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Some classes of smooth bimodules over tracial von Neumann algebras and their associated 1-cohomology spaces

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A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

Patrick Jon Hiatt

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ABSTRACT OF THE DISSERTATION

Some classes of smooth bimodules over tracial von Neumann algebras
and their associated 1-cohomology spaces

by

Patrick Jon Hiatt
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2024
Professor Sorin Popa, Chair

We study several classes of Banach bimodules over a II_1 factor M, endowed with topologies that make them "smooth" with respect to L^p-norms implemented by the trace τ on M. In particular, letting $M \subset \mathcal{B} = \mathcal{B}(L^2M)$, and $2 \leq p < \infty$, we first consider the space $\mathcal{B}(p)$, obtained as the completion of $\mathcal{B}$ in the norm $|||T|||_p$ defined as the supremum of $|\varphi(T)|$ over all linear functionals $\varphi \in \mathcal{B}^*$ such that $|\varphi(xYz)| \leq \|x\|_p \|Y\| \|z\|_p$, for all $x, y \in M$ and $Y \in \mathcal{B}$. We continue by examining the subspace $\mathcal{K}(p) \subset \mathcal{B}(p)$, obtained as the closure in $\mathcal{B}(p)$ of the space of compact operators $\mathcal{K}(L^2M)$. We then construct the space $\mathcal{K}_p \subset \mathcal{B}$ of operators that are $|||\ |||_p$-limits of bounded sequences of operators in $\mathcal{K}(L^2M)$. We prove that $\mathcal{K}_p$ are all equal to the *τ-rank-completion* of $\mathcal{K}(L^2M)$ in $\mathcal{B}$, denoted as $\mathrm{q}\mathcal{K}_M$, which is the subspace of all operators $K \in \mathcal{B}(L^2M)$ such that there exists a sequence operators $K_n \in \mathcal{K}(L^2M)$ and projections $p_n \in \mathcal{P}(M)$ with $\lim_n \|p_n(K - K_n)p_n\| = 0$ and $\lim_n \tau(1 - p_n) = 0$. We conclude by proving that any separable II_1 factor M admits non-inner derivations into $\mathrm{q}\mathcal{K}_M$, but that any derivation $\delta : M \to \mathrm{q}\mathcal{K}_M$ is a pointwise limit in τ-rank-topology of inner derivations. This gives us an example of a "non-vanishing" theory of 1-cohomology for II_1 factors M.

The dissertation of Patrick Jon Hiatt is approved.

Tim Austin

Mario Bonk

Dimitri Y. Shlyakhtenko

Sorin Popa, Committee Chair

University of California, Los Angeles

2024

To my family

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PUBLICATIONS

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CHAPTER 1

Introduction

While Hochschild introduced his cohomology theory for algebras in the mid 1940s, it was around 1970 that this theory started to be adapted and systematically studied in operator algebras framework (see the series of papers by Johnson, Kadison and Ringrose [Joh72], [KR71], [JKR72]). However, problems related to derivations of an operator algebra M with values in special M-bimodules, such as M itself, which amounts to the 1-cohomology group of M with coefficients in M, started to be investigated several years earlier, triggered by Kaplanski's interest in such problems (see e.g. [Kap53] or the footnote in [SW55]). In a pioneering result in this direction, it is shown in [SW55] that any derivation of a commutative Banach algebra must be equal to zero, while in [Kad66], [Sak66] is it shown that all derivations of a von Neumann algebra are inner. More general M-bimodules $\mathcal{B}$ were soon considered, such as algebras $\mathcal{B}$ that contain M, notably $M \subset \mathcal{B} = \mathcal{B}(\mathcal{H})$ (see e.g. [Chr82]), or classical ideals in $\mathcal{B}(\mathcal{H})$, like the Schatten-von Neumann p-class $c_p(\mathcal{H})$, $1 \leq p < \infty$ ([JP72], [Joh72], [Hoo77]), or the ideal of compact operators ([JP72]).

Most of the early results in this direction aimed at proving that all derivations of an algebra M into an M-bimodule $\mathcal{B}$ are inner, and more generally on showing that all cohomology groups of M with coefficients in $\mathcal{B}$ vanish, $\mathrm{H}^n(M, \mathcal{B}) = 0$, $\forall n$. But starting with the work of Johnson in ([Joh72], [Joh74]), an interest towards using the cohomology groups $\mathrm{H}^n(M, \mathcal{B})$ as effective invariants for a von Neumann algebra M has emerged. However, while the amenable-nonamenable dichotomy could soon be established this way, by showing that a tracial von Neumann algebra M is amenable if and only if $H^1(M, \mathcal{B}) = 0$ for any normal dual Banach M-bimodule $\mathcal{B}$ (cf [Joh72], [Con76a], [Con76b], [Eff88]), by early 1980s all ef-

forts in this direction have stalled. At the "Operator Algebra Summer School" in Kingston 1980, where the main directions of research in this area were presented, two cohomology problems were particularly emphasized: (1) whether $\mathrm{H}^n(M,M)=0$, $\forall n$, for any II_1 factor M; (2) whether any derivation of a II_1 factor M into $\mathcal{B}(\mathcal{H})$ is inner when $M\subset\mathcal{B}(\mathcal{H})$ has infinite coupling constant (the case when $\dim_M\mathcal{H}<\infty$ had been settled in the affirmative in [Chr82]).

These problems are still open, but there has been progress on both. On the one hand, problem (2) was shown to be equivalent to the similarity problem, asking whether any bounded representation of any C*-algebra A, $\pi:A\to\mathcal{B}(\mathcal{H})$, is similar to a *-representation (i.e., $\exists S\in\mathcal{B}(\mathcal{H})$ invertible such that $A\ni x\mapsto S^{-1}\pi(x)S$ is a *-representation), see [Pis04] for several equivalent formulations and a deep analysis of this problem. On the other hand, it was shown that $\mathrm{H}^n(M,M)=0$, $\forall n$, for many classes of II_1 factors with "good decomposability" features, such as the property Gamma of Murray and von Neumann, existence of Cartan subalgebras, and more generally existence of a "thin decomposition" of M with respect to a pair of amenable subalgebras (see [CPS03]). But the perception on these problems has changed: one now expects that there do exist II_1 factors M for which $H^2(M,M)\neq 0$ and $H^1(M,\mathcal{B}(L^2M\otimes\ell^2\mathbb{N}))\neq 0$, and that in fact this should be the case for the free group factors $M=L\mathbb{F}_n, 2\leq n\leq\infty$. However, these cohomology spaces are expected to be difficult to calculate, and to not be able to make "fine distinctions", such as to differentiate between the free group factors $L\mathbb{F}_n, 2\leq n\leq\infty$, or show that $L\mathbb{F}_\infty$ cannot be finitely generated.

A big impetus towards finding a different cohomology theory for II_1 factors, one that would be non-vanishing and calculable, providing an efficient invariant that would reflect fine structural properties of the algebras involved, came in 2001, triggered by Gaboriau's successful generalization to orbit equivalence relations $\mathcal{R}_\Gamma$ arising from actions of countable groups by measure preserving transformations $\Gamma\curvearrowright X$ of Atyiah's and Cheeger-Gromov notion of L^2-cohomology of groups, leading to his notion of L^2-Betti numbers for $\mathcal{R}_\Gamma$ satisfying $\beta_n^{(2)}(\mathcal{R}_\Gamma)=\beta_n^{(2)}(\Gamma)$, with the striking consequence that free groups of different rank cannot

be orbit equivalent ([Gab02]). Since Gaboriau's L^2-cohomology for $\mathcal{R}_\Gamma$ can be viewed as a cohomology theory for the corresponding Cartan inclusion $A = L^\infty(X) \subset L(\mathcal{R}_\Gamma) = M$, of the group measure space II_1 factor associated with the orbit equivalence relation $\mathcal{R}_\Gamma$, it is an invariant for factors M with unique Cartan decomposition (see [Pop06]), for which one can simply define associated L^2-Betti numbers as $\beta_n^{(2)}(M) = \beta_n^{(2)}(\mathcal{R}_\Gamma)$.

But a more interesting "wishful" L^2-cohomology theory along these lines would be for group factors $M = L\Gamma$ arising from ICC groups Γ, typically without Cartan subalgebras, for which one would like to have an identification between the L^2-cohomology of $L\Gamma$ and the L^2-cohomology of the group Γ, with the corresponding L^2-Betti number $\beta_n^{(2)}(L\Gamma)$ coinciding with Atyiah's L^2-Betti number of the group, $\beta_n^{(2)}(\Gamma)$. This problem was much emphasized by the well know mathematician Alain Connes in his talk at MSRI in the Spring of 2001 ([Con01]).

Several attempts were made in this direction: (a) Connes-Shlyakhtenko proposed in [CS05] an "everywhere defined" cohomology of M with coefficients in the Murray-von Neumann algebra $\mathrm{Aff}(M \otimes M^{op})$ of operators affiliated with $M \otimes M^{op}$; (b) Peterson considered in [Pet09] a "densely defined" L^2-cohomology theory for II_1 factors; (c) Galatan-Popa considered in [GP17] a generalized version of the 1-cohomology with coefficients in $\mathcal{K}(L^2M)$ in ([JP72], [Pop87]), based on the larger class of *smooth bimodules*, trying this way to avoid being always equal to 0, while still vanishing in "amenable directions".

All these attempts have shortcomings: [Pet09] encountered the difficulty of having to prove the independence of the cohomology on the dense domain of the derivations; [CS05] had to be adjusted with some continuity conditions in [Tho08], and that modified version was shown in [PV15] to always be equal to 0 (this was previously shown in [AK15] and [Ale14] to hold in certain cases, such as for free group factors); of the two classes of smooth bimodules proposed in [GP17], one was shown to produce a cohomology that's always 0 and the other one has not led so far to non-vanishing examples.

The work in this thesis serves as a new effort towards identifying a class of M-bimodules

$\mathcal{B}$ that would allow defining a viable cohomology theory, an effective isomorphism invariant, for the II_1 factors M. To begin with, since this approach is somewhat inspired by the L^2-cohomology of groups, one expects $\mathcal{B}$ to depend canonically on M and be related in some ways to the Hilbert space L^2M and the space of linear bounded operators acting on it $\mathcal{B}(L^2M)$.

Beyond that, a first priority for this thesis was that the 1-cohomology with coefficients in $\mathcal{B}$ should not always vanish, i.e, that there should exist II_1 factors M that admit non-inner derivations into $\mathcal{B}$, especially in the case $M = L\Gamma$ with $\beta_1^{(2)}(\Gamma) \neq 0$, like $\Gamma = \mathbb{F}_n$, $2 \leq n \leq \infty$. Another consideration was that $\mathcal{B}$ should host the derivations $\delta_c : L\Gamma \to \mathcal{B}$ coming from 1-cocycles $c : \Gamma \to \ell^2\Gamma$, which on the group algebra $\mathbb{C}\Gamma = \mathrm{span}\{u_g\}_g$ are of the form $\delta_c(u_g) = [T_f, u_g]$, where T_f is the diagonal operator implemented by $f \in \ell^\infty\Gamma$, obtained by "integrating" c over the Cayley graph of Γ (note that one can just take $f(g) = -c_g(g)$, $\forall g \in \Gamma$). This implicitly means that derivations of M into $\mathcal{B}$ should be uniquely determined by their values on weakly dense *-subalgebras. At the same time, one would like $\mathcal{B}$ to have an M^{op}-bimodule structure as well, commuting with its M-bimodule structure, potentially leading to a right $M\overline{\otimes}M^{op}$-module structure on $\mathcal{B}$. One would further hope that whenever $(u_k)_{k=1}^n \subset \mathcal{U}(M)$ is a finite set of unitaries generating M as a von Neumann algebra, the map $\delta \mapsto (\delta(u_k))_k$ gives an injective right-$M\overline{\otimes}M^{op}$-modular map from the space of derivations $\mathrm{Z}^1(M, \mathcal{B})$ into $\mathcal{B}^n$, that would behave well to the quotient by the space of inner derivations $\mathrm{B}^1(M, \mathcal{B})$, or by its closure $\overline{\mathrm{B}^1}$ under a suitable topology. If such requirements are met, this would allow associating a first L^2-Betti number for M, $\beta_1^{(2)}(M)$, as the Murray-von Neumann-Lueck dimension of $\tilde{\mathrm{H}}^1(M, \mathcal{B}) := \mathrm{Z}^1(M, \mathcal{B})/\overline{\mathrm{B}^1}$ viewed as a right $M\overline{\otimes}M^{op}$-module.

These considerations force $\mathcal{B}$ to be somewhat related to $\mathcal{K}(L^2M)$, the space where $[M_f, u_g]$ takes values. So having all this in mind, we consider here the following spaces.

For each $p \geq 2$, we consider the Banach space of "compact-like operators" $\mathcal{K}(p)$ defined as follows. We first let $\mathcal{B}^*(p)$ be the space of functionals φ on $\mathcal{B}(L^2M)$ with the property

that

$$\|\varphi\|_{\mathcal{B}^*(p)} := \sup\{|\varphi(xYz)| \mid Y \in (\mathcal{B})_1, x, y \in M \cap (L^pM)_1\} < \infty.$$

We then let $\mathcal{B}(p)$ denote the completion of $\mathcal{B}(L^2M)$ in the norm

$$|||T|||_p := \sup\{|\varphi(T)| \mid \varphi \in (\mathcal{B}^*(p))_1\}.$$

Finally, we denote by $\mathcal{K}(p) \subset \mathcal{B}(p)$ the closure in $\mathcal{B}(p)$ of the space of compact operators $\mathcal{K}(L^2M)$. It is immediate to see that $(\mathcal{K}(p), |||\cdot|||_p)$ is both a Banach M-bimodule and a Banach M^{op}-bimodule. It is also easy to see that for each $X \in \mathcal{K}(p)$ the left-right multiplications by elements in the unit ball of M is $\|\cdot\|_2 - |||\cdot|||_p$ continuous (smoothness).

Since the derivations of $M = L\Gamma$ arising from cocycles $c : \Gamma \to \ell^2\Gamma$ are often implemented by bounded operators $M_f \in \mathcal{B}(L^2M)$, we in fact expect that the M-bimodules of interest for us consist of bounded operators. We thus also consider the spaces $\mathcal{K}_p := \mathcal{K}(p) \cap \mathcal{B}(L^2M)$, $p \geq 2$. We prove that in fact all $\mathcal{K}_p$, $2 \leq p < \infty$, "collapse" to just one space, which we show to coincide with the closure in $\mathcal{B}(L^2M)$ of $\mathcal{K}(L^2M)$ in the so-called *τ-rank metric* q_M, given by its M-bimodule structure, $\mathrm{q}_M(S,T) = \inf\{\tau(1-p) + \|p(T-S)p\| \mid p \in \mathcal{P}(M)\}$.

Theorem 1.0.1. *For each $p \geq 2$ denote by $\mathcal{K}_p$ the space of operators $T \in \mathcal{B}(L^2M)$ for which there exists a sequence $K_n \in \mathcal{K}(L^2M)$ such that $\sup_n \|K_n\| < \infty$ and $\lim_n |||T - K_n|||_p = 0$. Then $\mathcal{K}_p$ coincides with the τ-rank-completion $\mathrm{q}\mathcal{K}_M$ of $\mathcal{K}(L^2M)$ in $\mathcal{B}(L^2M)$.*

We note that the τ-rank-completion $\mathrm{q}\mathcal{K}_M$ also coincides with the strong M-M-completion of $\mathcal{K}(L^2M)$ in the sense of [Mag00], although we do not take this perspective here.

Any derivation of M into $\mathrm{q}\mathcal{K}_M$ is indeed determined by its values on any weakly-dense *-subalgebra of M. In fact, any derivation of M into $\mathrm{q}\mathcal{K}_M$ is continuous from the unit ball of M with the $\|\cdot\|_2$-topology to $\mathrm{q}\mathcal{K}_M$ with its q_M-metric. Also, $\mathrm{q}\mathcal{K}_M$ is both a Banach M and M^{op}-bimodule and all derivations arising from non-vanishing 1-cocyles c of Γ into $\ell^2\Gamma$ described above give rise to non-inner derivations of $M = L\Gamma$ into $\mathrm{q}\mathcal{K}_M$. But in fact any separable II_1 factor M (so including the hyperfinite II_1 factor) admits non-inner derivations into the M-bimodule $\mathrm{q}\mathcal{K}_M$:

Theorem 1.0.2. *Given any separable diffuse tracial von Nuemann algebra M, there exist non-inner derivations of M into $q\mathcal{K}_M$.*

Thus, while the primary objective of getting a non-vanishing 1-cohomology is indeed being met by the M-bimodules $\mathrm{q}\mathcal{K}_M$, the above result shows that the associated (classic Hochschild) 1-cohomology space $\mathrm{H}^1(M, \mathrm{q}\mathcal{K}_M)$, obtained as the quotient of the space of derivations $\mathrm{Z}^1(M, \mathrm{q}\mathcal{K}_M)$ by the space of inner derivations $\mathrm{B}^1(M, \mathrm{q}\mathcal{K}_M)$, becomes too "wild", certainly un-calculable. This means one has to take instead the quotient of $\mathrm{Z}^1(M, \mathrm{q}\mathcal{K}_M)$ by a closure $\overline{\mathrm{B}^1(M, \mathrm{q}\mathcal{K}_M)}$ with respect to some suitable topology on the space of derivations, like one does for the L^2-cohomology of groups. This should however take into consideration that the closure of B^1 in the $\| \quad \|_2 - \mathrm{q}_M$ pointwise convergence on the unit ball of M is too weak for this purpose, as one has the following:

Theorem 1.0.3. *Let $\delta : M \to q\mathcal{K}_M$ be a derivation implemented by $T \in \mathcal{B}(L^2M)$. Then there exists a net of finite-rank operators K_ι with $\|K_\iota\| \leq \|T\|$ such that*

$$\lim_\iota q_M(\delta(x), [K_\iota, x]) = 0, \forall x \in M.$$

Moreover, if L^2M is separable, then the net can be taken a sequence.

This thesis is organized as follows. Chapter 2 begins by detailing some preliminaries in the theory of von Neumann algebras. In Section 2.1 and 2.2 we recall the definition of a tracial von Neumann algebra (M, τ) as well as their associated L^p-spaces. In Sections 2.3 and 2.4 we fix some of the notation and terminology for Banach bimodules over II_1 factors. Lastly, in Section 2.5 we recall the formal definition of Hoschild cohomology, primarily focusing on 1-cohomology.

