University of California

Ernest O. Lawrence Radiation Laboratory

A RELATION BETWEEN CRACK SURFACE DISPLACEMENTS AND THE STRAIN ENERGY RELEASE RATE

P. L. Key

June 1968

Submitted to International
Journal of Fracture Mechanics

UCRL-18065
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

A RELATION BETWEEN CRACK SURFACE DISPLACEMENTS AND THE STRAIN ENERGY RELEASE RATE

P. L. Key

June 1968

A RELATION BETWEEN CRACK SURFACE DISPLACEMENTS AND THE STRAIN ENERGY RELEASE RATE

P. L. Key

Inorganic Materials Research Division, Lawrence Radiation Laboratory, and the Department of Mineral Technology, College of Engineering, University of California, Berkeley, California

ABSTRACT

The relationship between the strain energy release rate, G, and the displacements of the surfaces of an extending crack in an elastic, tensile member is examined. It is shown that G can be expressed in terms of the volume of the deformed crack provided that any stresses applied to crack surfaces are uniform. This form is especially useful for superposition applications as it depends linearly on displacements.

The strain energy release rate is calculated from crack volumes for a crack in an infinite sheet and for two cases of a crack in an infinite solid: (a) a penny-shaped crack subjected to internal pressure as well as axial stress, and (b) an elliptical crack loaded by axial stress. The importance of the shape of the propagating crack is demonstrated by the elliptical crack by considering various shapes for the propagating crack such as preferential propagation along a diameter or propagation as an ellipse of invariant shape.

A discussion of the distinction between a fracture criterion based on the strain energy release rate and one based on the stress intensity factor is presented.

INTRODUCTION

Most recent theoretical fracture mechanics work and the majority of fracture toughness data involve the stress intensity factor K instead of the strain energy release rate G. This emphasis on K is due primarily to the development of techniques for its theoretical prediction from the stress distribution around the crack and is not due to any particular physical significance of this parameter. The strain energy release rate, G, has physical significance either in terms of an energy rate or a crack extension force and can be directly measured by the compliance technique.[1] However, the direct calculation of G usually involves complicated volume or surface integrals involving stresses and displacements. Thus, the theoretical values of G are usually obtained from the values of K related by the equations of the form[2]

$$G = \frac{K^2}{E}$$

(for a plane stress, mode I crack).

It is desirable, where possible, to cast the fracture mechanics parameters in terms of the local environment of the crack. For example, fracture criteria have been given in terms of local stresses by Irwin,[3] local strains by Krafft[4] and by Irwin and McClintock[5] and most recently, local displacement of the crack faces by Wells[6] and Cottrell.[7] The object of this paper is to formulate the strain energy release rate in terms of the displacements of the crack surfaces and to apply this formulation to two cases of a crack in an infinite solid loaded in tension: the elastic crack with internal pressure and the flat, elliptical crack.[8] The classical elastic crack in an infinite sheet is also treated.

ANALYSIS

Consider a right, cylindrical tensile member of uniform cross sectional area, Ag, containing a planar flaw lying in a plane perpendicular to the tensile axis. The member is loaded by tensile stresses distributed across the ends of the member and by stresses distributed over the flaw surface. The strain energy release rate, G, will be obtained by considering an energy balance on the tensile member for a small crack extension under the condition of constant axial load and crack surface stresses. As discussed by several authors,[9] the assumption of constant loads does not restrict the generality of the result. Referring to Fig. 1, a crack extension of ΔA_c is accompanied by the following energy changes:

$$\text{work done by F} = F\left(\frac{\partial \Delta}{\partial A_c}\right)\Delta A_c$$

$$\text{work done by p} = 2\int_{A_c} p\left(\frac{\partial w}{\partial A_c}\right)\Delta A_c\, dA$$

$$\text{strain energy released} = -\left(\frac{\partial U}{\partial A_c}\right)\Delta A_c$$

$$\text{total energy available for crack extension} = G\Delta A_c$$

Thus,

$$G = F\left(\frac{\partial \Delta}{\partial A_c}\right) + 2\int_{A_c} p\left(\frac{\partial w}{\partial A_c}\right) dA - \frac{\partial U}{\partial A_c} \tag{1}$$

where: Δ: axial displacement between the ends of the tensile member

w: axial displacement of the crack surface under the action of F and p

U: strain energy of the solid under the action of F and p

A_c: area of the flaw in the x,y plane

A: area measured in the x,y plane

p: normal stress applied to the flaw surface

F: total axial load on the member.

