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The Rigidity Theorems and Pointwise ∂-estimates

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The Rigidity Theorems and Pointwise $\overline{\partial}$-estimates

DISSERTATION

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in Mathematics

by

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DEDICATION

To my family Steve, DeDe, and Katie.

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ABSTRACT OF THE DISSERTATION

The Rigidity Theorems and Pointwise $\overline{\partial}$-estimates

By

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Doctor of Philosophy in Mathematics

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By Guan and Zhou's resolution of the Suita Conjecture, it is known that for any open, hyperbolic Riemann surface X, the Bergman kernel K, the logarithmic capacity c_β, and the analytic capacity c_B, are related by $\pi K \geq c_\beta \geq c_B$. When X is a domain in $\mathbb{C}$, we show that $c_B \geq \pi(Vol(X))^{-1}$ where Vol is the volume, and determine the conditions for when there exists a point z_0 such that $c_B(z_0) = \pi(Vol(X))^{-1}$, $c_\beta(z_0) = \pi(Vol(X))^{-1}$, and $\pi K(z_0) = \pi(Vol(X))^{-1}$. For open Riemann surfaces, we also determine equality conditions for $c_B \leq c_\beta$. A significant portion of this part of the thesis is based on joint work with Dong and Zhang.

The second part of the thesis is motivated by Henkin and Leiterer's question of whether uniform estimates for the $\overline{\partial}$-operator hold on the Cartan classical bounded symmetric domains. Using weighted L^2-methods initiated by Berndtsson, we obtain a pointwise estimate for the canonical solutions to the equation $\overline{\partial} u = f$ when f is bounded in a Bergman-type L^∞-norm. This part is based on joint work with Dong and Li.

In the third part of the thesis, we extend a theorem of M. Christ and S.-Y. Li on the $\overline{\partial}$-equation $\overline{\partial} u = f$. Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^∞ boundary which has a Stein neighborhood basis. We show that if f is a (p,q) form defined on Ω whose coefficients lie in a quasi-analytic class $C^L(\overline{\Omega})$, then there exists a solution u to $\overline{\partial} u = f$ such that the coefficients of u belong to the same quasi-analytic class.

Preface

In complex analysis of one variable and Riemann surface theory there are several canonical surface functions which have been used to prove many of the subject's major theorems. Every open, potentially-hyperbolic Riemann surface X has a Green's function G. The Green's function is essential to the proof of the uniformization theorem, and in the case of simply-connected domains, the Riemann map can be written in terms of the Green's function. In harmonic function theory, the Green's function is used to solve Laplace's equation with Dirichlet boundary conditions. The Green's function gives rise to two additional surface functions: the logarithmic capacity c_β, and for domains in $\mathbb{C}$, the Bergman kernel function. The Bergman kernel is central to the proof of the Fefferman Mapping Theorem, which when restricted to domains in $\mathbb{C}$ affirmatively answers the question *do biholomorphisms between bounded domains with C^∞ boundaries extend to the closures as C^∞-diffeomorphisms?* The final canonical surface function we investigate is the analytic capacity c_B, which was defined by Ahlfors to study the domains which do not admit bounded holomorphic functions.

In 1972, Suita proved using the Green's function and Riemann surface theory that $c_B^2 \leq \pi K$ and equality holds at a single point on the surface if and only if the surface is either potentially parabolic or biholomorphic to the unit disk less a relatively closed polar set. Suita conjectured that

$$c_B^2 \leq c_\beta^2 \leq \pi K, \tag{0.1}$$

and that equality holds in $c_\beta^2 \leq \pi K$ at a single point on the surface if and only if the surface is either potentially parabolic or biholomorphic to the unit disk less a relatively closed polar set. His conjecture, now eponymously referred to as the Suita conjecture, was studied by Ohsawa, Błocki, and ultimately proved in its entirety by Guan and Zhou. The theorems which determine the equality conditions in $c_B^2 \leq \pi K$ and $c_\beta^2 \leq \pi K$, due to Suita and Guan-Zhou respectively, may be called rigidity theorems because equality at a single point on the surface determines the surface, when it is hyperbolic, up to biholomorphism and a polar set. In the first part of the thesis, we show that the rigdity theorems due to Suita and Guan and Zhou are two among many rigidity theorems for these canonical surface functions. For domains in $\mathbb{C}$, we extend the work of Guan and Zhou and Suita (0.1) by showing

$$\frac{\pi}{v(\Omega)} \leq c_B^2 \leq c_\beta^2 \leq \pi K, \tag{0.2}$$

where $v(\Omega)$ is the volume of Ω, and prove several rigidity theorems which describe the remaining equality conditions between any two of the four quantities listed in (0.2). The rigidity theorem for the equality case in $c_\beta \geq c_B$ is also valid for open Riemann surfaces, and it is proved by proving a rigidity theorem for the Green's function. A significant part of this portion of the thesis is joint work with Dr. Dong of the University of Connecticut and Dr. Zhang of Purdue Fort-Wayne. I am grateful for our collaboration.

One of the most striking differences between holomorphic function theory in complex analysis of one variable and several variables is that by Hartogs' extension theorem, it is easy to construct domains in $\mathbb{C}^n$, $n \geq 2$, where every holomorphic function is the restriction of a holomorphic function on a larger domain. Such domains show that not all domains are natural domains for studying the holomorphic functions. The most natural ones are called domains of holomorphy (see [40] for the precise definition). A fundamental question in the early to mid-twentieth century was whether there was a geometric characterization of the domains of holomorphy? One was ultimately given by Oka, Bremermann and Norguet in the 1940s and 1950s. Their work showed that the domains of holomorphy are precisely the pseudoconvex domains, domains satisfying a complex convexity condition, and equivalently, the domains where the $\overline{\partial}$-problem is always solvable in the C^∞-setting, [16].

The $\overline{\partial}$-problem asks for any (p,q)-form $f = \sum f_{I,J} dz^I \wedge d\bar{z}^J$ does there exist a $(p,q-1)$-form u so that the $\overline{\partial}$-equation $\overline{\partial}u = f$ holds. When we require $f \in L^2_{(p,q)}(\Omega)$ and $u \in L^2_{(p,q-1)}(\Omega)$, we call this problem, the $\overline{\partial}$-problem in the L^2-setting. An analogous statement describes the $\overline{\partial}$-problem in the C^∞-setting. The $\overline{\partial}$-equation is the fundamental partial differential equation of several complex variables. It is the basis of the L^2-methods of several complex variables and the $\overline{\partial}$-Neumann problem. Through Kohn's formula, it is related to the Bergman kernel and metric, which are used to prove Fefferman's generalization of the Riemann Mapping Theorem: *Let $f : \Omega_1 \to \Omega_2$ be a biholomorphism between bounded strictly pseudoconvex domains with C^∞-boundary. Then f extends to a C^∞-diffeomorphism between the closures of the domains.*

Let $A^2(\Omega) = L^2(\Omega) \cap \ker(\overline{\partial})$ denote the Bergman space over Ω. Since the kernel of $\overline{\partial}$ contains the holomorphic functions, if the $\overline{\partial}$-problem is solvable in the L^2-setting and $A^2(\Omega) \neq \{0\}$, then the $\overline{\partial}$-equation $\overline{\partial}u = f$ has infinitely many solutions. However, there exists a unique solution to $\overline{\partial}u = f$ with $u \perp A^2(\Omega)$, which is called the canonical solution because it has minimal L^2-norm among all solutions. Hörmander [35] showed that if Ω is bounded and pseudoconvex and $f \in L^2_{(0,1)}(\Omega)$ is $\overline{\partial}$-closed, then the canonical solution u satisfies the estimate $\|u\|_{L^2} \leq C\|f\|_{L^2}$ for some constant C depending only on the diameter of Ω. In view of Hörmander's result, a natural question is when the canonical solution can be represented by a continuously-differentiable function, can pointwise estimates be given on the values of the canonical solution? In the second part of the thesis, we will investigate this question on certain convex domains. Let $g = (g_{j\bar{k}})_{j,k=1}^n$ be the Bergman metric on a domain Ω. For a $(0,1)$-form $f = \sum_{j=1}^n f_j d\bar{z}_j$, define

$$\|f\|^2_{g,\infty} := \operatorname{ess\,sup}\Big\{ \sum_{j,k=1}^n g^{j\bar{k}}(z) f_k(z) \overline{f_j(z)} : z \in \Omega \Big\},$$

where $(g^{j\bar{k}})^\tau = (g_{j\bar{k}})^{-1}$. Berndtsson used weighted L^2 estimates of Donnelly-Fefferman type to prove the following pointwise estimate.

