Title
Charge in Classical Gauge Theories

Author
Gilton, Marian Judith Rogers

Publication Date
2019

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

Charge in Classical Gauge Theories

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Philosophy

by

Marian Judith Rogers Gilton

Dissertation Committee:
Professor James Owen Weatherall, Chair
Chancellor's Professor Jeffrey A. Barrett
Professor JB Manchak

2019

DEDICATION

To my parents, Greg and Barbara, in gratitude for
the many ways in which they taught me to seek understanding.

TABLE OF CONTENTS

LIST OF FIGURES

Page

LIST OF TABLES

ACKNOWLEDGMENTS

The work that follows has benefited from the insights, suggestions, objections, and comments of many. I am chiefly grateful to my advisor, Jim Weatherall, for his guidance on this project over many years. In his reading of numerous drafts, in suggesting resources, in his research on the foundations of Yang-Mills theory, and in the example of his own tireless energy in incisive philosophical inquiry, he has taught me much. I am grateful also to JB Manchak and to Jeffrey Barrett for serving on my committee, and for their instruction, guidance, and support.

I am grateful to the members of the Southern California Philosophy of Physics reading group for engaging conversations regarding the material of chapter 4. I am particularly grateful to David Wallace for several productive conversations on this material, and for bringing to my attention the role of the Lie algebra in Noether charge currents, which is the springboard for the material in chapter 3. I owe my thanks to Charles Sebens, Benjamin Feintzeig, Samuel Fletcher, Sarita Rosenstock, Alisa Bokulich, and Katherine Brading for their thoughts on various parts of this dissertation. I am also grateful to David Allen Boozer for several productive conversations on the mathematical foundations of gauge theories.

It would be difficult to underestimate the value of the indirect impact that many others have had on my work while in graduate school. I am particularly grateful to Bonnie Kent, Sean Walsh, Penelope Maddy, and to Jeremy Hies for their contributions to the development of my skills as a writer. I have found further support in professional and personal conversations with Elliott Chen, Timothy Schmitz, Fabio Lampert, Kevin Kadowaki, Alysha Kassam, Helen Meskhidze, Christopher Mitsch, Michael Schneider, and Jeffrey Schatz.

I am indebted to Brian Davies, Victor Lee Austin, Christiana Zenner, Jada Strabbing, and especially to Nathan Ballantyne, for their inspiration and encouragement to pursue professional philosophy. I am also grateful to Marc Sher, Vassilios Fessatidis, Benjamin Shemmer, Melkana Brakalova, and especially to Christopher Aubin, for their mentorship and instruction in physics and mathematics while I was an undergraduate at Fordham University.

While at the University of California, Irvine, I have been supported by a Social Science Merit Fellowship from the School of Social Science. In Fall of 2018 and Fall of 2017, I was supported by an Associate Dean Fellowship. During Fall of 2014, Winter 2016, Spring 2016, and Winter 2017, I was supported by the National Science Foundation Grant (award no. 1331126), administered by James Weatherall. Much of the research related to this dissertation was conducted with the support of this grant, for which I am very grateful. Additionally, I am grateful for financial support from grant no. 61048 from the John Templeton Foundation during the Spring of 2019. The opinions expressed in this dissertation are those of the author and do not necessarily reflect the views of the John Templeton Foundation. From 2013 to 2016 I received further financial and professional support from the Lilly Fellows Program in Humanities and the Arts. I am especially grateful for the wisdom and leadership my Lilly mentors, Jane Kelley Rodeheffer and Arlin Migliazzo, and for the companionship of the other members of my fellowship cohort.

I feel that it is important to also acknowledge several additional forms support that I received while completing this work between the Spring of 2018 to the present. In March of 2018 I gave birth to my daughter, Zoe. I am especially grateful to the School of Social Sciences for expanding maternity leave benefits to graduate students beyond six weeks to ten weeks, and it is my hope that similar expansions of leave policies can be enacted to the benefit of a larger number of graduate students. I have also had crucial support—from family, friends, and from institutional funds—in conducting research related travel during a time when it was necessary for Zoe to travel with me. I am grateful for the support of a DECADE travel award, an Associated Graduate Students travel award, departmental support, and for a dependent care subsidy from the Philosophy of Science Association. Dania Theda, Ellen Rogers, Barbara Rogers, and Thomas Gilton each gave generously of their time in traveling with me and caring for Zoe.

Finally, I would not be the person or the scholar that I am today without the unwavering love and support of my husband, Thomas. He has meticulously edited the pages that follow, and in several cases, read numerous drafts. Most importantly, his companionship over many years has been a singular source of encouragement. My deepest thanks are to him.

CURRICULUM VITAE

Marian Judith Rogers Gilton

Appointments

Assistant Professor of History and Philosophy of Science — **From August 2019**
University of Pittsburgh

Education

Doctor of Philosophy in Philosophy — **2019**
University of California, Irvine — *Irvine, California*

Master of Arts in Philosophy — **2016**
University of California, Irvine — *Irvine, California*

Bachelor of Science in Physics and Theology — **2013**
Fordham University — *Bronx, New York*

Honors and Awards

Order of Merit award in Outstanding Graduate Student Scholarship — **2019**
School of Social Sciences, University of California, Irvine

Lilly Graduate Fellowship — **2013-2016**
The Lilly Fellows Program in Humanities and the Arts

Victor Hess Award in Physics — **2013**
Fordham University

University Service

DECADE PLUS Leadership Coach — **2017-2019**
University of California, Irvine

Competative Edge Mentor — **2017**
University of California, Irvine

Educational Quality Committee — **2011-2013**
Fordham University

ABSTRACT OF THE DISSERTATION

Charge in Classical Gauge Theories

By

Marian Judith Rogers Gilton

Doctor of Philosophy in Philosophy

University of California, Irvine, 2019

Professor James Owen Weatherall, Chair

This dissertation concerns the philosophical interpretation of charge in contemporary particle physics. The two most prominent examples of charge are electric charge and color charge. While these two charges are usually described as analogous, it is argued here that they have a number of significant physical and metaphysical differences. Drawing upon the various group representations used in particle physics as an interpretive basis, it is shown that electric charge in fact has the same mathematical group structure as color charge has, but in a degenerate way. This provides a way of recovering a reversed analogy between color charge and electric charge: it is electric charge that is like color charge, and not vice versa. This reversed analogy is defended in the fifth and final chapter of the dissertation.

This first chapter of the dissertation presents the motivation and scope of the project. The second chapter presents the mathematical framework necessary for drawing these distinctions. This framework includes the formulation of gauge theories on principal fiber bundles and their associated vector bundles, as well as foundational results in the theory of Lie group representations. The third chapter concerns the conservation of charge as a result of Noether's theorem. The Noether charge in non-Abelian gauge theories is interpreted as a union of the opposite properties of color and anti-color. In the fourth chapter, the metaphysical status of these charge properties is brought to the fore. It is often thought that fundamental properties from physics have a privileged status within a

scientifically informed ontology of the word. But here, too, the differences between electric charge and color charge have significant ramifications for our understanding of these fundamental properties. While color charge is shown to have a complex three-fold structure, the analogues of these three level collapse in a single level in the case of electric charge. Consequently, the most specific descriptions of color charge lack certain metaphysical virtues had by the most specific descriptions of electric charge.

Chapter 1

Introduction

The history of philosophy contains nothing earlier
than the observation, that yellow amber, when rubbed,
has the power of attracting light bodies.
—Joseph Priestley (1775) p. 1

1.1 Topic and Scope

Recall, if you can, the first time you experienced static electricity. Perhaps as a young child you shocked yourself touching a doorknob; or perhaps you can recall an early classroom demonstration of a teacher rubbing a balloon with a piece of woollen cloth, them moving the balloon inches above some small scraps of paper, which suddenly leapt from their place on a table and attached themselves to the balloon. Electrical phenomena of this sort have caught the attention of the curious for millennia. The earliest known discovery was due to Thales of Miletus (624–546 B. C.), a pre-Socratic astronomer, mathematician, and philosopher. He observed that amber, when rubbed, would attract small, light objects, such as pieces of straw. Thales and his successors suggested various theoretical explanations of this phenomenon. Some posited a soul or a god within the

Gauge Theory	Associated Charge
electrodynamics	electric charge
chromodynamics	color charge
weak dynamics	weak charge

Table 1.1: The three theories, and the three charges, of the Standard Model of particle physics.

amber; others believed that an effluence of atoms from the amber swirled out and gathered the straw back; and Plato, it seems, thought of this as an instance of natural motions in which the straw systematically exchanges places with the intervening air in order to arrive at its natural place aside the amber.

Today, physical science explains such events in terms of an attractive force between positive and negative electric charges. Our classroom demonstration is explained in terms of the willingness of wool, as a conductor, to give up some of its electrons and transfer them to the balloon. Then, the positive charges of the paper are attracted to the excess of negatively charged electrons on the surface of the balloon. The slight shocks we occasionally experience when touching things such as doorknobs are caused by the sudden jump of an excess of electrons between hand and object. Despite the vast distance of scientific progress between us and Thales, we still refer to these negatively charged subatomic particles by the Greek word for amber, ελεκτρον.

The science of charged subatomic particles, such as electrons, advanced considerably in the twentieth century. Since the discovery of the electron in the famous work of J. J. Thomson in 1897, and the pioneering work of R. Millikan measuring the charge of the electron in 1909, the field of particle physics has grown to include the study of a large storehouse of particles with various fundamental properties, each arranged and categorized across three theories of fundamental interactions. With the discovery of fundamental interactions different from the electromagnetic interaction came the introduction of new kinds of charge: in addition to electric charge, there is color charge and weak charge. These three kinds of charge correspond to the three fundamental gauge theories of the Standard Model of particle physics (see table 1.1).

What are these charge properties? How do they work? In what manner may we accurately predicate them of various particles? How are the three kinds of charge different from each other? Is there a more general sense in which they can be understood as instances of a unified concept of charge? Such questions are the subject of this dissertation.

While certainly of interest to physics itself, these questions are principally concerned with philosophical interpretations of the relevant scientific theories. Philosophers have long been interested in understanding the fundamental building blocks of matter, their properties, and their interactions. Charge properties are at the heart of these philosophical matters.

In the present age, the tools of theoretical physics available for the study of the charge properties in these gauge theories are mathematically elaborate. Wu and Yang (1975) famously showed that the theoretical apparatus of gauge theories as developed by physicists could be translated into the mathematician's language of *fiber bundles*. Subsequent researchers, most notably Trautman (1980), Bleecker (2013), Baez and Muniain (1994), Nakahara (2003), and recently Weatherall (2016a) and Hamilton (2017), have further developed this formalism for the foundational study of gauge theories. This formalism has several advantages. Chiefly for our purposes here, it makes evident that the mathematics of *group representations* is hardwired into the formalism of these theories. These group representations will be a key tool for addressing the interpretive questions of subsequent chapters.

To make the project of a reasonable and executable scope, the approach undertaken in the following chapters restricts attention in two ways. First, I shall focus on comparing and contrasting electric charge with color charge, while leaving interpretive questions about weak charge for separate work. All particles have non-zero weak charge. For this reason weak charge is not useful for classifying particles. In contrast, both electric charge and color charge are used to classify particles. Moreover, the study of weak dynamics in physics is closely related to the Higgs mechanism and to parity violation—two fascinatingly complex areas of physics that, if fruitfully addressed, would quickly expand this dissertation beyond a tractable scope.[1]

Second, I shall proceed in the context of *classical* (i.e. not quantum) gauge theories. That is, we will consider the theories before they are quantized.[2] As such, all of the answers to the above interpretative questions are, to some extent, provisional. This is because, presumably, some answers might be significantly different if asked of quantum gauge theories. And it is quite clear that a study of particle physics that does not account for quantum theory will be seriously lacking! So one might worry that this aim of answering interpretative questions in the classical context is premature since we already expect the classical field theories to have certain shortcomings that can be remedied only in a fully quantum theory.

In response to this concern, it is first worth underscoring the extent to which there is a live debate over the status of quantum gauge theory *mathematically*.[3] In contrast, the classical case is not subject to such a debate. Proceeding in the context of classical gauge theories thus insulates the project from the debate over formulations of quantum field theory suitable for philosophical interpretation.

But second, and more importantly, the precise nature of the mathematical difficulties faced by quantum gauge theory motivates this investigation of charge at the classical level. This is because the construction of a quantum gauge theory proceeds by first taking the classical version of the theory and then *quantizing* that classical version. This process of quantization has so far been successful for building rigorous models of quantum gauge theory in four spacetime dimensions so long as the theory is non-interacting. The as yet unmet goal of a rigorous quantization of gauge

[1] See, however, the brief discussion of the Higgs mechanims in Chapter 4.

[2] This, however, still allows for what we might think of as certain quantum aspects to creep in before the process of quantization. For instance, we will consider state vectors of particles quantum mechanically in that superpositions count as possible states. In addition, I include an invocation of color confinement, or color neutrality, which is supported by substantial computational evidence in *quantum* chromodynamics, but must be added by hand in a classical version of chromodynamics (Kosyakov (2006) p. 307 note 1, and §8.7). I will endeavor to highlight these and other such areas where the lines between the classical and the quantum versions of the theory are easily blurred.

[3] There are two prominent sides in this debate. On one side, Doreen Fraser defends Axiomatic quantum field theory as the appropriate mathematical context for interpretive work to take place. On the other side, David Wallace defends 'Lagrangian' or 'Conventional' quantum field theory as the preferred context. See Fraser (2011) and Wallace (2011) for their views. It is not clear to me that one must choose to join one of these two sides.

theory is to arrive at an interacting theory in four spacetime dimensions.[4] Thus we are in want of clarity as to how *interactions* may be rigorously studied in a quantum context. And *charge* as a property of particles is absolutely key to understanding interactions. Charge is at the heart of the precise place where our ability to move from a classical to a quantum theory hits a major roadblock. So perhaps the best way forward is to retreat back to the study of charge and interactions in the classical context, where we may find a new way to steer a route to a rigorous interacting quantum theory.

1.2 Synopsis

The remainder of this dissertation is divided into four principal chapters. Chapter 2 covers background material from mathematics and physics that undergirds the work of subsequent chapters. It reviews the physics of categorizing subatomic particles according to the three fundamental interactions. It then presents the fiber bundle formalism of gauge theories, setting notational conventions for the rest of the dissertation. Finally, this chapter covers several useful results from group representation theory that will guide the interpretive work that follows.

Chapter 3 focuses on the conservation laws for charge that we have as a result of Noether's theorem. It is well known that Noether's theorem applied to a gauge symmetry entails a conservation law for the charge associated to that symmetry. In the context of chromodynamics, Noether's theorem leads to a conservation law for an eight-dimensional quantity of charge. This is surprising, since we usually think of the color charge of chromodynamics in terms of the three dimensions of *red*, *green*, and *blue* colors.[5] What is this eight-dimensional Noether charge, and how does it relate to the three basic colors we usually think of as the charge properties?

---

[4]There exist rigorous models of an interacting theory in less than four spacetime dimensions, but the only models in four dimensions are all non-interacting. See Brunetti et al. (2015) ch. 1 for a recent review of this research program.

[5]These colors are related to the macroscopic colors of electromagnetic radiation only by way of analogy: just as white light is a combination of all the different colors of light, so too are certain 'white' states of neutral color charge a combination of 'red,' 'green,' and 'blue' color states.

To answer these questions, this chapter delves into the three key group representations of $SU(3)$ used in chromodynamics. The eight-dimensional Noether charge transforms according to the *adjoint representation*, which can be systematically built out of the two *fundamental representations* of $SU(3)$. The first of these representations is used for quarks, and the second is used for anti-quarks. As we work through the construction of the adjoint representation from these two fundamental representations, we come to see that Noether's charge is a union of colors and anti-colors. This is in contrast to the expectation that Noether's theorem would give us three conservation laws for the three basic colors.

Next, chapter 4 makes further use of the group representations to address the status of charge properties *vis a vis* the metaphysician's longstanding, traditional notion of a universal, namely, a fundamental property that 'carves nature at the joints' (*Phaedrus* (265e)). I develop a three-fold account of charge determinables and determinates. These three levels are most easily seen in the case of color charge. At the first level, particles either have color charge or they do not. If they do, then we may distinguish at the second level between kinds of color charge, or we may say 'ways in which' particles can have color charge. It is in this sense that quarks, anti-quarks, and gluons each carry color charge in different ways. Finally, these kinds of color charge may be made determinate at level three: a particular quark, anti-quark, or gluon may be in one specific color state. It is at this level that we may speak of a quark that is *red*, or of another quark that is *blue*.

I apply this account to a recent debate situated at the intersection of metaphysics and philosophy of physics. The metaphysical program of Lewis, Armstrong, and their followers famously looks to particle physics for an authoritative list of fundamental properties. Charge and mass seem like good candidates for being universals in this sense: we envision a scientifically informed ontology of fundamental particles 'carved' by their different values of mass, charge, and other such fundamental properties from physics. Maudlin (2007) argues that the fiber bundle formalism of gauge theories does not allow for either mass or charge to be universals of this sort, which are necessary for the Lewis-Armstrong project. The key for Maudlin is that comparisons of color properties such as the *red* and *blue* of quarks requires the presence of an additional mathematical object called a *connection*. This suggests that properties such as *red* and *blue* can only be predicated relative to a connection on the fiber bundle used to formulate a model of the relevant gauge theory.

I use my three-fold account of color charge to show that Maudlin's argument applies at the third and most determinate level, but not at the determinable first and second levels of description for color charge. Thus, if one is willing to accept that determinable properties may be fundamental, then Maudlin's argument does not succeed in showing that charge is not a candidate for being a fundamental property. Moreover, the three-fold account of color charge collapses when imported into the case of electric charge. In short, there is no electric charge analogue to the differences between *red* and *blue* for color charge. As a consequence, Maudlin's argument cannot show that electric charge is not a universal.

So much for the status of charge as a universal. What about mass? I present an argument of a different sort than Maudlin's to think that mass does not fit the traditional notion of a fundamental property. The role of the Higgs mechanism in accounting for the masses of subatomic particles is evidence that these generated masses are not the sort of property that Lewis and Armstrong had in mind. Moreover, the Higgs mechanism does not rely on the metaphysical interpretive status of connections on principal bundles. Thus, while I agree with Maudlin that mass is not what Lewis and Armstrong thought it would be, I disagree with him as to the appropriate reasons for drawing this conclusion. Finally, it is significant that mass and charge are treated differently in these gauge theories. Arguably, because of their shared status as foundational properties in developing these gauge theories, both mass and charge should be counted as fundamental properties. And yet, they are sufficiently different from each other that it now seems clear that there are a plurality of ways in which properties may be fundamental.

By way of conclusion in chapter 5, I draw together the lessons of this dissertation by advocating for the reversal of the usual analogy between electric charge and color charge: it is not color charge which like electric charge, but *vice versa*. Color charge exemplifies several complexities which are hidden in the electric charge case. Color charge demonstrates the more general notion of charge in a classical field theory, and the simpler electric charge is shown to fit within this more general conceptual scheme.

The foundational mathematical motivation for this reversal is clear. In one sense, Abelian groups (such as $U(1)$ used for electric charge) can be thought of as trivial versions of non-Abelian groups (such as $SU(3)$ used for color charge). Where non-Abelian groups are characterized by their non-zero commutators, Abelian groups have zero-valued commutators. From the perspective of these commutators, the Abelian case is a simplified version of the general, non-Abelian case.

This view of the relationship between Abelian and non-Abelian groups has subtle and varied philosophical consequences. In this chapter, I argue for an understanding of electric charge as a simplified and degenerate version of the far more general notion of charge exemplified by color charge. This is done on three accounts. First, as deployed in chapter 4, charge is in general a three-fold predicate. The three-fold conceptual structure of charge is made manifest in the color case. But in the electric case, these three collapse in on each other, leaving one flat sense in which a particle has electric charge. Second, the conservation of electric charge, as the net of positive and negative charges, is a simplified case of the more general form of charge conservation as a union of charge with anti-charge. As developed in detail in chapter 3, Noether's theorem applied in the context of the $SU(3)$ color symmetry of chromodynamics makes evident this complex foundational role of anti-charge in charge conservation laws. This significance of anti-charge is hidden in the electric charge case, since the group-theoretic relationship between charge and anti-charge is radically simplified. Third, the role of charge in force laws within a theory gives yet another sense in which electric charge is a simplified version of the notion of charge exemplified by color charge. In electrodynamics, the force law is the familiar Lorentz law, wherein the strength of the interaction is directly proportional to the amount of electric charge of the particle experiencing the force. This amount of electric charge is given by a real number. In chromodynamics, the force law is the far less studied Wong force law. In the Wong force law, the color charge factor is Lie algebra valued.

The Lie algebra of $U(1)$ is the real numbers, and so the real numbered value of electric charge in the Lorentz force law is also Lie algebra valued. But without comparing the Lorentz law to the Wong law, we would likely not notice this additional way in which electric charge is like color charge—not *vice versa*.

Our concept of charge, then, must be expanded and enriched in order to properly account for the complexities of color charge and the degeneracies it reveals in electric charge. Charge is, in the first instance, a classificatory property: it sorts particles according to their abilities to participate in fundamental interactions. Charge is also a dynamical quantity in the sense that it regiments the strength of these interactions via the appropriate force laws. This is done, in general, in terms of a charge current that unites specific values of charge with anti-charge. Finally, charge is itself subject to a multifaceted conservation law. From Noether's theorem, we come to understand the flow of charge currents in terms of a conserved union of charge and anti-charge. Charge, then, is much more than what it seems to be from the vantage point electric charge alone.

Chapter 2

Background and Technical Foundations

2.1 Introduction

Our topic is charge, a fundamental property of elementary particles. This chapter reviews background material from both physics and mathematics that is relevant to the discussion of charge in subsequent chapters. Section 2.2 reviews the basic classifications of elementary particles. This gives us our initial introduction to charge as a categorizing property. I then turn to the mathematical formulation of the field theories. Section 2.3 provides a brief introduction to key features of the fiber bundle formalism for gauge theories. Here we see the foundational role of group representations in specifying a matter field for a given kind of particle. Some details regarding these group representations are presented in section 2.4.

2.2 Particles

The basic model of an atom is familiar: there are negatively charged electrons surrounding the nucleus, composed of positively charged protons and electrically neutral neutrons. Within these protons and neutrons, there are quarks bound together by their interaction with gluons via the strong nuclear force. The quarks within a proton or neutron come in two kinds: up quarks and down quarks. Occasionally, another kind of particle called an ‘electron neutrino’ is emitted from an atom. This happens in the process of beta decay whereby a neutron decays into a proton.

The electrons, up quarks, down quarks, and electron neutrinos constitute the first generation of matter particles. There are two more generations of matter which consist of heavier cousins of these four basic matter particles. These are recorded in table 2.1. For example, the strange and bottom quarks are more massive cousins of the down quarks, and the muon and tauon are more massive cousins of the electron. The electron, muon, and tauon come paired with their own kinds of neutrinos.

These three generations constitute the elementary matter particles of the Standard Model of particle physics. As fermions, they are all spin-$\frac{1}{2}$ particles. In addition to these fermions, there are elementary bosons that mediate the three fundamental interactions; accordingly, they are referred to as *mediators*. Their properties are recorded in table 2.2. The spin-0 Higgs boson is the final particle in the standard model of particle physics.

Tables 2.1 and 2.2 give standard reference information for the fundamental particles.[1] Notice that color charge does not appear anywhere in these tables, while electric charge does. This points to a key difference in the metaphysical work of the two charges. Electric charge together with mass can be used as the defining properties for the quarks. We could *identify* the down quark, for instance, as the lightest particle with electric charge $-\frac{1}{3}$, and similarly for the other quarks. In contrast, quark colors *red*, *green*, and *blue* cannot be used for the metaphysical work of defining the quark flavors. We do not distinguish between the *red* and the *blue* down quarks, or between the *green* and the *red* up quarks, etc. for the purposes of classifying different kinds of quarks. Rather, we would say that

[1]Not included in these tables are the anti-particles. Given the state of any particle, one can use the operation of charge-conjugation to determine the corresponding anti-particle. See Baker and Halvorson (2010) for a detailed mathematical account of charge-conjugation and antimatter, and how this applies in the absence of a particle interpretation of quantum field theory.

