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Uniqueness of VOA Structure of 3C-Algebra and 5A-Algebra

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UNIQUENESS OF VOA STRUCTURE OF $3C$-ALGEBRA AND $5A$-ALGEBRA

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by

Wen Zheng

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Contents

Abstract

Uniqueness of VOA structure of $3C$-algebra and $5A$-algebra

by

Wen Zheng

In this thesis, we study the structure of $3C$-algebra and $5A$-algebra constructed by Lam-Yamada-Yamauchi. We mainly use relevant braiding matrices to establish the uniqueness of the vertex operator algebra structure of these two algebras. Besides, we also give the fusion rules for these two algebras.

Keywords: Vertex operator algebra, Fusion rule, Quantum dimension

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Chapter 1

Introduction

The Monster simple group $\mathbb{M}$ [18] is generated by some $2A$-involutions and the conjugacy class of the product $\tau\tau'$ of two $2A$-involutions is one of the nine classes $1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$ and $3C$ in $\mathbb{M}$ [5]. Moreover, each $2A$-involution τ defines a unique idempotent e_τ in the Monster Griess algebra, which is called an axis. The inner product of e_τ and $e_{\tau'}$ is uniquely determined by the conjugacy class of the product of two $2A$-involutions $\tau\tau'$. From the construction of the moonshine vertex operator algebra $V^\natural$ [17] we know that the Monster Griess algebra is the weight two subspace $V_2^\natural$ of the $V^\natural$. It was discovered in [13] that $V^\natural$ contains 48 Virasoro vectors, each Virasoro vector generates a Virasoro vertex operator algebra isomorphic to $L(\frac{1}{2}, 0)$ in $V^\natural$ and $L(\frac{1}{2}, 0)^{\otimes 48}$ is a conformal subalgebra of $V^\natural$. Such a Virasoro vector is called an Ising vector. Miyamoto [27] later realized that $2e_\tau$ is an Ising vector for any $2A$-involution τ. Conversely, for an Ising vector e in $V_2^\natural$, one can construct a $2A$-involution τ_e which is called Miyamoto involution. Thus there is a one-to-one correspondence between $2A$-involutions of $\mathbb{M}$ and Ising vectors of $V^\natural$. According to a result in [5], the structure of the subalgebra generated by two Ising vectors e and f in the algebra $V_2^\natural$ depends on only the conjugacy class of $\tau_e\tau_f$. For the nine classes $1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$ and

$3C$, the inner product $\langle e, f\rangle$ are $\frac{1}{4}, \frac{1}{2^5}, \frac{13}{2^{10}}, \frac{1}{2^7}, \frac{3}{2^9}, \frac{5}{2^{10}}, \frac{1}{2^8}, 0, \frac{1}{2^8}$, respectively.

It is natural to ask what the vertex operator subalgebra generated by two Ising vectors in an arbitrary vertex operator algebra is. A beautiful result given in [28] asserts that the inner product of any two different Ising vectors again take these 8 values as in the case of the moonshine vertex operator algebra. In [24], [25], for each of the nine cases, they constructed a subalgebra of $V_{\sqrt{2}E_8}$, which is generated by two Ising vectors. These vertex operator algebras generated by two Ising vectors are simply called $1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$ and $3C$-algebras, denoted by $\mathcal{U}_{nX}$. But this raises two more questions: (1) Is the vertex operator algebra structure of these algebras constructed in [24], [25] unique? (2) Is any vertex operator algebra generated by two Ising vectors isomorphic to one of these 9 algebras? The uniqueness of VOA structure of $\mathcal{U}_{6A}$ has been given in [7]. In [29], they proved the VOA generated by two Ising vectors whose inner product is $\frac{13}{2^{10}}$ has a unique VOA structure, so $3A$ case in question (2) has been solved thoroughly.

Now consider the uniqueness of VOA structure of $\mathcal{U}_{nX}$-algebra where $nX \neq 3A$ or $6A$. Note that $\mathcal{U}_{1A} \cong L\left(\frac{1}{2}, 0\right)$ and $\mathcal{U}_{2B} \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, 0\right)$. So the uniqueness of VOA structure of these two algebras is trivial. We also know that $\mathcal{U}_{2A} \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right)$ is a simple current extension of the subVOA $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right)$, $\mathcal{U}_{4B} \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right)$ is a simple current extension of the subVOA $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, 0\right)$, so the uniqueness of the VOA structure of $\mathcal{U}_{2A}$ and $\mathcal{U}_{4B}$ follows from [12], also see Remark 2.5.4. $\mathcal{U}_{4A} \cong V_{\mathcal{N}}^{+}$, So $\mathcal{U}_{4A}$, $\mathcal{U}_{5A}$ and $\mathcal{U}_{3C}$ are the three nontrivial cases left. In this paper, we only consider the uniqueness of VOA structure of $\mathcal{U}_{5A}$ and $\mathcal{U}_{3C}$. The main idea is to use relevant braiding matrices.

This thesis is organized as follows. In Chapter 2, we review some basic notions and some well known results in the vertex operator algebra theory. In Chapter 3, we study the structure of the $5A$-algebra and prove the uniqueness of the vertex operator algebra structure on $\mathcal{U}_{5A}$. In Chapter 4, we study the structure of the $3C$-algebra and prove the uniqueness of the vertex operator algebra structure on $\mathcal{U}_{3C}$. In Chapter 5, we give fusion rules of the $5A$-algebra and $3C$-algebra.

Chapter 2

Preliminaries

2.1 Basics

We first state the definition of vertex operator algebra from [4], [17].

Definition 2.1.1. *A vertex operator algebra* $(V, Y, \mathbf{1}, \omega)$ *is a* $\mathbb{Z}$*-graded vector space with a linear map*

$$Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$$
$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, v_n \in \mathrm{End}V$$

called the **vertex operator** *of* v *for each* $v \in V$ *satisfying:*

(1) For any $u, v \in V$, $u_n v = 0$ *for* n *sufficiently large.*

(2) There is a specific element $\mathbf{1} \in V_0$ *called the* **vaccum** *such that*

$$Y(\mathbf{1}, z) = \mathrm{Id}_V,$$
$$v_{-1}\mathbf{1} = v, \ \ v_n\mathbf{1} = 0 \ \text{ for } n \geq 0.$$

(3) There is a specific element $\omega \in V_2$ *called the* **Virasoro element** *such that*

(i) $\{L(n) := \omega_{n+1}\}$ *is a Virasoro algebra generator, that is , they satisfy*

$$[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}c$$

where $c \in \mathbb{C}$ *is called the* **rank** *of* V*.*

(ii) The $L(-1)$*-derivative property:*

$$[L(-1), Y(v,z)] = \frac{d}{dz}Y(v,z)$$

(iii) $L(0)|_{V_n} = n \cdot \mathrm{Id}_{V_n}$*.*

(4)Jacobi identity: For any $u, v \in V$*,*

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2),$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ *and all binomial expressions (here and below) are to be expanded in nonnegative integral powers of the second variable.*

From now on, we denote by $(V, Y, \mathbf{1}, \omega)$ a vertex operator algebra. Next we recall notions of weak, admissble and ordinary modules from [17], [31], [9].

Definition 2.1.2. *A* **weak** V**-module** M *is a vector space equipped with a linear map*

$$Y_M : V \to (\mathrm{End}M)[[z, z^{-1}]]$$
$$v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, v_n \in \mathrm{End}M,$$

which satisfies the following conditions: for $u, v \in V$, $w \in M$,

$$u_l w = 0 \text{ for } l \gg 0,$$

$$Y_M(\mathbf{1}, z) = \mathrm{Id}_M,$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_M(v, z_2)Y_M(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_M(Y(u, z_0)v, z_2).$$

We use $\mathbb{Z}_+$ to denote the set of nonnegative integers.

Definition 2.1.3. *An* **admissible V-module** *M is a weak V-module which carries a $\mathbb{Z}_+$-grading $M = \oplus_{n\in\mathbb{Z}_+} M(n)$ satisfying the following condition: for $m \in \mathbb{Z}, n \in \mathbb{Z}_+$ and homogeneous $v \in V$, $v_m M(n) \subseteq M(\mathrm{wt} v + n - m - 1)$.*

Definition 2.1.4. *An* **ordinary V-module** *M is a weak V-module which carries a $\mathbb{C}$-grading $M = \oplus_{\lambda\in\mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$ and $L(0)$ is the component operator of $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. We also require that $\dim M_\lambda$ is finite and for fixed λ, $M_{\lambda+n} = 0$ for small enough $n \in \mathbb{Z}$. From now on, if we say M is a V-module, we mean M is an ordinary V-module.*

Definition 2.1.5. *A vertex operator algebra V is said to be* **rational** *if the admissible module category is semisimple.*

Definition 2.1.6. *A vertex operator algebra V is called* **C_2-cofinite** *if it satisfies $\dim V/C_2(V) < \infty$, where $C_2(V) = \langle u_{-2}v | u, v \in V\rangle$.*

It is proved in [9] that if V is rational, then there are finitely many inequivalent irreducible admissible modules $M^0, M^1, \ldots, M^d$ and each irreducible admissible module is an ordinary module. Each M^i has weight space decomposition

$$M^i = \oplus_{n\in\mathbb{Z}_+} M^i_{\lambda_i+n},$$

where $\lambda_i \in \mathbb{C}$ is a complex number such that $M^i_{\lambda_i} \neq 0$ and $M^i_{\lambda_i+n}$ is the eigenspace of $L(0)$ with eigenvalue $\lambda_i + n$. The λ_i is called the conformal weight of M^i. If V is both rational and C_2-cofinite, then each λ_i and the central charge of V are rational numbers [10].

For a vertex operator algebra V, the following **skew symmetry** property [16]

$$Y(u,z)v = e^{zL(-1)}Y(v,-z)u$$

for $u, v \in V$ is useful later.

Definition 2.1.7. *A vertex operator algebra V is said to be* **CFT type** *if $V = \oplus_{n\in\mathbb{Z}_+} V_n$ and $V_0 = \mathbb{C}\mathbf{1}$.*

We say V is simple if as a V-module, V is irreducible. If V is both rational and CFT type, then it is easy to see that V is simple.

The following lemma from [7] will be used later.

Lemma 2.1.8. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and $\sigma : V \to V$ be a linear isomorphism such that $\sigma(\mathbf{1}) = \mathbf{1}, \sigma(\omega) = \omega$. Then $(V, Y^\sigma, \mathbf{1}, \omega)$ is a vertex operator algebra where*

$$Y^\sigma(u,z) = \sigma Y(\sigma^{-1}u, z)\sigma^{-1}$$

and $(V, Y, \mathbf{1}, \omega) \cong (V, Y^\sigma, \mathbf{1}, \omega)$.

2.2 Ising vector

Definition 2.2.1. *A vector $e \in V_2$ is called a* **conformal vector with central charge** c_e *if it satisfies $e_1 e = 2e$ and $e_3 e = \frac{c_e}{2}\mathbf{1}$. Then the operators $L_n^e := e_{n+1},\ n \in \mathbb{Z}$, satisfy the Virasoro commutation relation*

$$[L_m^e,\ L_n^e] = (m-n)\, L_{m+n}^e + \delta_{m+n,\ 0}\frac{m^3-m}{12}c_e$$

for $m,\ n \in \mathbb{Z}$. A conformal vector $e \in V_2$ with central charge $1/2$ is called an **Ising vector** *if e generates the simple Virasoro vertex operator algebra $L(1/2,\ 0)$.*

2.3 Invariant bilinear form

Let $M = \oplus_{\lambda\in\mathbb{C}} M_\lambda$ be a V-module. The restricted dual of M is defined by $M' = \oplus_{\lambda\in\mathbb{C}} M_\lambda^*$ where $M_\lambda^* = \mathrm{Hom}_{\mathbb{C}}(M_\lambda, \mathbb{C})$. It is proved in [16] that $M' = (M', Y_{M'})$ is naturally a V-module such that

$$\langle Y_{M'}(v,z) f, u\rangle = \left\langle f, Y_M\left(e^{zL(1)}\left(-z^{-2}\right)^{L(0)} v, z^{-1}\right) u\right\rangle,$$

for $v \in V$, $f \in M'$ and $u \in M$, and $(M')' \cong M$. Moreover, if M is irreducible, so is M'. A V-module M is said to be *self-dual* if $M \cong M'$.

Let $\mathbb{C}[z_1, z_2]$ be the polynomial ring and $\mathbb{C}(z_1, z_2)$ be the field of all rational functions in z_1 and z_2. Set $S = \{z_1, z_2, z_1 \pm z_2\}$. Let $\mathbb{C}[z_1, z_2]_S$ be the subalgebra of $\mathbb{C}(z_1, z_2)$ generated by $z_1^{\pm 1}$, $z_2^{\pm 1}$ and $(z_1 \pm z_2)^{-1}$. Define ι_{12} to be the linear map

$$\iota_{12} : \mathbb{C}[z_1, z_2]_S \to \mathbb{C}[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$$

such that $\iota_{12}(f(z_1, z_2))$ is the formal Laurent series expansion of $f(z_1, z_2)$ involving only finitely many negative powers of z_2. Analogously, we define the linear map $\iota_{21}(f(z_1, z_2))$ using the opposite expansion.

Proposition 2.3.1. *Let $u, v, w \in V$ and $w' \in V'$ be arbitrary. We have:*

*(a) (***rationality of products***) The formal series*

$$\langle w', Y(u, z_1) Y(v, z_2) w\rangle \left(= \sum_{m,n\in\mathbb{Z}} \langle w', u_m v_n w\rangle z_1^{-m-1} z_2^{-n-1} \right) \tag{2.3.1}$$

lies in the image of the map ι_{12}:

$$\langle w', Y(u, z_1) Y(v, z_2) w\rangle = \iota_{12} f(z_1, z_2), \tag{2.3.2}$$

where the (uniquely determined) element $f \in \mathbb{C}[z_1, z_2]_S$ is of the form

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{(z_1 - z_2)^k z_1^l z_2^m}$$

for some $g \in \mathbb{C}[z_1, z_2]$ and $k, l, m \in \mathbb{Z}$, where k depends only on u and v; it is independent of w and w'.

*(b) (***commutativity***) We also have*

$$\langle w', Y(v, z_2) Y(u, z_1) w\rangle = \iota_{21} f(z_1, z_2), \tag{2.3.3}$$

that is, in informal language,

"$Y(u, z_1)Y(v, z_2)$ agrees with $Y(v, z_2)Y(u, z_1)$

as operator-valued rational functions."

Definition 2.3.2. *A bilinear form $\langle\, ,\, \rangle$ on a V-module M is said to be* **invariant**

if it satisfies the condition

$$\langle Y(a,z)u,v\rangle = \langle u, Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v\rangle$$

for $a \in V, u, v \in M$.

The following result about invariant bilinear forms on V is from [26]:

Theorem 2.3.3. *The space of invariant bilinear forms on* V *is isomorphic to the space*

$$(V_0/L(1)V_1)^* = Hom_{\mathbb{C}}(V_0/L(1)V_1, \mathbb{C}).$$

In particular, if V *is a simple vertex operator algebra of CFT type with* $V_1 = 0$*, then there is a unique nondegenerate invariant bilinear form* $\langle\, , \rangle$ *on* V *satisfying* $\langle \mathbf{1}, \mathbf{1}\rangle = 1$ *and* $V \cong V'$.

2.4 Intertwining operators and fusion rules

Definition 2.4.1. *Let* V *be a vertex operator algebra and let* (M^i, Y^i), (M^j, Y^j), $\left(M^k, Y^k\right)$ *be three* V*-modules. An* **intertwining operator of type** $\begin{pmatrix} M^k \\ M^i \ M^j \end{pmatrix}$ *is a linear map*

$$\mathcal{Y}(\cdot, z): \ M^i \to Hom\left(M^j, \ M^k\right)\{z\}$$
$$u \mapsto \mathcal{Y}(u,z) = \sum_{n\in\mathbb{Q}} u_n z^{-n-1}$$

satisfying:

(1) For any $u \in M^i$ *and* $v \in M^j$, $u_n v = 0$ *for* n *sufficiently large;*

(2) $\mathcal{Y}(L_{-1}v, \ z) = \left(\frac{d}{dz}\right)\mathcal{Y}(v,z)$ *for* $v \in M^i$;

(3) (Jacobi Identity) For any $u \in V,\ v \in M^i$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y^k(u, z_1)\,\mathcal{Y}(v, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\mathcal{Y}(v, z_2)\,Y^j(u, z_1)$$
$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}\left(Y^i(u, z_0)\,v, z_2\right).$$

The space of all intertwining operators of type $\begin{pmatrix} M^k \\ M^i\ M^j \end{pmatrix}$ *is denoted by the symbol* $I_V\begin{pmatrix} M^k \\ M^i\ M^j \end{pmatrix}$. *Without confusion, we also denote it by* $I_{i,j}^k$. *Let* $N_{i,\ j}^k = \dim I_{i,j}^k$. *These integers* $N_{i,j}^k$ *are called the* fusion rules.