The strain energy can be expressed in terms of the surface stresses, T_i, and surface displacements, u_i:[10]

$$U = \frac{1}{2} \int_S T_i u_i d\sigma \tag{2}$$

where S is the total surface of the body. Thus, for F and p,

$$U = \frac{1}{2} F\Delta + 2\left(\frac{1}{2} \int_{A_c} pwdA\right) \tag{3}$$

Introducing Eq. (3) into (1)

$$G = \frac{1}{2} F \frac{\partial \Delta}{\partial A_c} + 2 \int_{A_c} p \left(\frac{\partial w}{\partial A_c}\right) dA - \frac{\partial}{\partial A_c} \int_{A_c} pwdA \tag{4}$$

The extension Δ can be expressed in terms of the displacements of the crack surface by an application of the reciprocal theorem of elasticity in a manner similar to Greenspan.[11] Specifically, consider the two stress and displacement fields shown in Fig. 2. The first state (a) consists of the actual stresses and displacements being considered while the second state (b) is a uniform stress field equal to the average axial stress actually applied to the member. Neglecting body forces, the reciprocal theorem (10) states that for these two stress states the mixed energies are equal:

$$\int_S T_{ai} u_{bi} d\sigma = \int_S T_{bi} u_{ai} d\sigma \tag{5}$$

Assuming the ends of the member remain plane and that the flaw deforms in the axial direction only, application of Eq. (5) to the stresses and displacements shown in Fig. 2 yields:

$$\Delta = \Delta(o) + \frac{2}{A_g} \int_{A_c} w dA - \frac{2}{EA_g} \int_{A_c} pz(o) dA \tag{6}$$

where: $\Delta(o) = \frac{FL}{EA_g}$

A_g: total area of tensile member

E: Young's modulus

z(o): initial unloaded axial position of the flaw surface (a non-planar flaw).

Introducing Eq. (6) into Eq. (4) for a planar flaw $(z(0) = 0)$ and letting $\sigma = \frac{F}{Ag}$,

$$G = \frac{\partial}{\partial A_c} \int_{A_c} (\sigma - p) w dA + 2 \int_{A_c} p \left(\frac{\partial w}{\partial A_c}\right) dA \tag{7}$$

Eq. (7) expresses the strain energy release rate in terms of the local environment of the crack: the crack surface displacements and stresses. The principle of superposition can be conveniently used with Eq. (7) by considering the displacements associated with the two loadings (F and p) separately. Hence,

$$w = w_1 + w_2 \qquad (8)$$

where w_1 is the displacement of the flaw surface obtained with F alone and w_2 is that for p alone. Several alternate forms of Eq. (7) are useful. For σ, p constant,

$$G = (\sigma + p) \frac{\partial}{\partial A_c} \int_{A_c} w dA$$

But this integral represents one-half the volume, V, of the deformed flaw

$$G = \frac{(\sigma + p)}{2} \frac{\partial V}{\partial A_c} = \frac{(\sigma + p)}{2} \frac{\partial}{\partial A_c} \left[V_1 + V_2 \right] \qquad (9)$$

For a plate containing a flaw of length 2a, Eqs. (7) and (9) become (10) and (11) respectively:

$$G = \frac{\partial}{\partial a} \int_o^a (\sigma - p) w dx + 2 \int_o^a p \frac{\partial w}{\partial a} dx \qquad (10)$$

$$G = \frac{(\sigma + p)}{4} \frac{\partial S}{\partial a} = \frac{(\sigma + p)}{4} \frac{\partial}{\partial a} \left[S_1 + S_2 \right] \qquad (11)$$

where S, S_1, S_2 are the areas of the loaded flaw in the plane of the sheet.