Quarks

Generation	Flavor	Electric Charge	Mass (MeV/c^2)
first	d (down)	-1/3	7
	u (up)	2/3	3
second	s (strange)	-1/3	120
	c (charmed)	2/3	1200
third	b (bottom)	-1/3	4300
	t (top)	2/3	174000

Leptons

Generation	Flavor	Electric Charge	Mass (MeV/c^2)
first	e (electron)	-1	.5
	ν_e (e neutrino)	0	≈ 0
second	μ (muon)	-1	105.659
	ν_μ (μ neutrino)	0	≈ 0
third	τ (tauon)	-1	1776.99
	ν_τ (τ neutrino)	0	≈ 0

Table 2.1: Elementary fermions (spin 1/2)
Neutrino masses are known to be non-zero but extremely small. For many purposes they are taken to be zero.

Interaction	Mediator	Electric Charge	Mass (MeV/c^2)
Strong	g (8 gluons)	0	0
Electromagnetic	γ photon	0	0
Weak	$W^\pm$ (charged) Z^0 (neutral)	± 1 0	80,420 91,190

Table 2.2: Elementary bosons (spin 1)

up, down, etc. quarks can be in any of the three color *states*, and a change in that color state does not amount to a change in the kind of quark.[2]

In another sense, however, color charge can be used to classify particles. The quarks and the gluons are all and only those particles that carry color charge *at all*, and it is because of this that they are the only particles which participate in the strong interaction. In particular, gluons can interact with each other because of their non-trivial color charge. In this same sense, all and only those particles with non-zero electric charge participate in the electromagnetic interaction. Photons cannot interact with each other via the electromagnetic interaction because they do not carry electric charge. Weak charge too has this classificatory role, albeit in a less useful way. Every elementary particle has non-trivial weak charge, and so all particles can participate in the weak interaction. This gives us the first major role of charge in particle physics: in this sense of having some unspecified non-trivial charge, charge properties classify particles according to their eligibility for participation in each of the fundamental interactions.

The three theories for these interactions (electrodynamics, chromodynamics, and weak dynamics) exhibit gauge symmetries. The symmetry group for electrodynamics is $U(1)$; chromodynamics uses $SU(3)$; and weak dynamics uses $SU(2)$. These groups describe rotations in the 'internal' charge spaces of these theories. The development of precise ways of formulating these rotations grew out of the work surrounding isotopic spin describing a (slightly broken) symmetry between protons and neutrons. Since protons and neutrons have nearly identical properties except for their electric charges, it was suggested that they could be viewed as two different states of one and the same kind of particle, and that certain isotopic spin rotations could transform a proton into a neutron and *vice versa*.

[2]Why then, one might wonder, do we distinguish between the different generations of quarks, rather than say that the down, strange, and bottom quarks are really all the same kind of quark which can be in three different flavor states? In a quark model with massless quarks, this is exactly what is done, and we refer to this as an $SU(3)$ flavor symmetry. This was Gell-Mann's original approach to the quark model, and $SU(3)$ was first used to describe this flavor symmetry before it was used to describe the symmetry between quark colors. However, we say that the $SU(3)$ flavor symmetry is 'broken', or that it is not an 'exact' symmetry, because the different flavors have different masses.

We find the seeds of contemporary gauge theory in a seminal 1954 paper by Yang and Mills. In that paper, Yang and Mills generalized the theory of isotopic spin in such a way that rotations in the internal isospin space could be enacted at different spacetime points in different ways. As they describe it,

> The conservation of isotopic spin is identical with the requirement of invariance of all interactions under isotopic spin rotation. This means that when electromagnetic interactions can be neglected, as we shall hereafter assume to be the case, the orientation of the isotopic spin is of no physical significance. The differentiation between a neutron and a proton is then a purely arbitrary process. As usually conceived, however, this arbitrariness is subject to the following limitation: once one chooses what to call a proton, what a neutron, at one space-time point, one is then not free to make any choices at other space-time points.
>
> It seems that this is not consistent with the localized field concept that underlies the usual physical theories. In the present paper we wish to explore the possibility of requiring all interactions to be invariant under *independent* rotations of the isotopic spin at all space-time points, so that the relative orientation of the isotopic spin at two space-time points becomes a physically meaningless quantity (the electromagnetic field being neglected). Yang and Mills (1954) p. 192.

Previously, a choice of 'directions' in the internal space to describe each state was fixed across spacetime. The standard for what to call a proton and what to call a neutron could not vary from point to point. Yang and Mills propose breaking this constraint and allowing 'rotations' between proton and neutron states at one point to be completely independent of those rotations at another point.

Two decades later, after the physics community had done much to develop and study the Yang-Mills theory of localized fields with independent isotopic spin, Yang in collaboration with Wu showed how this theory could be translated into the mathematician's language of fiber bundles. Quite explicitly, they presented a translation table between the languages of the two disciplines (see Wu and Yang (1975) p. 12). In the next section, I review several foundational pieces of contemporary fiber bundle formalism for classical gauge theories that are relevant for the subsequent chapters.

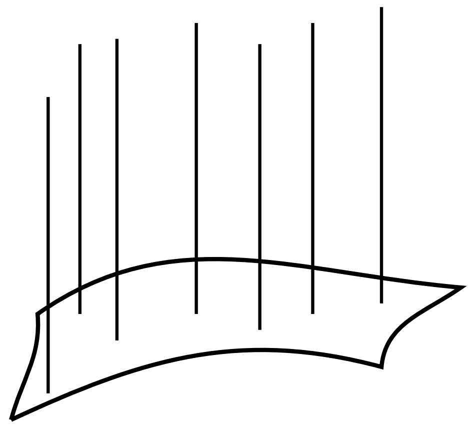

Figure 2.1: Fiber bundles can be visualized as something like a hairbrush.

2.3 Fiber Bundles

The foundational mathematical object for gauge theories is a *fiber bundle.*[3] These bundles are quite technical, and so before diving into their precise definitions, it will be useful to look at a rough picture of these objects. The intuitive picture of a fiber bundle is something like a hairbrush. There is a base space to which are attached at each point the bristles of the brush. These bristles are the fibers, and they are bundled together through their attachments to the base. See figure 2.1. However, this image should not be taken too literally. For instance, it is not the case in a fiber bundle that there is one point of each fiber which 'touches' the base space, while the other points of the fiber do not. Rather, each fiber in its entirety is attached to a single point at the base.

A fiber bundle is said to be *locally trivial*. Roughly, this means that locally (i.e. zoomed in), these bundles do indeed look like a hairbrush, but globally (i.e., zoomed out) this need not be the case. The Möbius strip is a useful example for illustrating this point. Zoomed in on the strip (figure 2.2 (b)), it looks like a two-sided comb: the vertical 'bristles' are aligned with each other and attached to the horizontal, one-dimensional, circular base. But zoomed out (figure 2.2 (a)), we see clearly

[3] Arguably, there are alternative formalisms for gauge theory which do not rely upon fiber bundles in this way. The most prominent alternative is the holonomy interpretation advanced by Healey (2007). I nonetheless choose to work in the fiber bundle formalism. See Rosenstock and Weatherall (2016) for considerations regarding the merits of fiber bundle and holonomy interpretations of gauge theories, and see Healey (2008) for more on the holonomy objection to the fiber bundle interpretation.

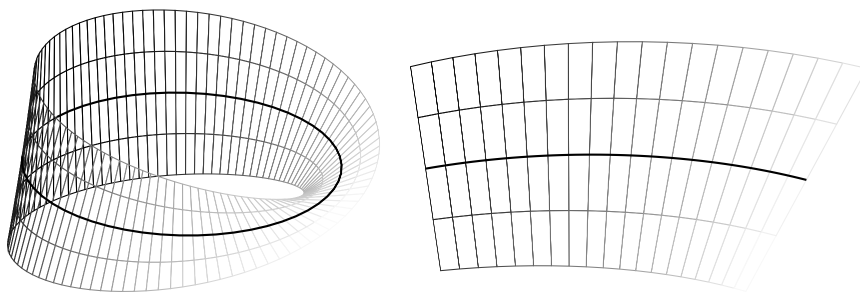
(a) (b)

Figure 2.2: The Möbius strip as a fiber bundle; (a) a 'zoomed out' view giving the global structure; (b) a 'zoomed in' view on the local structure.

the global property of the twist in the strip. Without this twist, the strip would wrap around to itself to form a cylinder instead. The cylinder is, in this formalism, a globally trivial fiber bundle.

To do field theory with these bundles, we use a special class of bundles. There are *principal fiber bundles* and their *associated vector bundles*. For these foundational definitions, I will adopt the notational conventions of Weatherall (2016a). Readers are strongly encouraged to review this work, especially the appendices for more details of the mathematical and conceptual foundations of the fiber bundle formalism. For an extensive treatment of the applications of fiber bundles to the Standard Model of particle physics, see Hamilton (2017). For a mathematical bootcamp on fiber bundles and Lie groups, see Svetlichny (1999).

Let us move forward, then, with a precise definition of a fiber bundle. It is assumed that the reader is familiar with some foundational notions from differential geometry as presented in such places as Wald (1984) and Malament (2012). In particular use the abstract index notion developed by Penrose and Rindler (1984), presented in Wald (1984) and Malament (2012), with the further notational conventions developed by Weatherall (2016a). In particular, vectors and tensors tangent to a base space M have lower-case Latin indices; vectors and tensors tangent to the total space P of a principal bundle have lower-case Greek indices; upper-case Fraktur indices are used for vectors with a Lie algebra structure.

Definition 2.3.1. *A (smooth) fiber bundle, denoted* $F \to B \xrightarrow{\pi} M$ *consists of a smooth, surjective* *map π from a manifold B to a manifold M such that the space B is locally isomorphic to the product manifold of F with M. In more detail, this means that, for any point p in M there exists an open neighborhood of $U \subseteq M$ containing p and a local trivialization of B over U. The local trivialization is a diffeomorphism $\xi : U \times F \to \pi^{-1}[U]$ such that $\pi \circ \xi : (q, f) \mapsto q$ for all (q, f) in $U \times F$.*

We call the map π the *projection map*. B is called the *total space* and M is called the *base space*. The space F is called the *typical fiber*. For any point p in M, we call $\pi^{-1}[p]$ the *fiber at p*. In physics, we use a relativistic spacetime as the base space for these bundles.

The principal fiber bundles, which are used for field theory, come equipped with an action of a Lie group on the bundle's total space.

Definition 2.3.2. *A Lie group $(G, \cdot)$ is a smooth manifold endowed with a group structure such that the group operations are smooth maps.*

Examples of Lie groups include the special linear groups $SL(n)$, unitary groups $U(n)$ and special unitary groups $SU(n)$, as well as orthogonal groups $O(n)$ and special orthogonal groups $SO(n)$. For our purposes, $U(1)$ and $SU(3)$ will be of considerable importance (see 2.4 below for more on these two groups). These are, respectively, the symmetry groups for electrodynamics and chromodynamics. Comparing and contrasting how these groups influence the metaphysical status of electric charge and color charge will be a central theme of the following chapters.

In general, there are two natural ways in which a Lie group might act on a manifold.

Definition 2.3.3. *Let $(G, \cdot)$ be a Lie group, and let M be a smooth manifold. A left-action of G on M is smooth map $\triangleright : G \times M \to M$ satisfying, for all p in M (i) $e \triangleright p = p$, and (ii) $g_2 \triangleright g_1 \triangleright p = (g_2 \cdot g_1) \triangleright p$.*

Definition 2.3.4. *Similarly, a right-action is a smooth map $\triangleleft : M \times G \to M$ such that, for all p in M, (i) $p \triangleleft e = p$, and (ii) $p \triangleleft g_1 \triangleleft g_2 = p \triangleleft (g_1 \cdot g_2)$.*

In a principal fiber bundle, the Lie group has a distinguished right action on the total space of the bundle.

Definition 2.3.5. *A principal (fiber) bundle is a smooth fiber bundle $G \to P \overset{\wp}{\to} M$ with G a Lie group, called the structure group of the bundle, subject to the following conditions:*

1. *There is a smooth right action $\lhd: P \times G \to P$ that is free, i.e., for any p in P, the only group element that maps p to itself is the identity element.*
2. *The right action preserves fibers: for all p in P and g in G, $\wp(pg) = \wp(p)$. Equivalently, each fiber is the orbit of G through p: $\wp^{-1}(\wp(p)) = \{p \lhd g : g \in G\}$. Thus each fiber $\wp^{-1}[x]$ for a point x in M is diffeomorphic to G.*
3. *The local trivializations play nicely with the group action: given a point x in M, there exists a neighborhood U of x and a local trivialization $\zeta : U \times G \to \wp^{-1}[U]$ such that for any y in U and any g, h in G, $\zeta(y, g) \lhd h = \zeta(y, g \cdot h)$.*

It is *not* correct to say that a principal bundle is a fiber bundle whose typical fiber is a Lie group. The Lie group G itself has a privilaged identity element, but a fiber $\wp^{-1}[x]$ of the principal bundle does *not* contain a unique point corresponding to this identity element. I am therefore adopting a slight abuse of notation in denoting a principal fiber bundle as $G \to P \overset{\wp}{\to} M$.

When we say that electrodynamics is a $U(1)$ gauge theory, part of what we mean is that the principal bundle used for electrodynamics has $G = U(1)$. Similarly, chromodynamics is an $SU(3)$ gauge theory in that its underlying principal bundle has $G = SU(3)$. In both theories, the base spaces M of their principal bundles are taken to be spacetime.

For each theory, the matter fields come with the mathematical structure of the fundamental representations of the theory's structure group. These group representations appear in a new sort of bundle, called an associated vector bundle. Before defining these associated bundles, we need to recall the definition of a group representation.

Definition 2.3.6. *A finite-dimensional representation of a group G is a group homomorphism $\rho : G \to GL(V)$, where V is a finite-dimensional complex vector space. We call this vector space V the carrier space of the representation.*

Given some principal bundle $G \to P \stackrel{\wp}{\to} M$, we define its various associated vector bundles using different representations of G.

Definition 2.3.7. *Let $G \to P \stackrel{\wp}{\to} M$ be a principal bundle. Let $\rho : G \to GL(V)$ be a representation of G. Define a right action of G on $P \times V$ by $(p,v) \lhd g = (p \lhd g, \rho(g^{-1})v)$ for all p in P and v in V. Then obtain the space $E = P \times_{\rho} V = (P \times V)/G$ by identifying points by this action. That is, $(p,v) \sim (p',v')$ if there exists a g in G such that $(p',v') = (p \lhd g, \rho(g^{-1})v)$. Denote the equivalence classes of each point (p,v) in E by $[p,v]$. We finish constructing the associated vector bundle by defining the projection map $\pi_{\rho} : E \to M$ by $\pi_{\rho}([p,v]) = \wp(p)$. This bundle is denoted by $E \stackrel{\pi_{\rho}}{\to} M$.*

Note that the right action of G on E in this definition really is a right action. This is because the representation ρ defines a left action of G on V, and we can use this to define a new right action. In general, one can define a right action from a left-action of a group G on a manifold M, and vice versa. Given a left-action $\rhd: G \times M \to M$, define $\lhd: M \times G \to M$ as $p \lhd g \mapsto g^{-1} \rhd p$. This is indeed a right-action since

(i) $p \lhd e = e^{-1} \rhd p = e \rhd p = p$, and

(ii) $p \lhd g_1 \lhd g_2 = g_2^{-1} \rhd g_1^{-1} \rhd p = g_2^{-1} \cdot g_1^{-1} \rhd p = (g_1 \cdot g_2)^{-1} \rhd p = p \lhd (g_1 \cdot g_2)$.

Note that taking the inverse of these actions is guaranteed to be smooth by virtue of the definition of a Lie group.

At this stage we can see that a given principal bundle can have several different associated bundles, identified by the different representations of the principal bundle's structure group. What we saw earlier about quarks and the representations of $SU(3)$ can now be situated within this larger bundle formalism. We specify a matter field by sections of the associated bundle with the appropriate group representation for the kind of matter at hand (e.g., for quarks we use the first fundamental representation of $SU(3)$). I note here the definition of a section.

Definition 2.3.8. *Let $F \to B \xrightarrow{\pi} M$ be a fiber bundle. Given an open set $U \subseteq M$, a local section on U of this bundle is a map $\sigma : U \to B$ such that $\pi \circ \sigma : U \to U$ is the identity map on U.*

In general, one cannot extend a local to a global section defined on all of M. This distinction between those bundles which admit of global sections and those which do not suffices to capture the relevant notion of triviality for principal bundles. This is given by the following definition and theorem.

Definition 2.3.9. *A principal bundle $G \to P \xrightarrow{\wp} M$ is called trivial if it is diffeomorphic to the product bundle $G \to P \times G \xrightarrow{\pi} M$, where $\pi(q, g) = q$.*

Theorem 2.3.10. *A principal bundle is trivial iff there exists a smooth (global) section on the bundle.*

For associated vector bundles, however, this theorem does not apply. In any vector bundle, there is a unique zero vector in each fiber. Thus, the section that assigns this zero vector to each point of the bases space will be a global section.

Definition 2.3.11. *An associated vector bundle is called trivial if its underlying principal bundle is trivial.*

Consider again the Möbius strip: as you follow along the non-zero section of your choice, after one full lap around the strip your section will become ill-defined. The exception is the complete circle in the middle, which is the zero section. The Möbius strip can be understood as a vector bundle associated to a non-trivial $O(1)$ principal bundle.

This distinction between trivial and non-trivial bundles is related to the technical details behind the discussion in Chapter 4. In that chapter, the central philosophical implications revolve around the

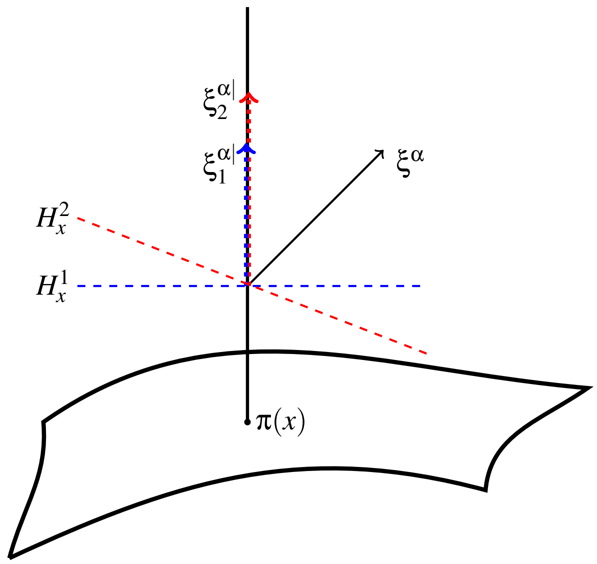

Figure 2.3: For a vector ξ^α in T_xB, its vertical projection $\xi^{\alpha|}$ depends upon the choice of H_x.

necessity of a *connection* on a fiber bundle; and connections behave differently on trivial bundles than they do on non-trivial bundles. A connection, roughly, is a choice of a standard of sameness of points across fibers. While each fiber in a bundle is isomorphic to the typical fiber, there need not be a canonical isomorphism. We will now continue to work our way up to the definition of a connection. The precise way of defining a connection relies upon the notion of vertical and horizontal subspaces to the tangent spaces T_xB of points x in B.

In a fiber bundle $F \to B \overset{\pi}{\to} M$, for any point x in B there is a unique vertical subspace of the tangent space T_xB. We write this vertical subspace as V_x, and it consists of all vectors ξ^α at x such that $\pi_*(\xi^\alpha) = \mathbf{0}$. The term 'vertical' is appropriate since the pushforward along π maps these vectors to the zero vector in the tangent space $T_{\pi(x)}M$ on the base manifold; in this sense, vertical vectors have no 'horizontal' projection. Next, a horizontal subspace of T_xB, denoted H_x, is any subspace such that any vector ξ^α at x may be decomposed into the sum of one vector in V_x and one vector in H_x. We denote the vertical competent of ξ^α as $\xi^{\alpha|}$. Even though the vertical space V_x is uniquely determined by π, both the vertical projection $\xi^{\alpha|}$ and its horizontal projection depend upon a choice of H_x. This illustrated in figure 2.3.

This brief discussion of vertical and horizontal projections helps to motivate the role of a connection, which encodes a choice of H_x at each point x in B.

Definition 2.3.12. *A connection on a fiber bundle $F \to B \xrightarrow{\pi} M$ is a smoothly varying choice of horizontal subspace at each point x in B encoded by the tensor $\omega^{\alpha|}_{\ \beta}$, which is subject to the following conditions:*

1. $\omega^{\alpha|}_{\ \beta}\omega^{\beta|}_{\ \kappa} = \omega^{\alpha|}_{\ \kappa}$, *and*
2. *given any vertical vector $\xi^{\alpha|}$ at x, $\omega^{\alpha|}_{\ \beta}\xi^{\beta|} = \xi^{\alpha|}$.*

We then define the chosen horizontal subspace H_x at x to be the kernel of $\omega^{\alpha|}_{\ \beta}$.

In the special case of principal fiber bundles, a connection must additionally play nicely with the Lie group structure. Such connections are called *principal connections.* In gauge theories, the gauge field is given by a principal connection.

Definition 2.3.13. *A principal connection on a principal bundle $G \to P \xrightarrow{\wp} M$ is a smoothly varying choice of horizontal subspace at each point x in B such that, for all group elements g in G, $(\lhd\, g)_*(H_xP) = H_{x\lhd g}P$. This means that the pushforward along $\lhd\, g$ maps the chosen horizontal subspace at one point x in a fiber to the chosen horizontal subspace at any other point $x \lhd g$ in the same fiber.*

A principal connection is encoded by a Lie algebra valued one-form on P, denoted by $\omega^{\mathfrak{A}}_{\ \alpha}$. I will sketch how this works, but see the appendix on principal connections in Weatherall (2016a) for more details. Recall that Lie groups are not just groups, but also manifolds. This means that we can consider vector fields on the group G. But because G is also a group, it has a privileged identity element e. Consider the tangent space T_eG, and choose a vector $\xi^{\mathfrak{A}}$ in T_eG (the rationale for the $\mathfrak{A}$ index will be apparent presently). Now take that vector and push it forward along the left action of G; that is, assign to each g in G the vector $(g \rhd)_*(\xi^{\mathfrak{A}})$. The resulting field on G is said to be *left-invariant*. The space of all left-invariant vector fields on G is called its associated *Lie algebra*, denoted $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is canonically isomorphic to T_eG.

Given not just G but a principal bundle $G \to P \xrightarrow{\wp} M$, the vertical subspaces V_x at each point x in P are isomorphic to T_eG. Denote this isomorphism by $i : T_eG \to V_x$. By means of this isomorphism, a principal connection $\omega^{\mathfrak{A}}_{\alpha}$ maps a tangent vector ξ^{α} in T_xP to the vector $\xi^{\mathfrak{A}} = i^{-1}(\xi^{\alpha|})$. In this way, the tensor $\omega^{\mathfrak{A}}_{\alpha}$ is a Lie algebra valued one form.

The thing to bear in mind is that connections are necessary for comparing tangent vectors across fibers. This is important in the context of using sections of certain bundles to represent properties of matter. More specifically, given a principal connection $\omega^{\mathfrak{A}}_{\alpha}$ on a bundle $G \to P \xrightarrow{\wp} M$, there is an induced, analogous determination for how to compare points across fibers in the associated vector bundles. This is given by a *covariant derivative operator*. A covariant derivative operator on an associated bundle $E \xrightarrow{\pi_\rho} M$ is the necessary standard of 'sameness' between fibers $\pi_\rho[p]$ and $\pi_\rho[q]$ for $p \neq q$ both in M.

This is the technical landscape behind much of the discussion in chapter 4 concerning the suitability of the theory of universals to a physical theory formulated using fiber bundles: the necessity of the connection for comparing vectors, which are used to represent physical properties, has consequences for the metaphysical status of these properties. The comparison of vector values across fibers is done relative to a connection on the bundle and relative to a path in the base space. Roughly, the choice of horizontal subspaces throughout the total space given by the connection gives the standard against which we can keep the vector 'level' as we move it from fiber to fiber. But suppose we began with just one vector ξ^{α} at x in P, and another vector η^{α} at $y \neq x$, and suppose further that we want to determine whether these are 'the same' vectors. Without a connection on P, we cannot answer this question.

In summary, the basic picture for a classical gauge theory is this. We work in the context of some relativistic spacetime M, over which is a principal fiber bundle. A principal connection on that bundle gives the gauge field. This principal connection induces a covariant derivative operation on the associated vector bundles. Sections of these vector bundles are interpreted as matter fields.