Theorem 0.1. *[8] There is a constant $C = C(n)$ such that for any $\bar{\partial}$-closed $(0,1)$-form f on the unit ball in $\mathbb{C}^n$, the canonical solution to $\overline{\partial} u = f$ satisfies*

$$|u(z)| \leq C\|f\|_{g,\infty} \log \frac{2}{1-|z|}. \tag{0.3}$$

The estimate (0.3) is sharp. In the second part of the thesis, we will derive a pointwise estimate for $u(z)$ on the simple convex domains and the Cartan classical bounded symmetric domains. Both classes of domains contain the unit ball in $\mathbb{C}^n$. The estimate on the Cartan classical domains restricted to the unit ball will agree with Berndtsson's estimate (0.3). Additionally, we will show that the estimate we derive is sharp on the Cartan classical bounded symmetric domain $II(2)$. This part of the thesis is based on joint work with Dr. Dong of the University of Connecticut and Dr. Li of the University of California, Irvine.

When $\Omega \subset \mathbb{C}^n$, $n \geq 2$, is bounded pseudoconvex with C^∞-boundary, Kohn proved that for any $f \in C^\infty_{(p,q)}(\overline{\Omega})$ where $0 \leq p \leq n$, $1 \leq q \leq n$ with $\overline{\partial} f = 0$, there exists $u \in C^\infty_{(p,q-1)}(\overline{\Omega})$ such that $\overline{\partial} u = f$, [16, Theorem 6.1.1]. This important result due to Kohn was investigated further in 1997 by Christ and Li, when they examined the special case where $f \in C^\omega_{(p,q)}(\overline{\Omega})$ and Ω has additionally real-analytic boundary. They showed that if $\Omega \subset \mathbb{C}^n, n \geq 2$, is bounded pseudoconvex with C^ω-boundary, then for any $f \in C^\omega_{(p,q)}(\overline{\Omega})$ where $0 \leq p \leq n$, $1 \leq q \leq n$ with $\overline{\partial} f = 0$, there exists $u \in C^\omega_{(p,q-1)}(\overline{\Omega})$ such that $\overline{\partial} u = f$.

The class of real-analytic functions on a domain Ω can be distinguished from the class of C^∞-functions by the important property that the real-analytic functions are the functions which locally have convergent power series expansions. As a result of this property, the real-analytic functions satisfy a second property that any real-analytic function which vanishes to infinite order at any point in Ω must be the zero function. This second property does not imply the first property, and the classes of C^∞-functions which have this property are called quasi-analytic. When appropriately defined, the quasi-analytic classes can be recognized by the Denjoy-Carleman Theorem, and can serve as intermediate classes of functions between the real-analytic functions and the C^∞ class of functions.

In the final chapter of my thesis, we extend Christ and Li's theorem by investigating Kohn's regularity theorem for the $\overline{\partial}$-problem for forms f belonging to quasi-analytic classes. Let $C^L(\Omega)$ be a quasi-analytic class defined as in Chapter 4, and let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded pseudoconvex with C^∞-boundary and a Stein neighborhood basis. We show that if $f \in C^L_{(p,q)}(\overline{\Omega})$ where $0 \leq p \leq n$, $1 \leq q \leq n$ with $\overline{\partial} f = 0$, then there exists $u \in C^L_{(p,q-1)}(\overline{\Omega})$ such that $\overline{\partial} u = f$. Christ and Li's theorem follows from this result.

We begin the thesis by reviewing the basic facts about the Bergman kernel. For a domain in $\mathbb{C}^n$, the Bergman kernel is the integral kernel which reproduces the L^2-holomorphic functions. The Bergman kernel is central to the results in both Chapters 2 and 3.

Chapter 1

Introductory Bergman kernel theory

1.1 Introduction

In this chapter, we review background material on the Bergman kernel. The Bergman kernel will be extensively used in both Chapters 2 and 3.

1.2 The Bergman kernel

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $L^2(\Omega)$ denote the Hilbert space of square-integrable functions with respect to the Euclidean volume measure dv, the inner product and norm

$$\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)}\, dv(z), \quad \|f\|_2^2 = < f, f > .$$

Let $\mathcal{O}(\Omega)$ denote the set of holomorphic functions on Ω. The Bergman space $A^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$ is the set of L^2-holomorphic functions on Ω. Using the mean-value property, one can show that $A^2(\Omega)$ is a Hilbert space and the linear-functional

$$ev_z(f) = f(z), \quad f \in A^2(\Omega),$$

is bounded. By the Riesz representation theorem, ev_z has a Riesz representative $k_z(w) \in A^2(\Omega)$; that is $k_z(w)$ satisfies

$$f(z) = \int_\Omega f(w)\overline{k_z(w)}\, dv(w).$$

The Bergman kernel $K : \Omega \times \Omega \to \mathbb{C}$ is the function $K(z, w) = k_w(z)$. It is the unique function on $\Omega \times \Omega$ which satisfies

1. (reproducing property of the Bergman kernel)

$$f(z) = \int_\Omega f(w)K(z, w)dv(w), \quad f \in A^2(\Omega), \tag{1.1}$$

2. $K(z, w) = \overline{K(w, z)}$

3. $K(\cdot, w) \in A^2(\Omega)$ for all $w \in \Omega$.

When we wish to emphasize the domain Ω, we will use the notation K_Ω. When it is clear from context which domain the Bergman kernel belongs to we shall omit the subscript.

By the uniqueness of the Bergman kernel and the change of variables formula from integral calculus, it follows that the Bergman kernel satisfies a transformation law under biholomorphic mappings.

Theorem 1.1. *If $h : \Omega_1 \to \Omega_2$ is biholomorphic between two domains, then*

$$K_{\Omega_1}(z, w) = det(J_\mathbb{C}h(z))\overline{det(J_\mathbb{C}h(w))}K_{\Omega_2}(h(z), h(w)), \quad z, w \in \Omega, \tag{1.2}$$

where $J_\mathbb{C}h$ denotes the complex Jacobian of h.

As $L^2(\Omega)$ is separable, its subspace $A^2(\Omega)$ is also separable and admits an orthonormal basis $\{\phi_n\}_{n=0}^{\infty}$. It can be shown that the Bergman kernel has an orthonormal series expansion

$$K(z,w) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)}, \quad z, w \in \Omega, \tag{1.3}$$

which converges uniformly on compact subsets of $\Omega \times \Omega$. The series expansion is independent of the choice of orthonormal basis. In this thesis, we will denote the balls in $\mathbb{C}$ and $\mathbb{C}^n$ for $n > 1$ respectively by

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}, \quad \mathbb{B}^n(z_0, r) = \{z \in \mathbb{C}^n : |z - z_0| < r\}.$$

When $n > 1$, if $z = (z_1, \ldots, z_n)$ and $r = (r_1, \ldots, r_n)$, we will use the notation

$$\mathbb{D}^n(z, r) = \prod_{k=1}^{n} D(z_k, r_k),$$

to denote the polydisk centered at z with polyradius r. Using the orthonormal series expansion of the Bergman kernel (1.3), one can derive the Bergman kernel for $D(0,1)$ as follows. By integrating in polar coordinates,

$$\langle \frac{z^k}{\pi^{1/2}(k+1)^{-1/2}}, \frac{z^j}{\pi^{1/2}(j+1)^{-1/2}} \rangle = \delta_{kj}, \quad \left\| \frac{z^k}{\pi^{1/2}(k+1)^{-1/2}} \right\|_2 = 1.$$

Since any holomorphic function in $D(0,1)$ has a power series expansion converging in $D(0,1)$, by integrating in polar coordinates, we can also check that $\{z_k\}_{k=0}^{\infty}$ is a complete orthogonal system. By (1.3),

$$K_{D(0,1)}(z,w) = \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1) z^k \bar{w}^k = \frac{1}{\pi} \frac{1}{(1 - z\bar{w})^2}. \tag{1.4}$$

A similar, but more complicated calculation gives

$$K_{\mathbb{B}^n(0,1)}(z,w) = \frac{n!}{\pi^n} \frac{1}{(1 - \sum_{k=1}^{n} z_k \bar{w}_k)^{n+1}}. \tag{1.5}$$

By the transformation property of the Bergman kernel (1.2), similar equations can be derived for the Bergman kernel for balls of other radii centered at other points in $\mathbb{C}^n$. For details on these calculations and the preceding Bergman kernel theory, see [40]. When Ω has finite volume, $\|v(\Omega)^{\frac{-1}{2}}\|_2 = 1$, and $\{v(\Omega)^{\frac{-1}{2}}\}$ can be completed to an orthonormal basis $\{v(\Omega)^{\frac{-1}{2}}\} \cup \{\phi_k\}_{k=1}^{\infty}$ of $A^2(\Omega)$. By (1.3),

$$K(z, z) = v(\Omega)^{-1} + \sum_{k=1}^{\infty} |\phi_k(z)|^2, \quad z \in \Omega, \tag{1.6}$$

which implies the lower bound for the Bergman kernel

$$K(z, z) \geq v(\Omega)^{-1}. \tag{1.7}$$

We will use the convention that $v(\Omega)^{-1} = 0$ when $v(\Omega) = \infty$. When we consider the Bergman kernel restricted to the diagonal, for brevity we often write $K(z) := K(z, z)$. This lower bound also holds for domains of infinite volume because by (1.3), $K(z) \geq 0$. It can be seen by (1.4) and (1.5) that this lower bound is sharp.