The above relations are applicable to both infinite bodies and finite bodies with no modification. The effect of specimen size is introduced through its effect on the crack surface displacements.

ELASTIC CRACK

For an elastic crack in an infinite plate with a uniform stress, σ, at infinity, Irwin[2] showed that the displacement along the crack borders for the case of plane stress is:

$$w(x) = \frac{2\sigma}{E}\left(a^2 - x^2\right)^{1/2} \qquad (12)$$

The crack thus deforms into an elliptical shape whose area is:

$$S = \pi a\left(\frac{2\sigma a}{E}\right)$$

Hence, using Eq. (11)

$$G = \frac{\sigma}{4}\frac{\partial S}{\partial a} = \frac{\pi\sigma^2 a}{E} \qquad (13)$$

which is the well-known result.

ELLIPTICAL CRACK

Green and Sneddon[8] showed that a flat, elliptical crack in an infinite body subjected to a uniaxial stress, σ, normal to the crack deformed into an ellipsoidal cavity:

$$w(x,y) = w(0,0)\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{c}\right)^2\right]^{1/2} \qquad (14a)$$

where

$$w(0,0) = \frac{2\left(1 - \nu^2\right)}{E} \frac{a\sigma}{E(k)} \tag{14b}$$

and

$$E(k) = \int_0^{\pi/2} \left[\cos^2\phi + \left(\frac{a}{c}\right)^2 \sin^2\phi\right]^{1/2} d\phi$$

where $k = \left[1 - (a/c)^2\right]^{1/2}$ and $E(k)$ is the complete elliptic integral of the second kind.

In Eq. (14), a, c are the minor and major axes of the elliptical crack, respectively. Note that as c becomes very large, this equation reduces to Eq. (12) for a one-dimensional crack except for the factor $(1 - \nu^2)$, because Eq. (14) represents a plane strain condition rather than the plane stress of Eq. (12).

The volume of the deformed crack is given by:

$$V = \frac{4}{3}\pi acw(0,0) = \frac{8\pi\left(1 - \nu^2\right)\sigma}{3E} \frac{a^2c}{E(k)}$$

Using Eq. (9):

$$G = \frac{4\pi}{3E}\left(1 - \nu^2\right)\sigma^2 \frac{\partial}{\partial A_c}\left(\frac{a^2c}{E(k)}\right) \tag{15}$$

To calculate the derivative in Eq. (15), an assumption must be made concerning the shape of the growing crack, i.e., the relationship between a and c.

<u>Case I</u> - Constant shape crack - $a/c = \alpha$, constant. The area of the

elliptical crack is given by

$$A_c = \pi ac = \pi\alpha c^2$$

Therefore,

$$\frac{\partial}{\partial A_c} = \frac{1}{2\pi\alpha c}\frac{\partial}{\partial c}$$

Hence, noting that E(k) is a constant for (a/c) constant, Eq. (15) reduces to

$$G = \frac{2\left(1 - \nu^2\right)\sigma^2}{E}\frac{a}{E(k)}$$

or

$$\frac{G}{Ge} = \frac{\pi}{2}\frac{1}{E(k)} \qquad (16)$$

where

$$Ge = \frac{4\sigma^2\left(1 - \nu^2\right)a}{\pi E} ,$$

the value for a penny shaped crack as obtained by Sack.[12]

For the case of a penny shaped crack, a/c = 1.0 and E(k) = $\pi/2$;

$$\frac{G}{Ge} = 1.0$$

as expected.

Case II - Constant major diameter - c = constant. As Irwin[13] showed, the local stress intensity factor varies around the crack border for an elliptical crack with the maximum value at the minor diameter. Thus, crack extension in this direction is probable.