In Chapter 3, we begin by defining for each $p \geq 2$ the space $\mathcal{B}^*(p)$ of functionals φ on $\mathcal{B} := \mathcal{B}(L^2M)$ with the property that

$$\|\varphi\|_{\mathcal{B}^*(p)} := \sup\{|\varphi(xYz)| \leq 1, \forall Y \in (\mathcal{B})_1, x, y \in M \cap (L^pM)_1\} < \infty.$$

We continue by considering its predual, $\mathcal{B}(p)$, obtained as the completion of $\mathcal{B}$ in the norm $|||T|||_p := \sup\{|\varphi(T)| \mid \varphi \in (\mathcal{B}^*(p))_1\}$. In Section 3.3 we define the subspace $\mathcal{K}(p) \subset \mathcal{B}(p)$, obtained as the closure in $\mathcal{B}(p)$ of the space of compact operators $\mathcal{K}(L^2M)$, whose dual identifies naturally to the "normal part" $\mathcal{B}^*_{\mathrm{n}}(p)$ of $\mathcal{B}^*(p)$.

In Chapter 4, we define the space $\mathcal{K}_p \subset \mathcal{B}$ of operators that are $|||\cdot|||_p$-limits of bounded sequences of operators in $\mathcal{K}(L^2M)$, define the τ-rank topology on M-bimodules, and prove Theorem 1.0.1, showing that all $\mathcal{K}_p$ coincide with the closure $\mathrm{q}\mathcal{K}_M$ of $\mathcal{K}(L^2M)$ in $\mathcal{B}(L^2M)$, in the τ-rank topology (see Theorem 4.1.5). Then in Section 4.1 we consider the space of derivations of M into $\mathrm{q}\mathcal{K}_M$ and prove Theorems 1.0.2 and 1.0.3 (see 4.2.3 and 4.2.7).

This thesis is based on [HPP24], of which the author of this dissertation was a primary investigator and author. Chapter 3 is based on the constructions in Sections 3, 4, and 5. The main results in Chapter 4 are based on sections 6 and 7.

CHAPTER 2

Preliminaries

In this chapter, we outline some of the basic preliminary results and definitions used throughout this thesis. We also fix some of the notation used later in the proof of the main results. For the sake of brevity, we forgo giving any proof. We refer to [AP17] for a full treatment of this material.

2.1 von Neumann Algebras

Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$. Equipped with the involution $x \mapsto x^*$, given by taking the adjoint, $\mathcal{B}(\mathcal{H})$ becomes a $*$-algebra under the operations of addition and composition.

Denote by $\| \ \|$ the operator norm on $\mathcal{B}(\mathcal{H})$. We call the *norm topology*, the topology on $\mathcal{B}(\mathcal{H})$ induced by $\| \ \|$. In addition to the norm topology, we recall the following two weaker topologies on $\mathcal{B}(\mathcal{H})$.

- The *weak operator topology* (wo-topology) is the locally convex topology on $\mathcal{B}(\mathcal{H})$ induced by the seminorms $p_{\xi,\eta}(T) = |\langle T\xi, \eta\rangle$, for $\xi, \eta \in \mathcal{H}$.
- The *strong operator topology* (so-topology) is the locally convex topology on $\mathcal{B}(\mathcal{H})$ induced by the seminorms $p_\xi(T) = \|T\xi\|$, for $\xi \in \mathcal{H}$.

A unital $*$-subalgebra $M \subset \mathcal{B}(\mathcal{H})$ is called a *von Neumann Algebra* if it closed in the weak operator topology. By a result of von Neumann known as the *bicommutant theorem*, the is equivalent to either

- M is closed in the strong operator topology.
- $M = M''$

where here S' denote the *commutant* of S, i.e. the set $\{T \in \mathcal{B}(\mathcal{H}) \mid Tx = xT, \forall x \in M\}$

For this thesis, we mainly consider *tracial von Neumann algebras*, i.e. a von Neumann algebra M with a faithful normal trace $\tau : M \to \mathbb{C}$. Given a tracial von Neumann algebra (M, τ), we denote by $\|x\|_2 = \tau(x^*x)^{1/2}$ the 2-norm on (M, τ). We denote by L^2M the completion of M with respect to $\| \ \|_2$. This space L^2M is a Hilbert space, and one can view M as a $*$-subalgebra of $\mathcal{B}(L^2M)$. In the particular case that a tracial von Neumann algebra (M, τ) is infinite-dimensional and has trivial center, i.e. $\mathcal{Z}(M) = \mathbb{C}1$, we call M a *II_1 factor*.

Below, we recall some natural examples of tracial von Neumann algebras resulting from groups and group actions due to Murray and von Neumann [MN36, MN43].

Example 2.1.1. (Group von Neumann algebra) Let Γ be a countable group. Denote by $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ the left regular representation of Γ on $\ell^2\Gamma$, given by $\lambda(g)\delta_h = \delta_{gh}$, for $g, h \in \Gamma$. The *group von Neumann algebra or* Γ is then the von Neumann algebra $L\Gamma$ generated by $\{\lambda(g) \mid g \in \Gamma\}$ in $\mathcal{B}(\ell^2\Gamma)$. The group von Neumann algebra $L\Gamma$ has a trace $\tau(x) = \langle x\delta_e, \delta_e \rangle$. Moreover, $L\Gamma$ is a II_1 factor precisely when all non-trivial conjugacy classes of Γ are infinite, i.e. when Γ is an icc group.

Example 2.1.2. (Group Measure Space von Neumann Algebra) Let $\sigma : \Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable group Γ on a standard probability space (X, μ). Then one has a representation $\pi : \Gamma \to \mathcal{B}(L^2(X, \mu) \otimes \ell^2\Gamma)$ given by $\pi(g)(f \otimes \delta_h) = \sigma(g)f \otimes \delta_{gh}$, for $f \in L^2(X, \mu)$ and $g, h \in \Gamma$. Then the von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma$, generated by $f \otimes 1$, for $f \in L^\infty(X, \mu)$ and $\pi(g)$, for $g \in \Gamma$, is called the *group measure space von Neumann algebra*. Here $L^\infty(X, \mu) \rtimes \Gamma$ has a tracial state τ given by $\tau(x) = \langle xu_e, u_e \rangle$, where $u_e \in L^2(X, \mu) \otimes \ell^2\Gamma$ is the vector $u_e = 1 \otimes \delta_e$. Moreover, in the case the action of Γ is free and ergodic, $L^\infty(X, \mu) \rtimes \Gamma$ will be a II_1 factor.

Example 2.1.3. (Hyperfinite II_1 Factor) Consider the tensor product $R_0 = \bigotimes_{n\geq 1} \mathbb{M}_2(\mathbb{C})$. This algebra has a tracial state $\tau = \bigotimes_{n\geq 1} \mathrm{tr}$, where tr is the normalized trace on $\mathbb{M}_2(\mathbb{C})$ with $\mathrm{tr}(1) = 1$. Call $\mathcal{H}$ the Hilbert space completion of R_0 with respect to the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1^2}$. Then the von Neumann algebra R in $\mathcal{B}(\mathcal{H})$ generated by R_0 is called the *hyperfinite II_1 factor*. By a theorem of Connes in [Con76a], R is the unique separable *amenable* II_1 factor. In particular, if Γ is any icc amenable group (i.e. an amenable group whose non-trivial conjugacy classes are infinite) then $L\Gamma \cong R$.

2.2 Non-commutative L^p-spaces

Let (M, τ) be a tracial von Neumann algebra. Then for any $1 \leq p < \infty$, $\|x\|_p = \tau(|x|^p)^{1/p}$ defines a norm on M. Here the norm $\|x\|_p$ is increasing in p, and the limit $\lim_{p\to\infty} \|x\|_p$ equal to the operator norm $\|x\|_\infty = \|x\|$ on M.

The completion of M in the norm $\| \ \|_p$ is denoted by L^pM, called the *non-commutative L^p-space of M*. Since $\| \ \|_p$ is increasing in p, one has $L^pM \supset L^{p'}M$ whenever $p' \geq p$. One can naturally identify L^pM with the space of densely defined closed operators T on L^2M that are affiliated with M and have the property that $|T|$ has spectral decomposition $|T| = \int \lambda \mathrm{d}e_\lambda$ satisfying $\int \lambda^p \mathrm{d}\tau(e_\lambda) < \infty$.

If $1 \leq p \leq \infty$, then $(L^pM)_1$ is closed in $L^{p'}M$, for any $1 \leq p' \leq p$. Moreover, all of the $\| \ \|_{p'}$-topologies on the unit ball $(M)_1$ of M for $1 \leq p' < \infty$ coincide with the *so*-topology on $(M)_1$ and if $p < \infty$, then all $\| \ \|_{p'}$-topologies on $(L^pM)_1$, $1 \leq p' \leq p$ coincide with the $\| \ \|_p$-topology.

Recall that if $1 \leq p < \infty$, then $(L^pM)^* \simeq L^qM$, where $q = \frac{p}{p-1}$ (with the usual convention $1/0 = \infty$), the duality being given by $(\xi, \zeta) \mapsto \tau(\zeta^*\xi)$ for $\xi \in L^pM$, $\zeta \in L^qM$, viewed as operators affiliated with M. This also shows that if $y \in M$ and $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\|y\|_p = \sup\{|\tau(yz)| \mid z \in (L^qM)_1\}$.

2.3 Banach Bimodules

Let M be a von Nuemann algebra. Then a *Banach M-bimodule* $\mathcal{B}$ is a Banach space with left and right multiplication operations $M \times \mathcal{B} \ni (x, T) \mapsto xT \in \mathcal{B}$ and $\mathcal{B} \times M \ni (T, x) \mapsto Tx \in \mathcal{B}$, which satisfy

- $x(yT) = (xy)T$ and $(Tx)y = T(xy)$, for all $x, y \in M$ and $T \in \mathcal{B}$.
- $1_M T = T 1_M = T$, for all $T \in \mathcal{B}$.
- $\|xT\|_{\mathcal{B}} \leq \|x\|_M \|T\|_{\mathcal{B}}$, $\|Tx\|_{\mathcal{B}} \leq \|T\|_{\mathcal{B}} \|x\|_M$, for all $x \in M$ and $T \in \mathcal{B}$.

Often times a Banach M-bimodule $\mathcal{B}$ is a assumed to have additional structural or topological properties. For example, if $\mathcal{B}$ is the dual of a Banach space $\mathcal{B}_*$ and for each $x \in M$ the maps $\mathcal{B} \ni T \mapsto xT \in \mathcal{B}$, $\mathcal{B} \ni T \mapsto Tx \in \mathcal{B}$ are continuous with respect to the $\sigma(\mathcal{B}, \mathcal{B}_*)$ topology (also called weak*-topology), then $\mathcal{B}$ is called a *dual M-bimodule*. Furthermore, if M is a von Neumann algebra, $\mathcal{B}$ is a dual M-bimodule, and for each $T \in \mathcal{B}$ the maps $M \ni x \mapsto xT \in \mathcal{B}$, $M \ni T \mapsto Tx \in \mathcal{B}$ are continuous from $(M)_1$ with the $\sigma(M, M_*)$-topology to $\mathcal{B}$ with the $\sigma(\mathcal{B}, \mathcal{B}_*)$-topology, then we say that the dual M-bimodule $\mathcal{B}$ is *normal*.

We conclude by listing several examples of Banach M-bimodules.

Example 2.3.1. Let $\mathcal{B}$ be any larger unital Banach algebra that contains M with $1_M = 1_{\mathcal{B}}$. Then $\mathcal{B}$ is a Banach M-bimodule with left and right products xT, Tx for $x \in M$, $T \in \mathcal{B}$, being the restrictions of the product in the larger algebra $\mathcal{B}$. In the case $M \subset \mathcal{B}$ is an inclusion of von Neumann algebras ${}_M\mathcal{B}_M$ is in fact a dual normal M-bimodule. For example, if (M, τ) is tracial then $\mathcal{B} = \mathcal{B}(L^2 M)$ becomes a dual normal M-bimodule.

Example 2.3.2. If $\mathcal{B}$ is a Banach M-bimodule, then any norm closed two sided ideal $\mathcal{J}$ in the Banach algebra $\mathcal{B}$, which will have a natural M-bimodule structure by restriction from $\mathcal{B}$. For example, if $M \subset \mathcal{B}(\mathcal{H})$, then the compact operators $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ becomes a Banach M-bimodule.

Example 2.3.3. Let $\mathcal{B}$ be a Banach M-bimodule. Then the dual space $\mathcal{B}^*$ naturally has an M-bimodule structure given by $\mathcal{B}^* \ni \varphi \mapsto x \cdot \varphi \cdot y \in \mathcal{B}^*$, $\forall x, y \in M$, which for $T \in \mathcal{B}$ is defined by $x \cdot \varphi \cdot y(T) = \varphi(yTx)$. Then with respect to operator norm $\| \ \|_{\mathcal{B}^*}$, $\mathcal{B}^*$ becomes a Bancah dual normal M-bimodule.

Example 2.3.4. Let (M, τ) be a tracial von Neumann algebra viewed in $\mathcal{B}(L^2M)$. Then the predual of $\mathcal{B}(L^2M)$ is the space all normal linear functionals $\varphi \in \mathcal{B}(L^2M)^*$, denoted as $\mathcal{B}^*_{\mathrm{n}}(L^2M)$. This forms a norm closed two sided ideal on $\mathcal{B}(L^2M)^*$, and hence is a M-bimodule. In fact, since $\mathcal{K}(L^2M)^* = \mathcal{B}^*_{\mathrm{n}}(L^2M)$, it is a dual normal M-bimodule.

Example 2.3.5. Let (M, τ) be a tracial von Neumann algebra. Then we note that if $x, y \in M$ and $\xi \in L^pM$, then $\|x\xi y\|_p \leq \|x\|\|\xi\|_p\|y\|$. This makes L^pM, for $1 \geq p \leq \infty$, a Banach M-bimodule. In the case $1 < p \leq \infty$, since $L^pM = (L^qM)^*$ for $q = \frac{p}{p-1}$, L^pM will be dual and normal.

2.4 Smooth Bimodules

Let (M, τ) be a tracial von Neumann algebra and $\mathcal{B}$ a Banach M-bimodule. Following Galatan and Popa in [GP17], we say an element $T \in \mathcal{B}$ is *smooth* if the maps $x \mapsto xT$ and $x \mapsto Tx$ are continuous from the unit ball of M with its $\| \cdot \|_2$-topology to $\mathcal{B}$ with its Banach norm topology. We call the *smooth part* of $\mathcal{B}$, denoted as $s^*(\mathcal{B})$, the set of all smooth elements in $\mathcal{B}$. In the case $s^*(\mathcal{B}) = \mathcal{B}$, we say that $\mathcal{B}$ is a *smooth bimodule*. Of course, $s^*(\mathcal{B})$ is itself a smooth M-bimodule.

One can check that the compact operators $\mathcal{K}(L^2M)$ are contained in the smooth part $s^*(\mathcal{B}(L^2M))$. And hence $\mathcal{K}(L^2M)$ is itself a smooth bimodule. This example is much emphasised in [Pop87], [GP17].

In general, $s^*(\mathcal{B}(L^2M))$ is strictly larger than $\mathcal{K}(L^2M)$ (see [GP17, Theorem 1.6]). For example, assume that M is diffuse and that $A \subset M$ is a separable diffuse abelian subalgebra. Then let $\{\xi_n\}_{n\in\mathbb{N}} \subset L^2(A, \tau)$ be a a sequence of independent identically distributed real

valued Gaussian random variables. Let p be the orthogonal projection of L^2M onto the subspace $\mathcal{H}_0$ generated by the $\{\xi_n\}_{n\in\mathbb{N}}$. Then one has $p \in s^*(\mathcal{B}(L^2M))$ but $p \notin \mathcal{K}(L^2M)$.

Another example, studied in [PR88], is when M is contained (as a von Neumann subalgbera) in a II_∞ factor $\mathcal{M}$ with a normal semifinite faithful trace Tr and $\mathcal{B}$ is the norm closed *-ideal of "compact operators" $\mathcal{J}(\mathcal{M}) \subset \mathcal{M}$, consisting of $T \in \mathcal{M}$ with the property that all spectral projections $e_{[t,\infty)}(T^*T)$ corresponding to $t > 0$ have finite trace, $\forall t > 0$.

One should mention that in both these examples, the norm $\| \ \|$ on the M-bimodule $\mathcal{B}$ is *operatorial* (see [JP72], [GP17]). That is if $T \in \mathcal{B}$, and $p \in \mathcal{P}(M)$ is a projection in M, then $\|pTp + (1-p)T(1-p)\| = \max\{\|pTp\|, \|(1-p)T(1-p)\|\}$. we note that the operator norm on M is itself operatorial. However, for $1 \leq p < \infty$, the $\| \ \|_p$-norm fails this operatorial property.

2.5 1-Cohomology group $H^1(M, \mathcal{B})$

We recall here the definition of the 1-cohomology group $H^1(M, \mathcal{B})$ in the sense of Hochschild. Let M be a von Neumann algebra and $\mathcal{B}$ a Banach M-bimodule. We say a linear map $\delta : M \to \mathcal{B}$ is a *derivation* if it satisfies the product rule, i.e. if $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in M$.

For example, for any fixed element $T \in \mathcal{B}$ the map $\mathrm{Ad}_T : M \to \mathcal{B}$ given by $\mathrm{Ad}_T(x) = [T, x] = Tx - xT$ is a derivation. Any such derivation $\delta : M \to \mathcal{B}$ of the form $\delta = \mathrm{Ad}_T$ for some $T \in \mathcal{B}$ is called an *inner derivation*.

Denote by $Z^1(M, \mathcal{B})$ the space of all derivations $\delta : M \to \mathcal{B}$. Call $B^1(M, \mathcal{B}) \subset Z^1(M, \mathcal{B})$ the subspace of all inner derivations. We denote by $H^1(M, \mathcal{B})$ the quotient space

$$H^1(M, \mathcal{B}) = Z^1(M, \mathcal{B})/B^1(M, \mathcal{B})$$

Note that $H^1(M, \mathcal{B})$ is the 1-cohomology group, in the sense of Hochschild (c.f. [Hoc45]).

Much of the initial work regarding 1-cohomology for von Neumann algebras revolves

around proving vanishing results, i.e. that $H^1(M, \mathcal{B}) = 0$ for certain classes of bimodule $\mathcal{B}$. This amounts to proving that all derivations $\delta : M \to \mathcal{B}$ are inner. We recall some of these vanishing results here.

Example 2.5.1. Let M be a von Neumann algebra. Then $H^1(M, M) = 0$. This follows from the results of Kadison [Kad66] and Sakai [Sak66].

Example 2.5.2. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ be the subset of compact operators (see 2.3.2). Then $H^1(M, \mathcal{K}(\mathcal{H})) = 0$. This was first proven in the case when M is amenable by Johnson and Parrot [JP72], and later for a general II_1 factor by Popa in [Pop87].

Example 2.5.3. Let (M, τ) be a tracial von Neuamnn algebra. Then $H^1(M, \mathcal{B}(L^2M)) = 0$. This follows from a result of Christensen [Chr82]. More generally, if $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $\mathcal{H}$ has finite coupling constant then $H^1(M, \mathcal{B}(\mathcal{H})) = 0$.