Note that, so far, we have discussed the formalism for a gauge theory using any Lie group whatsoever. Many of the crucial and interesting differences between specific gauge theories depend upon differences in the Lie groups used. Again, in electromagnetism the group is $U(1)$, whereas in in chromodynamics the group is $SU(3)$. As we will see in several places, key metaphysical differences between electric charge and color charge are rooted in the mathematical differences between $U(1)$ and $SU(3)$. The next section reviews useful information about these two Lie groups, their Lie algebras, and their representations.

2.4 Lie Groups, Algebras, and Representations

Recall that group representations encode the structure of the group multiplication into a set of matrices that can act on a vector space. The groups we will focus on here, namely $U(1)$ and $SU(3)$, are often defined as follows. We say that $U(1)$ is the set of unitary 1×1 matrices, and that $SU(3)$ is the set of special unitary 3×3 matrices. For our purposes, however, it will be useful to split hairs: thinking of the group elements as these 3×3 or 1×1 matrices is already to think of the group *in a representation.* Indeed, sometimes these are called the 'defining' representations. But there can be many other representations of these groups, and differences (or a lack thereof) between representations of a given group will prove to have interpretive consequences for our understanding of charge properties associated with $U(1)$ and $SU(3)$ gauge theories. Better, then, to think of the group itself abstractly and not as any given set of matrices.

The group $SU(3)$, and its various $SU(n)$ cousins (for $n \geq 2$), fall within the class of simple Lie groups. The theory of representations for simple Lie groups is a well-developed area of mathematics. See, for example, Hall (2015) and Cahn (2014) for mathematical presentations. For more on

Lie groups in their application to physics, see Hermann (1966), von Steinkirch (2011), Georgi and Jagannathan (1982), Cornwell (1997), and Baez and Huerta (2010).

The group $U(1)$, in contrast, is not a simple group (in the technical sense of the class of simple groups, that is; its group structure and representations are highly simplistic). The elaborate theory of classifying group representations for simple groups does not apply to the group $U(1)$. Instead, its representations can be characterized directly. $U(1)$ is the circle group, the set of numbers in the complex plane with unit modulus (see figure 2.4). We may write elements of this group as $e^{i\theta}$ for $\theta \in [0, 2\pi]$. Its complex irreducible representations are all of the form

$$\rho_n(e^{i\theta}) = e^{in\theta}, \tag{2.1}$$

where n is an integer. Because this group is Abelian, each of these representations ρ_n are one-dimensional. This result follows from Schur's Lemma. First, however, we need the notion of an intertwining map between two representations.

Definition 2.4.1. *Let G be a Lie group. Let ρ be a representation of G acting on a vector space V, and let $\tilde{\rho}$ be a representation of G acting on a vector space W. A linear map $\phi : V \to W$ is called an intertwining map of the representations if,*

$$\phi(\rho(g)v) = \tilde{\rho}(g)\phi(v) \tag{2.2}$$

for all g in G and all v in V.

Theorem 2.4.2. *(Schur's Lemma) Let ρ be an irreducible complex representation of a group or Lie algebra and let $\phi : V \to V$ be an intertwining map of V with itself. Then $\phi = cI$, for some $c \in \mathbb{C}$. [Hall (2015) Theorem 4.29]*

Corollary 2.4.3. *Let ρ be an irreducible complex representation of a matrix Lie group G. If g is in*

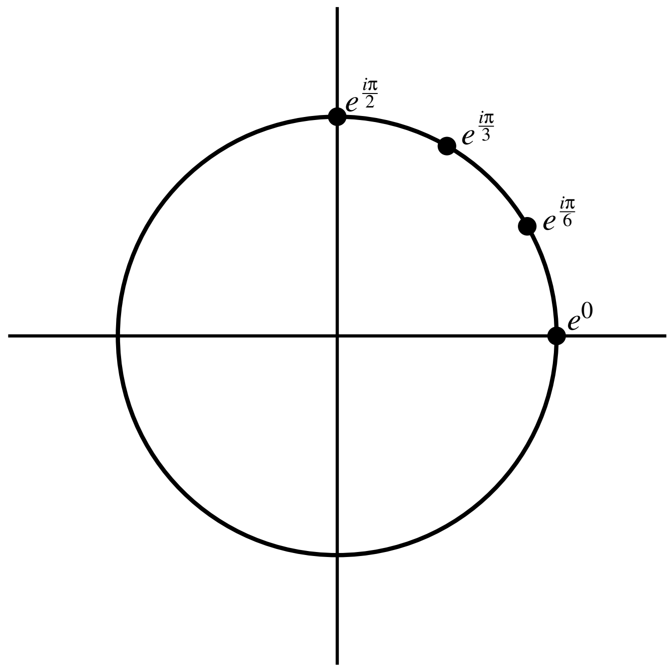

Figure 2.4: The circle group $U(1)$. For group elements z and w, their product zw under the group action is given by addition of the angles, e.g., $e^{\frac{i\pi}{2}} = (e^{\frac{i\pi}{3}} \cdot e^{\frac{i\pi}{6}}) = e^{(\frac{i\pi}{3}+\frac{i\pi}{6})}$. This action commutes.

the center (i.e., the set of all x in G such that x commutes with every member of the group) of G, then $\rho(g) = cI$, *for some* $c \in \mathbb{C}$.

Proof. Consider a group element g in the center of G. Then for all h in G we have,

$$\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g), \tag{2.3}$$

which shows that $\rho(g)$ is an intertwining map of the space with itself. Invoking Schur's Lemma as stated above, we have that $\rho(g) = cI$, for some $c \in \mathbb{C}$. □

Proposition 2.4.4. *Any irreducible representation* ρ *of* $U(1)$ *takes values in* $U(1)$.

Proof. Any representation of $U(1)$ will be of the form $\rho : U(1) \to GL(n, \mathbb{C})$. First, we prove that $n = 1$. Since $U(1)$ is Abelian, its *center* (i.e. the set of all x in G such that x commutes with every member of the group) is the entire group. By Schur's Lemma, $\rho(x)$ is a multiple of the identity for each $x \in U(1)$. Since, then, the group action cannot do anything beside re-scale, we have that *every* subspace of the carrier space V is invariant. However, we have supposed that ρ is irreducible, which means that there are *no* nontrivial invariant subspaces. The only way that V can meet these two constraints on its invariant subspaces is if V is one-dimensional.

So we have that any complex irreducible representation of $U(1)$ will be of the form $\rho : U(1) \to GL(1, \mathbb{C})$. Finally, since $U(1)$ is compact, each of its finite dimensional representations must be unitary (See Hall (2015) theorem 4.28). So any irreducible representation of $U(1)$ must be $\rho : U(1) \to U(1)$. □

This result significantly limits the space of complex irreducible representations of $U(1)$. Each non-trivial representation 'stretches' or 'compresses' the way in which the group elements are placed on this circle. The trivial representation ρ_0 maps every point of the circle to the identity element. The differences between these representations of $U(1)$ are rather slight. In contrast, the differences between representation of $SU(3)$ can be significant. This greater range between representations of $SU(3)$ leads to metaphysical differences between color charge and electric charge.

To see the range of differences between representations of $SU(3)$, we need to develop some foundational results from the representation theory of $SU(n)$. The ways in which this representation theory is implemented for both $SU(2)$ and $SU(3)$ is sketched in Chapter 4, and it is more fully presented in Chapter 3. For any simply connected Lie group G (such as $SU(n)$), there is a one-to-one correspondence between the group's representations and the representations of its Lie algebra $\mathfrak{g}$ (see Hall (2015) proposition 4.4). Using the Lie algebra is convenient because, as the generators of the group, it gives an infinitesimal description of the group. I will therefore focus attention on representations for Lie algebras, keeping in mind that these have unique partner representations of the group.

In thinking about representations of a Lie algebra $\mathfrak{g}$, it is helpful to have in mind the abstract characterization of a Lie algebra independent of any particular Lie group.

Definition 2.4.5. *An abstract Lie algebra is a vector space* $\mathfrak{g}$ *together with a Lie bracket* $[\![\cdot,\cdot]\!]$:

$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, *which is bilinear, skew symmetric, and satisfies the Jacobi identity.*

Any Lie algebra of a Lie group, understood either as T_eG or as the space of left-invariant vector fields on G as defined above, fulfills this definition. The defining characteristic of a representation of a Lie algebra is that it preserves the structure of the Lie bracket.

Definition 2.4.6. *Let* $(\mathfrak{g}, [\![\cdot,\cdot]\!])$ *be a Lie algebra. A (linear) representation of this Lie algebra is a map* $\rho : \mathfrak{g} \to End(V)$ *for* V*, some finite-dimensional vector space, such that, for all* X, Y *in* $\mathfrak{g}$*,* $\rho([\![X,Y]\!]) = [\rho(X), \rho(Y)] \equiv (\rho(X) \circ \rho(Y)) - (\rho(Y) \circ \rho(X))$.

To classify and distinguish representations of a Lie algebra, we use its *Cartan subalgebra*. In a matrix representation, a Lie algebra's Cartan subalgebra is the algebra of all diagonal matrices. For example, the Lie algebra $\mathfrak{su}(2)$ is spanned by the Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

Its Cartan subalgebra is spanned by σ_3. Similarly, the Lie algebra $\mathfrak{su}(3)$ can be represented by the set of 3×3 traceless Hermitian matrices, which is spanned by the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

In this representation, the Cartan subalgebra is spanned by λ_3 and λ_8.

More precisely, we define the Cartan subalgebra using the adjoint map.

Definition 2.4.7. *For all* X *in* $\mathfrak{g}$*, we define the adjoint map w.r.t.* X*,* $ad(X) : \mathfrak{g} \to \mathfrak{g}$ *as* $ad(X)(\cdot) := [\![X,\cdot]\!]$.

Definition 2.4.8. *Given a Lie algebra* $(\mathfrak{g}, [\![\cdot,\cdot]\!])$ *of dimension* d*, the Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is the maximal subalgebra of* $\mathfrak{g}$ *such that there exists a basis* $H_1, \ldots, H_m$ *of* $\mathfrak{h}$ *that can be extended to a basis* $H_1, \ldots, H_m, E_1, \ldots, E_{d-m}$ *of* $\mathfrak{g}$ *such that the* $E_1, \ldots, E_{d-m}$ *are eigenvectors for any* $ad(H)$ *where* H *in* $\mathfrak{h}$*. That is,*

$$ad(H)E_i = c_i(H)E_i, \tag{2.5}$$

where $c_i(H) \in \mathbb{C}$ *and* $i = 1, \ldots, d-m$*.*

In this definition, we make use of the adjoint map to determine the relevant action of $\mathfrak{h}$ on all of $\mathfrak{g}$. Of course, other members of $\mathfrak{g}$ not in $\mathfrak{h}$ can also act on $\mathfrak{g}$ in this way. This gives the adjoint representation of the Lie algebra $\mathfrak{g}$.

Definition 2.4.9. *The adjoint representation of a Lie algebra* $\mathfrak{g}$ *is given by* $\rho_{adj} : \mathfrak{g} \to End(\mathfrak{g})$ *defined by* $a \mapsto [\![a,\cdot]\!]$.

Note that a distinctive feature of the adjoint representation is that the Lie algebra $\mathfrak{g}$ is its own carrier space.

In any representation $\rho : \mathfrak{g} \to End(V)$, the simultaneous eigenvalues of $\rho(H_1)$, $\rho(H_2)$, …, $\rho(H_m)$ are called the *weights* of the representation. In equation 2.5, the $c_i(H)$ give a partial list of the weights of the adjoint representation. The list can be filled out by adding zero-valued weights for each basis element $H_1, \ldots, H_m$. The weights of these Cartan elements are zero because the Cartan subalgebra is, by definition, commutative.

The non-zero weights of the adjoint representation have important work to do in classifying other representations, and so we give them a special name. They are called *roots*. Let's continue with the concrete example of the Gell-Mann matrices. Of the six roots, we can select two of them, denoted $\alpha_i = (c_i(\lambda_3), c_i(\lambda_8))$ and $\alpha_j = (c_j(\lambda_3), c_j(\lambda_8))$, to be the *positive simple roots*. Any two roots α_i and α_j such that the other roots can be written as linear combinations of α_i and α_j with integer coefficients can serve as the set of positive simple roots.

We use this designated set of positive simple roots to define a partial order on weights of the representations of $\mathfrak{g}$. Given two weights μ_1 and μ_2 of some representation, we say that μ_1 is *higher* than μ_2 if their difference can be written in the form

$$\mu_1 - \mu_2 = a\alpha_i + b\alpha_j \tag{2.6}$$

with $a \geq 0$ and $b \geq 0$.

Using this ordering, it can be shown that every irreducible representation of $\mathfrak{su}(3)$ has a unique highest weight (see Hall (2015) theorem 6.7). This classification of representations in terms of their highest weight will be particularly useful in Chapter 3. In that chapter, we will be primarily concerned with understanding the relationship between three different representations of $SU(3)$. One of these representations is used for quark colors, another for anti-quark anti-colors, and the third is the adjoint representation, which is used for gluon colors and for the conserved Noether charge. From the weights of these representations, we will be able to grasp several things. We will see how to distinguish them one from another; we will see a clear sense in which colors and anti-colors are opposites; and I will develop an interpretation of the adjoint representation states as a union of colors and anti-colors.

Chapter 3

Noether's Theorem in non-Abelian Gauge Theories

3.1 Introduction

Emmy Noether's celebrated 1918 theorem gives a correspondence between the continuous symmetries of a physical system and the conserved quantities of that system (Noether (1918)). It is well-known that this theorem provides, when applied in the appropriate contexts, proofs of the conservation of energy, linear and angular momentum, and charge. More specifically, for example, the $U(1)$ gauge symmetry in classical electromagnetism is shown, by way of this theorem, to imply conservation of electric charge. The generalization of this application to the conservation of new kinds of charge in non-Abelian gauge theories, such as color charge for chromodynamics, is mathematically straightforward. However, the physical interpretation of the quantity which is conserved is not straightforward. This aim of this chapter is to clarify the physical interpretation of Noether's theorem for non-Abelian gauge theories. We find that the non-Abelian case exhibits the foundational role of the relationship between charge and anti-charge. This relationship between charge and anti-charge is obscured in the more familiar Abelian case of electrodynamics.

To make the discussion concrete, we will focus on (a classical version of) chromodynamics, the non-Abelian gauge theory of the strong interaction between quarks, anti-quarks, and gluons. In this theory, the charge property is color charge. Philosophers and physicists alike introduce color charge as analogous to electric charge, and as three colors, usually called 'red', 'green', and 'blue.'[1] Here are some examples of how color charge is introduced.

> Quantum chromodynamics is a gauge theory with the gauge symmetry group $SU(3)$ which refers to the color of quarks (i.e., the electric charge is replaced by the so-called color charge). Zeidler (2011) (p. 894)

> Color has two facets in particle physics. One is as a three-valued charge degree of freedom, analogous to electric charge as a degree of freedom in electromagnetism. The other is as a gauge symmetry, analogous to the $U(1)$ gauge theory of electromagnetism.... The union of the two contains the essential ingredients of Quantum Chromodynamics, QCD. Greenberg (2009) (p. 109)

> Take, for example, chromodynamics, the theory of the force that binds quarks together. The easiest way to begin to describe the theory employs the language of universals: there are three color 'charges' ('red', 'blue', and 'green'), which are analogous to, e.g. positive and negative electric charge. There is a force produced between colored particles, like the electric force. Maudlin (2007) (p. 94)

Quarks are the sort of particles that can have these three different colors. Moreover, quarks are the leptons in the theory, which means that they are the chromodynamic analogues of electrons (and their heavier cousins, the muons and tauons): as electrons are in electrodynamics, quarks are in chromodynamics. This analogy suggests that Noether's theorem in chromodynamics would provide conservation laws for *red* charge, for *blue* charge, and for *green* charge.

However, this is not what Noether's theorem yields in chromodynamics. In general, the conserved quantity given by this theorem takes values in the Lie algebra of the symmetry group. In chromodynamics, the relevant Lie algebra is eight-dimensional. So the conserved quantity cannot be

[1] These colors are related to the macroscopic colors of electromagnetic radiation only by way of analogy: just as white light is a combination of all the different colors of light, so too are certain 'white' states of neutral color charge a combination of 'red,' 'blue,' and 'green' color states.

valued in the three-dimensional space of the three basic color states. If not these three basic colors, what is the appropriate physical interpretation of this eight-dimensional Noether charge?

The answer is that Noetherian color charge is the same sort of color charge carried by gluons, namely, it is a combination of both color and anti-color. The goal of this chapter is to develop this interpretation explicitly. In so doing, several points of connection between the pure mathematics of Lie group representations and the applications of these representations to particle physics are laid out systematically. There are a number of excellent resources for the applications of group theory to particle physics, such as Cahn (2014), Georgi (1999), Baez and Huerta (2010), Lichtenberg (1978), and more recently in Woit (2017). However, the specific interpretive challenges of Lie algebra valued quantities, such as in the case of color charge, are not addressed in these resources. This chapter serves to fill in this gap in the literature concerning applications of group representations to particle physics.

Additionally, with this interpretation of the Lie algebra as combinations of color and anti-color clearly laid out, we come to see the philosophical significance of charge conservation in the case of non-Abelian gauge theories. In this class of theories, the foundational role of this relation between ordinary charge and anti-charge is made manifest. In non-Abelian theories, it becomes clear that Noether's theorem ensures not simply the conservation of ordinary charge on its own. Instead, Noether's theorem gives a conservation law for the union of charge with anti-charge.

The remainder of this chapter is structured as follows. Section 3.2 reviews the standard account of Noetherian conservation of electric charge in scalar electrodynamics, and it presents the generalization of this account to non-Abelian scalar field theories. Section 3.3 develops the group representation theory of $SU(3)$ necessary for understanding the sense in which Noether charge for chromodynamics is a combination of both color and anti-color. The further philosophical significance of this is discussed in section 3.4. Concluding remarks are given in section 3.5.

For a recent mathematical treatment of Noether's theorems and generalizations thereof, see Sardanashvily (2016). See also Kosmann-Schwarzbach (2011) for an authoritative historical account of Noether's work. For more on the history and philosophy of Noether's work, see Brading (2002), Brading and Brown (2003), and Brading and Brown (2000).

3.2 Noether's Theorem in Scalar Field Theories

In this section, we first briefly review the standard account of Noether's theorem as applied in classical scalar electrodynamics following the classic treatments in such places as Ryder (1996) and Brading and Brown (2000). We then consider the straightforward generalization of this account to classical scalar chromodynamics. Throughout, we restrict our attention to scalar matter for simplicity of presentation.

In classical electrodynamics, the conservation of electric charge is derived by applying Noether's theorem to the following Lagrangian. (The subscript E on the Lagrangian reminds us that we are considering electrodynamics.)

$$L_E = D_a\phi(D^a\phi)^* - m\phi\phi^*, \tag{3.1}$$

where ϕ is a scalar field of mass m carrying non-zero electric charge. D_μ is the covariant derivative,

$$D_a = \partial_a + iqA_a, \tag{3.2}$$

where A_a is the vector potential. The real number q has a double role: it is both the coupling constant and the fundamental unit of electric charge.

The charge-current density associated with this Lagrangian is

$$j^a = iq(\phi^* D^a\phi - \phi(D^a\phi)^*). \tag{3.3}$$

Noether's first theorem shows that the $U(1)$ symmetry of this Lagrangian implies that

$$\partial_a j^a = 0. \tag{3.4}$$

This is the so-called continuity equation, and it is the key result that leads to the conservation of electric charge. One defines the total charge Q as

$$Q := \int j^0 d^3x. \tag{3.5}$$

From the continuity of j^a given in equation (3.4), it follows that the time derivative of Q is zero. This is how Noether's theorem implies the conservation of total electric charge.

More generally, we define the total current J_α on the total space of the principal bundle $U(1) \to P \stackrel{\wp}{\to} M$. With a choice of section $\sigma : M \to P$, we can pull back the current along this section, resulting in a local representation J_a on spacetime. The current J_a is further more conserved in the sense that it is divergence-free. Because $U(1)$ is Abelian, this resulting current J_a on spacetime is independent of the choice of section, and thus the current in electromagnetism is gauge independent.

There are a number of subtleties in how this account of charge conservation generalizes to non-Abelian gauge theories. In particular, while a current for a non-Abelian theory defined on the total space of the bundle can still be pulled back along a choice of local section to the base space, the resulting local representation of the current depends upon the choice of section, making the current a gauge dependent quantity. This gauge dependence poses the usual interpretive challenges concerning the physical significance of a quantity that changes with arbitrary choices of gauge. But prior to pulling back the current along a choice of section, one is struck by a different interpretive challenge. In the case of non-Abelian gauge theories, Noether's conserved current is manifestly Lie algebra valued. In the Abelian case of electrodynamics, the current is also Lie algebra valued, but this is easily overlooked. This is because the Lie algebra of $U(1)$ is simply $\mathbb{R}$. We are accustomed to interpreting real valued quantities such as electric charge and mass without needed to give any special attention to the structure of Lie algebras. But in the non-Abelian case, special attention is necessary.

In the specific case of chromodynamics, the relevant symmetry group of $SU(3)$ has an eight-dimensional Lie algebra, denoted $\mathfrak{su}(3)$. To reflect the fact that the current is valued in the Lie algebra, we denoted it as $J^{\mathfrak{A}}_{\alpha}$ (for the definition of $J^{\mathfrak{A}}_{\alpha}$, see chapter 5 section 5.3).

This is surprising. We usually think of color charge in terms of a three-dimensional space of *red*, *blue*, and *green*. There is no obvious way of relating these three basic colors to the eight-dimensional Lie algebra. Further, we might have reasonably expected that the application of Noether's theorem in chromodynamics would give us a clear sense in which *red*, *blue*, and *green* are conserved. But instead Noether's theorem tells us that the conserved color current $J^{\mathfrak{A}}_{\alpha}$ associated with the $SU(3)$ symmetry is eight-dimensional. What, then, is the relationship between the conserved current $J^{\mathfrak{A}}_{\alpha}$ and the three colors of *red*, *blue*, and *green*?

To get some traction with this question, we need to appreciate the crucial role of different group representations in non-Abelian gauge theories. Matter fields ϕ are valued in the carrier space of the appropriate representation of the group for that kind of matter. For instance, quarks and anti-quarks, as different kinds of particles, transform according to different representations of $SU(3)$. Furthermore, Lie algebra valued quantities, such as $J^{\mathfrak{A}}_{\alpha}$, transform according to the adjoint representation of the group.

The sense in which these distinct group representations are hardwired into the equations of a non-Abelian gauge theory is even made evident in the fiber bundle formulation of these gauge theories. In that context, the gauge field is a connection on a principal fiber bundle, and all such connections are Lie algebra valued. Moreover, matter fields are sections of associated vector bundles. The typical fiber of an associated bundle is a carrier space for an irreducible representation of the principal bundle's Lie group. For more on this construction, see Kobayashi and Nomizu (1969), Bleecker (2013), Weatherall (2016a), Hamilton (2017), and references therein, or refer back to chapter 2.

The answer to our question lies in understanding the relationship between, on the one hand, the the role of the Lie algebra in the adjoint representation and, on the other hand, the description of the basic colors *red*, *blue*, and *green* in the fundamental representations of $SU(3)$. The next section fleshes out this suggestion by explicitly constructing the $SU(3)$ adjoint representation from the two representations fundamental used for quark colors and anti-quark anti-colors.

3.3 Representation Theory for $SU(n)$

Our target is to understand the relationship between the Lie algebra valued Noether current $J_a^{\mathfrak{A}}$ and three basic colors *red*, *blue*, and *green* of quarks and the corresponding anti-colors of anti-quarks. The key to understanding this relationship lies in showing how three distinct irreducible representations of $SU(3)$ relate to each other. These three irreducible representations are the two fundamental representations (used for quark and anti-quark color states) and the adjoint representation (used for gluon color states). This section presents the key definitions and results from Lie group theory necessary for understanding the interpretive significance of the Lie algebra in the context of Noether's theorem for non-Abelian gauge theories. Readers familiar with group representations may proceed to section 3.4.