For this case

$$\frac{\partial}{\partial A_c} = \frac{1}{\pi c}\frac{\partial}{\partial a}$$

Equation (15) becomes

$$G = \frac{4\left(1 - \nu^2\right)\sigma^2}{3E}\frac{\partial}{\partial a}\left(\frac{a}{E(k)}\right)$$

or

$$\frac{G}{Ge} = \frac{\pi}{3[E(K)]^2}\left\{E(k) + K(k) + \frac{E(k) - K(k)}{k^2}\right\} \tag{17}$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\left(1 - k^2 \sin^2\phi\right)^{1/2}} ,$$

the complete elliptic integral of the first kind and

$$Ge = \frac{4\left(1 - \nu^2\right)\sigma^2 a}{\pi E}$$

For an initially penny shaped crack, a/c = 1.0;

$$\frac{G}{Ge} = 1.0$$

as before.

Case III - Constant minor diameter - a = constant. For this case

$$\frac{\partial}{\partial A_c} = \frac{1}{\pi a}\frac{\partial}{\partial c}$$

Eq. (15) becomes

$$G = \frac{4\left(1-\nu^2\right)\sigma^2 a}{3E}\frac{\partial}{\partial c}\left(\frac{c}{E(k)}\right)$$

which reduces to

$$\frac{G}{Ge} = \frac{\pi}{3[E(K)]^2}\left\{E(k) + \frac{1-k^2}{k^2}\left[K(k) - E(k)\right]\right\} \tag{18}$$

where

$$Ge = \frac{4\left(1-\nu^2\right)\sigma^2 a}{\pi E}$$

Equations (16), (17) and (18) are plotted in Fig. 3.

CRACK WITH INTERNAL PRESSURE

Consider a penny shaped crack in an infinite solid stressed with an internal pressure p and axial stress σ. Green and Sneddon[8] showed that the penny shaped crack stressed with an axial stress σ deformed into an ellipsoidal cavity:

$$w_1(x,y) = w_1(0,0)\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2\right]^{1/2} \tag{19a}$$

and

$$w_1(0,0) = \frac{4\left(1 - \nu^2\right)a\sigma}{\pi E} \tag{19b}$$

The volume V_1 is given by

$$V_1 = \frac{4}{3}\pi a^2 w_1(0,0) = \frac{16\left(1 - \nu^2\right)a^3\sigma}{3E}$$

Sneddon[14] showed that the penny shaped crack stressed with an internal pressure p also deformed into an ellipsoidal cavity

$$w_2(x,y) = w_2(0,0)\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2\right]^{1/2} \tag{20a}$$

$$w_2(0,0) = \frac{4\left(1 - \nu^2\right)ap}{\pi E} \tag{20b}$$

Thus, the volume V_2 is

$$V_2 = \frac{16\left(1 - \nu^2\right)a^3 p}{3E}$$

The area of the crack is given by

$$A_c = \pi a^2$$

Hence,

$$\frac{\partial}{\partial A_c} = \frac{1}{2\pi a}\frac{\partial}{\partial a}$$

Using the above expressions for V_1 and V_2 in Eq. (9), the strain energy release rate becomes

$$G = \frac{4\left(1 - \nu^2\right)a}{\pi E}\left(\sigma + p\right)^2 \tag{21}$$

A similar result was obtained by Murrell[15] but a different one has been reported by Swedlow[16] in which an error may have resulted from the greater algebraic complexity of the method used by Swedlow.

DISCUSSION

General - As shown by the above example of a crack loaded by both internal pressure and axial stress, the relation between G and the volume of the deformed flaw is especially useful for applications of the superposition principle since it has a linear dependence on displacements. For example, Swedlow[16] has reexamined the case of fracture of a crack in a biaxial tension field. Griffith[17] originally concluded that fracture was independent of biaxiality but Swedlow derived a relation indicating that G was dependent upon biaxiality. The above theory can be used to examine this conflict. A transverse stress would not change the volume of the deformed crack and thus would not affect the terms of Eq. (9). In addition, the work done by the transverse stress is independent of the crack size and thus no additional terms must be added to Eq. (9). Hence, G should not depend upon biaxiality as has also been concluded by Rice.[18]