The proceeding chapter will in part investigate the question *is there a geometric characterization of the domains* Ω *so that* $K_\Omega(z) = v(\Omega)^{-1}$*?* A few more observations that will be relevant to this question can be made from the basic theory described in this chapter.

Lemma 1.2. *Let* $\Omega \subset \mathbb{C}^n$ *be a domain. If there exists* $z_0 \in \Omega$ *so that* $K(z_0, z_0) = v(\Omega)^{-1}$*, then* $K(\cdot, z_0) = v(\Omega)^{-1}$ *and* Ω *satisfies the mean-value property*

$$f(z_0) = \frac{1}{v(\Omega)} \int_\Omega f(w) dv(w), \quad f \in A^2(\Omega). \tag{1.8}$$

Proof. Suppose $v(\Omega) < \infty$ and let $\{v(\Omega)^{-\frac{1}{2}}\} \cup \{\phi_k\}_{k=1}^{\infty}$ denote an orthonormal basis of $A^2(\Omega)$. By (1.3), $\phi_k(z_0) = 0$. It follows that $K(\cdot, z_0)$ is a constant function. If $v(\Omega) = \infty$, then for any orthonormal basis $\{\phi_k\}_{k=1}^{\infty}$ of $A^2(\Omega)$, we have again that $\phi_k(z_0) = 0$ and $K(\cdot, z_0) = 0$. The mean-value property (1.8) holds by the reproducing property of the Bergman kernel (1.1).

□

Chapter 2

The rigidity theorems

2.1 Introduction

Let X be an open hyperbolic Riemann surface (that is, an open Riemann surface admitting a Green's function) and consider the negative Green's function $G(z, z_0)$, the logarithmic capacity c_β and the analytic capacity c_B for X. Here, with respect to a fixed local coordinate,

$$G(z, z_0) = \sup\{u(z) : u \in SH^-(X), \limsup_{z \to z_0} u(z) - \ln|z - z_0| < \infty\}, \tag{2.1}$$

where $SH^-(X)$ denotes the negative subharmonic functions on X not identically equal to $-\infty$,

$$c_\beta(z) = \lim_{z \to z_0} \exp(G(z, z_0) - \ln|z - z_0|) \tag{2.2}$$

and

$$c_B(z_0) = \sup\left\{\left|\frac{\partial f}{\partial z}(z_0)\right| : f \in \mathcal{O}(X), |f| < 1, f(z_0) = 0\right\}. \tag{2.3}$$

In 1972, Suita [52] determined the relationship between the analytic capacity and the Bergman kernel restricted to the diagonal $K(z) := K(z, z)$.

Theorem 2.1. [52] *Suppose X is an open Riemann surface. Then $\pi K(z) \geq c_B^2(z)$ and equality holds at some $z_0 \in X$ if and only if either X is biholomorphically equivalent to the unit disk less a (possibly empty) closed set of inner capacity zero, or X is a parabolic Riemann surface.*

A closed set of inner capacity zero is a relatively closed polar set. One may think of Suita's theorem as a rigidity theorem because equality at a single point between the two quantities determines the surface up to biholomorphism. In that same paper, Suita conjectured that the logarithmic capacity would satisfy a similar inequality and rigidity theorem.

Theorem 2.2 (Suita Conjecture)**.** [52] *Suppose X is an open Riemann surface. Then $\pi K(z) \geq c_\beta^2(z)$ and equality holds at some $z_0 \in X$ if and only if either X is biholomorphically equivalent to the unit disk less a (possibly empty) closed set of inner capacity zero, or X is a parabolic Riemann surface.*

The Suita conjecture was proved completely in 2015 by Guan and Zhou [31] and has been studied by several other outstanding mathematicians. In 1995, Ohsawa noted [49] that the Suita conjecture was connected to his Ohsawa-Takegoshi L^2-extension theorem. He was able to prove that $c_\beta^2 \leq 750\pi K$. Błocki lowered the constant from 750π to 2π, and in 2013, he proved the inequality $c_\beta^2 \leq \pi K$ for bounded domains in $\mathbb{C}$, see [10, 11]. In 2015, Guan and Zhou proved the Suita conjecture in its entirety. Their solution of the Suita conjecture was proved as an application of their proof of the Ohsawa-Takegoshi L^2-extension theorem with optimal constants. It should be noted in particular, their solution is a very deep theorem which used methods of several complex variables. Dong [23] reproved parts of Guan and Zhou's proof using Maitani and Yamaguchi's variational formula for the Bergman kernel.

It should be noted that it is straightforward to show that $c_B^2 \leq c_\beta^2$, see Lemma 2.26 below; hence the solution of the Suita conjecture does not follow from Suita's Theorem, Theorem 2.1.

Consequently, for any open hyperbolic Riemann surface

$$\pi K \geq c_\beta^2 \geq c_B^2, \tag{2.4}$$

and if $\pi K(z_0) = c_\beta^2(z_0)$ or $\pi K(z_0) = c_B^2(z_0)$, then the Riemann surface is biholomorphic to a disk less a relatively closed polar set.

In this chapter, we first establish a rigidity theorem which desribes the equality condition between c_B and c_β in (2.4). We then restrict our attention from surfaces to domains in $\mathbb{C}$ where we extend (2.4) by proving that for any domain $\Omega \subset \mathbb{C}$,

$$\pi K \geq c_\beta^2 \geq c_B^2 \geq \frac{\pi}{v(\Omega)}. \tag{2.5}$$

We establish the conditions for equality between the various quantities, and in particular, we give several proofs of the equality conditions for the inequality $\pi K(z_0) \geq \frac{\pi}{v(\Omega)}$. We begin by studying in more depth the Green's function, logarithmic capacity and analytic capacity.

This chapter is based on the works [25], [54], [26]. I thank my collaborators Dr. Dong and Dr. Zhang.

2.2 The Green's function of Riemann surfaces

In this chapter, we consider the (negative) Green's function (of the Laplacian).

Definition 2.3. *For an open Riemann surface X, the Green's function with pole at $z_0 \in X$, if it exists, is defined to be*

$$G(z, z_0) = \sup\{u(z) : u \in SH^-(X), \limsup_{z \to z_0} u(z) - \ln|z - z_0| < \infty\}, \tag{2.1 revisited}$$

where $SH^-(X)$ denotes the negative subharmonic functions not identically equal to $-\infty$. If the Green's function with a pole at z_0 exists for all $z_0 \in X$, then X is said to be **hyperbolic**. *Otherwise, it is said to be* **parabolic**.

Remark 2.4. *The definition of hyperbolic/parabolic follows the definition used in the classification theory of open Riemann surfaces, cf. [3]. There are other non-equivalent definitions of hyperbolic and parabolic used in other areas of geometry.*

From the definition, several properties of the Green's function can be derived.

Proposition 2.5. *An open Riemann surface is hyperbolic if and only if there exists a non-constant negative subharmonic function defined on it. Moreover*

1. *The Green's function with pole at $z_0 \in X$ exists if and only if $G_X(\cdot, z)$ exists for all $z \in X$.*

Consequently, we can refer to the Green's function $G_X : X \times X \to \mathbb{R}$ without referring to the particular pole. The Green's function satisfies additionally

2. $G_X(\cdot,\cdot)$ *is harmonic on* $X \times X \setminus \{(z,w) : z = w\}$, *and* $G_X(z, z_0) - \ln|z - z_0|$ *is harmonic in* z *in a neighborhood of* z_0.
3. $G_X(\cdot, z_0) \in SH^-(X), \quad z_0 \in X$.
4. *(Symmetry property)* $G_X(z,w) = G_X(w,z), \quad z, w \in X \times X$
5. $G_X(z,w) = -\infty$ *if and only if* $z = w$.
6. *(Monotonicity property) If* $X_1 \subset X_2$, *then* $G_{X_1}(z, z_0) \geq G_{X_2}(z, z_0)$.
7. *(Biholomorphic transformation law) If* $h : X_1 \to X_2$ *is biholomorphic and* X_2 *admits a Green's function, then* $G_{X_1}(z,w) = G_{X_2}(h(z), h(w))$.

Remark 2.6. *By Proposition 2.5(1), a Riemann surface is hyperbolic if and only if the Green's function with pole at z_0 exists for a single $z_0 \in X$.*

There are several equivalent definitions for the Green's function. The basic properties in Proposition 2.5 can be found in several books including [3] and [50].

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. The boundary values of the Green's function for domains in $\mathbb{C}_\infty$ are important for our applications.