Definition 2.4.2. *Let* M^1 *and* M^2 *be* V*-modules. A* **tensor product** *for the ordered pair* (M^1, M^2) *is a pair* $(M, \mathcal{Y}(\cdot, z))$ *which consists of a* V*-module* M *and an intertwining operator* $\mathcal{Y}(\cdot, z)$ *of type* $\begin{pmatrix} W \\ M^1\ M^2 \end{pmatrix}$ *satisfies the following universal property: For any* V*-module* W *and any intertwining operator* $\mathcal{I}(\cdot, z)$ *of type* $\begin{pmatrix} W \\ M^1\ M^2 \end{pmatrix}$, *there exists a unique* V*-homomorphism* ϕ *from* M *to* W *such that* $\mathcal{I}(\cdot, z) = \phi \circ \mathcal{Y}(\cdot, z)$. *From the definition it is easy to see that if a tensor product of* M^1 *and* M^2 *exists, it is unique up to isomorphism. In this case, we denote the* fusion product *by* $M^1 \boxtimes_V M^2$.

Let V^1 and V^2 be vertex operator algebras. Let $\{M^i\ ,\ i = 1, 2, 3\}$ be V^1-modules, and $\{N^i,\ i = 1, 2, 3\}$ be V^2-modules. Then $\{M^i \otimes N^i,\ i = 1, 2, 3\}$ are $V^1 \otimes V^2$-modules by [16]. The following property was given in [3]:

Proposition 2.4.3. *If* $N_{M^1,M^2}^{M^3} < \infty$ *or* $N_{N^1,N^2}^{N^3} < \infty$, *then*

$$N_{M^1 \otimes N^1, M^2 \otimes N^2}^{M^3 \otimes N^3} = N_{M^1,M^2}^{M^3} N_{N^1,N^2}^{N^3}.$$

2.5 Simple current extensions

Definition 2.5.1. *Let V be a simple VOA. An irreducible V module U is called a* **simple current V-module** *if for any irreducible V-module M, the fusion product $U \boxtimes M$ is also irreducible.*

Definition 2.5.2. *A VOA is graded by an abelian group G if $V = \oplus_{g \in G} V^g$, and $u_n v \in V^{g+h}$ for any $u \in V^g, v \in V^h$, and $n \in \mathbb{Z}$.*

Definition 2.5.3. *Let $V = \sum_{g \in G} V^g$ be a simple G-graded VOA such that $V^g \neq 0$ for all $g \in G$, then V is called a G-graded simple current extension if all $V^g, g \in G$ are simple current V^0-modules.*

By [12], we have the following proposition:

Proposition 2.5.4. *Let $V = \sum_{g \in G} V^g$ be a simple G-graded VOA which is a simple current extension of V^0. Then the VOA structure of V is determined uniquely by the V^0-module structure of V.*

2.6 Quantum dimensions

Definition 2.6.1. *An* **automorphism** *g of a vertex operator algebra V is a linear isomorphism of V satisfying $g(\omega) = \omega$ and $gY(v, z)g^{-1} = Y(gv, z)$ for any $v \in V$. We denote by $\mathrm{Aut}(V)$ the group of all automorphisms of V.*

For a subgroup $G \leq \mathrm{Aut}(V)$ the fixed point set $V^G = \{v \in V | g(v) = v, \text{for any } g \in G\}$ has a vertex operator algebra structure. By [11], [8], we have the following:

Theorem 2.6.2. *Suppose that V is a simple vertex operator algebra and that G is a finite group of automorphisms of V. Then the following hold:*

(i) $V = \oplus_{\chi \in Irr(G)} V^\chi$, *where* V^χ *is the subspace of* V *on which* G *acts according to the character* χ. *Each* V^χ *is nonzero;*

(ii) For $\chi \in Irr(G)$, *each* V^χ *is a simple module for the* G*-graded vertex operator algebra* $\mathbb{C}G \otimes V^G$ *of the form*

$$V^\chi = M_\chi \otimes V_\chi,$$

where M_χ *is the simple* G*-module affording* χ *and where* V_χ *is a simple* V^G*-module.*

(iii) The map $M_\chi \mapsto V_\chi$ *is a bijection from the set of inequivalent simple* G*-modules to the set of inequivalent simple* V^G*-modules which are contained in* V.

Now we recall the notion of quantum dimensions from [6]. Let $M = \oplus_{n \in \mathbb{Z}_+} M_{\lambda+n}$ be a V-module. The formal character of M is defined to be

$$\mathrm{ch}_q M = \mathrm{tr}_M q^{L(0)-c/24} = q^{\lambda - c/24} \sum_{n \in \mathbb{Z}_+} (\dim M_{\lambda+n}) q^n.$$

It is proved in [31] and [10] that $\mathrm{ch}_q M$ converges to a holomorphic function on the domain $|q| < 1$ if V is C_2-cofinite. We sometimes also use $Z_M(\tau)$ to denote the holomorphic function $\mathrm{ch}_q M$ with variable τ in the complex upper half-plane $\mathbb{H}$ and $q = e^{2\pi i \tau}$. By [6], we have the following:

Definition 2.6.3. *Let* M *be a* V*-module such that* $Z_V(\tau)$ *and* $Z_M(\tau)$ *exist. The* **quantum dimension** *of* M *over* V *is defined as*

$$q\dim_V M = \lim_{y \to 0} \frac{Z_M(iy)}{Z_V(iy)},$$

where y *is real and positive. Sometimes we use an alternative definition which* *involves the q-characters:*

$$q\dim_V M = \lim_{q\to 1^-} \frac{\mathrm{ch}_q M}{\mathrm{ch}_q V}.$$

We have the following results [6], [3]:

Theorem 2.6.4. *Let V be a rational and C_2-cofinite simple vertex operator algebra, G, V_χ and M_χ are defined as in Theorem 2.6.2. Assume V is g-rational and the conformal weight of any irreducible g-twisted V-module is positive except for V itself for all $g \in G$. Then*

$$q\dim_{V^G} V_\chi = \dim M_\chi.$$

Proposition 2.6.5. *Let V be a rational and C_2-cofinite simple vertex operator algebra of CFT type with $V \cong V'$. Let $M^0, M^1, \cdots, M^d$ be all the inequivalent irreducible V-modules with $M^0 \cong V$. The corresponding conformal weights λ_i satisfy $\lambda_i > 0$ for $0 < i \leq d$. Then*

(i) $q\dim_V (M^i \boxtimes M^j) = q\dim_V M^i \cdot q\dim_V M^j$, *for any* $i, j \in \{0, 1, \ldots, d\}$.

(ii) A V-module M^i is simple current if and only if $q\dim_V M^i = 1$.

(iii) $q\dim_V M^i \in \{2\cos(\pi/n) \,|n \geq 3\} \cup \{a|2 \leq a < \infty, a \text{ is algebraic}\}$.

Remark 2.6.6. *Let U and V be a vertex operator algebra under the same assumption with Proposition 2.6.5, M be a U-module and N be a V-module. Then*

$$q\dim_{U\otimes V} M \otimes N = q\dim_U M \cdot q\dim_V N.$$

2.7 The unitary series of the Virasoro VOAs

From now on we always assume $p, q \in \{2, 3, 4, ...\}$, and p, q are relatively prime.

Definition 2.7.1. *An ordered triple of pairs of integers* $((m,n),(m',n'),(m'',n''))$ *is called* **admissible** *if* $0<m,m',\ m''<p, 0<n,n',n''<q,\ m+m'+m''<2p$, $n+n'+n''<2q$, $m<m'+m''$, $m'<m+m''$, $m''<m+m'$, $n<n'+n''$, $n'<n+n''$, $n''<n+n'$, *and the sums* $m+m'+m''$, $n+n'+n''$ *are odd. We identify the triples* $((m,n),(m',n'),(m'',n''))$ *and* $((m,n),(p-m',q-n'),(p-m'',q-n''))$.

Let $c_{p,q}=1-\frac{6(p-q)^2}{pq}$, $h_{m,n}=\frac{(np-mq)^2-(p-q)^2}{4pq}$, $0<m<p, 0<n<q$. $L(c_{p,q},h_{m,n})$ is the irreducible highest weight representation of the Virasoro algebra L with highest weight $(c_{p,q},h_{m,n})$. Then $L(c_{p,q},0)$ has a VOA structure. Moreover, we have the following results [30]:

Theorem 2.7.2. *The vertex operator algebra* $L(c_{p,q},0)$ *is rational and the minimal modules* $L(c_{p,q},h_{m,n})$, $0<m<p, 0<n<q$ *are all irreducible representations of* $L(c_{p,q},0)$.

Theorem 2.7.3. *The fusion rules between* $L(c_{p,q},0)$*-modules* $L(c_{p,q},h_{m',n'})$ *and* $L(c_{p,q},h_{m'',n''})$ *are*

$$L(c_{p,q},h_{m',n'})\boxtimes L(c_{p,q},h_{m'',n''})=\sum_{(m,n)}N_{(m',n'),(m'',n'')}^{(m,n)}L(c_{p,q},h_{m,n}),$$

where $N_{(m',n'),(m'',n'')}^{(m,n)}$ *is* 1 *if and only if* $((m,n),(m',n'),(m'',n''))$ *is an admissible triple of pairs and* 0 *otherwise.*

2.8 Braiding matrices

Let V be a rational and C_2-cofinite vertex operator algebra of CFT type and $V\cong V'$. Let $\mathcal{A}=\{0,1,2,...,d\}$ and $\{M^i, i\in\mathcal{A}\}$ be the set of all inequivalent irreducible V-modules. Let $I_{a_1,a_2}^{a_3}$ be the space of all intertwining operators of type

$\begin{pmatrix} & M^{a_3} & \\ M^{a_1} & & M^{a_2} \end{pmatrix}$ for $a_1, a_2, a_3 \in \mathcal{A}$. It follows from [19] that for any $a_1, a_2, a_3, a_4 \in \mathcal{A}$ there is a **braiding isomorphism** $\mathcal{B}^{a_1,a_2}_{a_4,a_3}$ from $\coprod_{a\in\mathcal{A}} I^{a_4}_{a_1,a} \otimes I^{a}_{a_2,a_3}$ to $\coprod_{a\in\mathcal{A}} I^{a_4}_{a_2,a} \otimes I^{a}_{a_1,a_3}$. We choose a basis $\mathcal{Y}^{a_3}_{a_1,a_2;i}$ of the space $I^{a_3}_{a_1,a_2}$ for $a_1, a_2, a_3 \in \mathcal{A}$ and $i = 1, 2, \ldots, N^{a_3}_{a_1,a_2}$. Then we obtain the **braiding matrices** $B^{a_1,a_2}_{a_4,a_3}$ corresponding to the braiding isomorphisms $\mathcal{B}^{a_1,a_2}_{a_4,a_3}$, $a_1, a_2, a_3, a_4 \in \mathcal{A}$, whose entries are given by

$$(\mathcal{B}^{a_1,a_2}_{a_4,a_3})(\mathcal{Y}^{a_4}_{a_1,a_5;i} \otimes \mathcal{Y}^{a_5}_{a_2,a_3;j}) = \sum_{a\in\mathcal{A}} \sum_{k=1}^{N^{a_4}_{a_2,a}} \sum_{l=1}^{N^{a}_{a_1,a_3}} \left(B^{a_1,a_2}_{a_4,a_3}\right)^{i,j;k,l}_{a_5,a} (\mathcal{Y}^{a_4}_{a_2,a;k} \otimes \mathcal{Y}^{a}_{a_1,a_3;l}) \quad (2.8.1)$$

for $a_5 \in \mathcal{A}$, $i = 1, \ldots, N^{a_4}_{a_1,a_5}$, $j = 1, \ldots, N^{a_5}_{a_2,a_3}$.

Consider the simply connected regions in $\mathbb{C}^2$ obtained by cutting the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$ along the intersections of these regions with $\mathbb{R}^2 \subset \mathbb{C}^2$. We denote them by R_1 and R_2 respectively. For $a_1, a_2, a_3, a_4 \in \mathcal{A}$, let $\mathbb{G}^{a_1,a_2,a_3,a_4}$ be the space of multivalued analytic functions on

$$R = \{(z_1, z_2) \in \mathbb{C}^2 | z_1, z_2 \neq 0, z_1 \neq z_2\}$$

with a suitable choice of a single valued branch which is called **prefered branch** in [19] on the regions R_1 and R_2. Then we have linear maps

$$\iota_{12} : \mathbb{G}^{a_1,a_2,a_3,a_4} \to \coprod_{a\in\mathcal{A}} x_1^{\lambda_{a_4}-\lambda_{a_1}-\lambda_a} x_2^{\lambda_a-\lambda_{a_2}-\lambda_{a_3}} \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}][[x_2/x_1]]$$
$$\iota_{21} : \mathbb{G}^{a_1,a_2,a_3,a_4} \to \coprod_{a\in\mathcal{A}} x_2^{\lambda_{a_4}-\lambda_{a_2}-\lambda_a} x_1^{\lambda_a-\lambda_{a_1}-\lambda_{a_3}} \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}][[x_1/x_2]]$$

where $\lambda_i, i \in \mathcal{A}$ is the conformal weight of M^i. These maps are injective and generalize ι_{12}, ι_{21} discussed before. We have the following result [19]:

Proposition 2.8.1. *For any $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, any $w_{a_i} \in M^{a_i}, i = 1, 2, 3, w'_{a_4} \in (M^{a_4})'$, any $i, j \in \mathbb{Z}$ satisfying $1 \leq i \leq N^{a_4}_{a_1,a_5}$, $1 \leq j \leq N^{a_5}_{a_2,a_3}$, the (multivalued)* *analytic function*

$$\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_1,a_5;i}(w_{a_1},x_1)\mathcal{Y}^{a_5}_{a_2,a_3;j}(w_{a_2},x_2)w_{a_3}\rangle|_{x_1=z_1,x_2=z_2}$$

defined on the region $|z_1|>|z_2|>0$ *and the (multivalued) analytic function*

$$\sum_{a\in\mathcal{A}}\sum_{k=1}^{N^{a_4}_{a_2,a}}\sum_{l=1}^{N^{a}_{a_1,a_3}}\left(B^{a_1,a_2}_{a_4,a_3}\right)^{i,j;k,l}_{a_5,a}\cdot\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_2,a;k}(w_{a_2},x_2)\mathcal{Y}^{a}_{a_1,a_3;l}(w_{a_1},x_1)w_{a_3}\rangle|_{x_1=z_1,x_2=z_2}$$

on the region $|z_2|>|z_1|>0$ *are analytic extensions of each other. We can simply write this property in the following way:*

$$\begin{aligned}&\iota_{12}^{-1}\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_1,a_5;i}(w_{a_1},z_1)\mathcal{Y}^{a_5}_{a_2,a_3;j}(w_{a_2},z_2)w_{a_3}\rangle\\&=\sum_{a\in\mathcal{A}}\sum_{k=1}^{N^{a_4}_{a_2,a}}\sum_{l=1}^{N^{a}_{a_1,a_3}}\left(B^{a_1,a_2}_{a_4,a_3}\right)^{i,j;k,l}_{a_5,a}\cdot\iota_{21}^{-1}\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_2,a;k}(w_{a_2},z_2)\mathcal{Y}^{a}_{a_1,a_3;l}(w_{a_1},z_1)w_{a_3}\rangle.\end{aligned}$$

Remark 2.8.2. *By [20], for any* $a_1,a_2,a_3,a_4\in\mathcal{A}$*,*

$$\iota_{12}^{-1}\left\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_1,a_5;i}\left(w_{a_1},z_1\right)\mathcal{Y}^{a_5}_{a_2,a_3;j}\left(w_{a_2},z_2\right)w_{a_3}\right\rangle$$

where $i=1,\cdots,N^{a_4}_{a_1,a_5}, j=1,\cdots,N^{a_5}_{a_2,a_3}, a_5\in\mathcal{A}$ *is a linearly independent set.*

Now let's recall some formulas about minimal models of Virasoro vertex operator algebra given in [15]. Let $\alpha_-^2=\frac{p}{p'}$, $\alpha_+^2=\frac{p'}{p}$, here $p'=p+1$. Let $x=\exp\left(2\pi i\alpha_+^2\right)$, $y=\exp\left(2\pi i\alpha_-^2\right)$, $[l]=x^{l/2}-x^{-l/2}$, $[l]'=y^{l/2}-y^{-l/2}$. Denote $c_p=1-\frac{6}{p(p+1)}$ with $p=2,3,4,\cdots$, $h^{(p)}_{(i'\ i)}=\frac{1}{4}\left(i'^2-1\right)\alpha_-^2-\frac{1}{2}\left(i'i-1\right)+\frac{1}{4}\left(i^2-1\right)\alpha_-^{-2}=\frac{(pi'-(p+1)i)^2-1}{4p(p+1)}$. Now we fix the central charge c_p, denote $L\left(c_p,h^{(p)}_{(i',i)}\right)$ by (i',i). Note that here (i',i) is the same as (i,i') in Theorem 2.7.3. Let (a',a), (m',m), (n',n), (c',c), (b',b), (d',d) be irreducible $L\left(c_p,0\right)$-modules. The braiding matrices of screened vertex operators have the almost factorized form (cf. (2.19) of [15]):

$$\begin{aligned}
&\left(B^{(a',a),(c',c)}_{(m',m),(n',n)}\right)_{(b',b),(d',d)} \\
&= i^{-(m'-1)(n-1)-(n'-1)(m-1)} (-1)^{(a-b+c-d)(n'+m)/2+(a'-b'+c'-d')(n+m)/2} \\
&\cdot r'\left(a',m',n',c'\right)_{b',d'} \cdot r\left(a,m,n,c\right)_{b,d} ,
\end{aligned} \tag{2.8.2}$$

where the nonvanishing entries of r-matrices are

$$\begin{aligned}
r\left(a,1,n,c\right)_{a,c} = r\left(a,m,1,c\right)_{c,a} = 1, \\
r\left(l\pm 2,2,2,l\right)_{l\pm 1,l\pm 1} = x^{1/4}, \\
r\left(l,2,2,l\right)_{l\pm 1,l\pm 1} = \mp x^{-1/4\mp l/2}\frac{[1]}{[l]}, \\
r\left(l,2,2,l\right)_{l\pm 1,l\mp 1} = x^{-1/4}\frac{[l\pm 1]}{[l]},
\end{aligned} \tag{2.8.3}$$

and other entries of r-matrices are given by the recursive relation

$$\begin{aligned}
r\left(a,m+1,n,c\right)_{b,d} = \sum_{d_1\geq 1} r\left(a,2,n,d_1\right)_{a_1,d} \cdot r\left(a_1,m,n,c\right)_{b,d_1} , \\
r\left(a,m,n+1,c\right)_{b,d} = \sum_{d_1\geq 1} r\left(a,m,2,c_1\right)_{b,d_1} \cdot r\left(d_1,m,n,c\right)_{c_1,d} ,
\end{aligned} \tag{2.8.4}$$

for any choice of a_1 and c_1 compatible with the fusion rules. The r' matrices are given by the same formulas with the replacement $x \to y$, $[\quad] \to [\quad]'$.