Example 2.5.4. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Nuemann algebra and $\mathcal{B}_0$ be a closed subspace of the smooth part $s^*(\mathcal{B}(\mathcal{H}))$, (see Section 2.4). Then Galatan and Popa in [GP17] proved $H^1(M, \mathcal{B}_0)$. Since $\mathcal{K}(\mathcal{H}) \subset s^*(\mathcal{B}(\mathcal{H}))$, this can be seen as a generalization of $H^1(M, \mathcal{K}(\mathcal{H}))$ in Example 2.5.2. Galatan and Popa further showed such a vanishing result holds for a larger class of smooth and operatorial (in the sense of Section 2.4) bimodules.

CHAPTER 3

Classes of Smooth Bimodules

Let (M, τ) be a tracial von Neumann algebra, and view $M \subset \mathcal{B}(L^2M)$ as a subalgebra of $\mathcal{B}(L^2M)$. Then, as described in Section 2.3, we recall the following four canonical Banach M-bimodules associated with $\mathcal{B}(L^2M)$. Namely,

- $\mathcal{B}(L^2M)$ itself, which is a dual nomal M-bimodule (see example 2.3.1).
- The subspace $\mathcal{K}(L^2M) \subset \mathcal{B}(L^2M)$ of compact operators. (see example 2.3.2).
- The dual space $\mathcal{B}^*(L^2M) = (\mathcal{B}(L^2M))^*$, which equipped with operator norm $\| \ \|_{\mathcal{B}^*(L^2M)}$ is a dual nomal M-bimodule (see example 2.3.3).
- The subspace of normal linear functionals $\mathcal{B}^*_{\mathrm{n}}(L^2M) \subset \mathcal{B}^*(L^2M)$, i.e. the predual of $\mathcal{B}(L^2M)$ (see example 2.3.4).

We observe that if one begins with the M-bimodule $\mathcal{B}^*(L^2M)$, one can construct all four of the above bimodules by successively taking preduals. Namely, we observe that $\mathcal{B}^*(L^2M) = (\mathcal{B}(L^2M))^*$, $\mathcal{B}(L^2M) = (\mathcal{B}^*_{\mathrm{n}}(L^2M))^*$, and $\mathcal{B}^*_{\mathrm{n}}(L^2M) = (\mathcal{K}(L^2M))^*$.

In this chapter we define, for any $2 \leq p < \infty$, a sort of "L^p-smooth" generalization of each of these bimodules. In section 3.1, we define for $2 \leq p < \infty$ the smooth M-bimodule $\mathcal{B}^*(p)$. This space can be thought of as the space of linear functionals on $\mathcal{B}(L^2M)$ that are "L^p-smooth relative to M". In section 3.2, we construct a smooth bimodule $\mathcal{B}(p)$ analogous to $\mathcal{B}(L^2M)$ by taking the predual of $\mathcal{B}^*(p)$. Lastly in 3.3, we define an analogue of the compact operators $\mathcal{K}(L^2M)$ denoted as $\mathcal{K}(p)$.

The constructions in Sections 3.1, 3.2, and 3.3 are based on the constructions in Sections 3, 4, and 5 in [HPP24]

3.1 The dual Banach M-bimodules $\mathcal{B}^*(p)$, $2 \leq p < \infty$

Fix a tracial von Neumann algebra (M, τ) and we set $\mathcal{B} = \mathcal{B}(L^2M)$. We first consider a one parameter family of M sub-bimodules $\mathcal{B}^*(p) \subset \mathcal{B}^*$, $2 \leq p < \infty$, defined as spaces of functionals on $\mathcal{B}$ that are "L^p-smooth relative to M".

Definition 3.1.1. Let $2 \leq p < \infty$. We denote by $\mathcal{B}^*(p)$ the subspace of functionals $\varphi \in \mathcal{B}^* = \mathcal{B}(L^2M)^*$ with the property that

$$\|\varphi\|_{\mathcal{B}^*(p)} := \sup\{|\varphi(xTy)| \ \mid T \in (\mathcal{B})_1, x, y \in M, \|x\|_p, \|y\|_p \leq 1\}$$

is finite. Note right away that $\mathcal{B}^*(p)$ is a vector subspace of $\mathcal{B}^*$ and that $\|\cdot\|_{\mathcal{B}^*(p)}$ is a norm on it that majorizes the usual norm of functionals in $\mathcal{B}^*$.

Proposition 3.1.2. 1° *The space $\mathcal{B}^*(p)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{B}^*(p)}$.*

2° *If $2 \leq p' \leq p < \infty$, then $\mathcal{B}^*(p') \subset \mathcal{B}^*(p)$. Moreover, for any $\varphi \in \mathcal{B}^*$ we have $\|\varphi\| \leq \|\varphi\|_{\mathcal{B}^*(p)} \leq \|\varphi\|_{\mathcal{B}^*(p')}$. Thus, $\lim_{p\to\infty} \|\varphi\|_{\mathcal{B}^*(p)} = \inf_{p\to\infty} \|\varphi\|_{\mathcal{B}^*(p)} \geq \|\varphi\|$.*

Proof. (1) It remains to check that $\mathcal{B}^*(p)$ is complete with respect to the $\|\cdot\|_{\mathcal{B}^*(p)}$-norm. So take a Cauchy sequence (φ_n) in $(\mathcal{B}^*(p), \|\cdot\|_{\mathcal{B}^*(p)})$. Since the norm $\|\cdot\|_{\mathcal{B}^*(p)}$ majorizes the norm $\|\cdot\|_{\mathcal{B}^*}$, the sequence (φ_n) is also Cauchy in $\mathcal{B}^*$. Let φ be its $\|\cdot\|_{\mathcal{B}^*}$-norm limit in $\mathcal{B}^*$. We claim first that $\varphi \in \mathcal{B}^*(p)$. Take any $T \in (\mathcal{B})_1$ and $x, y \in M$ with $\|x\|_p, \|y\|_p \leq 1$. Since $\varphi_n \to \varphi$ with respect to the $\|\cdot\|_{\mathcal{B}^*}$-norm, we can find an m such that

$$\|\varphi - \varphi_m\|_{\mathcal{B}^*} \leq \|xTy\|^{-1}.$$

In particular, we see that

$$\begin{aligned} |\varphi(xTy)| &\leq |\varphi(xTy) - \varphi_m(xTy)| + |\varphi_m(xTy)| \\ &\leq \|xTy\|\|\varphi - \varphi_m\| + \sup_{n\geq 1} |\varphi_n(xTy)| \\ &\leq 1 + \sup_{n\geq 1} \|\varphi_n\|_{\mathcal{B}^*(p)}. \end{aligned} \tag{3.1}$$

This last quantity is finite since (φ_n) was assumed to be Cauchy in $\mathcal{B}^*(p)$. It follows then that $|\varphi(xTy)|$ is uniformly bounded over all $T \in (\mathcal{B})_1$ and $\|x\|_p, \|y\|_p \leq 1$, and so $\varphi \in \mathcal{B}^*(p)$.

It remains to check that $\varphi_n \to \varphi$ with respect to the $\|\cdot\|_{\mathcal{B}^*(p)}$ norm. To do this, let S be the set of elements X in $\mathcal{B}$ of the form xTy with $T \in (\mathcal{B})_1$ and $x, y \in M$, $\|x\|_p, \|y\|_p \leq 1$. Then we have

$$\begin{aligned} \lim_{n\to\infty} \|\varphi - \varphi_n\|_{\mathcal{B}^*(p)} &= \lim_{n\to\infty} \sup_{X\in S} |\varphi(X) - \varphi_n(X)| \\ &= \lim_{n\to\infty} \sup_{X\in S} \lim_{m\to\infty} |\varphi_m(X) - \varphi_n(X)| \\ &\leq \lim_{n\to\infty} \lim_{m\to\infty} \sup_{X\in S} |\varphi_m(X) - \varphi_n(X)| \\ &= \lim_{n,m\to\infty} \|\varphi_m - \varphi_n\|_{\mathcal{B}^*(p)}. \end{aligned} \tag{3.2}$$

Since (φ_n) was Cauchy with respect to the $\|\cdot\|_{\mathcal{B}^*(p)}$ norm, it follows that (φ_n) also converge to φ with respect to the $\|\cdot\|_{\mathcal{B}^*(p)}$ norm. This shows $\mathcal{B}^*(p)$ is complete, and thus is a Banach space.

(2) Now suppose $2 \leq p' \leq p < \infty$. For any $x \in M$ we have that $\|x\|_{p'} \leq \|x\|_p$, so the set $\{xTy \mid T \in (\mathcal{B})_1, x, y \in M, \|x\|_p, \|y\|_p \leq 1\}$ is a subset of $\{xTy \mid T \in (\mathcal{B})_1, x, y \in M, \|x\|_{p'}, \|y\|_{p'} \leq 1\}$. Taking supremums in the definition of $\|\cdot\|_{\mathcal{B}^*(p)}$, we conclude $\|\varphi\|_{\mathcal{B}^*(p)} \leq \|\varphi\|_{\mathcal{B}^*(p')}$. The rest of the statement follows immediately.

□

Proposition 3.1.3. 1° *Let $2 \leq p < \infty$. If $x, y \in M$ and $\varphi \in \mathcal{B}^*(p)$, then*

$$\|x \cdot \varphi \cdot y\|_{\mathcal{B}^*(p)} \leq \|x\|\|y\|\|\varphi\|_{\mathcal{B}^*(p)}, \ \|x^{op} \cdot \varphi \cdot y^{op}\|_{\mathcal{B}^*(p)} \leq \|x^{op}\|\|y^{op}\|\|\varphi\|_{\mathcal{B}^*(p)}.$$

Thus, the M and M^{op} bimodule structures on $\mathcal{B}^*$ leave $\mathcal{B}^*(p)$ invariant and implement Banach M-bimodule and M^{op}-bimodule structures on $(\mathcal{B}^*(p), \|\cdot\|_{\mathcal{B}^*(p)})$.

2° *The unit ball $(\mathcal{B}^*(p))_1$ is compact in the $\sigma(\mathcal{B}^*, \mathcal{B})$ topology.*

3° *The unit ball $(\mathcal{B}^*(p))_1$ is norm closed in $\mathcal{B}^*$.*

4° *For each $x, y \in M$, the map $\mathcal{B}^*(p) \ni \varphi \mapsto x \cdot \varphi \cdot y \in \mathcal{B}^*(p)$ is continuous with respect to the $\sigma(\mathcal{B}^*, \mathcal{B})$-topology.*

Proof. 1° Take some $2 \leq p < \infty$. Fix elements $x, y \in M$ and a functional $\varphi \in \mathcal{B}^*(p)$. Let $T \in (\mathcal{B})_1$ and $x', y' \in M$ with $\|x'\|_p, \|y'\|_p \leq 1$. Then, if we apply $x \cdot \varphi \cdot y$ to $x'Ty'$, we get

$$|[x \cdot \varphi \cdot y](x'Ty')| = |\varphi(yx'Ty'x)| = \|x\|\|y\| \left|\varphi\left(\frac{yx'}{\|y\|} T \frac{y'x}{\|x\|}\right)\right|.$$

Notice that we have the bounds $\|yx'/\|y\|\|_p \leq 1$ and $\|y'x/\|x\|\|_p \leq 1$. It follows by definition then that

$$|[x \cdot \varphi \cdot y](x'Ty')| = \|x\|\|y\| \left|\varphi\left(\frac{yx'}{\|y\|} T \frac{y'x}{\|x\|}\right)\right| \leq \|x\|\|y\|\|\varphi\|_{\mathcal{B}^*(p)}.$$

Taking the supremum over all $T \in (\mathcal{B})_1$ and all $x', y' \in M$ with $\|x'\|_p, \|y'\|_p \leq 1$ gives the bound $\|x \cdot \varphi \cdot y\|_{\mathcal{B}^*(p)} \leq \|x\|\|y\|\|\varphi\|_{\mathcal{B}^*(p)}$ as desired.

Let us now fix $x^{op}, y^{op} \in M^{op}$. Take $T \in (\mathcal{B})_1$ and $x', y' \in M$ with $\|x'\|_p, \|y'\|_p \leq 1$. As we did before, if we apply $x^{op} \cdot \varphi \cdot y^{op}$ to $x'Ty'$, we get

$$|[x^{op} \cdot \varphi \cdot y^{op}](x'Ty')| = |\varphi(y^{op}x'Ty'x^{op})| = \|x^{op}\|\|y^{op}\| \left|\varphi\left(x' \frac{y^{op}}{\|y^{op}\|} T \frac{x^{op}}{\|x^{op}\|} y'\right)\right|.$$

Here this operator $y^{op}Tx^{op}/\|x^{op}\|\|y^{op}\|$ has norm at most 1, so by definition we get

$$|[x^{op} \cdot \varphi \cdot y^{op}](x'Ty')| \leq \|x^{op}\|\|y^{op}\|\|\varphi\|_{\mathcal{B}^*(p)}.$$

Taking supremums over all T, x', y' will give $\|x^{op} \cdot \varphi \cdot y^{op}\|_{\mathcal{B}^*(p)} \leq \|x^{op}\|\|y^{op}\|\|\varphi\|_{\mathcal{B}^*(p)}$.

2° For $x, y \in M$, let $S_{x,y} \subset \mathcal{B}^*$ be the set of all functionals $\varphi \in \mathcal{B}^*$ such that $\|x \cdot \varphi \cdot y\|_{\mathcal{B}^*} \leq 1$. It is clear from the definitions that

$$(\mathcal{B}^*(p))_1 = \bigcap_{\|x\|_p, \|y\|_p \leq 1} S_{x,y}.$$

Now each of these sets $S_{x,y}$ is closed in the $\sigma(\mathcal{B}^*, \mathcal{B})$ topology so $(\mathcal{B}^*(p))_1$ is also closed in this topology. Furthermore, notice since the norm $\|\cdot\|_{\mathcal{B}^*(p)}$ majorizes the operator norm on $\mathcal{B}^*$ that $(\mathcal{B}^*(p))_1 \subset (\mathcal{B}^*)_1$. The Banach-Alaoglu theorem then gives us that $(\mathcal{B}^*(p))_1$ is compact.

3° This is just a consequence of 2°.

4° Fix elements $x, y \in M$. From 1°, we know that the map $\varphi \mapsto x \cdot \varphi \cdot y$ is a well defined linear map from $\mathcal{B}^*(p)$ to itself. It is also a $\sigma(\mathcal{B}^*, \mathcal{B})$ continuous map on the whole space $\mathcal{B}^*$, so restricting to $\mathcal{B}^*(p)$ proves the claim.

□

Lemma 3.1.4. *Let $2 \leq p < \infty$ and $\varphi \in \mathcal{B}^*$. Assume $\varphi = \omega_{\xi,\eta}$ for some $\xi, \eta \in L^2M$. Then $\varphi \in \mathcal{B}^*(p)$ if and only if $\xi, \eta \in L^qM$, where $q = \frac{2p}{p-2}$, with the conventions $\frac{1}{0} = \infty$. Moreover, if this is the case, then $\|\omega_{\xi,\eta}\|_{\mathcal{B}^*(p)} = \|\xi\|_q \|\eta\|_q$.*

Proof. Take elements $x, y \in M$ with $\|x\|_p, \|y\|_p \leq 1$ and an operator $T \in (\mathcal{B})_1$. By Cauchy-Schwartz, we have a bound

$$|\omega_{\xi,\eta}(xTy)| = |\langle xTy\xi, \eta\rangle| = |\langle Ty\xi, x^*\eta\rangle| \leq \|Ty\xi\|_2 \|x^*\eta\|_2 \leq \|y\xi\|_2 \|x^*\eta\|_2.$$

This gives a bound

$$\|\omega_{\xi,\eta}\|_{\mathcal{B}^*(p)} \leq \sup_{\|y\|_p \leq 1} \|y\xi\|_2 \sup_{\|x\|_p \leq 1} \|x^*\eta\|_2 = \sup_{\|y\|_p \leq 1} \|y\xi\|_2 \sup_{\|x\|_p \leq 1} \|x\eta\|_2.$$

Notice that the reverse inequality also holds. For if $x, y \in M$ are fixed with $\|x\|_p, \|y\|_p \leq 1$, consider the rank-one partial isometry $T_{x,y} \in (\mathcal{B})_1$ that maps $y\xi$ to $\frac{\|y\xi\|}{\|x^*\eta\|} x^*\eta$. Then

$$\|y\xi\|_2 \|x^*\eta\|_2 = |\omega_{\xi,\eta}(xT_{x,y}y)| \leq \|\omega_{\xi,\eta}\|_{\mathcal{B}^*(p)}.$$

Taking the supremum over x and y gives us the reverse inequality.

So far we have that

$$\|\omega_{\xi,\eta}\|_{\mathcal{B}^*(p)} = \sup_{\|y\|_p \leq 1} \|y\xi\|_2 \sup_{\|x\|_p \leq 1} \|x\eta\|_2.$$

If we now use the non-commutative version of Hölder's inequality, then

$$\sup_{\|y\|_p \leq 1} \|y\xi\|_2 = \|\xi\|_q,$$

where $1/p + 1/q = 1/2$, or $q = \frac{2p}{p-2}$. Similarly,

$$\sup_{\|x\|_p \leq 1} \|x\eta\|_2 = \|\eta\|_q.$$

This gives the desired result $\|\omega_{\xi,\eta}\|_{\mathcal{B}^*(p)} = \|\xi\|_q \|\eta\|_q$.

□

A version of the previous lemma actually works for arbitrary positive finite-rank functionals $\varphi \in \mathcal{B}^*(p)$.

Lemma 3.1.5. *Fix $2 \leq p < \infty$, and let $q = \frac{2p}{p-2}$ be as in the last lemma. Let $\varphi \in B^*(p)$ be of the form $\varphi = \sum_{i=1}^n \omega_{\xi_i,\xi_i}$ where $\xi_1, \xi_2, \ldots, \xi_n$ are in $L^q M$. Then*

$$\|\varphi\|_{\mathcal{B}^*(p)} = \left\| \left| \sum_{i=1}^n \xi_i \xi_i^* \right| \right\|_{q/2}.$$

Proof. First we derive a lower bound for $\|\varphi\|_{\mathcal{B}^*(p)}$. Recall that $\|\varphi\|_{\mathcal{B}^*(p)}$ is a supremum over the values $|\varphi(xTy)|$, where $x, y \in M$ are such that $\|x\|_p, \|y\|_p \leq 1$ and $T \in \mathcal{B}(L^2 M)$ is such that $\|T\| \leq 1$. In particular, if we make T the identity on $\mathcal{B}(L^2 M)$,

$$\begin{aligned} \|\varphi\|_{\mathcal{B}^*(p)} \geq \sup_{\|x\|_p, \|y\|_p \leq 1} |\varphi(xy)| &= \sup_{\|x\|_p, \|y\|_p \leq 1} \left| \sum_{i=1}^n \langle xy\xi_i, \xi_i \rangle \right| \\ &= \sup_{\|x\|_p, \|y\|_p \leq 1} \left| \sum_{i=1}^n \tau(xy\xi_i\xi_i^*) \right| \\ &= \sup_{\|x\|_p, \|y\|_p \leq 1} \left| \tau\left(xy \sum_{i=1}^n \xi_i \xi_i^* \right) \right|. \end{aligned} \tag{3.3}$$

As x and y range over all elements with p norm at most 1, xy can be any element of M with $p/2$ norm at most 1. By density, the above supremum is equal to $\sup_{\eta \in (L^{p/2})_1} \tau\left(\eta \sum_{i=1}^n \xi_i \xi_i^*\right)$, which by duality is the same as $\left|\left|\sum_{i=1}^n \xi_i \xi_i^*\right|\right|_r$, where r is the Hölder conjugate of $p/2$. A quick calculation gives

$$\frac{1}{r} = 1 - \frac{1}{p/2} = \frac{p-2}{p} = \frac{1}{q/2}.$$

So we get a lower bound $\left|\left|\sum_{i=1}^n \xi_i \xi_i^*\right|\right|_{q/2}$ for $\|\varphi\|_{\mathcal{B}^*(p)}$.

