Using Instantaneous Normalized Receive SNR for Diversity Gain Calculation

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Abstract

We propose a technique to calculate the diversity gain for a vector channel when the noises have any nonsingular distribution and the fading coefficients have arbitrary distribution. This technique uses the serial expansion of the outage probability of the instantaneous normalized receive SNR to obtain the diversity gain. The approach is simpler than using the pairwise symbol error rate to analyze the diversity gain.

1 Main Results

Consider the equivalent system equation

$$\mathbf{y} = \sqrt{P}\mathbf{h}s + \mathbf{n}, \tag{1}$$

where $\mathbf{y}, P, s, \mathbf{h}, \mathbf{n}$ denote the $N \times 1$ received vector, the transmit power, the information symbol from a fixed constellation, the $N \times 1$ channel vector, and the $N \times 1$ noise vector, respectively. We assume that $\mathbf{h}$ has any distribution with limited variance on each entry. The covariance matrix of $\mathbf{n}$, denoted as $\mathbf{\Sigma}$, is nonsingular so that it can be inverted.

Conventionally, the diversity gain is defined and analyzed by

$$d = -\frac{\log\ \mathrm{Prob}(s \to s')}{\log\ P}, \tag{2}$$

where $\mathrm{Prob}(s \to s')$ denotes the pairwise symbol error rate (SER) of decoding s to s'. Using (2) to calculate the diversity requires averaging over the distribution of the channel fading. The analysis might be intractable when the equivalent channels have complicated distribution, e.g., in the case of cooperative networks. Motivated by making the analysis simpler, we present the following theorem:

Theorem 1. *Define the* instantaneous normalized receive SNR *as* $\gamma = \mathbf{h}^* \mathbf{\Sigma}^{-1} \mathbf{h}$. *The diversity gain can be calculated by*

$$d = \lim_{\epsilon \to 0^+} \frac{\log \text{Prob}(\gamma < \epsilon)}{\log \epsilon}. \tag{3}$$

Proof. The pairwise SER of decoding s to s' can be written as

$$\text{Prob}(s \to s') = \mathop{\mathbb{E}}_{\gamma} Q\left(\sqrt{P|s-s'|^2\gamma/2}\right), \tag{4}$$

where $Q(x)$ denotes Gaussian Q function. Note that $Q(x) \geq \min\left\{\frac{1}{5}e^{-\frac{x^2}{2}}, \frac{1}{3x}e^{-\frac{x^2}{2}}\right\}$ and both $\frac{1}{3x}e^{-\frac{x^2}{2}}$ and $\frac{1}{5}e^{-\frac{x^2}{2}}$ are decreasing functions. Thus, for any $\epsilon \geq 0$,

$$\text{Prob}(s \to s') \geq \text{Prob}(\gamma \leq \epsilon) \min\left\{\frac{1}{5}e^{-P\Delta^2\epsilon/2}, \frac{e^{-P\Delta^2\epsilon/2}}{3\sqrt{P\Delta^2\epsilon/2}}\right\} \tag{5}$$

where Δ denotes the minimum distance between any two constellation points. Let $\epsilon = P^{-1}$. Then, we have $P(s \to s') \geq P(\gamma \leq \epsilon)\underbrace{\min\left\{\frac{1}{5}e^{-\Delta^2/2}, \frac{e^{-\Delta^2/2}}{3\Delta/\sqrt{2}}\right\}}_{c}$, where c is a constant. Thus, the diversity gain can be calculated using (2) as

$$d = -\lim_{P \to \infty} \frac{\log\ \text{Prob}(s \to s')}{\log\ P} \leq \lim_{\epsilon \to 0^+} \frac{\log\ \text{Prob}(\gamma < \epsilon) + \log\ c}{\log\ \epsilon} = \lim_{\epsilon \to 0^+} \frac{\log\ \text{Prob}(\gamma < \epsilon)}{\log \epsilon}. \tag{6}$$

This shows that the diversity is upperbounded by the RHS of (3). Next, we show that the diversity is also lowerbounded by the RHS of (3). Denote β as the maximum distance between any two constellation points. For any $\epsilon \geq 0$, the Chernoff bound on symbol error rate can be broken into

$$\text{Prob}(s \to s') < \mathop{\mathbb{E}}_{\gamma} e^{-\frac{\beta^2 P\gamma}{4}} = \int_0^{\epsilon} e^{-\frac{\beta^2 P\gamma}{4}} f(\gamma) d\gamma + \int_{\epsilon}^{\infty} e^{-\frac{\beta^2 P\gamma}{4}} f(\gamma) d\gamma < \text{Prob}(\gamma < \epsilon) + e^{-\frac{\beta^2 P\epsilon}{4}}, \tag{7}$$

where $f(\gamma)$ denotes the probability density function of γ. The second inequality holds because $e^{-\frac{\beta^2 P\gamma}{4}}$ is a decreasing function with γ. Let $\epsilon = P^{-a}$ where a is a number approaching 1 from the left side. When P approaches infinity, the RHS of (7) is dominated by $\text{Prob}(\gamma < \epsilon)$. Thus, the diversity gain can be lowerbounded as

$$d = -\lim_{P \to \infty} \frac{\log\ \text{Prob}(s \to s')}{\log\ P} \geq \lim_{a \to 1^-, \epsilon \to 0^+} \frac{a \log \text{Prob}(\gamma < \epsilon)}{\log\ \epsilon} = \lim_{\epsilon \to 0^+} \frac{\log \text{Prob}(\gamma < \epsilon)}{\log\ \epsilon}. \tag{8}$$

From (8) and (6), the lowerbound and upperbound converge. □

Theorem 1 says that diversity can be calculated using the outage probability of the instantaneous normalized receive SNR. Eq. (3) is applicable for any vector channels with nonsingular noises. Application of this theorem for multi-source cooperative networks can be found in [1, 2].

References

[1] L. Li, Y. Jing, and H. Jafarkhani, "Interference cancellation at the relay in multi-access wireless relay networks," *submitted to IEEE Trans. on Wireless Comm., also available on http://arxiv.org/abs/1004.3807*, Apr. 2010.

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