Fracture Criteria - The distinction between a local criterion of fracture in terms of the stress intensity factor, K, and an average criterion defined by the strain energy release rate, G, is not usually made clear. In fact, these criteria are frequently considered equivalent and related by the expression:[3]

$$G = \left(\frac{1 - \nu^2}{E}\right) K^2 \tag{22}$$

However, Eq. (22) applies only for a straight crack boundary with constant K and thus can apply only locally to a flaw with a curved boundary which is the usual case of a natural flaw. In general, one would not expect equivalence of average and local criteria except for cases where the local parameter does not vary spatially. The value of K obtained from Eq. (22) is only an average value and thus the maximum local value may be substantially larger. If critical values of K or G are determined with a specimen in which K is constant (most fracture specimens approximate this condition) and these values are then applied to a design problem involving a crack with varying K, the local criterion will be satisfied at some point on the crack boundary before the critical G level of the specimen is reached. To interpret the effect of this distinction between K and G on design, one must decide which criterion is physically correct and, to the author's knowledge, this has not yet been done. The magnitude of this effect is illustrated below for the elliptical crack.

Elliptical Crack - The stress intensity factor for elliptical cracks has been discussed by Irwin.[13] Considering only the case of crack propagation in the constant shape mode, Irwin showed that the maximum stress intensity factor occurred at the end of the minor diameter with a value of:

$$K_{max} = \frac{(\pi a)^{1/2}}{E(k)} \tag{23}$$

In order to compare this value with the values of G for the entire specimen, it is converted to a local G value using Eq. (22):

$$\left(\frac{G_{max}^{loc}}{Ge}\right) = \left(\frac{\pi}{2E(k)}\right)^2 \tag{24}$$

which is plotted in Fig. 3. This local criterion lies above the average values for all three modes of crack propagation considered but should only be compared with the values for constant shape propagation. There is an obvious significant difference between local and average criterion for this mode of crack propagation. It is expected that similar differences would occur for the other modes of crack propagation if the local values were available. Since the strain energy release rate is greatest for the case of crack propagation along the minor diameter, this mode is preferred from either an average or a local criterion. Therefore, the elliptical crack is expected to be unstable and to grow into a circular crack. Even though the mode of crack propagation can be determined, the critical stress for initiation of propagation depends on knowledge of the correct physical fracture criterion and thus cannot presently be predicted.

Although the local values of G defined by Eq. (22) have little physical significance, they can be used to obtain an average G value for the entire body by integration. Thus, considering an element of crack length $d\ell$,

$$G = \frac{1 - \nu^2}{E} \int_c K^2 d\ell \qquad (25)$$

where c is the boundary of the crack. Using Irwin's expression[13] for K for an elliptical flaw propagating in a constant shape mode in Eq. (25), the relation for G previously derived (Eq. (16)) is obtained as expected. The greater ease of the approach used here is obvious. In addition, the method developed above allows different modes of crack propagation (constant shape, constant diameter) to be readily investigated.

CONCLUSIONS

1. The strain energy release rate of an elastic body containing a propagating flaw can be related to the displacements of the flaw surfaces and the stresses on the flaw surfaces.

2. The strain energy release rate for an elliptical crack in an infinite solid depends on the shape of the propagating crack; the greatest value occurs for an elliptical crack propagating along its minor diameter.

3. In general, fracture criteria based on local quantities such as K are not equivalent to a criterion based on the strain energy release rate, G. The G criterion prevails for cases where the Griffith criterion of fracture is applicable. A G criterion requires a higher load for fracture than the local criterion.

ACKNOWLEDGMENT

This work was done under the auspices of the United States Atomic Energy Commission. Discussions with E. R. Parker, W. W. Gerberich, and Y. Katz on this paper are gratefully acknowledged.

