [LP93, Corollary 6.3], T' admits quantifier elimination in the language $\mathcal{L}'$. Hence it suffices to show that every atomic $\mathcal{L}'$-formula $\varphi(x, y)$, with x, y arbitrary finite tuples of variables, is equivalent in T' to a Boolean combination of $(2,1)$-semi-equations.

By the proof of Theorem 4.3 in [And21], every atomic $\mathcal{L}'$-formula $\varphi(x, y)$ is equivalent in T' to a Boolean combination of atomic formulas of the form $f(x) \Box g(y) + c$, where $\Box \in \{<, =, >\}$,

$f : M^{|x|} \to M, g : M^{|y|} \to M$ are total multivariate $\mathcal{L}'(\emptyset)$-definable homomorphisms and $c \in \mathrm{dcl}_{\mathcal{L}'}(\emptyset)$. Every formula of this form clearly defines a basic relation on $M^{|x|} \times M^{|y|}$, hence is a $(2,1)$-semi-equation by Proposition 3.2.26(1).

(2) By the o-minimal trichotomy theorem (see [PS98] and Remark 2 after the statement of Theorem 1.7 there), if $\mathcal{M}$ is not linear, then it defines an infinite field. But then Remark 3.2.27(1) implies that T is not $(2,1)$-semi-equational. $\square$

Problem 3.3.2. Is every o-minimal 1-semi-equational structure linear? A a positive answer would follow from a positive answer to Problem 3.2.29.

Problem 3.3.3. Which o-minimal theories are semi-equational? In particular, is $\mathrm{Th}(\mathbb{R}, +, \times)$ semi-equational?

3.3.2 Colored linear orders

Definition 3.3.4. Given a linearly ordered set $(S, <)$, a binary relation $R \subseteq S^2$ is *monotone* if $(x, y) \in R$, $x' \leq x$, and $y \leq y'$ implies $(x', y') \in R$.

Fact 3.3.5. *Let* $\mathcal{M} = \left(M, <, (C_i)_{i \in I}, (R_j)_{j \in J}\right)$ *be a linear order expanded by arbitrary unary relations* C_i *and monotone binary relations* R_j*. Then* $T := \mathrm{Th}(\mathcal{M})$ *is* $(2,1)$*-semi-equational.*

Proof. Let $\mathcal{M}'$ be an expansion of $\mathcal{M}$ obtained by naming all $\mathcal{L}_{\mathcal{M}}(\emptyset)$-definable unary and monotone binary relations, then a subset of M^n is $\emptyset$-definable in $\mathcal{M}$ if and only if it is $\emptyset$-definable in $\mathcal{M}'$, so it suffices to show that $T' := \mathrm{Th}(\mathcal{M}')$ is $(2,1)$-semi-equational. By [Sim11, Proposition 4.1], T' eliminates quantifiers. Note that if $R(x, y)$ is monotone, then it is a $(2,1)$-semi-equation (given any $b_1 \leq b_2 \in M$, for any $a \in M$ we have $\models R(a, b_1) \to R(a, b_2)$ by monotonicity, hence $R(M, b_1) \subseteq R(M, b_2)$). And any unary relation $C_i(x)$ is trivially a $(2,1)$-semi-equation, hence T' is $(2,1)$-semi-equational. $\square$

3.3.3 Cyclic orders

Definition 3.3.6. (see e.g. [CK12, Section 5] or [TW17]). A *cyclic order* is a structure with a single ternary relation $\circlearrowright$ satisfying

1. cyclicity: if $\circlearrowright(a, b, c)$, then $\circlearrowright(b, c, a)$;
2. antisymmetry: if $\circlearrowright(a, b, c)$, then not $\circlearrowright(c, b, a)$;
3. transitivity: if $\circlearrowright(a, b, c)$ and $\circlearrowright(a, c, d)$, then $\circlearrowright(a, b, d)$;
4. totality: if a, b, c are distinct, then $\circlearrowright(a, b, c)$ or $\circlearrowright(c, b, a)$.

A cyclic order $\circlearrowright$ is *dense* if its underlying set is infinite and for every distinct a, b, there is c such that $\circlearrowright(a, b, c)$, and d such that $\circlearrowright(d, b, a)$.

The following is standard, but we include a proof for completeness.

Proposition 3.3.7. *The theory $T_{\circlearrowright}$ of dense cyclic orders is complete and has quantifier elimination.*

Proof. For a dense cyclic order $\mathcal{M}$, let $c \in \mathcal{M}$ be arbitrary. Then there is a dense linear order $<$ on $\mathcal{M} \setminus \{c\}$ defined by $a < b \iff \circlearrowright(a, b, c)$. The theory of dense linear orders is complete, and $\circlearrowright$ can be recovered from $<$.

Let $\mathcal{M}, \mathcal{N}$ be dense cyclic orders, $A \subset \mathcal{M}$ finite, $m \in \mathcal{M} \setminus A$, and $f : A \to \mathcal{N}$ a partial isomorphism. Then f can be extended to a partial isomorphism $A \cup \{m\} \to \mathcal{N}$ as follows. If $A = \emptyset$, then m can be sent to any element of $\mathcal{N}$. Otherwise, there are elements $\ell, r \in A$ closest to m on either side, in the sense that for $a \in A \setminus \{\ell\}$, $\mathcal{M} \models \circlearrowright(\ell, m, a)$, and for $a \in A \setminus \{r\}$, $\mathcal{M} \models \circlearrowright(a, m, r)$. By density of $\mathcal{N}$, there is $n \in \mathcal{N}$ such that $\mathcal{N} \models \circlearrowright(f(\ell), n, b)$ for $b \in f(A \setminus \{\ell\})$ and $\mathcal{N} \models \circlearrowright(b, n, f(r))$ for $b \in f(A \setminus \{r\})$. Then m can be sent to n. $\square$

Proposition 3.3.8. *1. Dense cyclic orders are not semi-equational. In particular, the partitioned formula $\psi(x_1, x_2; y) := \circlearrowright(x_1, x_2; y)$ is not a Boolean combination of semi-equations.*

2. Dense cyclic orders expanded with one constant symbol c are $(2,1)$-semi-equational.

Proof. (1) By quantifier elimination, the formulas $\circlearrowright(x_1, x_2; y)$ and $\circlearrowright(x_2, x_1; y)$ each isolate a complete 3-type (over $\emptyset$). Any Boolean combination of formulas that is equivalent to $\circlearrowright(x_1, x_2; y)$ must contain some formula $\varphi(x_1, x_2; y)$ that is implied by $\circlearrowright(x_1, x_2; y)$ and is inconsistent with $\circlearrowright(x_2, x_1; y)$, or vice versa. Assume the former. Let $(c_i)_{i\in\mathbb{Z}}$ be such that $\models\circlearrowright(c_k, c_i, c_j)$ for $i < j < k$. Let $a_{1,i} = c_{2i}$, $a_{2,i} = c_{2i+2}$, and $b_i = c_{2i+1}$. Then $\models\circlearrowright(a_{2,i}, a_{1,i}; b_j) \iff i = j$ and $\models\circlearrowright(a_{1,i}, a_{2,i}; b_j) \iff i \neq j$, so $\models \varphi(a_{1,i}, a_{2,i}; b_j) \iff i \neq j$, so $\varphi(x_1, x_2; y)$ is not a semi-equation. If instead, $\varphi(x_1, x_2; y)$ is implied by $\circlearrowright(x_2, x_1; y)$ and inconsistent with $\circlearrowright(x_1, x_2; y)$, then we can switch the roles of x_1 and x_2 to get the same result.

(2) Let $<$ be defined by $x < y \iff \circlearrowright(x, y, c)$. Then $<$ is a dense linear order on the complement of $\{c\}$, and thus $x < y$ is a $(2,1)$-semi-equation. We have

$$\models\circlearrowright(x, y, z) \leftrightarrow$$
$$\Big(x < y < z \vee y < z < x \vee z < x < y \vee (z = c \wedge x < y)$$
$$\vee (y = c \wedge z < x) \vee (x = c \wedge y < z)\Big).$$

Hence $\circlearrowright(x, y, z)$ is a Boolean combination of $(2,1)$-semi-equations (with c as a parameter), under any partition of the variables. By quantifier elimination, it follows that every formula is a Boolean combination of $(2,1)$-semi-equations (using c as a parameter). $\square$

The significance of this example is that it shows that a theory being semi-equational, or 1-semi-equational, is not preserved under forgetting constants (note that naming constants clearly preserves (k-)semi-equationality). This is in contrast to equationality ([Jun00, Proposition 3.5]) and distality ([Sim13, Corollary 2.9]), which are invariant under naming or forgetting constants. This is also an example of a distal, non-semi-equational theory.

Problem 3.3.9. Is weak semi-equationality of theories preserved by forgetting constants?

3.3.4 Ordered abelian groups

We consider ordered abelian groups, as structures in the language $\mathcal{L}_{\mathrm{CH}}$ introduced in [CH11]. Given an ordered abelian group $(G, +, <)$ and prime p, for $a \in G \setminus pG$ we let $G_p(a)$ be the largest convex subgroup of G such that $a \notin G_p(a) + pG$, and for $a \in pG$ let $G_p(a) := \{0\}$. Let $\mathcal{S}_p := \{G_p(a) : a \in G\}$. Then the $\mathcal{L}_{\mathrm{CH}}$-structure $\bar{G}$ corresponding to G consists of the main sort $\mathcal{G}$ for G, an auxiliary sort $\mathcal{S}_p$ for each p, along with countably many further auxiliary sorts and relations between them. A relative quantifier elimination result is obtained for such structures in [CH11], to which we refer for the details (see also [ACGZ22, Section 3.2] for a quick summary).

Here we only consider the case of ordered abelian groups with the sorts $\mathcal{S}_p$ finite for all prime p, in which case this relative quantifier elimination simplifies. This includes Presburger arithmetic, and in fact any ordered abelian group with a strongly dependent theory (by [CKS15, DG18, Far17, HH19]).

Proposition 3.3.10. *Every ordered abelian group (either as a pure ordered abelian group, or the corresponding structure $\bar{G}$) with finite auxiliary sorts $\mathcal{S}_p$ for all p is 1-semi-equational.*

Proof. Since every auxiliary sort is finite and linearly ordered by a (definable) relation in $\mathcal{L}_{\mathrm{CH}}$, all auxiliary sorts are contained in $\mathrm{dcl}(\emptyset)$. Hence we only need to verify that every formula $\varphi(x, y)$ with x, y tuples of the main sort $\mathcal{G}$ is a Boolean combination of 1-semi-equations in the expansion with every element of every auxiliary sort named by a new constant symbol (countably many in total).

As explained in [ACGZ22, Proposition 3.14], it then follows from the relative quantifier elimination that $\varphi(x, y)$ is equivalent to a Boolean combination of atomic formulas of the form $\pi_\alpha(f(x)) \diamond_\alpha \pi_\alpha(g(y)) + k_\alpha$, where $\diamond \in \{=, <, \equiv_m\}$, $k \in \mathbb{Z}$, α is an element of an auxiliary sort, f, g are $\mathbb{Z}$-linear functions on $\mathcal{G}$, G_α is a corresponding convex subgroup of G, $\pi_\alpha : G \to G/G_\alpha$ is the quotient map, 1_α is the minimal positive element of G/G_α if it is discrete or $0 \in G/G_\alpha$ otherwise, and $k_\alpha = k \cdot 1_\alpha$ in G/G_α, and for $g, h \in G/G_\alpha$ we have

$g \equiv_m h$ if $g - h \in m\,(G/G_\alpha)$ (note that these relations on G are expressible in the pure language of ordered abelian groups).

The following general claim is straightforward from Definition 3.2.17:

Claim 3.3.11. *If $\varphi(x, y)$ is a (k, n)-semi-equation and $f(x), g(y)$ are $\emptyset$-definable functions, then the formula $\psi(x, y) := \varphi(f(x), g(y))$ is also a (k, n)-semi-equation.*

Using the claim (in an expansion of $\bar{G}$ naming π_α, and the ordered group structure on G/G_α together with the constants for k_α), we only have to show that the relations $x = y, x < y, x \in y + m\,(G/G_\alpha)$ on G/G_α are $(2, 1)$-semi-equations, which is straightforward. □

Problem 3.3.12. Is every ordered abelian group 1-semi-equational, or at least (weakly) semi-equational? We expect a negative answer, by interpreting a variant of the example from Section 3.5.2 on some quotient sorts.

3.3.5 Trees

In this section we use "$\wedge$" to denote "meet", and "&" to denote conjunction.

Definition 3.3.13.

1. By a *tree* we mean a meet-semilattice $(M, \wedge)$ with an associated partial order $\leq$ (defined by $x \leq y \iff x \wedge y = x$) so that all of its initial segments are linear orders.
2. An *infinitely-branching dense tree* is a tree whose initial segments are dense linear orders and such that for each element x, there are infinitely many elements any two of which have meet x.

The following lemma is standard, we include a proof for completeness.

Lemma 3.3.14. *The theory of infinitely-branching dense trees is complete and eliminates quantifiers in the language $\{\wedge\}$.*

Proof. By back-and-forth. Let $\mathcal{M}, \mathcal{N}$ be infinitely-branching dense trees, $A \subseteq M$ finite, $m \in M$, and $f : A \to \mathcal{N}$ a partial isomorphism. It is enough to extend f to a partial isomorphism $f' : A \cup \{m\} \to \mathcal{N}$. We may assume that A is a substructure (i.e. closed under meets) taking $f'\left(\bigwedge X\right) := \bigwedge \{f(a) \mid a \in X\}$ for every $\emptyset \neq X \subseteq A$. Let $m' := \max_{a \in A}(m \wedge a)$. This maximum exists because the initial segment below m is linearly ordered and A is finite. Note that if A contains an element above m, then $m' = m$, but $m' < m$ otherwise. We will first extend f to $A \cup \{m'\}$. Let $b := \bigwedge \{a \in A \mid m' \leq a\}$. This is well-defined because m' was defined so that there must be some $a \in A$ such that $m' \leq a$. If $m' = b$, extend f to send m' to $f(b)$. Otherwise, $\{a \in A \mid a \leq m'\} \subseteq \{x \in M \mid x < b\}$, so $\{f(a) \mid m' \geq a \in A\} \subseteq \{x \in N \mid x < f(b)\}$. As $\{x \in N \mid x < f(b)\}$ is a dense linear order, it contains elements above every element of the finite subset $\{f\left(\bigwedge X\right) \mid \emptyset \neq X \subseteq A,\, m' \geq \bigwedge X\}$. Then we can define $f'(m')$ to be any such element. If $m' < m$, then we still must extend f to $A \cup \{m', m\}$. But $f'(m)$ can be defined to be any element of N above $f'(m')$ whose meet with every element of $f(A)$ is at most $f'(m')$. $\square$

Lemma 3.3.15. *In any tree $\mathcal{M} = (M, \wedge)$ with no additional structure, if every formula of the form $\varphi(x; y_1, y_2)$ with x, y_1, y_2 singletons is a Boolean combination of semi-equations, then every formula is a Boolean combination of semi-equations.*

Proof. By [Sim11, Corollary 4.6] (using that $x \leq y \iff x \wedge y = x$), in any tree $\mathcal{M} = (M, \wedge)$ we have: two tuples $\bar{a} = (a_i : i \in [n]), \bar{b} = (b_j : j \in [n]) \in M^n$ have the same type if and only if (a_i, a_j, a_k) and (b_i, b_j, b_k) have the same type for every $i, j, k \in [n]$. Hence for any $\bar{a}, \bar{b}$, $\mathrm{tp}\left(\bar{a}\bar{b}\right)$ is implied by the set of formulas satisfied by 3-element subtuples of $\bar{a}\bar{b}$. So if every partitioned formula with 3 total free variables is a Boolean combination of semi-equations, then $\mathrm{tp}\left(\bar{a}\bar{b}\right)$ is implied by a Boolean combination of semi-equations. It is enough that every formula of the form $\varphi(x; y_1, y_2)$ is a Boolean combination of semi-equations, because then by symmetry, every formula of the form $\varphi(x_1, x_2; y)$ is as well, and every partitioned formula with one of the parts empty (i.e. $\varphi(; y_1, y_2, y_3)$ or $\varphi(x_1, x_2, x_3;))$ is automatically a semi-equation. $\square$

Theorem 3.3.16. *The theory of infinitely-branching dense trees is semi-equational.*

Proof. Let $\mathcal{M} = (M, \wedge)$ be an infinitely-branching dense tree. By Lemma 3.3.15, it is enough to check that every formula $\varphi(x; y_1, y_2)$ is a Boolean combination of semi-equations, and, by Lemma 3.3.14, it is enough to check this for positive atomic formulas $\varphi(x; y_1, y_2)$. Using the fact that $\wedge$ is associative, commutative, and idempotent, there are only finitely many such formulas up to equivalence, since each such formula is equivalent to a formula of the form $\bigwedge A = \bigwedge B$ for non-empty $A, B \subseteq \{x, y_1, y_2\}$. Since there are 7 such subsets, that gives us 49 formulas to check; 7 of them are tautologies ($\bigwedge A = \bigwedge A$ for some non-empty $A \subseteq \{x, y_1, y_2\}$), hence trivially semi-equations. Of the remaining 42 formulas, 6 do not mention x, hence are automatically semi-equations, and 6 more do not mention y_1, hence are redundant with formulas that do not mention y_2. This leaves us with 30 formulas which come in 15 pairs of equivalent formulas by symmetry of $=$. These 15 formulas are listed below:

1. $x = y_1$,
2. $x = x \wedge y_1$ (i.e. $x \leq y_1$),
3. $x = y_1 \wedge y_2$,
4. $x = x \wedge y_1 \wedge y_2$ (i.e. $x \leq y_1$ & $x \leq y_2$, hence redundant with (2))
5. $x \wedge y_1 = y_1$ (i.e. $x \geq y_1$, hence redundant with (2)),
6. $x \wedge y_1 = y_2$ (i.e. $x \wedge y_1 \leq y_2$ & $x \geq y_2$ & $y_1 \geq y_2$, hence redundant with (2) and (9)),
7. $x \wedge y_1 = x \wedge y_2$,
8. $x \wedge y_1 = y_1 \wedge y_2$ (i.e. $x \wedge y_1 \leq y_2$ & $y_1 \wedge y_2 \leq x$, hence redundant with (9) and (12)),
9. $x \wedge y_1 = x \wedge y_1 \wedge y_2$ (i.e. $x \wedge y_1 \leq y_2$),
10. $x \wedge y_1 \wedge y_2 = y_1$ (i.e. $x \geq y_1$ & $y_2 \geq y_1$, hence redundant with (2)),

11. $x \wedge y_1 \wedge y_2 = y_2$ (redundant with (10)),

12. $x \wedge y_1 \wedge y_2 = y_1 \wedge y_2$ (i.e. $x \geq y_1 \wedge y_2$),

13. $x \wedge y_2 = y_1$ (redundant with (6)),

14. $x \wedge y_2 = y_1 \wedge y_2$ (redundant with (8)),

15. $x \wedge y_2 = x \wedge y_1 \wedge y_2$ (redundant with (9)).