Definition 2.7. *Let $\Omega \subset \mathbb{C}_\infty$ be a domain admitting a Green's function. A point $z_0 \in \partial\Omega$ is said to be a regular boundary point if*

$$\lim_{z \to z_0} G(z, w) = 0, \quad w \in \Omega,$$

otherwise, z_0 is said to be an irregular boundary point.

The set of irregular boundary points is small.

Definition 2.8. *A Borel set $E \subset \mathbb{C}_\infty$ is said to be* **polar** *if there is an open set U containing E and a subharmonic function $u : U \to \mathbb{R} \cup \{-\infty\}$ such that*

$$E \subset \{z \in U : u(z) = -\infty\}.$$

Proposition 2.9. *If E is a polar set, then E is totally disconnected and has Hausdorff dimension 0; hence its two-dimensional Lebesgue measure is 0.*

From Proposition 2.9, since polar sets are totally-disconnected, a domain Ω satisfies that $\partial\Omega$ is polar if and only if $\mathbb{C}_\infty \setminus \Omega$ is polar. In that case $\mathbb{C}_\infty \setminus \Omega = \partial\Omega$.

Theorem 2.10. [Kellogg's Theorem] *Let $\Omega \subset \mathbb{C}_\infty$ be a domain which admits a Green's function. The set of irregular boundary points form an F_σ-polar set (a polar set which is the countable union of closed sets.)*

Using the notion of polar sets, we can classify the parabolic domains in $\mathbb{C}_\infty$.

Theorem 2.11. *A domain $\Omega \subset \mathbb{C}_\infty$ is parabolic if and only if $\mathbb{C}_\infty \setminus \Omega$ is polar if and only if $\partial\Omega$ is polar.*

Using Theorem 2.11, we can prove a subordination property of the Green's function, well-known for domains, cf. [50]. The proof in [50] needs to be modified to work for Riemann surfaces with our choice of definition of the Green's function. Therefore, we prove the subordination property below in its entirety.

Lemma 2.12 (Subordination Property)**.** *Let X be an open Riemann surface and let f be a holomorphic function on X such that $\partial f(X)$ is not a polar set. Then*

$$G_X(z, z_0) \geq G_{f(X)}(f(z), f(z_0)), \quad (z, z_0) \in X \times X. \tag{2.6}$$

Moreover, if there exists a $(z, z_0) \in X \times X$ with $z \neq z_0$ so that equality holds, then

$$G_X(z, z_0) = G_{f(X)}(f(z), f(z_0)), \quad (z, z_0) \in X \times X, \tag{2.7}$$

and f is injective.

Proof. Since $\partial f(X)$ is not polar, $f(X)$ admits a Green's function $G_{f(X)}$. If $u : f(X) \to \mathbb{C}$ is a non-constant, negative subharmonic function, then $u \circ f$ is also non-constant, negative and subharmonic. Thus, X admits a Green's function G_X. First assume that $\frac{\partial f}{\partial z}(z_0) \neq 0$. Since

$$G_{f(X)}(f(z), f(z_0)) - \ln|f(z) - f(z_0)| = \mathrm{O}(1), \quad z \to z_0,$$

$$G_{f(X)}(f(z), f(z_0)) - \ln|z - z_0| = \ln\left|\frac{\partial f}{\partial z}(z_0)\right| + \mathrm{O}(1) = \mathrm{O}(1), \quad z \to z_0.$$

Thus, $G_{f(X)}(f(z), f(z_0))$ is in the defining set of $G_X(\cdot, z_0)$, which implies that (2.6) holds for z_0

such that $\frac{\partial f}{\partial z}(z_0) \neq 0$. Since the zero set of df is discrete and Green's functions are continuous off their diagonal, the inequality (2.6) holds for all $(z, z_0) \in X \times X$.

Suppose $G_X(z, z_0) = G_{f(X)}(f(z), f(z_0))$ for some $z \neq z_0$. Define

$$u_1(\cdot) := G_{f(X)}(f(\cdot), f(z_0)) - G_X(\cdot, z_0), \quad u_2(\cdot) := G_{f(X)}(f(z), f(\cdot)) - G_X(z, \cdot).$$

Then

$$u_1 \in SH(X \setminus \{z_0\}), \quad u_2 \in SH(X \setminus \{z\}),$$

and they attain a maximum at z and z_0 respectively. By the maximum principle,

$$G_{f(X)}(f(\cdot), f(z_0)) = G_X(\cdot, z_0), \quad G_{f(X)}(f(z), f(\cdot)) = G_X(z, \cdot),$$

which proves (2.7).

If $f(z) = f(z_0)$, then by (2.7), $G_X(z, z_0) = -\infty$. Thus, $z = z_0$. □

One of the most famous applications of the Green's function is the uniformization theorem [3, Theorem III.11G]. We state one of its consequences.

Theorem 2.13. *Let X be an open, hyperbolic Riemann surface. Then there exists a holomorphic covering map $p : D(0, 1) \to X$.*

2.3 Logarithmic capacity

Definition 2.14. *Let X be an open hyperbolic Riemann surface. The logarithmic capacity $c_\beta : X \to \mathbb{R}$ is defined as*

$$c_\beta(z) = \lim_{z \to z_0} \exp(G(z, z_0) - \ln|z - z_0|) \qquad \text{(2.2 revisited)}$$

where z is a local coordinate. If X is an open parabolic Riemann surface, then we define $c_\beta \equiv 0$.

When we wish to emphasize the surface under consideration, we will use the notation $c_{\beta,X}$.

Remark 2.15. *By Proposition 2.5(2), if X is hyperbolic, then $c_\beta > 0$. Thus, for a domain $\Omega = \mathbb{C}_\infty \setminus E$, by Theorem 2.11, E is polar if and only if $c_{\beta,\mathbb{C}_\infty\setminus E} \equiv 0$ if and only if $c_{\beta,\mathbb{C}_\infty\setminus E}$ vanishes at a single point.*

One can show that the logarithmic capacity satisfies a transformation law under biholomorphic mappings.

Proposition 2.16. [Transformation law of the logarithmic capacity]

Let $h : X_1 \to X_2$ be a biholomorphic mapping between open Riemann surfaces such that X_2 is hyperbolic. Then

$$c_{\beta,X_1}(z_0) = c_{\beta,X_2}(h(z_0)) \left| \frac{\partial h}{\partial z}(z_0) \right|,$$

where z is a local coordinate.

Proof. If $u : X_2 \to \mathbb{R} \cup \{-\infty\}$ is a negative, subharmonic function, then so is $u \circ h$. Thus, X_1 is also hyperbolic. Notice that

$$\begin{aligned} c_{\beta,X_1}(z_0) &= \lim_{z\to z_0} \exp(G_{X_1}(z, z_0) - \ln|z - z_0|) \\ &= \lim_{z\to z_0} \exp(G_{X_2}(h(z), h(z_0)) - \ln|h(z) - h(z_0)| + \ln|h(z) - h(z_0)| - \ln|z - z_0|) \\ &= c_{\beta,X_2}(h(z_0)) \left| \frac{\partial h}{\partial z}(z_0) \right|. \end{aligned}$$

□

Polar sets are removable singularities for bounded harmonic functions. Since the Green's function with a pole at w is harmonic and bounded away from w, closed polar sets are removable singularities for the Green's function.

Proposition 2.17. *Let $\Omega \subset \mathbb{C}$ be a domain and $P \subset \Omega$ be a relatively closed polar set. Then*

$$G_{\Omega\setminus P}(z,w) \equiv G_{\Omega}(z,w), \quad z,w \in \Omega \setminus P.$$

We can also use that polar sets are removable singularities for bounded harmonic functions, to extend Liouville's Theorem.