3.3.1 Representations of $SU(2)$ and spin

It is instructive to first review the standard application of the classification of representations in the case of spin and $SU(2)$. This case exemplifies the basic framework for distinguishing between representations, and for interpreting the physical properties associated with various aspects of the representations. In physics, we often use terms such as 'doublet' and 'triplet' to indicate a set of spin states transforming according to two and three-dimensional representations of $SU(2)$. Thus spin-$\frac{1}{2}$ particles, such as electrons, transform according to $SU(2)$ doublets,

$$|\uparrow\rangle = +\frac{1}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = -\frac{1}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.6}$$

whereas spin-1 particles, such as photons, are described by an $SU(2)$ triplet of states,

$$1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad -1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.7}$$

These sets of states transform according to two different representations of the same group. While the group $SU(2)$ is often defined as the group of 2×2 special unitary matrices, thinking of the group this way is, strictly speaking, to already think of it *in a representation*. Indeed, writing out elements of the group as special unitary matrices is sometimes called the “defining” representation of the group. There is a more abstract characterization of the set of group elements which gives us the sense in which the 2×2 matrices which act on the states in equation (3.6) can implement the same symmetries as the 3×3 matrices which act on the states in equation (3.7).

So far, this is enough to illustrate the two main ingredients to any (matrix) representation of a group. We need a vector space V (to be interpreted as a space of physical states), and an appropriate mapping $\rho : G \to \mathsf{GL}(V)$ of the abstract group elements into matrices which act on that vector space. This mapping must preserve the relationships between the abstract group elements, so that the same relationships are held between the representative matrices. The space V on which the represented group elements act is called the *carrier space*. (For more formal definitions, refer back to Chapter 2.)

For any Lie group, such as $SU(2)$, it is often advantageous for many purposes to work with its associated Lie algebra, $\mathfrak{su}(2)$. These Lie algebra elements are also called "generators" of the group. This is because elements of the group are generated by exponentiation of the Lie algebra elements. Indeed, one way of defining the Lie algebra of a Lie group is that it consists of all matrices X such that e^{tX} is an element of the group, for all real numbers t (see, for example, Hall (2015) §3.3).[2,3] For any simply connected Lie group G (such as $SU(n)$), there is a one-to-one correspondence between the group's representations, and the representations of its Lie algebra $\mathfrak{g}$ (see Hall (2015) proposition 4.4). Thus, by working with representations of the more convenient algebra, we also gain understanding of group representations.

Just as for the Lie group $SU(2)$, the Lie algebra $\mathfrak{su}(2)$ has both a defining representation as the set 2×2 traceless Hermitian matrices, and a more abstract characterization. In the defining representation, $\mathfrak{su}(2)$ is spanned by the Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.8}$$

This is an arbitrary choice of basis for $\mathfrak{su}(2)$. However, for any choice of basis, only one such basis matrix can be diagonalized at a time. In any basis, the diagonalized matrix (here given by σ_3) has a special role to play in distinguishing representations of the group.

More abstractly, the key properties of $\mathfrak{su}(2)$ as a Lie algebra are captured by the commutation relations

$$[\sigma_i, \sigma_j] = \varepsilon_{ij}^{k} \sigma_k, \tag{3.9}$$

where ε_{ijk} is totally anti-symmetric. The components of ε_{ijk} are called the *structure constants* of $\mathfrak{su}(2)$. Any set of matrices which obey these commutation relations (together with an appropriate vector space) can serve as a representation for $\mathfrak{su}(2)$. For instance, in the triplet representation, the

[2] Many physics books use an alternative convention such that group elements correspond to e^{itX} for real numbers t.

[3] There are several alternative, equivalent, definitions of Lie algebras. From a more geometric point of view, one can consider the set of left-invariant vector fields on the Lie group understood as a manifold. These left-invariant vector fields are another way of defining the Lie algebra. Refer back to Chapter 2.

Pauli matrices may be written as

$$\rho(\sigma_1) = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(\sigma_2) = \frac{\sqrt{2}i}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

In this context, we use the group $SU(2)$ to describe the rotational symmetry of spin properties which may be aligned in either the x, y, or z directions. For spin properties aligned in each of these three directions, we use eigenvectors of the Pauli matrices,

$$J_x = \frac{1}{2}\sigma_1, \quad J_y = \frac{1}{2}\sigma_2, \quad J_z = \frac{1}{2}\sigma_3, \tag{3.11}$$

where the factors of $\frac{1}{2}$ are conventional. The eigenvalues of J_z give the conventional numerical values for labeling spin states. For instance, in a doublet, the z-spin up state is labeled by $+\frac{1}{2}$, and the z-spin down state is labeled by $-\frac{1}{2}$. That is, the eigenvalues of the diagonalized matrix in the chosen basis for the Lie algebra serve as labels for distinct determinate states (z-spin up or z-spin down) of the general property (spin-$\frac{1}{2}$) under study. These eigenvalues of the diagonzalized Lie algebra element are called *weights*.

Finally, complex sums of the two non-diagonal operators are used to define *raising* and *lowering* operators,

$$J_+ = (J_x + iJ_y), \quad J_- = (J_x - iJ_y). \tag{3.12}$$

In the doublet representation, the action of J_+ on a spin down state raises it to a spin up state, and similarly, the action of J_- on a spin up state lowers it to a spin down state. Analogous raising and lowering operators are available in the triplet representation as well by taking complex sums of the non-diagonal elements in equation (3.10).

Much of the physical interpretation of these representations can be usefully summarized by dia-

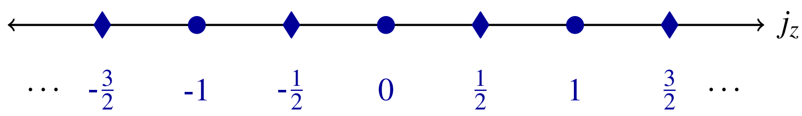

Figure 3.1: Weight diagram from $SU(2)$.

gramming the weights of the various spin states. We call such a diagram a *weight diagram*. Certain subsets of this diagram (e.g. $\{-\frac{1}{2}, \frac{1}{2}\}$) correspond to possible spin states of various classes of particles. The highest weight in such a set labels the corresponding class of particles. In this way the highest doublet weight $+\frac{1}{2}$ labels spin-$\frac{1}{2}$ particles, and the highest triplet weight $+1$ labels spin-1 particles.

For $SU(2)$, the ordering on the space of weights which determines which weight within a set is 'highest' is obvious. This will not be the case for weights in $SU(3)$. Additionally, in $SU(2)$, no two inequivalent irreducible representations have the same dimension. It is therefore not misleading to use terminology such as "doublet" and "triplet" to name distinct representations. That is, for $SU(2)$ the number of dimensions of an irreducible representation suffices to distinguish it from all other irreducible representations. This, also, is not the case in $SU(3)$.

To summarize, the interpretation of spin in $SU(2)$ that have clear analogues in $SU(3)$ consists of the following three key steps. Points in the weight space label distinct spin states for particles of various kinds of spin. Different classes of particles correspond to different representations of the group. These different representations are distinguished by their highest weight. These key lessons of the weight diagram for spin generalize to the case of color charge.

3.3.2 Representations of $SU(3)$ and color charge

In its defining representation, elements of $SU(3)$ are written as 3×3 special unitary matrices. In this representation, the associated Lie algebra $\mathfrak{su}(3)$ is given by the set of 3×3 traceless Hermitian matrices, which is spanned by the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

By convention, and in analogy to Pauli spin matrices, we adjust each matrix by a factor of $\frac{1}{2}$ to redefine these Lie algebra elements as

$$T_i = \frac{1}{2}\lambda_i, \tag{3.13}$$

for $i = 1$ to $i = 8$. For ease of reference later, this gives,

$$T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_8 = \frac{\sqrt{3}}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.14}$$

Just as before, by writing out these elements as explicit 3×3 matrices, we are in a sense already thinking of the Lie algebra *in a representation*. More abstractly, the Lie algebra is spanned by a set of eight ordered elements T_a that obey the following commutation relations,

$$[T_a, T_b] = i f_{ab}^{c} T_c, \tag{3.15}$$

where the real numbers f^{abc} are the structure constants. They are the generalization of ε_{ijk} in equation (3.9). The constants f^{abc} are completely antisymmetric. A minimal, defining number of non-zero constants are listed in equation (3.16). The remaining non-zero constants can be determined by antisymmetry, and all others are zero. Any set of matrices, of any size, which obey these same commutation relations, can be used to represent $\mathfrak{su}(3)$.

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \quad f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}. \tag{3.16}$$

In this defining representation, these Gell-Mann matrices can act upon a three-dimensional carrier space. Basis vectors for the carrier space of this defining representation are used to describe the three basic quark colors, *red*, *blue*, and *green*.

$$r = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ \mathbf{1} \\ 0 \end{pmatrix}, \ g = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1} \end{pmatrix}. \tag{3.17}$$

In this sense, these three basic colors are analogous to the properties of z-spin up and z-spin down.

As noted above, in $\mathfrak{su}(2)$ only one of the Pauli matrices can be diagonalized at a time. But here in $\mathfrak{su}(3)$, both T_3 and T_8 are diagonal. For any Lie algebra in general, the subspace of simultaneously diagonalizable matrices is called the *Cartan subalgebra*. The diagonal matrices T_3 and T_8 (and their counterparts in other representations) which span the $\mathfrak{su}(3)$ Cartan subalgebra have the same special role to play in classifying and distinguishing all of the irreducible representations of $SU(3)$ as did J_z in $SU(2)$. Each basis vector of the carrier space of the representation is labeled by the pair of its simultaneous eigenvalues (t_3, t_8) of these diagonal matrices. A partial order (given below) is placed on these pairs of simultaneous eigenvalues. Finally, the highest weight according to this partial order distinguishes between inequivalent representations.

Returning, now, to this defining representation of $SU(3)$, direct and simple calculation gives the weights for each state. We label the b color state as $(-1, \frac{\sqrt{3}}{6})$, the green color state by $(0, -\frac{1}{\sqrt{3}})$, and the red color state by $(1, \frac{\sqrt{3}}{6})$. These are the weights for the first fundamental representation of $SU(3)$. They are plotted in figure 3.2 (a).

Next, we need a way of determining the highest weight. In general, one first designates a subset of weights, called *positive simple roots*, which can serve as a sort of basis for all of the other weights. These positive simple roots are use to define a partial ordering on the space of weights.[4] Fortunately, the fine details of this process are unnecessary for our purposes of looking at just three significant representations of $SU(3)$. To this end, it suffices to set the ordering according to the first entry. Under this ordering, $(1, \frac{\sqrt{3}}{6}) \succ (0, -\frac{1}{\sqrt{3}}) \succ (-1, \frac{\sqrt{3}}{6})$. Thus the *red* state has the highest weight, which will serve as a suitable label to distinguish this representation for color states from other representations of $SU(3)$.

In further analogy with the case of spin, we have at our disposal a notion of raising and lowering operators which move states of a higher/lower weight to states of a lower/higher weight. As with the Pauli spin matrices, these are given by complex sums of the remaining non-Cartan Gell-Mann matrices:

$$T_{\pm} = (T_1 \pm iT_2), \;\; U_{\pm} = (T_6 \mp iT_7), \;\; V_{\pm} = (T_4 \pm iT_5). \tag{3.18}$$

Explicitly, in the defining representation, this gives:

$$T_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$T_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

One may easily verify that $T_{\pm}$ raises and lowers between *red* and *blue* states, whereas $U_{\pm}$ raises and lowers between *blue* and *green* states, and $V_{\pm}$ raises and lowers between *green* and *red*.

In general for $SU(n)$, there are $(1-n)$-many fundamental representations, so called because all other irreducible representations of $SU(n)$ may be systematically constructed from these fundamental representations. Thus, $SU(3)$ has two fundamental representations. What we have been

[4]See Hall (2015) §6.1 - 6.3 for more on weights and roots.

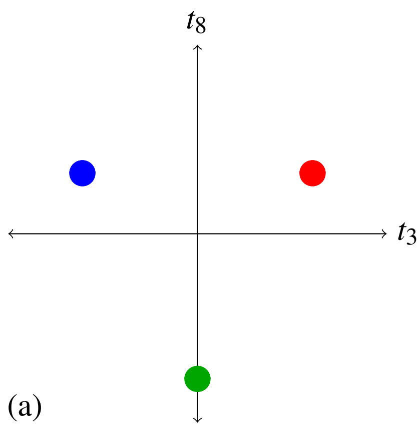

Figure 3.2: (a) Weight space for the quark color states given by a basis for the carrier space of the first fundamental representation of $SU(3)$. (b) Weight space for the anti-color states of anti-quarks given by a basis for the carrier space of the second fundamental representation of $SU(3)$.

calling the defining representation is one of them, and it is used for quark color charge states. The other is used to describe anti-color states. This second fundamental representation of $SU(3)$ is dual to the first. Consequently the carrier space for this representation is $\mathbb{C}^{3*}$, the dual space of $\mathbb{C}^3$. Following Hall (2015) §6.5, we use the following basis for $\mathbb{C}^{3*}$,

$$\bar{r} = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \end{pmatrix}, \quad \bar{b} = -\begin{pmatrix} 0 \\ \mathbf{1} \\ 0 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1} \end{pmatrix}. \tag{3.19}$$

We further exploit the analogy with the colors of ordinary light using complementary hues to denote anti-colors: thus anti-red is red's compliment, cyan, and similarly anti-blue is yellow, and finally anti-green is magenta.

Next we need to define the action of the Lie algebra elements on $\mathbb{C}^{3*}$ to complete the definition of this dual representation. This action is given by $\overline{Z} = -\rho(Z)^{tr}$ for all $Z \in \mathfrak{su}(3)$. Consequently,

$$\overline{T}_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \overline{T}_8 = \frac{\sqrt{3}}{6}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{3.20}$$

The simultaneous eigenvalues of $\overline{T}_3$ and $\overline{T}_8$ give the weights for this second fundamental representation. Each weight designates a different anti-color state. These are plotted in 3.2 (b). Notice the way in which the weight diagram captures a sense of a relation of 'opposites' between the color and anti-color states. A color label by (t_3, t_8) has a corresponding anti-color labeled by $(-t_3, -t_8)$.

In this second fundamental representation we again have instances of (or better yet 'representatives' of) the raising and lowering operators. Thus, for example,

$$\overline{T}_- = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \overline{U}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \overline{V}_- = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.21}$$

and one easily verifies that $\overline{T}_-$ lowers $\bar{b}$ states into $\bar{r}$ states, while $\overline{U}_-$ lowers $\bar{b}$ into $\bar{g}$, and $\overline{V}_-$ lowers $\bar{g}$ into -$\bar{r}$. The weight $(\frac{1}{2}, -\frac{\sqrt{3}}{6})$ of the $\bar{b}$ state is the highest weight of this representation.

So far we have distinguished between the two fundamental representations of $SU(3)$ and identified their carrier spaces with the spaces of color and anti-color states of quarks and anti-quarks, respectively. We now turn to the adjoint representation, recalling our aim of showing the relationship between this adjoint representation (and the Noether charge which transforms according to it) and the two fundamental representations.

As we have already seen, a key feature of a representation is its carrier space. The carrier space may be any vector space. The adjoint representation uses the Lie algebra itself as its underlying carrier space. In this case, then, the Lie algebra's role becomes twofold: it is both the generators for the group elements, and it is the carrier space on which those group elements act. This is why the adjoint representation of $SU(3)$ is necessarily eight-dimensional.

To determine the weights of the adjoint representation, we need the simultaneous eigenvalues of the Cartan subalgebra. Using the ordered basis T_+, T_-, T_3, U_+, U_-, V_+, V_-, and T_8, the Cartan subalgebra of $\mathfrak{su}(3)$ in the adjoint representation is

$$\rho_{adj}(T_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{adj}(T_8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.22}$$

Plotting their simultaneous eigenvalues gives the weights for the elements of the Lie algebra in its role as the carrier space of the adjoint representation (the black dots in figure 3.3 (a)).

We may now compare the weights for each of our three representations. For ease of comparison the weights of the two fundamental representations are included in figure 3.3 (a). *This diagrammatic comparison of weights among the three representations is the first step in answering our question concerning the physical interpretation of the Noether current* $J_a^{\mathfrak{A}}$. The comparison shows that each non-zero weight of the adjoint representation is precisely the vector sum of a weight from the first fundamental representation and a weight from the second fundamental representation. This suggests an interpretation of each state of the adjoint representation as precisely those combinations of color and anti-color given by the appropriate vector sum. This is noted in figure 3.3 (b).

While suggestive, this diagrammatic explanation of the relationship between our three representations is insufficient in two ways. First, it needs to be clarified how it is that these vector sums are not mere coincidence: *why* is it that adding *red* with *anti-blue* results in an adjoint-representation state? We need a more principled reason to identify elements of $\mathfrak{su}(3)$ with combinations of color and anti-color. Second, there are two weight (0, 0) states of the adjoint representation whose interpretation is underdetermined by the diagram alone. To remedy these two insufficiencies, we turn in

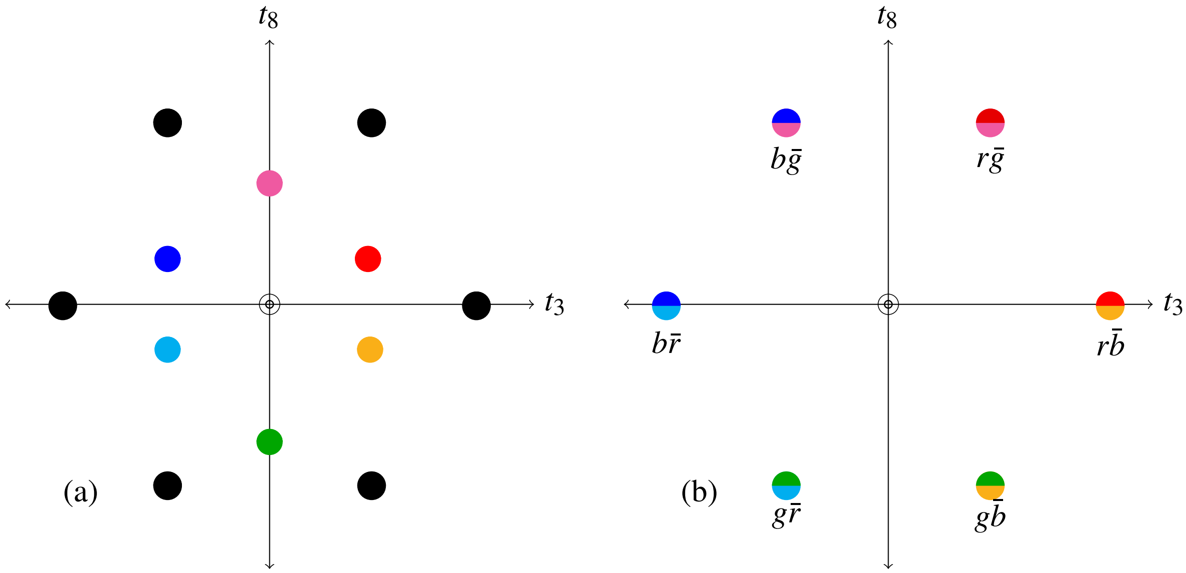

Figure 3.3: (a) Color states labeled in the weight space of first the fundamental representation (*red*, *blue*, and *green*), the second fundamental representation (*magenta*, *yellow*, and *cyan*), and of the adjoint representation (*black*) of $SU(3)$. (b) Initial interpretation of adjoint states as combinations of color and anti-color.

the next section to the formal construction of the adjoint representation out of the two fundamental representations.

3.4 Construction of the Adjoint Representation

In the previous section, we saw that, after first determining the weights of the adjoint representation, it turned out that these weights could be associated with pairs of weights from the two fundamental representations. In this section, we reverse directions. We start instead with pairs of states from the two fundamental representations, and we ultimately arrive at a principled way of identifying these pairs with elements of $\mathfrak{su}(3)$. The process begins by building a basis for the carrier space for the adjoint representation, and it then proceeds by defining the action of the group on that space. Next, we identify the state of highest weight. Finally, successive applications of the lowering operators to this state produce all the remaining states of the adjoint representation.

To build a basis for the carrier space for the adjoint representation, we start with pairs of states from the two fundamental representations, understood as elements of the space $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$. The group acts on this space by

$$\rho_{adj}(Z) = Z \otimes I + I \otimes \overline{Z}. \tag{3.23}$$

For example, $\rho_{adj}(T_-)(r \otimes \bar{g}) = b \otimes \bar{g}$.

It can be shown that, in a higher-dimensional representation built out of lower-dimensional representations, the state of the higher-dimensional representation with the highest weight is the state built from the tensor product of the states of highest weight from the lower dimensional representations (see Hall (2015) prop. 6. 17). Thus, the state of highest weight in the adjoint representation is $r \otimes \bar{b}$, since r is the highest weight state of the first fundamental representation and $\bar{b}$ is the highest weight state of the second fundamental representation. Successive application of the lowering operators to this $r \otimes \bar{b}$ state produce each of the other basis elements of the adjoint representation. It suffices to consider just two lowering operators since $[U_-, V_-] = T_-$.

We can summarize the results of this process in the diagram given in figure 5.1. To save space, we omit the tensor product symbol and simply write the states as, e.g. $r\bar{b}$. Arrows to the left indicate the action of U_- on the previous state, and arrows to the right indicate the action of V_-. From the diagram, we see that the resulting eight states are: $r\bar{b}$, $r\bar{g}$, $g\bar{b}$, $g\bar{g} - r\bar{r}$, $b\bar{b} + g\bar{g}$, $b\bar{g}$, -$g\bar{r}$, and -$b\bar{r}$.

At this stage we have eight linearly independent combinations of color and anti-color. We now need a principled way of associating each of these eight states with elements of $\mathfrak{su}(3)$. To do this we need to say more about how $\mathfrak{su}(3)$ can act on itself. In general, a Lie algebra acts on itself via the adjoint map: $Ad_Z(\cdot) = [Z, \cdot]$ for all Lie-algebra elements Z. This adjoint action corresponds to those foundational commutation relations, such as equation (3.9) for $\mathfrak{su}(2)$ and equation (3.15) for $SU(3)$. These commutation relations for $\mathfrak{su}(3)$ are recorded in table 3.1.

Our ordering on the weights continues to be determined by the first entry t_3 in a weight (t_3, t_8). Inspecting the T_3 row of table 3.1, we find that T_+ is the operator that corresponds to the state

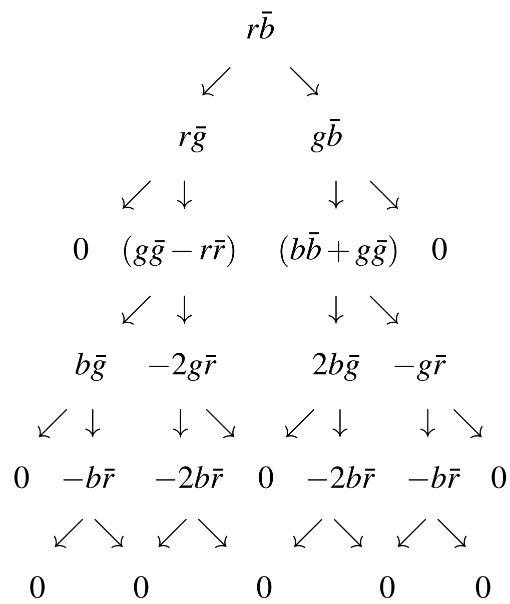

Figure 3.4: Construction of the adjoint representation in color/anti-color states.