Remark 2.8.3. *Using the above notation, we see that the central charge of the model* $L\left(\frac{25}{28},0\right)$ *corresponds to the parameter* $\alpha_-^2 = \frac{7}{8}$ *with* $p = 7$, $p' = 8$. *The pairs* $(1,1)$, $(3,1)$, $(5,1)$ *and* $(7,1)$ *correspond to the highest weights 0,* $\frac{3}{4}$, $\frac{13}{4}$ *and* $\frac{15}{2}$ *respectively. The central charge of the model* $L\left(\frac{21}{22},0\right)$ *corresponds to the parameter* $\alpha_-^2 = \frac{11}{12}$ *with* $p = 11$, $p' = 12$. *The pairs* $(1,1)$, *and* $(7,1)$ *correspond* *to the highest weights 0 and* 8 *respectively.*

Now we will prove two lemmas which will be used in the proof of the uniqueness of $5A$ and $3C$ algebras. First we consider braiding matrix for $L\left(\frac{25}{28},0\right)$-modules. Note that $P_2 = L\left(\frac{25}{28},\frac{15}{2}\right)$, $P_3 = L\left(\frac{25}{28},\frac{3}{4}\right)$,$P_4 = L\left(\frac{25}{28},\frac{13}{4}\right)$ are irreducible $L\left(\frac{25}{28},0\right)$-modules. For convenience, we will denote $\left(B^{P_c,P_d}_{P_a,P_b}\right)_{P_e,P_f}$ by $\left(B^{c,d}_{a,b}\right)_{e,f}$, for $a, b, c,\ d,$ $e,\ f \in \{2,3,4\}$.

Lemma 2.8.4. *$\left(B^{4,4}_{3,3}\right)_{4,4} \cdot \left(B^{4,4}_{3,3}\right)_{2,3} - \left(B^{4,4}_{3,3}\right)_{4,3} \cdot \left(B^{4,4}_{3,3}\right)_{2,4} \neq 0$, and $\left(B^{4,4}_{3,3}\right)_{3,2} \cdot \left(B^{4,4}_{3,3}\right)_{4,4} - \left(B^{4,4}_{3,3}\right)_{4,2} \cdot \left(B^{4,4}_{3,3}\right)_{3,4} \neq 0$.*

Proof. Using (2.8.2), (2.8.3) and Remark 2.8.3 , we have $\left(B^{4,4}_{3,3}\right)_{4,4} = r'\left(5,3,3,5\right)_{5,5}$. Let $[l]' = 2i\sin\left(\frac{7}{8}\pi l\right)$, $y = \exp\left(\frac{7}{4}\pi i\right)$. Using (2.8.3) and (2.8.4) we obtain:

$$\begin{aligned}
r'\left(5,3,3,5\right)_{5,5} &= r'\left(5,2,3,4\right)_{4,5} \cdot r'\left(4,2,3,5\right)_{5,4} + r'\left(5,2,3,6\right)_{4,5} \cdot r'\left(4,2,3,5\right)_{5,6}\,,\\
r'\left(5,2,3,4\right)_{4,5} &= r'\left(5,2,2,3\right)_{4,4} \cdot r'\left(4,2,2,4\right)_{3,5} = y^{\frac{1}{4}} \cdot y^{-\frac{1}{4}} \frac{[3]'}{[4]'} = \frac{[3]'}{[4]'},\\
r'\left(4,2,3,5\right)_{5,4} &= r'\left(4,2,2,6\right)_{5,5} \cdot r'\left(5,2,2,5\right)_{6,4} = y^{\frac{1}{4}} \cdot y^{-\frac{1}{4}} \frac{[6]'}{[5]'} = \frac{[6]'}{[5]'},\\
r'\left(5,2,3,6\right)_{4,5} &= r'\left(5,2,2,5\right)_{4,4} \cdot r'\left(4,2,2,6\right)_{5,5} + r'\left(5,2,2,5\right)_{4,6} \cdot r'\left(6,2,2,6\right)_{5,5}\\
&= y^{5/2} \frac{[1]'\,[6]' + [4]'\,[1]'}{[5]'\,[6]'},\\
r'\left(4,2,3,5\right)_{5,6} &= r'\left(4,2,2,4\right)_{5,5} \cdot r'\left(5,2,2,5\right)_{4,6} = -y^{-5/2} \frac{[1]'}{[5]'}.
\end{aligned}$$

Then we have:

$$\left(B^{4,4}_{3,3}\right)_{4,4} = r'\left(5,3,3,5\right)_{5,5} = \frac{[6]'\,[3]'}{[5]'\,[4]'} - \frac{[1]'^2\left([4]' + [6]'\right)}{[5]'^2\,[6]'}. \tag{2.8.5}$$

Similarly, we obtain:

$$
\begin{aligned}
\left(B_{3,3}^{4,4}\right)_{2,3} &= r'\left(5,3,3,5\right)_{7,3} = y^{-1}\frac{[6]'\,[7]'}{[4]'\,[5]'},\\
\left(B_{3,3}^{4,4}\right)_{4,3} &= r'\left(5,3,3,5\right)_{5,3} = y^{2}\frac{[1]'\,[6]'}{[4]'\,[5]'},\\
\left(B_{3,3}^{4,4}\right)_{2,4} &= r'\left(5,3,3,5\right)_{7,5} = -y^{-3}\left(\frac{[1]'\,[7]'\left([4]'+[6]'\right)}{[5]'^{2}\,[4]'}+\frac{[1]'\,[7]'\left([5]'+[7]'\right)}{[6]'^{2}\,[5]'}\right).
\end{aligned}
\tag{2.8.6}
$$

From (2.8.5), (2.8.6) and a direct computation, we obtain:

$$
\left(B_{3,3}^{4,4}\right)_{4,4}\cdot\left(B_{3,3}^{4,4}\right)_{2,3}-\left(B_{3,3}^{4,4}\right)_{4,3}\cdot\left(B_{3,3}^{4,4}\right)_{2,4}=\frac{\sqrt{2}-1}{2}(1+i)\neq 0. \tag{2.8.7}
$$

By a similar process, we obtain:

$$
\begin{aligned}
\left(B_{3,3}^{4,4}\right)_{3,2}\cdot\left(B_{3,3}^{4,4}\right)_{4,4} &= y^{-1}\frac{[4]'\,[3]'}{[6]'\,[5]'}\left(\frac{[6]'\,[3]'}{[5]'\,[4]'}-\frac{[1]'^{2}\left([4]'+[6]'\right)}{[5]'^{2}\,[6]'}\right)\\
&= y^{-1}\left(\sqrt{2}-1\right),
\end{aligned}
\tag{2.8.8}
$$

$$
\begin{aligned}
\left(B_{3,3}^{4,4}\right)_{4,2}\cdot\left(B_{3,3}^{4,4}\right)_{3,4} &= -y^{-1}\frac{[1]'\,[4]'}{[5]'\,[6]'}\left(\frac{[1]'\,[3]'\left([3]'+[5]'\right)}{[5]'\,[4]'^{2}}+\frac{[1]'\,[3]'\left([4]'+[6]'\right)}{[5]'^{2}\,[6]'}\right)\\
&= -y^{-1}.
\end{aligned}
\tag{2.8.9}
$$

From (2.8.8) and (2.8.9), we have

$$
\left(B_{3,3}^{4,4}\right)_{4,4}\cdot\left(B_{3,3}^{4,4}\right)_{2,3}-\left(B_{3,3}^{4,4}\right)_{4,3}\cdot\left(B_{3,3}^{4,4}\right)_{2,4}=\sqrt{2}y^{-1}=1+i\neq 0. \tag{2.8.10}
$$

□

Next we consider the braiding matrix for $L\left(\frac{21}{22},0\right)$-modules. Note that $U^1 = L\left(\frac{21}{22},0\right)$, $U^2 = L\left(\frac{21}{22},8\right)$ are irreducible $L\left(\frac{21}{22},0\right)$-modules. For convenience, we will denote $\left(B_{U^a,U^b}^{U^c,U^d}\right)_{U^e,U^f}$ by $\left(B_{a,b}^{c,d}\right)_{e,f}$, for $a, b, c,\ d,\ e,\ f \in \{1,2\}$. Now we are ready to give the following lemma:

Lemma 2.8.5. $\left(B_{2,2}^{2,2}\right)_{2,1} \neq 0$.

Proof. Let $[l]' = 2i\sin\left(\frac{11}{12}\pi l\right)$, $y = \exp\left(\frac{11}{6}\pi i\right)$. By a careful computation which is similar to Lemma 2.8.4, we obtain

$$\left(B_{2,2}^{2,2}\right)_{2,1} = y^6 \cdot \frac{[1]'^3 \cdot ([2]' + [4]') \cdot ([2]'[3]' + [2]'[5]' + [5]'[4]')}{[3]'^2 \cdot [4]' \cdot [5]'^2 \cdot [6]'} \neq 0.$$

□

Chapter 3

Uniqueness of VOA structure of the $5A$-algebra $\mathcal{U}_{5A}$

As in [24], we denote the irreducible module $L\left(\frac{1}{2},h_1\right)\otimes L\left(\frac{25}{28},h_2\right)\otimes L\left(\frac{25}{28},h_3\right)$ by $[h_1,h_2,h_3]$ for simplicity. Let

$$V^1=[0,0,0],\ V^2=[0,\frac{15}{2},\frac{15}{2}],\ V^3=[0,\frac{3}{4},\frac{13}{4}],\ V^4=[0,\frac{13}{4},\frac{13}{4}],$$
$$V^5=[\frac{1}{2},0,\frac{15}{2}],\ V^6=[\frac{1}{2},\frac{15}{2},0],\ V^7=[\frac{1}{2},\frac{3}{4},\frac{3}{4}],\ V^8=[\frac{1}{2},\frac{13}{4},\frac{13}{4}],$$
$$V^9=[\frac{1}{16},\frac{5}{32},\frac{57}{32}],\ V^{10}=[\frac{1}{16},\frac{57}{32},\frac{5}{32}],\ V^{11}=[\frac{1}{16},\frac{57}{32},\frac{165}{32}],\ V^{12}=[\frac{1}{16},\frac{165}{32},\frac{57}{32}].$$

Then the $5A$-algebra

$$\mathcal{U}_{5A}\cong V^1\oplus V^2\oplus ...\oplus V^{11}\oplus V^{12}$$

as V^1-modules. Since V^1 is rational and C_2-cofinite, by [1] and [22], it is easy to see that $\mathcal{U}_{5A}$ is a simple, rational and C_2-cofinite VOA.

Lemma 3.0.1. *Let $W=V^1+V^2+V^3+V^4$, then W is a subVOA of $\mathcal{U}_{5A}$ and the VOA structure of $5A$-algebra $\mathcal{U}_{5A}$ is uniquely determined by W.*

Proof. By the fusion rules of $L\left(\frac{1}{2},0\right)$ modules and $L\left(\frac{25}{28},0\right)$ modules, we see that $W = V^1 + V^2 + V^3 + V^4$ is a subVOA of $V^1 + V^2 + ... + V^7 + V^8$ and $V^1+V^2+...+V^7+V^8$ is a subVOA of $\mathcal{U}_{5A}$. By [27], we can define an involution τ of $\mathcal{U}_{5A}$ by letting it act by -1 on $V^9+V^{10}+V^{11}+V^{12}$ and by 1 on $V^1+V^2+...+V^7+V^8$. Then it follows from [6] that $V^9+V^{10}+V^{11}+V^{12}$ is a simple current module over $V^1+V^2+...+V^7+V^8$. Similarly, one can define an involution σ of $V^1+V^2+...+V^7+V^8$ which acts by 1 on $W = V^1+V^2+V^3+V^4$ and by -1 on $V^5+V^6+V^7+V^8$. Then again by [6] we see that $M = V^5+V^6+V^7+V^8$ is a simple current module over W.

Claim 1. *The W-module structure of M is unique. That is, if M^1 is also a W-module such that $M^1 \cong M$ as V^1-modules, then $M^1 \cong M$ as W-modules.*

We need to use some category theory to prove that claim. Recall some basics on category theory from [23], [14]. An object A in a braided fusion category $\mathcal{C}$ is called regular commutative algebra if there are morphisms $\mu : A \boxtimes A \to A$ and $\eta : \mathbf{1}_{\mathcal{C}} \to A$ such that $\mu \circ (\mu \boxtimes \mathrm{id}_A) \circ \alpha_{A,A,A} = \mu \circ (\mathrm{id}_A \boxtimes \mu)$, $\mu \circ (\eta \boxtimes \mathrm{id}_A) \circ l_A^{-1} = \mathrm{id}_A = \mu \circ (\mathrm{id}_A \boxtimes \eta) \circ r_A^{-1}$, $\mu = \mu \circ c_{A,A}$ and $\dim \hom(\mathbf{1}_{\mathcal{C}}, A) = 1$ where $\alpha_{A,A,A} : A \boxtimes (A \boxtimes A) \to (A \boxtimes A) \boxtimes A$ is the associative isomorphism, $l_A : \mathbf{1}_{\mathcal{C}} \boxtimes A \to A$ is the left unit isomorphism, $r_A : A \boxtimes \mathbf{1}_{\mathcal{C}} \to A$ is the right unit isomorphism and $c_{A,A} : A \boxtimes A \to A \boxtimes A$ is the braiding isomorphism. A left A-module N is an object in $\mathcal{C}$ with a morphism $\mu_N : A \boxtimes N \to N$ such that $\mu_N \circ (\mu \boxtimes \mathrm{id}_N) \circ \alpha_{A,A,N} = \mu_N \circ (\mathrm{id}_A \boxtimes \mu_N)$. We denote the left A-module category by $\mathcal{C}_A$. Let $N_1, N_2 \in \mathcal{C}_A$. Define $N_1 \boxtimes_A N_2 = N_1 \boxtimes N_2 / \mathrm{Im}(\mu_1 - \mu_2)$ where $\mu_1, \mu_2 : A \boxtimes N_1 \boxtimes N_2 \to N_1 \boxtimes N_2$ are defined by $\mu_1 = \mu_{N_1} \boxtimes \mathrm{id}_{N_2}$, $\mu_2 = (\mathrm{id}_A \boxtimes \mu_{N_2}) \circ c_{A,N_1} \boxtimes \mathrm{id}_{N_2}$. Then $\mathcal{C}_A$ is a fusion category with tensor product $N_1 \boxtimes_A N_2$. An A-module N is called local if $\mu_N \circ c_{N,A} \circ c_{A,N} = \mu_N$. We denote the local A-module categoy by $\mathcal{C}_A^0$. Then $\mathcal{C}_A^0$ is a braided fusion category. Moreover, if $\mathcal{C}$ is modular tensor category, so is $\mathcal{C}_A^0$

[23]. For any $N \in \mathcal{C}_A$ and $X \in \mathcal{C}$, $N \boxtimes X \in \mathcal{C}_A$. From [21] we know that the module category $\mathcal{C}_V$ of any rational, C_2-cofinite, selfdual vertex operator algebra V is a modular tensor category with tensor product $\boxtimes$. Thus both $\mathcal{C}_{V^1}$ and $\mathcal{C}_W$ are modular tensor categories. Moreover, W is a regular commutative algebra in $\mathcal{C}_{V^1}$ [22]. From the discussion above, $(\mathcal{C}_{V^1})_W$ is a fusion category and $(\mathcal{C}_{V^1})_W^0$ is exactly $\mathcal{C}_W$ [22].

We now can prove $M \cong M^1 \cong W \boxtimes_{V^1} V^5$ as W-modules. Note that

$$\mathrm{qdim}_{V^1}(W \boxtimes_{V^1} V^5) = \mathrm{qdim}_{V^1}(W)\mathrm{qdim}_{V^1}(V^5) = \mathrm{qdim}_{V^1}(W)$$

by Proposition 2.6.5 as V^5 is a simple current. Thus

$$\mathrm{qdim}_W(W \boxtimes_{V^1} V^5) = \frac{\mathrm{qdim}_{V^1}(W \boxtimes_{V^1} V^5)}{\mathrm{qdim}_{V^1}(W)} = 1.$$

From Theorem 1.6 of [23] we have isomorphisms

$$\mathrm{Hom}_{(\mathcal{C}_{V^1})_W}(W \boxtimes_{V^1} V^5, M) \cong \mathrm{Hom}_{V^1}(V^5, M) \cong \mathrm{Hom}_{(\mathcal{C}_{V^1})_W}(W \boxtimes_{V^1} V^5, M^1).$$

Since $(\mathcal{C}_{V^1})_W$ is a semisimple category, M and M^1 are irreducible W-submodules of $W \boxtimes_{V^1} V^5$. Using the fact that $\mathrm{qdim}_W(M) = \mathrm{qdim}_W(M^1) = \mathrm{qdim}_W(W \boxtimes_{V^1} V^5) = 1$ we immediately conclude that $M \cong M^1 \cong W \boxtimes_{V^1} V^5$ as W-modules.