Of the 6 formulas left, (1) and (3) are clearly equations.

(2) is a semi-equation: given $(a_i, b_i)_{i \in \mathbb{Z}}$ such that $\models a_i \leq b_j \iff i \neq j$, $a_i \leq b_0$ for $i \neq 0$, so $(a_i)_{i \neq 0}$ forms a chain. This is not consistent with $a_1 \leq b_2$, $a_2 \leq b_1$, $a_1 \not\leq b_1$, $a_2 \not\leq b_2$.

(12) is a semi-equation for the same reason.

(7) is a negated semi-equation: given $(a_i, b_i, b'_i)_{i \in \mathbb{Z}}$ such that $\models a_i \wedge b_j = a_i \wedge b'_j \iff i = j$, for every $i \neq 0$ we have: either $a_i \wedge b_0 > a_0 \wedge b_0$ or $a_i \wedge b'_0 > a_0 \wedge b_0$. By pigeonhole, there are $i_1 \neq i_2$ such that the same case holds for both. Without loss of generality, $a_1 \wedge b_0 > a_0 \wedge b_0$ and $a_2 \wedge b_0 > a_0 \wedge b_0$. But then $a_1 \wedge a_2 > a_0 \wedge b_0 = a_0 \wedge a_1$, so a_1 and a_2 meet strictly closer to each other than to a_0. But, since $a_1 \wedge b_1 \leq a_0 \wedge b_1$ and $a_1 \wedge b_1 \leq a_2 \wedge b_1$, it also must be true that $a_0 \wedge a_2 \geq a_1 \wedge b_1 = a_1 \wedge a_0$, so a_0 and a_2 meet at least as closely to each other as to a_1. These are inconsistent.

(9) is a negated semi-equation: given $(a_i, b_i, b'_i)_{i \in \mathbb{Z}}$ such that $\models a_i \wedge b_j \leq b'_j \iff i = j$, in particular $a_0 \wedge b_0 \leq b'_0$ and $a_i \wedge b_0 \not\leq b'_0$ for $i \neq 0$. Since the initial segment below b_0 is totally ordered, it follows that $a_0 \wedge b_0 < a_i \wedge b_0$ for $i \neq 0$. $a_1 \wedge a_2 \geq (a_1 \wedge b_0) \wedge (a_2 \wedge b_0) > a_0 \wedge b_0 = a_0 \wedge a_1$. That is, a_1 and a_2 meet strictly closer together with each other than with a_0. But, by switching the roles of the indices 0 and 2 in that argument, a_0 and a_1 must meet strictly closer together with each other than with a_2 as well, a contradiction. $\square$

Theorem 3.3.17. *In an infinitely-branching dense tree $\mathcal{M} = (M, \wedge)$, the formula $x < y$ is not a Boolean combination of 1-semi-equations (without parameters).*

Proof. By quantifier elimination, there are 3 complete 2-types over $\emptyset$ axiomatized by $\{x = y,\ x > y,\ x < y,\ x \perp y\}$, where $\perp$ denotes incomparable elements. Thus, up to equivalence, there are only 16 formulas $\varphi(x, y)$ with x, y singletons without parameters:

1. $x \neq x$ — a $(1, 1)$-semi-equation;

2. $x = y$ — a $(2, 1)$-semi-equation;

3. $x > y$ — a $(2, 1)$-semi-equation;

4. $x < y$ — not an $(n, 1)$-semi-equation for any n (since given pairwise incomparable $y_1, \ldots, y_n$, $x < y_i$ is consistent with $x \perp y_j$ for any distinct $i, j \in [n]$);

5. $x \perp y$ — not an $(n, 1)$-semi-equation for any n (since given pairwise incomparable $y_1, \ldots, y_n$, $x \perp y_i$ is consistent with $x = y_j$ for any distinct $i, j \in [n]$);

6. $x \leq y$ — not an $(n, 1)$-semi-equation for any n (for the same reason as (4));

7. $x \geq y$ — a $(2, 1)$-semi-equation;

8. $x = y \vee x \perp y$ — not a $(n, 1)$-semi-equation for any n (since given pairwise incomparable $y_1, \ldots, y_n$, $x \perp y_i$ is consistent with $x < y_j$ for any distinct $i, j \in [n]$);

9. $x > y \vee x < y$ — not an $(n, 1)$-semi-equation for any n (for the same reason as (4)).

10. $x > y \vee x \perp y$ — not an $(n, 1)$-semi-equation for any n (for the same reason as (5));

11. $x < y \vee x \perp y$ — not an $(n, 1)$-semi-equation for any n (for the same reason as (5));

12. $x \not\leq y$ — not an $(n, 1)$-semi-equation for any n (for the same reason as (4));

13. $x \not< y$ — not an $(n, 1)$-semi-equation for any n (for the same reason as (8));

14. $x \not> y$ — not an $(n, 1)$-semi-equation for any n (since given pairwise incomparable $y_1, \ldots, y_n$, $x \perp y_i$ is consistent with $x > y_j$ for any distinct $i, j \in [n]$);

15. $x \neq y$ — not an $(n,1)$-semi-equation for any n (for the same reason as (5));

16. $x = x$ — a $(2,1)$-semi-equation.

Of the five $(2,1)$-semi-equations on this list, none of them separate $x < y$ from $x \perp y$, so any Boolean combination of them implied by $x < y$ must also be implied by $x \perp y$, thus $x < y$ is not equivalent to a Boolean combination of them. □

Corollary 3.3.18. *In any expansion of an infinitely-branching dense tree $\mathcal{M} = (M, \wedge)$ by naming constants, the formula $x < y$ is not a Boolean combination of 1-semi-equations.*

Proof. Suppose $x < y$ is equivalent to a Boolean combination of 1-semi-equations with parameters $c = (c_1, \ldots, c_n)$. Say $x < y \iff \Phi(\varphi_1(x,y,c), \ldots, \varphi_k(x,y,c))$, where Φ is a Boolean formula in k variables, and $\varphi_1(x,y,c), \ldots, \varphi_k(x,y,c)$ are 1-semi-equations. Let d be an element such that $d \perp \bigwedge_{i \leq n} c_i$. For each i, let

$$\psi_i(x,y) := \exists z \left(\mathrm{tp}(z) = \mathrm{tp}(c) \,\&\, \left(x \wedge y \perp \bigwedge_{i \leq n} z_i \right) \&\, \varphi_i(x,y,z) \right).$$

Note that $\mathrm{tp}(c)$ is isolated by quantifier elimination, so this is indeed a first-order formula. For $a, b > d$, clearly if $\models \varphi_i(a,b,c)$, then $\models \psi_i(a,b)$. By quantifier elimination and [Sim11, Lemma 4.4], the converse also holds. Thus, for $a, b > d$,

$$\models a < b \iff \models \Phi(\varphi_1(a,b,c), \ldots, \varphi_k(a,b,c)) \iff \models \Phi(\psi_1(a,b), \ldots, \psi_k(a,b)).$$

Since all singletons have the same type, it follows that this holds for all a, b. It thus remains to show that each $\psi_i(x,y)$ is a 1-semi-equation, contradicting Theorem 3.3.17.

If this were not the case for some $i \leq k$, then there would be $(b_j)_{j \in \mathbb{N}}$ and a such that $\models \psi_i(a, b_j)$ for all $j \in \mathbb{N}$, but such that for every $j \neq \ell \in \mathbb{N}$, there is $a_{j,\ell}$ such that $\models \psi_i(a_{j,\ell}, b_j)$ but $\not\models \psi_i(a_{j,\ell}, b_\ell)$. But, again because all singletons have the same type, and every finite set of elements has a lower bound, it is consistent that furthermore all of these elements are above d. But then this would also provide a counterexample to $\varphi_i(x,y)$ being a 1-semi-equation. □

Remark 3.3.19. Since $x > y$ is a $(2,1)$-semi-equation and $x < y$ is not, this shows that being an (n,k)-semi-equation for fixed n,k (or even being a Boolean combination of them) is not preserved under exchanging the roles of the variables (while being a semi-equation is preserved).

Remark 3.3.20. Note also that every tree admits an expansion in which $x < y$ is a Boolean combination of $(2,1)$-semi-equations. In a tree, let $\leq_{\text{lex}}$ be a linear order refining $\leq$ such that for a, b, b' such that $a \perp b$ and $b \wedge b' > b \wedge a$, $a \leq_{\text{lex}} b \iff a \leq_{\text{lex}} b'$. Then let $\leq_{\text{revlex}}$ be given by $x \leq_{\text{revlex}} y \ :\iff \ x \leq y \vee (x \perp y \& y \leq_{\text{lex}} x)$. Then $\leq_{\text{revlex}}$ satisfies the same conditions as $\leq_{\text{lex}}$ (so both are $(2,1)$-semi-equations as both are linear orders), and $x \leq y \iff x \leq_{\text{lex}} y \& x \leq_{\text{revlex}} y$.

Problem 3.3.21. Is every theory of trees semi-equational? Is every theory of trees (expanded by constants) not 1-semi-equational?

3.4 Weak semi-equations and strong honest definitions

In this section we discuss how (weak) semi-equationality can be naturally viewed as a generalization of both distality and equationality.

Definition 3.4.1. Given a formula $\varphi(x,y) \in \mathcal{L}$ and a type p, we denote by

$$p^+_\varphi := \{\varphi(x,b) : \varphi(x,b) \in p\}$$

the positive φ-part of the type p.

Definition 3.4.2. Given small sets $A, B, C \subseteq \mathbb{M}$, let $A \mathop{\smile\!\!\!\!\!\mid}^u_C B$ denote that $\operatorname{tp}(A/BC)$ is finitely satisfiable in C.

We recall the following characterization of equations from [MPZ20, Lemma 2.4], which in turn is a variant of [Sro88a, Theorem 2.5]. Note that Fact 3.4.3(3) below is equivalent to [MPZ20, Lemma 2.4(3)] since in stable theories non-forking is symmetric and equivalent to finite satisfiability over models. Existence of k in Fact 3.4.3(2) is not stated explicitly in [MPZ20, Lemma 2.4(2)], but is immediate from the proof.

Fact 3.4.3. *Given a formula $\varphi(x,y)$ in a stable theory T, the following are equivalent.*

1. *The formula $\varphi(x,y)$ is an equation (equivalently, $\varphi^*(y,x) := \varphi(x,y)$ is an equation).*
2. *There is some $k \in \mathbb{N}$ such that for any $a \in \mathbb{M}^x$ and small $B \subseteq \mathbb{M}^y$, there is a subset B_0 of B of size at most k such that $\mathrm{tp}^+_\varphi(a/B_0) \vdash \mathrm{tp}^+_\varphi(a/B)$.*

On the other hand, we recall that theory is distal if and only if every formula $\varphi(x,y)$ is *distal*, that is, for any I_L and I_R infinite linear orders, $b \in \mathbb{M}^y$ and indiscernible sequence $(a_i)_{i \in I_L+(0)+I_R}$ with $a_i \in \mathbb{M}^x$ such that $(a_i)_{i \in I_L+I_R}$ is indiscernible over b, $\models \varphi(a_0,b) \iff \models \varphi(a_i,b)$ for $i \in I_L + I_R$. There is a straightforward relationship between weak semi-equationality as defined in Definition 3.2.1 and distality: a formula $\varphi(x,y)$ is distal if and only if both $\varphi(x,y)$ and $\neg\varphi(x,y)$ are weak semi-equations. An NIP theory is distal if and only every formula admits a *strong honest definition*:

Fact 3.4.4. *[CS15, Theorem 21] A theory T is distal if and only if for every formula $\varphi(x,y)$ there is a formula $\theta(x;y_1,\ldots,y_k)$, called a* strong honest definition *for $\varphi(x,y)$, such that for any finite set $C \subseteq \mathbb{M}^y$ ($|C| \geq 2$) and $a \in \mathbb{M}^x$, there is $b \in C^k$ such that $\models \theta(a;b)$ and $\theta(x;b) \vdash \mathrm{tp}_\varphi(a/C)$.*

We now show that in an NIP theory, weak semi-equationality is equivalent to the existence of *one-sided* strong honest definitions, which is also a generalization of Fact 3.4.3 (replacing a conjunction of finitely many instances of φ by some formula θ).

Theorem 3.4.5. *Let T be NIP, and let $\varphi(x,y)$ be a formula. The following are equivalent:*

1. *The formula $\varphi^*(y,x) := \varphi(x,y)$ is a weak semi-equation.*
2. *For every small $B \subseteq \mathbb{M}^y$ and $a \in \mathbb{M}^x$ such that $\models \varphi(a,b)$ for all $b \in B$ there exists $\theta(x;y_1,\ldots,y_k)$ and $c \in (\mathbb{M}^y)^k$ such that $c \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{B} a$, $\models \theta(a,c)$ and $\theta(x,c) \vdash \mathrm{tp}^+_\varphi(a/B)$.*

3. *There is some formula $\theta(x; y_1, \ldots, y_k)$ and number N such that for any finite set $B \subseteq \mathbb{M}^y$ with $|B| \geq 2$ and $a \in \mathbb{M}^x$, there is some $B_0 \subseteq B$ with $|B_0| \leq N$ such that $\mathrm{tp}^+_\theta(a/B_0) \vdash \mathrm{tp}^+_\varphi(a/B)$.*

Proof. (1) implies (2). We follow closely the proof of [CS15, Proposition 19]. Let a, B be such that $\models \varphi(a, b)$ for all $b \in B$. Let $\mathcal{M} \preceq \mathbb{M}$ contain a, B, let $(\mathcal{M}', B') \succ (\mathcal{M}, B)$ be a $\kappa := |M|^+$-saturated elementary extension (with B named by a new predicate), we may assume $\mathcal{M}' \prec \mathbb{M}$ is a small submodel. Let $p(x) := \mathrm{tp}(a/B')$.

Claim 3.4.6. *Let $q(y) \in S_y(B')$ be any type finitely satisfiable in B. Then $p(x) \cup q(y) \vdash \varphi(x, y)$.*

Proof. The complete type

$$q^{(\omega)}(y_1, y_2, \ldots) := \bigcup \{\mathrm{tp}(b_1, b_2, \ldots/C) : B \subseteq C \subseteq B', |C| < \kappa, b_i \models q|_{Cb_{<i}}\}$$

over B' is finitely satisfiable in B. As T is NIP, by [CS15, Lemma 5] there is some D with $B \subseteq D \subseteq B', |D| < \kappa$ such that for any two realizations $I, I' \subseteq B'$ of $q^{(\omega)}|_D$ we have $aI \equiv_D aI'$. Fix some $I \models q^{(\omega)}|_D$ in B' (exists by saturation of $(\mathcal{M}', B')$ and finite satisfiability of $q^{(\omega)}$ in B) and $J \models q^{(\omega)}|_{\mathbb{M}}$ (in some larger monster model $\mathbb{M}' \succ \mathbb{M}$, here $q^{(\omega)}|_{\mathbb{M}}$ is an arbitrary type over $\mathbb{M}$ extending $q^{(\omega)}$ and finitely satisfiable in B).

We claim that $I + J$ is indiscernible over aB. Indeed, as $q^{(\omega)}|_{\mathbb{M}}$ is finitely satisfiable in B, by compactness and saturation of $(\mathcal{M}', B')$ there is some $J' \models q^{(\omega)}|_{aDI}$ in B'. If $I + J$ is not aB-indiscernible, then $I' + J'$ is not aB-indiscernible for some finite subsequence I' of I. As by construction both $I' + J'$ and J' realize $q^{(\omega)}|_D$ in B', it follows by the choice of D that J' is not indiscernible over aB — contradicting the choice of J'.

Now let $b^* \in \mathbb{M}$ be any realization of q, then the sequence $I + (b^*) + J$ is Morley in q over B, hence indiscernible (even over B). And $I + J$ is indiscernible over a (even over aB) by the previous paragraph. Note also that $\models \varphi(a, b)$ for every $b \in B'$ (by assumption we had $\models \varphi(a, b)$ for all $b \in B$, but $a \in \mathcal{M}$ and $(\mathcal{M}', B') \succ (\mathcal{M}, B)$). Hence $\models \varphi(a, b)$ for every

$b \in I + J$. And since $\varphi^{*}(y, x)$ is a weak semi-equation, this implies $\models \varphi(a, b^{*})$. That is, for any $a \models p$ and $b^{*} \models q$, we have $\models \varphi(a, b^{*})$, as wanted. $\square$

Now let S' be the set of types over B' finitely satisfiable in B, it is a closed subset of $S_y(B')$. By the claim, for every $q \in S'$ we have $p(x) \cup q(y) \vdash \varphi(x, y)$, hence by compactness $\theta_q(x) \cup \psi_q(y) \vdash \varphi(x, y)$ for some formulas $\theta_q(X) \in p, \psi_q(y) \in q$. As $\{\psi_q(y) : q \in S'\}$ is a covering of the closed set S', it has a finite sub-covering $\{\psi_{q_k} : k \in K\}$. Let $\theta(x) := \bigwedge_{k \in K} \theta_{q_k}(x) \in p(x)$. As in particular $\mathrm{tp}(b/B) \in S'$ for every $b \in B$, we thus have $\theta(x) \in \mathcal{L}(B')$ (and $B' \mathop{\smile\!\!\!\!\!\!\;\;|}^{u}_{B} a$), $\models \theta(a)$ and $\theta(x) \vdash \mathrm{tp}^{+}_{\varphi}(a/B)$.

(2) implies (3). Given a, B, we either have that $\models \neg\varphi(a, b)$ for all $b \in B$, in which case $\mathrm{tp}^{+}_{\theta}(a/B_0) = \mathrm{tp}^{+}_{\varphi}(a/B) = \emptyset$, and $\emptyset \vdash \emptyset$ trivially. Or we replace B by $\{b \in B \mid\models \varphi(a, b)\}$, and follow the proof of (1) implies (2) in [CS15, Theorem 21], we provide the details.

By (2), given small $B \subseteq \mathbb{M}^{y}$ and $a \in \mathbb{M}^{x}$ such that $\models \varphi(a, b)$ for all $b \in B$, we have $\theta(x; y_1, \ldots, y_\ell)$ and $c \in (\mathbb{M}^{y})^{\ell}$ such that $c \mathop{\smile\!\!\!\!\!\!\;\;|}^{u}_{B} a$, $\models \theta(a, c)$, and $\theta(x, c) \vdash tp^{+}_{\varphi}(a/B)$. Given finite $B_0 \subseteq B$, there is $d \in B^{\ell}$ such that $d \equiv_{aB_0} c$, so $\models \theta(a, d)$ and $\theta(x, d) \vdash tp^{+}_{\varphi}(a/B_0)$.