Now we prove the reverse inequality. By definition, $\|\varphi\|_{\mathcal{B}^*(p)}$ is the supremum over all sums

$$\varphi(xTy) = \sum_{i=1}^n \langle xTy\xi_i, \xi_i\rangle = \sum_{i=1}^n \langle Ty\xi_i, x^*\xi_i\rangle,$$

where $\|x\|_p, \|y\|_p \leq 1$ and $\|T\| \leq 1$. By Cauchy Schwartz, any one of the inner products $\langle Ty\xi_i, x^*\xi_i\rangle$ is bounded by

$$\langle Ty\xi_i, x^*\xi_i\rangle \leq \|x^*\xi_i\|_2 \|Ty\xi_i\|_2 \leq \|x^*\xi_i\|_2 \|y\xi_i\|_2.$$

Thus, we get the upper bound

$$\|\varphi\|_{\mathcal{B}^*(p)} \leq \sup_{\|x\|_p, \|y\|_p \leq 1} \sum_{i=1}^n \|x^*\xi_i\|_2 \|y\xi_i\|_2 = \sup_{\|x\|_p, \|y\|_p \leq 1} \sum_{i=1}^n \|x\xi_i\|_2 \|y\xi_i\|_2.$$

If we use Hölder's inequality, then

$$\begin{aligned} \|\varphi\|_{\mathcal{B}^*(p)} &\leq \sup_{\|x\|_p, \|y\|_p \leq 1} \left(\sum_{i=1}^n \|x\xi_i\|_2^2\right)^{1/2} \left(\sum_{i=1}^n \|y\xi_i\|_2^2\right)^{1/2} \\ &\leq \sup_{\|x\|_p \leq 1} \sum_{i=1}^n \|x\xi_i\|_2^2. \end{aligned} \tag{3.4}$$

Now, we can write $\|x\xi_i\|_2^2$ in terms of τ so that

$$\|\varphi\|_{\mathcal{B}^*(p)} \leq \sup_{\|x\|_p \leq 1} \sum_{i=1}^n \tau(x^* x \xi_i \xi_i^*) \leq \sup_{\|x\|_p \leq 1} \tau\left(x^* x \sum_{i=1}^n \xi_i \xi_i^*\right).$$

By the same duality argument, this is equal to $\left|\left|\sum_{i=1}^n \xi_i \xi_i^*\right|\right|_{q/2}$. This completes the proof. □

We remark that one can calculate an upper bound for $\|\varphi\|_{\mathcal{B}^*(p)}$ for an arbitrary finite-rank functional $\varphi \in \mathcal{B}^*(p)$ by using the polarization identity combined with Lemma 3.1.5. For a general, not necessarily finite-rank, φ one has the following.

Proposition 3.1.6. *Let $2 \leq p < \infty$ and $\varphi \in \mathcal{B}^*$.*

1° *If $\varphi \in \mathcal{B}^*(p)$, then $\varphi^* \in \mathcal{B}^*(p)$ and $\|\varphi^*\|_{\mathcal{B}^*(p)} = \|\varphi\|_{\mathcal{B}^*(p)}$. Thus, $\Re\varphi, \Im\varphi \in \mathcal{B}^*(p)$ and $\|\Re\varphi\|_{\mathcal{B}^*(p)}$, $\|\Im\varphi\|_{\mathcal{B}^*(p)} \leq \|\varphi\|_{\mathcal{B}^*(p)}$.*

2° *If $\varphi \in \mathcal{B}^*(p)$, then its normal and singular parts (as functionals in $\mathcal{B}^*$) φ_n, φ_s, belong to $\mathcal{B}^*(p)$, with $\|\varphi_n\|_{\mathcal{B}^*(p)}$, $\|\varphi_s\|_{\mathcal{B}^*(p)} \leq \|\varphi\|_{\mathcal{B}^*(p)}$.*

Proof. 1° By the definitions, one obviously have $\|\varphi\|_{\mathcal{B}^*(p)} = \|\varphi^*\|_{\mathcal{B}^*(p)}$ for each $\varphi \in \mathcal{B}^*$. Thus, $\varphi \in \mathcal{B}^*(p)$ implies $\varphi^* \in \mathcal{B}^*(p)$, and hence also the real and imaginary parts of any such φ, lie in $\mathcal{B}^*(p)$. The given upper bounds then follow from the triangle inequality.

2° Let φ be any element of $\mathcal{B}^*(p)$, and let φ_{n} and φ_{s} be the normal and singular parts of φ respectively. Recalling the construction of these functionals, let p_M be the central projection in $\mathcal{B}^{**}$ such that $\varphi_{\mathrm{n}} = p_M \cdot \varphi$ and $\varphi_{\mathrm{s}} = (1 - p_M) \cdot \varphi$. If x and y are any elements of M such that $\|x\|_p, \|y\|_p \leq 1$, then by using the fact that p_M commutes with M we get

$$x \cdot \varphi_{\mathrm{n}} \cdot y = x \cdot (p_M \cdot \varphi) \cdot y = p_M(x \cdot \varphi \cdot y).$$

If we then apply the usual norm from $\mathcal{B}^*$ we have that

$$\|x \cdot \varphi_{\mathrm{n}} \cdot y\| = \|p_M(x \cdot \varphi \cdot y)\| \leq \|x \cdot \varphi \cdot y\| \leq \|\varphi\|_{\mathcal{B}^*(p)}.$$

Taking the supremum over all x and y with $\|x\|_p, \|y\|_p \leq 1$, gives us then that $\|\varphi_{\mathrm{n}}\|_{\mathcal{B}^*(p)} \leq \|\varphi\|_{\mathcal{B}^*(p)}$. The same argument with $1 - p_M$ shows that $\|\varphi_{\mathrm{s}}\|_{\mathcal{B}^*(p)} \leq \|\varphi\|_{\mathcal{B}^*(p)}$.

□

Corollary 3.1.7. *Let $2 \leq p < \infty$ and denote $\mathcal{B}^*_n(p) = \{\varphi \in \mathcal{B}^*(p) \mid \varphi = \varphi_n\}$. Then $\mathcal{B}^*_n(p)$ is norm closed and $\sigma(\mathcal{B}^*, \mathcal{B})$-dense in $(\mathcal{B}^*(p), \|\cdot\|_{\mathcal{B}^*(p)})$.*

Proof. Take any $2 \leq p \leq \infty$. Since the space $(\mathcal{B}^*_{\mathrm{n}})^* = \mathcal{B}$, the space is B^*_{n} is $\sigma(\mathcal{B}^*, \mathcal{B})$ dense in $\mathcal{B}^*$. Moreover, the space $\mathcal{L} \subset \mathcal{B}^*_{\mathrm{n}}$ obtained as the span of functionals of the form $\omega_{\xi,\eta}$ with $\xi, \eta \in \hat{M} \subset L^2M$ is clearly dense in $\mathcal{B}^*_{\mathrm{n}}$ with respect to the usual norm in $\mathcal{B}^*$. Since $\mathcal{L}$ is contained in $\mathcal{B}^*(p)$, this implies that $\mathcal{B}^*_{\mathrm{n}}(p)$ is $\sigma(\mathcal{B}^*, \mathcal{B})$ dense in $\mathcal{B}^*(p)$.

Next, consider a Cauchy sequence $\{\varphi_n\}$ in $\mathcal{B}^*_{\mathrm{n}}(p)$. Since $\mathcal{B}^*(p)$ is complete, the sequence converges to some $\varphi \in \mathcal{B}^*(p)$. But for all p, the $\|\cdot\|_{\mathcal{B}^*(p)}$ norm dominates the usual norm of functionals in $\mathcal{B}^*$. Thus, φ is the usual norm limit in $\mathcal{B}^*$ of the normal functionals φ_n, and hence it is normal itself, $\varphi \in \mathcal{B}^*_{\mathrm{n}}$, showing that $\mathcal{B}^*_{\mathrm{n}}(p)$ is norm closed.

□

We end this section by noticing that the norm $\|\cdot\|_{\mathcal{B}^*(p)}$ on the M-bimodules $\mathcal{B}^*(p)$ satisfies an interesting property with respect to direct sums, which we will however not use in this paper.

Proposition 3.1.8. *Let (M, τ) be a tracial von Neumann algebras and $2 \leq p < \infty$. Assume $\varphi_1, \varphi_2 \in \mathcal{B}^*(p)$ are supported by mutually orthogonal projections in $\mathcal{Z}(M)$, i.e., there exist $z_1, z_2 \in \mathcal{P}(\mathcal{Z}(M))$ such that $\varphi_i = \varphi_i(z_i \cdot z_i)$, $i = 1, 2$. Then for $p = 2$ we have $\|\varphi_1 + \varphi_2\|_{\mathcal{B}^*(2)} = \max\{\|\varphi_1\|_{\mathcal{B}^*(2)}, \|\varphi_2\|_{\mathcal{B}^*(2)}\}$ and for $2 < p < \infty$ we have $\|\varphi_1 + \varphi_2\|_{\mathcal{B}^*(p)} = (\|\varphi_1\|^q_{\mathcal{B}^*(p)} + \|\varphi_1\|^q_{\mathcal{B}^*(p)})^{1/q}$, where $q = \frac{p}{p-2}$.*

Proof. Let $\varphi = \varphi_1 + \varphi_2$. By definition, $\|\varphi\|_{\mathcal{B}^*(p)}$ is the supremum of $|\varphi(xTy)|$ for $x, y \in M$ with p-norm at most 1 and $T \in \mathcal{B}(L^2M)$ with norm at most 1. Since φ is supported on $z_1 + z_2$, we can restrict the values of x and y we take to only those in $M(z_1 + z_2)$, and operators T we take to those supported on $(z_1 + z_2)L^2M$. With this in mind, consider such a triple x, y, and T. We can decompose $x = x_1 + x_2$ where $x_1 = z_1xz_1$ and $x_2 = z_2xz_2$. Similarly, we can decompose $y = y_1 + y_2$ where y_1 and y_2 are defined in the same manner. We then define the operator

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where here $T_{ij} = z_i T z_j$. Under this decomposition, we have

$$|\varphi(xTy)| = |\varphi_1(x_1 T_{11} y_1) + \varphi_2(x_2 T_{22} y_2)|.$$

We now wish to maximize this quantity given $\|T\| \leq 1$ and $|x|_p, |y|_p \leq 1$. First, it is clear that it is optimal make the off diagonal terms of T equal to 0, and have the diagonal terms T_{11} and T_{22} have norm 1. Next, we see by properties of the p-norm in M that $|x_1|_p^p + |x_2|_p^p = |x|_p^p$ and $|y_1|_p^p + |y_2|_p^p = |y|_p^p$. While varying the x_i, y_i, T_{ii} under these constraints, we calculate the norm $\|\varphi\|_{\mathcal{B}^*(p)}$ to be the supremum of

$$\alpha_1\beta_1\|\varphi_1\|_{\mathcal{B}^*(p)} + \alpha_2\beta_2\|\varphi_1\|_{\mathcal{B}^*(p)},$$

where the α_i and β_i are in $[0,1]$ and satisfy $\alpha_1^p + \alpha_2^p = 1$ and $\beta_1^p + \beta_2^p = 1$. Now let q be such that $1/q + 2/p = 1$, i.e. the Hölder conjugate of $p/2$. Then the discrete version of Hölder's inequality gives us

$$\begin{aligned}\alpha_1\beta_1\|\varphi_1\|_{\mathcal{B}^*(p)} + \alpha_2\beta_2\|\varphi_1\|_{\mathcal{B}^*(p)} &\leq (\alpha_1^p + \alpha_2^p)^{1/p}(\beta_1^p + \beta_2^p)^{1/p}(\|\varphi_1\|_{\mathcal{B}^*(p)}^q + \|\varphi_2\|_{\mathcal{B}^*(p)}^q)^{1/q} \\ &= (\|\varphi_1\|_{\mathcal{B}^*(p)}^q + \|\varphi_2\|_{\mathcal{B}^*(p)}^q)^{1/q}.\end{aligned} \tag{3.5}$$

Moreover, equality is guaranteed to be achieved for some values of α_i and β_i. This gives us

$$\|\varphi\|_{\mathcal{B}^*(p)} = (\|\varphi_1\|_{\mathcal{B}^*(p)}^q + \|\varphi_2\|_{\mathcal{B}^*(p)}^q)^{1/q}.$$

Raising both sides to the q^{th} power then completes the proof.

□

3.2 The Banach bimodules $\mathcal{B}(p)$, $2 \leq p < \infty$

We now consider the natural preduals of the spaces $\mathcal{B}^*(p)$ introduced in the previous section.

Definition 3.2.1. Let $2 \leq p < \infty$. For each $T \in \mathcal{B} = \mathcal{B}(L^2M)$, denote $|||T|||_p = \sup\{|\varphi(T)| \mid \varphi \in (\mathcal{B}^*(p))_1\}$. Noticing that $|||\cdot|||_p$ is a norm on $\mathcal{B}$, we denote by $\mathcal{B}(p)$ the completion of $\mathcal{B}$ in this norm.

Lemma 3.2.2. 1° *For each $T \in \mathcal{B}$, the norms $|||T|||_p$ are increasing in p and majorized by the operator norm $\|T\|$, with $\lim_{p\to\infty} |||T|||_p = \sup_p |||T|||_p = \|T\|$.*

2° *If $T \in \mathcal{B}$ and $x, y \in M$, then*

$$|||xTy|||_p \leq \|x\|_p \|T\| \|y\|_p, \ |||xTy|||_p \leq \|x\| |||T|||_p \|y\|,$$

$$|||x^{op} T y^{op}|||_p \leq \|x^{op}\| |||T|||_p \|y^{op}\|.$$

Proof. 1° Take $2 \leq p \leq p' < \infty$ and $T \in \mathcal{B}$. Since $(\mathcal{B}^*(p))_1 \subset (\mathcal{B}^*(p'))_1$ we have that

$$\begin{aligned} |||T|||_p &= \sup\{|\varphi(T)| \ \mid \varphi \in (\mathcal{B}^*(p))_1\} \\ &\leq \sup\{|\varphi(T)| \ \mid \varphi \in (\mathcal{B}^*(p'))_1\} \\ &= |||T|||_{p'}. \end{aligned} \tag{3.6}$$

So the norms $|||\cdot|||_p$ are increasing as p increases. For any finite $2 \leq p < \infty$, we also have a bound $|||T|||_p \leq \|T\|$, since the unit ball $(\mathcal{B}^*(p))_1$ is a subset of $(\mathcal{B}^*)_1$. So it follows that $|||T|||_p$ converges as p tends to infinity.

To find the limit of these norms, take x and y any elements of M. Let $\hat{x}$ and $\hat{y}$ be the associated elements of $L^2 M$. By Lemma 3.1.4 we have

$$|\langle T\hat{x}, \hat{y}\rangle| = |\omega_{\hat{x},\hat{y}}(T)| \leq |||T|||_p \|\omega_{\xi,\eta}\|_{\mathcal{B}^*(p)} = |||T|||_p \|x\|_q \|y\|_q,$$

where $q = \frac{2p}{p-2}$. Letting p tend to infinity gives us the bound

$$|\langle T\hat{x}, \hat{y}\rangle| \leq \|x\|_2 \|y\|_2 \lim_{p\to\infty} |||T|||_p.$$

If we take the supremum over all $x, y \in M$ with $\|x\|_2, \|y\|_2 \leq 1$, we get that $\|T\| \leq \lim_{p\to\infty} |||T|||_p$. The result then follows.

2° First consider when $2 \leq p < \infty$. Fix $x, y \in M$ and $T \in \mathcal{B}$. If φ is an element of $\mathcal{B}^*(p)$, then we have a bound

$$\begin{aligned} |\varphi(xTy)| &= \|x\|_p \|y\|_p \|T\| \cdot \left|\varphi\left(\frac{x}{\|x\|_p} \frac{T}{\|T\|} \frac{y}{\|y\|_p}\right)\right| \\ &\leq \|x\|_p \|y\|_p \|T\| \cdot \|\varphi\|_{\mathcal{B}^*(p)}. \end{aligned} \tag{3.7}$$

If we take the supremum over all φ with $\|\varphi\|_{\mathcal{B}^*(p)} \leq 1$, this gives the

$$\|\|xTy\|\|_p = \sup_{\|\varphi\|_{\mathcal{B}^*(p)} \leq 1} |\varphi(xTy)| \leq \|x\|_p \|T\| \|y\|_p,$$

which is the first desired inequality. On the other hand, one could also note that

$$\|\|xTy\|\|_p = \sup_{\|\varphi\|_{\mathcal{B}^*(p)} \leq 1} |\varphi(xTy)| = \sup_{\|\varphi\|_{\mathcal{B}^*(p)} \leq 1} |(y \cdot \varphi \cdot x)(T)|.$$

From Proposition 3.1.3, we know that $\|y \cdot \varphi \cdot x\|_{\mathcal{B}^*(p)} \leq \|x\| \|y\| \|\varphi\|_{\mathcal{B}^*(p)}$. Thus, it follows that

$$\|\|xTy\|\|_p \leq \sup_{\|\varphi\|_{\mathcal{B}^*(p)} \leq \|x\| \|y\|} |\varphi(T)| = \|x\| \|\|T\|\|_p \|y\|.$$

This gives the second desired inequality. The case when x and y are elements of M^{op} follows by the exact same reasoning. □

Proposition 3.2.3. *Let $q = \frac{2p}{p-2}$ as before, and let $q' = \frac{2p}{p+2}$ be the Hölder conjugate of q. If $T \in \mathcal{B}(L^2M)$ satisfies $\|\|T\|\|_p \leq 1$, then T takes the unit ball of L^qM into the unit ball of $L^{q'}M$, thus defining an element $\tilde{T} \in (\mathcal{B}(L^qM, L^{q'}M))_1$. The map $T \mapsto \tilde{T}$ extends uniquely to a contractive linear map from $\mathcal{B}(p)$ into $\mathcal{B}(L^qM, L^{q'}M)$, which is injective when restricted to $\mathcal{B}(L^2M)$.*

Proof. Noticing that for any $2 \leq p \leq \infty$ one has $q \leq 2 \leq q'$, if $T \in \mathcal{B}(L^2M)$, then for any vector $\xi \in L^qM \subset L^2M$ we have

$$\|T\xi\|_{q'} \leq \|T\xi\|_2 \leq \|T\| \|\xi\|_2 \leq \|T\| \|\xi\|_q.$$

Hence, T restricts to a bounded operator $\tilde{T} \in \mathcal{B}(L^qM, L^{q'}M)$. Moreover, we notice by Lemma 3.1.4 that if ξ, η are vectors in L^qM, then

$$|\langle T\xi, \eta \rangle| = |\omega_{\xi,\eta}(T)| \leq \|\omega_{\xi,\eta}\|_{\mathcal{B}^*(p)} \|\|T\|\|_p = \|\xi\|_q \|\eta\|_q \|\|T\|\|_p.$$

Thus, the bilinear form $u : L^2M \times L^2M \to \mathbb{C}$ given by $u(\xi, \eta) = \langle T\xi, \eta \rangle$ restricts to a bilinear form on $L^qM \times L^qM$ with norm at most $\|\|T\|\|_p$. But notice by the noncommutative version of Hölder's inequality

$$\sup_{\|\xi\|_q, \|\eta\|_q \leq 1} |\langle T\xi, \eta \rangle| = \|T\|_{L^qM \to L^{q'}M}$$

where here, this norm represents the operator norm in $\mathcal{B}(L^qM, L^{q'}M)$. Thus we conclude that $\|\tilde{T}\|_{L^qM\to L^{q'}M} \leq |||T|||_p$. By the $|||\cdot|||_p$-density of $\mathcal{B}(L^2M)$ in $\mathcal{B}(p)$, it follows that the map $T \mapsto \tilde{T}$ extends uniquely to a contractive linear map on all $\mathcal{B}(p)$.