Table 3.1: Commutation Relations for $\mathfrak{su}(3)$. The label on the row gives the first entry in commutator, and the column label gives the second entry. For example, $[T_+, T_-] = 2T_3$

	T_+	T_-	T_3	U_+	U_-	V_+	V_-	T_8
T_+	0	$2T_3$	$-T_+$	0	V_+	0	$-U_+$	0
T_-	$-2T_3$	0	T_-	$-V_-$	0	U_-	0	0
T_3	T_+	$-T_-$	0	$\frac{1}{2}U_+$	$-\frac{1}{2}U_-$	$\frac{1}{2}V_+$	$-\frac{1}{2}V_-$	0
U_+	0	V_-	$-\frac{1}{2}U_+$	0	$T_3 - \frac{3}{2}T_8$	$-T_+$	0	U_+
U_-	$-V_+$	0	$\frac{1}{2}U_-$	$\frac{3}{2}T_8 - T_3$	0	0	T_-	$-U_-$
V_+	0	$-U_-$	$-\frac{1}{2}V_+$	T_+	0	0	$T_3 + \frac{3}{2}Y$	$-V_+$
V_-	U_+	0	$\frac{1}{2}V_-$	0	$-T_-$	$-T_3 - \frac{3}{2}T_8$	0	V_-
T_8	0	0	0	$-U_+$	U_-	V_+	$-V_-$	0

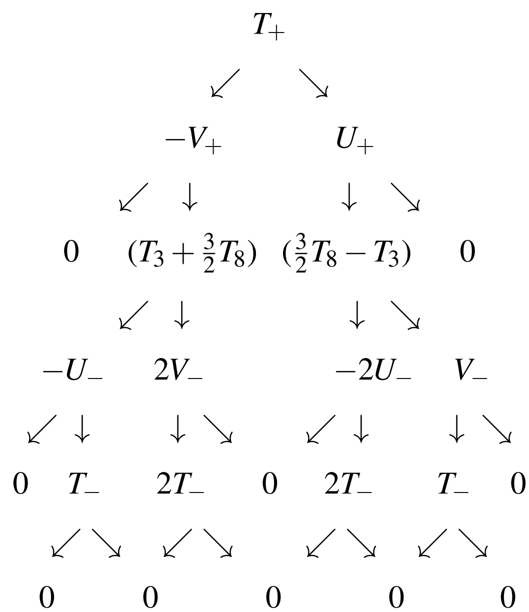

Figure 3.5: Construction of the adjoint representation in operators.

of highest weight. But we have already seen that the state of the highest weight for the adjoint representation is $r\bar{b}$. Thus, the raising operator T_+ corresponds to the $r\bar{b}$ state in the adjoint representation. Successive application of the lowering operators on T_+ in the adjoint representation will produce the remaining states, as recorded in figure 3.5. Again, arrows to the left indicate the action of $\rho_{adj}(U_-)$ on the previous state, and arrows to the right indicate the action of $\rho_{adj}(V_-)$. For example, $\rho_{adj}(U_-)(T_+) = [U_-, T_+] = -V_+$.

We are now in position to remedy the two insufficiencies of the diagramatic comparison of representation given in the previous section. Using the operations summarized in figures 5.1 and 3.5, we now have a principled way of associating a unique combination of color and anti-color with each element of $\mathfrak{su}(3)$. These combinations are recorded for the six raising and lowering operators in table 3.2. Furthermore, we can also now associate specific combinations of color and anti-color states with the two weight $(0,0)$ states of the adjoint representation, which correspond to the Cartan subalgebra elements T_3 and T_8. A simple calculation shows that $T_3 = \frac{1}{2}(-r\bar{r} - b\bar{b})$ and

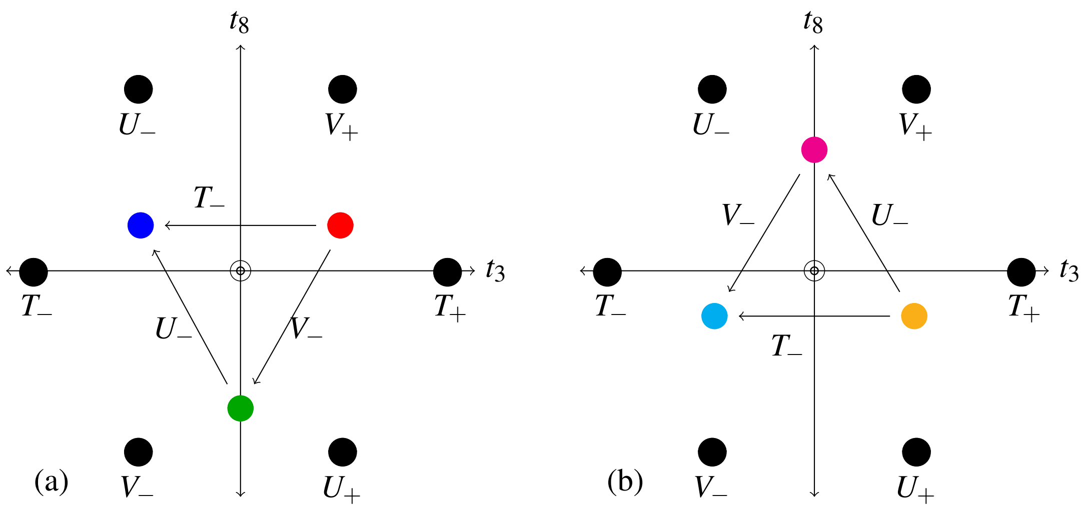

Figure 3.6: Action of lowering operators in (a) the first fundamental representation and (b) the second fundamental representation.

$T_8 = \frac{1}{3}(2g\bar{g} - r\bar{r} + b\bar{b})$. Any element of the Cartan subalgebra will have weight $(0,0)$, and so will be 'white' in that it has some combination of $r\bar{r}$, $b\bar{b}$, and $g\bar{g}$. But individual elements of the Cartan subalgebra can be distinguished from each other by the specific amounts of $r\bar{r}$, $b\bar{b}$, and $g\bar{g}$ present in that state.

Notice, however, there is no element corresponding to $(r\bar{r} + b\bar{b} + g\bar{g})$ within $\mathfrak{su}(3)$. This particular combination is not a state of the adjoint representation. Adding this element would turn $\mathfrak{su}(3)$ into $\mathfrak{u}(3)$, the Lie algebra for the group $U(3)$. This additional generator would be written as the identity matrix. It is physically significant that this additional generator is not allowed. If it were, it would allow for a symmetry transformation such that a quark in the *red* state could be transformed into a state with any multiple of *red*. It is therefore fitting that we use $\mathfrak{su}(3)$ and not $\mathfrak{u}(3)$ so that symmetry transformations cannot be used to create or destroy color charges in this way. This observation provides one type of explanation for why it is that Noetherian color current, as well as gluon color states, have eight dimensions rather than nine. A ninth dimension would lead to the generation of unphysical symmetry transformations.

Table 3.2: Physical interpretation of $\mathfrak{su}(3)$ raising and lowering operators

Lowering Operator	Color/Anti-color	Raising Operator	Color/Anti-color
T_-	-$b\bar{r}$	T_+	$r\bar{b}$
U_-	-$b\bar{g}$	U_+	$g\bar{b}$
V_-	-$g\bar{r}$	V_+	-$r\bar{g}$

3.5 Conclusion

We have seen how to interpret basis elements of the Lie algebra $\mathfrak{su}(3)$ in terms of the interpretation given for the basis elements of carrier spaces for the two fundamental representations of $SU(3)$. This explains the relationship between Noether's color current $J_a^{\mathfrak{A}}$—which takes values in $\mathfrak{su}(3)$—and the properties *red*, *green*, and *blue* given in the defining representation of $SU(3)$. $J_a^{\mathfrak{A}}$ is interpreted as combinations of these three basic colors and their anti-color counterparts.

In this way, the conservation of color charge involves a complex process of accounting for both color and anti-color—more complex, that is, than the conservation of electric charge. To account for a conserved amount of electric charge in some physical process, it suffices to simply add the initial positive and negative charges and ensure that this is equivalent to the sum of the final positive and negative charges. This process of adding electric charges is straightforward since we use real numbers. But for the conservation of color charge, the analogous sense of 'adding' colors and anti-colors relies upon the far less straightforward relationships between the two fundamental representations and the adjoint representation of $SU(3)$.

For example, a quark of a particular color state is transformed into a different color state in such a way that each unit of color and each unit of anti-color is accounted for at each stage in the process. The same is true for transformations of anti-quarks. Suppose an anti-quark in the $\bar{g}$ state is transformed into an $\bar{b}$ state via a $\overline{U}_+$ transformation. The $\overline{U}_+$ transformation is understood as the combination $g\bar{b}$, thereby exchanging the initial $\bar{g}$ of the anti-quark for the final $\bar{b}$ of the anti-quark by means of its unit of ordinary g color charge.

Thus Noether's theorem in chromodynamics does not imply the conservation of the colors of ordinary matter—those *red*, *green*, and *blue* properties which we so readily think of as *the* color charges—on their own. Rather, the conserved quantity is a union of charge and anti-charge. This foundational role of anti-charge, in concert with ordinary charge, is obscured in the case of electrodynamics. In that case, the relevant group is the Abelian group $U(1)$, which is the set of numbers in the complex plane with unit modulus, $e^{i\theta}$ for $\theta \in [0, 2\pi]$. Its complex irreducible representations are all of the form

$$\rho_n(e^{i\theta}) = e^{in\theta} \tag{3.24}$$

where n is an integer. The value of n labeling the representation gives the amount of electric charge had by particles which transform according to that representation. Thus electrons transform in ρ_{-1}, neutrinos in ρ_0, positrons in ρ_1, etc. This labeling of the irreducible representations is the $U(1)$ analog of the weights used to distinguish representations in $SU(n)$.

The ρ_n and the ρ_{-n} representations are dual to each other, just as the two fundamental representations of $SU(3)$ are dual to each other. Thus, particles in the ρ_{-n} representation carry anti-charge relative to the particles in the ρ_n representation. The conservation of electric charge is, of course, understood as net of positive and negative charges in the system. Consequently, conservation of electric charge is also a law regarding a union of charge with anti-charge.

And yet the nature of this union in the electric case is far simpler than in the color case. For color charge, the union of colors with anti-colors in the adjoint representation resulted in a novel space of eight states. For electric charge, the union of charge with anti-charge is garden variety addition. Noether's theorem in non-Abelian gauge theories reveals that the general notion of charge conservation is not simply the conservation of ordinary charge: rather, it is the conservation of a union of ordinary charge with anti-charge.

Chapter 4

Could Charge and Mass be Universals?

4.1 Introduction

There is a tradition in contemporary metaphysics of looking to particle physics to inform our understanding of fundamental ontology. For instance, Armstrong endorses an "*a posteriori* or scientific realism" about properties and relations, where the properties of particle physics are taken to at least approximate the true ontology of the world.[1] In this vein, Lewis lists mass, electric charge, and the colors and flavors of quarks as his prime examples of "perfectly natural" properties.[2] The metaphysician following Armstrong and Lewis in this tradition invokes physics to warrant her realism about a small, special class properties that can serve as a basis for her metaphysical theory.

Maudlin (2007) argues against this approach to traditional metaphysics on the grounds that the most general notion of a universal is precluded by the mathematical structure of fundamental phys-

[1] "If we combine an *a posteriori* or scientific realism about properties (and relations) with the speculative but attractive thesis of physicalism, then we shall look to physics, the most mature science of all, for *our best predicates so far*. Physics. . . shows promise of giving an explanatory account of the workings of the whole space-time realm, and thus, perhaps, the whole of being. And it shows promise of doing this in terms of a quite restricted range of fundamental properties and relations." (Armstrong (1997), p. 167.)

[2] See Lewis (1997) p. 179. Lewis uses this exact list numerous times. See also Lewis (1997) p. 186 (footnote 6) and Lewis (1986) p. 14, 67 (footnote 47), and 178.

ical theories. While Lewis and Armstrong look to fundamental physics to provide an accurate list of universals, Maudlin thinks that fundamental physics instead provides good reason to think that there are no such properties. So we have the following main question.

The Main Question: could properties like mass, electric charge, color charge, etc. be universals?

Maudlin thinks that his argument settles the question in the negative. His argument proceeds by taking the color charge of quarks—one property off of Lewis's list—as his primary example. Maudlin intends for his argument to generalize from the one case of color charge to the other properties from Lewis's list, concluding that there are *no* universals. He concludes that, because he has "eliminated" all of the properties from Lewis's list, "a wholesale revision" of that traditional picture of universals is in order (p. 102).

If successful, Maudlin's total elimination of universals would be a landmark development in metaphysics. For centuries, various metaphysical theories have relied upon, in one version or another, the idea that at least some things in the world can be understood in themselves, without reference to any other things, in terms of participating in, or instantiating, or exemplifying, etc. certain universals. A theory of universals is usually employed to account for objective similarity between two things. The details of how this works are the subjects of numerous metaphysical debates, but the broad structure of universals has remarkably wide reach in the field of metaphysics. Maudlin aims to bring the authority of mathematical physics in force against all such theories of universals. As Baker (2010) puts it, Maudlin's position "dictates a surprisingly revisionist ontology" (p 1165). Demarest (2015) shows how challenges for traditional accounts of universals from physics, such as Maudlin's, have further implications for our understanding of physical laws. For these reasons, Maudlin's argument against universals is of considerable interest for many issues in both metaphysics and philosophy of science.

Maudlin's argument against universals is presented as a part of a more specific project concerned with Lewis's full metaphysical picture of Humean supervenience. For the purposes of this paper, however, I will focus on Maudlin's claim that charge and mass cannot possibly be universals. Maudlin's argument relies crucially on the assumption that his analysis of color charge carries over intact to the cases of both electric charge and mass. The chief aim of this paper is to show precisely where this assumption goes wrong. In so doing, I develop a novel analysis of charge properties, showing that charge properties generally have a three-fold conceptual structure that is degenerate in the special case of electric charge.

My criticism of Maudlin should not be read as a defense of either Lewis or Armstrong, and certainly not as a defense of Humean supervenience. For Lewis and Armstrong, every property from particle physics is thought to be fundamental in some univocal way. Yet, as described below, there are physically important senses in which electric charge is not at all like color charge, and yet both are clearly good candidates for being fundamental properties, in the sense that they play certain roles within fundamental physics. There is more than one way to be a fundamental property. In this sense, Lewis, Armstrong, and Maudlin all make the mistake of expecting that color charge, electric charge, and mass share the exact same metaphysical character.

The remainder of the paper is structured as follows. Section 4.2 reviews the traditional metaphysical approach to fundamental properties as presented by Lewis and Armstrong, and it then clarifies Maudlin's argument against it. The import of this section is that we arrive at Maudlin's criterion for settling the Main Question. In section 4.3 I argue that Maudlin's argument against the existence of universals is successful on only one notion of color charge. There are at least two other physically operative notions of color charge *qua* property that a particle might have, on which Maudlin's argument fails. In section 4.4, I show that Maudlin's argument does not generalize to the properties of electric charge and mass. Concluding remarks are given in section 4.5.

4.2 Maudlin's Argument

Maudlin's argument is aimed against a large swath of metaphysical theories of fundamental ontology, including trope theory, Lewis's theory of sparse properties, theories of primitive naturalness, and all variants of universals. The argument is meant to be so general that even the radical differences between Aristotelian and Platonic universals are irrelevant (Maudlin (2007) p. 80, note 1). Maudlin chooses to take the theory of universals as representative. In concluding that there are no universals, he means to also conclude that there are no tropes, no primitively natural sets, etc. I will here follow Maudlin in using the term 'universal' as representative of this larger collection of notions of fundamental properties.

Since Maudlin wants to think of fundamental properties at such a general level, it is difficult to see what exactly it would take for a given property to be a universal. But, at the very least, any good universal should 'carve nature at the joints', an image that originates with Plato. In the *Phaedrus*, Socrates says that it is a principle of good discourse that divisions of species be made "according to the natural formation, where the joint is, not breaking any part as a bad carver might" (265e). Lewis and Armstrong both make allusions to this image. Lewis says this work of carving nature at the joints is done by what he calls "sparse"[3] properties, which he also calls "perfectly natural" properties.

> Sharing of [sparse properties] makes for qualitative similarity, they carve at the joints, they are intrinsic, they are highly specific, the set of their instances are *ipso facto* not entirely miscellaneous, there are only just enough of them to characterize things completely and without redundancy. (Lewis (1997) p. 178)[4]

[3]Sparse, that is, in contrast to the 'abundant' properties. A set of things sharing any given abundant property may be as miscellaneous and arbitrary as you please. For example, the union of the set of *things that are in my house* with the set of *penguins in Antarctica with spots on their left wings* shares one of these abundant properties. Exercising a little imagination to find other such arbitrary unions quickly shows that such properties truly are abundant. See Lewis (1986) p. 59 ff.

[4]Similarly, Armstrong writes, "And here, I think, we are led on to Plato's marvelous image of carving the beast (the great beast of reality) at the joints. The carving may be more or less precise, so we reach predicates that are of greater and greater theoretical value, predicates more and more fit to appear in the formulations of an exact science." (Armstrong (1997), p. 166)

Moreover, both Lewis and Armstrong take it that fundamental physics is the best place to look for those properties that will delineate the natural joints. Lewis says that "physics has undertaken...an inventory of the *sparse* properties of this-worldly things" (Lewis (1997), p. 178). Armstrong also takes the mature sciences to be the best place to look for the most fundamental properties:

> How do we determine what these ontological properties are?… With difficulty… But in the present age we take ourselves to have advanced… and to have sciences that we speak of as 'mature'. There we will find the predicates that constitute our most educated guess about what are the true properties and relations. (Armstrong (1997) p. 166-167)

According to this view, it is the mature science of physics that shows the most promise of providing the metaphysician with an accurate short list of fundamental properties—properties like mass, electric charge, and color charge.

Maudlin argues that, given the mathematical formalism of contemporary fundamental physics, it is not possible to interpret these properties from physics as universals. The key for Maudlin's argument is a criterion that he calls "metaphysical purity." If, for a given property, it can be shown that it is not metaphysically pure, then the property cannot be a universal. He gives the following necessary condition for metaphysical purity.

> [I]f a relation is metaphysically pure, then it is at least *possible* that the relation be instantiated in a world in which only the relata of the relation exist… if [this condition] fails, then the condition for the holding of the relation must make implicit reference to items other than the relata, so the relation is not just a matter of how the relata directly stand to each other. (p. 86)

Similarly, a metaphysically pure property is such that it is at least possible that it be instantiated in a world inhabited by only one thing with said property.

For example, *having the same electric charge* and *having the same mass* are candidates for metaphysically pure internal relations. Imagine a world that contains only two electrons. Lewis and

Armstrong would want to say that those two electrons instantiate both of these relations, and that they do so in virtue of sharing the metaphysically pure properties *having electric charge -1* and *having mass .51 MeV*.[5] These are the sorts of properties that are supposed to 'carve' at the natural joints. Maudlin intends to show that, in light of how these properties are treated mathematically in fundamental physics, none of them can be instantiated in a metaphysically pure way. This gives us the following criterion for settling the Main Question.

Criterion: If a given property is not metaphysically pure, then it is not a universal.

So why is it, according to Maudlin, that the properties on Lewis's list do not meet this criterion of metaphysical purity? His argument takes the form of a reductio: he first argues that there are no metaphysically pure intrinsic *relations*; consequently, there are no metaphysically pure intrinsic *properties*, for, if there were any, they would be sufficient to determine the would-be intrinsic relations.[6]

The argument draws upon the fiber bundle formalism used in gauge theories such as quantum electrodynamics (QED) and quantum chromodynamics (QCD).Maudlin sketches the intuitive ideas of the fiber bundle formalism without presenting the technical details. I'll rehearse his intuitive example here. You are asked to imagine two different arrows situated at two different points on a sphere. The arrows would each seem to have the property of *pointing in a certain direction*. (The claim will be that properties in physics are just like directions.) And given those properties, the two arrows should stand in one of two relations: either they point in the same direction (they are parallel) or they point in different directions (they are not parallel). The trouble is that neither of these relations (being parallel or not) is metaphysically pure. Without a connection—a mathematical object in addition to the two arrows—there is no fact of the matter as to whether or not the two arrows are parallel.

[5] There are, to be sure, substantive metaphysical debates about just *how* this sharing of a fundamental property works, with the three most prominent contenders being trope theory, primitive naturalness, and universals. Maudlin's argument is intended to refute all of these at a very general level.

[6] His argument against the existence of metaphysically pure external relations is outside the scope of this paper.

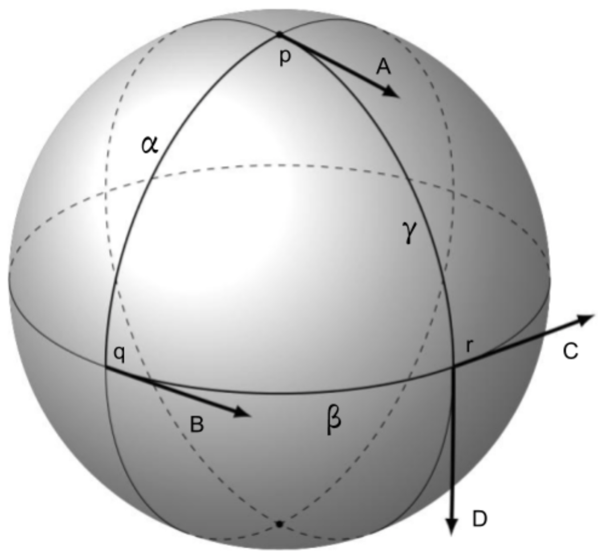

Figure 4.1: Parallel transport on a sphere.

Why is it that we need a connection? Imagine moving one of the arrows over to the other so that you can check to see whether or not they are parallel. In general, the answer turns out to be different depending upon which path you take. Compare vectors A and D tangent to the sphere depicted in figure 4.1. If we move vector A from point p to point r along path γ, then the result will be parallel to vector D. If instead we move vector A from point p to point q along path α, and from there along path β to point r, then the result will be parallel to vector C. So we have to relativize the property of *being parallel*: relative to one path A and D are parallel, but relative to another path they are not.

The connection is part of the mathematics necessary for defining the transport of one vector along a path. At each point on the sphere, there is a space of all possible tangent vectors at that point. So, for example, at point r in figure 4.1, vectors D and C are both members of this tangent space, pointing in different directions. The collection of all these tangent spaces connected to the sphere defines a specific kind of fiber bundle, called a 'tangent bundle.' Each tangent space is one 'fiber,' and collecting all of them makes the 'bundle.' The tangent spaces at different points are all isomorphic, but there is no canonical isomorphism between them. We cannot simply compare the vector spaces at points p and r and determine which vector at r is the same as a given vector at p. Consequently, in order to compare vectors A and D, we have to 'move' A into D's tangent space, while keeping A, in a sense, constant. The connection gives the relevant standard of constancy by defining which vectors in the neighboring tangent spaces count as 'parallel to' vector A. Moving a vector along a path while keeping the vector constant as defined by the connection is called parallel transport.[7]

If the connection on a given bundle is curved, as is the case of the sphere, then parallel transport as defined by that connection is path-dependent. But if the connection is flat, as in Euclidean space, then parallel transport is path-independent. Note carefully, however, that Maudlin's position is *not* that spaces with flat connections can give rise to metaphysically pure relations of *being parallel*, while only those spaces with curvature pose a threat to metaphysical purity. As he puts it,

> In a perfectly flat Euclidean space, the result is the same no matter which path is taken, but even in this case the *metaphysics* of parallelism is not that of a metaphysically pure relation: two distant arrows are only parallel in virtue of the affine connection along paths which connect their locations, even if the result of the parallel transport will be the same along any path. (p. 92)

Maudlin's concern is not just that the result of comparing two arrows might be dependent upon the path taken to bring the two arrows together. Even in cases where the result is the same no matter which path is taken, the comparison still cannot be made in the absence of a connection. The notion of metaphysical purity is operative: in a world with just the two arrows—and so *without* a connection—it is not possible to instantiate the relation of *being parallel*. So *being parallel* is not a metaphysically pure relation.

This point about the necessity of a connection for comparing directions is supposed to bear upon the question of universals, because, Maudlin says, "[g]auge theories apply exactly the sort of structures that we have used to explicate comparison of directions to other sorts of fundamental physical

[7] For more technical details, see, for example, Kobayashi and Nomizu (1969) or Hamilton (2017).

‘properties’ or ‘magnitudes’ ” (p. 94). The idea is that attributing a fundamental property to a fundamental particle is just like attributing a specific direction to an arrow: comparisons of fundamental properties cannot be done in the absence of a connection any more than comparisons of directions can be done in the absence of a connection. Since the relation *having the same property* cannot be instantiated in a world without a connection, that relation is not metaphysically pure.

To make his argument, Maudlin takes up the case of quark colors as a concrete and illustrative example, intending for this argument to generalize to such cases as mass, electric charge, and any other would-be universals. *Prima facie*, it would seem that QCD “ employs the language of universals: there are three color ‘charges’ (‘red’, ‘blue’, and ‘green’)” but, since chromodynamics is a gauge theory formulated using fiber bundles, “at base color charge is treated completely analogously to directions” (p. 94). Consequently, comparisons of color require a connection:

> [E]ach point in the base space has a space of possible color states associated with it, but we have no means of comparing the states at *different* points with each other. In order to do this, we need to add something more, something which, intuitively, ties together the fiber at every point with the fibers of points that are infinitesimally close. This something more we need is a *connection* on the fiber bundle. . . And just as for directions, the results of the comparison will in general depend upon the particular path chosen: there is no path independent fact about whether vectors in different fibers are ‘the same’ or ‘different’. (p. 95-6)

Again, we have the threat of path-relativity, and it may seem that the path-relativity of quark colors is the reason that they cannot be universals. But, in footnote 8, Maudlin clarifies that this is not his chief concern:

> If the connection were (in the appropriate sense) flat, the result of transporting a vector might be the same no matter which path is chosen. But this sort of path *invariance* should not fool us into thinking that the comparison is metaphysically path *independent*: comparisons can only be made if there are paths connecting the points.