In particular, if we assign to each formula $\theta(x; y_1, \ldots, y_\ell)$ a number $n_\theta \in \mathbb{N}$, we get that for any $B \subseteq \mathbb{M}^{y}$ and $a \in \mathbb{M}^{x}$ such that $\models \varphi(a, b)$ for all $b \in B$, there is $\theta(x; y_1, \ldots, y_\ell)$ such that for every $B_0 \subseteq B$ with $|B_0| \leq n_\theta$, $\exists c \in B^{\ell}$ such that $\models \theta(a, c)$ and $\theta(x, c) \vdash tp^{+}_{\varphi}(a/B_0)$. That is, if we expand the language with a predicate symbol for B and a constant symbol for a, and we expand the theory with axioms saying $\forall x \in B\, \varphi(a, x)$ and, for every formula $\theta(x; y_1, \ldots, y_\ell)$, an axiom $\exists b_1, \ldots, b_{n_\theta} \in B\, \forall c \in B^{\ell}\, (\neg\theta(a, c)) \vee \exists x\, \left(\theta(x, c) \wedge \bigvee_{i \leq n_\theta} \neg\varphi(a, b_i)\right)$, this theory is inconsistent. By compactness, the inconsistency only requires finitely many of these formulas $\theta_1, \ldots, \theta_k$.

Thus there are $\theta_1(x; y_1, \ldots, y_{\ell_1}), \ldots, \theta_k(x; y_1, \ldots, y_{\ell_k})$ such that given $B \subseteq \mathbb{M}^{y}$ and $a \in \mathbb{M}^{x}$ such that $\models \varphi(a, b)$ for all $b \in B$, there is $i \leq k$ such that for all $B_0 \subseteq B$ with $|B_0| \leq n_{\theta_i}$, there is $c \in B^{\ell_i}$ such that $\models \theta_i(a; c)$ and $\theta_i(x; c) \vdash tp^{+}_{\varphi}(a/B_0)$.

For each formula $\theta(x; y_1, \ldots, y_\ell)$, let $\rho_\theta(x, y; z) := \theta(x; z) \wedge \forall w\, \theta(w; z) \to \varphi(w, y)$, and let $n_\theta := \mathrm{VC}(\rho_\theta) + 1$ in the above argument (where VC is the VC-dimension). For $a \in \mathbb{M}^x$ and finite $B \subseteq \mathbb{M}^y$ such that $\models \varphi(a;b)$ for $b \in B$, let $i_{a,B} \leq k$ be such that for all $B_0 \subseteq B$ with $|B_0| \leq n_{\theta_{i_{a,B}}}$, there is $c \in B^{\ell_{i_{a,B}}}$ such that $\models \theta_{i_{a,B}}(a;c)$ and $\theta_{i_{a,B}}(x;c) \vdash tp^+_\varphi(a/B_0)$. For $b \in B$, let

$$\begin{aligned} F^b_{a,B} &:= \left\{c \in B^{\ell_{i_{a,B}}} \mid \models \theta_{i_{a,B}}(a;c), \theta_{i_{a,B}}(x;c) \vdash \varphi(x;b)\right\} \\ &= \left\{c \in B^{\ell_{i_{a,B}}} \mid \models \rho_{\theta_{i_{a,B}}}(a,b;c)\right\}, \end{aligned}$$

and let $\mathcal{F}_{a,B} := \left\{F^b_{a,B} \mid b \in B\right\}$. By the (p,k)-theorem (see [CS15, Fact 6]) applied to $\mathcal{F}_{a,B}$, with $p = k = n_{\theta_{a,B}}$, there is $N_{i_{a,B}}$ (depending on $i_{a,B}$ but not otherwise depending on a, B) such that if every $n_{\theta_{i_{a,B}}}$ sets from $\mathcal{F}_{a,B}$ intersect, then there is $B_0 \subseteq B^{\ell_{i_{a,B}}}$ with $|B_0| \leq N_{i_{a,B}}$ intersecting all sets from $\mathcal{F}_{a,B}$. Furthermore, by choice of $i_{a,B}$, the condition that every $n_{\theta_{i_{a,B}}}$ sets from $\mathcal{F}_{a,B}$ intersect holds. And there are only finitely many possible values of $i_{a,B}$, so we have a uniform finite bound $N := \max_i N_i$.

We thus have $N \in \mathbb{N}$ such that for all $a \in \mathbb{M}^x$ and finite $B \subseteq \mathbb{M}^y$ for which $\models \varphi(a;b)$ for all $b \in B$, there is $i \leq k$ and $B_1 \subseteq B^{\ell_{i_{a,B}}}$ with $|B_1| \leq N$ intersecting all sets from $\mathcal{F}_{a,B}$, meaning that for every $b \in B$ there is $c \in B_1$ such that $\models \theta_{i_{a,B}}(a;c)$ and $\theta_{i_{a,B}}(x;c) \vdash \varphi(x;b)$. That is, $tp^+_{\theta_{i_{a,B}}}(a/B_1) \vdash tp^+_\varphi(a/B)$.

Let $\theta(x;y_1,\ldots,y_\ell)$ be a formula that can code for any $\theta_i(x;y_1,\ldots,y_{\ell_i})$ when parameters range over a set with at least two elements. For all $a \in \mathbb{M}^x$ and finite $B \subseteq \mathbb{M}^y$ with $|B| \geq 2$, for which $\models \varphi(a;b)$ for all $b \in B$, there is $B_0 \subseteq B$ with $2 \leq |B_0| \leq \ell N + 2$ (consisting of the coordinates of B_1 from the previous paragraph, and two points for coding) such that $tp^+_\theta(a/B_0) \vdash tp^+_\varphi(a/B)$, as desired.

(3) implies (1). This follows almost verbatim from the proof of (2) implies (1) in [CS15, Theorem 21]. Let $I+d+J$ be an indiscernible sequence in $\mathbb{M}^y$, with I and J infinite, and $I+J$ indiscernible over $a \in \mathbb{M}^x$, and suppose $\models \varphi(a,b)$ for $b \in I+J$. Let $I_1 \subset I$ with $|I_1| = N+1$. Then there is some $I_0 \subseteq I_1$ such that $|I_0| \leq N$ and $tp^+_\theta(a/I_0) \vdash tp^+_\varphi(a/I_1)$. Let $b \in I_1 \setminus I_0$. By indiscernibility of $I+d+J$, there is some $\sigma \in \mathrm{Aut}(\mathbb{M})$ such that $\sigma(I_1) \subset I+d+J$

and $\sigma(b) = d$. We have $\sigma(I_0) \subseteq I + J$, so by a-indiscernibility of $I + J$, $\models \theta(a, \sigma(c))$ for every $c \in I_0^k$ for which $\models \theta(a, c)$, and hence $a \models \sigma\left(tp_\varphi^+(a/I_1)\right)$. And $\varphi(x, b) \in tp_\varphi^+(a/I_1)$, so $\varphi(x, d) \in \sigma\left(tp_\varphi^+(a/I_1)\right)$, and hence $\models \varphi(a, d)$. $\square$

Problem 3.4.7. Can the assumption that T is NIP be omitted? (Note that the proof of (3) implies (1) does not use it.)

Proposition 3.2.20 immediately implies an analog of Fact 3.4.3 for semi-equations, telling us that $\varphi(x; y)$ is a one-sided strong honest definition for itself:

Corollary 3.4.8. *A formula $\varphi(x, y)$ (equivalently, $\varphi^*(y, x)$) is a semi-equation if and only if there is some $k \in \mathbb{N}$ such that: for every finite $B \subset \mathbb{M}^y$ and $a \in \mathbb{M}^x$ there is some $B_0 \subseteq B$ with $|B_0| \leq k$ such that $\mathrm{tp}_\varphi^+(a/B_0) \vdash tp_\varphi^+(a/B)$.*

Example 3.4.9.

1. In a dense cyclic order (see Section 3.3.3), which is distal, the formula $\circlearrowright(x; y, y')$ has a one-sided strong honest definition $\theta(x; y_1, y_1', y_2, y'_2) := \circlearrowright(x, y_1, y_2')$. This is because for any nonempty finite set B of pairs such that $\models \circlearrowright(a, b, b')$ for $(b, b') \in B$, we can let $(b_1, b_1') \in B$ be such that b_1 is as close to a as possible, and $(b_2, b_2') \in B$ be such that b_2' is as close to a as possible. Then $\circlearrowright(x, b_1, b_2') \vdash \mathrm{tp}_\circlearrowright^+(a/B)$. This illustrates a subtlety in the fact that a formula is a semi-equation if and only if a conjunction of its instances gives a one-sided strong honest definition for itself (Corollary 3.4.8): as $\circlearrowright(x; y, y')$ is not a semi-equation (Proposition 3.3.8), but has a one-sided strong honest definition which is the same formula, but with its variables split differently. That is, there is $B_0 \subseteq B$ with $|B_0| \leq 2$ such that $\mathrm{tp}_\theta^+(a/B_0) \vdash \mathrm{tp}_\circlearrowright^+(a/B)$, but $\mathrm{tp}_\theta^+(a/B_0) \neq \mathrm{tp}_\circlearrowright^+(a/B_0)$ even though they are the same as a non-partitioned formula (thus not satisfying the condition that Corollary 3.4.8 says is equivalent to semi-equationality).

2. Partitioning the same formula as in (1) the other way, $\circlearrowright(x_1, x_2; y)$ has a one-sided strong honest definition $\theta(x_1, x_2; y_1, y_2) = \circlearrowright(x_1, x_2, y_1) \wedge \circlearrowright(x_1, x_2, y_2) \wedge \circlearrowright(y_2, y_1, x_1)$.

Given (a_1, a_2) and a nonempty finite set B such that $\models \circlearrowleft (a_1, a_2; b)$ for $b \in B$, let $b_1 \in B$ closest to a_1 and $b_2 \in B$ closest to a_2. Then $\theta(x_1, x_2; b_1, b_2) \vdash \mathrm{tp}^+_{\circlearrowleft}(a_1, a_2/B)$.

3. In a linear order, $x \neq y$ has a one-sided strong honest definition $\theta(x; y_1, y_2, y_3, y_4)$ that uses coding techniques with the variables y_3, y_4 to represent any of the formulas $\theta_1(x; y_1) :\iff x < y_1$, $\theta_2(x; y_1, y_2) :\iff y_1 < x < y_2$, and $\theta_3(x; y_1) :\iff y_1 < x$. Unlike in (1) or (2), $x \neq y$ does not have a one-sided strong honest definition that is defined entirely in terms of Boolean combinations of instances of $x \neq y$ with the variables shifted around. If this were possible, then it would also work in any reduct, but $y \neq x$ is not a weak semi-equation in all reducts of a linear order.

We end the section with some remarks about forking for weak semi-equations.

Definition 3.4.10. We say that a formula $\varphi(x, y)$ satisfies the *definable* (p, q)*-property* if for every small model M and b, if $\varphi(x, b)$ does not fork over M then there is some $\psi(y) \in \mathrm{tp}(b/M)$ such that the set of formulas $\{\varphi(x, b') : b' \in \psi(M)\}$ is consistent (equivalently, $\{\varphi(x, b') : b' \in \psi(\mathbb{M})\}$ is consistent).

It was asked in [CS15, Section 2] if every formula in an NIP theory satisfies the definable (p, q)-property. It is known in some special cases (see the introduction in [BK18]), in particular:

Fact 3.4.11. *[BK18] If T is distal, then every formula $\varphi(x, y)$ satisfies the definable (p, q)-property.*

The proof in [BK18] in fact demonstrates the following:

Proposition 3.4.12. *If T is NIP and $\varphi(x, y)$ a formula such that $\varphi^*(x, y)$ is a weak semi-equation, then $\varphi(x, y)$ satisfies the definable (p, q)-property.*

Proof. Given a formula $\varphi(x, y)$, distality of T (as opposed to just NIP) is only used in the proof in [BK18] to say that $\varphi(x, y)$ satisfies the conclusion of [BK18, Proposition 2.4], i.e. that there is some formula $\theta(x; y_1, \ldots, y_k)$ such that for any finite set $B \subseteq \mathbb{M}^y$, $|B| \geq 2$ and $a \in \mathbb{M}^x$, if $\models \varphi(a, b)$ for all $b \in B$, then there is some $c \in B^k$ such that $\models \theta(a, c)$ and for all $b \in B$ we have $\models \theta(x, c) \to \varphi(x, b)$. By Theorem 3.4.5, this holds assuming that $\varphi^*(x, y)$ is a weak semi-equation. □

3.5 Examples of non weakly semi-equational NIP theories

3.5.1 Boolean combinations of weak semi-equations

We provide a sufficient criterion for when a formula is not a Boolean combination of weak semi-equations (analogous to a criterion for equations from [MS17]).

Lemma 3.5.1. *If $\varphi(x, y)$ and $\psi(x, y)$ are weak semi-equations, then there is no $b \in \mathbb{M}^y$ and array $(a_{i,j})_{i,j \in \mathbb{Z}}$ with $a_{i,j} \in \mathbb{M}^x$ such that:*

- *every row (i.e. $(a_{i,j} : j \in \mathbb{Z})$ for a fixed $i \in \mathbb{Z}$) and every column (i.e. $(a_{i,j} : i \in \mathbb{Z})$ for a fixed $j \in \mathbb{Z}$) is indiscernible (over $\emptyset$);*
- *rows and columns without their 0-indexed elements (i.e. $(a_{i,j})_{j \neq 0}$ for fixed i, and $(a_{i,j})_{i \neq 0}$ for fixed j) are b-indiscernible;*
- $\models \varphi(a_{i,j}, b) \land \neg\psi(a_{i,j}, b) \iff i = 0 \lor j \neq 0$.

Proof. Assume there exists an array $(a_{i,j} : i, j \in \mathbb{Z})$ and b with these properties. For any fixed $i \neq 0$, we have $\models \varphi(a_{i,j}, b)$ for all $j \neq 0$, $(a_{i,j})_{j \in \mathbb{Z}}$ is indiscernible and $(a_{i,j})_{j \neq 0}$ is b-indiscernible, so, by weak semi-equationality of φ, $\models \varphi(a_{i,0}, b)$. But $\not\models \varphi(a_{i,0}, b) \land \neg\psi(a_{i,0}, b)$, so $\models \psi(a_{i,0}, b)$. Now the sequence $(a_{i,0})_{i \in \mathbb{Z}}$ is indiscernible, $(a_{i,0})_{i \neq 0}$ is b-indiscernible and $\models \psi(a_{i,0}, b)$ for all $i \neq 0$, so, by weak semi-equationality of ψ, $\models \psi(a_{0,0}, b)$ — contradicting $\models \varphi(a_{0,0}, b) \land \neg\psi(a_{0,0}, b)$. □

Lemma 3.5.2. *If $\varphi(x, y)$ is a Boolean combination of weak semi-equations, then there is no $b \in \mathbb{M}^y$ and array $(a_{i,j})_{i,j \in \mathbb{Z}}$ with $a_{i,j} \in \mathbb{M}^x$ such that:*

- *rows and columns of* $(a_{i,j})_{i,j\in\mathbb{Z}}$ *are indiscernible;*

- *rows and columns without their* 0*-indexed elements (i.e.* $(a_{i,j})_{j\neq 0}$ *for fixed* i*, and* $(a_{i,j})_{i\neq 0}$ *for fixed* j*) are* b*-indiscernible;*

- $\models \varphi(a_{i,j}, b) \iff i = 0 \vee j \neq 0$*;*

- *all* $a_{i,j}$ *with* $i = 0$ *or* $j \neq 0$ *have the same type over* b*.*

Proof. Any conjunction of finitely many weak semi-equations and negations of weak semi-equations is of the form $\psi(x,y) \wedge \neg\theta(x,y)$ for some weak semi-equations $\psi(x,y)$ and $\theta(x,y)$, because weak semi-equations are closed under conjunction and under disjunction (Proposition 3.2.6(3)), so negations of weak semi-equations are also closed under conjunction. Thus any Boolean combination of weak semi-equations is equivalent, via its disjunctive normal form, to $\bigvee_{k\in I}(\psi_k(x,y) \wedge \neg\theta_k(x,y))$ for some finite index set I and weak semi-equations $\psi_k(x,y)$ and $\theta_k(x,y)$ for $k \in I$. Given b and $(a_{i,j})_{i.j\in\mathbb{Z}}$ as above, since $i = 0 \vee j \neq 0 \iff \models \varphi(a_{i,j}, b) \iff \models \bigvee_{k\in I}(\psi_k(a_{i,j}, b) \wedge \neg\theta_k(a_{i,j}, b))$, and all $a_{i,j}$ with $i = 0$ or $j \neq 0$ have the same type over b, there is some k such that $\models \psi_k(a_{i,j}, b) \wedge \neg\theta_k(a_{i,j}, b) \iff i = 0 \vee j \neq 0$, contradicting Lemma 3.5.1. □

3.5.2 Valued trees are not weakly semi-equational

We are following the notation and terminology of Section 3.3.5.

Definition 3.5.3. 1. A *valued tree* $(M, \wedge, \preccurlyeq)$ is a tree $(M, \wedge)$ equipped with a *height-order* $\preccurlyeq$ which is a total preorder which refines the tree-order $\leq$, and whose equivalence classes are antichains of the tree-order. Then $x \preccurlyeq y$ can be thought of as expressing that y is at least as high above the root as x is, whether or not y actually extends x in the tree-order. The strict version of the height order (i.e. $x \not\succcurlyeq y$), will be denoted by $x \prec y$, and the equal-height equivalence relation ($x \preccurlyeq y \& x \succcurlyeq y$) will be denoted by $x \approx y$). The words "above" and "below" will be used to refer to the tree-order, while "higher than" and "lower than" will be used to refer to the height-order.

2. An *infinitely-branching dense valued tree* $(M, \wedge, \preccurlyeq)$ is an infinitely-branching dense tree $(M, \wedge)$ which is also a valued tree, with the additional assumptions that for every $x, y \in M$ if $x \succcurlyeq y$, then there is $z \leq x$ such that $z \approx y$, and if $x \preccurlyeq y$, then there is $z \geq x$ such that $z \approx y$. This assumption essentially says that for every initial segment and every final segment, there are elements of every possible height below, and above, respectively, the height of the endpoint.

Extending Lemma 3.3.14, we have:

Lemma 3.5.4. *The theory of infinitely branching dense valued trees is complete, and eliminates quantifiers in the language with the binary function symbol* $\wedge$ *and the binary relation symbol* $\preccurlyeq$.