REFERENCES

1. G. R. Irwin, Fracture Mechanics, Structural Mechanics (Pergamon Press, London, 1960), p. 557.
2. G. R. Irwin, Relation of Stresses Near a Crack to the Crack Extension Force, Proc. Ninth Congress of Applied Mechanics, Brussels, 1956, p. 245.
3. G. R. Irwin, Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate, Journal of Applied Mechanics, 24, 361 (1957).
4. J. M. Krafft, Correlation of a Plane Strain Crack Toughness with Strain Hardening Characteristics of a Low, a Medium, and a High Strength Steel, Applied Materials Research, 3, 88 (1964).
5. F. A. McClintock and G. R. Irwin, Plasticity Aspects of Fracture Mechanics, Fracture Toughness Testing and Its Applications (ASTM, Philadelphia, 1965), p. 84.
6. A. A. Wells, Unstable Crack Propagation in Metals: Cleavage and Fast Fracture, Crack Propagation Symposium, College of Aeronautics and the Royal Aeronautical Society, Cranfield, 1961, p. 210.
7. A. H. Cottrell, Theoretical Aspects of Radiation Damage and Brittle Fracture in Steel Pressure Vessels, Steels for Reactor Pressure Circuits, Iron and Steel Institute, Special Report No. 69, 1960, p. 281.
8. A. E. Green and I. N. Sneddon, The Distribution of Stresses in the Neighborhood of a Flat Elliptical Crack in an Elastic Solid, Proceedings of the Cambridge Philosophical Society, 46, 159 (1950).
9. P. C. Paris and G. C. Sih, Stress Analysis of Cracks, in Fracture Toughness Testing and Its Applications (ASTM, Philadelphia, 1965), p. 30.

10. I. S. Sokolnikoff, Mathematical Theory of Elasticity (McGraw-Hill, New York, 1956).

11. M. Greenspan, Axial Rigidity of Perforated Structural Members, Journal of Research of the National Bureau of Standards, 31, 305 (1943).

12. R. A. Sack, Extension of Griffith's Theory of Rupture to Three Dimensions, Proceedings of the Physical Society, 58, 729 (1946).

13. G. R. Irwin, Crack Extension Force for a Part-Through Crack in a Plate, Journal of Applied Mechanics, 29, 651 (1962).

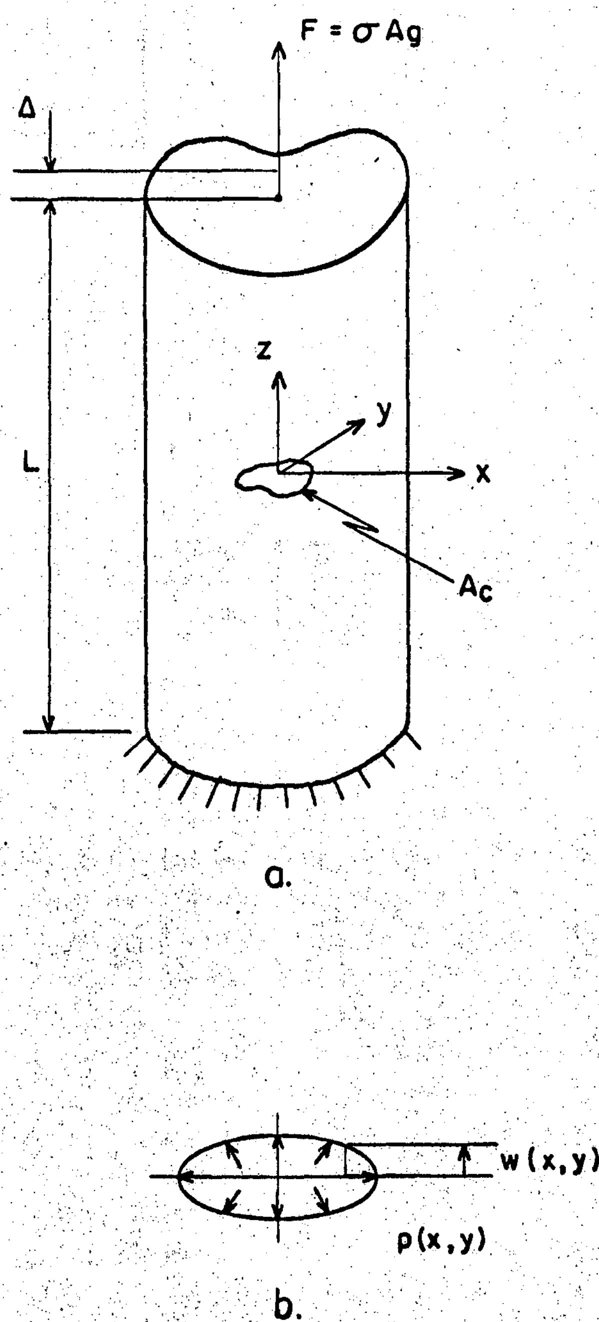
14. I. N. Sneddon, The Distribution of Stress in the Neighborhood of a Crack in an Elastic Solid, Proceedings of the Royal Society, 187A, 229 (1946).