Claim 2. *The vertex operator algebra structure on $W + M$ is unique.*

From Claim 1 and the discussion above, $W + M$ is a simple current extension of W. The Claim follows from Proposition 2.5.4.

Claim 3. *The W-module structure of $N = V^9 + V^{10} + V^{11} + V^{12}$ is unique. That is, if N^1 is also a W-module such that $N^1 \cong N$ as V^1-modules, then $N^1 \cong N$ as W-modules.*

Consider $W \boxtimes_{V^1} V^9 \in (\mathcal{C}_{V^1})_W$. By Theorem 1.6 of [23] again we have isomorphisms

$$\begin{aligned}\mathrm{Hom}_{(\mathcal{C}_{V^1})_W}(W \boxtimes_{V^1} V^9, W \boxtimes_{V^1} V^9) &\cong \mathrm{Hom}_{V^1}(V^9, W \boxtimes_{V^1} V^9) \\ &\cong \mathrm{Hom}_{V^1}(V^9 \boxtimes_{V^1} (V^9)', W)\end{aligned}$$

where $(V^9)'$ is the restricted dual of V^9 and is isomorphic V^9. It is easy to see that the projection $V^9 \boxtimes_{V^1} V^9$ to $W + M$ is isomorphic to $V^1 + V^3 + V^5 + V^7$. Thus $\mathrm{Hom}_{V^1}(V^9 \boxtimes_{V^1} (V^9)', W)$ is 2-dimensional and $W \boxtimes_{V^1} V^9$ is a direct sum of two inequivalent irreducible W-modules in $(\mathcal{C}_{V^1})_W$. As before one can compute that $\mathrm{qdim}_W(W \boxtimes_{V^1} V^9) = \mathrm{qdim}_{V^1}(V^9) = 4 + 2\sqrt{2}$. Also, both N and N^1 are submodules of $\mathrm{qdim}_W(W \boxtimes_{V^1} V^9)$. If N and N^1 are inequivalent W-modules then $\mathrm{qdim}_W(W \boxtimes_{V^1} V^9) = N \oplus N^1$. But this is a contradiction as $\mathrm{qdim}_W(N + N^1) = 2\mathrm{qdim}_W(N) = 4$.

Claim 4. *There are two inequivalent $W + M$-module structures on N.*

Define an automorphism σ of $W + M$ such that $\sigma = 1$ on W and -1 on M. Following [10] $N \circ \sigma$ is also a $W + M$-module such that $Y_{N\circ\sigma}(v, z) = Y_N(\sigma v.z)$ for $v \in W + M$ where $N \circ \sigma = N$ as vector space. Since N is an irreducible module, $N \circ \sigma$ and N are inequivalent $W + M$-modules [12]. Note that $Y_{N\circ\sigma}(v, z) = Y_N(v, z)$ if $v \in W$ and $Y_{N\circ\sigma}(v, z) = -Y_N(v, z)$ if $v \in M$.

Now let $N^1 = (N^1, Y_1)$ be any irreducible $W + M$-module such that $N^1 \cong N$ as W-modules. Then $Y_N(v, z) = Y_1(v, z)$ for $v \in W$ by Claim 3. Since M is a simple current, the space of intertwining operators $I_W \begin{pmatrix} N \\ M\ N \end{pmatrix}$ is one dimensional and spanned by Y_N. Then there is a nonzero constant λ such that $Y_1(u, z) = \lambda Y(u, z)$

for all $u \in M$. Using the associativity

$$
\begin{aligned}
&Y_N(Y(u,z_0)v,z_2)\\
&= \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_1(u,z_1)Y_1(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_1(v,z_2)Y_1(u,z_1)\right)\\
&= \lambda^2 \cdot\\
&\quad \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_N(u,z_1)Y_N(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_N(v,z_2)Y_N(u,z_1)\right)\\
&= \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_N(u,z_1)Y_N(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_N(v,z_2)Y_N(u,z_1)\right)
\end{aligned}
$$

for $u, v \in M$, we see that $\lambda = \pm 1$. So N^1 is either isomorphic to N or $N \circ \sigma$. We denote these two module structures by N^+, N^-.

Claim 5. *VOA structure of $5A$-algebra $\mathcal{U}_{5A}$ is uniquely determined by W.*

Let V be a vertex operator algebra such that $V \cong \mathcal{U}_{5A}$ as V^1-modules. From the discussion above we see that $V \cong W + M + N^+$ or $W + M + N^-$ as $W + M$-modules. Note that both $W + M + N^+$ and $W + M + N^-$ are simple current extensions of $W + M$. By Proposition 2.5.4, the vertex operator algebra structures on both $W + M + N^+$ and $W + M + N^-$ are unique.

Finally we show that $W + M + N^+$ and $W + M + N^-$ are isomorphic vertex operator algebras. Note that as vector space $W + M + N^+ = W + M + N^- = W + M + N$. We now extend the action of σ from $W + M$ to $W + M + N^+$ so that $\sigma = 1$ on N. Then σ is a linear isomorphism of $W + M + N^+$ satisfying $\sigma(\mathbf{1}) = \mathbf{1}$ and $\sigma(\omega) = \omega$. Let Y defines the vertex operator structure on $W + M + N^+$. Then $Y^\sigma(v,z) = \sigma Y(\sigma v, z)\sigma$ defines a new vertex operator algebra structure on $W + M + N^+$ and $(W + M + N^+, Y, \mathbf{1}, \omega) \cong (W + M + N^+, Y^\sigma, \mathbf{1}, \omega)$ by Lemma 2.1.8. It is easy to verify that $Y^\sigma(u,z) = Y(u,z)$ for $u \in W$ and $Y^\sigma(u,z) = Y(u,z)$ on $W + M$ and $Y^\sigma(u,z) = -Y(u,z)$ on N for $u \in M$. Thus,

$(N, Y^{\sigma}) \cong (N \circ \sigma, Y_{N \circ \sigma})$ where $Y_N(a, z) = Y(a, z)$ on N for $a \in W + M$. Using the uniqueness of the vertex operator algebra structure on $W + M + N^-$ we conclude that $(W + M + N^+, Y^{\sigma}, \mathbf{1}, \omega) \cong W + M + N^-$. As a result, $W + M + N^+$ and $W + M + N^-$ are isomorphic vertex operator algebras. The proof is complete. $\square$

Let $U^1 = L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{25}{28}, 0\right)$, $U^2 = L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{25}{28}, \frac{15}{2}\right)$, $U^3 = L\left(\frac{25}{28}, \frac{3}{4}\right) \otimes L\left(\frac{25}{28}, \frac{13}{4}\right)$, $U^4 = L\left(\frac{25}{28}, \frac{13}{4}\right) \otimes L\left(\frac{25}{28}, \frac{3}{4}\right)$ and $U = U^1 + U^2 + U^3 + U^4$. Then $W = L\left(\frac{1}{2}, 0\right) \otimes U$ and U admits a VOA structure. Next we will prove the vertex operator algebra structure on U over $\mathbb{C}$ is unique.

Remark 3.0.2. *By [1] and [15], U is rational and C_2-cofinite. Since $U_1 = 0$ and* $\dim U_0 = 1$ *by Theorem 2.3.3, there is a unique nondegenerate invariant bilinear form on U and thus $U' \cong U$. Without loss of generality, we can identify U with U'.*

Set

$$P_1 = Q_1 = L\left(\frac{25}{28}, 0\right), \ P_2 = Q_2 = L\left(\frac{25}{28}, \frac{15}{2}\right),$$
$$P_3 = Q_4 = L\left(\frac{25}{28}, \frac{3}{4}\right), \ P_4 = Q_3 = L\left(\frac{25}{28}, \frac{13}{4}\right).$$

Then $U^i = P_i \otimes Q_i$ for $i = 1, 2, 3, 4$, and

$$U \cong P_1 \otimes Q_1 \oplus P_2 \otimes Q_2 \oplus P_3 \otimes Q_3 \oplus P_4 \otimes Q_4 = U^1 \oplus U^2 \oplus U^3 \oplus U^4.$$

For convenience, we list fusion rules $I_{P_1}\binom{P_c}{P_a \ P_b}$ and $I_{U^1}\binom{U^c}{U^a \ U^b}$ with $a, b, c \in \{1, 2, 3, 4\}$ in the following table.

P_1	P_2	P_3	P_4
P_2	P_1	P_4	P_3
P_3	P_4	$P_1+P_3+P_4$	$P_2+P_3+P_4$
P_4	P_3	$P_2+P_3+P_4$	$P_1+P_3+P_4$

U^1	U^2	U^3	U^4
U^2	U^1	U^4	U^3
U^3	U^4	$U^1+U^3+U^4$	$U^2+U^3+U^4$
U^4	U^3	$U^2+U^3+U^4$	$U^1+U^3+U^4$

Let $I_{Q_1}\binom{Q_c}{Q_a\ Q_b} = \mathbb{C}\overline{\mathcal{Y}}^c_{a,b}$ and $I_{P_1}\binom{P_c}{P_a\ P_b} = \mathbb{C}\mathcal{Y}^c_{a,b}$. Then $I_{U^1}\binom{U^c}{U^a\ U^b} = \mathbb{C}\mathcal{I}^c_{a,b}$ where $\mathcal{I}^c_{a,b} = \mathcal{Y}^c_{a,b} \otimes \overline{\mathcal{Y}}^c_{a,b}$. Let (U, Y) be a vertex operator algebra structure on U with

$$Y(u,z) = \sum_{a,b,c\in\{1,2,3,4\}} \lambda^c_{a,b} \cdot \mathcal{I}^c_{a,b}(u^a, z)\, u^b$$

where $\lambda^c_{a,b} \in \mathbb{C}$.

Lemma 3.0.3. *$\lambda^c_{a,b} \neq 0$ if $N^c_{a,b} = \dim I_{U^1}\binom{U^c}{U^a\ U^b} \neq 0$.*

Proof. The proof consists of several claims.

Claim 1. $\lambda^k_{k,1} \neq 0,$ for $k = 2, 3, 4$.

For any $u^k \in U^k$, $k = 1, 2, 3, 4$, using skew symmetry of $Y(\cdot, z)$, we have

$$\begin{aligned} Y(u^k, z)u^1 &= e^{zL(-1)}Y\left(u^1, -z\right)u^k = \lambda^k_{1,k} \cdot e^{zL(-1)}\mathcal{I}^k_{1,k}\left(u^1, -z\right)u^k \\ &= \lambda^k_{k,1} \cdot \mathcal{I}^k_{k,1}\left(u^k, z\right)u^1. \end{aligned}$$

Since U^k is an irreducible U^1-module, we have $\lambda^k_{1,k} \neq 0$, for $k = 1, 2, 3, 4$. So $\lambda^k_{k,1} \neq 0$, for $k = 2, 3, 4$.

Claim 2. $\lambda_{k,k}^{1} \neq 0$, for $k = 2, 3, 4$.

By Remark 3.0.2, U has a unique nondegenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ with $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. For $u^k, v^k \in U^k$, $k = 1, 2, 3, 4$, we have

$$\left\langle Y\left(u^k, z) v^k\right), u^1 \right\rangle = \left\langle v^k, Y\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)} u^k, z^{-1}\right) u^1 \right\rangle.$$

That is,

$$\left\langle \lambda_{k,k}^{1} \cdot \mathcal{I}_{k,k}^{1}\left(u^k, z\right) v^k, u^1 \right\rangle = \left\langle v^k, \lambda_{k,1}^{k} \cdot \mathcal{I}_{k,1}^{k}\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)} u^k, z^{-1}\right) u^1 \right\rangle.$$

By Claim 1, we see that $\lambda_{k,1}^{k} \neq 0$,. Hence $\lambda_{k,k}^{1} \neq 0$, for $k = 2, 3, 4$.

Claim 3. $\lambda_{3,2}^{4}$, $\lambda_{2,3}^{4}$, $\lambda_{2,4}^{3}$, $\lambda_{4,2}^{3}$, $\lambda_{3,4}^{2}$, $\lambda_{4,3}^{2}$ are all nonzero .

Let $u^2 \in U^2$, $u^3 \in U^3$, $u^4 \in U^4$. It follows from the skew symmetry of Y that

$$\left\langle Y\left(u^2, z\right) u^3, u^4 \right\rangle = \left\langle e^{zL(-1)} Y\left(u^3, -z\right) u^2, u^4 \right\rangle.$$

That is,

$$\left\langle \lambda_{2,3}^{4} \cdot \mathcal{I}_{2,3}^{4}\left(u^2, z\right) u^3, u^4 \right\rangle = \left\langle \lambda_{3,2}^{4} \cdot e^{zL(-1)} \mathcal{I}_{3,2}^{4}\left(u^3, -z\right) u^2, u^4 \right\rangle. \tag{3.0.1}$$

So $\lambda_{2,3}^{4}$ and $\lambda_{3,2}^{4}$ are both zero or nonzero. Similarly, $\lambda_{2,4}^{3}$ and $\lambda_{4,2}^{3}$ are both zero or nonzero, $\lambda_{4,3}^{2}$ and $\lambda_{3,4}^{2}$ are both zero or nonzero. For any $u^1 \in U^1$, $u^2, v^2 \in U^2$, $u^3, v^3 \in U^3$ and $u^4 \in U^4$, commutativity of Y implies

$$\iota_{12}^{-1}\left\langle u^1, Y\left(u^2, z_1\right) Y\left(u^3, z_2\right) u^4 \right\rangle = \iota_{21}^{-1}\left\langle u^1, Y\left(u^3, z_1\right) Y\left(u^2, z_1\right) u^4 \right\rangle,$$

$$\iota_{12}^{-1}\left\langle u^1, Y\left(u^4, z_1\right) Y\left(u^3, z_2\right) u^2\right\rangle = \iota_{21}^{-1}\left\langle u^1, Y\left(u^3, z_1\right) Y\left(u^4, z_1\right) u^2\right\rangle,$$

$$\iota_{12}^{-1}\left\langle v^2, Y\left(u^2, z_1\right) Y\left(u^3, z_2\right) v^3\right\rangle = \iota_{21}^{-1}\left\langle v^2, Y\left(u^3, z_1\right) Y\left(u^2, z_1\right) v^3\right\rangle.$$

Equivalently,

$$\begin{aligned}&\iota_{12}^{-1}\left\langle u^1, \lambda_{2,2}^1\lambda_{3,4}^2\cdot\mathcal{I}_{2,2}^1\left(u^2, z_1\right)\mathcal{I}_{3,4}^2\left(u^3, z_2\right) u^4\right\rangle\\ &\quad= \iota_{21}^{-1}\left\langle u^1, \lambda_{3,3}^1\lambda_{2,4}^3\cdot\mathcal{I}_{3,3}^1\left(u^3, z_2\right)\mathcal{I}_{2,4}^3\left(u^2, z_1\right) u^4\right\rangle,\end{aligned} \tag{3.0.2}$$

$$\begin{aligned}&\iota_{12}^{-1}\left\langle u^1, \lambda_{4,4}^1\lambda_{3,2}^4\cdot\mathcal{I}_{4,4}^1\left(u^4, z_1\right)\mathcal{I}_{3,2}^4\left(u^3, z_2\right) u^2\right\rangle\\ &\quad= \iota_{21}^{-1}\left\langle u^1, \lambda_{3,3}^1\lambda_{4,2}^3\cdot\mathcal{I}_{3,3}^1\left(u^3, z_2\right)\mathcal{I}_{4,2}^3\left(u^4, z_1\right) u^2\right\rangle,\end{aligned} \tag{3.0.3}$$

$$\begin{aligned}&\iota_{12}^{-1}\left\langle v^2, \lambda_{2,1}^2\lambda_{3,3}^1\cdot\mathcal{I}_{2,1}^2\left(u^2, z_1\right)\mathcal{I}_{3,3}^1\left(u^3, z_2\right) v^3\right\rangle\\ &\quad= \iota_{21}^{-1}\left\langle v^2, \lambda_{3,4}^2\lambda_{2,3}^4\cdot\mathcal{I}_{3,4}^2\left(u^3, z_2\right)\mathcal{I}_{2,3}^4\left(u^2, z_1\right) v^3\right\rangle.\end{aligned} \tag{3.0.4}$$

Claim 3 now follows from (3.0.1), (3.0.2), (3.0.3), (3.0.4) and Claims 1 and 2.

Claim 4. $\lambda_{3,4}^3$, $\lambda_{4,3}^3$, $\lambda_{3,3}^4$, $\lambda_{4,4}^4$, $\lambda_{3,4}^3$, $\lambda_{3,4}^4$, $\lambda_{4,3}^4$, $\lambda_{3,3}^3$, $\lambda_{4,4}^3$ are all nonzero .