Proof. Let $\mathcal{M}, \mathcal{N}$ be infinitely-branching dense valued trees, $A \subseteq M$ finite, $m \in M$, and $f : A \to \mathcal{N}$ a partial isomorphism. For $\emptyset \neq X \subseteq A$, letting $f'\left(\bigwedge X\right) := \bigwedge \{f(a) \mid a \in X\}$ we may assume that A is closed under $\wedge$. It is enough to extend f to a partial isomorphism $f' : A \cup \{m\} \to \mathcal{N}$. Let $m' := \max_{a \in A}(m \wedge a)$. This maximum exists because the initial segment below m is linearly ordered and A is finite. Note that if A contains an element above m, then $m' = m$, but $m' < m$ otherwise. We will first extend f to $A \cup \{m'\}$. Let $b := \bigwedge \{a \in A \mid m' \leq a\}$. This is well-defined because m' was defined such that there must be some $a \in A$ such that $m' \leq a$. Clearly $m' \leq b$. If $m' = b$, extend f to send m' to $f(b)$. Otherwise, we must extend f to send m' to something below $f(b)$. If $m' \approx \bigwedge X$ for some $\emptyset \neq X \subseteq A$, then extend f to send m' to the element below $f(b)$ with the same height as $f\left(\bigwedge X\right)$ (i.e. such that $f(m') \approx f\left(\bigwedge X\right)$), which exists by the assumption of initial segments containing elements of every possible height. Otherwise, m' is either higher than all, lower than all, or between two of $\{\bigwedge X \mid \emptyset \neq X \subseteq A\}$ in height, and density of the tree order implies density of the height order, so there are elements below $f(b)$ that are higher than all, lower than all, and between any two of $\{\bigwedge X \mid \emptyset \neq X \subseteq A\}$, and m' can be sent to such an element.

If $m' < m$, then we still must extend f to $A \cup \{m', m\}$. Let $c \in \mathcal{N}$ be such that $c > f(m')$, $c \succ f(a)$ for every $a \in A$, and the meet of c with each element of $f(A)$ is at most $f(m')$ (which is possible by the assumptions of infinite branching and final segments containing elements of all possible heights). We can extend f to send m to an element between c and m', such that, if $m \approx \bigwedge X$ for some $\emptyset \neq X \subseteq A$, $f'(m) \approx f(\bigwedge X)$ (which is possible by the fact that initial segments contain elements of all possible heights), and otherwise, $f(m)$ should be higher than $f(\bigwedge X)$ for each X such that $m \succ \bigwedge X$, and lower than $f(\bigwedge X)$ for each X such that $m \prec \bigwedge X$ (which is possible by density of the tree-order). $\square$

Theorem 3.5.5. *The theory of infinitely-branching dense valued trees is not weakly semi-equational. Namely, the partitioned formula $\psi(x_1, x_2; y_1, y_2) := x_1 \wedge y_1 \preccurlyeq x_2 \wedge y_2$ is not equivalent to a Boolean combination of weak semi-equations.*

Proof. To show that $\psi(x_1, x_2; y_1, y_2)$ is not equivalent to a Boolean combination of weak semi-equations, by Lemma 3.5.2 (noting that the condition there is symmetric under exchanging the roles of i and j), it suffices to find b, b', and $(a_i)_{i\in\mathbb{Z}}, \left(a'_j\right)_{j\in\mathbb{Z}}$ such that:

1. the sequences $(a_i)_{i\in\mathbb{Z}}$ and $\left(a'_j\right)_{j\in\mathbb{Z}}$ are mutually indiscernible (which implies that rows and the columns of the array $\left(a_i a'_j\right)_{i,j\in\mathbb{Z}}$ are indiscernible);

2. the sequence $(a_i)_{i\neq 0}$ is indiscernible over $bb'a'_j$ for every $j \in \mathbb{Z}$ and the sequence $\left(a'_j\right)_{j\neq 0}$ is indiscernible over $bb'a_i$ for every $i \in \mathbb{Z}$ (which implies that the rows and the columns of the array $\left(a_i a'_j\right)_{i,j\in\mathbb{Z}}$ with their 0-indexed elements removed are indiscernible over bb');

3. $\models a_i \wedge b \preccurlyeq a'_j \wedge b' \iff i \neq 0 \vee j = 0$;

4. all $\left(a_i, a'_j\right)$ with $i \neq 0$ or $j = 0$ have the same type over bb'.

To find these elements, working in a saturated model let m, m' be $\leq$-incomparable elements such that $m \prec m'$. Let $(a_i)_{i\in\mathbb{Z}}, \left(a'_j\right)_{j\in\mathbb{Z}}, b, b'$ all have the same height (which is higher than m'), such that the meet of any two of the $(a_i)_{i\in\mathbb{Z}}$ is m, the meet of any two of $\left(a'_j\right)_{j\in\mathbb{Z}}$ is m', and $m' \prec (a_0 \wedge b) \prec (a'_0 \wedge b')$.

Since $a_i \wedge b = m$ for $i \neq 0$, $a'_j \wedge b' = m'$ for $j \neq 0$, and $m \prec m' \prec (a_0 \wedge b) \prec (a'_0 \wedge b')$, clearly $\models (a_i \wedge b) \preccurlyeq \left(a'_j \wedge b'\right) \iff i \neq 0 \vee j = 0$, so the condition (3) is satisfied. It remains to verify the conditions (1), (2) and (4).

Claim 3.5.6. *The sequences $(a_i)_{i\in\mathbb{Z}}$ and $\left(a'_j\right)_{j\in\mathbb{Z}}$ are mutually totally indiscernible.*

Proof. By Lemma 3.5.4, every formula $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_k)$ is equivalent to a Boolean combination of formulas of the form $t_1 \preccurlyeq t_2$ or $t_1 = t_2$, where t_1 and t_2 are terms. For $I \subseteq [n]$, $J \subseteq [k]$, not both empty, let

$$t_{I,J}(x_1, \ldots, x_n; y_1, \ldots, y_k) := \bigwedge_{i\in I} x_i \wedge \bigwedge_{j\in J} y_j.$$

Every term in the variables $x_1, \ldots, x_n; y_1, \ldots, y_k$ is of this form. We now calculate $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right)$ for $I \subseteq [n]$, $J \subseteq [k]$, not both empty, distinct $i_1, \ldots, i_n \in \mathbb{Z}$, and distinct $j_1, \ldots, j_k \in \mathbb{Z}$:

- if I and J are both nonempty, then $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right) = m \wedge m'$;
- if $J = \emptyset$ and $|I| \geq 2$, then $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right) = m$.
- if $I = \emptyset$ and $|J| \geq 2$, then $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right) = m'$;
- if $J = \emptyset$ and $I = \{\ell\}$, then $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right) = a_{i_\ell}$;
- if $I = \emptyset$ and $J = \{\ell\}$, then $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right) = a'_{j_\ell}$.

Thus, the value of $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right)$ does not depend on $i_1, \ldots, i_n, j_1, \ldots, j_k$ unless $|I| + |J| = 1$. And the height of $t_{I,J}\left(a_{i_1}, \ldots, a_{i_n}; a'_{j_1}, \ldots, a'_{j_k}\right)$ does not depend on $i_1, \ldots, i_n$, $j_1, \ldots, j_k$ for any I, J. So for the formula $\varphi(x_1, \ldots x_n; y_1, \ldots, y_k) := t_{I_1,J_1} \preccurlyeq t_{I_2,J_2}$, the truth value of the formula $\varphi\left(a_{i_1},\ldots,a_{i_n};a'_{j_1},\ldots,a'_{j_k}\right)$ does not depend on $i_1,\ldots,i_n,j_1,\ldots,j_k$. And for the formula $\varphi := \left(t_{I_1,J_1} = t_{I_2,J_2}\right)$ we have:

- if $|I_1|+|J_1|=1$ or $|I_2|+|J_2|=1$, then $\varphi\left(a_{i_1},\ldots,a_{i_n};a'_{j_1},\ldots,a'_{j_k}\right)$ holds if and only if t_{I_1,J_1} and t_{I_2,J_2} are equal as terms (i.e. $I_1=I_2$ and $J_1=J_2$) — this does not depend on $i_1,\ldots,i_n,j_1,\ldots,j_k$;
- and if $|I_1|+|J_1|\neq 1$ and $|I_2|+|J_2|\neq 1$, then since the values of neither of the terms depend on $i_1,\ldots,i_n,j_1,\ldots,j_k$, the truth value of $\varphi\left(a_{i_1},\ldots,a_{i_n};a'_{j_1},\ldots,a'_{j_k}\right)$ does not either.

Thus, by quantifier elimination, the truth value of $\varphi\left(a_{i_1},\ldots,a_{i_n};a'_{j_1},\ldots,a'_{j_k}\right)$ does not depend on $i_1,\ldots,i_n,j_1,\ldots,j_k$ for any formula φ, since we have established this for atomic, and thus for quantifier-free formulas. That is, $(a_i)_{i\in\mathbb{Z}}$ and $\left(a'_j\right)_{j\in\mathbb{Z}}$ are mutually totally indiscernible. $\square$

Claim 3.5.7. *The sequence $(a_i)_{i\neq 0}$ is indiscernible over $bb'a'_j$ for every $j\in\mathbb{Z}$, and the sequence $\left(a'_j\right)_{j\neq 0}$ is indiscernible over $bb'a_i$ for every $i\in\mathbb{Z}$.*

Proof. As the sequences $(a_i)_{i<0}+(b)+(a_i)_{i>0}$ and $\left(a'_j\right)_{j<0}+(b')+\left(a'_j\right)_{j>0}$ satisfy all the assumptions that were made about the sequences $(a_i)_{i\in\mathbb{Z}}$ and $\left(a'_j\right)_{j\in\mathbb{Z}}$ respectively, having proven that $(a_i)_{i\in\mathbb{Z}}$ and $\left(a'_j\right)_{j\in\mathbb{Z}}$ are mutually totally indiscernible in Claim 3.5.6, it follows for the same reason that $(a_i)_{i<0}+(b)+(a_i)_{i>0}$ and $\left(a'_j\right)_{j<0}+(b')+\left(a'_j\right)_{j>0}$ are mutually totally indiscernible, which implies the claim, except for the part that $(a_i)_{i\neq 0}$ is indiscernible over $bb'a'_0$ and $\left(a'_j\right)_{j\neq 0}$ is indiscernible over $bb'a_0$, which can be proved in a similar manner as follows.

Every formula $\varphi\left(x_1,\ldots,x_n;b,b',a'_0\right)$ is a Boolean combination of formulas of the form $t_1=t_2$ and $t_1\preccurlyeq t_2$, where t_1 and t_2 are terms in the variables $x_1,\ldots,x_n,b,b',a'_0$. These terms are all of the form $t_{I,J} := \bigwedge_{i\in I} x_i \wedge \bigwedge_{c\in J} c$ for $I\subseteq[n]$ and $J\subseteq\{b,b',a'_0\}$, not both empty. The value of $t_{I,J}\left(a_{i_1},\ldots,a_{i_n}\right)$ (for pairwise distinct $i_1,\ldots,i_n$) depends on $a_{i_1},\ldots,a_{i_n}$

only in the case where $|I| = 1$ and $J = \emptyset$. Even then, each a_i has the same height, so for formulas of the form $\varphi(x_1, \ldots, x_n) := t_{I_1,J_1} \preccurlyeq t_{I_2,J_2}$, the truth-value of $\varphi(a_{i_1}, \ldots, a_{i_n})$ does not depend on $i_1, \ldots, i_n$. And for $\varphi(x_1, \ldots, x_n) := t_{I_1,J_1} = t_{I_2,J_2}$, if $|I_1| = 1$ and $J_1 = \emptyset$, then $\varphi(a_{i_1}, \ldots, a_{i_n})$ holds if and only if t_{I_1,J_1} and t_{I_2,J_2} are equal as terms. So the truth values of these formulas do not depend on $i_1, \ldots, i_n$ either, hence $(a_i)_{i \neq 0}$ is indiscernible over $bb'a'_0$.

The sequence $\left(a'_j\right)_{j \neq 0}$ is indiscernible over $bb'a_0$ for the same reason, replacing a'_0 with a_0 and replacing $(a_i)_{i \neq 0}$ with $\left(a'_j\right)_{j \neq 0}$ in the above argument. □

Claim 3.5.8. *All pairs $\left(a_i, a'_j\right)$ with $i \neq 0$ or $j = 0$ have the same type over bb'.*

Proof. By quantifier elimination, every formula $\varphi(x, y; z, z')$ with x, y, z, z' singletons is equivalent to a Boolean combination of formulas of the form $t_1 \preccurlyeq t_2$ or $t_1 = t_2$, where t_1 and t_2 are terms. For $\emptyset \neq X \subseteq \{x, y, z, z'\}$, let $t_X(x, y; z, z') := \bigwedge_{w \in X} w$. Every term in the variables $x, y; z, z'$ is of this form. We have the following observations:

- if X intersects both $\{x, z\}$ and $\{y, z'\}$, then $t_X\left(a_i, a'_j; b, b'\right) = m \wedge m'$ for every $i, j \in \mathbb{Z}$;
- $t_{\{x,z\}}\left(a_i, a'_j; b, b'\right)$ is equal to $a_i \wedge b = m$ for $i \neq 0$, and to $a_0 \wedge b$ for $i = 0$;
- $t_{\{y,z'\}}\left(a_i, a'_j; b, b'\right)$ is equal to $a'_j \wedge b' = m'$ for $j \neq 0$, and to $a'_0 \wedge b'$ for $j = 0$;
- $t_{\{w\}} = w$ for $w \in \{x, y, z, z'\}$.

Thus the formula $t_X\left(a_i, a'_j; b, b'\right) = t_Y\left(a_i, a'_j; b, b'\right)$ is true if $X = Y$ or if both X and Y intersect both $\{x, z\}$ and $\{y, z'\}$, and false otherwise, with no dependence on i, j. The formula $t_X\left(a_i, a'_j; b, b'\right) \preccurlyeq t_Y\left(a_i, a'_j; b, b'\right)$ is true if X intersects both $\{x, z\}$ and $\{y, z'\}$, false if Y intersects both $\{x, z\}$ and $\{y, z'\}$ but X does not, true if $|Y| = 1$, false if $|X| = 1$ but $|Y| > 1$, and true if $X = Y$. None of those depend on i, j, and the only remaining case is when one of X and Y is $\{x, z\}$ and the other is $\{y, z'\}$. We have:

$$\models t_{\{x,z\}}\left(a_i, a'_j; b, b'\right) \preccurlyeq t_{\{y,z'\}}\left(a_i, a'_j; b, b'\right) \iff \models a_i \wedge b \preccurlyeq a'_j \wedge b' \iff i \neq 0 \vee j = 0;$$

$$\models t_{\{y,z'\}}\left(a_i, a'_j; b, b'\right) \preccurlyeq t_{\{x,z\}}\left(a_i, a'_j; b, b'\right) \iff \models a'_j \wedge b' \preccurlyeq a_i \wedge b \iff i = 0 \& j \neq 0.$$

Thus all atomic formulas with parameters from $\{b, b'\}$ have constant truth-value on every pair in $\left\{\left(a_i, a'_j\right) \mid i \neq 0 \vee j = 0\right\}$. That is, these pairs all have the same quantifier-free type over bb', and hence, by quantifier elimination, the same type. $\square$

This concludes the proof of Theorem 3.5.5. $\square$

Problem 3.5.9. Is there a semi-equational expansion of the theory of infinitely-branching dense valued trees?

3.5.3 Non weakly-semi-equational valued fields

In this section we demonstrate that many valued fields are not weakly semi-equational. By an ac-*valued field field* we mean a three-sorted structure $(K, k, \Gamma, \nu, \mathrm{ac})$ in the Denef-Pas language, where K is a field, $\nu : K \to \Gamma$ is a valuation, with (ordered) value group Γ and residue field k, and $\mathrm{ac} : K \to k$ the angular component map. As usual, $\mathcal{O} = \mathcal{O}_\nu$ denotes the valuation ring of ν, and for $x \in \mathcal{O}$, $\bar{x}$ denotes the residue of x in k.

Theorem 3.5.10. *Let K be an ac-valued field for which the residue field k contains a non-constant totally indiscernible sequence (for instance, if k is infinite and stable), and which eliminates quantifiers of the main field sort (for example, a Henselian ac-valued field of equicharacteristic 0 with an algebraically closed residue field). Then K is not weakly semi-equational.*

Towards the proof of this theorem, it will be useful to consider the following notion:

Definition 3.5.11. Let K be a field with valuation ν.

1. We say that $a_1, \ldots, a_n \in K$ are *valuationally independent* if, for every polynomial $f(x_1, \ldots, x_n) = \sum_i c_i x_1^{\alpha_{1,i}} \ldots x_n^{\alpha_{n,i}}$ (where i runs over some finite index set, $c_i, \alpha_{1,i}, \ldots, \alpha_{n,i} \in \mathbb{Z}$, and $(\alpha_{1,i}, \ldots, \alpha_{n,i}) \neq (\alpha_{1,j}, \ldots, \alpha_{n,j})$ for $i \neq j$) we have
$$\nu\left(f\left(a_1, \ldots, a_n\right)\right) = \min_i \nu\left(c_i a_1^{\alpha_{1,i}} \ldots a_n^{\alpha_{n,i}}\right).$$

That is, if the valuation of every polynomial applied to $a_1, \ldots, a_n$ is the minimum of the valuations of its monomials (including their coefficients).

2. An infinite set is *valuationally independent* if every finite subset is.

Example 3.5.12. 1. A set of elements with valuation 0 is valuationally independent if and only if their residues are algebraically independent.

2. In a valued field of pure characteristic, every set of elements whose valuations are $\mathbb{Z}$-linearly independent is valuationally independent. In mixed characteristic $(0,p)$, every set of elements whose valuations, together with $\nu(p)$, are $\mathbb{Z}$-linearly independent, is valuationally independent. In an ac-valued field, this is the only way for a set of elements with angular component 1 to be valuationally independent.

The rest of this section constitutes a proof of Theorem 3.5.10. We will show that the partitioned formula $\psi(x_1, x_2; y_1, y_2) := \nu(x_1 - y_1) < \nu(x_2 - y_2)$ is not a Boolean combination of weak semi-equations (this can be viewed as a strengthening of Theorem 3.5.5).

Without loss of generality we may assume that K is a monster model. By Lemma 3.5.2, it suffices to find b, b' and $(a_i)_{i\in\mathbb{Z}}, \left(a'_j\right)_{j\in\mathbb{Z}}$ in K such that the sequences $(a_i)_{i\in\mathbb{Z}}$ and $\left(a'_j\right)_{j\in\mathbb{Z}}$ are mutually indiscernible (so that rows and columns of the array $\left(a_i a'_j\right)_{i,j\in\mathbb{Z}}$ are indiscernible), $(a_i)_{i\neq 0}$ is indiscernible over $bb'\left(a'_j\right)_{j\in\mathbb{Z}}$, $\left(a'_j\right)_{j\neq 0}$ is indiscernible over $bb'(a_i)_{i\in\mathbb{Z}}$ (so that the rows and the columns of the array $\left(a_i a'_j\right)_{i,j\in\mathbb{Z}}$ with their 0-indexed elements removed are indiscernible over bb'), $\models \nu(a_i - b) < \nu\left(a'_j - b'\right) \iff i \neq 0 \vee j = 0$, and all pairs $\left(a_i, a'_j\right)$ with $i \neq 0$ or $j = 0$ have the same type over bb'.