15. S. A. F. Murrell, The Theory of the Propagation of Elliptical Griffith Cracks under Various Conditions of Plane Strain or Plane Stress, British Journal of Applied Physics, 15, 1195 (1964).

16. J. L. Swedlow, On Griffith's Theory of Fracture, International Journal of Fracture Mechanics, 1, 210 (1965).

17. A. A. Griffith, The Phenomena of Rupture and Flow in Solids, Philosophical Transactions of the Royal Society (London), 221A, 163 (1921).

18. J. R. Rice, An Examination of the Fracture Mechanics Energy Balance From the Point of View of Continuum Mechanics, Proceedings of the First International Conference on Fracture, Sendai, Japan, 1965, 1, 309.



XBL 683-364

Fig. 1 (a) Right cylindrical tensile member containing a planar flaw

(b) Detail of flaw

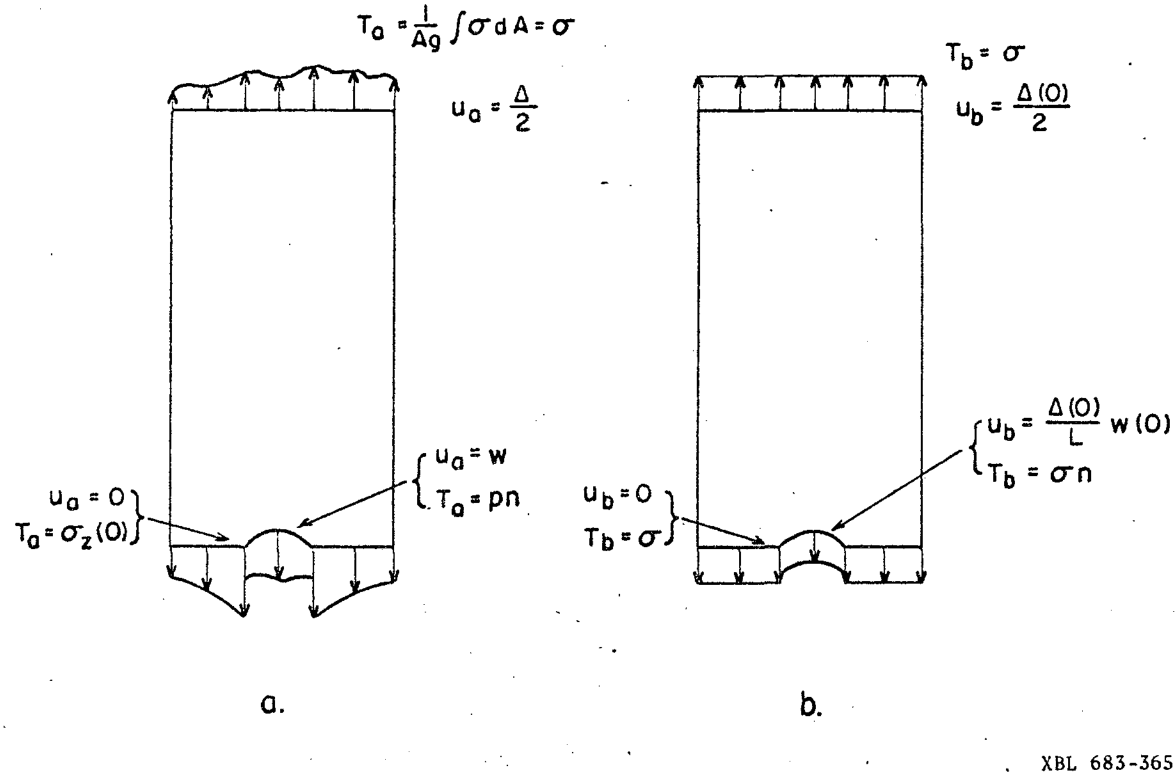
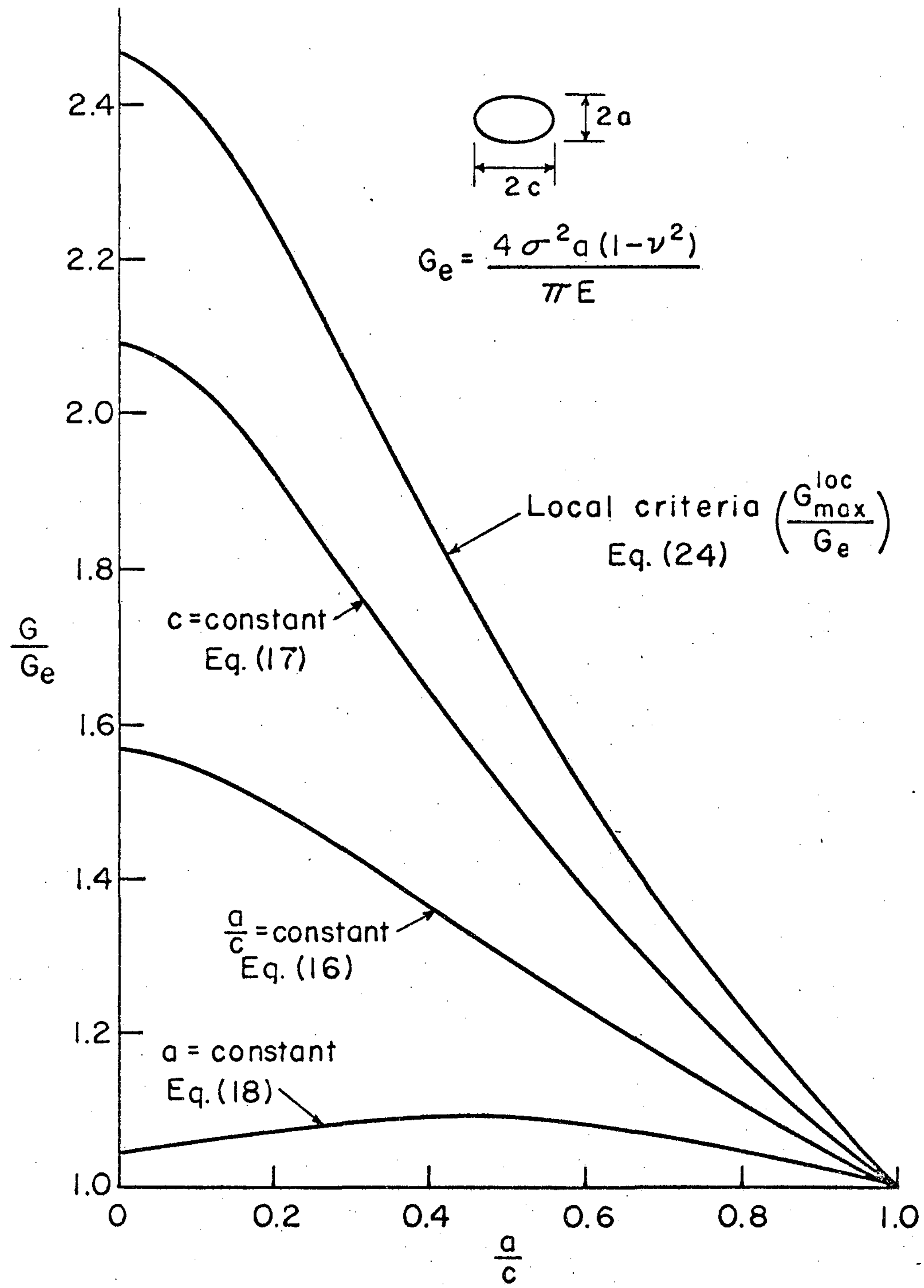


Fig. 2 Application of reciprocal theorem

(a) Actual stresses and displacements

(b) Uniform stress field



XBL 682-181-A

Fig. 3 Strain energy release rate for an elliptical crack