Proposition 2.18. *Let $P \subset \mathbb{C}$ be a closed polar set. If $h : \mathbb{C}\setminus P \to \mathbb{C}$ is a bounded holomorphic function, then h is constant.*

Theorem 2.19. *Let $P \subset D(0,1)$ be a relatively closed polar set. If $h : D(0,1)\setminus P \to X$ is biholomorphic, then*

$$c_{\beta,X}(h(z)) = \frac{1}{1-|z|^2}\left|\frac{\partial h}{\partial z}(z)\right|^{-1}. \tag{2.8}$$

Proof. The Green's function with pole 0 for $D(0,1)$ is given by $G_{D(0,1)}(z,0) = \ln|z|$. By the biholomorphic transformation law of the Green's function and Proposition 2.17,

$$G_{D(0,1)\setminus P}(z,w) = \ln\left|\frac{w-z}{1-\bar{w}z}\right|, \quad z,w \in D(0,1)\setminus P.$$

Thus,

$$\begin{aligned} c_{\beta,D(0,1)\setminus P}(z) &= \lim_{w\to z}\exp(\ln\left|\frac{z-w}{1-\bar{z}w}\right| - \ln|w-z|) \\ &= \lim_{w\to z}\exp(-\ln|1-\bar{z}w|) \\ &= \frac{1}{1-|z|^2}. \end{aligned}$$

The result follows from Proposition 2.16. □

2.4 Analytic capacity

In classical complex analysis, one of the fundamental questions is Painlevé's question: *which compact sets E in the Riemann sphere are removable for the bounded holomorphic functions.* We can think of E as the complement of the domain $\mathbb{C}_\infty \setminus E$ and reformulate Painlevé's question as *which domains in $\mathbb{C}$ do not admit non-constant, bounded holomorphic functions?*

From Liouville's theorem and Riemann's removable singularity theorem, if E is a discrete closed set, then any bounded holomorphic function on $\mathbb{C}_\infty \setminus E$ is constant. On the other hand, if E is a compact connected set with at least two points, then $\mathbb{C}_\infty \setminus E$ is simply-connected and by the Riemann mapping theorem, there exists a bounded holomorphic function mapping $\mathbb{C}_\infty \setminus E$ into $D(0,1)$. From these examples, we see that the domains which do not admit bounded, non-constant holomorphic functions exist and their complements are totally-disconnected.

Ahlfors introduced the analytic capacity

$$c_B(z_0) = \sup\left\{\left|\frac{\partial f}{\partial z}(z_0)\right| : f \in \mathcal{O}(\Omega), |f| < 1, f(z_0) = 0\right\},$$

for domains Ω, [2], [1] in order to study Painlevé's question. The definition of analytic capacity extends to open Riemann surfaces as a conformally-invariant metric $c_B(z)|dz|$. Compact Riemann surfaces are not of any interest because all holomorphic functions on a compact Riemann surface are constant. The domain or surface under consideration will usually be understood from context, but when it is not, we will use the notation $c_{B;X}$ to clarify that the analytic capacity refers to the domain or surface X. It is obvious that for a domain Ω, $c_B \equiv 0$ if and only if Ω does not admit non-constant, bounded holomorphic functions. In this section, we introduce results about the analytic capacity needed for subsequent sections. We begin by defining the Painlevé null sets, cf. [1].

Definition 2.20. *A compact set $E \subset \mathbb{C}_\infty$ is said to be a null set of class $\mathcal{N}_B$ if*

$$c_{B,\mathbb{C}_\infty \setminus E} \equiv 0.$$

Remark 2.21. *Equivalently, a compact set $E \in \mathcal{N}_B$ if the set of bounded holomorphic functions on $\mathbb{C}_\infty \setminus E$ consists of only the constant functions.*

Remark 2.22. *By Proposition 2.18, if P is a compact polar set, then $P \in \mathcal{N}_B$. However, there are sets $Q \in \mathcal{N}_B$ which are not polar, cf. [30].*

The next proposition leads to another equivalent definition of a null set of class $\mathcal{N}_B$.

Proposition 2.23. [1] *Let $\Omega \subset \mathbb{C}_\infty$ be a domain. Then $c_B \equiv 0$ if and only if there exists a $z_0 \in \Omega$ so that $c_B(z_0) = 0$.*

Proof. We prove the non-trivial direction. After an automorphism of $\mathbb{C}_\infty$ we may assume $z_0 \neq \infty$. We first claim that

$$c_B(z_0) = \sup \left\{ \left| \frac{\partial f}{\partial z}(z_0) \right| : f \in \mathcal{O}(\Omega, D(0,1)) \right\}. \tag{2.9}$$

Notice that for $a \in D(0,1)$, the automorphism $\phi_a(z) = (a-z)/(1-\bar{a}z)$ of $D(0,1)$ satisfies $\phi_a'(a) \geq 1$. Thus, if f is in the defining set of the right hand side, then $\phi_{f(z_0)} \circ f$ is in the defining set of $c_B(z_0)$ and

$$|(\phi_{f(z_0)} \circ f)'(z_0)| \geq \left| \frac{\partial f}{\partial z}(z_0) \right|,$$

which proves

$$c_B(z_0) \geq \sup \left\{ \left| \frac{\partial f}{\partial z}(z_0) \right| : f \in \mathcal{O}(\Omega, D(0,1)) \right\}. \tag{2.10}$$

The reverse inequality is trivial, which establishes the claim. Suppose that $c_B(z_0) = 0$. If $f \in \mathcal{O}(\Omega, D(0,1))$ is non-constant, then in a neighborhood of z_0, there exists $k > 1$ so that

$$f(z) = f(z_0) + a_k(z-z_0)^k + O((z-z_0)^{k+1}), \quad z \to z_0, \quad a_k \neq 0.$$

The function $g(z) = c(f(z) - f(z_0))/(z - z_0)^{k-1}$ for sufficiently small $c > 0$ is in the defining set of $c_B(z_0)$ and $g'(z_0) > 0$, which is a contradiction. Thus, $c_B \equiv 0$. □

As an example we calculate the analytic capacity for certain domains.

Proposition 2.24. *Let $h : X \to h(X)$ be biholomorphic. Then*

$$c_{B,X}(z_0) = c_{B,h(X)}(h(z_0)) \left| \frac{\partial h}{\partial z}(z_0) \right|.$$

Proof. If f is in the defining set of $c_{B,h(X)}(h(z_0))$, then $f \circ h$ is in the defining set of $c_{B,X}(z_0)$. Since

$$\left| \frac{\partial f \circ h}{\partial z}(z_0) \right| = \left| \frac{\partial f}{\partial w}(h(z_0)) \right| \left| \frac{\partial h}{\partial z}(z_0) \right| \le c_{B,X}(z_0),$$

$$c_{B,h(X)}(h(z_0)) \left| \frac{\partial h}{\partial z}(z_0) \right| \le c_{B,X}(z_0). \tag{2.11}$$

Similarly,

$$c_{B,X}(z_0) \left| \frac{\partial h^{-1}}{\partial w}(h(z_0)) \right| \le c_{B,h(X)}(h(z_0)).$$

Thus,

$$c_{B,h(X)}(h(z_0)) \left| \frac{\partial h}{\partial z}(z_0) \right| \ge c_{B,X}(z_0). \tag{2.12}$$

The result follows by considering (2.11) and (2.12). □

Lemma 2.25. *Let $Q \subset D(z_0, r)$ be relatively closed such that*

$$Q \cap \overline{D(z_0, s)} \in \mathcal{N}_B, \quad s < r.$$

Then

$$c_{B,D(z_0,r)\setminus Q}(z_0) = \sqrt{\frac{\pi}{v(D(z_0, r))}} = \frac{1}{r}. \tag{2.13}$$

Additionally, let $P \subset D(0,1)$ be a relatively closed polar set and $h : D(0,1) \setminus P \to X$ be biholo-

morphic. Then

$$c_{\beta,X} \equiv c_{B,X}. \tag{2.14}$$

Proof. By the Schwarz lemma, if f is in the defining set of $c_{B,D(0,1)}(0)$, then $|f'(0)| \leq 1$ and equality holds for $f(z) = e^{i\theta}z$. Thus, $c_{B,D(0,1)}(0) = 1$. By Proposition 2.24, since $\phi_a = (a-z)/(1-\bar{a}z)$ where $|a| < 1$ is an automorphism of $D(0,1)$,

$$c_{B,D(0,1)}(w) = c_{B,D(0,1)}(0)|\phi_w'(w)| = \frac{1}{1-|w|^2}. \tag{2.15}$$

By Proposition 2.24,

$$c_{B,D(z_0,r)}(w) = \frac{1}{r(1-\left|\frac{w-z_0}{r}\right|^2)}.$$

Since $\mathcal{N}_B$ sets are removable for bounded analytic functions, plugging in $w = z_0$ proves (2.13). Since $P \in \mathcal{N}_B$,

$$c_{B,D(0,1)\setminus P}(w) = \frac{1}{1-|w|^2}, \tag{2.16}$$

If $h : D(0,1) \setminus P \to X$ is biholomorphic, then by (2.16) and Proposition 2.24,

$$c_{B,X}(h(z)) = \frac{1}{1-|z|^2}\left|\frac{\partial h}{\partial z}(z)\right|^{-1}, \quad z \in D(0,1) \setminus P.$$

Comparing this formula with (2.8) proves (2.14). □

In general, $c_B \leq c_\beta$.

Lemma 2.26. *For an open Riemann surface $c_B \leq c_\beta$.*

Proof. First assume X is hyperbolic. Then $c_\beta > 0$ by Remark 2.15. Without loss of generality, we may assume $c_B \neq 0$. It suffices to show that if f is in the defining set of the analytic capacity, (2.3), and $\frac{\partial f}{\partial z}(z_0) \neq 0$, then

$$\ln \left| \frac{\partial f}{\partial z}(z_0) \right| \leq \lim_{z \to z_0} G(z, z_0) - \ln |z - z_0|. \tag{2.17}$$

Notice that

$$-\infty \neq \ln \left| \frac{\partial f}{\partial z}(z_0) \right| = \ln \left| \lim_{z \to z_0} \frac{f(z)}{z - z_0} \right| = \lim_{z \to z_0} \ln |f(z)| - \ln |z - z_0|. \tag{2.18}$$

Since $\ln |f|$ is in the defining set of the Green's function (2.1), (2.17) follows from (2.18).