To find these elements, first let $0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_5 < \gamma_6 \in \Gamma$ be an increasing *indiscernible* sequence of positive elements of the value group (exists by Ramsey and saturation). Let $a_\infty, a'_\infty \in K$ be such that $\nu(a_\infty) = \gamma_0$, $\nu(a'_\infty) = \gamma_1$, and $\mathrm{ac}(a_\infty) = \mathrm{ac}(a'_\infty) = 1$. Let $(\tilde{a}_i)_{i\in\mathbb{Z}} + \left(\tilde{b}\right)$ and $\left(\tilde{a}'_j\right)_{j\in\mathbb{Z}} + \left(\tilde{b}'\right)$ be arbitrary mutually totally indiscernible sequences in k (which exist by assumption on k and saturation, splitting a totally indiscernible sequence into two disjoint subsequences), and let $a_i := a_\infty + \alpha\,\mathrm{lift}(\tilde{a}_i)$

and $a'_j := a'_\infty + \beta \operatorname{lift}\left(\tilde{a}'_j\right)$ for $i, j \in \mathbb{Z}$, for some $\alpha, \beta \in K$ with $\nu(\alpha) = \gamma_2$, $\nu(\beta) = \gamma_3$, and $\operatorname{ac}(\alpha) = \operatorname{ac}(\beta) = 1$, where $\operatorname{lift}(x)$ is some arbitrary element of $\mathcal{O}$ such that $\overline{\operatorname{lift}(x)} = x$. Let b, b' be such that $\nu(a_0 - b) = \gamma_4$, $\nu(a'_0 - b') = \gamma_5$ and $\operatorname{ac}(a_0 - b) = \tilde{b} - \tilde{a}_0$, and $\operatorname{ac}(a'_0 - b') = \tilde{b}' - \tilde{a}'_0$. All of these elements are fixed for the rest of the section.

It is clear that $\models \nu(a_i - b) < \nu\left(a'_j - b'\right) \iff i \neq 0 \vee j = 0$, because $\nu(a_0 - b) = \gamma_4$, $\nu(a_i - b) = \gamma_2$ for $i \neq 0$, $\nu(a'_0 - b') = \gamma_5$, and $\nu\left(a'_j - b'\right) = \gamma_3$ for $j \neq 0$.

We will prove the following two claims. Given a sequence $(x_i)_{i \in I}$ and $J \subseteq I$, we will write x_J to denote the subsequence $(x_i : i \in J)$.

Claim 3.5.13. *1. Let $\varphi(x; z; w; b', a'_J)$ be a formula with parameters b' and a'_J for some $J \subseteq \mathbb{Z}$, tuples of variables x of sort K, z of sort k, and w of sort Γ_∞. Let I_1, I_2 be tuples of distinct indices from $\mathbb{Z}$, with $|I_1| = |I_2| = |x|$. Let $\sigma \in \operatorname{Aut}(k)$ be such that $\sigma(\tilde{a}_{I_1}) = \tilde{a}_{I_2}$ (preserving the ordering of the tuples), $\sigma(\tilde{a}'_J) = \tilde{a}'_J$, and $\sigma\left(\tilde{b}'\right) = \tilde{b}'$. Then for any tuples $c \in k^z, d \in \Gamma_\infty^w$ we have*

$$\models \varphi(a_{I_1}; c; d; b'; a'_J) \iff \models \varphi(a_{I_2}; \sigma(c); d; b'; a'_J).$$

2. Likewise, let $\varphi(y; z; w; b, a_I)$ be a formula with parameters b and a_I for some $I \subseteq \mathbb{Z}$, tuples of variables y of sort K, z of sort k, and w of sort Γ_∞. Let J_1, J_2 be tuples of distinct indices from $\mathbb{Z}$, with $|J_1| = |J_2| = |y|$, and let $\sigma \in \operatorname{Aut}(k)$ be such that $\sigma\left(\tilde{a}'_{J_1}\right) = \tilde{a}'_{J_2}$, $\sigma(\tilde{a}_I) = \tilde{a}_I$, and $\sigma\left(\tilde{b}\right) = \tilde{b}$. Then for any tuples $c \in k^z, d \in \Gamma_\infty^w$ we have

$$\models \varphi\left(a'_{J_1}; c; d; b; a_I\right) \iff \models \varphi\left(a'_{J_2}; \sigma(c); d; b; a_I\right).$$

Claim 3.5.14. *Let $\varphi(x; y; z; w; b; b')$ be a formula with parameters b, b', where x and y are single variables of sort K, and z and w are tuples of variables of sort k and Γ_∞, respectively. Let $\sigma_i \in \operatorname{Aut}(k)$ be such that $\sigma_i(\tilde{a}_i) = \tilde{b}$, $\sigma_i(\tilde{a}_0) = \tilde{a}_0$, $\sigma_i(\tilde{a}'_0) = \tilde{a}'_0$, and $\sigma_i\left(\tilde{b}'\right) = \tilde{b}'$. Let $\sigma'_j \in \operatorname{Aut}(k)$ be such that $\sigma'_j\left(\tilde{b}'\right) = \tilde{a}'_j$, $\sigma'_j(\tilde{a}'_0) = \tilde{a}'_0$, $\sigma'_j(\tilde{a}_i) = \tilde{a}_i$, and $\sigma'_j\left(\tilde{b}\right) = \tilde{b}$. Let $\pi \in \operatorname{Aut}(\Gamma_\infty)$ be such that $\pi(\gamma_2) = \gamma_4$, $\pi(\gamma_0) = \gamma_0$, $\pi(\gamma_1) = \gamma_1$, and $\pi(\gamma_5) = \gamma_5$, and let $\tau \in \operatorname{Aut}(\Gamma_\infty)$ be such that $\tau(\gamma_5) = \gamma_3$, $\tau(\gamma_0) = \gamma_0$, $\tau(\gamma_1) = \gamma_1$, and $\tau(\gamma_2) = \gamma_2$. Then, for*

$i,j\neq 0,\ c\in k^{z},\ \textit{and}\ d\in\Gamma_{\infty}^{w},\ \models\varphi\left(a_0;a_0';\sigma_i\left(c\right);\pi\left(d\right);b;b'\right)\iff\models\varphi\left(a_i;a_0';c;d;b;b'\right)\iff\models\varphi\left(a_i;a_j';\sigma_j'\left(c\right);\tau\left(d\right);b;b'\right)$.

Assuming these claims, from the $|z|=|w|=0$ case of Claim 3.5.13, we get that $\left(a_i\right)_{i\in\mathbb{Z}}$ is totally indiscernible over $b'\left(a_j'\right)_{j\in\mathbb{Z}}$, and $\left(a_j'\right)_{j\in\mathbb{Z}}$ is totally indiscernible over $b\left(a_i\right)_{i\in\mathbb{Z}}$. In particular $\left(a_i\right)_{i\in\mathbb{Z}}$ and $\left(a_j'\right)_{j\in\mathbb{Z}}$ are mutually totally indiscernible.

In describing $\left(a_i\right)_{i\in\mathbb{Z}},\left(a_j'\right)_{j\in\mathbb{Z}},b,b'$, we have made exactly the same assumptions about a_0 as about b, and the same assumptions about a_0' as about b', in the sense that if we replace a_0 with b or replace a_0' with b', the resulting elements $\left(a_i\right)_{i\in\mathbb{Z}},\left(a_j'\right)_{j\in\mathbb{Z}},b,b'$ could have come from the same construction. Thus, as Claim 3.5.13 implies that $\left(a_i\right)_{i\neq 0}$ is totally indiscernible over $a_0b'\left(a_j'\right)_{j\in\mathbb{Z}}$, and $\left(a_j'\right)_{j\neq 0}$ is totally indiscernible over $a_0'b\left(a_i\right)_{i\in\mathbb{Z}}$, it must also be the case that $\left(a_i\right)_{i\neq 0}$ is totally indiscernible over $bb'\left(a_j'\right)_{j\in\mathbb{Z}}$, and $\left(a_j'\right)_{j\neq 0}$ is totally indiscernible over $b'b\left(a_i\right)_{i\in\mathbb{Z}}$.

From the $|z|=|w|=0$ case of Claim 3.5.14, we get that

$$\operatorname{tp}\left(a_i,a_j'/b,b'\right)=\operatorname{tp}\left(a_i,a_0'/b,b'\right)=\operatorname{tp}\left(a_0,a_0'/b,b'\right)$$

for $i,j\neq 0$, hence all $\left(a_i,a_j'\right)$ with $i\neq 0$ or $j=0$ have the same type over bb'.

Thus these two claims establish the conditions needed for Lemma 3.5.2 to imply that $\nu\left(x_1-y_1\right)<\nu\left(x_2-y_2\right)$ is not a Boolean combination of weak semi-equations.

Both claims will be proved by induction on the parse tree of the formula φ (without parameters). There are five cases that must be considered:

Case 1. The formula φ is of the form $t_1\leq t_2$, where t_1,t_2 are terms of sort Γ_∞. Such terms are $\mathbb{N}$-linear combinations of variables of sort Γ_∞ and valuations of polynomials in variables of sort K; i.e. of the form $\boldsymbol{n}\cdot x+\boldsymbol{m}\cdot\nu\left(f\left(y\right)\right)$, where $x=\left(x_1,\ldots,x_{\ell_1}\right)$ is a tuple of variables of sort Γ_∞, y is a tuple of variables of sort K, f is a tuple of polynomials $(f_1(y),\ldots,f_{\ell_2}(y))$, $\boldsymbol{n}=\left(n_1,\ldots,n_{\ell_1}\right)\in\mathbb{N}^{|x|}$, $\boldsymbol{m}=\left(m_1,\ldots,m_{\ell_2}\right)\in\mathbb{N}^{|f|}$, $\nu\left(f(y)\right)$ is an abbreviation for the tuple $\left(\nu\left(f_1(y)\right),\ldots,\nu\left(f_\ell(y)\right)\right)$, and "$\cdot$" is the dot product.

Case 2. φ is of the form $t_1 =_k t_2$, where t_1, t_2 are terms of sort k. Terms of sort k are polynomials applied to variables of sort k and angular components of terms of sort K; i.e. of the form $f(x, \mathrm{ac}(g(y)))$, where f is a polynomial, $g = (g_1, \ldots, g_\ell)$ is a tuple of polynomials, x is a tuple of variables of sort k, y is a tuple of variables of sort K, and $\mathrm{ac}(g(y))$ is an abbreviation for the tuple $(\mathrm{ac}(g_1(y)), \ldots, \mathrm{ac}(g_\ell(y)))$. Since $t_1 =_k t_2$ if and only if $t_1 - t_2 =_k 0$, every formula of this form is equivalent to a formula of the form $f(x, \mathrm{ac}(g(y))) =_k 0$.

Case 3. φ is a Boolean combination of formulas for which the claim holds.

Case 4. φ is of the form $\exists u\, \psi$, where u is a variable of sort k, and the claim holds for ψ.

Case 5. φ is of the form $\exists u\, \psi$, where u is a variable of sort Γ_∞, and the claim holds for ψ.

There are four more cases for how φ could be constructed, which will not be checked individually because they follow from the previous five cases:

Case 6. φ is of the form $t_1 =_\Gamma t_2$, where t_1, t_2 are terms of sort Γ_∞. This is equivalent to $t_1 \le t_2 \wedge t_2 \le t_1$, and is thus redundant with Cases 1 and 3.

Case 7. φ is of the form $t_1 =_K t_2$, where t_1, t_2 are terms of sort K. This is equivalent to $\nu(t_1 - t_2) = \nu(0)$, and is thus redundant with Case 6.

Case 8. φ is of the form $\forall u\, \psi$, where u is a variable of sort k or Γ_∞. This is redundant with Cases 3, 4, and 5.

Case 9. φ is of the form $\exists u\, \psi$, or $\forall u\, \psi$, where u is a variable of sort K. This case can be neglected by quantifier elimination, since we can always pick a formula equivalent to φ which has no quantifiers of sort K.

The following auxiliary claims will be used.

Claim 3.5.15. *The elements $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ are $\mathbb{Z}$-linearly independent (in Γ viewed as a $\mathbb{Z}$-module). If K has mixed characteristic $(0, p)$, then $\nu(p), \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ are $\mathbb{Z}$-linearly independent.*

Proof. If $n_0\gamma_0 + \ldots + n_5\gamma_5 = 0$ with $n_0, \ldots, n_5 \in \mathbb{Z}$ not all 0, let $i \le 5$ be maximal such that

$n_i \neq 0$. Now $n_0\gamma_0+\ldots+n_i\gamma_i = 0$, and $n_i \neq 0$. By indiscernibility of the sequence $(\gamma_1, \ldots, \gamma_6)$, $n_0\gamma_0+\ldots+n_{i-1}\gamma_{i-1}+n_i\gamma_6 = 0$, but then $n_i(\gamma_i - \gamma_6) = 0$, contradicting that $n_i \neq 0$, $\gamma_i \neq \gamma_6$, and Γ is ordered and thus torsion-free. In mixed characteristic, the same argument can be repeated starting from $n_0\gamma_0 + \ldots + n_5\gamma_5 = m\nu(p)$ with $n_0, \ldots, n_5, m \in \mathbb{Z}$. $\square$

Claim 3.5.16. *The elements* $a_\infty, a'_\infty, (\alpha \operatorname{lift}(\tilde{a}_i))_{i\in\mathbb{Z}}, \left(\beta \operatorname{lift}\left(\tilde{a}'_j\right)\right)_{j\in\mathbb{Z}}, b - a_0, b' - a'_0$ *are valuationally independent.*

Proof. Define a valuation $\nu^* : \mathbb{Z}[u, v, x, y, z, w] \to \Gamma_\infty$ (with $|u| = |v| = |z| = |w| = 1$, $|x|$,$|y|$ arbitrary), by, for monomials (which in case of mixed characteristic is taken to include its coefficient),

$$\nu^*\left(nu^{r_\infty}x_1^{r_1}\ldots x_{|x|}^{r_{|x|}}v^{s_\infty}y_1^{s_1}\ldots y_{|y|}^{s_{|y|}}z^{t_1}w^{t_2}\right)$$
$$:= \nu(n) + r_\infty\gamma_0 + s_\infty\gamma_1 + \left(r_1 + \ldots + r_{|x|}\right)\gamma_2 + \left(s_1 + \ldots + s_{|y|}\right)\gamma_3 + t_1\gamma_4 + t_2\gamma_5,$$

and the valuation of a polynomial is the minimum of the valuations of its monomials. That way, for any $I, J \subseteq \mathbb{Z}$ with $|I| = |x|$ and $|J| = |y|$ we have:

$$\nu^*(f(u, v, x, y, z, w)) = \nu\left(f\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}(\tilde{a}_I), \beta \cdot \operatorname{lift}(\tilde{a}'_J), b - a_0, b' - a'_0\right)\right)$$

when f is a monomial (where $\alpha \cdot \operatorname{lift}(\tilde{a}_I) := (\alpha \operatorname{lift}(\tilde{a}_i))_{i\in I}$), and we need to prove that this holds for all polynomials f. Given a polynomial $f(u, v, x, y, z, w)$,

$$\nu^*(f) = \nu(n) + m_0\gamma_0 + m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 + m_4\gamma_4 + m_5\gamma_5$$

for some $n, m_0, m_1, m_2, m_3, m_4, m_5 \in \mathbb{N}$ (with $\nu(n), m_0, m_1, m_2, m_3, m_4, m_5$ unique by Claim 3.5.15). Let $\tilde{f}(u, v, x, y, z, w)$ be the sum of monomials in f of the same valuation as f, so that every monomial appearing in $\tilde{f}(u, v, x, y, z, w)$ has degree m_0 in u, degree m_1 in v, total degree m_2 in x, total degree m_3 in y, degree m_4 in z, degree m_5 in w, and has leading coefficient with valuation $\nu(n)$, and $\nu^*\left(f - \tilde{f}\right) > \nu^*(f)$. Thus

$$\frac{\tilde{f}(u, v, x, y, z, w)}{nu^{m_0}v^{m_1}z^{m_4}w^{m_5}}$$

is a non-zero polynomial in x, y, all coefficients having valuation 0, so it reduces under the residue map to a nonzero polynomial in x, y. Since the set of elements in the tuples $\tilde{a}_I, \tilde{a}'_J$ is algebraically independent (they come from an infinite indiscernible sequence), it follows that

$$\frac{\tilde{f}\left(u, v, \tilde{a}_I, \tilde{a}'_J, z, w\right)}{n u^{m_0} v^{m_1} z^{m_4} w^{m_5}} \neq 0,$$

and thus a lift of it,

$$\frac{\tilde{f}\left(u, v, \operatorname{lift}\left(\tilde{a}_I\right), \operatorname{lift}\left(\tilde{a}'_J\right), z, w\right)}{n u^{m_0} v^{m_1} z^{m_4} w^{m_5}},$$

has valuation 0. Thus

$$\nu\left(\tilde{f}\left(a_\infty, a'_\infty, \operatorname{lift}\left(\tilde{a}_I\right), \operatorname{lift}\left(\tilde{a}'_J\right), b - a_0, b' - a'_0\right)\right) = \nu\left(n\right) + m_0\gamma_0 + m_1\gamma_1 + m_4\gamma_4 + m_5\gamma_5$$

and, by homogeneity of $\tilde{f}$,

$$\begin{gathered}\nu\left(\tilde{f}\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}\left(\tilde{a}_I\right), \beta \cdot \operatorname{lift}\left(\tilde{a}'_J\right), b - a_0, b' - a'_0\right)\right) \\ = \nu\left(n\right) + m_0\gamma_0 + m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 + m_4\gamma_4 + m_5\gamma_5 = \nu^*\left(f\right).\end{gathered}$$

We have

$$\nu\left(\left(f - \tilde{f}\right)\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}\left(\tilde{a}_I\right), \beta \cdot \operatorname{lift}\left(\tilde{a}'_J\right), b - a_0, b' - a'_0\right)\right) \geq \nu^*\left(f - \tilde{f}\right) > \nu^*\left(f\right)$$

(the first inequality holds by the ultrametric property, combined with the fact that it holds for monomials), so it follows that

$$\nu\left(f\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}\left(\tilde{a}_I\right), \beta \cdot \operatorname{lift}\left(\tilde{a}'_J\right), b - a_0, b' - a'_0\right)\right) = \nu^*\left(f\right)$$

as well, as desired. □

We are ready to prove the two claims.