Even if the results of a comparison are the same for any path, Maudlin still thinks that the comparisons are not metaphysically pure. Imagine a world with just two quarks in it.[8] Is there any fact of the matter as to whether or not they have the same color, or the same electric charge? On Maudlin's account there is not, because such a world *lacks* a connection.[9] We must first add a connection to that world, and only then can we parallel transport the vector representing the property of the one quark along the chosen path to reach the vector representing the property of the second quark, and then compare the two. Maudlin concludes,

> If we adopt the metaphysics of the fiber bundle to represent chromodynamics, then we must reject the notion that quark color is a universal, or that there are color tropes which can be duplicates... So it seems that there are no color properties and no metaphysically pure internal relations between quarks. And if one believes that fundamental physics is the place to look for the truth about universals (or tropes or natural sets), then one may find that physics is telling us there are no such things. (p. 96)

Maudlin clearly intends for this argument to generalize from the case of color charge to other properties such as electric charge and mass

> Since metaphysicians like Armstrong have focused on examples like electric charge and mass in explicating the theory of universals, eliminating them requires a wholesale revision of that picture of universals (p. 102).

In summary, Maudlin's argument is this:

[8]This world is not actually physically possible (given QCD), since quarks always come in bound states of three quarks, or in a bound state with an anti-quark. But for the sake of argument, we may pretend that it is.

[9]There is an ambiguity here: is the worry over metaphysical purity that the *path* is metaphysically 'something more', or that the *connection* is metaphysically 'something more'? In footnote 8 Maudlin seems to think that it is the path. But surely positing a world to begin with means to posit a spacetime, and spacetimes come with paths. So why should we think that the existence of a continuous *path* between the two points counts as an extra *thing* in the world? Perhaps what is really going on is this: since the base space of the fiber bundle is spacetime, that gives us a continuous path through spacetime connecting the two points in the base space, above which 'arrows' hang up in the total space. But the comparison we really want to make is between those two arrows. And without a connection, we have no way of bringing the arrow at one point in the total space over to the other arrow at another point in the total space, even though there are continuous paths down in the base space. That is, without a connection, we cannot utilize the extant paths for the purpose of transporting one vector over to the other.

1. All universals must be metaphysically pure.
2. In gauge theories, fundamental property attribution amounts to attributing a specific direction to a vector.
3. Directions are not metaphysically pure.
4. Therefore, there are no universals.

There are a number of things one might say in response to this argument. For one thing, we might want a story about why the spacetime points at which the arrows are located do not count as extra things that exist over and above the arrows, yet the paths between the points and the connection both seem to count as extra things that exist.[10] How do we pick out which parts of the mathematical formalism are ontologically relevant? (See Hirsch (2017) for a response to Maudlin focusing on this question.) Additionally, one might want clarification as to precisely what metaphysical purity amounts to. At first pass, it may seem that metaphysically pure relations are what are usually called internal relations, that is, they supervene on the intrinsic natures of their relata. But that cannot be right, since Maudlin considers the possibility of metaphysically pure *external* relations as well as internal relations.

What is clear is Maudlin's test for metaphysical purity. In order to test some property for metaphysical purity, we first posit a world, mathematically represented by a fiber bundle, inhabited with just two things. We represent properties of these things by vectors at points in the fiber bundle. We then compare the vectors to see whether or not the two things both instantiate the property in question. If we can settle this question *without* positing any additional things in the world (such as

[10]There is a case to be made for the metaphysical significance of the connection, since it is used to represent the gauge field, but it is much harder to see why a path within the manifold should count as an extra thing that exists in the world. To be sure, the *math* requires that such paths exist within the fiber bundle we are using, but it is not clear that this means that the path through the manifold corresponds to some physical object in the world. It is especially puzzling that Maudlin is willing to ascribe this level of metaphysical weight to the mathematical paths, given his expressed concern to warn against "illegitimately projecting the structure of our language onto the world," thereby "mistak[ing] grammatical form for ontological structure" (p. 79). If, following Maudlin, fiber bundles are the appropriate mathematical language for gauge theories, then the fact that the formalism of that language requires the existence of a certain mathematical object—and so in this context, a *linguistic* object—does not necessarily indicate that the theory requires the existence of a corresponding *physical* object in the world. See Muntean (2012) for further discussion.

connections on the bundle, or paths in spacetime), then the property passes the test and remains a candidate universal. But if we cannot settle this question regarding the sameness of the property without additional things in the world, then the property fails the test, and therefore cannot be a universal.

At the end of the chapter, Maudlin revises his strong position, conceding that the fiber bundle formalism does allow for what he calls "universals of pure form":

> Each fiber, though, will retain a sort of *geometrical structure*, as it were, a generic structure that the fibers all share. This structure *is* a universal, and so provides an exception to our overgeneralization ....(p. 98)

> Furthermore, we have not really eliminated all notions of universals or metaphysically pure internal relations. When we constructed the fiber bundles, we began by associating to each point in the base space fibers *with the same geometrical structure*.... [W]e might say that although there are no universals that correspond to *matter* or to *physical magnitudes*, there are geometrical universals of pure form. But since metaphysicians like Armstrong have focused on examples like electric charge and mass in explicating the theory of universals, eliminating these requires a wholesale revision of that picture of universals. (p. 102)

What precisely is Maudlin conceding here?[11] In allowing for "geometrical universals of pure form" while denying that they might correspond to either "matter" or "physical magnitudes," he seems to think that the only way for fiber bundles to accommodate a universal is at a level of abstraction that is physically irrelevant. Moreover, he clearly means to exclude both electric charge and mass from the realm of these universals of pure form. In what follows, I argue that electric charge, and two senses of color charge, are represented in the fiber bundle formalism at precisely this more abstract level of shared structure across fibers. Consequently, these properties are, by Maudlin's own criterion, good candidates for being universals.

[11] Read straightforwardly, he is committed to the view that certain geometrical structures are in fact universals. It is clear that the shared geometrical structure possessed by each fiber passes the metaphysical purity test, but it is unclear what additional sufficient conditions (if any) for universal-hood are satisfied by these geometrical structures.

4.3 Three Levels of Color Charge

It will be useful to first consider the case of color charge in more detail. There are three distinct, physically significant levels of description for color charge, and Maudlin's argument only applies to one of those levels. At the other two levels of description, color charge passes the metaphysical purity test. In order to illustrate the distinction between these three notions, it will be instructive to consider color charge in analogy, not with electric charge as is usually done, but with the property of spin.

Color charge and spin are both treated in particle physics using the representation theory of the Lie groups $SU(N)$: for color $N = 3$ and for spin $N = 2$. This relationship suggests that one can use interpretive principles that seem sensible for spin as a guide for finding analogous interpretive principles for color charge. This is not to say that we should think of color charge in exactly the same way that we think of spin; of course the analogy between the two will break down at some point. But the mathematical similarities make spin a useful starting point for investigating the appropriate interpretation for color charge.

So, what do we mean when we ask, for a given particle, "What is the spin of this particle?" There are several possible meanings. First, we might mean to ask for what *kind* of spin it has, or *in what way* it is a particle with spin. In this case, the answer for all of the leptons (e.g. electrons, muons, quarks), is that these are spin-$\frac{1}{2}$ particles. Meanwhile the answer for photons is that they are spin-1 particles; Δ-baryons are spin-$\frac{3}{2}$ particles; Higgs bosons are spin-0 particles; and so on.

But at other times when we ask for a particle's spin, we mean to ask for a particular particle's spin state, e.g., 'Is *this* electron in the z-spin-up state or the z-spin-down state?' Once we specify the first notion of spin, namely, what kind of spin the particle has, we can determine the number of possible spin states that any particle of that kind of spin might occupy. While the spin-$\frac{1}{2}$ particles can be in one of two z-spin states, the spin-$\frac{3}{2}$ particles can be in one of four different z-spin states. Mathematically, this number of different possible states is given by the dimension of the representation of $SU(2)$ used for particles with that kind of spin.[12] Transforming according to a different representation of $SU(2)$ corresponds to having spin in a different way or of a different kind. Finally, at a still more general level, we distinguish between particles with spin at all and particles without spin. Thus, the spin-0 Higgs is said to be spin*less*, while all those that are not spin-0 are particles with spin.

This gives us three different levels at which we can specify the spin of a given particle: we determine (1) whether the particle is spinless or otherwise has some non-zero spin; if the latter, then we determine (2) in what *way* it has spin, or, to put it another way, what *kind* of spin it has (i.e., spin-1, spin-$\frac{1}{2}$, etc.); and we can ask (3) which of the possible spin states, for a given kind of spin, a particular particle happens to be in.[13]

We have precisely the same three levels of description for color charge. At level 1 we may distinguish between particles with color charge *at all* and particles with *no* color charge. Gluons, quarks, and anti-quarks all have color charge, while the rest of the subatomic world carries no color charge at all. But at level 2 the gluons, quarks, and anti-quarks all carry color charge in different ways, and this is shown mathematically by the fact that they each transform according to different representations of $SU(3)$. The distinct representations used for the quarks and the anti-quarks are both three-dimensional. So at level 3 there are three different possible color states for the quarks (red, blue and green) and three different possible color states for the anti-quarks (anti-red, anti-blue, anti-green). Meanwhile, the representation for the gluons is eight-dimensional, and there are in this sense eight different states for a gluon, corresponding to various combinations of color and

[12] A representation is a group homomorphism from the group G to the general linear group, $\mathsf{GL}(V)$, of some vector space V. The dimension of the representation is the same as the dimension of V, and we refer to V itself as the 'carrier space' for the representation. See Hall (2015) for an introduction to group representations.

[13] Arguably, the distinction between (1) and (2) is rather thin. We might instead collapse the two into one level, and then say that spinless particles *do* have a value for spin–it is just that that value is zero. I don't think much turns on whether we take spinless and colorless to be ways of having spin and color, respectively, or if we maintain spinless and colorless as separate categories from having spin at all and having color charge at all. However, I have chosen the more pedantic route of maintaining this distinction between levels 1 and 2 because it is useful for understanding the metaphysical differences between electric charge and color charge: it is noteworthy that gluons carry color charge at level 1 while photons do not carry electric charge at level 1. Moreover, it is at level 1 where quarks and gluons are said to both carry color charge; at level 2 they have color charge in different ways.

anti-color.

At this stage, it is clear that these three levels of description for spin and color charge have the conceptual structure of *determinables* and *determinates*: level 3 is a determinate of level 2, which in turn in a determinate of level 1, in the same way that the *isosceles triangle* is a determinate of *triangle*, which in turn is a determinate of *polygon*. (See Funkhouser (2006) for a mathematical model of the determinable-determinate relation.) A central question in metaphysics, which we will return to below, considers the status of determinable properties.

For a spin-$\frac{1}{2}$ particle, we use the fundamental representation of $SU(2)$, so called because there is a way to construct other representations from this fundamental representation. The fundamental representation of $SU(2)$ is two-dimensional. A set of basis vectors for the carrier space of this representation is used for the z-spin-up and z-spin-down states, and we usually choose to write them as:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.1}$$

In precisely the same way, the three quark color states correspond to basis vectors of the carrier space for the first fundamental representation of $SU(3)$, whose carrier space is $\mathbb{C}^3$. We can write such a set of basis vectors as:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.2}$$

The color states of the anti-quark are given by basis vectors for the carrier space for the second fundamental representation of $SU(3)$, which is dual to the quark color space. Thus, for both spin and color charge, the third level of description for specific spin or color states is given by a set of linearly independent directions in the carrier space of the relevant group representation.

How do these three levels relate to the fiber bundle formalism of gauge theories? A gauge theory is formulated using a principal bundle, which comes equipped with the structure of a Lie group, such as $SU(3)$. Matter fields are sections of vector bundles associated to the principal bundle, and these associated vector bundles are constructed using a specific representation of the Lie group. For example, the matter field for a quark is a section of an associated bundle built using the first fundamental representation of $SU(3)$. (For details about fiber bundles and their application to particle physics, see Hamilton (2017).)

In light of these distinctions, let's rerun the metaphysical purity test for color charge. Posit a world (fiber bundle) without a connection, and with just two quarks, whose color states are represented by two vectors at different points.[14] Each fiber has the structure of the carrier space for the first fundamental representation of $SU(3)$, but, unless we also have a connection, we do not have a way of identifying elements across fibers. Metaphysically, this means that there is no way of determining whether or not the two color states of the quarks are the same. But we can determine sameness of color charge at levels 1 and 2: since the quarks' color states transform according to the same non-trivial representation of $SU(3)$, they instantiate sameness of color charge at these two levels.

If we run the test for color charge using particles of two different kinds, we can see the physical significance of sameness of color at level 1 and 2. Suppose we want to know if one quark and one gluon instantiate the relation of *sameness of color charge*. As before, the quark's state is a vector in a copy of the three-dimensional carrier space of the first fundamental representation of $SU(3)$. The gluon, in contrast, has a state in the eight-dimensional carrier space of the adjoint representation of $SU(3)$.[15] Suppose now that we wanted to ask the same question of the one quark and the one gluon that we previously asked concerning the two quarks: are the quark and the gluon in the same color state? If we try to answer this question using Maudlin's test of seeing if the two vectors are

[14] It must be acknowledged that there are deep conceptual issues regarding how we can get something like a particle interpretation out of these gauge theories to begin with (see especially Ruetsche (2011) Ch. 9). For the sake of argument, I'll follow Maudlin in glossing over the details, and assume that we can, at a minimum, think of property instantiations as state vectors of some Hilbert space at a point in space-time.

[15] In this case, there will also be a connection in the background, but it does not count as 'something more.' The connection *is* the gluon field, so it is not something more over and above the gluon field.

the same, we cannot make sense of the result. No matter how we might try to transport one vector over to the other, there is no isomorphism between their respective vector spaces. So there is no sense in which the two vectors could possibly be the same vectors.

Nevertheless, we can make sense of this question: are the quark and the gluon both *colored particles*, i.e., do they both have color charge at level 1? Their two color state vectors are both members of carrier spaces for non-trivial representations of $SU(3)$. Consequently, both spaces are such that all of their vectors represent particles with the property *being colored*. We can also make sense of this question: do the quark and the gluon both have the same *kind* of color charge? Here the answer is No, because the fundamental representation is inequivalent to the adjoint representation. The ways in which they have color charge is different. For both of these questions, the relevant mathematical facts are properties of the entire vector space, not individual vectors. Consequently, we do not need the extra structure of a connection, or of a continuous path in space time connecting the points, or any other such 'something more'. This shows that the properties *having color charge at all* and *having color charge in a certain way* meet the necessary condition for being metaphysically pure. So these properties are candidates to be universals.

At the third level of description, Maudlin is correct: these color state properties are treated *exactly* like directions, and a connection is indeed necessary in order to meaningfully compare color states at different points. But not so for the levels 1 and 2. Indeed, levels 1 and 2 are given by the shared group structure of each fiber, and so they correspond to the type of universals which Maudlin dismisses as "pure form."

Recall that, for Maudlin, universals of this kind could not correspond to either "matter" or to "physical magnitudes." But charge at level 2 is precisely how we distinguish between matter and anti-matter: anti-matter transforms according to the representation that is conjugate to the representation for matter, and conjugation takes place at the level of distinguishing between irreducible representations (see Baker and Halvorson (2010)). Charge at level 1 is physically significant as well: it is at this level that we can say that both quarks and gluons carry color charge. The fact that gluons carry color charge is vital to the physics of QCD; indeed it is the hallmark of all non-Abelian gauge theories that the gauge boson carries the relevant charge, and this accounts for many of the radical differences between Abelian and non-Abelian gauge theories. So we should not dismiss charge at levels 1 and 2 as properties of "pure form" without physical significance.

Let us step back for a moment, and recall the main question: could properties such as mass, electric charge, and color charge be universals? The three-fold structure of color charge, and the metaphysical impurity at the bottom level, will have different implications for this question depending upon one's further commitments regarding universals. For those who, like Lewis and Armstrong, expected that these properties would fit with the ideal of sparse properties (maximally determinate, highly specific, and metaphysically pure), it may be surprising and discouraging that the most determinate sense of color charge is metaphysically impure. If one is of the persuasion that only the most basic, in the sense of maximally determinate, properties are of greatest metaphysical significance, then the metaphysical purity of color charge at levels 1 and 2 is of little interest: those determinable properties are simply less important.[16]

However, the larger metaphysical tradition of universals included in the intended scope of Maudlin's argument (such as Platonic and Aristotelian conceptions) is not so wedded to maximal specificity. Aristotle, for his part, says that, "I call universal that which is by its nature predicated of a number of things, and particular that which is not; man, for instance, is a universal, Callias a particular" (*De Interpretatione* 17a37, translation by Barnes (1995)). The universal 'man' is a determinable: its determinates include specifications with respect to height, age, character, etc., as well as indi-

[16]What does this mean for Humean supervenience? At its core, the doctrine of Humean supervenience is that "all there is to the world is a vast mosaic of local matters of particular fact, just one little thing and then another" (Lewis (1987) p. *ix*). Arguably, the fiber bundle formalism *does* allow for a weak version this bare-bones characterization of Humean supervenience, since each fiber at a point in spacetime has its own complete set of vectors. There is no impediment to letting one vector in each fiber represent these local matters of fact: just one little vector and then another. But, as Maudlin makes clear, the challenge of the fiber bundle formalism is that—without the additional structure of the connection—one cannot establish resemblance or similarity between these vectors. Unable to account for similarity, these local matters of fact are of little use to metaphysics. Moreover, a vector in one fiber at a point in spacetime cannot be duplicated at another point in spacetime, since we do not have a way of saying which vector in this new fiber is *the same* as the first vector in the first fiber. That is, level 3 properties cannot be freely recombined, another blow to Lewisian metaphysics (Maudlin (2007) p. 103).

vidual persons. Additionally, ‘man’ is also a determinate of, for instance, ‘animal.’ Thus, on the Aristotelian conception, both determinables and determinates can be universals.

Some recent work in metaphysics indicates that the tides might be turning back to this older and wider conception of universals that can ascribe fundamentality to determinables. For example, Wilson (2012) defends the view that determinables can be fundamental. More broadly, philosophers of science have moved away from thinking of fundamentality in terms of the very small toward thinking of it in terms of comprehensiveness, or wideness of applicability.[17] In this vein, the electromagnetic interaction, for example, is not fundamental on account of the fact that it occurs at small length scales, but because of its wide scope of applicability. And the boundaries of that scope are defined by the fundamental property of electric charge: all and only those particles with electric charge participate in this fundamental interaction. Along similar lines, French (2014) concludes forcefully, “those who exclude determinables from the fundamental base need to show what specificity has to do with fundamentality” (284). Thus, while the metaphysical purity of levels 1 and 2 for color charge is of no use to Humean supervenience, it does accord with a number of other metaphysical views from both the past and the present.[18]

4.4 Why Maudlin’s Argument Does not Generalize

In the previous section, I argued that color charge should be understood at three different levels of description, and that Maudlin’s argument only succeeds with respect to one of those levels. In this section, I turn to addressing whether or not Maudlin’s argument can generalize to the properties of electric charge and mass as he claims. To anticipate: the directional quality of color charge at the third level of description does not carry over to these other properties, and consequently Maudlin’s

[17] This was a prominent theme at a symposium at the 2018 Philosophy of Science Association Biennial Meeting, entitled “What (if anything) is fundamental about physics?” See Carroll (2018), Ladyman (2018), Miller (2018), and Ney (2018).

[18] I am grateful to two anonymous reviewers for helpful comments related to fundamentality, determinables, and universals.

argument does not generalize.

4.4.1 Electric Charge is not like Color Charge

Having seen three levels of description for color charge, one might expect that electric charge would have analogous levels of description, since these properties are simply two different kinds of charge corresponding to two different gauge theories. In principle, much of what we have said about color charge does carry over to the case of electric charge, where the relevant group is $U(1)$: we can again distinguish between the trivial and non-trivial representations, and distinguish the non-trivial representations from each other, etc. However, in this case, the mathematical theory used to characterize the various representations is significantly different from that used to characterize $SU(N)$. These differences can be used to show important ways in which electric charge is not like color charge. In particular, electric charge has no meaningful analog of the third level of description for color charge.

Some initial oberservations about the group $U(1)$ are in order. $U(1)$ is the circle group, the set of numbers in the complex plane with unit modulus. We may write elements of this group as $e^{i\theta}$ for $\theta \in [0, 2\pi]$. Its complex irreducible representations are all of the form $\rho_n(e^{i\theta}) = e^{in\theta}$ where n is an integer. Because this group is Abelian (that is, the result of multiplying two elements does not depend upon the order in which they are written) each of these representations ρ_n are one-dimensional. Every carrier space for a complex, non-trivial, irreducible representation of $U(1)$ is isomorphic to $\mathbb{C}$, the complex numbers.

At the first level we distinguish between those particles with electric charge at all, and those with no electric charge at all using the distinction between trivial and non-trivial representations. The trivial representation of $U(1)$ is ρ_0. Any other ρ_n is non-trivial. At the second level we distinguish between the various non-trivial representations themselves by different non-zero integers labeling the representations.

What about the third level of description? For color charge and spin, the third level describes specific property states picked out by different basis vectors in the carrier space of the representations. But for $U(1)$ the carrier space is one-dimensional, and so it does not have multiple directions in which its vectors could point. Since the carrier spaces are all isomorphic to $\mathbb{C}$, individual unit vectors in these spaces can differ at most by a phase. While the full extent of the physical significance of a phase factor is a matter of some debate, differences of phase have never been taken to correspond to differences of electric charge in any sense. Electrically charged particles cannot be in various different 'electric states' in the way that color-carrying particles can be in various different color charge states. Thus, level 3 simply does not apply in the case of electric charge.

This leaves levels 1 and 2 as the only relevant levels of description for electric charge. Yet, for electric charge, the conceptual distinction between these two levels can be cashed out in the single notion of *net amount* of charge, which is encoded in the integer labels for the representations. *Having electric charge at all* is the same thing as *having a non-zero net amount*, and *having no charge at all* is the same thing as *having net amount zero*. Moreover, we can replace the notion of *having electric charge in different ways* with simply *having different amounts of electric charge*. Whereas it was expedient to distinguish between the way in which gluons have color and the way in which quarks have color, no conceptual clarity is gained by similarly distinguishing between 'ways of having' electric charge. So electric charge at level 2 is very different from color charge at level 2.

Let's run the metaphysical purity test for electrically charged particles at these two levels. Imagine a world (fiber bundle) with just two electrons in it, one at point p and one at a different point, q. Electron states transform according to the ρ_{-1} representation of $U(1)$. Thus, there is one copy of $\mathbb{C}$ at p and another copy at q, and the states of the electrons are given by vectors within these carrier spaces for ρ_{-1}. Do the two electrons share the property *having electric charge at all*? Yes they do, because they each transform according to a non-trivial representation of $U(1)$. Do the two electrons have electric charge *in the same way*? Yes they do, because they each transform according to the same representation of $U(1)$. Now consider a world with one electron and one proton. Proton states transform according to the ρ_1 representation. Thus, the electron and the proton share the notion of electric charge at level 1, but not at level 2. What if we were to compare an electron and a neutron? Neutron states transform according to ρ_0 since this particle is electrically neutral. Thus, the electron and the neutron do not share the property of electric charge at either levels 1 or 2.

In this manner we can run the metaphysical purity test for electric charge at these two levels *without* determining whether the fiber bundle is equipped with a connection or with continuous paths, or anything else beyond the mathematical structure used to represent the two particles. In positing the existence of the particles, we must use the associated vector bundle whose sections are matter fields for that kind of particle. These bundles come automatically equipped with the structure of the group representation used to describe the electric charge of the particles in question. This structure is sufficient for determining sameness or difference of electric charge at the first and second levels of description, which are the only relevant levels in this case. Therefore, electric charge passes the metaphysical purity test.