First we will show that $\lambda_{4,3}^3$, $\lambda_{3,3}^4$, $\lambda_{4,4}^4$, $\lambda_{3,4}^3$ are all zero or all nonzreo. Let

$u^1 \in U^1, u^2 \in U^2,\ v^3, u^3 \in U^3,\ v^4, u^4 \in U^4$. By skew symmetry of Y, we have

$$\left\langle Y\left(u^3, z\right) u^4, v^3\right\rangle = \left\langle e^{zL(-1)} Y\left(u^4, -z\right) u^3, v^3\right\rangle .$$

Equivalently,

$$\left\langle \lambda_{3,4}^3 \cdot \mathcal{I}_{3,4}^3\left(u^3, z\right) u^4, v^3\right\rangle = \left\langle \lambda_{4,3}^3 \cdot e^{zL(-1)} \mathcal{I}_{4,3}^3\left(u^4, -z\right) u^3, v^3\right\rangle . \tag{3.0.5}$$

So $\lambda_{3,4}^3$ and $\lambda_{4,3}^3$ are both zero or nonzero. By commutativity of Y, we have

$$\iota_{12}^{-1}\left\langle u^1, Y\left(u^3, z_1\right) Y\left(u^4, z_2\right) v^3\right\rangle = \iota_{21}^{-1}\left\langle u^1, Y\left(u^4, z_1\right) Y\left(u^3, z_1\right) v^3\right\rangle ,$$

$$\iota_{12}^{-1}\left\langle u^2, Y\left(u^3, z_1\right) Y\left(u^4, z_2\right) v^4\right\rangle = \iota_{21}^{-1}\left\langle u^2, Y\left(u^4, z_1\right) Y\left(u^3, z_1\right) v^4\right\rangle .$$

That is,

$$\begin{aligned} &\iota_{12}^{-1}\left\langle u^1, \lambda_{3,3}^1 \lambda_{4,3}^3 \cdot \mathcal{I}_{3,3}^1\left(u^3, z_1\right) \mathcal{I}_{4,3}^3\left(u^4, z_2\right) v^3\right\rangle \\ &\quad = \iota_{21}^{-1}\left\langle u^1, \lambda_{4,4}^1 \lambda_{3,3}^4 \cdot \mathcal{I}_{4,4}^1\left(u^4, z_2\right) \mathcal{I}_{3,3}^4\left(u^3, z_1\right) v^3\right\rangle , \end{aligned} \tag{3.0.6}$$

$$\begin{aligned} &\iota_{12}^{-1}\left\langle u^2, \lambda_{3,4}^2 \lambda_{4,4}^4 \cdot \mathcal{I}_{3,4}^2\left(u^3, z_1\right) \mathcal{I}_{4,4}^4\left(u^4, z_2\right) v^4\right\rangle \\ &\quad = \iota_{21}^{-1}\left\langle u^2, \lambda_{4,3}^2 \lambda_{3,4}^3 \cdot \mathcal{I}_{4,3}^2\left(u^4, z_2\right) \mathcal{I}_{3,4}^3\left(u^3, z_1\right) v^4\right\rangle . \end{aligned} \tag{3.0.7}$$

Combining with Claims 1-3 and (3.0.5), (3.0.6), (3.0.7) we have $\lambda_{4,3}^3$ and $\lambda_{3,3}^4$ are both zero or nonzero, $\lambda_{4,4}^4$ and $\lambda_{3,4}^3$ are both zero or nonzero. In total, $\lambda_{4,3}^3$, $\lambda_{3,3}^4, \lambda_{4,4}^4, \lambda_{3,4}^3$ are all zero or all nonzero. Similarly, $\lambda_{3,4}^4$, $\lambda_{4,3}^4, \lambda_{3,3}^3, \lambda_{4,4}^3$ are all zero or all nonzero.

Next we use braiding matrices to establish this Claim 4. Let $p_1^3 \otimes p_2^3, t_1^3 \otimes t_2^3 \in U^3$, $u_1^4 \otimes u_2^4, v_1^4 \otimes v_2^4 \in U^4$. Let $B_{3,3}^{4,4}$, $\tilde{B}_{3,3}^{4,4}$ be as defined in (2.8.1). Then by Proposition 2.8.1 we have

$$\begin{aligned}
&\iota_{12}^{-1} \left\langle t_1^3 \otimes t_2^3, Y\left(v_1^4 \otimes v_2^4, z_1\right) Y\left(u_1^4 \otimes u_2^4, z_2\right) p_1^3 \otimes p_2^3 \right\rangle \\
&= \iota_{12}^{-1} \langle t_1^3 \otimes t_2^3, \lambda_{4,2}^3 \lambda_{4,3}^2 \cdot \mathcal{Y}_{4,2}^3 \otimes \overline{\mathcal{Y}}_{4,2}^3 \left(v_1^4 \otimes v_2^4, z_1\right) \cdot \mathcal{Y}_{4,3}^2 \otimes \overline{\mathcal{Y}}_{4,3}^2 \left(u_1^4 \otimes u_2^4, z_2\right) \cdot p_1^3 \otimes p_2^3 \\
&\quad + \lambda_{4,3}^3 \lambda_{4,3}^3 \cdot \mathcal{Y}_{4,3}^3 \otimes \overline{\mathcal{Y}}_{4,3}^3 \left(v_1^4 \otimes v_2^4, z_1\right) \cdot \mathcal{Y}_{4,3}^3 \otimes \overline{\mathcal{Y}}_{4,3}^3 \left(u_1^4 \otimes u_2^4, z_2\right) \cdot p_1^3 \otimes p_2^3 \rangle \\
&\quad + \lambda_{4,4}^3 \lambda_{4,3}^4 \cdot \mathcal{Y}_{4,4}^3 \otimes \overline{\mathcal{Y}}_{4,4}^3 \left(v_1^4 \otimes v_2^4, z_1\right) \cdot \mathcal{Y}_{4,3}^4 \otimes \overline{\mathcal{Y}}_{4,3}^4 \left(u_1^4 \otimes u_2^4, z_2\right) \cdot p_1^3 \otimes p_2^3 \rangle \\
&= \iota_{12}^{-1} \langle t_1^3 \otimes t_2^3, \lambda_{4,2}^3 \lambda_{4,3}^2 \cdot \mathcal{Y}_{4,2}^3 \left(v_1^4, z_1\right) \mathcal{Y}_{4,3}^2 \left(u_1^4, z_2\right) p_1^3 \otimes \overline{\mathcal{Y}}_{4,2}^3 \left(v_2^4, z_1\right) \overline{\mathcal{Y}}_{4,3}^2 \left(u_2^4, z_2\right) p_2^3 \\
&\quad + \lambda_{4,3}^3 \lambda_{4,3}^3 \cdot \mathcal{Y}_{4,3}^3 \left(v_1^4, z_1\right) \mathcal{Y}_{4,3}^3 \left(u_1^4, z_2\right) p_1^3 \otimes \overline{\mathcal{Y}}_{4,3}^3 \left(v_2^4, z_1\right) \overline{\mathcal{Y}}_{4,3}^3 \left(u_2^4, z_2\right) p_2^3 \\
&\quad + \lambda_{4,4}^3 \lambda_{4,3}^4 \cdot \mathcal{Y}_{4,4}^3 \left(v_1^4, z_1\right) \mathcal{Y}_{4,3}^4 \left(u_1^4, z_2\right) p_1^3 \otimes \overline{\mathcal{Y}}_{4,4}^3 \left(v_2^4, z_1\right) \overline{\mathcal{Y}}_{4,3}^4 \left(u_2^4, z_2\right) p_2^3 \rangle \\
&= \iota_{21}^{-1} \langle t_1^3 \otimes t_2^3, \lambda_{4,2}^3 \lambda_{4,3}^2 A + \lambda_{4,3}^3 \lambda_{4,3}^3 B + \lambda_{4,4}^3 \lambda_{4,3}^4 C \rangle
\end{aligned} \tag{3.0.8}$$

where

$$\begin{aligned}
A = &\sum_{i=2,3,4} \left(B_{3,3}^{4,4}\right)_{2,i} \mathcal{Y}_{4,i}^3 \left(u_1^4, z_2\right) \mathcal{Y}_{4,3}^i \left(v_1^4, z_1\right) p_1^3 \\
&\otimes \sum_{j=2,3,4} \left(\tilde{B}_{3,3}^{4,4}\right)_{2,j} \overline{\mathcal{Y}}_{4,j}^3 \left(u_2^4, z_2\right) \overline{\mathcal{Y}}_{4,3}^j \left(v_2^4, z_1\right) p_2^3,
\end{aligned}$$

$$\begin{aligned}
B = &\sum_{i=2,3,4} \left(B_{3,3}^{4,4}\right)_{3,i} \mathcal{Y}_{4,i}^3 \left(u_1^4, z_2\right) \mathcal{Y}_{4,3}^i \left(v_1^4, z_1\right) p_1^3 \\
&\otimes \sum_{j=2,3,4} \left(\tilde{B}_{3,3}^{4,4}\right)_{3,j} \overline{\mathcal{Y}}_{4,j}^3 \left(u_2^4, z_2\right) \overline{\mathcal{Y}}_{4,3}^j \left(v_2^4, z_1\right) p_2^3,
\end{aligned}$$

$$
\begin{aligned}
C = &\sum_{i=2,3,4} \left(B^{4,4}_{3,3}\right)_{4,i} \mathcal{Y}^{3}_{4,i}\left(u^4_1, z_2\right) \mathcal{Y}^{i}_{4,3}\left(v^4_1, z_1\right) p^3_1 \\
&\otimes \sum_{j=2,3,4} \left(\tilde{B}^{4,4}_{3,3}\right)_{4,j} \overline{\mathcal{Y}}^{3}_{4,j}\left(u^4_2, z_2\right) \overline{\mathcal{Y}}^{j}_{4,3}\left(v^4_2, z_1\right) p^3_2.
\end{aligned}
$$

In the meantime, we have

$$
\begin{aligned}
&\iota_{21}^{-1}\left\langle t^3_1 \otimes t^3_2, Y\left(u^4_1 \otimes u^4_2, z_2\right) Y\left(v^4_1 \otimes v^4_2, z_1\right) p^3_1 \otimes p^3_2\right\rangle \\
&= \iota_{21}^{-1}\langle t^3_1 \otimes t^3_2, \lambda^3_{4,2}\lambda^2_{4,3} \cdot \mathcal{Y}^3_{4,2}\left(u^4_1, z_2\right) \mathcal{Y}^2_{4,3}\left(v^4_1, z_1\right) p^3_1 \otimes \overline{\mathcal{Y}}^3_{4,2}\left(u^4_2, z_2\right) \overline{\mathcal{Y}}^2_{4,3}\left(v^4_2, z_1\right) p^3_2 \\
&\quad + \lambda^3_{4,3}\lambda^3_{4,3} \cdot \mathcal{Y}^3_{4,3}\left(u^4_1, z_2\right) \mathcal{Y}^3_{4,3}\left(v^4_1, z_1\right) p^3_1 \otimes \overline{\mathcal{Y}}^3_{4,3}\left(u^4_2, z_2\right) \overline{\mathcal{Y}}^3_{4,3}\left(v^4_2, z_1\right) p^3_2 \\
&\quad + \lambda^3_{4,4}\lambda^4_{4,3} \cdot \mathcal{Y}^3_{4,4}\left(u^4_1, z_2\right) \mathcal{Y}^4_{4,3}\left(v^4_1, z_1\right) p^3_1 \otimes \overline{\mathcal{Y}}^3_{4,4}\left(u^4_2, z_2\right) \overline{\mathcal{Y}}^4_{4,3}\left(v^4_2, z_1\right) p^3_2\rangle.
\end{aligned} \tag{3.0.9}
$$

By Proposition 2.3.1, we have

$$
\begin{aligned}
&\iota_{12}^{-1}\left\langle t^3_1 \otimes t^3_2, Y\left(v^4_1 \otimes v^4_2, z_1\right) Y\left(u^4_1 \otimes u^4_2, z_2\right) p^3_1 \otimes p^3_2\right\rangle \\
&= \iota_{21}^{-1}\left\langle t^3_1 \otimes t^3_2, Y\left(u^4_1 \otimes u^4_2, z_2\right) Y\left(v^4_1 \otimes v^4_2, z_1\right) p^3_1 \otimes p^3_2\right\rangle.
\end{aligned} \tag{3.0.10}
$$

Then by (3.0.8), (3.0.9), (3.0.10) and Remark 2.8.2, comparing the coefficients of

$$
\begin{aligned}
&\iota_{21}^{-1}\langle t^3_1 \otimes t^3_2, \mathcal{Y}^3_{4,2}\left(u^4_1, z_2\right) \mathcal{Y}^2_{4,3}\left(v^4_1, z_1\right) p^3_1 \otimes \overline{\mathcal{Y}}^3_{4,2}\left(u^4_2, z_2\right) \overline{\mathcal{Y}}^2_{4,3}\left(v^4_2, z_1\right) p^3_2\rangle, \\
&\iota_{21}^{-1}\langle t^3_1 \otimes t^3_2, \mathcal{Y}^3_{4,3}\left(u^4_1, z_2\right) \mathcal{Y}^3_{4,3}\left(v^4_1, z_1\right) p^3_1 \otimes \overline{\mathcal{Y}}^3_{4,2}\left(u^4_2, z_2\right) \overline{\mathcal{Y}}^2_{4,3}\left(v^4_2, z_1\right) p^3_2\rangle, \\
&\iota_{21}^{-1}\langle t^3_1 \otimes t^3_2, \mathcal{Y}^3_{4,4}\left(u^4_1, z_2\right) \mathcal{Y}^4_{4,3}\left(v^4_1, z_1\right) p^3_1 \otimes \overline{\mathcal{Y}}^3_{4,2}\left(u^4_2, z_2\right) \overline{\mathcal{Y}}^2_{4,3}\left(v^4_2, z_1\right) p^3_2\rangle
\end{aligned}
$$

in (3.0.8) and (3.0.9), we have

$$\begin{cases} \lambda_{4,2}^3\lambda_{4,3}^2 \left(B_{3,3}^{4,4}\right)_{2,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,2} + \lambda_{4,3}^3\lambda_{4,3}^3 \cdot \left(B_{3,3}^{4,4}\right)_{3,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,2} \\ +\lambda_{4,4}^3\lambda_{4,3}^4 \cdot \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,2} = \lambda_{4,2}^3\lambda_{4,3}^2 \\ \lambda_{4,2}^3\lambda_{4,3}^2 \left(B_{3,3}^{4,4}\right)_{2,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,2} + \lambda_{4,3}^3\lambda_{4,3}^3 \cdot \left(B_{3,3}^{4,4}\right)_{3,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,2} \\ +\lambda_{4,4}^3\lambda_{4,3}^4 \cdot \left(B_{3,3}^{4,4}\right)_{4,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,2} = 0 \\ \lambda_{4,2}^3\lambda_{4,3}^2 \left(B_{3,3}^{4,4}\right)_{2,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,2} + \lambda_{4,3}^3\lambda_{4,3}^3 \cdot \left(B_{3,3}^{4,4}\right)_{3,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,2} \\ +\lambda_{4,4}^3\lambda_{4,3}^4 \cdot \left(B_{3,3}^{4,4}\right)_{4,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,2} = 0 \end{cases} \tag{3.0.11}$$

If $\lambda_{4,3}^3 = 0$, then (3.0.11) becomes

$$\begin{cases} \lambda_{4,2}^3\lambda_{4,3}^2 \left(B_{3,3}^{4,4}\right)_{2,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,2} + \lambda_{4,4}^3\lambda_{4,3}^4 \cdot \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,2} = \lambda_{4,2}^3\lambda_{4,3}^2 & \text{(3.0.12a)} \\ \lambda_{4,2}^3\lambda_{4,3}^2 \left(B_{3,3}^{4,4}\right)_{2,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,2} + \lambda_{4,4}^3\lambda_{4,3}^4 \cdot \left(B_{3,3}^{4,4}\right)_{4,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,2} = 0 & \text{(3.0.12b)} \\ \lambda_{4,2}^3\lambda_{4,3}^2 \left(B_{3,3}^{4,4}\right)_{2,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,2} + \lambda_{4,4}^3\lambda_{4,3}^4 \cdot \left(B_{3,3}^{4,4}\right)_{4,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,2} = 0. & \text{(3.0.12c)} \end{cases}$$

Replacing (3.0.12b) and (3.0.12c) by the following:

$$\left(B_{3,3}^{4,4}\right)_{4,4} \cdot (3.0.12b) - \left(B_{3,3}^{4,4}\right)_{4,3} \cdot (3.0.12c),$$
$$\left(B_{3,3}^{4,4}\right)_{2,3} \cdot (3.0.12c) - \left(B_{3,3}^{4,4}\right)_{2,4} \cdot (3.0.12b),$$

then we get:

$$\begin{cases} \lambda_{4,2}^{3}\lambda_{4,3}^{2}\left(B_{3,3}^{4,4}\right)_{2,2}\cdot\left(\tilde{B}_{3,3}^{4,4}\right)_{2,2}+\lambda_{4,4}^{3}\lambda_{4,3}^{4}\cdot\left(B_{3,3}^{4,4}\right)_{4,2}\cdot\left(\tilde{B}_{3,3}^{4,4}\right)_{4,2}=\lambda_{4,2}^{3}\lambda_{4,3}^{2} \\ \lambda_{4,2}^{3}\lambda_{4,3}^{2}\left(\left(B_{3,3}^{4,4}\right)_{4,4}\cdot\left(B_{3,3}^{4,4}\right)_{2,3}-\left(B_{3,3}^{4,4}\right)_{4,3}\cdot\left(B_{3,3}^{4,4}\right)_{2,4}\right)\cdot\left(\tilde{B}_{3,3}^{4,4}\right)_{2,2}=0 \\ \lambda_{4,4}^{3}\lambda_{4,3}^{4}\left(\left(B_{3,3}^{4,4}\right)_{4,4}\cdot\left(B_{3,3}^{4,4}\right)_{2,3}-\left(B_{3,3}^{4,4}\right)_{4,3}\cdot\left(B_{3,3}^{4,4}\right)_{2,4}\right)\cdot\left(\tilde{B}_{3,3}^{4,4}\right)_{4,2}=0. \end{cases} \tag{3.0.13}$$