Proof of Claim 3.5.13. Let $\varphi\left(x; z; w; b'; a'_J\right)$ with $x = \left(x_1, \ldots, x_{|x|}\right)$ and I_1, I_2, σ, c, d be as in the statement of the claim, and we analyze the five cases described above. We will assume without loss of generality that $j_1 = 0$, where $J = (j_1, \ldots, j_{|J|})$ (since if 0 appears somewhere else in J, J may be re-ordered, and if 0 does not appear in J, it may be added).

The proof for the part regarding a formula $\varphi(y; z; w; b; a_I)$ is identical, switching the roles of $(a_i)_{i\in\mathbb{Z}}$ and $\left(a'_j\right)_{j\in\mathbb{Z}}$, replacing b' with b', and replacing γ_5 with γ_4.

Case 1. $\varphi(x; z; w; b'; a'_J)$ is of the form $\boldsymbol{n}_1 \cdot w + \boldsymbol{m}_1 \cdot \nu(g(x, b', a'_J)) \leq \boldsymbol{n}_2 \cdot w + \boldsymbol{m}_2 \cdot \nu(h(x, b', a'_J))$.

It is enough to show that for any polynomial $f(x, q, y)$ (with $|x| = |I_1|, |y| = |J|, |q| = 1$), we have $\nu(f(a_{I_1}, b', a'_J)) = \nu(f(a_{I_2}, b', a'_J))$, because then

$$\boldsymbol{m}_1 \cdot \nu(g(a_{I_1}, b', a'_J)) = \boldsymbol{m}_1 \cdot \nu(g(a_{I_2}, b', a'_J)) \text{ and}$$
$$\boldsymbol{m}_2 \cdot \nu(h(a_{I_1}, b', a'_J)) = \boldsymbol{m}_2 \cdot \nu(h(a_{I_2}, b', a'_J)),$$

so

$$\models \boldsymbol{n}_1 \cdot d + \boldsymbol{m}_1 \cdot \nu(g(a_{I_1}, b', a'_J)) \leq \boldsymbol{n}_2 \cdot d + \boldsymbol{m}_2 \cdot \nu(h(a_{I_1}, b', a'_J)) \iff$$
$$\models \boldsymbol{n}_1 \cdot d + \boldsymbol{m}_1 \cdot \nu(g(a_{I_2}, b', a'_J)) \leq \boldsymbol{n}_2 \cdot d + \boldsymbol{m}_2 \cdot \nu(h(a_{I_2}, b', a'_J)).$$

Given a polynomial $f(x, q, y)$, let

$$f^*(u, v, x, y, q) := f\left(x_1 + u, \ldots, x_{|x|} + u, q + y_1 + v, y_1 + v, \ldots, y_{|y|} + v\right),$$

with $|u| = |v| = |q| = 1$, $|x| = |I_1|$, $|y| = |J|$, so that

$$f^*(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}(\tilde{a}_{I_i}), \beta \cdot \operatorname{lift}(\tilde{a}'_J), b' - a'_0) = f(a_{I_i}, b', a'_J)$$

for $i \in \{1, 2\}$ (using that $a_i = a_\infty + \alpha \cdot \operatorname{lift}(\tilde{a}_i)$ and $a'_j = a'_\infty + \beta \cdot \operatorname{lift}\left(\tilde{a}'_j\right)$ and $j_1 = 0$). Since

$$\nu\Bigg(n a_\infty^{r_\infty} (\alpha \operatorname{lift}(\tilde{a}_{i_1}))^{r_1} \ldots \left(\alpha \operatorname{lift}\left(\tilde{a}_{i_{|x|}}\right)\right)^{r_{|x|}} (a'_\infty)^{s_\infty} \cdot$$
$$\cdot \left(\beta \operatorname{lift}\left(\tilde{a}'_{j_1}\right)\right)^{s_1} \ldots \left(\beta \operatorname{lift}\left(\tilde{a}'_{j_{|y|}}\right)\right)^{s_{|y|}} (b' - a'_0)^t\Bigg)$$
$$= \nu(n) + r_\infty \gamma_0 + s_\infty \gamma_1 + \left(r_1 + \ldots + r_{|x|}\right) \gamma_2 + \left(s_1 + \ldots + s_{|y|}\right) \gamma_3 + t\gamma_5,$$

regardless of $i_1, \ldots, i_{|x|}$, if we let

$$nu^{r_\infty} v^{s_\infty} x_1^{r_1} \ldots x_{|x|}^{r_{|x|}} y_1^{s_1} \ldots y_{|y|}^{s_{|y|}} q^t$$

be a monomial in $f^* (u, v, x, y, q)$ minimizing

$$\nu (n) + r_\infty \gamma_0 + s_\infty \gamma_1 + \left(r_1 + \ldots + r_{|x|}\right) \gamma_2 + \left(s_1 + \ldots + s_{|y|}\right) \gamma_3 + t\gamma_5,$$

then by Claim 3.5.16,

$$\begin{aligned} &\nu \left(f \left(a_{I_i}, b', a'_J\right)\right) = \nu \left(f^* \left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift} \left(\tilde{a}_{I_i}\right), \beta \cdot \operatorname{lift} \left(\tilde{a}'_J\right), b' - a'_0\right)\right) \\ &= \nu (n) + r_\infty \gamma_0 + s_\infty \gamma_1 + \left(r_1 + \ldots + r_{|x|}\right) \gamma_2 + \left(s_1 + \ldots + s_{|y|}\right) \gamma_3 + t\gamma_5 \end{aligned}$$

for $i \in \{1, 2\}$.

Case 2. $\varphi (x; z; w; b'; a'_J)$ is of the form $f (z, \operatorname{ac} (g (x, b, a'_J))) =_k 0$.

It is enough to show that $f (\sigma (c), \operatorname{ac} (g (a_{I_2}, b', a'_J))) = \sigma (f (c, \operatorname{ac} (g (a_{I_1}, b', a'_J))))$, for which it is in turn enough to show that $\operatorname{ac} (g (a_{I_2}, b', a'_J)) = \sigma (\operatorname{ac} (g (a_{I_1}, b', a'_J)))$. Since $a_i = a_\infty + \alpha \cdot \operatorname{lift} (\tilde{a}_i)$ and $a'_j = a'_\infty + \beta \cdot \operatorname{lift} \left(\tilde{a}'_j\right)$, there is a polynomial $h (u, v, x, y, q)$ (with $|u| = |v| = |q| = 1$, $|x| = |I_1|$, $|y| = |J|$) such that

$$h \left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift} \left(\tilde{a}_{I_i}\right), \beta \cdot \operatorname{lift} \left(\tilde{a}'_J\right), b' - a'_0\right) = g \left(a_{I_i}, b', a'_J\right)$$

for $i \in \{1, 2\}$. As in the proof of Case 1, there are $n, m_0, m_1, m_2, m_3, m_5 \in \mathbb{N}$ such that

$$\begin{aligned} &\nu \left(h \left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift} \left(\tilde{a}_{I_i}\right), \beta \cdot \operatorname{lift} \left(\tilde{a}'_J\right), b' - a'_0\right)\right) \\ &= \nu (n) + m_0 \gamma_0 + m_1 \gamma_1 + m_2 \gamma_2 + m_3 \gamma_3 + m_5 \gamma_5 \end{aligned}$$

for $i \in \{1, 2\}$. Let $\tilde{h} (u, v, x, y, q)$ be the sum of monomials in h with degree m_0 in u, degree m_1 in v, total degree m_2 in x, total degree m_3 in y, degree m_5 in q, and whose coefficient has valuation $\nu(n)$. That way

$$\begin{aligned}
&\nu\left(\tilde{h}\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}\left(\tilde{a}_{I_i}\right), \beta \cdot \operatorname{lift}\left(\tilde{a}'_J\right), b' - a'_0\right)\right) \\
&= \nu(n) + m_0\gamma_0 + m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 + m_5\gamma_5, \text{ and} \\
&\nu\left(\left(h - \tilde{h}\right)\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}\left(\tilde{a}_{I_i}\right), \beta \cdot \operatorname{lift}\left(\tilde{a}'_J\right), b' - a'_0\right)\right) \\
&> \nu(n) + m_0\gamma_0 + m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 + m_5\gamma_5.
\end{aligned}$$

Then

$$h^*(x, y) := \frac{\tilde{h}(u, v, x, y, q)}{n u^{m_0} v^{m_1} q^{m_5}}$$

is a non-zero polynomial in x, y, all coefficients having valuation 0, so it reduces under the residue map to a nonzero polynomial in x, y. Since $\tilde{a}_{I_i}, \tilde{a}'_J$ are algebraically independent (by indiscernibility), it follows that $\overline{h^*}\left(\tilde{a}_{I_i}, \tilde{a}'_J\right) \neq 0$, so $h^*\left(\operatorname{lift}\left(\tilde{a}_{I_i}\right), \operatorname{lift}\left(\tilde{a}'_J\right)\right)$ has valuation 0, and hence its angular component is its residue, $\overline{h^*}\left(\tilde{a}_{I_i}, \tilde{a}'_J\right)$. We have

$$\overline{h^*}\left(\tilde{a}_{I_2}, \tilde{a}'_J\right) = \overline{h^*}\left(\sigma\left(\tilde{a}_{I_1}\right), \sigma\left(\tilde{a}'_J\right)\right) = \sigma\left(\overline{h^*}\left(\tilde{a}_{I_1}, \tilde{a}'_J\right)\right).$$

Thus

$$\begin{aligned}
&\operatorname{ac}\left(\tilde{h}\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}\left(\tilde{a}_{I_2}\right), \beta \cdot \operatorname{lift}\left(\tilde{a}'_J\right), b' - a'_0\right)\right) \\
&= \operatorname{ac}\left(n a_\infty^{m_0}\left(a'_\infty\right)^{m_1} \alpha^{m_2} \beta^{m_3}\left(b' - a'_0\right)^{m_5}\right) \operatorname{ac}\left(\frac{\tilde{h}\left(a_\infty, a'_\infty, \operatorname{lift}\left(\tilde{a}_{I_2}\right), \operatorname{lift}\left(\tilde{a}'_J\right), b' - a'_0\right)}{n a_\infty^{m_0}\left(a'_\infty\right)^{m_1}\left(b' - a'_0\right)^{m_5}}\right) \\
&= \operatorname{ac}(n)\left(\tilde{a}'_0 - \tilde{b}'\right)^{m_5} \overline{h^*}\left(\tilde{a}_{I_2}, \tilde{a}'_J\right) = \operatorname{ac}(n)\left(\tilde{a}'_0 - \tilde{b}'\right)^{m_5} \sigma\left(\overline{h^*}\left(\tilde{a}_{I_1}, \tilde{a}'_J\right)\right) \\
&= \sigma\left(\operatorname{ac}(n)\left(\tilde{a}'_0 - \tilde{b}'\right)^{m_5} \overline{h^*}\left(\tilde{a}_{I_1}, \tilde{a}'_J\right)\right) \\
&= \sigma\Bigg(\operatorname{ac}\left(n a_\infty^{m_0}\left(a'_\infty\right)^{m_1} \alpha^{m_2} \beta^{m_3}\left(b' - a'_0\right)^{m_5}\right) \cdot \\
&\qquad \cdot \operatorname{ac}\left(\frac{\tilde{h}\left(a_\infty, a'_\infty, \operatorname{lift}\left(\tilde{a}_{I_1}\right), \operatorname{lift}\left(\tilde{a}'_J\right), b' - a'_0\right)}{n a_\infty^{m_0}\left(a'_\infty\right)^{m_1}\left(b' - a'_0\right)^{m_5}}\right)\Bigg) \\
&= \sigma\left(\operatorname{ac}\left(\tilde{h}\left(a_\infty, a'_\infty, \alpha \cdot \operatorname{lift}\left(\tilde{a}_{I_1}\right), \beta \cdot \operatorname{lift}\left(\tilde{a}'_J\right), b' - a'_0\right)\right)\right).
\end{aligned}$$

Since

$$\nu\left(\left(h-\tilde{h}\right)\left(a_{\infty},a'_{\infty},\alpha\cdot\operatorname{lift}\left(\tilde{a}_{I_i}\right),\beta\cdot\operatorname{lift}\left(\tilde{a}'_J\right),b'-a'_0\right)\right)$$
$$>\nu\left(h\left(a_{\infty},a'_{\infty},\alpha\cdot\operatorname{lift}\left(\tilde{a}_{I_i}\right),\beta\cdot\operatorname{lift}\left(\tilde{a}'_J\right),b'-a'_0\right)\right),$$

we have

$$\begin{aligned}\operatorname{ac}\left(g\left(a_{I_i},b',a'_J\right)\right)&=\operatorname{ac}\left(h\left(a_{\infty},a'_{\infty},\alpha\cdot\operatorname{lift}\left(\tilde{a}_{I_i}\right),\beta\cdot\operatorname{lift}\left(\tilde{a}'_J\right),b'-a'_0\right)\right)\\&=\operatorname{ac}\left(\tilde{h}\left(a_{\infty},a'_{\infty},\alpha\cdot\operatorname{lift}\left(\tilde{a}_{I_i}\right),\beta\cdot\operatorname{lift}\left(\tilde{a}'_J\right),b'-a'_0\right)\right),\end{aligned}$$

hence

$$\operatorname{ac}\left(g\left(a_{I_2},b',a'_J\right)\right)=\operatorname{ac}\left(\tilde{h}\left(a_{\infty},a'_{\infty},\alpha\cdot\operatorname{lift}\left(\tilde{a}_{I_2}\right),\beta\cdot\operatorname{lift}\left(\tilde{a}'_J\right),b-a'_0\right)\right)$$
$$=\sigma\left(\operatorname{ac}\left(\tilde{h}\left(a_{\infty},a'_{\infty},\alpha\cdot\operatorname{lift}\left(\tilde{a}_{I_1}\right),\beta\cdot\operatorname{lift}\left(\tilde{a}'_J\right),b'-a'_0\right)\right)\right)=\sigma\left(\operatorname{ac}\left(g\left(a_{I_1},b',a'_J\right)\right)\right)$$

as desired.

Case 3. Clear.

Case 4. $\varphi\left(x;z;w;b';a'_J\right)$ is of the form $\exists u\,\psi\left(x;z,u;w;b';a'_J\right)$, where u is a variable of sort k, and the claim holds for ψ.

If $\models\varphi\left(a_{I_1};c;d;b';a'_J\right)$, then $\models\psi\left(a_{I_1};c,e;d;b';a'_J\right)$ for some $e\in k$. Then we have $\models\psi\left(a_{I_2};\sigma\left(c\right),\sigma\left(e\right);d;b';a'_J\right)$, so $\models\varphi\left(a_{I_2};\sigma\left(c\right);d;b';a'_J\right)$.

Case 5. $\varphi\left(x;z;w;b';a'_J\right)$ is of the form $\exists u\,\psi\left(x;z;w,u;b';a'_J\right)$, where u is a variable of sort Γ_{∞}, and the claim holds for ψ.

If $\models\varphi\left(a_{I_1};c;d;b';a'_J\right)$, then $\models\psi\left(a_{I_1};c;d,e;b';a'_J\right)$ for some $e\in\Gamma_{\infty}$. Then we have $\models\psi\left(a_{I_2};\sigma\left(c\right);d,e;b';a'_J\right)$, so $\models\varphi\left(a_{I_2};\sigma\left(c\right);d;b';a'_J\right)$. □

Proof of Claim 3.5.14. Let $\varphi\left(x;y;z;w;b;b'\right),\sigma_i,\sigma'_j,\pi,\tau$ be as in the claim.

Case 1. $\varphi\left(x;y;z;w;b;b'\right)$ is of the form $\boldsymbol{n}_1\cdot w+\boldsymbol{m}_1\cdot\nu\left(g\left(x,y,b,b'\right)\right)\leq\boldsymbol{n}_2\cdot w+\boldsymbol{m}_2\cdot\nu\left(h\left(x,y,b,b'\right)\right)$.

It is enough to show that for any polynomial $f(x,y,u,u')$,

$$\pi^{-1}\left(\nu\left(f\left(a_0,a_0',b,b'\right)\right)\right)=\nu\left(f\left(a_i,a_0',b,b'\right)\right)=\tau^{-1}\left(\nu\left(f\left(a_i,a_j',b,b'\right)\right)\right)$$

for $i,j\neq 0$, because then

$$\boldsymbol{n}_1\cdot\pi(d)+\boldsymbol{m}_1\cdot\nu\left(g\left(a_0,a_0',b,b'\right)\right)=\pi\left(\boldsymbol{n}_1\cdot d+\boldsymbol{m}_1\cdot\nu\left(g\left(a_i,a_0',b,b'\right)\right)\right)\text{ and}$$
$$\boldsymbol{n}_1\cdot\tau(d)+\boldsymbol{m}_1\cdot\nu\left(g\left(a_i,a_j',b,b'\right)\right)=\tau\left(\boldsymbol{n}_1\cdot d+\boldsymbol{m}_1\cdot\nu\left(g\left(a_i,a_0',b,b'\right)\right)\right)$$

for $i,j\neq 0$, and likewise for $\boldsymbol{n}_2,\boldsymbol{m}_2,h$, and, as π and τ preserve order, this implies

$$\models\varphi\left(a_0;a_0';\sigma_i(c);\pi(d);b;b'\right)\iff\models\varphi\left(a_i;a_0';c;d;b;b'\right)$$
$$\iff\models\varphi\left(a_i;a_j';\sigma_j'(c);\tau(d);b;b'\right).$$

To show this, let $f^*(x,y,u,v):=f(x+u,y+v,u,v)$. By Claim 3.5.15 and the choice of these elements, for $i,j\in\mathbb{Z}$, the valuations of a_i-b, $a_j'-b'$, b, and b' are $\mathbb{Z}$-linearly independent (together with $\nu(p)$ if the characteristic is mixed), and hence these are valuationally independent. Let $nx^{e_1}y^{e_2}u^{e_3}v^{e_4}$ be the monomial in $f^*(x,y,u,v)$ minimizing $\nu(n)+e_1\gamma_2+e_2\gamma_5+e_3\gamma_0+e_4\gamma_1$, so that by valuational independence,

$$\nu\left(f^*\left(a_i-b,a_0'-b',b,b'\right)\right)=\nu(n)+e_1\gamma_2+e_2\gamma_5+e_3\gamma_0+e_4\gamma_1.$$

This monomial is unique by linear independence (Claim 3.5.15). Since π and τ preserve order, this monomial also minimizes

$$\pi\left(\nu(n)+e_1\gamma_2+e_2\gamma_5+e_3\gamma_0+e_4\gamma_1\right)=\nu(n)+e_1\gamma_4+e_2\gamma_5+e_3\gamma_0+e_4\gamma_1$$
$$=\nu\left(f^*\left(a_0-b,a_0'-b',b,b'\right)\right),\text{ and}$$
$$\tau\left(\nu(n)+e_1\gamma_2+e_2\gamma_5+e_3\gamma_0+e_4\gamma_1\right)=\nu(n)+e_1\gamma_2+e_2\gamma_3+e_3\gamma_0+e_4\gamma_1$$
$$=\nu\left(f^*\left(a_i-b,a_j'-b',b,b'\right)\right)$$

for $i,j\neq 0$.