□

Proposition 3.2.4. 1° *The restriction of the norm $|||\cdot|||_p$ to $M \subset \mathcal{B}$ is equal to the norm $\|\cdot\|_{p/2}$ for $L^{\frac{p}{2}}M$.*

2° *If M is assumed to be a factor, then the restriction of the norm $|||\cdot|||_p$ to $M^{op} \subset \mathcal{B}$ is equal to the operator norm $\|\cdot\|$ on M^{op}.*

3° *If M^{op} is viewed as a subset of $\mathcal{B}(L^qM, L^{q'}M)$, then the restriction of the norm $\|\cdot\|_{L^qM\to L^{q'}M}$ to M^{op} is equal to the norm $\|\cdot\|_{p/2}$.*

Proof. 1° Fix an element of $x \in M$. Let $x = u|x|$ be the polar decomposition of x. Let $1_{\mathcal{B}}$ be identity operator in $\mathcal{B}$. Then for any $\varphi \in \mathcal{B}^*(p)$

$$\begin{aligned} |\varphi(x)| &= |\varphi(u|x|^{1/2}1_{\mathcal{B}}|x|^{1/2})| \\ &\leq \|u|x|^{1/2}\|_p \cdot \||x|^{1/2}\|_p \cdot \|1_{\mathcal{B}}\| \cdot \|\varphi\|_{\mathcal{B}^*(p)} \\ &= \||x|^{1/2}\|_p^2 \cdot \|\varphi\|_{\mathcal{B}^*(p)} \\ &= \|x\|_{p/2} \cdot \|\varphi\|_{\mathcal{B}^*(p)}. \end{aligned} \tag{3.8}$$

Taking the supremum over all φ in $(\mathcal{B}^*(p))_1)$ gives the inequality $|||x|||_p \leq \|x\|_{p/2}$.

Now we prove the reverse inequality. Let $q = \frac{2p}{p-2}$, as in Lemma 3.1.4. Then note that if ξ and η are vectors in L^2M such that $\|\xi\|_q = \|\eta\|_q = 1$, Lemma 3.1.4 implies that $|\langle x\xi, \eta\rangle| \leq |||x|||_p$. Thus, we have that

$$|||x|||_p \geq \sup_{\|\xi\|_q=\|\eta\|_q=1} |\langle x\xi, \eta\rangle|. \tag{3.9}$$

If we choose q' such that $\frac{1}{q} + \frac{1}{q'} = 1$ and choose r such that $\frac{1}{r} + \frac{1}{q} = \frac{1}{q'}$, then by duality we

have

$$\sup_{\|\xi\|_q=\|\eta\|_q=1} |\langle x\xi, \eta\rangle| = \sup_{\|\xi\|_q=1} \|x\xi\|_{q'} = \|x\|_r,$$

so we have a bound $\|x\|_r \leq |||x|||_p$. Now by our chosen definition of r we check that

$$\frac{1}{r} = \frac{1}{q'} - \frac{1}{q} = 1 - \frac{2}{q} = \frac{2}{p}.$$

So indeed, we have $r = p/2$, and the reverse inequality $\|x\|_{p/2} \leq |||x|||_p$ holds. This completes the proof.

2° Assume that M is a factor, and take an element $x^{op} \in M^{op}$. By Lemma 3.2.2, we already know that $|||x^{op}|||_p \leq \|x^{op}\|$, so it suffices to check that $|||x^{op}|||_p \geq \|x^{op}\|$. To do this, we will construct a family of functionals $\varphi \in \mathcal{B}^*(p)$ such that $|\varphi(x^{op})|/\|\varphi\|_{\mathcal{B}^*(p)}$ can come arbitrarily close to $\|x^{op}\|$. From here the result will follow since $|||x^{op}|||_p \geq |\varphi(x^{op})|/\|\varphi\|_{\mathcal{B}^*(p)}$ for all $\varphi \in \mathcal{B}^*(p)$

With this in mind, let's say we choose a self adjoint element $m \in M$ and a finite list of unitaries $u_1, u_2, \ldots, u_n \in M$. Then we can define a linear functional $\varphi \in \mathcal{B}^*(p)$ by

$$\varphi(T) = \frac{1}{n}\sum_{i=1}^{n} \langle T(u_i m), u_i m\rangle.$$

By Lemma 3.1.5, we know that

$$\|\varphi\|_{\mathcal{B}^*(p)} = \left\| \frac{1}{n}\sum_{i=1}^{n} (u_i m)(u_i m)^* \right\|_{q/2} = \left\| \frac{1}{n}\sum_{i=1}^{n} u_i m^2 u_i^* \right\|_{q/2}.$$

If we apply this φ to x^{op}, we get

$$\varphi(x^{op}) = \frac{1}{n}\sum_{i=1}^{n} \langle x^{op}(u_i m), u_i m\rangle = \frac{1}{n}\sum_{i=1}^{n} \langle u_i m x, u_i m\rangle.$$

Using that this inner product comes from the trace τ, we can simplify this to be

$$\varphi(x^{op}) = \frac{1}{n}\sum_{i=1}^{n} \tau(u_i m x (u_i m)^*) = \frac{1}{n}\sum_{i=1}^{n} \tau(x m^2) = \langle x, m^2\rangle.$$

Now, using that $|||x|||_p \geq |\varphi(x^{op})|/\|\varphi\|_{\mathcal{B}^*(p)}$, we get the following lower bound

$$|||x^{op}|||_p \geq |\langle x, m^2\rangle| \left\| \frac{1}{n}\sum_{i=1}^{n} u_i m^2 u_i^* \right\|_{q/2}^{-1}.$$

Note that this holds for any self adjoint $m \in M$ and any choice of unitaries $u_1, u_2, \cdots, u_n$. But by Diximier's averaging property, we know the $\|\cdot\|$-norm closure of the convex hull of the set $\{um^2u^* : u \in \mathcal{U}(M)\}$ intersects the center of M. In this case, since M was assumed to be a factor, the center of M is trivial. In particular, since the trace of any element in $\{um^2u^* : u \in \mathcal{U}(M)\}$ is $\tau(m^2)$, it follows that $\tau(m^2)$ is in the $\|\cdot\|$-norm closure of the convex hull of this set. Now the operator norm $\|\cdot\|$ majorizes the norm $\|\cdot\|_{q/2}$ norm, so the same averaging result is true in the space $L^{q/2}M$. It follows then from the above lower bound that

$$|||x^{op}|||_p \geq |\langle x, m^2\rangle| \frac{1}{\tau(m^2)} = |\langle x, \frac{m^2}{\tau(m^2)}\rangle|,$$

where m^2 can be an arbitrary positive element of M. Thus, we conclude that

$$|||x^{op}|||_p \geq \sup_{m \geq 0,\ \|m\|_1 \leq 1} |\langle x, m\rangle|,$$

where here this supremum runs over all positive $m \in M$ with $\|m\|_1 = \tau(m) \leq 1$.

Now if x was assumed to be positive, duality would force this supremum to be $\|x\|$, which would give us the desired reverse inequality $\|x^{op}\| \leq |||x^{op}|||_p$. In general, we can write $x^{op} = u^{op}|x^{op}|$ to be the polar decomposition of x^{op}. Note by part 1° of Proposition 3.1.3, that the map $T \mapsto u^{op}T$ is an isometry on $\mathcal{B}(p)$ with respect to the $|||\cdot|||_p$-norm. In particular, it follows from the positive case that

$$|||x^{op}|||_p = ||| \, |x^{op}| \, |||_p \geq \| \, |x^{op}| \, \| = \|x^{op}\|,$$

so the reverse inequality holds for a general x^{op}, which completes the proof of the first claim.

3° We see by Hölder's inequality that $\|x^{op}\|_{L^qM \to L^{q'}M}$ is equal to $\|x^{op}\|_r$, where r is the solution to the equation $q^{-1} + r^{-1} = (q')^{-1}$. Using the definition of q and q', one gets $r = p/2$, as desired.

□

Corollary 3.2.5. *If M is a II_1 factor, then the map $\mathcal{B}(p) \ni T \mapsto \tilde{T} \in \mathcal{B}(L^qM, L^{q'}M)$ is not a homeomorphism of Banach spaces.*

Proof. Let $x_n \in (M)_1$ be so that $\|x_n\| = 1$ but $\|x_n\|_{p/2} \to 0$. If we take $T_n = x_n^{op}$, then by Proposition 3.2.4.2° we have $|||T_n|||_p = \|T_n\| = \|x_n\| = 1$, while by 3.2.4.3° we have $\|\tilde{T_n}\|_{L^q M \to L^{q'} M} = \|x_n\|_{p/2} \to 0$.

□

The next result, which is crucial in proving Theorem 4.1.5 later in this paper (Theorem 1.0.1 in the introduction) should be compared to [Oza10] and Proposition 3.1 in [DEP23] where similar decompositions are considered. The previous corollary shows however that the proof strategy employed there will not apply to our current situation.

Lemma 3.2.6. *For any $T \in \mathcal{B}(L^2 M)$*

$$|||T|||_p = \inf\{\|a\|_p \|S\| \|b\|_p : a, b \in M, S \in \mathcal{B}(L^2 M), aSb = T\}.$$

Proof. Using Lemma 3.2.2, we have that for any decomposition $T = aSb$ with $a, b \in M$ and $S \in \mathcal{B}(L^2 M)$ that $|||T|||_p \leq \|a\|_p \|S\| \|b\|_p$. This shows at the very least that $|||T|||_p$ is smaller than this infimum.

To obtain the reverse inequality, we use a convexity argument. To this end, for the remainder of the proof we consider the following subsets of $\mathcal{B}(L^2 M)$. For any positive number α, let $\mathcal{C}_\alpha$ be the set of operators $T \in \mathcal{B}(L^2 M)$ such that we can find a decomposition $T = aSb$ with $a, b \in M$ and $S \in \mathcal{B}(L^2 M)$ such that $\|a\|_p \|S\| \|b\|_p \leq \alpha$. We claim first that $\mathcal{C}_\alpha$ is convex.

For let's say we have operators $T_1, T_2 \in \mathcal{C}_\alpha$. Then by definition, we can find decompositions $T_1 = a_1 S_1 b_1$ and $T_2 = a_2 S_2 b_2$ with the $a_i, b_i \in M$ and the $S_i \in \mathcal{B}(L^2 M)$ such that $\|a_i\| \|S_i\| \|b_i\|_p \leq \alpha$. After rescaling, we may assume without loss of generality that $\|S_i\| = 1$ and $\|a_i\|_p = \|b_i\|_p \leq \alpha^{1/2}$. For any $\lambda \in (0, 1)$, we can form the decomposition $\lambda T_1 + (1 - \lambda) T_2 = aSb$ as follows. First, factor $\lambda T_1 + (1 - \lambda) T_2$ as a chain of operators through $L^2 M \oplus L^2 M$ by noting

$$\lambda a_1 S_1 b_1 + (1 - \lambda) a_2 S_2 b_2 = \begin{pmatrix} \lambda^{1/2} a_1 & (1 - \lambda)^{1/2} a_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} \lambda^{1/2} b_1 \\ (1 - \lambda)^{1/2} b_2 \end{pmatrix}.$$

Let au be the polar decomposition of $(\lambda^{1/2}a_1 \ (1-\lambda)^{1/2}a_2)$, where $a \in M$ is positive, and let ub be the polar decomposition of $\binom{\lambda^{1/2}b_1}{(1-\lambda)^{1/2}b_2}$, where here $b \in M$ is positive. If we call $S \in \mathcal{B}(L^2M)$ the product

$$S = u \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} v.$$

We arrive at a decomposition $T_1 + T_2 = aSb$.

Now we can calculate that

$$aa^* = \lambda a_1 a_1^* + (1-\lambda) a_2 a_2^*.$$

So that

$$\begin{aligned} \|a\|_p^2 = \|aa^*\|_{p/2} &\leq \lambda \|a_1 a_1^*\|_{p/2} + (1-\lambda)\|a_2 a_2^*\|_{p/2} \\ &= \lambda \|a_1\|_p^2 + (1-\lambda)\|a_2\|_p^2 \\ &\leq \alpha. \end{aligned} \tag{3.10}$$

By the same logic we also have $\|b\|_p^2 \leq \alpha$. Lastly, we see by inspection that $\|S\| = \max\{\|S_1\|, \|S_2\|\} = 1$. Putting this together, we have then

$$\|a\|_p \|S\| \|b\|_p \leq \alpha \tag{3.11}$$

It follows then that $\lambda T_1 + (1-\lambda)T_2 \in \mathcal{C}_\alpha$, so $\mathcal{C}_\alpha$ is convex.

Next, let's say we have an operator $T \in \mathcal{B}(L^2M)$ with $|||T|||_p = \alpha$. We claim that T is in the $|||\cdot|||_p$ norm closure of $\mathcal{C}_\alpha$. For otherwise, since $\mathcal{C}_\alpha$ is convex, we can find using Hahn-Banach a functional $\varphi \in \mathcal{B}^*(p)$ with $\|\varphi\|_{\mathcal{B}^*(p)} = 1$ such that

$$\mathrm{Re}(\varphi(T)) > \sup_{S \in \mathcal{C}_\alpha} \mathrm{Re}(\varphi(S)).$$

Now, by definition we have

$$\mathrm{Re}(\varphi(T)) \leq |||T|||_p = \alpha.$$

Moreover, since $\|\varphi\|_{\mathcal{B}^*(p)}$ was equal to the supremum of all $|\varphi(aSb)|$ where $\|a\|_p, \|b\|_p \leq 1$ and $\|S\| \leq 1$, it follows that

$$\sup_{S \in \mathcal{C}_\alpha} \mathrm{Re}(\varphi(S)) = \alpha.$$

But this leads to a contradiction, so we must have $T \in \overline{\mathcal{C}_\alpha}^{\|\|\|\cdot\|\|\|_p}$.

On the other hand, we claim that the closure of $\mathcal{C}_\alpha$ in the $\sigma(\mathcal{B}(L^2M), \mathcal{B}^*(p))$ topology is $\bigcap_{\beta>\alpha} \mathcal{C}_\beta$. For consider

$$p(T) = \inf\{\beta : T \in \mathcal{C}_\beta\}$$

the seminorm corresponding to the convex sets $\mathcal{C}_\beta$. Note that a linear functional φ on $\mathcal{B}(L^2M)$ will be bounded with respect to the seminorm p if and only if φ is bounded on any set $\mathcal{C}_\beta$. But as we observed already $\sup_{S\in\mathcal{C}_\beta} |\varphi(S)| = \beta\|\varphi\|_{\mathcal{B}^*(p)}$, so the only such functional are in $\mathcal{B}^*(p)$. It follows then that the $\sigma(\mathcal{B}(L^2M), \mathcal{B}^*(p))$ topology and the weak topology for $(\mathcal{B}(L^2M), p)$ coincide. Hence, since $\mathcal{C}_\alpha$ was convex, the closure of $\mathcal{C}_\alpha$ in the $\sigma(\mathcal{B}(L^2M), \mathcal{B}^*(p))$ topology will be the same as the closure of $\mathcal{C}_\alpha$ with respect to the seminorm p, which is indeed $\bigcap_{\beta>\alpha} \mathcal{C}_\beta$.

To complete the proof, we notice by convexity that the $\|\|\|\cdot\|\|\|_p$ norm closure and the $\sigma(\mathcal{B}(L^2M), \mathcal{B}^*(p))$ closure of $\mathcal{C}_\alpha$ are the same. Hence, for any $T \in \mathcal{B}(L^2M)$ such that $\|\|T\|\|_p = \alpha$ we have

$$T \in \overline{\mathcal{C}_\alpha}^{\|\|\|\cdot\|\|\|_p} = \overline{\mathcal{C}_\alpha}^{\sigma(\mathcal{B}(L^2M),\mathcal{B}^*(p))} = \bigcap_{\beta>\alpha} \mathcal{C}_\beta.$$

It follows the that we can find decomposition $T = aSb$ with $\|a\|_p\|S\|\|b\|_p$ arbitrarily close to α, and the lemma follows immediately.