Since by Lemma 2.8.4

$$\left(B_{3,3}^{4,4}\right)_{4,4}\cdot\left(B_{3,3}^{4,4}\right)_{2,3}-\left(B_{3,3}^{4,4}\right)_{4,3}\cdot\left(B_{3,3}^{4,4}\right)_{2,4}\neq 0,$$

from the last two equations of (3.0.13) we have

$$\lambda_{4,2}^{3}\lambda_{4,3}^{2}\left(\tilde{B}_{3,3}^{4,4}\right)_{2,2}=0,\lambda_{4,4}^{3}\lambda_{4,3}^{4}\left(\tilde{B}_{3,3}^{4,4}\right)_{4,2}=0.$$

But then by the first equation of (3.0.13) we get $0=\lambda_{4,2}^{3}\lambda_{4,3}^{2}$, contradicting Claim 3. So $\lambda_{4,3}^{3}\neq 0$. By a similar procedure, we get $\lambda_{4,4}^{3}\neq 0$. Hence $\lambda_{3,4}^{3}$, $\lambda_{4,3}^{3}$, $\lambda_{3,3}^{4}, \lambda_{4,4}^{4}, \lambda_{3,4}^{3}, \lambda_{3,4}^{4}$, $\lambda_{4,3}^{4}$, $\lambda_{3,3}^{3}, \lambda_{4,4}^{3}$ are all nonzero . □

Let $(U,\ Y)$ be a vertex operator algebra structure on U. First we fix a basis $\left\{\overline{\mathcal{Y}}_{a,b}^{c}(\cdot,z)\,|a,b,c=1,2,3,4\right\}$ for space of intertwining operators of type $\left(\begin{array}{c} Q_c \\ Q_a\ Q_b \end{array}\right)$, $a,b,c\in\{1,2,3,4\}$ as in [15]. By Lemma 3.0.3, without loss of generality, we can choose a basis $\left\{\mathcal{Y}_{a,b}^{c}(\cdot,z)\,|a,b,c=1,2,3,4\right\}$ for space of intertwining operators of type $\left(\begin{array}{c} P_c \\ P_a\ P_b \end{array}\right)$, $a,b,c\in\{1,2,3,4\}$ such that the coefficients $\lambda_{a,b}^{c}=1$ if $N_{a,b}^{c}\neq 0$. Now we have (U,Y), a vertex operator algebra structure on $U=U^1\oplus U^2\oplus U^3\oplus U^4$ such that for any $u^k,v^k\in U^k$ with $k=1,2,3,4$, we have

$$Y\left(u^k,z\right)u^1=\mathcal{I}_{k,1}^{k}\left(u^k,z\right)u^1, k\in\{2,3,4\},$$
$$Y\left(u^2,z\right)u^a=\mathcal{I}_{2,a}^{b}\left(u^2,z\right)u^a, \{a,b\}=\{3,4\},$$
$$Y\left(u^a,z\right)u^2=\mathcal{I}_{a,2}^{b}\left(u^a,z\right)u^2, \{a,b\}=\{3,4\},$$
$$Y\left(u^2,z\right)v^2=\mathcal{I}_{2,2}^{1}\left(u^2,z\right)v^2,$$
$$Y\left(u^k,z\right)v^k=\mathcal{I}_{k,k}^{1}\left(u^k,z\right)v^k+\mathcal{I}_{k,k}^{3}\left(u^k,z\right)v^k+\mathcal{I}_{k,k}^{4}\left(u^k,z\right)v^k, k\in\{3,4\},$$
$$Y\left(u^a,z\right)u^b=\mathcal{I}_{a,b}^{2}\left(u^a,z\right)u^b+\mathcal{I}_{a,b}^{3}\left(u^a,z\right)u^b+\mathcal{I}_{a,b}^{4}\left(u^a,z\right)u^b, \{a,b\}=\{3,4\}, \tag{3.0.14}$$

where $\mathcal{I}_{a,b}^{c}\in I_{U^1}\binom{U^c}{U^a\ U^b}$, $a,b,c\in\{1,2,3,4\}$ are nonzero intertwining operators. Furthermore, for each $u^i\in U^i$, we write $u^i=u_1^i\otimes u_2^i$ where $u_1^i\in P_i$ and $u_2^i\in Q_i$, $\mathcal{I}_{a,b}^{c}=\mathcal{Y}_{a,b}^{c}\otimes\overline{\mathcal{Y}}_{a,b}^{c}$ where $\mathcal{Y}_{a,b}^{c}\in I_{P_1}\binom{P_c}{P_a\ P_b}$, $\overline{\mathcal{Y}}_{a,b}^{c}\in I_{Q_1}\binom{Q_c}{Q_a\ Q_b}$ with $a,b,c\in\{1,2,3,4\}$.

Theorem 3.0.4. *The vertex operator algebra structure on U over $\mathbb{C}$ is unique.*

Proof. Let (U,Y) be the vertex operator algebra structure as given in (3.0.14). Suppose $\left(U,\overline{Y}\right)$ is another vertex operator algebra structure on U. Without loss of generality, we may assume $Y(u,z)=\overline{Y}(u,z)$ for all $u\in U^1$. From our settings above, there exist nonzero constants $\lambda_{i,1}^{i}$, $\lambda_{2,2}^{1}$, $\lambda_{2,j}^{k}$, $\lambda_{j,2}^{k}$ $\lambda_{3,4}^{p}$,$\lambda_{4,3}^{p}$, λ_{33}^{l}, $\lambda_{4,4}^{l}$ where $i,p=2,3,4$, $\{j,k\}=\{3,4\}$, $l=1,3,4$ such that for any $u^i,v^i\in U^i$, $i=1,2,3,4$, we have

$$\overline{Y}\left(u^k,z\right)u^1=\lambda_{k,1}^{k}\cdot\mathcal{I}_{k,1}^{k}(u^k,z)u^1, \text{for } k\in\{2,3,4\},$$

$$\overline{Y}\left(u^2,z\right)v^2=\lambda_{2,2}^{1}\cdot\mathcal{I}_{2,2}^{1}(u^2,z)v^2,$$

$$\overline{Y}\left(u^2,z\right)u^a=\lambda_{2,a}^{b}\cdot\mathcal{I}_{2,a}^{b}(u^2,z)u^a, \text{for } \{a,b\}=\{3,4\},$$

$$\overline{Y}\left(u^a,z\right)u^2=\lambda_{a,2}^{b}\cdot\mathcal{I}_{a,2}^{b}(u^a,z)u^2, \text{for } \{a,b\}=\{3,4\},$$

$$\overline{Y}\left(u^k,z\right)v^k=\lambda_{k,k}^1\cdot\mathcal{I}_{k,k}^1\left(u^k,z\right)v^k+\lambda_{k,k}^3\cdot\mathcal{I}_{k,k}^3\left(u^k,z\right)v^k+\lambda_{k,k}^4\cdot\mathcal{I}_{k,k}^4\left(u^k,z\right)v^k$$

for $k\in\{3,4\}$,

$$\overline{Y}\left(u^a,z\right)u^b=\lambda_{a,b}^2\cdot\mathcal{I}_{a,b}^2\left(u^a,z\right)u^b+\lambda_{a,b}^3\cdot\mathcal{I}_{a,b}^3\left(u^a,z\right)u^b+\lambda_{a,b}^4\cdot\mathcal{I}_{a,b}^4\left(u^a,z\right)u^b$$

for $\{a,b\}=\{3,4\}$,
where $\mathcal{I}_{a,b}^c\in I_{U^1}\binom{U^c}{U^a\ U^b}$, for $a,b,c\in\{1,2,3,4\}$ are nonzero intertwining operators.

The rest of proof is similar to that of Lemma 3.0.3. In the proof of Lemma we need to show certain constants are nonzero. But we need to determine these constants explicitly here.

Claim 1. $\lambda_{k,1}^k=1$, *for* $k\in\{2,3,4\}$.

For any $u^1\in U^1$, $u^k\in U^k,k\in\{2,3,4\}$, skew symmetry of $Y(\cdot,z)$ and $\overline{Y}(\cdot,z)$ imply

$$\begin{aligned}\overline{Y}(u^k,z)u^1&=e^{zL(-1)}\overline{Y}\left(u^1,-z\right)u^k=e^{zL(-1)}Y(u^1,-z)u^k\\&=Y\left(u^k,z\right)u^1=\mathcal{I}_{k,1}^k\left(u^k,z\right)u^1.\end{aligned}$$

In the meantime, $\overline{Y}\left(u^k,z\right)u^1=\lambda_{k,1}^k\cdot\mathcal{I}_{k,1}^k(u^k,z)u^1$. Thus we get $\lambda_{k,1}^k=1$.

Claim 2. $\lambda_{k,k}^1=1$, *for* $k\in\{2,3,4\}$.

Note from Remark 3.0.2 that U has a unique nondegenerate invariant bilinear form $\langle\cdot,\cdot\rangle$ with $\langle\mathbf{1},\mathbf{1}\rangle=1$. For $u^1\in U^1$ and $u^k,v^k\in U^k,k\in\{2,3,4\}$, we have

$$\left\langle Y\left(u^k,z)v^k\right),u^1\right\rangle=\left\langle v^k,Y\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)}u^k,z^{-1}\right)u^1\right\rangle.$$

This implies that

$$\left\langle \mathcal{I}_{k,k}^{1}\left(u^{k},z\right)v^{k},u^{1}\right\rangle = \left\langle v^{k},\mathcal{I}_{k,1}^{k}\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)}u^{k},z^{-1}\right)u^{1}\right\rangle .$$

The invariant bilinear form on $\left(U,\overline{Y}\right)$ gives

$$\left\langle \lambda_{k,k}^{1}\cdot\mathcal{I}_{k,k}^{1}\left(u^{k},z\right)v^{k},u^{1}\right\rangle = \left\langle v^{k},\lambda_{k,1}^{k}\cdot\mathcal{I}_{k,1}^{k}\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)}u^{k},z^{-1}\right)u^{1}\right\rangle .$$

Using Claim 1, we get $\lambda_{k,k}^{1}=1$.

Claim 3. $\lambda_{2,3}^{4}=\lambda_{3,2}^{4}=\lambda_{2,4}^{3}=\lambda_{4,2}^{3}=\lambda_{4,3}^{2}=\lambda_{3,4}^{2}=\lambda$, *for* $\lambda^{2}=1$.

Let $u^2\in U^2$, $u^3\in U^3$, $u^4\in U^4$. By skew symmetry of Y we have

$$\left\langle Y\left(u^{2},z\right)u^{3},u^{4}\right\rangle = \left\langle e^{zL(-1)}Y\left(u^{3},-z\right)u^{2},u^{4}\right\rangle .$$

That is,

$$\left\langle \mathcal{I}_{2,3}^{4}\left(u^{2},z\right)u^{3}u^{4}\right\rangle = \left\langle e^{zL(-1)}\mathcal{I}_{3,2}^{4}\left(u^{3},-z\right)u^{2},u^{4}\right\rangle .$$

Skew symmetry of $\overline{Y}$ gives

$$\lambda_{2,3}^{4}\left\langle \mathcal{I}_{2,3}^{4}\left(u^{2},z\right)u^{3},u^{4}\right\rangle = \lambda_{3,2}^{4}\left\langle e^{zL(-1)}\mathcal{I}_{3,2}^{4}\left(u^{3},-z\right)u^{2},u^{4}\right\rangle .$$

The above two identities give $\lambda_{2,3}^{4}=\lambda_{3,2}^{4}$. Similarly, we can prove $\lambda_{2,4}^{3}=\lambda_{4,2}^{3}$, $\lambda_{3,4}^{2}=\lambda_{4,3}^{2}$. Then for any $u^1\in U^1$, $u^2,v^2\in U^2$, $u^3,v^3\in U^3$ and $u^4\in U^4$, commutativity of Y implies

$$\begin{aligned}&\iota_{12}^{-1}\left\langle u^{1},\mathcal{I}_{2,2}^{1}\left(u^{2},z_{1}\right)\mathcal{I}_{3,4}^{2}\left(u^{3},z_{2}\right)u^{4}\right\rangle\\&\quad=\iota_{21}^{-1}\left\langle u^{1},\mathcal{I}_{3,3}^{1}\left(u^{3},z_{2}\right)\mathcal{I}_{2,4}^{3}\left(u^{2},z_{1}\right)u^{4}\right\rangle ,\end{aligned}$$

commutativity of $\overline{Y}$ implies

$$\begin{aligned}&\iota_{12}^{-1}\left\langle u^1, \lambda_{2,2}^1\lambda_{3,4}^2 \cdot \mathcal{I}_{2,2}^1\left(u^2, z_1\right)\mathcal{I}_{3,4}^2\left(u^3, z_2\right)u^4\right\rangle \\ &\quad = \iota_{21}^{-1}\left\langle u^1, \lambda_{3,3}^1\lambda_{2,4}^3 \cdot \mathcal{I}_{3,3}^1\left(u^3, z_2\right)\mathcal{I}_{2,4}^3\left(u^2, z_1\right)u^4\right\rangle.\end{aligned}$$

The above two identities and Claim 2 together imply

$$\lambda_{3,4}^2 = \lambda_{2,4}^3. \tag{3.0.15}$$

Similarly, using (3.0.3) and (3.0.4) gives

$$\lambda_{3,2}^4 = \lambda_{4,2}^3, \tag{3.0.16}$$

$$\lambda_{3,4}^2 \cdot \lambda_{2,3}^4 = 1. \tag{3.0.17}$$

So $\lambda_{2,3}^4 = \lambda_{3,2}^4 = \lambda_{2,4}^3 = \lambda_{4,2}^3 = \lambda_{4,3}^2 = \lambda_{3,4}^2 = \lambda$, for $\lambda^2 = 1$.

Claim 4. $\lambda_{3,4}^3 = \lambda_{4,3}^4 = \lambda_{3,3}^4 = \lambda_{4,4}^4 = \mu$, $\lambda_{3,4}^4 = \lambda_{4,3}^4 = \lambda_{4,4}^3 = \lambda_{3,3}^3 = \gamma$, *for* $\mu^2 = \gamma^2 = 1$.

The proof of equalities $\lambda_{3,4}^3 = \lambda_{4,3}^4 = \lambda_{3,3}^4 = \lambda_{4,4}^4$, $\lambda_{3,4}^4 = \lambda_{4,3}^4 = \lambda_{4,4}^3 = \lambda_{3,3}^3$ is similar to Claim 3, we denote them as μ, γ respectively. Now we prove $\mu^2 = \gamma^2 = 1$. For (U, Y) and $\left(U, \overline{Y}\right)$, similar to (3.0.11) we have

$$\begin{cases}\left(B_{3,3}^{4,4}\right)_{2,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,3} + \left(B_{3,3}^{4,4}\right)_{3,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + \left(B_{3,3}^{4,4}\right)_{4,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 1 \\ \left(B_{3,3}^{4,4}\right)_{2,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,3} + \left(B_{3,3}^{4,4}\right)_{3,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0 \\ \left(B_{3,3}^{4,4}\right)_{2,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,3} + \left(B_{3,3}^{4,4}\right)_{3,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + \left(B_{3,3}^{4,4}\right)_{4,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0,\end{cases}$$

$$\begin{cases} \left(B_{3,3}^{4,4}\right)_{2,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,3} + \mu^2 \cdot \left(B_{3,3}^{4,4}\right)_{3,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + \gamma^2 \cdot \left(B_{3,3}^{4,4}\right)_{4,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = \mu^2 \\ \left(B_{3,3}^{4,4}\right)_{2,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,3} + \mu^2 \cdot \left(B_{3,3}^{4,4}\right)_{3,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + \gamma^2 \cdot \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0 \\ \left(B_{3,3}^{4,4}\right)_{2,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{2,3} + \mu^2 \cdot \left(B_{3,3}^{4,4}\right)_{3,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + \gamma^2 \cdot \left(B_{3,3}^{4,4}\right)_{4,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0. \end{cases}$$

If $1 - \mu^2 \neq 0$, by the two systems above, we have

$$(1-\mu^2) \cdot \left(B_{3,3}^{4,4}\right)_{3,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + (1-\gamma^2) \cdot \left(B_{3,3}^{4,4}\right)_{4,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 1 - \mu^2, \quad (3.0.18)$$

$$(1-\mu^2) \cdot \left(B_{3,3}^{4,4}\right)_{3,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + (1-\gamma^2) \cdot \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0, \quad (3.0.19)$$

$$(1-\mu^2) \cdot \left(B_{3,3}^{4,4}\right)_{3,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + (1-\gamma^2) \cdot \left(B_{3,3}^{4,4}\right)_{4,4} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0. \quad (3.0.20)$$

Replacing (3.0.19) and (3.0.20) by the following:

$$\left(B_{3,3}^{4,4}\right)_{4,4} \cdot (3.0.19) - \left(B_{3,3}^{4,4}\right)_{4,2} \cdot (3.0.20),$$
$$\left(B_{3,3}^{4,4}\right)_{3,2} \cdot (3.0.20) - \left(B_{3,3}^{4,4}\right)_{3,4} \cdot (3.0.19),$$

then we get

$$\begin{cases} (1-\mu^2) \cdot \left(B_{3,3}^{4,4}\right)_{3,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} + (1-\gamma^2) \cdot \left(B_{3,3}^{4,4}\right)_{4,3} \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 1 - \mu^2 \\ (1-\mu^2) \cdot \left(\left(B_{3,3}^{4,4}\right)_{3,2} \cdot \left(B_{3,3}^{4,4}\right)_{4,4} - \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(B_{3,3}^{4,4}\right)_{3,4}\right) \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} = 0 \\ (1-\gamma^2) \cdot \left(\left(B_{3,3}^{4,4}\right)_{3,2} \cdot \left(B_{3,3}^{4,4}\right)_{4,4} - \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(B_{3,3}^{4,4}\right)_{3,4}\right) \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0. \end{cases} \quad (3.0.21)$$

By Lemma 2.8.4, $\left(B_{3,3}^{4,4}\right)_{3,2} \cdot \left(B_{3,3}^{4,4}\right)_{4,4} - \left(B_{3,3}^{4,4}\right)_{4,2} \cdot \left(B_{3,3}^{4,4}\right)_{3,4} \neq 0$, so from the last two equations of (3.0.21) we have $(1-\mu^2)\left(\tilde{B}_{3,3}^{4,4}\right)_{3,3} = 0$ and $(1-\gamma^2) \cdot \left(\tilde{B}_{3,3}^{4,4}\right)_{4,3} = 0$. But then from the first equation of (3.0.21), we get $1-\mu^2 = 0$, which is a contradiction. So $\mu^2 = 1$. Similarly, we have $\gamma^2 = 1$.