Suppose now that X is parabolic. If f is a bounded, nonconstant holomorphic function, then $|f| - \sup_X |f|$, is a negative, non-constant subharmonic function. Since X is parabolic, such an f cannot exist. Thus, $c_B \equiv 0$, which completes the proof of the lemma. □

Definition 2.27. *A function $f_0 : X \to D(0,1)$ is said to be an extremal function for the defining set of the analytic capacity or an extremal function of $c_B(z_0)$ if*

$$f_0(z_0) = 0, \qquad \left| \frac{\partial f_0}{\partial z}(z_0) \right| = c_B(z_0).$$

A normal family argument shows that extremal functions for c_B exist.

Proposition 2.28. *Let X be an open Riemann surface and $z_0 \in X$. There exists an extremal function f_0 for $c_B(z_0)$.*

Proof. If $c_B(z_0) = 0$, then any constant function is an extremal function. Thus, we may assume that $0 < c_B(z_0) \leq c_\beta(z_0)$. Since $c_\beta(z_0) > 0$, X is hyperbolic. By the uniformization theorem, there exists a holomorphic covering map $p : D(0,1) \to X$. Let $\{f_n\}$ be in the defining set of $c_B(z_0)$ so that $\{\frac{\partial f_n}{\partial z}(z_0)\}$ is monotone increasing to $c_B(z_0)$. By Montel's theorem, there is a holomorphic function $g : D(0,1) \to D(0,1)$ and a sequence $\{n_m\}_{m=1}^{\infty}$ so that $f_{n_m} \circ p$ converges to g uniformly on compact subsets of $D(0,1)$. Let $x \in X$ and consider a neighborhood U_x of x and a local inverse

p_1 of p such that p_1 is biholomorphic on a neighborhood of $\overline{U_x}$ onto its image in $D(0,1)$.

Since $\{f_{n_m} \circ p\}$ converges uniformly on $p_1(\overline{U_x})$ to g, $\{f_{n_m}\}$ converges uniformly on $\overline{U_x}$ to $g \circ p_1$. As $x \in X$ was arbitrary, it follows that $\{f_{n_m}\}$ converges uniformly on compact subsets of X. Let $f_0(z)$ equal g evaluated at any element of $p^{-1}(z)$. Since $x \in X$ was arbitrary and the choice of the local inverse of p_1 of p was arbitrary, it follows that f_0 is a well-defined, holomorphic function on X. Thus, f_{n_m} converges uniformly on compact sets of X to f_0. By the uniform convergence, $|\frac{\partial f_0}{\partial z}(z_0)| = c_B(z_0)$. □

If f_0 is an extremal function, then so is $e^{i\theta} f_0$ for $\theta \in [0, 2\pi]$. Havinson [32] described the images of the extremal functions in $D(0,1)$.

Proposition 2.29. [32, Theorem 28] *Let f_0 be an extremal function of the analytic capacity of Ω such that $c_B \not\equiv 0$. Then $f_0(\Omega) = D(0,1) \setminus P$ where P satisfies that*

$$P \cap \overline{D(0,r)} \in \mathcal{N}_B, \quad 0 \le r < 1.$$

The analytic capacity is often defined in the literature, equivalently, in terms of compact sets.

Definition 2.30. [29] *The analytic capacity $\gamma(E)$ of a compact subset $E \subset \mathbb{C}$ is*

$$\gamma(E) = \sup\{|g'(\infty)| : g \in \mathcal{O}(\mathbb{C}_\infty \setminus E), g(\infty) = 0, |g(z)| \le 1\}$$

where

$$g'(\infty) = \lim_{z \to \infty} z(g(z) - g(\infty)).$$

A compact set $E \subset \mathbb{C}$ satisfies $\gamma(E) = 0$ if and only if every bounded holomorphic function on $\mathbb{C} \setminus E$ is constant. The two definitions of analytic capacity given are related as follows.

Definition 2.31. *Let $f_{z_0}(z) = \frac{1}{z - z_0}$. For ease of notation, when $z_0 = 0$, let $f = f_{z_0}$.*

Lemma 2.32. *For any domain $\Omega \subset \mathbb{C}$, $c_B(z_0) = \gamma(\mathbb{C}_\infty \setminus f_{z_0}(\Omega))$.*

Proof. Since $f(\Omega - \{z_0\}) = f_{z_0}(\Omega)$, by Proposition 2.24, $c_{B,\Omega}(z_0) = c_{B,\Omega-\{z_0\}}(0)$, it suffices to prove the lemma when $z_0 = 0 \in \Omega$. Notice that for g holomorphic in a neighborhood of ∞ with $g(\infty) = 0$,

$$\begin{aligned}
(g \circ f)'(\infty) &= \lim_{w\to\infty} w((g \circ f)(w) - (g \circ f)(\infty)) \\
&= \lim_{w\to 0} \frac{g \circ f(\frac{1}{w}) - g(0)}{w - 0} \\
&= \lim_{w\to 0} \frac{g(w) - g(0)}{w - 0} \\
&= g'(0).
\end{aligned}$$

Let $E = \mathbb{C}_\infty \setminus f(\Omega)$. Then

$$\begin{aligned}
c_B(0) &= \sup\{|g'(0)| : g \in \mathcal{O}(\Omega), g(0) = 0, |g(z)| \le 1\} \\
&= \sup\{|(g \circ f)'(\infty)| : g \circ f \in \mathcal{O}(f(\Omega)), (g \circ f)(\infty) = 0, |g \circ f| \le 1\} \\
&\le \gamma(E) \\
&= \sup\{|g'(\infty)| : g \in \mathcal{O}(\mathbb{C}_\infty \setminus E), g(\infty) = 0, |g| \le 1\} \\
&= \sup\{|(g \circ f)'(0)| : g \circ f \in \mathcal{O}(\Omega), (g \circ f)(0) = 0, |g \circ f| \le 1\} \\
&\le c_B(0).
\end{aligned}$$

□

The analytic capacity is difficult to compute in general for most domains. It does however satisfy the following lower bound due to Ahlfors and Beurling, see [1], [29, Theorem 4.6].

Theorem 2.33 (Ahlfors-Beurling Inequality)**.** *For any compact set $E \subset \mathbb{C}$,*

$$\gamma^2(E) \ge \frac{v(E)}{\pi}.$$

2.5 Rigidity theorem of the relation $c_B = c_\beta$

In this section, we prove that if X is an open Riemann surface, then $c_B(z_0) = c_\beta(z_0)$ for some $z_0 \in X$ if and only if X is biholomorphic to the unit disk less a relatively closed polar set, or X is parabolic. Towards this end, we establish first a rigidity theorem for the Green's function.

Theorem 2.34. *On an open, hyperbolic Riemann surface X, the Green's function with a pole $z_0 \in X$ equals*

$$G(z, z_0) = \log |f(z)| \tag{2.19}$$

for some holomorphic function f on X if and only if f is a biholomorphism from X to a disk possibly less a relatively closed polar subset.

Proof of Theorem 2.34. Since $G_X(z, z_0) < 0$, $f(X) \subset D(0,1)$. Moreover, $f(z_0) = 0$. The image $f(X)$ admits a Green's function because $\partial f(X)$ is not polar. By Lemma 2.12 and the monotonicity property of the Green's function, Proposition 2.5(6),

$$\log |f(z)| = G_X(z, z_0) \geq G_{f(X)}(f(z), 0) \geq G_{D(0,1)}(f(z), 0) = \log |f(z)|.$$

By Lemma 2.12, f is injective and $G_{f(X)}(\zeta, 0) = \log |\zeta|$ for $\zeta \in f(X)$. Let $\eta \in \partial f(X) \cap D(0,1)$ and $\zeta_n \to \eta$ with $\zeta_n \in f(X)$. Since

$$G_{f(X)}(\eta, 0) = \log |\eta| = \lim_{n\to\infty} \log |\zeta_n| = \lim_{n\to\infty} G_{f(X)}(\zeta_n, 0) < 0,$$

η is an irregular boundary point. By Kellogg's Theorem, Theorem 2.10, $P = \partial f(X) \cap D(0,1)$ is a polar set relatively closed in $D(0,1)$. Suppose $z_0 \in D(0,1) \setminus \overline{f(X)}$. Then for some $\epsilon > 0$, $f(X) \subset D(0,1) \setminus \overline{D(z_0, \epsilon)}$. Let k be the harmonic function defined on the latter set with Dirichlet boundary data

$$k(z) = \begin{cases} 0, & z \in \partial D(0,1) \\ -\log(|z_0| + \epsilon), & z \in \partial D(z_0, \epsilon). \end{cases}$$

Since $G_{f(X)}(z,0) \leq G_{f(X)}(z,0) + k(z)$ and $G_{f(X)}(z,0) + k(z)$ is in the defining set of the Green's function for the domain $f(X)$, (2.1), we have arrived at a contradiction. Thus, $D(0,1) = f(X) \sqcup P$.