□

Lemma 3.2.7. *Assume $\{u_n\}_n \subset \mathcal{U}(M)$ is a sequence of unitary elements in M with $\tau(u_n^* u_m) = 0$ for all $n \neq m$. For each $T \in \mathcal{B}(L^2M)$ let $\mathcal{E}_0(T) \in \mathcal{B}$ be the operator that acts as 0 on $\mathcal{H}_0^\perp$, where $\mathcal{H}_0 = \overline{sp}(\{\hat{u}_n\}_n) \subset L^2M$, and as the diagonal operator that takes $\hat{u}_n$ to $\langle T(\hat{u}_n), \hat{u}_n\rangle\hat{u}_n$. Then $\|\|\mathcal{E}_0(T)\|\|_p \leq \|\|T\|\|_p$, for all $2 \leq p \leq \infty$. Moreover, if $T \in \mathcal{B}(L^2M)$ is diagonal with respect to $\{\hat{u}_n\}_n$, i.e., $\mathcal{E}_0(T) = T$, then $\|\|T\|\|_p$ is equal to the operator norm of T in $\mathcal{B}(L^2M)$, $\forall p$.*

Proof. By Proposition 3.1.2, one has $\|\|T\|\|_p \leq \|T\|$, $\forall T \in \mathcal{B}(L^2M)$. If in addition T is diagonal with respect to $\{\hat{u}_n\}_n$ and equal to 0 on $\mathcal{H}_0^\perp$, then $\|T\| = \sup_n |\langle T(\hat{u}_n), \hat{u}_n\rangle|$. But

by Lemma 3.1.4 and the definition of $|||T|||_p$, the right hand side is larger than or equal to $|||T|||_p$, showing that $|||T|||_p \geq \|T\|$ as well, so altogether $|||T|||_p = \|T\|$.

For an arbitrary $T \in \mathcal{B}(L^2M)$, by the definition of $|||T|||_p$ and Lemma 3.1.4 one has $|||T|||_p \geq |\langle T(\hat{u}_n), \hat{u}_n\rangle|$, $\forall n$. Thus, $|||T|||_p \geq \sup_n |\langle T(\hat{u}_n), \hat{u}_n\rangle| = \|\mathcal{E}_0(T)\| = |||\mathcal{E}_0(T)|||_p$.

□

Corollary 3.2.8. *If $M = L\Gamma$ and we denote $\mathcal{D} = \ell^\infty\Gamma \subset \mathcal{B}(\ell^2\Gamma) = \mathcal{B}(L^2M)$, $\mathcal{C} = c_0(\Gamma) \subset \ell^\infty\Gamma$, then for each $T \in \mathcal{D}$ we have $|||T|||_p = \|T\|$, $\forall p \geq 2$. Thus, $\mathcal{C} \subset \mathcal{D}$ are $|||\cdot|||_p$-closed in $\mathcal{B}(p)$.*

Proof. Since for $M = L\Gamma$ we have $L^2M = \ell^2\Gamma$, with $\{\hat{u_g}\}_g$ as orthonormal basis, the previous lemma implies that $|||\cdot|||_p$ restricted to $\mathcal{D} = \ell^\infty\Gamma$ coincided with the operator norm.

□

3.3 The Banach bimodules $\mathcal{K}(p)$, $2 \leq p < \infty$

Since the ideal of compact operators $\mathcal{K}(L^2M)$ is a Banach bimodule over both M, M^{op}, its $|||\cdot|||_p$-completions, $2 \leq p < \infty$, give rise to a one parameter family of bimodules that we now consider.

Definition 3.3.1. For each $2 \leq p < \infty$, we denote by $\mathcal{K}(p)$ the closure of $\mathcal{K} = \mathcal{K}(L^2M)$ in $\mathcal{B}(p)$.

Lemma 3.3.2. *Let $2 \leq p < \infty$ and denote $q = \frac{2p}{p-2}$ and $q' = \frac{2p}{p+2}$ as before. Following Proposition 3.2.4, for each $K \in \mathcal{B}(p)$ we denote by $\tilde{K}$ the element it induces in $\mathcal{B}(L^qM, L^{q'}M)$.*

1° If $K \in \mathcal{K}(p)$, then $\tilde{K}$ takes the unit ball $(L^qM)_1$ into a $\|\cdot\|_{q'}$-compact subset of $L^{q'}M$.

2° If $K \in \mathcal{K}(p)$, then for any sequence of unitary elements $\{u_n\}_n \subset M$ that converges weakly to 0, one has $\|\tilde{K}(\hat{u}_n)\|_{q'} \to 0$.

Proof. 1° Note first that if $K \in \mathcal{K}(p)$ is a finite-rank operator, then $\tilde{K}$ is in $\mathcal{K}(L^qM, L^{q'}M)$. If K is a general element of $\mathcal{K}(p)$, then we can find a sequence (K_n) of finite-rank operators in $\mathcal{B}(p)$ such that $\mathcal{K}_n \to K$ with respect to the $|||\cdot|||_p$ norm. Since the mapping $K \mapsto \tilde{K}$ is contractive, we have $\tilde{K_n} \to \tilde{K}$ in $\mathcal{B}(L^qM, L^{q'}M)$. Since the space $\mathcal{K}(L^qM, L^{q'}M)$ is closed in $\mathcal{B}(L^qM, L^{q'}M)$, and each finite-rank K_n was in $\mathcal{K}(L^qM, L^{q'}M)$, it follows that $K \in \mathcal{K}(L^qM, L^{q'}M)$ as well.

2° As in part 1°, if $K \in \mathcal{K}(p)$ is a assumed to be a finite-rank operator the claim follows immediately. Let now $K \in \mathcal{K}(p)$ be arbitrary and let $(K_n)_n$ be a sequence of finite-rank operators in $\mathcal{B}(p)$ such that $K_n \to K$ with respect to the $|||\cdot|||_p$ norm. Now if $\epsilon > 0$ we can find an m such that

$$\|\tilde{K} - \tilde{K_m}\|_{L^qM \to L^{q'}M} \leq |||K - K_m|||_p < \epsilon/2.$$

Now since K_m is finite-rank, we can also find an N such that for all $n \geq N$ we have $\|K_m(\hat{u}_n)\|_{q'} < \epsilon$. Combining these we get for any $n \geq N$

$$\begin{aligned} \|K(\hat{u}_n)\|_{q'} &\leq \|(K - K_m)(\hat{u}_n)\|_{q'} + \|K_m(\hat{u}_n)\|_{q'} \\ &< \|\hat{u}_n\|_q \|K - K_m\|_{L^qM \to L^{q'}M} + \epsilon/2 \\ &< \epsilon. \end{aligned} \tag{3.12}$$

Thus we conclude $\|\tilde{K}(\hat{u}_n)\|_{q'} \to 0$ as desired.

□

Proposition 3.3.3. *For each $2 \leq p < \infty$, $\mathcal{K}(p)$ endowed with the norm $|||\cdot|||_p$ is a Banach M-bimodule and Banach M^{op}-bimodule.*

Proof. The fact that $\mathcal{K}(p)$ is a Banach space is clear from the fact that it is a norm closed subspace of $\mathcal{B}(p)$. The M-bimodule and M^{op}-bimodule structure also follows by restricting from $\mathcal{B}(p)$, and by taking into account that the $|||\cdot|||_p$-dense subset $\mathcal{K}(L^2M)$ of $\mathcal{K}(p)$ is invariant under left and right multiplication by elements of M, M^{op}.

□

Recalling that $\mathcal{K}(L^2M)^* = \mathcal{B}^*_{\mathrm{n}}$, we now prove the analogous result for the spaces $\mathcal{K}(p)$ and $\mathcal{B}^*_{\mathrm{n}}(p)$.

Theorem 3.3.4. *For all $p \geq 2$, $\mathcal{K}(p)^* = \mathcal{B}_n(p)$. Also, for each $2 \leq p < \infty$, $\mathcal{K}(p)$ endowed with its norm $|||\cdot|||_p$ is a smooth M-bimodule, in the sense of definition* **??**.

Proof. We use the description of preduals in [Kai77] to prove the result. Note first that the compact operators $\mathcal{K}(L^2M)$, form a subspace of $(\mathcal{B}_{\mathrm{n}}(p))^*$. Furthermore, the space $\mathcal{K}(L^2M)$ separates points of $\mathcal{B}_{\mathrm{n}}(p)$, i.e. for any distinct $\varphi, \varphi' \in \mathcal{B}_{\mathrm{n}}(p)$ there is a $K \in \mathcal{K}(L^2M)$ such that $\varphi(K) \neq \varphi'(K)$. This is because the space $\mathcal{K}(L^2M)$, which is strictly smaller than $\mathcal{K}(p)$, separates the points of its dual $\mathcal{B}(L^2M)_*$, which is strictly larger than $\mathcal{B}_{\mathrm{n}}(p)$.

Now we claim that the unit ball of $\mathcal{B}^*_{\mathrm{n}}(p)$ is compact in the $\sigma(\mathcal{B}^*_{\mathrm{n}}(p), \mathcal{K}(L^2M))$ topology. To see this, consider a net (φ_α) in the unit ball $(\mathcal{B}^*_{\mathrm{n}}(p))_1$. Recall that the norm $\|\cdot\|_{\mathcal{B}^*(p)}$ majorizes the usual norm on $\mathcal{B}^*$ and that $(\mathcal{B}^*_{\mathrm{n}}(p))_1$ is a subset of $(\mathcal{B}^*_{\mathrm{n}})_1$. Since the predual of the space $\mathcal{B}^*_{\mathrm{n}}$ of normal linear functionals on $\mathcal{B}$ is $\mathcal{K}(L^2M)$, we have that $(\mathcal{B}^*_{\mathrm{n}})_1$ is compact in the $\sigma(\mathcal{B}^*_{\mathrm{n}}, \mathcal{K}(L^2M))$ topology. Thus, there exists a subnet (φ_β) of our original net that converges to some $\varphi \in (\mathcal{B}^*_{\mathrm{n}})_1$ in the $\sigma(\mathcal{B}^*_{\mathrm{n}}, \mathcal{K}(L^2M))$ topology.

We claim further that this φ is actually in $\mathcal{B}^*_{\mathrm{n}}(p)$. Let x and y be any elements of M. Since the space $\mathcal{B}^*_{\mathrm{n}}$ is a dual normal Banach M-bimodule, it follows the net $x\cdot\varphi_\beta\cdot y$ converges to $x\cdot\varphi\cdot y$ in the $\sigma(\mathcal{B}^*_{\mathrm{n}}, \mathcal{K}(L^2M))$ topology. Note that this implies $\|x\cdot\varphi\cdot y\| \leq \sup_\beta \|x\cdot\varphi_\beta\cdot y\|$. In particular, notice that if x and y are chosen so that $\|x\|_p, \|y\|_p \leq 1$, then, since the net (φ_β) lies in $(\mathcal{B}^*_{\mathrm{n}}(p))_1$, we would have $\|x\cdot\varphi\cdot y\| \leq \sup_\beta \|x\cdot\varphi_\beta\cdot y\| \leq 1$. Varying over all x and y with p-norm less than 1, we gather that $\|\varphi\|_{\mathcal{B}^*(p)} \leq 1$, so indeed our φ lies in $(\mathcal{B}^*_{\mathrm{n}}(p))_1$. It follows then that the unit ball $(\mathcal{B}^*_{\mathrm{n}}(p))_1$ is compact in the $\sigma(\mathcal{B}^*_{\mathrm{n}}(p), \mathcal{K}(L^2M))$ topology. Using the description of preduals in [K77], we find that a predual of $\mathcal{B}^*_{\mathrm{n}}(p)$ is the norm closure of $\mathcal{K}(L^2M)$ in the dual space $(\mathcal{B}^*_{\mathrm{n}}(p))^*$. By definition this is $\mathcal{K}(p)$.

It remains then to check that $\mathcal{K}(p)$ is a smooth bimodule. Note that if $T \in \mathcal{K}(p)$ lies in

$\mathcal{B}(L^2M)$, then by Lemma 3.2.2 the maps $x \mapsto Tx$ and $x \mapsto xT$ are $\|\cdot\|_2$ to $|||\cdot|||_p$ continuous on the unit ball of M. If $T \in \mathcal{K}(p)$ is an arbitrary element, then we for any $\epsilon > 0$ there exists $S \in \mathcal{K}(p) \cap \mathcal{B}(L^2M)$ such that $|||T - S|||_p < \epsilon$. But then for any net (x_ι) in the unit ball of M such that $\|x_\iota\|_2 \to 0$ we have

$$|||Tx_\iota|||_p \leq |||(T-S)x_\iota|||_p + |||Sx_\iota|||_p < \epsilon + |||Sx_\iota|||_p.$$

Hence $\limsup_\iota |||Tx_\iota|||_p \leq \epsilon$. Since ϵ was arbitrary, it follows $|||Tx_\iota|||_p$ tends to 0. Hence the map $x \mapsto Tx$ is still $\|\cdot\|_2$ to $|||\cdot|||_p$ continuous on the unit ball of M. A similar argument shows the same for the map $x \mapsto xT$. It follows then that $\mathcal{K}(p)$ is smooth.

□

CHAPTER 4

Non-vanishing 1-Cohomology Results

4.1 The q_M-topology and the bimodule $\mathrm{q}\mathcal{K}_M$

In this section we consider a new topology on Banach bimodules over tracial von Neumann algebras (M, τ), which we will denote q_M, that takes into consideration the trace on M, and which we will refer to as the *τ-rank topology* (sometimes also called the topology of convergence in measure). When applied to the Banach M-bimodule $\mathcal{B}(L^2M)$, the restriction of the q_M-topology to the unit ball $(\mathcal{B}(L^2M))_1$ is "almost the same" as the topology given by $|||\cdot|||_p$-norms, but finer. However, the q_M-closure in $\mathcal{B}(L^2M)$ of the unit ball of compact operators $(\mathcal{K}(L^2M))_1$ coincides with its $|||\cdot|||_p$-closure, thus giving rise to an interesting Banach M-bimodule of "almost-compact" operators denoted $\mathrm{q}\mathcal{K}_M$.

Definition 4.1.1. Let $\mathcal{B}$ be a Banach M-bimodule. We say that a net $(T_i)_i \subset \mathcal{B}$ is q_M-convergent to $T \in \mathcal{B}$ if the following conditions are satisfied: $\sup_i \|T_i\| < \infty$; for any $\varepsilon > 0$, there exists i_0 such that for any $i \geq i_0$ there exists a projection $p \in \mathcal{P}(M)$ with $\tau(1-p) < \varepsilon$, $\|p(T_i - T)p\| < \varepsilon$.

Note that if these conditions are satisfied, then $\|T\| \leq \limsup_i \|T_i\|$. Thus, for any finite $r > 0$, the q_M-convergent nets in $(\mathcal{B})_r$ define a topology on $(\mathcal{B})_r$, that we will also denote by q_M. Note also that if $r' \geq r > 0$, then the restriction to $(\mathcal{B})_r$ of the q_M-topology on $(\mathcal{B})_{r'}$, coincides with the q_M-topology on $(\mathcal{B})_r$.

Note that the q_M-topology on any bounded subset of $\mathcal{B}$ is implemented by the metric given by $\mathrm{q}_M(T, S) = \inf\{\tau(1-p) + \|p(T-S)p\| \mid p \in \mathcal{P}(M)\}$.

Given a linear subspace $\mathcal{B}_0 \subset \mathcal{B}$, we denote $\overline{\mathcal{B}_0}^{\mathrm{q}_M}$ the union over all $r > 0$ of the q_M-closures of $(\mathcal{B}_0)_r$ in $(\mathcal{B})_r$. Equivalently, $\overline{\mathcal{B}_0}^{\mathrm{q}_M}$ is the set of all scalar multiples of elements in $\overline{(\mathcal{B}_0)_1}^{\mathrm{q}_M}$.

The q_M-topology on $(\mathcal{B})_r$ is obviously weaker than the norm topology. A typical example of a Banach M-bimodule $\mathcal{B}$ that we consider is the algebra $\mathcal{B}(\mathcal{H})$ of all linear bounded operators on a Hilbert space $\mathcal{H}$ on which M acts normally and faithfully, with the M-bimodule structure given by left-right multiplication by elements in M. More generally, we consider (linear) subspaces $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ with $M\mathcal{B}M \subset \mathcal{B}$, such as the space of compact operators $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$. For this class of examples, another natural topology on $\mathcal{B}$ is the s*-topology. If $\mathcal{B} = M$, then this is easily seen to coincide with the q_M-topology on bounded sets. But in general, the s*-topology is strictly weaker than the q_M-topology on $(\mathcal{B})_1$ (notably if $\mathcal{B} = \mathcal{K}(\mathcal{H})$ and M is infinite dimensional, see below).

Proposition 4.1.2. *Let $\mathcal{B}$ be a dual normal M-bimodule and $\mathcal{B}_0 \subset \mathcal{B}$ a norm closed sub-bimodule.*

1° *For any $T \in \mathcal{B}$, the maps $(M)_1 \ni x \mapsto xT, Tx \in \mathcal{B}$ are $\|\cdot\|_2 - q_M$ continuous.*

2° *$(\mathcal{B})_1$ is complete in the q_M-metric (and thus so is $\overline{(\mathcal{B}_0)_1}^{q_M} \subset (\mathcal{B})_1$).*

3° *$\overline{\mathcal{B}_0}^{q_M}$ is a Banach M-bimodule.*

4° *Given any norm-separable subspace $\mathcal{E} \subset \overline{\mathcal{B}_0}^{q_M}$, there exists an increasing sequence of projections $p_n \in M$ with $p_n \to 1$ such that $p_n T p_n \in \mathcal{B}_0$, for all $T \in \mathcal{E}$.*

Proof. 1° If $\varepsilon > 0$ and $\|x\| \leq 1$ satisfies $\|x\|_2 \leq \varepsilon$, then the spectral projection p of xx^* corresponding to the interval $[0, \varepsilon]$ has trace at least $1 - \varepsilon$, or else we have $\|x\|_2^2 = \|px\|_2^2 + \|(1-p)x\|_2^2 > \|(1-p)x\|_2^2 \geq \varepsilon^2$, a contradiction. Thus, we have $\|pxT\| \leq \|px\|\|T\| \leq \varepsilon$ and $\tau(1-p) \leq \varepsilon$. This shows that $(M)_1 \ni x \mapsto xT \in \mathcal{B}$ is $\|\cdot\|_2 - \mathrm{q}_M$ continuous. The proof for $(M)_1 \ni x \mapsto Tx \in \mathcal{B}$ is similar.

2° If $T_n \in (\mathcal{B})_1$ is q_M-Cauchy, then for any $k \geq 1$, there exists n_k such that for any

$n, m \geq n_k$ there exists a projection $p_k \in \mathcal{P}(M)$ with the property that $\tau(1-p_k) + \|p_k(T_m - T_n)p_k\| \leq 2^{-k}$. Thus, the sequence of projections $P_k = \wedge_{l \geq k} p_l$, $k \geq 1$, is increasing and satisfies $\tau(1-P_k) \leq 2^{-k+1}$, $\|P_k(T_n - T_m)P_k\| \leq 2^{-k}$ for any $n, m \geq n_k$. By the inferior semicontinuity of the norm on $\mathcal{B}$ with respect to the w^*-topology, it follows that any w^*-limit point $T \in (\mathcal{B})_1$ of the sequence $\{T_m\}_m$ satisfies $\|P_k(T - T_n)P_k\| \leq 2^{-k}$ for any $n \geq n_k$. This shows that $\{T_m\}_m$ is q_M-convergent to T.