Claim 5. *(U, Y) is isomorphic to $\left(U, \overline{Y}\right)$.*

Define a linear map σ such that

$$\sigma|_{U^1} = 1,\ \sigma|_{U^2} = \lambda\mu\gamma,\ \sigma|_{U^3} = \gamma,\ \sigma|_{U^4} = \mu,$$

where $\lambda^2 = \mu^2 = \gamma^2 = 1$. It is clear that σ is a linear isomorphism of U. Using Lemma 2.1.8, σ gives a vertex operator algebra structure (U, Y^σ) with $Y^\sigma(u, z) = \sigma Y(\sigma^{-1}u, z)\sigma^{-1}$ which is isomorphic to (U, Y). It is easy to verify that $Y^\sigma(u, z) = \overline{Y}(u, z)$ for all $u \in U$. Thus we proved the uniqueness of the vertex operator algebra structure on U . □

Theorem 3.0.5. *The vertex operator algebra structure on $5A$-algebra $\mathcal{U}_{5A}$ over $\mathbb{C}$ is unique.*

Proof. Recall that $\mathcal{U}_{5A} = W + V^5 + V^6 + V^7 + V^8 + V^9 + V^{10} + V^{11} + V^{12}$. Assume there are two VOA structures $(\mathcal{U}_{5A}, Y^1)$, $(\mathcal{U}_{5A}, Y^2)$ on $\mathcal{U}_{5A}$. By Lemma 3.0.1, W is a subalgebra of both $(\mathcal{U}_{5A}, Y^1)$ and $(\mathcal{U}_{5A}, Y^2)$, so there are two VOA sturctures $(W, Y^1|_W)$ and $(W, Y^2|_W)$ on W. The unique VOA structure on U by Theorem 3.0.4 implies that the VOA structure on $W = L\left(\frac{1}{2}, 0\right) \otimes U$ is unique. So we have $Y^1|_W = Y^2|_W$. Then again by Lemma 3.0.1, we have $Y^1 = Y^2$ on $\mathcal{U}_{5A}$, as desired. □

Chapter 4

Uniqueness of VOA structure of the $3C$-algebra $\mathcal{U}_{3C}$

Set

$$
\begin{aligned}
V^1 &= L\left(\frac{1}{2},0\right)\otimes L\left(\frac{21}{22},0\right),\ V^2 = L\left(\frac{1}{2},0\right)\otimes L\left(\frac{21}{22},8\right),\\
V^3 &= L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{21}{22},\frac{45}{2}\right),\ V^4 = L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{21}{22},\frac{7}{2}\right),\\
V^5 &= L\left(\frac{1}{2},\frac{1}{16}\right)\otimes L\left(\frac{21}{22},\frac{31}{16}\right),\ V^6 = L\left(\frac{1}{2},\frac{1}{16}\right)\otimes L\left(\frac{21}{22},\frac{175}{16}\right).
\end{aligned}
$$

Then from [24], the $3C$-algebra

$$\mathcal{U}_{3C} \cong V^1 \oplus V^2 \oplus V^3 \oplus V^4 \oplus V^5 \oplus V^6.$$

Since V^1 is rational and C_2-cofinite, by [1] and [22], it is easy to see that $\mathcal{U}_{3C}$ is a simple, rational and C_2-cofinite VOA.

Lemma 4.0.1. *Let $W = V^1 + V^2$, then W is a subVOA of $\mathcal{U}_{3C}$ and the VOA structure of $3C$-algebra $\mathcal{U}_{3C}$ is uniquely determined by W.*

Proof. The proof is similar to that of Lemma 3.0.1. By the fusion rules of $L\left(\frac{1}{2},0\right)$

modules and $L\left(\frac{21}{22},0\right)$ modules, we see that $W = V^1 + V^2$ is a subVOA of $V^1 + V^2 + V^3 + V^4$ and $V^1 + V^2 + V^3 + V^4$ is a subVOA of $\mathcal{U}_{3C}$. By [27], we can define an involution τ of $\mathcal{U}_{3C}$ by letting it act by -1 on $V^5 + V^6$ and by 1 on $V^1 + V^2 + V^3 + V^4$. Then it follows from [6] that $V^5 + V^6$ is a simple current module over $V^1 + V^2 + V^3 + V^4$. Similarly, one can define an involution σ of $V^1+V^2+V^3+V^4$ which acts by 1 on $W = V^1+V^2$ and by -1 on $M = V^3+V^4$. Then again by [6] we see that M is a simple current module over W.

Claim 1. *The W-module structure of M is unique. That is, if M^1 is also a W-module such that $M^1 \cong M$ as V^1-modules, then $M^1 \cong M$ as W-modules.*

We need to use category theory to prove this claim. For the notations, see Lemma 3.0.1 or [23], [14]. We prove $M \cong M^1 \cong W \boxtimes_{V^1} V^3$ as W-modules. Note that

$$\text{qdim}_{V^1}(W \boxtimes_{V^1} V^3) = \text{qdim}_{V^1}(W)\text{qdim}_{V^1}(V^3) = \text{qdim}_{V^1}(W)$$

by Proposition 2.6.5 as V^3 is a simple current. Thus

$$\text{qdim}_{W}(W \boxtimes_{V^1} V^3) = \frac{\text{qdim}_{V^1}(W \boxtimes_{V^1} V^3)}{\text{qdim}_{V^1}(W)} = 1.$$

Theorem 1.6 of [23] gives isomorphisms

$$\text{Hom}_{(\mathcal{C}_{V^1})_W}(W \boxtimes_{V^1} V^3, M) \cong \text{Hom}_{V^1}(V^3, M) \cong \text{Hom}_{(\mathcal{C}_{V^1})_W}(W \boxtimes_{V^1} V^3, M^1).$$

Since $(\mathcal{C}_{V^1})_W$ is a semisimple category, M and M^1 are irreducible W-submodules of $W \boxtimes_{V^1} V^3$. Noting that $\text{qdim}_W(M) = \text{qdim}_W(M^1) = \text{qdim}_W(W \boxtimes_{V^1} V^3) = 1$ we immediately conclude that $M \cong M^1 \cong W \boxtimes_{V^1} V^3$ as W-modules.

Claim 2. *The vertex operator algebra structure on $W + M$ is unique.*

From Claim 1 and the discussion above, $W + M$ is a simple current extension of W. The Claim follows from Proposition 2.5.4.

Claim 3. *The W-module structure of $N = V^5 + V^6$ is unique. That is, if N^1 is also a W-module such that $N^1 \cong N$ as V^1-modules, then $N^1 \cong N$ as W-modules.*

First it is easy to see that N is an irreducible W-module. Consider $W \boxtimes_{V^1} V^5 \in (\mathcal{C}_{V^1})_W$. By Theorem 1.6 of [23] we have isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{(\mathcal{C}_{V^1})_W}(W \boxtimes_{V^1} V^5, W \boxtimes_{V^1} V^5) &\cong \mathrm{Hom}_{V^1}(V^5, W \boxtimes_{V^1} V^5)\\ &\cong \mathrm{Hom}_{V^1}(V^5 \boxtimes_{V^1} (V^5)', W)\end{aligned}$$

where $(V^5)'$ is the restricted dual of V^5 and is isomorphic V^5. It is easy to see that the projection $V^5 \boxtimes_{V^1} V^5$ to $W + M$ is isomorphic to $V^1 + V^2 + V^4$. Thus $\mathrm{Hom}_{V^1}(V^5 \boxtimes_{V^1} (V^5)', W)$ is 2-dimensional and $W \boxtimes_{V^1} V^5$ is a direct sum of two inequivalent irreducible W-modules in $(\mathcal{C}_{V^1})_W$. As before one can compute that $\mathrm{qdim}_W(W \boxtimes_{V^1} V^5) = \mathrm{qdim}_{V^1}(V^5) = 3 + \sqrt{3}$. Also, both N and N^1 are submodules of $\mathrm{qdim}_W(W \boxtimes_{V^1} V^5)$. If N and N^1 are inequivalent W-modules then $\mathrm{qdim}_W(W \boxtimes_{V^1} V^5) = N \oplus N^1$. But this is a contradiction as $\mathrm{qdim}_W(N + N^1) = 2\mathrm{qdim}_W(N) = 4$.

Claim 4. *There are two inequivalent $W + M$-module structures on N.*

The proof is exactly same as Claim 4 in Lemma 3.0.1.

Claim 5. *VOA structure of $3C$-algebra $\mathcal{U}_{3C}$ is uniquely determined by W.*

The proof is exactly same as Claim 5 in Lemma 3.0.1. The proof is complete.

□

Let $U = U^1 + U^2$, for $U^1 = L\left(\frac{21}{22}, 0\right)$, $U^2 = L\left(\frac{21}{22}, 8\right)$. Then $W = L\left(\frac{1}{2}, 0\right) \otimes U$ and U admits a VOA structure. By [1] and [22], U is simple, rational and C_2-

cofinite. Next we will prove the vertex operator algebra structure on U over $\mathbb{C}$ is unique.

Remark 4.0.2. *Since $U_1 = 0$ and $\dim U_0 = 1$ by Theorem 2.3.3, there is a unique nondegenerate invariant bilinear form on U and thus $U' \cong U$. Without loss of generality, we can identify U with U'.*

Let (U, Y) be a vertex operator algebra structure on U with

$$Y(u, z) = \sum_{a,b,c \in \{1,2\}} \lambda_{a,b}^{c} \cdot \mathcal{I}_{a,b}^{c} (u^a, z) u^b,$$

where $I_{U^1} \binom{U^c}{U^a \; U^b} = \mathbb{C}\mathcal{I}_{a,b}^{c}$, for $a, b, c \in \{1, 2\}$.

Lemma 4.0.3. $\lambda_{a,b}^{c} \neq 0$ *if* $N_{a,b}^{c} = \dim I_{U^1} \binom{U^c}{U^a \; U^b} \neq 0$.

Proof. The proof consists of three claims.

Claim 1. $\lambda_{2,1}^{2} \neq 0$.

For any $u^k \in U^k$, $k = 1, 2$, using skew symmetry of $Y(\cdot, z)$, we have

$$\begin{aligned} Y(u^2, z)u^1 &= e^{zL(-1)} Y\left(u^1, -z\right) u^2 = \lambda_{1,2}^{2} \cdot e^{zL(-1)} \mathcal{I}_{1,2}^{2} \left(u^1, -z\right) u^2 \\ &= \lambda_{2,1}^{2} \cdot \mathcal{I}_{2,1}^{2} \left(u^2, z\right) u^1. \end{aligned}$$

Since U^2 is an irreducible U^1-module, we have $\lambda_{1,2}^{2} \neq 0$. So $\lambda_{2,1}^{2} \neq 0$.

Claim 2. $\lambda_{2,2}^{1} \neq 0$.

By Remark 4.0.2, U has a unique invariant bilinear form $\langle \cdot, \cdot \rangle$ with $\langle 1, 1 \rangle = 1$. For $u^k, v^k \in U^k$, $k = 1, 2$, we have

$$\left\langle Y\left(u^2, z)v^2\right), u^1 \right\rangle = \left\langle v^2, Y\left(e^{zL(-1)} \left(-z^{-2}\right)^{L(0)} u^2, z^{-1}\right) u^1 \right\rangle.$$

This implies that

$$\left\langle \lambda_{2,2}^{1} \cdot \mathcal{I}_{2,2}^{1}\left(u^{2}, z\right) v^{2}, u^{1}\right\rangle=\left\langle v^{2}, \lambda_{2,1}^{2} \cdot \mathcal{I}_{2,1}^{2}\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)} u^{2}, z^{-1}\right) u^{1}\right\rangle .$$

Applying Claim 1 gvies $\lambda_{2,1}^{2} \neq 0$, hence $\lambda_{2,2}^{1} \neq 0$.

Claim 3. $\lambda_{2,2}^{2} \neq 0$.

Assume $\lambda_{2,2}^{2}=0$. Let X_1, X_2 be two subspaces of U. Define $X_1 \cdot X_2$ to be the linear span of $u_n v$ for $u \in X_1, v \in X_2$ and $n \in \mathbb{Z}$. Then we have $U^1 \cdot U^2 = U^2$, $U^2 \cdot U^1 = U^2$, $U^2 \cdot U^2 = U^1$. Define $\sigma : U^1 + U^2 \to U^1 + U^2$ such that $\sigma|_{U^1} = 1$ and $\sigma|_{U^2} = -1$. Then σ is an order 2 automorphism of $U^1 + U^2$ with $(U^1 + U^2)^{\sigma} = U^1$ and U^2 is a U^1-module. By Theorems 2.6.2 and 2.6.4, $q\dim_{U^1} U^2 = 1$ because any irreducible representation of the group generated by σ is 1-dimensional, contradicting the fact that $q\dim_{U^1} U^2 = \frac{\sin(\frac{5\pi}{12})}{\sin(\frac{\pi}{12})} \neq 1$. Therefore, $\lambda_{2,2}^{2} \neq 0$. ∎

Let $(U,\ Y)$ be a vertex operator algebra structure on U. Without loss of generality, we can choose nonzero $\mathcal{I}_{a,b}^{c} \in I_{U^1}\binom{U^c}{U^a\ U^b}$ for $a, b, c \in \{1, 2\}$ such that the coefficients $\lambda_{a,b}^{c} = 1$ if $N_{a,b}^{c} \neq 0$. Now we have $(U,\ Y)$, a vertex operator algebra structure on $U = U^1 \oplus U^2$ such that for any $u^k, v^k \in U^k$ with $k = 1, 2$,

$$\begin{aligned}
Y\left(u^{2}, z\right) u^{1} &= \mathcal{I}_{2,1}^{2}\left(u^{2}, z\right) u^{1}, \\
Y\left(u^{2}, z\right) v^{2} &= \mathcal{I}_{2,2}^{2}\left(u^{2}, z\right) v^{2}, \\
Y\left(u^{2}, z\right) v^{2} &= \mathcal{I}_{2,2}^{1}\left(u^{2}, z\right) v^{2}.
\end{aligned} \tag{4.0.1}$$

Theorem 4.0.4. *The vertex operator algebra structure on U over $\mathbb{C}$ is unique.*

Proof. Let (U, Y) be the vertex operator algebra structure as given in (4.0.1).

Suppose $\left(U, \overline{Y}\right)$ is another vertex operator algebra structure on U. Without loss of generality, we may assume $Y(u,z) = \overline{Y}(u,z)$ for all $u \in U^1$. From our settings above, there exist nonzero constants $\lambda_{2,1}^2$, $\lambda_{2,2}^1$, $\lambda_{2,2}^2$ such that for any $u^i, v^i \in U^i$, $i = 1, 2$, we have

$$\overline{Y}\left(u^2, z\right) u^1 = \lambda_{2,1}^2 \cdot \mathcal{I}_{2,1}^2(u^2, z)u^1,$$

$$\overline{Y}\left(u^2, z\right) v^2 = \lambda_{2,2}^1 \cdot \mathcal{I}_{2,2}^1(u^2, z)v^2,$$

$$\overline{Y}\left(u^2, z\right) u^2 = \lambda_{2,2}^2 \cdot \mathcal{I}_{2,2}^2(u^2, z)u^2,$$

where $\mathcal{I}_{a,b}^c \in I_{U^1}\binom{U^c}{U^a\ U^b}$ for $a, b, c \in \{1, 2\}$ are nonzero intertwining operators.

Claim 1. $\lambda_{2,1}^2 = 1$.