Without the third level, electric charge escapes Maudlin's argument. Electromagnetism is of course a gauge theory—in many ways, it is the paradigm gauge theory—and yet its corresponding charge property is not directional. Moreover, electric charge at levels 1 and 2 is fully captured by the structure that is shared by each fiber in the relevant bundles, since each fiber is a copy of the carrier space for the representation used to construct the associated bundle. It follows that, by Maudlin's own standard for geometrical structures, electric charge *is* a universal. (Or at the very least, it has met the necessary condition of metaphysical purity. Recall note 11). But again, as with color charge, electric charge as described at levels 1 and 2 is not physically insignificant, geometrical pure form. The net amount of electric charge which distinguishes the irreducible representation of $U(1)$ is, *contra* Maudlin, a physical magnitude that is used to categorize various kinds of matter. As far as the fiber bundle formalism is concerned, electric charge appears to be an excellent candidate for being a universal.

4.4.2 Mass Is Not Like Color Charge

Maudlin takes his argument against universals to generalize beyond color charge and electric charge to include mass as well. I have argued that the mathematical structure used to describe electric charge reveals an important disanalogy between color charge and electric charge, in light of which Maudlin's argument does not carry over from color charge to electric charge. The sense in which mass is not like color charge is even stronger in that group representations do not have the same role to play for mass as they do for both color and electric charge. As we will presently see, mass appears within the fiber bundle formalism in a different way than charge does.

In the initial formulation of any gauge theory, the fermion particles described by the theory have some quantity of mass, either positive or zero. The mass is identified in the Lagrangian of the theory as a constant parameter in a term of the form $m\psi\bar{\psi}$. The field ψ is a section of an associated bundle, and it is the sort of mathematical object that transforms according to the appropriate group representation. But the parameter m, which is interpreted as mass, is not at all affected by the group transformations. Mathematically, it is just some real number multiplying another real number, $\psi\bar{\psi}$. And, unlike the vectors of $\mathbb{C}^3$ used to describe color states, there is nothing inherently 'directional' about a real number.

Moreover, if we take a gauge theoretic approach to relativity, mass can be understood against the backdrop of the group representations of the Poincaré group, which is the group of Minkowski spacetime isometries. As famously developed by Wigner (1939), a continuous parameter for mass, together with a discrete parameter for spin are sufficient to characterize the irreducible representations of this group. These parameters are eigenvalues of the Poincaré Lie algebra's two Casimir operators. (For a contemporary treatment of the Poincaré group and its representations, see Woit (2017) chapter 42.) In this way of looking at mass, it is on par with charge and spin at level 2: it corresponds to real numbers used to differentiate between different irreducible representations of a physically significant Lie group, in this case, the Poincaré group. This is further reason to think that mass is not like color charge at level 3.[19]

This is why Maudlin's argument for the metaphysical impurity of color charge does not generalize to the case of mass. His argument rests on the claim that using fiber bundles as the mathematical setting for gauge theories implies that all properties are directional in the way that color charge at the third level is directional. But mass is a property in gauge theories that is not directional.

This is not to say, however, that mass is therefore a metaphysically pure property. Indeed, the ultimate notion of mass we find in particle physics is not metaphysically pure, but for entirely different reasons than those given for color charge. In the Standard Model of particle physics, the Higgs boson is used to account for the mass of all particles that have any mass at all. Moreover, renormalization and regularization of the gauge theories in the Standard Model lead to a notion of mass that is often explained in terms of interactions with a cloud of virtual particles. Any property that depends upon the existence both of the Higgs and of a number of additional virtual particles is certainly not metaphysically pure.

The Higgs accounts for the masses of other particles in two different ways, one for the fundamental fermions (such as quarks and electrons) and one for the intermediate vector bosons (the $W^{\pm}$ and the Z). The way in which the Higgs gives mass to the intermediate vector bosons is through the celebrated Higgs mechanism. Spontaneous symmetry breaking leads to massless Goldstone bosons, and the massive $W^{\pm}$ and Z are said to result from the Higgs 'eating' the Goldstone bosons. The expressions for their masses are given as follows:

$$m_{W^{\pm}} = \frac{gv}{2}, \quad m_Z = \frac{m_{W^{\pm}}}{\cos\theta_W} \tag{4.3}$$

where g is the weak coupling constant, v is the Higgs vacuum expectation value, and θ_W is the weak-mixing angle. This θ_W is the angle of rotation in a two-dimensional vector boson plane that gives rise to the photon and the Z bosons as orthogonal directions in an $SU(2)$ representation.

[19] I am grateful to an anonymous reveiwer for raising this point.

So there clearly is *some* sense in which directional properties are important to this theory. In particular, we might take the photon and the Z bosons to be represented precisely by directions in the carrier space for a two-dimensional $SU(2)$ representation used here. However, while this is in the background behind how we derive quantities of mass, the properties of *mass* that we attribute to the $W^{\pm}$ and the Z are not these directional quantities, but instead real numbers determined by equation 4.3. So again, mass in this context is not a directional property in need of a connection in order to be meaningfully compared. Rather, the masses of the $W^{\pm}$ and the Z are dependent upon the existence of the Higgs and its ability to transform Goldstone bosons. Since the intermediate vector bosons can only have their masses in a world with a Higgs, their masses are likely not metaphysically pure.

The fermions, in contrast, gain their mass through a type of interaction with the Higgs called a 'Yukawa' interaction. While directly writing down fermion mass terms as above is the simplest way to get mass in a general gauge theory, this approach runs into problems in the context of the full Standard Model. The Standard Model must include an account of parity violation, the phenomenon that the weak interaction discriminates on the basis of handedness: left-handed particles and right-handed antiparticles participate in the weak interaction, while right-handed particles and left-handed antiparticles do not. The way that we account for parity violation spoils the simple approach to getting mass terms, since those terms are no longer invariant under the symmetry of the electro-weak gauge theory. The interaction with the Higgs is necessary for restoring invariance. Using the interaction with the Higgs, the fermion masses are given by expressions of the form $m = (gv)/\sqrt{2}$, where g is now a constant known as the 'Yukawa coupling' of the particle to the Higgs field, and v is again the vacuum expectation value of the Higgs. The key for our purposes here, is that the result is a real number value for mass. Thus, in this context of fermion masses, the mass property is not directional in the sense that color charge states are directional, but it still seems to be metaphysically impure since fermion masses depend upon the existence of the Higgs.[20] So both bosonic and fermionic masses seem to be metaphysically impure, but not

[20]There is a further complication in that quark states 'mix': the actual quark fields that interact with the Higgs can

because they are directional like color charge.

4.5 Conclusion

I have argued that color charge at the third level of description fails the metaphysical purity test, while the other two levels pass the test. Electric charge does not display the same three-fold conceptual complexity of color charge, and can instead be understood simply in terms of the net amount of electric charge. This notion of electric charge is metaphysically pure. Mass also is unlike color charge, lacking any directional quality. But all of the particles in the Standard Model depend upon the Higgs for their masses in a metaphysically impure way. So the use of fiber bundles in gauge theories allows for both the metaphysically pure and the metaphysically impure.

My argument is largely based on an unexpected analogy between color charge and spin and a disanalogy between color charge and electric charge. One might wonder why this analogy between spin and color charge is easy to miss. So I want to sketch an explanation of why this is, and, consequently, why it is easy to miss the three-fold notions of color charge that I have identified here.

In physics and philosophy discussions alike, one usually introduces color charge as an analogue of electric charge. From a pedagogical standpoint, introducing color charge in this way is a natural place to start. In a physics class especially, it makes good sense to begin with the simpler theory, QED, and then introduce the more complicated theory of QCD in reference to what is already understood from the study of QED. Moreover, the theorists who first constructed QCD did so with a conscious effort to generalize the methods of QED. So we should not think that there is anything inherently misleading about introducing color charge as an analogue of electric charge. However, the analogy is (in one sense) concerned with the functional role played by the charges within their

include, for instance, combinations of both up and down quarks fields. This sort of mixing seems to constitute an additional threat to the metaphysical purity of the individual types of quarks.

respective theories. As electric charge is *in QED*, so color charge is *in QCD*. This does not imply that the metaphysical character of these two color charge must be the same. Indeed, since QCD and QED are in certain crucial respects very different theories, the nature of the property that plays the role of charge in each theory need not be exactly the same.

Indeed, here is another dissimilarity between color charge and electric charge. The values of electric charge, such as +1 and -1, are gauge invariant quantities, whereas the individual color charges *red*, *blue*, and *green* are not gauge invariant.[21] Usually, a failure of gauge invariance is taken to entail a lack of physical significance. If that is right, then further work is needed to better understand the gauge invariant characterization of the relations between the color states at level 3 as actually used in physics. (For an entry to the literature on gauge, see for example Earman (2003), Weatherall (2016b), Weatherall (2016a), and references therein.) This difference of gauge invariance between electric charge and color charge suggests that we once more compare the structure of their respective group representations. As we saw above, the values of electric charge correspond to integers that label the distinct irreducible representations of $U(1)$. For $SU(3)$, the analogous labeling of representations is given by a pair of simultaneous eigenvalues for the two diagonalizable elements of the Lie algebra. These eigenvalues can be used to label color states under the $SU(3)$ color symmetry in the same way that the two quantities of strong isospin and strong hyper-charge are used to label baryons and mesons under the $SU(3)$ flavor symmetry. This puts another wrinkle in the analogy between color charge and electric charge, indicating that more work is needed to fully understand $SU(3)$ color symmetry.

Let us now return to the Main Question: could properties like mass, electric charge, and color charge be universals? If we continue to accept metaphysical purity as a necessary condition for universals, then mass cannot be a universal, while electric charge might be. Color at levels 1 and 2 might be universals, but color at level 3 is not. Much more work is needed in order to understand the full ramifications of the metaphysical impurity of the most specific level of the fundamental

[21] I am grateful to an anonymous reviewer for raising this point.

property of color charge. In particular, a full account of the metaphysical significance of the connection, and its associated curvature and derivative operators, is clearly called for.

Moreover, the considerations presented in this paper have only scratched the surface of the many interesting features of color charge which deserve further philosophical study. For one thing, there is the principle of color neutrality that says that all detectable particles are 'white' in the sense that they have no overall color charge. We never detect individual quarks, anti-quarks, or gluons, but only composite particles such as protons (composed of three quarks) and mesons (composed of one quark and one anti-quark) whose overall color charge is zero. Consequently, unlike electric charge or spin, color charge is never measured. In a further oddity, the level 3 properties of *red*, *blue*, and *green* were first hypothesized as a way of saving the Pauli exclusion principle after the discovery of the Δ^{++}, a baryon which was known to be composed of three up quarks with aligned spin states. Considering only the electric charge, mass, and spin of these three quarks, the Δ^{++} would have to be a fermion with a symmetric wavefunction, in contradiction to the Pauli exclusion principle. If, however, we add in color charge, the three quarks could each be in a different level 3 color state, thereby restoring asymmetry to the overall wavefunction for the Δ^{++}. So in practice these color states are not used to account for sameness of color charge, but they are used to account for differences of color charge. What a strange property! Color states do not obviously fit into any traditional notion of fundamental property, and so it may indeed be that the physics community has discovered or developed a genuinely novel notion of property. The metaphysics community would do well to study it further.

Chapter 5

Color Charge is not Like Electric Charge

5.1 Introduction

One first comes across the property of color charge as a new kind of charge. It is a property in some sense 'like' electric charge, doing the same sort of work in the theory of chromodynamics that electric charge does in the theory of electrodynamics. Knowing only this much about color charge, one might mistakenly expect that color charge will have many of the same metaphysical features of electric charge. Indeed, some metaphysicians take them to be on equal footing, not only with each other, but also with other fundamental properties such as mass or quark flavor.[1] However, a closer look at the theories of chromodynamics and electrodynamics reveals that the analogy between the two properties does not license such thorough metaphysical similarities.

Despite these differences between the two properties (differences which will be made precise below), we can recover an analogy between them by comparing them in the reverse direction: electric

[1]Most prominently, Lewis lists mass, electric charge, color charge, and quark flavors as prime examples of what he calls "perfectly natural" properties (Lewis (1997) p. 179. Cf. Lewis (1997) p. 186 (footnote 6) as well as Lewis (1986) p. 14, 67 (footnote 47), and 178.) Maudlin (2007) similarly treats mass, color charge, and electric charge as having the same metaphysical status (though, to be clear, his purpose here is to argue that the physics does not actually warrant thinking of these properties as universal properties in the traditional way).

charge *is* like color charge, in the sense that electric charge is an instance of a more general notion of charge at work in any gauge theory. Considering electrodynamics as a specific instance of a general gauge theory reveals that electric charge shares many of the features of a general notion of charge which are exemplified by color charge, but in a degenerate way. In this sense, it is electric charge that is like color charge, and not the other way around.

Charge has at least the following three key roles to play in a gauge theory. First, it is an attribute of the particles that participate in the interaction of the gauge theory. Particles with electric charge participate in the electromagnetic interaction, those with color charge participate in the strong interaction, and those with weak charge in the weak interaction. Second, in classical gauge theory, charge is a sort of measure of the strength of the force, in that the more charge a particle has, the greater its acceleration under this force. Here, charge is closely related to the charge-current density, which is a source for the Yang-Mills equation. Third, charge is a conserved quantity as a result of Noether's theorem. In each of these cases, the way in which color charge plays its roles demonstrates a richness to the general notion of charge in a classical gauge theory that is hidden in the case of electric charge. The remainder of this paper discusses these three roles of charge in detail.

5.1.1 Definitions and notation

We proceed in the context of classical rather than quantum field theory because the classical theory enjoys a mathematically precise formulation not yet achieved for quantum field theory. A final say on the nature of charge would of course require the full quantum theory. In a classical gauge theory, we fix the following mathematical structures:

- a relativistic spacetime (M, g_{ab});
- a principal G-bundle $P \overset{\wp}{\to} M$ over M, where G is a Lie group;
- a principal connection $\omega^{\mathfrak{A}}_{\alpha}$ on P;

- an inner product $k_{\mathfrak{A}\mathfrak{B}}$ on the Lie algebra $\mathfrak{g}$ associated to G.

Recall from chapter 2 that we use the abstract index notion developed in Wald (1984) with the further notational conventions of Weatherall (2016a). In particular, vectors and tensors tangent to M have lower-case Latin indices a, b, etc.; vectors and tensors tangent to the total space P have lower-case Greek indices; upper-case Fraktur indices are used for vectors with a Lie algebra structure. In addition, indices i, j, etc. will be used to label vectors in the carrier space V of a representation of G used to construct an associated bundle.

The curvature of the principal connection is interpreted as the field strength. This curvature is

$$\Omega^{\mathfrak{A}}{}_{\alpha\beta} = d_\alpha \omega^{\mathfrak{A}}_\beta + \frac{1}{2}[\omega^{\mathfrak{A}}_\alpha, \omega^{\mathfrak{A}}_\beta], \tag{5.1}$$

where d_α is the exterior derivative on P, and the bracket $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$.

The various types of matter fields correspond to sections of different associated bundles $P \times_G V \overset{\pi_\rho}{\to} M$, where V is the carrier space for an irreducible representation of G. Thus, a matter field $\Psi : M \to P \times_G V$ maps points $x \in M$ to equivalence classes $[p,v]$ for some $p \in P$ and $v \in V$. Here $[p,v] \cong [p',v']$ just in case there exists a $g \in G$ such that $(pg, \rho(g^{-1})v) = (p',v')$ where $\rho : G \to \mathsf{GL}(V)$ is an irreducible representation of G. These vectors v describe states in the internal charge space for an elementary particle of this kind of matter field.

As a useful technical fact, the sections $\Psi : M \to P \times_G V$ are in one-to-one correspondence with V-valued, G-equivariant maps ψ^i on P. Given such a $\Psi(x) = [p,v]$, we define $\psi^i : P \to V$ by $\psi^i(p) \mapsto \lambda_p^{-1}(\Psi(\wp(p)))$, where the map $\lambda_p : V \to \pi_\rho^{-1}[\wp(p)]$ is defined by $\lambda_p(v) = [p,v]$. In certain contexts, it is sometimes more convenient to use ψ^i on P rather than Ψ on M.

In particular, it is technically simpler to define the action of the covariant derivative induced by the connection $\omega^{\mathfrak{A}}_\alpha$ on ψ^i than on Ψ. For a field $\psi^i : P \to V$, its covariant derivative is the ordinary exterior derivative d_α on P following the horizontal projection. That is, for any vector ξ^α on P, the covariant derivative induced by the connection is,

$$\overset{\omega}{D}_\alpha \psi^i \xi^\alpha = d_\alpha \psi^i \xi^{\alpha|}. \tag{5.2}$$

This covariant derivative operator gives the standard of constancy for matter fileds ψ^i on P, or equivalently, for parallel transport amongst fibers in an associated bundle $P \times_G V \overset{\pi_\rho}{\to} M$.

5.2 Charge as a property of particles

If we ask for the electric charge of a given particle, the answer is some amount of charge in units of the charge of the positron e, e.g., the up quark has electric charge $+\frac{2}{3}e$. If we ask for the color charge of a particle, the answer is some type of charge state, e.g., this quark is in a 'red' state, or this gluon is in a 'blue/anti-green' state. These are the usual ways in which the electric and color charge properties are predicated of various particles. But if we take the analogy between electric charge and color charge to mean that these predicate are analogous (i.e., that having $+\frac{2}{3}e$ electric charge is somehow analogous to having 'red' color charge), we will be consistently mislead. The aim of this section is to clarify three different levels of description for color charge as it can be predicated of a particle. Electric charge is shown to have these same levels of description, but in a degenerate way.

The group $SU(n)$ has $n-1$ many inequivalent fundamental representations, and (for $n > 2$) two of those fundamental representations are n-dimensional. So $SU(3)$, the group used for chromodynamics, has two inequivalent fundamental representations which are both three-dimensional. Quark states transform according to the first of these fundamental representations, and in this representation $V = \mathbb{C}^3$. The three quark colors are identified with the standard basis vectors for $\mathbb{C}^3$, using the labels r, g, and b for 'red', 'green', and 'blue' respectively.

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.3}$$

Meanwhile, the anti-quarks transform according to the second fundamental representation, whose carrier space $\mathbb{C}^{3*}$ is dual to the first, and their anti-color states are identified with the corresponding dual basis. The color charge of a gluon is similarly given by (basis) vectors of the carrier space of the adjoint representation which is the Lie algebra itself, $V = \mathfrak{g}$. This space can be built up from the quark and anti-quark spaces in such a way that we think of gluon color charges as combinations of both color and anti-color (see section 5.3). The adjoint representation of $SU(3)$ is eight-dimensional, and this is why it is said that there are eight gluons.

This structure from the group representations gives us three different senses in which particles may be said to 'have' or 'carry' color charge. First, we have the sense in which quarks, anti-quarks, and gluons all alike have color charge, namely, that they each transform according to non-trivial representations of $SU(3)$. All other particles, such as electrons or neutrinos, transform trivially under $SU(3)$, and in this sense they do not have color charge.

Second, these each have color charge in different ways given by the differences in their respective representations. Quarks have color charge in the red/blue/green way, whereas anti-quarks have it in the anti-red/anti-blue/anti-green way, and gluons have color charge as a combination of both color and anti-color. The different ways of having charge correspond to different representations of the group.

Third, a given color-charged particle can be said to be in a specific color state. At this level, we have a fixed representation and distinguish states by different basis vectors for that representation. It is in this sense that we distinguish between, for example, a quark in the red or green state, or between an anti-quark in the anti-blue state and one in the anti-green state, or between a blue/anti-red gluon and a green/anti-blue gluon.

In principle, we can make all the same distinctions for electric charge, where now the relevant group is $U(1)$. This is the circle group, the set of numbers in the complex plane with unit modulus, $e^{i\theta}$ for $\theta \in [0, 2\pi]$. Its complex irreducible representations are all of the form

$$\rho_n(e^{i\theta}) = e^{in\theta} \tag{5.4}$$

where n is an integer. The trivial representation is ρ_0. The value of n labeling the representation gives the amount of electric charge had by particles which transform according to that representation. Thus electrons transform in ρ_{-1}, neutrinos in ρ_0, positrons in ρ_1, etc.

In the first sense, we distinguish between electrically charged particles and electrically neutral particles by whether or not their representation of $U(1)$ is trivial. At the second level, we distinguish between non-trivial representations by different n's, and at the third level we can distinguish between the different vector states within a given representation.

While these distinctions can be made with mathematical precision in the same way for both $SU(3)$ and $U(1)$, the interpretation of each sense of charge does not fully carry over to electric charge. Because $U(1)$ is Abelian, each of these representations ρ_n is one-dimensional. Consequently, the third level of description of color charge does not clearly apply in the case of electric charge. We distinguished the *red* and *blue* color states by different vectors in a fixed basis of the quark representation, and so these color properties correspond to different directions in the state space. But for the $U(1)$ representations, a fixed basis can only have one vector, and so there is no analogous sense in which two electrons, say, could be in different electrically charged states. Since $V = \mathbb{C}$, individual unit vectors can differ at most by a phase, but differences of phase have never been taken to correspond to differences of electric charge in any sense.

Moreover, the stark difference between the way in which quarks have color and gluons have color given at the second level of description does not carry over to electric charge. It seems inapt to similarly think of, say, electrons and protons as having electric charge in different 'ways' or of different 'kinds.' Rather, they simply have different amounts of precisely the same kind (and the only kind there is) of electric charge. So here too, color charge is not like electric charge since it has a meaningful distinction at the second level that is not shared by electric charge.

This can be seen more clearly by considering the difference between how color and electric charges combine to form an overall charge for a composite particle. For example, let's consider the proton. In order to determine the color-state p_c of a proton, we take the tensor product of each of the three quarks within the proton:

$$p_c \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cong \mathbb{C}^1 \oplus \mathbb{C}^8 \oplus \mathbb{C}^8 \oplus \mathbb{C}^{10}. \tag{5.5}$$

On the right hand side, we have the three different mathematical possibilities for which representation of $SU(3)$ the proton could be in: $\mathbb{C}^1$, $\mathbb{C}^8$, or $\mathbb{C}^{10}$.[2] So the three colors of the quarks can in principle combine in a variety of different ways: their combination could result in a $\mathbb{C}^8$ or a $\mathbb{C}^{10}$ way of having charge, or it could combine into the $\mathbb{C}^1$ way of having no charge, or being in the 'white' color charge state.

What about the proton's electric charge, p_e? We follow the same process of taking the tensor product of the electric charge carrier spaces for the quarks. So we have

$$p_e \in \mathbb{C}^1 \otimes \mathbb{C}^1 \otimes \mathbb{C}^1 \cong \mathbb{C}^1. \tag{5.6}$$

Again, the state of the proton with respect to electric charge will be represented by a vector in one of the mathematically possible spaces given on the right hand side of equation 5.6. But of course now there is only one possibility. What's more, there is no difference between this one option and the initial electric charge states of the quarks given on the left hand side. Electric charge does not come in different kinds, and so there is only one way for individual electric charges to combine.

[2] It turns out that the first option, the one-dimensional representation, is the only *physical* possibility. This is because of the principle of color confinement, which says that the overall color must be zero or 'white.' Mathematically, this means that the state vector for any observable particle made up of quarks must be invariant under the action of $SU(3)$.

5.3 Charge is Lie algebra valued

The previous section described three senses in which charge is predicated of both fundamental and composite particles. But this is certainly not the only role of charge in a gauge theory. In addition, it is a conserved quantity as a result of Noether's theorem, and it plays a crucial role in the force laws of classical field theory. In both of these two new roles, the mathematical quantity interpreted as *charge* takes values in the Lie algebra of the relevant group. The presence of the Lie algebra in defining these roles for charge is made manifest in chromodynamics, but it is obscured in electrodynamics since the Lie algebra of $U(1)$ is $\mathbb{R}$. Consequently, the usual interpretation of charge in electromagnetism does not carry over to chromodynamics, giving another sense in which color charge is not like electric charge.