The converse direction follows by the biholomorphic transformation law of the Green's function, Proposition 2.5(7) and that $G_{D(0,1)\setminus P}(z,0) = \ln|z|$ for any relatively closed polar set P. The proof of the theorem is complete.

□

Guan and Zhou prove in Lemma 4.25 of [31]

Lemma 2.35. *If there is a holomorphic function g on Ω, which satisfies $|g(z)| = \exp G(z, z_0)$, then we have $c_\beta^2(z_0) = c_B^2(z_0)$.*

In the conclusion of their proof of the Suita conjecture, [31, Theorem 3.1, page 1196], they show that if $\pi K(z_0, z_0) = c_\beta^2(z_0)$, then the hypotheses of Lemma 2.35 are satisfied. Consequently, by Suita's Theorem, Theorem 2.1, Ω is biholomorphic to a disk less a relatively closed polar set. The main result of this subsection improves this line of argument. We show that the hypothesis $c_\beta(z_0) = c_B(z_0)$ is enough to conclude that Ω is biholomorphic to a disk less a relatively closed polar set.

Theorem 2.36. *For an open Riemann surface X, $c_\beta(z_0) = c_B(z_0)$ for some $z_0 \in X$ if and only if either*

1. *X is parabolic;*

2. *X is biholomorphic to the unit disk possibly less a relatively closed polar subset.*

Moreover, in case 2, the set of biholomorphisms equals the set of extremal functions of the analytic capacity, $c_B(\cdot)$.

Proof. $c_\beta(z_0) = 0$ if and only if X is parabolic, and by Lemma 2.26, $c_B \leq c_\beta$. So we may assume $c_\beta > 0$. If X is biholomorphic to the unit disk less a relatively closed polar set, then by Lemma 2.25, $c_B \equiv c_\beta$. It remains to consider the case where $c_\beta(z_0) = c_B(z_0) > 0$ and show that X is biholomorphic to a disk less a relatively closed polar set.

Let $u(z) := \log(|f_0(z)|)$ for $z \in X$, where f_0 is an extremal function of the analytic capacity. Then $u \in SH^-(X)$ and

$$\lim_{z \to z_0} \log|f_0(z)| - \ln|z - z_0| = \ln\left|\frac{\partial f_0}{\partial z}(z_0)\right| = \ln c_B(z_0) > -\infty. \tag{2.20}$$

By the definition of the Green's function, $u(z) - G(z, z_0) \leq 0$. Furthermore,

$$\begin{aligned} \lim_{z \to z_0} u(z) - G(z, z_0) &= \lim_{z \to z_0} \ln(|f_0(z)|) - \ln|z - z_0| - (G_{z_0}(z) - \ln|z - z_0|) \\ &= \ln c_B(z_0) - \ln c_\beta(z_0) \\ &= 0. \end{aligned}$$

By the maximum principle,

$$G(z, z_0) = \ln|f_0(z)|.$$

By Theorem 2.34, f_0 is a biholomorphism to the unit disk less a relatively closed polar set P. By Lemma 2.25,

$$c_B(z) \equiv c_\beta(z), \quad z \in X.$$

Repeating the argument with a fixed $z \in X$ in place of z_0, any extremal function for $c_B(z)$ is a biholomorphism to the unit disk less a relatively closed polar set.

If f is any biholomorphism from X to the unit disk less a relatively closed polar set, then for $z_0 = f^{-1}(0)$, $G(z, z_0) = \log|f(z)|$. For any $h \in \mathcal{O}(X, D(0,1))$ with $h(z_0) = 0$, $\log|h| \in SH^-(X)$. By the definition of the Green's function, $\log|h| \leq G(z, z_0) = \log|f|$. Thus, $|h'(z_0)| \leq |f'(z_0)|$. By the definition of the analytic capacity (2.3), $|f'(z_0)| = c_B(z_0)$.

□

2.6 Equality conditions for $\frac{\pi}{v(\Omega)} \leq c_B^2 \leq c_\beta^2$

In this subsection, we restrict our attention from Riemann surfaces X to domains $\Omega \subset \mathbb{C}$. With this restriction we will be able to examine the relationship between the domain functions c_β, c_B and the volume of the domain $v(\Omega)$.

We establish $c_B^2 \geq \frac{\pi}{v(\Omega)}$, including for the case where $v(\Omega) = \infty$. Consequently, we will have proved

$$K \geq c_\beta^2 \geq c_B^2 \geq \frac{\pi}{v(\Omega)}. \qquad \text{(2.5 revisited)}$$

We will then establish the equality cases for the inequalities $c_\beta^2 \geq \frac{\pi}{v(\Omega)}$ and $c_B^2 \geq \frac{\pi}{v(\Omega)}$.

Lemma 2.37. *Let $\Omega \subset \mathbb{C}$ be a domain with $z_0 \in \Omega$ and $v(\Omega) < \infty$. Then*

$$\frac{\pi}{v(\Omega)} \leq \frac{v(\mathbb{C}_\infty \setminus f_{z_0}(\Omega))}{\pi} \qquad (2.21)$$

and equality holds if and only if Ω is a disk centered at z_0 less a relatively closed set of measure 0.

Proof. If $M \subset \mathbb{C}$ is a set with $v(M) = 0$, then $v(f_{z_0}(M)) = 0$. With this fact it is straightforward to verify that equality holds for a disk less a relatively closed set of measure 0. Since $v(\Omega) = v(\Omega - \{z_0\})$ and $f_{z_0}(\Omega) = f(\Omega - \{z_0\})$, without loss of generality we may suppose $z_0 = 0 \in \Omega$.

Since f is an automorphism of $\mathbb{C}_\infty$,

$$f(\Omega) \sqcup (\mathbb{C}_\infty \setminus f(\Omega)) = \mathbb{C}_\infty = \Omega \sqcup (\mathbb{C}_\infty \setminus \Omega),$$

where $\sqcup$ denotes the disjoint union. Hence

$$\mathbb{C}_\infty \setminus f(\Omega) = f(\mathbb{C}_\infty \setminus \Omega). \tag{2.22}$$

By (2.22), (2.21) is equivalent to

$$v(f(\mathbb{C}_\infty \setminus \Omega))v(\Omega) \geq \pi^2. \tag{2.23}$$

Let $D = D(0, a)$ denote the disk centered at 0 of radius a where a is chosen so that $v(D) = v(\Omega)$. Since

$$\pi^2 = v(D)v(f(\mathbb{C}_\infty \setminus D)), \tag{2.24}$$

(2.23) is equivalent to

$$v(f(\mathbb{C}_\infty \setminus \Omega)) \geq v(f(\mathbb{C}_\infty \setminus D)). \tag{2.25}$$

Let

$$S_1 = (\mathbb{C}_\infty \setminus \Omega) \cap D, \quad S_2 = (\mathbb{C}_\infty \setminus D) \cap \Omega.$$

Since

$$\mathbb{C}_\infty \setminus \Omega = \mathbb{C}_\infty \setminus (\Omega \cup D) \sqcup (\mathbb{C}_\infty \setminus \Omega) \cap D$$

and f is an automorphism of $\mathbb{C}_\infty$,

$$v(f(\mathbb{C}_\infty \setminus \Omega)) = v(f((\mathbb{C}_\infty \setminus (\Omega \cup D)))) + v(f(S_1)).$$

Similarly,

$$v(f(\mathbb{C}_\infty \setminus D)) = v(f((\mathbb{C}_\infty \setminus (D \cup \Omega)))) + v(f(S_2)).$$

Thus, (2.25) is equivalent to

$$v(f(S_1)) \geq v(f(S_2)). \tag{2.26}$$

For all $z \in S_1$, $|z| \leq a$ and for all $z \in S_2$, $|z| \geq a$. Hence

$$\min_{z \in S_1} \frac{1}{|z|^4} \geq \frac{1}{a^4} \geq \max_{z \in S_2} \frac{1}{|z|^4}.$$

Notice also that

$$v(S_1) = v(D \setminus \Omega) = v(\Omega \setminus D) = v(S_2).$$

Thus, (2.26) is equivalent to

$$\begin{aligned} v(f(S_1)) &= \int_{f(S_1)} dv(z) \\ &= \int_{S_1} \frac{1}{|z|^4} dv(z) \\ &\geq \frac{1}{a^4} v(S_1) \\ &= \frac{1}{a^4} v(S_2) \\ &\geq \int_{S_2} \frac{1}{|z|^4} dv(z) \\ &= v(f(S_2)). \end{aligned} \tag{2.27}$$