Case 2. $\varphi(x;y;z;w;b;b')$ is of the form $f\left(z,\operatorname{ac}\left(g\left(x,y,b,b'\right)\right)\right)=_k 0$.

It is enough to show that

$$\sigma_i\left(f\left(c, \operatorname{ac}\left(g\left(a_i, a_0', b, b'\right)\right)\right)\right) = f\left(\sigma_i\left(c\right), \operatorname{ac}\left(g\left(a_0, a_0', b, b'\right)\right)\right) \text{ and}$$
$$\sigma_j'\left(f\left(c, \operatorname{ac}\left(g\left(a_i, a_0', b, b'\right)\right)\right)\right) = f\left(\sigma_j'\left(c\right), \operatorname{ac}\left(g\left(a_i, a_j', b, b'\right)\right)\right),$$

for which it is in turn enough to show that

$$\sigma_i\left(\operatorname{ac}\left(g\left(a_i, a_0', b, b'\right)\right)\right) = \operatorname{ac}\left(g\left(a_0, a_0', b, b'\right)\right) \text{ and}$$
$$\sigma_j'\left(\operatorname{ac}\left(g\left(a_i, a_0', b, b'\right)\right)\right) = \operatorname{ac}\left(g\left(a_i, a_j', b, b'\right)\right).$$

Let $h\left(x, y, u, v\right) := g\left(x + u, y + v, u, v\right)$. Let $nx^{m_1}y^{m_2}u^{m_3}v^{m_4}$ be the (unique, by Claim 3.5.15) monomial in $h\left(x, y, u, v\right)$ minimizing $\nu\left(n\right) + m_1\gamma_2 + m_2\gamma_5 + m_3\gamma_0 + m_4\gamma_1$, so that by valuational independence, $\nu\left(h\left(a_i - b, a_0' - b', b, b'\right)\right) = \nu\left(n\right) + m_1\gamma_2 + m_2\gamma_5 + m_3\gamma_0 + m_4\gamma_1$. Since π and τ preserve order, this monomial also minimizes $\nu\left(n\right) + m_1\gamma_4 + m_2\gamma_5 + m_3\gamma_0 + m_4\gamma_1$ and $\nu\left(n\right) + m_1\gamma_2 + m_2\gamma_3 + m_3\gamma_0 + m_4\gamma_1$. For $i \neq 0$,

$$\begin{aligned} &\operatorname{ac}\left(n\left(a_i - b\right)^{m_1}\left(a_0' - b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\ &= \operatorname{ac}\left(n\right)\left(\operatorname{ac}\left(\alpha\right)\left(\tilde{a}_i - \tilde{a}_0\right)\right)^{m_1}\left(\tilde{b}' - \tilde{a}_0'\right)^{m_2} \operatorname{ac}\left(a_\infty\right)^{m_3} \operatorname{ac}\left(a_\infty'\right)^{m_4} \\ &= \operatorname{ac}\left(n\right)\left(\tilde{a}_i - \tilde{a}_0\right)^{m_1}\left(\tilde{b}' - \tilde{a}_0'\right)^{m_2}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\operatorname{ac}\left(n\left(a_0 - b\right)^{m_1}\left(a_0' - b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\ &= \operatorname{ac}\left(n\right)\left(\tilde{b} - \tilde{a}_0\right)^{m_1}\left(\tilde{b}' - \tilde{a}_0'\right)^{m_2} \operatorname{ac}\left(a_\infty\right)^{m_3} \operatorname{ac}\left(a_\infty'\right)^{m_4} \\ &= \operatorname{ac}\left(n\right)\left(\tilde{b} - \tilde{a}_0\right)^{m_1}\left(\tilde{b}' - \tilde{a}_0'\right)^{m_2} \\ &= \sigma_i\left(\operatorname{ac}\left(n\right)\left(\tilde{a}_i - \tilde{a}_0\right)^{m_1}\left(\tilde{b}' - \tilde{a}_0'\right)^{m_2}\right). \end{aligned}$$

And for $i, j \neq 0$ we have

$$\begin{aligned}
&\operatorname{ac}\left(n\left(a_i-b\right)^{m_1}\left(a_j'-b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\
&\operatorname{ac}(n)\left(\operatorname{ac}(\alpha)\left(\tilde{a}_i-\tilde{a}_0\right)\right)^{m_1}\left(\operatorname{ac}(\beta)\left(\tilde{a}_j'-\tilde{a}_0'\right)\right)^{m_2} \operatorname{ac}\left(a_\infty\right)^{m_3} \operatorname{ac}\left(a_\infty'\right)^{m_4} \\
&=\operatorname{ac}(n)\left(\tilde{a}_i-\tilde{a}_0\right)^{m_1}\left(\tilde{a}_j'-\tilde{a}_0'\right)^{m_2} \\
&=\sigma_j'\left(\operatorname{ac}(n)\left(\tilde{a}_i-\tilde{a}_0\right)^{m_1}\left(\tilde{b}'-\tilde{a}_0'\right)^{m_2}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\nu\left(h\left(a_i-b, a_0'-b', b, b'\right)-n\left(a_i-b\right)^{m_1}\left(a_0'-b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\
&>\nu\left(h\left(a_i-b, a_0'-b', b, b'\right)\right),
\end{aligned}$$

we have

$$\begin{aligned}
\operatorname{ac}\left(h\left(a_i-b, a_0'-b', b, b'\right)\right) &=\operatorname{ac}\left(n\left(a_i-b\right)^{m_1}\left(a_0'-b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\
&=\operatorname{ac}(n)\left(\tilde{a}_i-\tilde{a}_0\right)^{m_1}\left(\tilde{b}'-\tilde{a}_0'\right)^{m_2}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\nu\left(h\left(a_0-b, a_0'-b', b, b'\right)-n\left(a_0-b\right)^{m_1}\left(a_0'-b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\
&>\nu\left(h\left(a_0-b, a_0'-b', b, b'\right)\right),
\end{aligned}$$

so

$$\begin{aligned}
\operatorname{ac}\left(h\left(a_0-b, a_0'-b', b, b'\right)\right) &=\operatorname{ac}\left(n\left(a_0-b\right)^{m_1}\left(a_0'-b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\
&=\sigma_i\left(\operatorname{ac}(n)\left(\tilde{a}_i-\tilde{a}_0\right)^{m_1}\left(\tilde{b}'-\tilde{a}_0'\right)^{m_2}\right).
\end{aligned}$$

And

$$\begin{aligned}
&\nu\left(h\left(a_i-b, a_j'-b', b, b'\right)-n\left(a_i-b\right)^{m_1}\left(a_j'-b'\right)^{m_2} b^{m_3}\left(b'\right)^{m_4}\right) \\
&>\nu\left(h\left(a_i-b, a_j'-b', b, b'\right)\right),
\end{aligned}$$

so

$$\begin{aligned}\operatorname{ac}\left(h\left(a_i-b,a'_j-b',b,b'\right)\right)&=\operatorname{ac}\left(n\left(a_i-b\right)^{m_1}\left(a'_j-b'\right)^{m_2}b^{m_3}\left(b'\right)^{m_4}\right)\\&=\sigma'_j\left(\operatorname{ac}\left(n\right)\left(\tilde{a}_i-\tilde{a}_0\right)^{m_1}\left(\tilde{b}'-\tilde{a}'_0\right)^{m_2}\right).\end{aligned}$$

Since $g\left(a_i,a'_0,b,b'\right)=h\left(a_i-b,a'_0-b',b,b'\right)$, $g\left(a_0,a'_0,b,b'\right)=h\left(a_0-b,a'_0-b',b,b'\right)$, and $g\left(a_i,a'_j,b,b'\right)=h\left(a_i-b,a'_j-b',b,b'\right)$, this is what we wanted to show.

Case 3. Clear.

Case 4. $\varphi\left(x;y;z;w;b;b'\right)$ is of the form $\exists u\,\psi\left(x;y;z,u;w;b;b'\right)$, where u is a variable of sort k, and the claim holds for ψ.

For $i,j\neq 0$, if $\models\varphi\left(a_i;a'_0;c;d;b;b'\right)$, then $\models\psi\left(a_i;a'_0;c,e;d;b;b'\right)$ for some $e\in k$. Then $\models\psi\left(a_0;a'_0;\sigma_i\left(c\right),\sigma_i\left(e\right);\pi\left(d\right);b;b'\right)$ and $\models\psi\left(a_i;a'_j;\sigma'_j\left(c\right),\sigma'_j\left(e\right);\tau\left(d\right);b;b'\right)$, so $\models\varphi\left(a_0;a'_0;\sigma_i\left(c\right);\pi\left(d\right);b;b'\right)$ and $\models\varphi\left(a_i;a'_j;\sigma'_j\left(c\right);\tau\left(d\right);b;b'\right)$. Note that each of these implications are reversible.

Case 5. $\varphi\left(x;y;z;w;b;b'\right)$ is of the form $\exists u\,\psi\left(x;y;z;w,u;b;b'\right)$, where u is a variable of sort Γ_∞, and the claim holds for ψ.

For $i,j\neq 0$, if $\models\varphi\left(a_i;a'_0;c;d;b;b'\right)$, then $\models\psi\left(a_i;a'_0;c;d,e;b;b'\right)$ for some $e\in\Gamma_\infty$. Then $\models\psi\left(a_0;a'_0;\sigma_i\left(c\right);\pi\left(d\right),\pi\left(e\right);b;b'\right)$ and $\models\psi\left(a_i;a'_j;\sigma'_j\left(c\right);\tau\left(d\right),\tau\left(e\right);b;b'\right)$, so $\models\varphi\left(a_0;a'_0;\sigma_i\left(c\right);\pi\left(d\right);b;b'\right)$ and $\models\varphi\left(a_i;a'_j;\sigma'_j\left(c\right);\tau\left(d\right);b;b'\right)$. Since π and τ are bijective, each of these implications is reversible. $\square$

This concludes the proof of Theorem 3.5.10.

Remark 3.5.17. This proof of Theorem 3.5.10 also applies to any reduct of an ac-valued field K whose residue field has a non-constant totally indiscernible sequence to a language $\mathcal{L}\subseteq\mathcal{L}_{\text{Denef-Pas}}$ such that $\mathcal{L}$ contains the relation $\nu\left(x_1-y_1\right)<\nu\left(x_2-y_2\right)$, and every $\mathcal{L}$-formula is equivalent to a Boolean combination of $\mathcal{L}_{\text{Denef-Pas}}$-formulas with no quantifiers of the main sort. This gives us further examples of NIP theories that are not weakly semi-equational, such as:

1. A Henselian valued field of equicharacteristic 0 with algebraically closed residue field.
2. An algebraically closed valued field (of any characteristic).
3. The reduct of either of the above to a valued vector space or valued abelian group.
4. A generic abstract ultrametric space: a two-sorted structure $(\mathcal{M}, \Gamma_\infty)$, with a linear order $\leq$ on Γ_∞ that is dense with maximal element $\infty \in \Gamma_\infty$ and no minimal element, and a function $\nu : \mathcal{M}^2 \to \Gamma_\infty$, such that $\nu(x, y) = \infty \iff x = y$, $\nu(x, y) = \nu(y, x)$, and $\nu(x, z) \geq \max(\nu(x, y), \nu(y, z))$, and such that for every $\gamma \in \Gamma$ and $a \in \mathcal{M}$, there are $(b_i)_{i \in \mathbb{N}}$ in $\mathcal{M}$ such that $\nu(a, b_i) = \nu(b_i, b_j) = \gamma$ for $i, j \in \mathbb{N}$.

Remark 3.5.18. If K is a valued field with infinite residue field k, which eliminates quantifiers of the main sort in the 3-sorted language with sorts K, k, and Γ_∞, field structures on K and k, an ordered monoid structure on Γ_∞, the valuation map $\nu : K \to \Gamma_\infty$, and a residue map $\mathrm{res} : K \to k$ which sends elements of $\mathcal{O}$ to their residue and everything else to 0, then a similar argument to the proof of Theorem 3.5.10 can be carried out without the assumption that k has a non-constant totally indiscernible sequence. Then $(\tilde{a}_i)_{i \in \mathbb{Z}}, \tilde{b}, (\tilde{a}'_j)_{j \in \mathbb{Z}}$,
$\tilde{b}' \in k$ need only be assumed to be algebraically independent, and the automorphisms σ, σ_i, and σ'_j of k mentioned in Claims 3.5.13 and 3.5.14 are replaced with the identity on k. The only steps in the proof affected by these changes are Case 2 in the proofs of each of Claims 3.5.13 and 3.5.14. But these steps become much simpler, because the relevant formulas, rather than $f(x, \mathrm{ac}(g(y))) =_k 0$ (for variables x of sort k and y of sort K), are $f(x, \mathrm{res}(g(y))) =_k 0$, and in both claims, the only conditions we need to verify involve plugging in elements with residue 0 into the polynomials g, making the claims easy to verify. It is unknown to the authors whether this remark applies to any valued fields not already covered by Theorem 3.5.10 and Remark 3.5.17.

Problem 3.5.19. Is the field $\mathbb{Q}_p$ semi-equational? (It is weakly semi-equational by distality.)

3.6 Weak semi-equationality in expansions by a predicate

3.6.1 Context

We recall the setting and some results from [CS13] (as usual, below x, y, z denote arbitrary finite tuples of variables). We start with a theory T in a language $\mathcal{L}$, and let $\mathcal{L}_{\mathrm{P}} := \mathcal{L} \cup \{\mathrm{P}(x)\}$, where P is a new unary predicate. Let $T_{\mathrm{P}} := \mathrm{Th}_{\mathcal{L}_{\mathrm{P}}}(M, A)$, where A is some subset of M (interpreted as P). We fix some monster model $(M', A') \succ (M, A)$ of T_{P}. An $\mathcal{L}_{\mathrm{P}}$-formula $\psi(x)$ is *bounded* if it is of the form $Q_0 y_0 \in \mathrm{P} \ldots Q_n y_n \in \mathrm{P}\, \varphi(x, y)$, where $Q_i \in \{\exists, \forall\}$ and $\varphi(x, y) \in \mathcal{L}$. We denote the set of all bounded $\mathcal{L}_{\mathrm{P}}$-formulas by $\mathcal{L}_{\mathrm{P}}^{\mathrm{bdd}}$ and say that the theory T_{P} is *bounded* if every $\mathcal{L}_{\mathrm{P}}$ formula is equivalent modulo T_{P} to a bounded one. Finally, for $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{L}_{\mathrm{P}}(M)$ we denote by $A_{\mathrm{ind}(\mathcal{L}')}$ the $\mathcal{L}'(\emptyset)$-induced structure on A.

Fact 3.6.1.
1. *[CS13, Corollary 2.5] Assume that T is NIP, $A_{\mathrm{ind}(\mathcal{L})}$ is NIP and T_{P} is bounded. Then T_{P} is NIP.*
2. *[CS13, Corollary 2.6] In particular, if T is NIP, $A \preceq M$ and T_{P} is bounded, then T_{P} is NIP.*

Some results on preservation of equationality under naming a set by a predicate are obtained in [MPZ20]. As pointed out in [HN17], the exact analog with distality in place of NIP is false:

Fact 3.6.2. *([HN17, Theorem 5.1] and the examples after it) The theory of dense pairs of o-minimal structures expanding a group is not distal (even though it is bounded and the induced structure on the submodel is distal). More precisely, their proof shows that the formula $\varphi(x, y) = \neg\exists u \in \mathrm{P}\,(x = u + y)$ is not a weak semi-equation in the theory of dense pairs.*

In this section we show that at least weak semi-equationality of T_{P} can be salvaged, in the following form.

Definition 3.6.3. A theory T_{P} is *almost model complete* if, modulo T_{P}, every $\mathcal{L}_{\mathrm{P}}$-formula $\psi(x)$ is equivalent to a Boolean combination of formulas of the form $\exists y_0 \in \mathrm{P} \ldots \exists y_{n-1} \in \mathrm{P}\, \varphi(x, y)$, where $\varphi(x, y)$ is an $\mathcal{L}$-formula.

Theorem 3.6.4. *Assume that T is distal, $A_{\mathrm{ind}(\mathcal{L})}$ is distal and T_{P} is almost model complete. Then T_{P} is weakly semi-equational.*

Corollary 3.6.5. *Dense pairs of o-minimal structures, as well as the other examples discussed after [HN17, Theorem 5.1], are weakly semi-equational.*

We will need the following properties of indiscernible sequences and definable sets with distal induced structure.

Fact 3.6.6. *[ACGZ22, Proposition 1.17] Let T be NIP, and let D be an $\emptyset$-definable set. Assume that D_{ind} is distal. Let $(c_i : i \in \mathbb{Q})$ be an indiscernible sequence of tuples in $\mathbb{M}$ and let a tuple b from D be given. Assume that $(c_i : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over b, then $(c_i : i \in \mathbb{Q})$ is indiscernible over b as well.*

Lemma 3.6.7. *Let T be NIP, let D be an $\emptyset$-definable set with D_{ind} distal. Let $(a_i : i \in \mathbb{Q})$ be an $\emptyset$-indiscernible sequence, and let b be such that $(a_i : i \in \mathbb{Q} \setminus \{0\})$ is b-indiscernible. Let $c \in D$ be arbitrary. Then we can find a sequence $(c_i : i \in \mathbb{Q})$ such that:*

- *$a_i c_i \equiv_b a_1 c$ for all $i \in \mathbb{Q} \setminus \{0\}$,*
- *$(a_i c_i : i \in \mathbb{Q})$ is $\emptyset$-indiscernible, and*
- *$(a_i c_i : i \in \mathbb{Q} \setminus \{0\})$ is b-indiscernible.*

Proof. By b-indiscernibility of $(a_i : i \in \mathbb{Q} \setminus \{0\})$, we can find a sequence $(c_i : i \in \mathbb{Q} \setminus \{0\})$ in D such that $(a_i c_i : i \in \mathbb{Q} \setminus \{0\})$ is b-indiscernible and $a_i c_i \equiv_b a_1 c$ for all $i \neq 0$. Indeed, fix any formula φ such that $\models \varphi(a_1, b, c)$, and let $(c_i : i \in \mathbb{Q} \setminus \{0\})$ be arbitrary with $\models D(c_i) \wedge \varphi(a_i, b, c_i)$ for all $i \in \mathbb{Q} \setminus \{0\}$ (exist by indiscernibility of $(a_i : i \neq 0)$ over b). For any finite set of formulas Δ, let the formula $\Psi_\Delta(x_0z_0,\ldots,x_nz_n,b)$ express that $(x_iz_i : i<n)$ is Δ-indiscernible over b and $\varphi(x_i,b,z_i)$ holds. Then by Ramsey we have some $0<i_0<\ldots<i_n\in\mathbb{Q}$ such that $\left(a_{i_j}c_{i_j} : j<n\right)$ satisfies Ψ_Δ. By b-indiscernibility of $(a_i : i\in\mathbb{Q}\setminus\{0\})$ it follows that for any $i_0<\ldots<i_n\in\mathbb{Q}\setminus\{0\}$ we have

$$\models \exists c_0\in D\ldots c_n\in D\Psi_\Delta(a_{i_0}c_0,\ldots,a_{i_n}c_n;b).$$

By compactness we can conclude.