3° If T_n is a sequence in $\overline{\mathcal{B}_0}^{\mathrm{q}_M}$ that converges in norm to some $T \in \mathcal{B}$, then T_n is automatically bounded and by 2° we have $T \in \overline{\mathcal{B}_0}^{\mathrm{q}_M}$ as well. The invariance of $\overline{\mathcal{B}_0}^{\mathrm{q}_M}$ to left-right multiplication by elements in M is obvious.

4° It is sufficient to show the existence of such projections for a countable subset $\{T_i\}_i \subset \overline{(\mathcal{B}_0)_1}^{\mathrm{q}_M}$. For each $i \leq n \leq k$, there exists a projection $p_{i,k} \in M$ and $S_{i,k} \in (\mathcal{B}_0)_1$ such that $\|p_{i,k}(T_i - S_{i,k})p_{i,k}\| \leq 2^{-k}$ and $\tau(1-p_{i,k}) \leq 2^{-k}/n$. Thus, if we let $P_{n,k} = \wedge_{i \leq n} p_{i,k}$, then $\|P_{n,k}(T_i - S_{i,k})P_{n,k}\| \leq 2^{-k}$, $\forall i \leq n$, and $\tau(1-P_{n,k}) \leq 2^{-k}$. If we now put $P_n = \wedge_{k \geq n} P_{n,k}$, then P_n is increasing, $\tau(1-P_n) \leq 2^{-n+1}$, and $\|P_n(T_i - S_{i,k})P_n\| \leq 2^{-n}$, $\forall i \leq n \leq k$. For each fixed m, by applying this to $k = n \geq m$ and taking into account that $P_m P_n = P_m$, it follows that $\|P_m(T_i - S_{i,n})P_m\| \leq 2^{-n}$, $\forall i \leq m$. This shows in particular that $\{P_m S_{i,n} P_m\}_n \subset (\mathcal{B}_0)_1$ is norm-Cauchy and thus convergent to some $X_{i,m} \in (\mathcal{B}_0)_1$. It also shows that $X_{i,m}$ satisfy $P_m T_i P_m = X_{i,m}$, while $\tau(1-P_m) \leq 2^{-m+1}$, $\forall i \leq m$.

□

Definition 4.1.3. Given a tracial von Neumann algebra (M, τ) in its standard representation on L^2M, we denote by $\mathrm{q}\mathcal{K}_M$ the q_M-closure of $\mathcal{K}(L^2M)$ in $\mathcal{B}(L^2M)$ and call its elements q_M-*compact operators*.

Notice that besides its M-bimodule structure, the algebra $\mathcal{B}(L^2M)$ also has an $JMJ = M'$ bimodule structure, where $J : L^2M \to L^2M$ is the canonical conjugacy defined by $J(x) = x^*$, $x \in M \subset L^2M$, and M' denotes as usual the commutant of M in $\mathcal{B}(L^2M)$. The algebra $JMJ = M'$ can be naturally identified with the opposite algebra M^{op} of M, and we will retain this notation for JMJ.

Proposition 4.1.4. *The space $q\mathcal{K}_M$ is a norm closed *-subspace of $\mathcal{B}(L^2M)$, which is both an M-bimodule and an M^{op}-bimodule.*

Proof. By applying Proposition 4.1.2 to $\mathcal{B}_0 = \mathcal{K}(L^2M) \subset \mathcal{B}(L^2M) = \mathcal{B}$, it follows that q_M is a norm closed M-bimodule. It is clearly an M^{op}-bimodule and closed under the *-operation.

□

Theorem 4.1.5. *For each $2 \leq p < \infty$ denote by $\mathcal{K}_p$ the space of all operators $T \in \mathcal{B} = \mathcal{B}(L^2M)$ with the property that there exists a sequence of compact operators $K_n \in \mathcal{K}_p$ such that $\sup_n \|K_n\| < \infty$ and $\lim_n |||T - K_n|||_p = 0$. Then $\mathcal{K}_p = q\mathcal{K}_M$.*

Proof. To see that $\mathrm{q}\mathcal{K}_M \subset \mathcal{K}_p$ let us show that the q_M-topology on the unit ball of $\mathcal{B}(L^2M)$ is stronger than the $|||\cdot|||_p$-topology, $\forall 2 \leq p < \infty$. Indeed, by Lemma 3.2.2, if $T \in \mathcal{B}(L^2M)$ and $P \in \mathcal{P}(M)$, then we have

$$\begin{aligned} |||T|||_p &\leq |||PTP|||_p + |||PT(1-P)|||_p + |||(1-P)T|||_p \\ &\leq \|PTP\| + 2\|1-P\|_p\|T\| = \|PTP\| + 2(\tau(1-P))^{1/p}\|T\|. \end{aligned} \tag{4.1}$$

This shows that $|||T|||_p \leq 2\inf\{\|PTP\| + (\tau(1-P))^{1/p}\|T\|\}$, implying that the q_M-toplogy on $(\mathcal{B}(L^2M))_1$ is stronger than the $|||\cdot|||_p$-topology.

To show that $\mathcal{K}_p \subset \mathrm{q}\mathcal{K}_M$, let T be an operator in K_p. Then we can find a sequence of uniformly bounded compact operators $K_n \in \mathcal{K}(L^2M)$ such $|||T - K_n|||_p$ tends to 0. Since $T - K_n$ is still in $\mathcal{B}(L^2M)$, Lemma 3.2.6 says we can find $a_n, b_n \in M$ and $S_n \in B(L^2M)$ such that $T - K_n = a_nS_nb_n$ and $\|a_n\|_p\|S_n\|\|b_n\|_p$ tends to 0. Taking spectral projections of $|a_n|$ and $|b_n^*|$ we can find a sequence of projections p_n with $\|p_n\|_p$ tending to 1 such that

$$\|p_n(T - K_n)p_n\| = \|p_na_nS_nb_np_n\| \to 0$$

Thus, $\mathcal{K}_p \subset \mathrm{q}\mathcal{K}_M$, which combined with the first part shows that $\mathcal{K}_p = \mathrm{q}\mathcal{K}_M$.

□

It is useful to note that due to their "compact nature", elements in the spaces $\mathrm{q}\mathcal{K}_M$ cannot intertwine diffuse subalgebras of M. This fact will be used later to deduce that an operator in $\mathcal{B}(L^2M)$ that commutes with M modulo $\mathrm{q}\mathcal{K}_M$ and commutes with a diffuse subalgebra of M, must in fact commute with all of M.

Lemma 4.1.6. *Let $B \subset eMe$ be a diffuse von Neumann subalgebra and $\sigma : B \to fMf$ be a unital faithful *-homomorphism, for some non-zero projections $e, f \in M$. If $K \in q\mathcal{K}_M$ satisfies $Kb = \sigma(b)K$, $\forall b \in B$, then $K = (1-f)K(1-e)$.*

Proof. Note that $fK(1-e) = 0$ and $(1-f)Ke = 0$. By replacing K by $K-(1-f)K(1-e)$, we may also assume $(1-f)K(1-e) = 0$. So we have to prove that if K satisfies the condition in the hypothesis and $K = fKe$, then $K = 0$.

Let u be a Haar unitary in B and $x \in eM$. Since $u^n x$ tends weakly to 0 and $\|v(\xi)\|_1 = \|\xi\|_1$ for any unitary $v \in fMf$ and $\xi \in L^1(fM)$, we get

$$0 = \lim_n \|K(\widehat{u^n x})\|_1 = \lim_n \|\sigma(u^n)(K(\hat{x}))\|_1 = \|K(\hat{x})\|_1,$$

where the first equality follows easily from the definition of $\mathrm{q}\mathcal{K}_M$. This shows that $K = fKe$ satisfies $K(\widehat{eM}) = 0$, thus $K = 0$.

□

4.2 Derivations of M into $\mathrm{q}\mathcal{K}_M$

Recall that if M is a Banach algebra (always assumed unital) and $\mathcal{B}$ is a Banach M-bimodule, then a derivation of M into $\mathcal{B}$ is a linear map $\delta : M \to \mathcal{B}$ satisfying the property $\delta(xy) = x\delta(y) + \delta(x)y$, for all $x, y \in M$.

It is immediate to check that if $T \in \mathcal{B}$, then the map $\mathrm{ad}T : M \to \mathcal{B}$ defined by $\mathrm{ad}T(x) = [T, x] := Tx - xT$, $x \in \mathcal{B}$, is a derivation. Such derivations are called *inner*.

It is useful to note that if $F \subset M$ is a set, then $\delta_{|F}$ determines the values of δ on all the algebra $\mathrm{Alg}(F)$ generated by F.

Recall from [Rin72] that a derivation is automatically norm-continuous. Moreover, if M is a von Neumann algebra and $\mathcal{B}$ is a dual normal M-bimodule, then any derivation is automatically continuous from M with the ultra-weak topology to $\mathcal{B}$ with its $\sigma(\mathcal{B}, \mathcal{B}_*)$ topology.

Thus, if $F = F^* \subset M$ is a set that generates M as a von Neumann algebra and M_0 is the norm closure of the *-algebra generated by F, then any derivation δ of M into a Banach M-bimodule is uniquely determined on M_0 by the values it takes on F, $\delta_{|F}$. If in addition $\mathcal{B}$ is a dual normal Banach bimodule, then all of δ is uniquely determined by $\delta_{|F}$.

Let us first notice an automatic continuity (smoothness) result for derivations, with respect to the q_M-metric and the $|||\cdot|||_p$ norms.

Theorem 4.2.1. *Let (M, τ) be a tracial von Neumann algebra, $\mathcal{B}$ a Banach M-bimodule and $\delta : M \to \mathcal{B}$ a derivation. Then δ is automatically $\|\cdot\|_2$-q_M continuous on $(M)_1$. More precisely, if $\varepsilon > 0$, then given any $x \in (M)_1$ with $\|x\|_2 \leq (\varepsilon/2)^{3/2}$, there exists $p \in \mathcal{P}(M)$ such that $\tau(1-p) \leq \varepsilon$ and $\|p\delta(x)p\| \leq \varepsilon\|\delta\|$.*

In particular, if $\mathcal{B} = q\mathcal{K}_M$, then δ is automatically continuous from $(M)_1$ with the $\|\cdot\|_2$-topology to $q\mathcal{K}_M$ with the topology given by the q_M-metric.

Proof. By [R72], δ is automatically norm continuous and without loss of generality we may assume $\|\delta\| = 1$. Let $\varepsilon > 0$. Let $x \in (M)_1$ be so that $\|x\|_2 \leq (\varepsilon/2)^{3/2}$. Denote by e the spectral projection of xx^* corresponding to $[0, \varepsilon^2/4]$. Then e satisfies $\|ex\| \leq \varepsilon/2$ and $(1-e)xx^* \geq (\varepsilon/2)^2(1-e)$. Thus we have:

$$(\varepsilon/2)^3 \geq \|x\|_2^2 = \|ex\|_2^2 + \|(1-e)x\|_2^2$$

$$\geq \|(1-e)x\|_2^2 = \tau((1-e)xx^*) \geq (\varepsilon/2)^2\tau(1-e).$$

This implies that $\tau(1-e) \leq \varepsilon/2$. Similarly, if e' denotes the spectral projection of x^*x

corresponding to $[0, \varepsilon^2/4]$, we have $\tau(1-e') \leq \varepsilon/2$ and $\|xe'\| \leq \varepsilon/2$. Thus, if we denote $p = e \wedge e'$, then $\tau(1-p) \leq \varepsilon$ and $\|p\delta(x)p\| = \|\delta(px)p - \delta(p)xp\| \leq \varepsilon$.

□

Lemma 4.2.2. *Assume $T \in \mathcal{B}(L^2M)$ is so that $[T, M_0] \subset q\mathcal{K}_M$ for some weakly dense *-subalgebra $M_0 \subset M$. Then we have:*

1° $[T, M] \subset q\mathcal{K}_M$.

2° *If, in addition, $T = e$ is a projection and there exists a Haar unitary $u \in M$ such that $[e, u] \in \mathcal{K}(L^2M)$ with eue having Fredholm index $\neq 0$ in $\mathcal{B}(e(L^2M))$, then $[T, M] \subset q\mathcal{K}_M$, and the derivation $\delta_e : M \to q\mathcal{K}_M$ defined by $\delta_e(x) = [e, x], x \in M$, is not inner, i.e., there exists no $K \in q\mathcal{K}_M$ such that $\delta_e = adK$.*

Proof. 1° By Theorem 4.2.1, the derivation $\delta = \mathrm{ad}T : M \to \mathcal{B}(L^2M)$ is $\|\cdot\|_2 - \mathrm{q}_M$ continuous. Since M_0 is $\|\cdot\|_2$-dense in M, $[T, M_0] \subset \mathrm{q}\mathcal{K}_M$ and $\mathrm{q}\mathcal{K}_M$ is q_M-closed in $\mathcal{B}(L^2M)$, it follows that $[T, M] \subset \mathrm{q}\mathcal{K}_M$.

2° Let $A = \{u\}''$. Since u is a Haar unitary, one can view the restriction of the action of u on L^2A as the bilateral shift on $\ell^2\mathbb{Z}$. Denote $u = v \oplus w$ where v is the restriction of u to $L^2A = \ell^2\mathbb{Z}$ and w its restriction to $L^2M \ominus L^2A$. By [BDF06], there exist compact operators $K_0, K_1 \in \mathcal{K}(L^2M)$ such that $(u + K_0, e + K_1)$ are unitary conjugate to (u, f), where f is the orthogonal projection of L^2M onto $\ell^2\mathbb{Z}_+ \subset \ell^2\mathbb{Z} = L^2A$.

Thus, if $\delta_e = \mathrm{ad}K$ for some $K \in \mathcal{K}(M, L^1M)$, then $\delta_f = \mathrm{ad}f = \mathrm{ad}(e + K_1) = \mathrm{ad}(K')$ with $K' = K + K_1 \in \mathcal{K}(M, L^1M)$. This implies $f - K' \in M' = M^{op}$, so there must exist $x_0 \in M$ such that $f(\hat{y}) = K'(\hat{y}) + y\hat{x}_0$, for all $y \in M$. Since $\lim_{|n|\to\infty} K'(\hat{u^n}) = 0$ and $f(\hat{u^n})$ is equal to $\hat{u^n}$ for $n > 0$ and is equal to 0 for $n < 0$, this shows on the one hand that $0 = \lim_{n\to\infty} \|f(\hat{u^{-n}})\|_2 = \|u^{-n}x_0\|_2 = \|x_0\|_2$, on the other hand $1 = \lim_{n\to\infty} \|f(\hat{u^n})\|_2 = \|u^n x_0\|_2 = \|x_0\|_2$, a contradiction.

□

Theorem 4.2.3. *For any separable diffuse finite von Neumann algebra M, there exists a non inner derivations of M into $q\mathcal{K}_M$.*

Proof. Since M is separable, we can fix a weakly dense sequence of x_n in M. By [Arv77], the closed ideal $\mathcal{K}(L^2M)$ of $\mathcal{B}(L^2M)$ has a quasicentral approximate unit. In particular, for any $\epsilon > 0$ and any operators $T_1, T_1, \ldots, T_k \in \mathcal{B}(L^2M)$, we can find an operator $K \in \mathcal{K}(L^2M)_+$ from such a quasicentral approximate unit such that $\|K\| \leq 1$ and $\|[K, T_i]\| < \epsilon$ for all $1 \leq i \leq k$. Moreover, since such a quasicentral approximate unit weakly tends to the identity, for any $0 < \alpha < 1$ such an operator K can be chosen to satisfy $\langle K\hat{1}, \hat{1}\rangle > \alpha$. Thus, we can find a sequence of operators K_n in $\mathcal{K}(L^2M)$ with $\|K_n\| \leq 1$ for all $n \geq 1$ such that $\|[K_n, x_k]\| < 2^{-n}$ for all $1 \leq k \leq n$ and $\langle K_n\hat{1}, \hat{1}\rangle > 1/2$ for all $n \geq 1$.

Now fix a sequence of unitaries u_n in M that are weakly tending to 0. We claim there exists a subsequence $(u_{n_i})_{i=1}^{\infty}$ such that

1. $\left\|\sum_{i=1}^{n} Ju_{n_i}JK_iJu_{n_i}^*J\right\| < 2$ for all $n \geq 1$;
2. $|\langle K_i u_{n_j}^* u_{n_i}\hat{1}, u_{n_j}^* u_{n_i}\hat{1}\rangle| < 2^{-i-1}$ for all $i \neq j$.

We construct such a subsequence inductively. First, let $u_{n_1} = u_1$. Next, assume for some $k \geq 1$ we have found $u_{n_1}, u_{n_2}, \ldots, u_{n_k}$ such that the above condition 1 occurs for all $1 \leq n \leq k$ and condition 2 occurs for all $1 \leq i, j \leq k$ with $i \neq j$. Then notice that for any compact operators $T, S \in K(L^2M)$ and any sequence of unitaries v_n in M converging weakly to 0 we have $\|T + v_nSv_n^*\| \to \max\{\|T\|, \|S\|\}$ as n tends to infinity. Since $\sum_{i=1}^{k} Ju_{n_i}JK_iJu_{n_i}^*J$ and K_{k+1} are compact operators of norm less than 2, it follows there is an N_1 such that for all $n \geq N_1$

$$\left\| Ju_nJK_{k+1}Ju_n^*J + \sum_{i=1}^{k} Ju_{n_i}JK_iJu_{n_i}^*J \right\| < 2.$$

Next, note that for each fixed $1 \leq i \leq k$ we have that $u_nu_{n_i}K_{k+1}u_{n_i}^*u_n^*$ converges weakly to 0 as n tends to infinity. Thus, there is an N_2 such that for all $1 \leq i \leq k$ and all $n \geq N_2$

$$|\langle K_{k+1}u_{n_i}^*u_n\hat{1}, u_{n_i}^*u_n\hat{1}\rangle| = |\langle u_nu_{n_i}K_{k+1}u_{n_i}^*u_n\hat{1}, \hat{1}\rangle| < 2^{-k-2}.$$

Similarly, for each fixed $1 \leq i \leq k$ we have that $u_n K_i u_n^*$ converges weakly to 0 as n tends to infinity. Hence, there is an N_3 such that for all $1 \leq i \leq k$ and all $n \geq N_3$

$$|\langle K_i u_n^* u_{n_i} \hat{1}, u_n^* u_{n_i} \hat{1} \rangle| = |\langle (u_n K_i u_n^*) u_{n_i} \hat{1}, u_{n_i} \hat{1} \rangle| < 2^{-i-1}.$$

If we take $n_{k+1} = \max\{N_1, N_2, N_3\}$, then the terms $u_{n_1}, u_{n_2}, \ldots, u_{n_{k+1}}$ will satisfy the above condition 1 for all $1 \leq n \leq k+1$ and condition 2 occurs for all $1 \leq i, j \leq k+1$ with $i \neq j$. By induction, it follows that the desired subsequence $(u_{n_i})_{i=1}^{\infty}$ exists.