The generalization of the Lorentz force law to non-Abelian gauge theory is known as the Wong force law, first given by Wong (1970). It can be mathematically derived as follows, although the status of this derivation as a physical argument is unclear.[3] Using the inner product $k_{\mathfrak{A}\mathfrak{B}}$, the metric g_{ab} on M, and our principal connection $\omega^{\mathfrak{A}}_{\alpha}$, we can construct a metric on the total space known as the 'bundle metric.' First, we take g_{ab} on the base space M, and we pull this back along the projection to get a symmetric rank 2 tensor $(\wp^* g)_{\alpha\beta}$ on the total space P. That is,

$$(\wp^* g)_{\alpha\beta}\sigma^{\alpha}\eta^{\beta} = g_{ab}(\wp_*\sigma^{\alpha})(\wp_*\eta^{\beta}) \tag{5.7}$$

for all vectors $\sigma^{\alpha}, \eta^{\alpha}$ at a point p in P. There is another symmetric rank 2 tensor on P, denoted $k_{\alpha\beta}$, defined in terms of the connection $\omega^{\mathfrak{A}}_{\alpha}$. It is given by

$$k_{\alpha\beta}\sigma^{\alpha}\eta^{\beta} = k_{\mathfrak{A}\mathfrak{B}}\omega^{\mathfrak{A}}{}_{\alpha}\sigma^{\alpha}\omega^{\mathfrak{B}}{}_{\beta}\eta^{\beta}. \tag{5.8}$$

[3] In particular, it is unclear what physical significance we should attribute to geodesics in the total space. The mathematical derivation rehearsed here was first given by Kerner (1968), who explicity sought to generalize an equivalence between the theories of Kaluza (1921) and Utiyama (1956). A similar derivation is given in Bleecker (2013) chapter 10 section 1. Alternative derivations can be found in Storchak (2014), Sternberg (1977), and Weinstein (1978). Wong (1970) himself arrives at the expression by extracting a classical limit of of the equations of motion for the quantum fields.

For any two vectors σ^{α} and η^{β} at a point p in P, $k_{\alpha\beta}$ gives the inner product of their vertical projections.Adding these two tensors gives us our bundle metric $h_{\alpha\beta}$:

$$h_{\alpha\beta} = (\wp^{*} g)_{\alpha\beta} + k_{\alpha\beta}. \tag{5.9}$$

Let $\gamma : [0,1] \to P$ be a geodesic relative to $h_{\alpha\beta}$ with tangent field $\xi^{\alpha}(t)$. It follows that $\omega^{\mathfrak{A}}_{\alpha}\xi^{\alpha}(t) = Q^{\mathfrak{A}} \in \mathfrak{g}$ is independent of t (see Bleecker (2013) Theorem 10.1.5). Let $\tilde{\gamma} = \wp \circ \gamma$ be the projection of γ down to the base space with tangent field ξ^{a}. It follows that the acceleration of this curve obeys the Wong force law (see Bleecker (2013) Theorem 10.1.6),

$$\xi^{n}\nabla_{n}\xi^{b} = k_{\mathfrak{A}\mathfrak{B}}g^{cb}Q^{\mathfrak{A}}\Omega^{\mathfrak{B}}_{\ ac}\xi^{a} = Q^{\mathfrak{A}}\Omega_{\mathfrak{A}a}{}^{b}\xi^{a} \tag{5.10}$$

where $\Omega^{\mathfrak{A}}_{\ ab} = \sigma^{*}(\Omega^{\mathfrak{A}}_{\ \alpha\beta})$, for a choice of section $\sigma : U \to P$, is the field strength. We interpret $\tilde{\gamma}$ as the world-line for a particle of mass m and charge $q^{\mathfrak{A}} = Q^{\mathfrak{A}}/m$. If $G = U(1)$, this reduces to the familiar Lorentz force law for electromagnetism. In this case we interpret $q^{\mathfrak{A}} \in \mathbb{R}$ as the amount of electric charge carried by the particle whose world line is $\tilde{\gamma}$. That is, $q^{\mathfrak{A}}$ is the charge property as usually predicated of electrically charged particles.

How may we interpret $q^{\mathfrak{A}}$ for chromodynamics? We usually predicate color charge of a particle in the sense of a basis vector in a representation of $SU(3)$, that is, a specific color *state*. Using the electromagnetic interpretation as a guide, one might then expect that $q^{\mathfrak{A}}$ gives the color state of the particle under the influence of the Wong force. And since gluons are the only type of particle which have Lie algebra valued color states, one might conclude that $\tilde{\gamma}$ is the world line of a gluon.

However, such reasoning would be a mistake rooted in our expectations that color charge is like electric charge. Again, we need to reverse the analogy. The color-charged matter fields subject to this Wong force law couples to the gauge field via the charge-current density, which is a Lie algebra valued 1-form $J^{\mathfrak{A}}_{\alpha}$ on the total space P. The charge $q^{\mathfrak{A}}$ is Lie algebra valued because the charge current density is Lie algebra valued. It is only in the case of electromagnetism that the charge $q^{\mathfrak{A}}$ reduces to a real number which can be interpreted as the amount of charge carried by an electrically charged particle.

The definition of $J^{\mathfrak{A}}_a$ relies upon the inner product $k_{\mathfrak{A}\mathfrak{B}}$ on $\mathfrak{g}$, as well as an inner product h_{ij} on the carrier space V of the representation of G used to describe the matter field. Fix a basis $\{e^{\mathfrak{A}}\}$ of the Lie algebra $\mathfrak{su}(3)$. Then, following Bleecker (2013) 5.1.2., the current $J^{\mathfrak{A}}_a$ is given by,

$$J^{\mathfrak{A}}_{\alpha} = k^{\mathfrak{A}\mathfrak{B}} e_{\mathfrak{B}} h_{ij} \tilde{\psi}^j \overset{\omega}{D}_{\alpha} \psi^i \tag{5.11}$$

where $\tilde{\psi}^j = \rho_*(e_{\mathfrak{A}}) \rhd \psi^j$. That is, $\tilde{\psi}^j$ is the result of transforming ψ^i under the representation ρ_* of $\mathfrak{g}$ on V induced by the representation ρ of G on V (see Hamilton (2017) 2.1.12).

This general expression for the current associated with a charged matter field ψ^i in a non-Abelian gauge theory reduces to the more familiar current of electromagnetism as follows. Suppose now that $G = U(1)$. Further, choose $i = \sqrt{-1}$ as a basis for $\mathfrak{g} = \mathbb{R}$, and choose $k_{\mathfrak{A}\mathfrak{B}}$ such that $k_{\mathfrak{A}\mathfrak{B}} e^{\mathfrak{A}} e^{\mathfrak{B}} = 1$. Then the current $J^{\mathfrak{A}}_{\alpha}$ becomes,

$$J^{\mathfrak{A}}_{\alpha} = i h_{ij} (i\psi)^j \overset{\omega}{D}_{\alpha} \psi^i \tag{5.12}$$

Now fix a choice of local section $\sigma : U \to P$. The vector potential is $\sigma^*(\omega^{\mathfrak{A}}_{\alpha}) = A_a$ is the pullback of the connection along the choice of section σ. Similarly, the complex scalar field on M is $\Psi = \sigma^*(\psi^i)$. With this notation, a local representation of the current on M is,

$$J_a = i((i\Psi)^* \nabla_a \Psi + i\Psi(\nabla_a \Psi)^*) \tag{5.13}$$

$$= \Psi^* \nabla_a \Psi - \Psi(\nabla_a \Psi)^*, \tag{5.14}$$

where $\nabla_a = \partial_a - iA_a$ is the covariant derivative on M, and the star * indicates complex conjugation. Since the general expression for a non-Abelian charge current density (eq. 5.11) reduces to the current for electromagnetism, this gives yet another instance in which the analogy between electric and color charge ought to be reversed. The electric charge-current density is like the color charge-current density.

The definition of the current in eq. (5.11) gives us mathematical reason to acknowledge that the charge values $q^{\mathfrak{A}}$ which contribute to $J^{\mathfrak{A}}_{\alpha}$ are Lie algebra valued. However, more needs to be said to address the physical significance of the Lie algebra in giving values of non-Abelian charges in this context. The Lie algebra turns out to be the mathematical object suited to giving appropriate sums of color and anti-color charges. The explicit construction of the adjoint representation (whose carrier space is the Lie algebra) was developed in detail in chapter 3. Here, we will review enough of those details to support the claim that the Lie algebra values of $q^{\mathfrak{A}}$ and $J^{\mathfrak{A}}_{\alpha}$ make sense as a describing the appropriate notion of *total* non-Abelian charge.

In general, matter fields, anti-matter fields, and the gauge field can contribute to the total charge-current density $\sigma^*(J^{\mathfrak{A}}_{\alpha}) = J^{\mathfrak{A}}_{a}$. In the exceptional case of electromagnetism, the gauge field (representing the photon) is not electrically charged, and so it does not contribute to the current. What can contribute to J_a for electromagnetism? Charged matter fields for particles such as electrons, positrons, quarks, anti-quarks, etc. The contributions of both positive and negative electric charges is a simplified case of combining charges with their corresponding anti-charges. In the more general non-Abelian case of chromodynamics, there are contributions to $J^{\mathfrak{A}}_{\alpha}$ from matter fields representing quarks, from their corresponding fields for anti-quarks, and from the color-charged gauge field itself. The total charge-current density needs to be able to add together all three of these different kinds of color charges.

The Lie algebra $\mathfrak{su}(3)$ is suitable for adding these three kinds of charge for two reasons: first, because the Lie algebra is the carrier space for the adjoint representation (used to describe gluon color states), and second, because the adjoint representation can be built from the two fundamental representations (used to describe quark and anti-quark color and anti-color states). This second reason supports the interpretation of Lie algebra valued quantities as combinations of color and anti-color. It is in this way that both quarks and anti-quarks can contribute to the total color charge-current density.

Recall that, in the first fundamental representation, a basis for the Lie algebra $\mathfrak{su}(3)$ is given by the Gell-Mann matrices.

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since λ_3 and λ_8 are diagonal, they can serve as the Cartan sub-algebra. Appropriate complex sums of the remaining, non-Cartan elements give three sets of "raising and lowering" operators. In the first fundamental representation these are

$$\rho_1(T_\pm) = \frac{1}{2}(\lambda_1 \pm i\lambda_2), \;\; \rho_1(U_\pm) = \frac{1}{2}(\lambda_6 \mp i\lambda_7), \;\; \rho_1(V_\pm) = \frac{1}{2}(\lambda_4 \pm i\lambda_5). \tag{5.15}$$

For example, applying the $\rho_1(T_-)$ lowering operator to the red state produces a blue state, whereas applying $\rho_1(V_-)$ to the red state produces a green state.

We construct the carrier space V_{adj} as follows. Let r, b, and g denote basis vectors for the carrier space V_1 of the first fundamental representation as given in equation 5.3. Let $\bar{g} = r$, $\bar{b} = -b$, $\bar{r} = g$ be a basis for the carrier space V_2 of the second fundamental representation. We set $\rho_2(Z) = \overline{Z} = -\rho_1(Z)^{tr}$ for all $Z \in \mathfrak{su}(3)$. The carrier space for the adjoint representation is the tensor product of the fundamental representations carrier spaces, $V_{adj} = V_1 \otimes V_2$, and ρ_{adj} is given by $\rho_{adj}(Z) = \rho_1(Z) \otimes I + I \otimes \rho_2(Z)$.

Now that we have ρ_{adj}, we can determine a basis for V_{adj} by successive applications of the lowering operators to the state of the highest weight. It can be shown that the state of the highest

Figure 5.1: Construction of the adjoint representation in combinations of color and anti-color states.

$$
\begin{array}{c}
r\bar{b} \\
\swarrow \quad \searrow \\
r\bar{g} \qquad\qquad g\bar{b} \\
\swarrow \ \downarrow \qquad\qquad \downarrow \ \searrow \\
0 \quad g\bar{g}-r\bar{r} \qquad b\bar{b}+g\bar{g} \quad 0 \\
\swarrow \ \downarrow \qquad\qquad \downarrow \ \searrow \\
b\bar{g} \quad -2g\bar{r} \qquad 2b\bar{g} \quad -g\bar{r} \\
\swarrow \downarrow \quad \downarrow \searrow \quad \swarrow \downarrow \quad \downarrow \searrow \\
0 \quad -b\bar{r} \quad -b\bar{r} \quad 0 \quad -b\bar{r} \quad -b\bar{r} \quad 0
\end{array}
$$

weight of a representation of a Lie group G on a space $V_1 \otimes V_2$ is the state $v_1 \otimes v_2$, for v_1 the state of highest weight for the fundamental representation on V_1 and for v_2 the state of highest weight for the fundamental representation on V_2 (see Hall (2015) prop. 6. 17). Thus, since r has the highest weight of ρ_1 and $\bar{b}$ has highest weight of ρ_2 (refer back to chapter 3), the state of highest weight of ρ_{adj} is $(r \otimes \bar{b})$. So successive application of the lowering operators $\rho_{adj}(T_-)$, $\rho_{adj}(U_-)$, and $\rho_{adj}(V_-)$ to the state $(r \otimes \bar{b})$ will result in all of the other states of the adjoint representation. However, it suffices to consider just $\rho_{adj}(U_-)$ and $\rho_{adj}(V_-)$ since $[U_-, V_-] = T_-$. We can summarize the results of this process in the diagram given in figure 5.1. To save space, we omit the tensor product symbol and simply write the states as, e.g. $r\bar{b}$. Arrows to the left indicate the action of U_- on the previous state, and arrows to the right indicate the action of V_-. From the diagram, we see that the resulting states give the following basis for the adjoint representation: $r\bar{b}$, $r\bar{g}$, $g\bar{b}$, $g\bar{g} - r\bar{r}$, $b\bar{b} + g\bar{g}$, $b\bar{g}$, $g\bar{r}$, and $b\bar{r}$. In this way, every element of the Lie algebra corresponds to a combination of both color and anti-color.

Recall the interpretation developed in chapter 3 of the Lie algebra in the context of Noether's theorem. In the case of color charge, the Lie-algebra values of the Noether charge shows that the conserved quantity is a combinations of color and anti-color. Something similar holds for electric charge: negative electric charge is conjugate to positive electric charge in just the same way that anti-color is conjugate to color. Since it is *net* electric charge that is conserved, the conservation of electric charge can also be understood as a conservation law for a combination of the one property and its conjugate property. Similarly, the role of the Lie algebra in the current $J^{\mathfrak{A}}_{\alpha}$ serves to unite all of the relevant contributions of color charges from quarks, anti-quarks, and from gluons. So in this sense, too, it is electric charge that is like color charge, and not *vice versa*.

5.4 Conclusion

I have argued that, on a number of accounts, color charge is not like electric charge. First, in so far as these properties may be predicated of particles, color charge reveals a rich, three-fold structure that is hidden in the case of electric charge. Whereas particles that have color charge may do so in a least three different ways (the quark way, the anti-quark way, and the gluon way), particles that have electric charge do so in only one way. Moreover, colored particles can be in different color charge states corresponding to different basis vectors for the representation that designates their kind of charge. But electrically charged particles transform according to one-dimensional representations, and so they do not have a corresponding notion of different electric charge states.

Additionally, color charge exemplifies the essential role of the Lie algebra in defining the charge-current density, and in the Noether charge, and in the charge factor of the Wong force law. These roles for the Lie algebra are obscured in the electromagnetism. Consequently, if we start with electric charge and attempt to understand color charge in its light, we will be consistently misled. The explanation goes the other way: it is electric charge that is like color charge, albeit in a degenerate sort of way.

However, this still leaves room to recover a sense in which color charge and electric charge truly are alike. We may say that those matter fields that transform according to a non-trivial representation of the group have charge *simpliciter*. Any such matter field contributes to the charge current density. In this sense the charge of that field is part of the overall conserved quantity of charge. Understood in this way, electric charge and color charge have exactly analogous roles to play in their respective theories.

Bibliography

Armstrong, D. M. (1997). "Properties." *Properties*. Ed. D. H. Mellor and Alex Oliver. Oxford University Press, 160–172.

Baez, John and Javier P Muniain (1994). *Gauge fields, knots and gravity*. Volume 4. World Scientific Publishing Co Inc.

Baez, John C. and John Huerta (2010). "The Algebra of Grand Unified Theories." *Bull. Am. Math. Soc.*, *47*, 483–552.

Baker, David and Hans Halvorson (2010). "Antimatter." *British Journal for the Philosophy of Science*, *61*(1), 93–121.

Baker, David John (2010). "Symmetry and the Metaphysics of Physics." *Philosophy Compass*, *5*(12), 1157–1166.

Barnes, J. (1995). *The Complete Works of Aristotle: The Revised Oxford Translation*. Bollingen series. Princeton University Press.

Bleecker, D. (2013). *Gauge Theory and Variational Principles*. Dover Books on Mathematics. Dover Publications.

Brading, Katherine and Harvey R Brown (2000). "Noether's theorems and gauge symmetries." *arXiv preprint hep-th/0009058*.

Brading, Katherine and Harvey R Brown (2003). "Symmetries and Noether?s theorems." *Symmetries in physics: Philosophical reflections*. . Cambridge University Press, 89–109.

Brading, Katherine A (2002). "Which symmetry? Noether, Weyl, and conservation of electric charge." *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, *33*(1), 3–22.

Brunetti, Romeo, Claudio Dappiaggi, Klaus Fredenhagen, and Jakob Yngvason (2015). *Advances in algebraic quantum field theory*. Springer.

Cahn, Robert N (2014). *Semi-simple Lie algebras and their representations*. Courier Corporation.

Carroll, Sean (2018). "What is Means to Be Fundamental.".

Cornwell, J.F. (1997). *Group Theory in Physics*. Techniques of Physics. San Diego: Academic Press.

Demarest, Heather (2015). “Fundamental properties and the laws of nature.” *Philosophy Compass*, *10*(5), 334–344.

Earman, John (2003). “Tracking down gauge: An ode to the constrained Hamiltonian formalism.” *Symmetries in physics: Philosophical reflections*. Ed. Katherine Brading and Elena Castellani. Cambridge University Press, Cambridge, UK, 140–162.

Fraser, Doreen (2011). “How to take particle physics seriously: A further defence of axiomatic quantum field theory.” *Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics*, *42*(2), 126–135.

French, Steven (2014). *The structure of the world: Metaphysics and representation*. Oxford University Press.

Funkhouser, Eric (2006). “The determinable-determinate relation.” *Noûs*, *40*(3), 548–569.

Georgi, Howard (1999). *Lie algebras in particle physics: from isospin to unified theories*. Volume 54. Westview Press.

Georgi, Howard and Kannan Jagannathan (1982). “Lie algebras in particle physics.” *American Journal of Physics*, *50*(11), 1053–1053.

Greenberg, Oscar Wallace (2009). “Color Charge Degree of Freedom in Particle Physics.” *Compendium of Quantum Physics*. Ed. Daniel Greenberger, Klaus Hentschel, and Friedel Weinert. Springer Berlin Heidelberg, 109–111.

Hall, Brian C (2015). *Lie groups, Lie algebras, and representations: an elementary introduction*. Volume 222. Springer.

Hamilton, Mark JD (2017). *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics*. Springer.

Healey, Richard (2007). *Gauging What’s Real*. Oxford University Press.

Healey, Richard (2008). “Review of *The Metaphysics Within Physics*, by Tim Maudlin.” *Notre Dame Philosophical Reviews*.

Hermann, Robert (1966). *Lie groups for physicists*. Volume 5. WA Benjamin New York.

Hirsch, Robert J. (2017). “Maudlin’s Mathematical Maneuver: A Case Study in the Metaphysical Implications of Mathematical Representations.” *Philosophy and Phenomenological Research*, *91*(2).

Kaluza, Theodor (1921). “Zum unitätsproblem der physik.” *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, *1921*(arXiv: 1803.08616), 966–972.

Kerner, R. (1968). "Generalization of the Kaluza-Klein Theory for an Arbitrary Nonabelian Gauge Group." *Ann. Inst. H. Poincare Phys. Theor.*, *9*, 143–152.

Kobayashi, Shoshichi and Katsumi Nomizu (1969). *Foundations of differential geometry*. Volume 1 and 2. Wiley New York.

Kosmann-Schwarzbach, Yvette (2011). "The Noether theorems." *The Noether Theorems*. . Springer, 55–64.

Kosyakov, Boris (2006). *Introduction to the Classical Theory of Particles and Fields*. 2007 edition. Springer.

Ladyman, James (2018). "What is Fundamental Physics?.".

Lewis, David (1997). "Modal Realism at Work: Properties." *Properties*. Ed. D. H. Mellor and Alex Oliver. Oxford University Press, 160–172.

Lewis, David K. (1986). *On the Plurality of Worlds*. Blackwell Publishers.

Lewis, David Kellogg (1987). *Philosophical Papers: Volume Il*. Oxford University Press.

Lichtenberg, DB (1978). *Unitary symmetry and elementary particles*. 2 edition. New York: Academic Press.

Malament, D.B. (2012). *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. Chicago Lectures in Physics. University of Chicago Press.

Maudlin, Tim (2007). *The Metaphysics Within Physics*. Oxford University Press.

Miller, Michael (2018). "Fundamental, Yet Imprecise?.".

Muntean, Ioan (2012). "The Fiber Bundle at the Gates of Metaphysics. Challenging Tim Maudlin?s Proposal." *EPSA Philosophy of Science: Amsterdam 2009*. . Springer, 239–251.

Nakahara, Mikio (2003). *Geometry, topology and physics*. CRC Press.

Ney, Alyssa (2018). "Funding and Fundamental Physics.".

Noether, Emmy (1918). "Invarianten beliebiger Differentialausdrücke." *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse*, *1918*, 37–44.

Penrose, Roger and Wolfgang Rindler (1984). *Spinors and space-time: Volume 1, Two-spinor calculus and relativistic fields*. Volume 1. Cambridge University Press.

Priestley, Joseph (1775). *The history and present state of electricity with original experiments, by Joseph Priestley LL.D. F.R.S.* Third edition. Volume 1. London : Printed for C. Bathurst, and T. Lowndes ; J. Rivington, and J. Johnson ; S. Crowder and 4 others in London.

Rosenstock, Sarita and James Owen Weatherall (2016). "A categorical equivalence between generalized holonomy maps on a connected manifold and principal connections on bundles over that manifold." *Journal of Mathematical Physics*, *57*(10), 102902.

Ruetsche, Laura (2011). *Interpreting Quantum Theories: The Art of the Possible*. Oxford University Press.

Ryder, Lewis H (1996). *Quantum field theory*. Cambridge university press.

Sardanashvily, Gennadi (2016). *Noether's Theorems*. Springer.

Sternberg, Shlomo (1977). "Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field." *Proceedings of the National Academy of Sciences*, *74*(12), 5253–5254.

Storchak, Sergey N. (2014). "Wong's equations in Yang-Mills theory." *Central European Journal of Physics*, *Apr*, 2014, *12*(4), 233–244.

Svetlichny, George (1999). "Preparation for gauge theory.".

Trautman, A. (1980). "Fiber Bundles Gauge Fields and Gravitation." *General Relativity and Gravitation. Vol. 1. One hundred years after the birth of Albert Einstein. Edited by A. Held. New York, NY: Plenum Press, p.287, 1980*. Ed. A. Held 287.

Utiyama, Ryoyu (1956). "Invariant theoretical interpretation of interaction." *Physical Review*, *101*(5), 1597.

Steinkirch, Marinavon (2011). "Introduction to group theory for physicists.".

Wald, Robert M (1984). *General relativity*. Chicago, IL: Chicago Univ. Press.

Wallace, David (2011). "Taking particle physics seriously: A critique of the algebraic approach to quantum field theory." *Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics*, *42*(2), 116–125.

Weatherall, James Owen (2016a). "Fiber bundles, Yang–Mills theory, and general relativity." *Synthese*, *193*(8), 2389–2425.

Weatherall, James Owen (2016b). "Understanding gauge." *Philosophy of Science*, *83*(5), 1039–1049.

Weinstein, Alan (1978). "A universal phase space for particles in Yang-Mills fields." *Letters in Mathematical Physics*, *2*(5), 417–420.

Wigner, E. (1939). "On Unitary Representations of the Inhomogeneous Lorentz Group." *Annals of Mathematics*, *40*(1), 149–204.

Wilson, Jessica M. (2012). "Fundamental Determinables." *Philosophers' Imprint*, *12*.

Woit, Peter (2017). *Quantum Theory, Groups and Representations: An Introduction*. Springer.

Wong, SK (1970). “Field and particle equations for the classical Yang-Mills field and particles with isotopic spin.” *Il Nuovo Cimento A*, *65*(4), 689–694.

Wu, Tai Tsun and Chen Ning Yang (1975). “Concept of Nonintegrable Phase Factors and Global Formulation of Gauge Fields.” *Phys. Rev.*, *D12*, 3845–3857.

Yang, C. N. and R. L. Mills (1954). “Conservation of Isotopic Spin and Isotopic Gauge Invariance.” *Phys. Rev.*, *Oct*, 1954, *96*, 191–195.

Zeidler, Eberhard (2011). *Quantum Field Theory III: Gauge Theory: A Bridge between Mathematicians and Physicists*. Springer Science & Business Media.