It remains to find a $c_0\in D$ such that $(a_ic_i : i\in\mathbb{Q})$ is $\emptyset$-indiscernible. Let $I\subseteq\mathbb{Q}\setminus\{0\}$ be an arbitrary finite set and let $\bar{a}_0 := (a_i : i\in I)$. Let $\varepsilon>0$ in $\mathbb{Q}$ be such that $I\subseteq\mathbb{Q}\setminus(-\varepsilon,\varepsilon)$. For each $i\in\mathbb{Q}$, let $a_i' := (a_i,\bar{a}_0)$ and consider the sequence $(a_i' : i\in(-\varepsilon,\varepsilon))$. It is $\emptyset$-indiscernible since the sequence $(a_i : i\in\mathbb{Q})$ is, and moreover $(a_i' : i\in(-\varepsilon,\varepsilon)\setminus\{0\})$ is indiscernible over $(c_i : i\in I)\subseteq D$ (since the sequence of pairs $(a_ic_i : i\in\mathbb{Q}\setminus\{0\})$ is indiscernible). Then by Fact 3.6.6 we have that $(a_i' : i\in(-\varepsilon,\varepsilon))$ is indiscernible over $(c_i : i\in I)$. In particular, there exists an automorphism σ sending $a'_{\frac{\varepsilon}{2}}$ to a_0' and fixing $(c_i : i\in I)$; hence sending $a_{\frac{\varepsilon}{2}}$ to a_0 and fixing $(a_ic_i : i\in I)$. As by assumption $(a_ic_i : i\in I, i<-\varepsilon)+\left(a_{\frac{\varepsilon}{2}}c_{\frac{\varepsilon}{2}}\right)+(a_ic_i : i\in I, i>\varepsilon)$ is indiscernible, applying σ we have that there is $\tilde{c}_0 := \sigma\left(c_{\frac{\varepsilon}{2}}\right)\in D$ such that $(a_ic_i : i\in I, i<-\varepsilon)$ $+(a_0\tilde{c}_0)+(a_ic_i : i\in I, i>\varepsilon)$ is indiscernible. As I was arbitrary, we can then find c_0 as wanted by compactness. $\square$

Proof of Theorem 3.6.4. We know by Fact 3.6.1 that T_{P} is NIP. As T_{P} is almost model complete, so in particular bounded, we have that the structures $A_{\mathrm{ind}(\mathcal{L}_{\mathrm{P}})}$ and $A_{\mathrm{ind}\left(\mathcal{L}_{\mathrm{P}}^{\mathrm{bdd}}\right)}$ have the same definable subsets of A^n, for all n. But also note that for any set A, the structures $A_{\mathrm{ind}\left(\mathcal{L}_{\mathrm{P}}^{\mathrm{bdd}}\right)}$ and $A_{\mathrm{ind}(\mathcal{L})}$ have the same definable subsets of A^n, for all n. Hence the full structure induced on P in T_{P} is distal, so the lemmas on the indiscernible sequences above can be applied in T_{P} with $D := \mathrm{P}$.

Let (M',A') be a sufficiently saturated elementary extension of $(M,A)\models T_{\mathrm{P}}$. As T_{P} is almost model complete by assumption, it is enough to show that every formula in $\mathcal{L}_{\mathrm{P}}$ of the form

$$\varphi(x,y) = \exists z_0 \in \mathrm{P} \ldots \exists z_{n-1} \in \mathrm{P}\, \psi(x,y,z),$$

where $\psi(x,x,z) \in \mathcal{L}$, is a weak semi-equation in T_{P}.

To check Definition 3.2.1, assume (using Remark 3.2.5) that we are given an $\mathcal{L}_{\mathrm{P}}$-indiscernible sequence of finite tuples $(a_i : i \in \mathbb{Q})$ and a finite tuple b, both in M', such that the sequence $(a_i : i \in \mathbb{Q} \setminus \{0\})$ is $\mathcal{L}_P(b)$-indiscernible and $\models \varphi(a_i, b)$ for all $i \neq 0$.

In particular, there is some tuple c in P such that $\models \psi(a_1, b, c)$ holds.

By Lemma 3.6.7 applied in T_{P}, it follows that there is some sequence $(c_i : i \in \mathbb{Q})$ with $c_i \in \mathrm{P}$ such that $(a_i c_i : i \in \mathbb{Q})$ is $\mathcal{L}_{\mathrm{P}}$-indiscernible, $(a_i c_i : i \neq 0)$ is $\mathcal{L}_{\mathrm{P}}(b)$-indiscernible and $a_i c_i \equiv_b^{\mathcal{L}_{\mathrm{P}}} a_1 c$ for all $i \neq 0$. In particular $\models \psi(a_i, b, c_i)$ for all $i \neq 0$. But $\psi'(x,z;y) := \psi(x,y,z) \in \mathcal{L}$ is a semi-equation in T as T is distal, hence we must have $\models \psi(a_0, b, c_0)$, and so $\models \varphi(a_0, b)$ holds — as wanted. $\square$

Remark 3.6.8. Unlike in the general NIP case, there is a reasonable sufficient criterion for the boundedness of T_{P} for distal T. We say that T satisfies dnfcp$'$ (*definable nfcp*) if for any $M \prec N \models T$ and any $\varphi(x,y), \psi(y,z) \in \mathcal{L}$ there is $k < \omega$ such that for any $b \in N$, if the set $\{\varphi(x,a) : a \in \psi(M,b)\}$ is k-consistent, then it is consistent. By [CS15, Theorem 43], if T be distal, $A \subseteq M$ is small and uniformly stably embedded, and A_{ind} has dnfcp$'$, then T_{P} is bounded.

Problem 3.6.9.

1. In Theorem 3.6.4, can we relax the assumption to "T and $A_{\mathrm{ind}(\mathcal{L})}$ are weakly semi-equational"?

2. Is there an analog of Theorem 3.6.4 for semi-equationality? Even a general result for equationality seems to be missing (the argument in [MPZ20] for belle pairs of algebraically closed fields appears to be specific to algebraically closed fields).

3.7 Some results on Boolean combinations of semi-equations

It is tempting to adapt Lemmas 3.5.1 and 3.5.2 by replacing weak semi-equations with semi-equations, and removing the assumption that rows and columns without their 0-indexed elements are b-indiscernible, thus getting criteria that might be usable for showing that some formula is not equivalent to a Boolean combination of semi-equations. This works just fine for Lemmas 3.5.1, but the proof of Lemma 3.5.2 relies on the fact that weak semi-equations are closed under disjunction, and semi-equations are not closed under disjunction. So we must first study disjunctions of semi-equations.

Definition 3.7.1. An n-dimensional array $(a_{i_1,\ldots,i_n} : i_1, \ldots, i_n \in \mathbb{Z})$ is *path-indiscernible* if, for every $f_1, \ldots, f_n : \mathbb{Z} \to \mathbb{Z}$ such that each f_j is either constant or strictly monotone, the sequence $\left(a_{f_1(i),\ldots,f_n(i)}\right)_{i\in\mathbb{Z}}$ is indiscernible.

Remark 3.7.2.

1. Note that a one-dimensional path-indiscernible array is just the same as an indiscernible sequence.
2. Any indiscernible array, in the sense of e.g. [BYC14, Definition 1.1(1)], is path-indiscernible (but the converse is not necessarily true).

The following is a standard partite version of Ramsey's theorem.

Fact 3.7.3. *For any $A, n, k \in \mathbb{N}$, there is some $B \in \mathbb{N}$ such that for every $f : [B]^n \to [k]$ there are some $X_1, \ldots, X_n \subseteq [B]$ with $|X_i| = A$ such that f is constant on $X_1 \times \ldots \times X_n$.*

Lemma 3.7.4. *Let $n, m \in \mathbb{N}$ with $m < 2^n$. If $\varphi_1(x, y), \ldots, \varphi_m(x, y)$ are semi-equations, let $\psi(x, y) := \varphi_1(x, y) \vee \ldots \vee \varphi_m(x, y)$. Then there is no element b and path-indiscernible array $(a_{i_1,\ldots,i_n})_{i_1,\ldots,i_n\in\mathbb{Z}}$ such that $\models \psi(a_{i_1,\ldots,i_n}, b) \iff i_1 \neq 0 \wedge \ldots \wedge i_n \neq 0$.*

Proof. Divide $\{(i_1, \ldots, i_n) \mid i_j \in \mathbb{Z} \setminus \{0\}\}$ into 2^n orthants based on the signs of each of $i_1, \ldots, i_n$, one for each element of $\{1, -1\}^n$. By assumption on ψ, for each orthant $(s_1, \ldots, s_n)$ $(s_j \in \{1, -1\})$, we can choose a function $f_{s_1,\ldots,s_n} : \mathbb{N}_{\neq 0} \to [m]$ such that

$\models \varphi_{f_{s_1,\ldots,s_n}(i_1,\ldots,i_n)}(a_{s_1 i_1,\ldots,s_n i_n})$ for $i_1,\ldots,i_n \in \mathbb{N}_{\neq 0}$. By Fact 3.7.3, for every $A \in \mathbb{N}$ there is some $B \in \mathbb{N}$ such that for every $Y_1,\ldots,Y_n \subseteq \mathbb{N}$ with $|Y_i| = B$, there are $X_i \subseteq Y_i$ with $|X_i| = A$ such that $f_{s_1,\ldots,s_n}$ is constant on $X_1 \times \ldots \times X_n$. Let $N \in \mathbb{N}$, and let $A_0,\ldots,A_{2^n} \in \mathbb{N}$ be defined recursively by $A_0 := N$, and A_{k+1} is large enough that for every $f : [A_{k+1}]^n \to [m]$ there are $X_1,\ldots,X_n \subseteq [A_{k+1}]$ with $|X_i| = A_k$ and f constant on $X_1 \times \ldots \times X_n$. Then, let $g : [2^n] \to \{1,-1\}^n$ be a bijection, and define $(X_{i,j} \subseteq \mathbb{N}_{\neq 0})_{0 \le i \le 2^n, j \in [n]}$ recursively by $X_{0,i} := [A_{2^n}]$ and $X_{k+1,i} \subseteq X_{k,i}$ such that $|X_{k+1,i}| = A_{2^n-(k+1)}$ and $f_{g(k+1)}$ is constant on $X_{k+1,1} \times \ldots \times X_{k+1,n}$. Then $|X_{2^n,i}| = N$ and every $f_{s_1,\ldots,s_n}$ is constant on $X_{2^n,1} \times \ldots \times X_{2^n,n}$. That is, we have found an arbitrarily large finite path-indiscernible array $\left(a'_{i_1,\ldots,i_n}\right)_{-N \le i_1,\ldots,i_n \le N}$ such that $\models \psi\left(a'_{i_1,\ldots,i_n}, b\right) \iff i_1 \neq 0 \wedge \ldots \wedge i_n \neq 0$, and for each orthant $(s_1,\ldots,s_n)$, there is $k \in [m]$ such that $\models \varphi_k\left(a'_{s_1 i_1,\ldots,s_n i_n}, b\right)$ for $i_1,\ldots,i_n \in [N]$. Because $m < 2^n$, by the pigeonhole principle there is some $k \in [m]$ such that there are two distinct orthants $(s_1,\ldots,s_n) \neq (t_1,\ldots,t_n)$ such that $\models \varphi_k\left(a'_{s_1 i_1,\ldots,s_n i_n}, b\right)$ and $\models \varphi_k\left(a'_{t_1 i_1,\ldots,t_n i_n}, b\right)$ for $i_1,\ldots,i_n \in [N]$. For $\ell \in [n]$ and $-N \le j \le N$, let $i_{j,\ell} := s_\ell$ if $s_\ell = t_\ell$, and $i_{j,\ell} := s_\ell j$ otherwise. That way $(i_{j,1},\ldots,i_{j,n})_{-N \le j \le N}$ is constant or strictly monotonic in each coordinate, in orthant $(s_1,\ldots,s_n)$ for $j < 0$, and in orthant $(t_1,\ldots,t_n)$ for $j > 0$, so $\left(a'_{i_{j,1},\ldots,i_{j,n}}\right)_{-N \le j \le N}$ is indiscernible by path-indiscernibility of the array, and $\models \varphi_k\left(a'_{i_{j,1},\ldots,i_{j,n}}, b\right)$ for $j \neq 0$, but $\not\models \psi\left(a'_{i_{0,1},\ldots,i_{0,n}}, b\right)$ and hence $\not\models \varphi_k\left(a'_{i_{0,1},\ldots,i_{0,n}}, b\right)$. Since for every $N \in \mathbb{N}$ we found some $k \in [m]$ for which this holds, there is some $k \in [m]$ such that this holds for infinitely many N, and hence by compactness there are $(a^*_i)_{i \in \mathbb{Z}}$ such that $\models \varphi_k(a^*_i, b) \iff i \neq 0$, contradicting that $\varphi_k(x,y)$ is a semi-equation. $\square$

Lemma 3.7.5. *Let $n, m \in \mathbb{N}$ with $m < 2^n$. If $\varphi(x,y), \psi_1(x,y), \ldots, \psi_m(x,y)$ are semi-equations, let $\rho(x,y) := \psi_1(x,y) \vee \ldots \vee \psi_m(x,y)$, and $\theta(x,y) := \varphi(x,y) \wedge \neg\rho(x,y)$. Then there is no b and $(n+1)$-dimensional array $(a_{i,j_1,\ldots,j_n})_{i,j_1,\ldots,j_n \in \mathbb{Z}}$ such that:*

- *the sequence $(a_{i,j_1,\ldots,j_n})_{i \in \mathbb{Z}}$ is indiscernible for each fixed $j_1,\ldots,j_n$,*
- *the n-dimensional array $(a_{0,j_1,\ldots,j_n})_{j_1,\ldots,j_n \in \mathbb{Z}}$ is path-indiscernible,*

- $\models \theta\left(a_{i,j_1,\ldots,j_n}\right) \iff i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0$.

Proof. For any $j_1, \ldots, j_n \in \mathbb{Z}$, $\models \varphi\left(a_{i,j_1,\ldots,j_n}, b\right)$ for $i \neq 0$ and $\varphi(x, y)$ is a semi-equation, so $\models \varphi\left(a_{0,j_1,\ldots,j_n}, b\right)$. Then $\left(a_{0,j_1,\ldots,j_n}\right)_{j_1,\ldots,j_n \in \mathbb{Z}}$ is a path-indiscernible array such that $\models \rho\left(a_{0,j_1,\ldots,j_n}\right) \iff j_1 \neq 0 \wedge \ldots \wedge j_n \neq 0$, contradicting Lemma 3.7.4. □

Note that the $m = n = 1$ case of Lemma 3.7.5 is the direct analog of Lemma 3.5.1 for semi-equations. From this, we obtain a sufficient criterion for failure of a theory to be semi-equational, which we hope will find future applications.

Proposition 3.7.6. *If $\varphi(x, y)$ is a Boolean combination of semi-equations, then there is some $n \in \mathbb{N}$ such that there is no b and $(n+1)$-dimensional array $\left(a_{i,j_1,\ldots,j_n}\right)_{i,j_1,\ldots,j_n \in \mathbb{Z}}$ such that:*

- *$\left(a_{i,j_1,\ldots,j_n}\right)_{i \in \mathbb{Z}}$ is indiscernible for each fixed $j_1, \ldots, j_n$,*
- *the n-dimensional array $\left(a_{0,j_1,\ldots,j_n}\right)_{j_1,\ldots,j_n \in \mathbb{Z}}$ is path-indiscernible,*
- *$\models \varphi\left(a_{i,j_1,\ldots,j_n}, b\right) \iff i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0$,*
- *all $a_{i,j_1,\ldots,j_n}$ with $i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0$ have the same type over b.*

Proof. Because semi-equations are closed under conjunction (Proposition 3.2.6(2)), any conjunction of finitely many semi-equations and negations of semi-equations is of the form $\psi(x, y) \wedge \neg\theta(x, y)$ for some semi-equation $\psi(x, y)$ and disjunction of semi-equations $\theta(x, y)$. Thus any Boolean combination of semi-equations is equivalent, via its disjunctive normal form, to $\bigvee_{k \in I}\left(\psi_k(x, y) \wedge \neg\theta_k(x, y)\right)$ for some finite index set I, semi-equations $\psi_k(x, y)$ for $k \in I$, and disjunctions of semi-equations $\theta_k(x, y)$ for $k \in I$. Let $n \in \mathbb{N}$ be such that each θ_k is the disjunct of fewer than 2^n semi-equations, and suppose there is b and array $\left(a_{i,j_1,\ldots,j_n}\right)_{i,j_1,\ldots,j_n \in \mathbb{Z}}$ such that $\left(a_{i,j_1,\ldots,j_n}\right)_{i \in \mathbb{Z}}$ is indiscernible for each $j_1, \ldots, j_n$, $\left(a_{0,j_1,\ldots,j_n}\right)_{j_1,\ldots,j_n \in \mathbb{Z}}$ is path-indiscernible, $\models \theta\left(a_{i,j_1,\ldots,j_n}\right) \iff i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0$, and all $a_{i,j_1,\ldots,j_n}$ with

$i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0$ have the same type over b. Since

$$\begin{aligned} i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0 &\iff \models \varphi\left(a_{i,j_1,\ldots,j_n}, b\right) \\ &\iff \models \bigvee_{k \in I} \psi_k\left(a_{i,j_1,\ldots,j_n}, b\right) \wedge \neg\theta_k\left(a_{i,j_1,\ldots,j_n}, b\right) \end{aligned}$$

and all $a_{i,j_1,\ldots,j_n}$ with $i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0$ have the same type over b, there is some k such that $\models \psi_k\left(a_{i,j_1,\ldots,j_n}, b\right) \wedge \neg\theta_k\left(a_{i,j_1,\ldots,j_n}, b\right) \iff i \neq 0 \vee j_1 = 0 \vee \ldots \vee j_n = 0$, contradicting Lemma 3.7.5. $\square$

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