Now we define an operator T by letting

$$T = \sum_{i=1}^{\infty} J u_{n_{2i}} J K_{2i} J u_{n_{2i}}^* J.$$

Note because of how we chose the unitaries u_{n_i} that T will indeed be a well-defined operator in $\mathcal{B}(L^2 M)$ with $\|T\| \leq 2$. Moreover, for any x_j from our weakly dense sequence of M we have

$$[T, x_j] = \sum_{i=1}^{\infty} [J u_{n_{2i}} J K_{2i} J u_{n_{2i}}^* J, x_j] = \sum_{i=1}^{\infty} J u_{n_{2i}} J [K_{2i}, x_j] J u_{n_{2i}}^* J.$$

Each summand $J u_{n_{2i}} J [K_{2i}, x_j] J u_{n_{2i}}^* J$ in this series is a compact operator and, because of how we chose the operators K_n, for all $i \geq j/2$ we have

$$\|J u_{n_{2i}} J [K_{2i}, x_j] J u_{n_{2i}}^* J\| = \|[K_{2i}, x_j]\| \leq 2^{-2i}.$$

Thus, this is a $\|\cdot\|$-norm convergent series of compact operators, and in turn $[T, x_j]$ is a compact for each x_j. By Lemma 4.2.2, $\mathrm{ad}T$ is a derivation of M into $\mathrm{q}\mathcal{K}_M$.

We claim, however, that $\mathrm{ad}T$ is not inner. Otherwise, assume for sake of contradiction there is an $S \in \mathrm{q}\mathcal{K}_M$ such that $\mathrm{ad}S = \mathrm{ad}T$. Take any sequence of unitaries v_n in M that weakly converge to 0. Then we note that since $(T - S)$ commutes with M

$$\begin{aligned} \langle v_n T v_n^* \hat{1}, \hat{1} \rangle &= \langle v_n (T - S) v_n^* \hat{1}, \hat{1} \rangle + \langle v_n S v_n^* \hat{1}, \hat{1} \rangle \\ &= \langle (T - S) \hat{1}, \hat{1} \rangle + \langle v_n S v_n^* \hat{1}, \hat{1} \rangle. \end{aligned} \tag{4.2}$$

Using lemma 3.3.2, $\langle v_n S v_n^* \hat{1}, \hat{1}\rangle$ must converge to 0 as n tends to infinity. Thus, we observe that $\langle v_n T v_n^* \hat{1}, \hat{1}\rangle$ must converge as n tends to infinity.

It follows then that $\langle u_{n_j} T u_{n_j}^* \hat{1}, \hat{1}\rangle$ converges as j tends to infinity. However, for even terms of this sequence, we notice that

$$\begin{aligned}
|\langle u_{n_{2j}} T u_{n_{2j}}^* \hat{1}, \hat{1}\rangle| &= \left|\sum_{i=1}^{\infty} \langle u_{n_{2j}} J u_{n_{2i}} J K_{2i} J u_{n_{2i}}^* J u_{n_{2j}}^* \hat{1}, \hat{1}\rangle\right| \\
&= \left|\sum_{i=1}^{\infty} \langle K_{2i} u_{n_{2j}}^* u_{n_{2i}} \hat{1}, u_{n_{2j}}^* u_{n_{2i}} \hat{1}\rangle\right| \\
&\geq |\langle K_{2j} \hat{1}, \hat{1}\rangle| - \sum_{1\leq i\neq j} \left|\langle K_{2i} u_{n_{2j}}^* u_{n_{2i}} \hat{1}, u_{n_{2j}}^* u_{n_{2i}} \hat{1}\rangle\right|.
\end{aligned}$$

Because of how we chose the operators K_n, we have $|\langle K_{2j}\hat{1}, \hat{1}\rangle| > 1/2$, whereas by construction $\left|\langle K_{2i} u_{n_{2j}}^* u_{n_{2i}} \hat{1}, u_{n_{2j}}^* u_{n_{2i}} \hat{1}\rangle\right| < 2^{-2i-1}$ for all $i \neq j$. We then get a lower bound

$$|\langle u_{n_{2j}} T u_{n_{2j}}^* \hat{1}, \hat{1}\rangle| > 1/2 - \sum_{1\leq i\neq j} 2^{-2i-1} \geq 1/3.$$

Conversely, for any odd term of this sequence

$$\begin{aligned}
|\langle u_{n_{2j+1}} T u_{n_{2j+1}}^* \hat{1}, \hat{1}\rangle| &= \left|\sum_{i=1}^{\infty} \langle u_{n_{2j+1}} J u_{n_{2i}} J K_{2i} J u_{n_{2i}}^* J u_{n_{2j+1}}^* \hat{1}, \hat{1}\rangle\right| \\
&= \left|\sum_{i=1}^{\infty} \langle K_{2i} u_{n_{2j+1}}^* u_{n_{2j}} \hat{1}, u_{n_{2j+1}}^* u_{n_{2i}} \hat{1}\rangle\right| \\
&\leq \sum_{i=1}^{\infty} \left|\langle K_{2i} u_{n_{2j+1}}^* u_{n_{2i}} \hat{1}, u_{n_{2j+1}}^* u_{n_{2i}} \hat{1}\rangle\right|.
\end{aligned} \tag{4.3}$$

Again, using that $\left|\langle K_{2i} u_{n_{2j+1}}^* u_{n_{2i}} \hat{1}, u_{n_{2j+1}}^* u_{n_{2i}} \hat{1}\rangle\right| < 2^{-2i-1}$ we get a bound

$$|\langle u_{n_{2j+1}} T u_{n_{2j+1}}^* \hat{1}, \hat{1}\rangle| < \sum_{i=1}^{\infty} 2^{-2i-1} = 1/6.$$

It follows then that the sequence $\langle u_{n_k} T u_{n_k}^* \hat{1}, \hat{1}\rangle$ does not converge. Hence, by contradiction, ad$T$ must be a non inner derivation.

□

Proposition 4.2.4. *Let Γ be a countable group, set $M = L\Gamma$ and let $f \in \ell^\infty\Gamma$ be so that ${}_gf - f \in c_0(\Gamma)$, $\forall g \in \Gamma$. Denote $T_f \in \mathcal{B}(L^2M)$ the diagonal operator corresponding to f.*

1° We have $[T_f, M] \subset q\mathcal{K}_M$, and thus $\delta_f := adT_f$ defines a derivation of M into $q\mathcal{K}_M$.

2° If $f \notin \mathbb{C} + c_0(\Gamma)$, then the derivation δ_f is outer, i.e, there exists no $K \in q\mathcal{K}_M$ such that $\delta_f = adK$.

Proof. 1° The condition ${}_gf - f \in c_0(\Gamma)$, $\forall g \in \Gamma$, amounts to $[M_0, T_f] \subset \mathcal{K}(L^2M) \subset \mathrm{q}\mathcal{K}_M$, where $M_0 = \mathbb{C}\Gamma$. Since M_0 is a weakly dense *-subalgebra of M, by Lemma 4.2.2 it follows that $[M, T_f] \subset \mathrm{q}\mathcal{K}_M$.

2° Assume there exists $K \in \mathrm{q}\mathcal{K}_M$ such that $\mathrm{ad}(K) = \mathrm{ad}(T_f)$ on M. We let $\mathcal{E}_0 : \mathcal{B}(L^2M) \to \ell^\infty\Gamma$ denote the conditional expectation to the diagonal operators given by $\mathcal{E}_0(T)(g) = \langle T\hat{u}_g, \hat{u}_g\rangle$. Notice that $\mathcal{E}_0$ implements the canonical trace on both $L\Gamma$ and $R\Gamma$. By Lemma 3.2.7 if $p \geq 2$, then we have $\|\mathcal{E}_0(T)\| \leq |||T|||_p$, and so from Theorem 4.1.5 it follows that $\mathcal{E}_0(K) \in c_0(\Gamma)$.

Since $K - T_f \in M' = M^{op}$, we then have $\mathcal{E}_0(K) - f = \mathcal{E}_0(K - T_f) \in \mathbb{C}$, contradicting the fact that $f \notin \mathbb{C} + c_0(\Gamma)$.

□

Corollary 4.2.5. *If Γ is any infinite group, then there exists a non-inner derivation of $M = L\Gamma$ into $q\mathcal{K}_M$ of the form $\delta_f = adT_f$ where $f \in \ell^\infty\Gamma$ is given as in Proposition 4.2.4.*

Proof. From an argument very similar to the one used in Theorem 4.2.3 it follows that there always exist $f \in \ell^\infty\Gamma$ so that ${}_gf - f \in c_0(\Gamma)$ for all $g \in \Gamma$, but such that $f \notin \mathbb{C} + c_0(\Gamma)$. One simply starts with an asymptotically Γ-invariant approximate identity in $c(\Gamma)$ and proceeds as in the proof of Theorem 4.2.3.

□

Lemma 4.2.6. *Assume $\delta : M \to q\mathcal{K}_M$ is implemented by $T \in \mathcal{B}(L^2M)$. If $K_n \in q\mathcal{K}_M$ are so that $\|K_n\| \leq \|T\|$, $\forall n$, and $\lim_n q_M([K_n, x], \delta(x)) = 0$ for all x in some weakly dense*

*-*subalgebra* M_0 *of* M*, then this limit holds true for all* $x \in M$*.*

Proof. Let $y \in (M)_1$. We have to prove that given any $\varepsilon > 0$ there exists n_0 such that for any $n \geq n_0$ there exists $p \in \mathcal{P}(M)$ satisfying $\tau(1-p) \leq \varepsilon$ and $\|p(\delta(y) - [K_n, y])p\| \leq \varepsilon$.

By Kaplanski's theorem, we can take $y_0 \in (M_0)_1$ with $\|y_0 - y\|_2 \leq (\varepsilon/2)^{3/2}/2$. By applying the hypothesis to this $y_0 \in M_0$, there exists n_0 such that $\forall n \geq n_0$, $\exists p_0 \in \mathcal{P}(M)$ with $\tau(1-p_0) \leq \varepsilon/2$ and $\|p_0(\delta(y_0) - [K_n, y_0])p_0\| \leq \varepsilon/3$. On the other hand, by applying Theorem 4.2.1 to $x = y - y_0$ and the derivations $\delta, \mathrm{ad}(K_n)$, we get a projection $p_1 \in M$ such that $\tau(1-p_1) \leq \varepsilon/2$ and $\|p_1\delta(y-y_0)p_1\|\varepsilon/3$, $\|p_1[K_n, (y-y_0)]p_1\| \leq \varepsilon/3$. Thus, if we let $p = p_0 \wedge p_1$, then $\tau(1-p) \leq \varepsilon$ and for each $n \geq n_0$ we have

$$\|p(\delta(y) - [K_n, y])p\|$$

$$\leq \|p\delta(y-y_0)p\| + \|p[K_n, (y-y_0)]p\| + \|p(\delta(y_0) - [K_n, y_0])p\| \leq \varepsilon.$$

□

Theorem 4.2.7. *Let* $\delta : M \to q\mathcal{K}_M$ *be a derivation implemented by* $T \in \mathcal{B}(L^2M)$*. Then there exists a net of finite-rank operators* K_ι *with* $\|K_\iota\| \leq \|T\|$ *such that* $\lim_\iota q_M(\delta(x), [K_\iota, x]) = 0$ *for all* $x \in M$*. Moreover, if* L^2M *is separable, then the net can be taken a sequence.*

Proof. Let $F = \{x_1, x_2, \ldots, x_n\}$ be an arbitrary finite subset of M and $\epsilon > 0$. Since δ is a derivation into $\mathrm{q}\mathcal{K}_M$, we can find a projection $p \in M$ with $\tau(1-p) < \epsilon/2$ such that $p\delta(x_i)p \in \mathcal{K}(L^2M)$ for $1 \leq i \leq n$. Consider then the convex subset $C \subset \mathcal{K}(L^2M)^n$ consisting of all n-tuples of the form

$$(p\delta(x_1)p - p[K, x_1]p, p\delta(x_2)p - p[K, x_2]p, \ldots, p\delta(x_n)p - p[K, x_n]p),$$

where K runs over all finite-rank operators in $\mathcal{B}(L^2M)$ such that $\|K\| \leq \|T\|$.

The set $C \subset \mathcal{K}(L^2M)^n$ can be viewed as a subset of $(\mathcal{K}(L^2M)^n)^{**} = \mathcal{B}(L^2M)^n$. Note that, since $\delta = \mathrm{ad}(T)$, if we plug in T for K in the above n-tuple viewed as an element in

$\mathcal{B}(L^2M)^n$, then one gets $(0,...,0)$. Note also that T is a *wo*-limit of finite-rank operators with norm at most $\|T\|$ and that this implies $(0,...,0) = (p\delta(x_1)p - p[T,x_1]p, p\delta(x_2)p - p[T,x_2]p, \ldots, p\delta(x_n)p - p[T,x_n]p)$ is in the $\sigma(\mathcal{B}(L^2M)^n, \mathcal{B}_{\mathrm{n}}^*(L^2M)^n)$-closure of C in $\mathcal{B}(L^2M)^n$. But since $C \subset \mathcal{K}(L^2M)^n$ is convex, its norm closure in $\mathcal{K}(L^2M)^n$ coincides with its closure in the $\sigma(\mathcal{K}(L^2M)^n, \mathcal{B}_{\mathrm{n}}^*(L^2M)^n)$ topology. Hence $(0,0,\ldots,0)$ is in the norm closure of C. In particular, there exists a finite-rank operator $K \in \mathcal{B}(L^2M)$ such that $\|K\| \leq \|T\|$ and $\|p\delta(x_i)p - p[K,x_i]p\| < \epsilon/2$ for all $1 \leq i \leq n$. But then we see that

$$\sup_{1\leq i\leq n} \mathrm{q}_M(\delta(x_i),[K,x_i]) \leq \sup_{1\leq i\leq n} \tau(1-p) + \|p(\delta(x_i) - [K,x_i])p\| < \epsilon.$$

This shows that for any set $F \subset M$ we can find a finite-rank operator $K_F \in \mathcal{B}(L^2M)$ such that $\|K_F\| \leq \|T\|$ and $\mathrm{q}_M(\delta(x_i) - [K_F,x_i]) < 1/|F|$. This net $(K_F)_F$, indexed over all finite subsets will then satisfy the condition.

The fact that this net can be taken to be a sequence when L^2M is separable follows from Lemma 4.2.6.

□

The next result shows if $B_0 \subset M$ is a weakly quasi-regular diffuse von Neumann subalgebra of M (in the sense of [GP17]), then the derivations of M into any of the bimodule $\mathrm{q}\mathcal{K}_M$, are uniqueley determined by their restriction to B_0.

Proposition 4.2.8. *Let M be a tracial von Neumann algebra with a diffuse weakly quasi-regular von Neumann subalgebra $B_0 \subset M$. If a derivation $\delta : M \to q\mathcal{K}_M$ vanishes on B_0, then $\delta = 0$ on all M.*

Proof. Since δ is automatically $\|\cdot\|_2$-q_M-continuous, it follows that the space $\tilde{B}$ of elements in M on which δ vanishes (which contains the diffuse algebra B_0, by hypothesis) is a von Neumann subalgebra of M. Let u be a unitary element in M such that $B := u^*\tilde{B}u \cap \tilde{B}$ is diffuse and denote $\sigma : B \to M$ the isomorphism of B into $\tilde{B}$ given by $\sigma(b) = ubu^*$,

$b \in B$. Since $ub = \sigma(b)u$, by applying δ it follows that $\delta(u)b = \sigma(b)\delta(u)$, $\forall b \in B$. Thus, $K = \delta(u) \in \mathrm{q}\mathcal{K}_M$ satisfies the conditions in Lemma 4.1.6, implying that $\delta(u) = 0$. Since $B_0 \subset M$ is weakly quasi-regular, this shows that $\tilde{B} = M$.

□

We end this section by mentioning some q_M-approximation properties of derivations of a tracial von Neumann algebra M into Banach M-bimodules endowed with the q_M-metric.

Proposition 4.2.9. *Let (M, τ) be a tracial von Neumann algebra, $\mathcal{B}$ a Banach M-bimodule and $\delta : M \to \mathcal{B}$ a derivation.*

1° *Let $M_0 \subset M$ be a weakly dense C^*-subalgebra and $\mathcal{B}_0 \subset \mathcal{B}$ an M sub-bimodule (not necessarily norm-closed). Assume $p\delta(M_0)p \subset \mathcal{B}_0$, for some projection $p \in M$. Then, for any countable subset $\mathcal{X} \subset M$ and any $\varepsilon_0 > 0$, there exists $p_0 \in \mathcal{P}(pMp)$ such that $\tau(p - p_0) \leq \varepsilon\tau(p)$ and $p_0\delta(x)p_0 \in \mathcal{B}_0$, $\forall x \in \mathcal{X}$.*

2° *If $\mathcal{B} = q\mathcal{K}_M$, then given any separable C^*-subalgebra $M_0 \subset M$ and any $\varepsilon > 0$, there exists $p_0 \in \mathcal{P}(M)$ such that $\tau(1 - p_0) \leq \varepsilon$ and $p_0\delta(x)p_0 \in \mathcal{K}(L^2M)$, $\forall x \in M_0$.*

Proof. 1° Let $\mathcal{X} = \{x_n\}_{n\geq 1}$ be an enumeration of $\mathcal{X}$. By Pedersen's Lusin-type Theorem, for each n there exists $p_n \in \mathcal{P}(M)$ and $y_n \in M_0$ such that $x_np_n = y_np_n$, $p_nx_n = p_ny_n$ and $\tau(p_n) \geq 1 - \tau(p)\varepsilon/2^{n+1}$, $\forall n$. Since $\delta(x_np_n) = y_n\delta(p_n) + \delta(y_n)p_n$, we have

$$p_n\delta(x_n)p_n = p_n\delta(x_np_n)p_n - p_nx_n\delta(p_n)p_n$$

$$= p_ny_n\delta(p_n)p_n + p_n\delta(y_n)p_n - p_nx_n\delta(p_n)p_n = p_n\delta(y_n)p_n \in \mathcal{B}_0.$$

Thus, if we let $p_0 = \wedge_{n\geq 1}p_n \wedge p$, then $p_0\delta(x)p_0 \in \mathcal{B}_0$, $\forall x \in \mathcal{X}$. Moreover, we have $\tau(\wedge_{n\geq 1}p_n) \geq (1 - \Sigma_{n\geq 1}\tau(1 - p_n)) = 1 - \varepsilon\tau(p)$ and thus $\tau(p_0) \geq (1 - \varepsilon\tau(p)) + \tau(p) - 1 = (1 - \varepsilon)\tau(p)$, implying that $\tau(p - p_0) \leq \varepsilon\tau(p)$.

2° This is trivial by Proposition 4.1.2.

□

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