For any $u^1 \in U^1$, $u^2 \in U^2$, skew symmetry of $Y(\cdot, z)$ and $\overline{Y}(\cdot, z)$ imply

$$\begin{aligned}\overline{Y}(u^2, z)u^1 &= e^{zL(-1)}\overline{Y}\left(u^1, -z\right) u^2 = e^{zL(-1)}Y(u^1, -z)u^2 \\ &= Y\left(u^2, z\right) u^1 = \mathcal{I}_{2,1}^2\left(u^2, z\right) u^1.\end{aligned}$$

In the meantime, $\overline{Y}(u^2, z)u^1 = \lambda_{2,1}^2 \cdot \mathcal{I}_{2,1}^2(u^2, z)u^1$. Thus we get $\lambda_{2,1}^2 = 1$.

Claim 2. $\lambda_{2,2}^1 = 1$.

Note from Remark 4.0.2 that U has a unique nondegenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ with $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. For $u^1 \in U^1$ and $u^2, v^2 \in U^2$, we have

$$\left\langle Y\left(u^2, z)v^2\right), u^1 \right\rangle = \left\langle v^2, Y\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)} u^2, z^{-1}\right) u^1 \right\rangle.$$

Consequently,

$$\left\langle \mathcal{I}_{2,2}^1\left(u^2, z\right) v^2, u^1 \right\rangle = \left\langle v^2, \mathcal{I}_{2,1}^2\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)} u^2, z^{-1}\right) u^1 \right\rangle.$$

The invariant bilinear form on $\left(U, \overline{Y}\right)$ gives

$$\left\langle \lambda_{2,2}^{1} \cdot \mathcal{I}_{2,2}^{1}\left(u^{2}, z\right) v^{2}, u^{1}\right\rangle = \left\langle v^{2}, \lambda_{2,1}^{2} \cdot \mathcal{I}_{2,1}^{2}\left(e^{zL(-1)}\left(-z^{-2}\right)^{L(0)} u^{2}, z^{-1}\right) u^{1}\right\rangle .$$

Using Claim 1 obtains $\lambda_{2,2}^{1} = 1$.

Claim 3. $\lambda_{2,2}^{2} = \pm 1$.

For simplicity, we denote $\lambda_{2,2}^{2} := \lambda$. For $\left(U, \overline{Y}\right)$, let $p^2, t^2, u^2, v^2 \in U^2$, we have

$$\begin{aligned}
&\iota_{12}^{-1}\left\langle t^{2}, \overline{Y}\left(v^{2}, z_{1}\right) \overline{Y}\left(u^{2}, z_{2}\right) p^{2}\right\rangle \\
&= \iota_{12}^{-1}\langle t^{2}, \mathcal{I}_{2,1}^{2}\left(v^{2}, z_{1}\right) \mathcal{I}_{2,2}^{1}\left(u^{2}, z_{2}\right) \cdot p^{2} + \lambda^{2} \mathcal{I}_{2,2}^{2}\left(v^{2}, z_{1}\right) \mathcal{I}_{2,2}^{2}\left(u^{2}, z_{2}\right) \cdot p^{2}\rangle \\
&= \iota_{21}^{-1}\langle t^{2}, \sum_{i=1,2}\left(B_{2,2}^{2,2}\right)_{1,i} \mathcal{I}_{2,i}^{2}\left(u^{2}, z_{2}\right) \mathcal{I}_{2,2}^{i}\left(v^{2}, z_{1}\right) \cdot p^{2} \\
&\quad + \lambda^{2} \cdot \sum_{i=1,2}\left(B_{2,2}^{2,2}\right)_{2,i} \mathcal{I}_{2,i}^{2}\left(u^{2}, z_{2}\right) \mathcal{I}_{2,2}^{i}\left(v^{2}, z_{1}\right) \cdot p^{2}\rangle. \qquad (4.0.2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\iota_{21}^{-1}\left\langle t^{2}, \overline{Y}\left(u^{2}, z_{2}\right) \overline{Y}\left(v^{2}, z_{1}\right) p^{2}\right\rangle \\
&= \iota_{21}^{-1}\langle t^{2}, \mathcal{I}_{2,1}^{2}\left(u^{2}, z_{2}\right) \mathcal{I}_{2,2}^{1}\left(v^{2}, z_{1}\right) \cdot p^{2} + \lambda^{2} \mathcal{I}_{2,2}^{2}\left(u^{2}, z_{2}\right) \mathcal{I}_{2,2}^{2}\left(v^{2}, z_{1}\right) \cdot p^{2}\rangle. \quad (4.0.3)
\end{aligned}$$

By Proposition 2.3.1, (4.0.2), (4.0.3) and Remark 2.8.2, comparing the coefficients of

$$\iota_{21}^{-1}\langle t^{2}, \mathcal{I}_{2,1}^{2}\left(u^{2}, z_{2}\right) \mathcal{I}_{2,2}^{1}\left(v^{2}, z_{1}\right) \cdot p^{2}\rangle$$

in (4.0.2) and (4.0.3), we have

$$\left(B_{2,2}^{2,2}\right)_{1,1} + \lambda^{2} \cdot \left(B_{2,2}^{2,2}\right)_{2,1} = 1.$$

Similarly, for (U, Y), we have

$$\left(B_{2,2}^{2,2}\right)_{1,1} + \left(B_{2,2}^{2,2}\right)_{2,1} = 1.$$

From these two equations, we get

$$(1-\lambda^2) \cdot \left(B_{2,2}^{2,2}\right)_{2,1} = 0.$$

By Lemma 2.8.5, we have $\left(B_{2,2}^{2,2}\right)_{2,1} \neq 0$, which implies $\lambda^2 = 1$.

Claim 4. *(U, Y) is isomorphic to $\left(U, \overline{Y}\right)$.*

Define a linear map σ such that

$$\sigma|_{U^1} = 1, \ \sigma|_{U^2} = \lambda,$$

where $\lambda^2 = 1$. It is clear that σ is a linear isomorphism of U. Using Lemma 2.1.8, σ gives a vertex operator algebra structure (U, Y^σ) with $Y^\sigma(u, z) = \sigma Y(\sigma^{-1}u, z)\sigma^{-1}$ which is isomorphic to (U, Y). It is easy to verify that $Y^\sigma(u, z) = \overline{Y}(u, z)$ for all $u \in U$. Thus we proved the uniqueness of the vertex operator algebra structure on U . □

Theorem 4.0.5. *The vertex operator algebra structure on $3C$-algebra $\mathcal{U}_{3C}$ over $\mathbb{C}$ is unique.*

Proof. Recall that $\mathcal{U}_{3C} = W + V^3 + V^4 + V^5 + V^6$. Assume there are two VOA structures $(\mathcal{U}_{3C}, Y^1)$, $(\mathcal{U}_{3C}, Y^2)$ on $\mathcal{U}_{3C}$. By Lemma 4.0.1, W is a subalgebra of both $(\mathcal{U}_{3C}, Y^1)$ and $(\mathcal{U}_{3C}, Y^2)$, so there are two VOA sturctures $(W, Y^1|_W)$ and $(W, Y^2|_W)$ on W. By Theorem 4.0.4 the VOA structure on U is unique, we conclude that the VOA structure on $W = L\left(\frac{1}{2}, 0\right) \otimes U$ is unique. So we have

$Y^1|_W = Y^2|_W$. By Lemma 4.0.1 again, we have $Y^1 = Y^2$ on $\mathcal{U}_{3C}$. The proof is complete. $\square$

Chapter 5

Fusion rules

In this section, we will use the following result:

Proposition 5.0.1. *[2] Let V be a vertex operator algebra and let W^1, W^2, W^3 be V-modules among which W^1 and W^2 are irreducible. Suppose that V_0 is a vertex operator subalgebra of V (with the same Virasoro element) and that N^1 and N^2 are irreducible V_0-modules of W^1 and W^2, respectively. Then the restriction map from $I_V\binom{W^3}{W^1\ W^2}$ to $I_{V_0}\binom{W^3}{N^1\ N^2}$ is injective. In particular,*

$$\dim I_V\binom{W^3}{W^1\ W^2} \leq \dim I_{V_0}\binom{W^3}{N^1\ N^2}.$$

5.1 Fusion rules of the $5A$-algebra $\mathcal{U}_{5A}$

First we need the following theorem:

Theorem 5.1.1. *(Theorem 3.19 in [24]) There are exactly nine irreducible modules $\mathcal{U}_{5A}(i,j)$, $i,j = 1,3,5$, for $\mathcal{U}_{5A}$. As $L\left(\frac{1}{2},0\right) \otimes L\left(\frac{25}{28},0\right) \otimes L\left(\frac{25}{28},0\right)$-modules,* *they are of the following form:*

$$\begin{aligned}\mathcal{U}_{5A}(i,j) \cong & [0, h_{i,1}, h_{j,1}] \oplus [0, h_{i,3}, h_{j,5}] \oplus [0, h_{i,5}, h_{j,3}] \oplus [0, h_{i,7}, h_{j,7}] \\ & \oplus [\frac{1}{2}, h_{i,1}, h_{j,7}] \oplus [\frac{1}{2}, h_{i,3}, h_{j,3}] \oplus [\frac{1}{2}, h_{i,5}, h_{j,5}] \oplus [\frac{1}{2}, h_{i,7}, h_{j,1}] \\ & \oplus [\frac{1}{16}, h_{i,2}, h_{j,4}] \oplus [\frac{1}{16}, h_{i,4}, h_{j,2}] \oplus [\frac{1}{16}, h_{i,6}, h_{j,4}] \oplus [\frac{1}{16}, h_{i,4}, h_{j,6}],\end{aligned}$$

where $h_{m,n} = \frac{(7n-8m)^2-1}{4\cdot 7\cdot 8}$.

Now we can state our theorem:

Theorem 5.1.2. $\dim I_{\mathcal{U}_{5A}} \binom{\mathcal{U}_{5A}(i'',j'')}{\mathcal{U}_{5A}(i,j)\ \mathcal{U}_{5A}(i',j')} = 1$ *if and only if both* $((i,1),\ (i',1)$ $,(i'',1))$ *and* $((j,1)\,,(j',1)\,,(j'',1))$ *are admissible triples of pairs for* $p=7, q=8$ *(see Definition 2.7.1) and* 0 *otherwise.*

Proof. Theorem 5.0.1 implies the following inequality:

$$\begin{aligned}\dim I_{\mathcal{U}_{5A}} \binom{\mathcal{U}_{5A}(i'',j'')}{\mathcal{U}_{5A}(i,j)\ \mathcal{U}_{5A}(i',j')} &\le \dim I_{[0,0,0]} \binom{\mathcal{U}_{5A}(i'',j'')}{[0,h_{i,1},h_{j,1}]\ [0,h_{i',1},h_{j',1}]} \\ &= \dim I_{[0,0,0]} \binom{[0,h_{i'',1},h_{j'',1}]}{[0,h_{i,1},h_{j,1}]\ [0,h_{i',1},h_{j',1}]}.\end{aligned}$$

On the other hand, by directly computation, we have

$$q\dim_{\mathcal{U}_{5A}} \mathcal{U}_{5A}(i,j) = q\dim_{[0,0,0]}[0,h_{i,1},h_{j,1}] = \frac{\sin(\frac{8i\pi}{7})}{\sin(\frac{8\pi}{7})} \cdot \frac{\sin(\frac{8j\pi}{7})}{\sin(\frac{8\pi}{7})}.$$

So we have

$$\begin{aligned}\dim I_{\mathcal{U}_{5A}} \binom{\mathcal{U}_{5A}(i'',j'')}{\mathcal{U}_{5A}(i,j)\ \mathcal{U}_{5A}(i',j')} &= \dim I_{[0,0,0]} \binom{\mathcal{U}_{5A}(i'',j'')}{[0,h_{i,1},h_{j,1}]\ [0,h_{i',1},h_{j',1}]} \\ = \dim I_{L\left(\frac{25}{28},0\right)} \binom{L\left(\frac{25}{28},h_{i'',1}\right)}{L\left(\frac{25}{28},h_{i,1}\right)\ L\left(\frac{25}{28},h_{i',1}\right)} &\cdot \dim I_{L\left(\frac{25}{28},0\right)} \binom{L\left(\frac{25}{28},h_{j'',1}\right)}{L\left(\frac{25}{28},h_{j,1}\right)\ L\left(\frac{25}{28},h_{j',1}\right)}.\end{aligned}$$

Then we can conclude our theorem by using Theorem 2.7.3. □

5.2 Fusion rules of the $3C$-algebra $\mathcal{U}_{3C}$

First we need the following theorem:

Theorem 5.2.1. *(Theorem 3.38 in [24]) There are exactly five irreducible $\mathcal{U}_{3C}$-modules $\mathcal{U}_{3C}(2k)$, for $0 \le k \le 4$. In fact, $\mathcal{U}_{3C}(0) = \mathcal{U}_{3C}$ and as $L\left(\frac{1}{2},0\right) \otimes L\left(\frac{21}{22},0\right)$-modules,*

$$
\begin{aligned}
\mathcal{U}_{3C}(2) &\cong [0,\frac{13}{11}] \oplus [0,\frac{35}{11}] \oplus [\frac{1}{2},\frac{15}{22}] \oplus [\frac{1}{2},\frac{301}{22}] \oplus [\frac{1}{16},\frac{21}{176}] \oplus [\frac{1}{16},\frac{901}{176}],\\
\mathcal{U}_{3C}(4) &\cong [0,\frac{6}{11}] \oplus [0,\frac{50}{11}] \oplus [\frac{1}{2},\frac{1}{22}] \oplus [\frac{1}{2},\frac{155}{22}] \oplus [\frac{1}{16},\frac{85}{176}] \oplus [\frac{1}{16},\frac{261}{176}],\\
\mathcal{U}_{3C}(6) &\cong [0,\frac{1}{11}] \oplus [0,\frac{111}{11}] \oplus [\frac{1}{2},\frac{35}{22}] \oplus [\frac{1}{2},\frac{57}{22}] \oplus [\frac{1}{16},\frac{5}{176}] \oplus [\frac{1}{16},\frac{533}{176}],\\
\mathcal{U}_{3C}(8) &\cong [0,\frac{20}{11}] \oplus [0,\frac{196}{11}] \oplus [\frac{1}{2},\frac{7}{22}] \oplus [\frac{1}{2},\frac{117}{22}] \oplus [\frac{1}{16},\frac{133}{176}] \oplus [\frac{1}{16},\frac{1365}{176}].
\end{aligned}
$$

Now we can state our theorem:

Theorem 5.2.2. $\dim I_{\mathcal{U}_{3C}(0)}\binom{\mathcal{U}_{3C}(k)}{\mathcal{U}_{3C}(i)\ \mathcal{U}_{3C}(j)} = 1$ *if and only if* $((i+1,1),\ (j+1,1),\ (k+1,1))$ *is an admissible triple of pairs for* $p = 11, q = 12$ *(see Definition 2.7.1) and* 0 *otherwise.*

Proof. Let $h_{m,n} = \frac{(11n-12m)^2-1}{4\cdot 11\cdot 12}$. Then the irreducible $L\left(\frac{21}{22},0\right)$-module $L\left(\frac{21}{22},h_{m,n}\right)$ with $h_{m,n} = 0, \frac{13}{11}, \frac{50}{11}, \frac{111}{11}, \frac{196}{11}$ corresponds to $(m,n) = (1,1),(3,1),(5,1),(7,1),(9,1)$ respectively. If we use the pair (m,n) to denote the irreducible $L\left(\frac{1}{2},0\right) \otimes L\left(\frac{21}{22},0\right)$-module $[0,h_{m,n}]$, then by Theorem 5.0.1, we have

$$
\begin{aligned}
\dim I_{\mathcal{U}_{3C}(0)}\binom{\mathcal{U}_{3C}(k)}{\mathcal{U}_{3C}(i)\ \mathcal{U}_{3C}(j)} &\le \dim I_{L\left(\frac{1}{2},0\right)\otimes L\left(\frac{21}{22},0\right)}\binom{\mathcal{U}_{3C}(k)}{[0,h_{i+1,1}]\ [0,h_{j+1,1}]}\\
&= \dim I_{L\left(\frac{1}{2},0\right)\otimes L\left(\frac{21}{22},0\right)}\binom{[0,h_{k+1,1}]}{[0,h_{i+1,1}]\ [0,h_{j+1,1}]}.
\end{aligned}
$$

On the other hand, by directly computation, we have

$$q\dim_{\mathcal{U}_{3C}(0)}\mathcal{U}_{3C}(i) = q\dim_{L\left(\frac{1}{2},0\right)\otimes L\left(\frac{21}{22},0\right)}[0,h_{i+1,1}] = \frac{\sin(\frac{(i+1)\pi}{11})}{\sin(\frac{\pi}{11})}.$$

So we have

$$\begin{aligned}\dim I_{\mathcal{U}_{3C}(0)}\binom{\mathcal{U}_{3C}(k)}{\mathcal{U}_{3C}(i)\ \mathcal{U}_{3C}(j)} &= \dim I_{L\left(\frac{1}{2},0\right)\otimes L\left(\frac{21}{22},0\right)}\binom{[0,h_{k+1,1}]}{[0,h_{i+1,1}]\ [0,h_{j+1,1}]}\\ &= \dim I_{L\left(\frac{21}{22},0\right)}\binom{L\left(\frac{21}{22},h_{k+1,1}\right)}{L\left(\frac{21}{22},h_{i+1,1}\right)\ L\left(\frac{21}{22},h_{j+1,1}\right)}.\end{aligned}$$

Then we can conclude our theorem by using Theorem 2.7.3. □

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