This completes the proof of the inequality part. For the equality part, first replace the inequalities in (2.23)-(2.26) with equalities. By (2.26) and (2.27),

$$\int_{\mathbb{C}} \frac{1}{|z|^4} (\chi_{S_2}(z) - \chi_{S_1}(z)) dv(z) = 0.$$

Since $v(S_1) = v(S_2)$,

$$\int_{\mathbb{C}} (\chi_{S_2}(z) - \chi_{S_1}(z)) dv(z) = 0.$$

Thus, for all $\alpha \in \mathbb{C}$,

$$\int_{\mathbb{C}} (\frac{1}{|z|^4} + \alpha)(\chi_{S_2}(z) - \chi_{S_1}(z))dv(z) = 0. \tag{2.28}$$

Without loss of generality, there exists r so that

$$S_1 \subset D(0,a) \setminus D(0,r), \quad S_2 \subset \{z \in \mathbb{C} : |z| \geq a\}.$$

With this r,

$$\begin{aligned} z \in S_1 &\Rightarrow \frac{1}{a^4} < \frac{1}{|z|^4} < \frac{1}{r^4} \\ z \in S_2 &\Rightarrow \frac{1}{|z|^4} \leq \frac{1}{a^4}. \end{aligned}$$

Let $\alpha = -a^{-4}$. Then

$$(\frac{1}{|z|^4} + \alpha)\chi_{S_2}(z) \leq 0 \tag{2.29}$$

$$(\frac{1}{|z|^4} + \alpha)(-\chi_{S_1}(z)) \leq 0 \tag{2.30}$$

It follows from (2.28)-(2.39) with this α that $v(S_1) = v(S_2) = 0$. Furthermore, since Ω is open, $S_2 = \Omega \setminus D = \emptyset$. Thus, $\Omega = D \setminus S_1$. Notice $S_1 = D \setminus \Omega$ is closed in D.

□

Theorem 2.38. *Let $\Omega \subset \mathbb{C}_\infty$ be a domain. Then*

$$c_B^2(z) \geq \frac{\pi}{v(\Omega)}, \quad z \in \Omega, \tag{2.31}$$

where we use the convention that if $v(\Omega) = \infty$, then $v(\Omega)^{-1} = 0$. Moreover,

1. *If $v(\Omega) = \infty$, then equality holds at some z_0 if and only if $\Omega = \mathbb{C}_\infty \setminus P$ where $P \in \mathcal{N}_B$. If*

additionally $c_\beta^2(z_0) = \frac{\pi}{v(\Omega)}$*, then equality holds if and only if* $\Omega = \mathbb{C}_\infty \setminus P$ *where* P *is a closed polar set.*

2. *If* $v(\Omega) < \infty$*, then equality holds at some* z_0 *if and only if* $\Omega = D\left(z_0, \sqrt{v(\Omega)\pi^{-1}}\right) \setminus P$ *where* P *satisfies*

$$P \cap \overline{D(z_0, s)} \in \mathcal{N}_B, \quad s < r.$$

If additionally $c_\beta^2(z_0) = \frac{\pi}{v(\Omega)}$*, then equality holds if and only if* $\Omega = D\left(z_0, \sqrt{v(\Omega)\pi^{-1}}\right) \setminus P$ *where* P *is a relatively closed polar set.*

Proof. If $v(\Omega) = \infty$, then the inequality is trivial. By Proposition 2.23 and Definition 2.20, $c_B(z_0) = 0$ if and only if $\Omega = \mathbb{C}_\infty \setminus P$ where $P \in \mathcal{N}_B$. By Remark 2.15, $c_\beta(z_0) = 0$ if and only if $\Omega = \mathbb{C}_\infty \setminus P$ where P is polar. We now assume $v(\Omega) < \infty$.

By Lemma 2.32 and the Ahlfors-Beurling Inequality, Theorem 2.33,

$$c_B^2(z_0) \geq \frac{v(\mathbb{C}_\infty \setminus f_{z_0}(\Omega))}{\pi} \geq \frac{\pi}{v(\Omega)},$$

which establishes (2.31).

If equality holds at some z_0, then by Lemma 2.37, $\Omega = D(z_0, r) \setminus P$ where $r = \sqrt{\frac{v(\Omega)}{\pi}}$ and P is a relatively closed set of measure 0.

The function $h(z) = r^{-1}(z - z_0)$ is in the defining set of the analytic capacity and $h'(z_0) = c_B(z_0) = r^{-1}$. By Proposition 2.29, $h(\Omega) = D(0,1) \setminus Q$ where

$$Q = h(P), \quad Q \cap \overline{D(0,r)} \in \mathcal{N}_B, \quad 0 \leq r < 1.$$

If additionally $c_\beta^2(z_0) = \frac{\pi}{v(\Omega)}$, then by Theorem 2.36, h is a biholomorphism from Ω to $D(0,1) \setminus Q$ where Q is a relatively closed polar set.

The converse directions of case 2 follow from Lemma 2.25 and that if P is relatively closed and polar, then

$$P \cap \overline{D(z_0, s)} \in \mathcal{N}_B, \quad s < r.$$

□

Remark 2.39. *If $c^2(z_0) > c_B^2(z_0) = \frac{\pi}{v(\Omega)}$, then P, in the preceeding theorem, may not be polar as the next example shows.*

Example 2.40. *Let K be the compact four-corner Cantor set defined in [30]. As shown therein, $K \in \mathcal{N}_B$, but is not polar. Let $\Omega = D(z_0, r) \setminus K$ where z_0 and r are chosen such that $z_0 \notin K \subset D(z_0, r)$. Since $K \in \mathcal{N}_B$, all bounded holomorphic functions on Ω extend to holomorphic functions on $D(z_0, r)$. Thus,*

$$c_{B;\Omega}(z_0) = c_{B;D(z_0,r)}(z_0) = \sqrt{\frac{\pi}{v(D(z_0, r))}} = \sqrt{\frac{\pi}{v(\Omega)}},$$

where the last equality used that sets of class $\mathcal{N}_B$ have two-dimensional Lebesgue measure 0, cf. [53].

Corollary 2.41. *Let $\Omega \subset \mathbb{C}$ be a domain. Then,*

$$\pi K \geq c_\beta^2 \geq c_B^2 \geq \frac{\pi}{v(\Omega)} \tag{2.5 revisited}$$

Proof. This follows from Theorems 2.2, Lemma 2.26 and Theorem 2.38. □

2.7 Rigidity theorem of the Bergman kernel

In this section, we complete describing the equality conditions of the quantities

$$\pi K \geq c_\beta^2 \geq c_B^2 \geq \frac{\pi}{v(\Omega)},$$

by characterizing the domains $\Omega \subset \mathbb{C}$ which satisfy $K(z_0) = v(\Omega)^{-1}$ for some $z_0 \in \Omega$.

Theorem 2.42. *Let $\Omega \subset \mathbb{C}$ be a domain. Suppose there exists a $z_0 \in \Omega$ such that*

$$K(z_0) = \frac{1}{v(\Omega)}, \tag{2.32}$$

where we use the convention $v(\Omega)^{-1} = 0$ if $v(\Omega) = \infty$.

(i) If $v(\Omega) = \infty$, then $\Omega = \mathbb{C} \setminus P$ where P is a possibly empty, closed polar set.

(ii) If $v(\Omega) < \infty$, then $\Omega = D(z_0, r) \setminus P$ where P is a possibly empty, polar set closed in the relative topology of $D(z_0, r)$ with $r^2 = v(\Omega)\pi^{-1}$.

We give several proofs of Theorem 2.42 This first one is based most closely on the preceeding subsections and is the simplest.

Proof of Theorem 2.42. If $\pi K(z_0) = \pi v(\Omega)^{-1}$, then by Corollary 2.41, $c_\beta^2(z_0) = \pi v(\Omega)^{-1}$. The proof now follows from Theorem 2.38. □

The preceeding proof used the inequality part of the Suita conjecture when we cited Corollary 2.41. In a sense, this is undesirable as Guan and Zhou's (respectively Błocki's [11]) solution [31] for surfaces (respectively bounded domains) used methods of several complex variables, whereas Theorem 2.42 is a statement about domains in one-dimensional complex space.

The case when $v(\Omega) = \infty$ for Theorem 2.42 can be proved without the logarithmic or analytic capacity and essentially follows from an argument of Wiegerinck [56], who showed that $dim(A^2(\Omega)) = 0$ or ∞ for all domains $\Omega \subset \mathbb{C}$.

Lemma 2.43. *Suppose $\Omega \subset \mathbb{C}$ is a domain such that $\{z : |z| > R\} \subset \Omega$ for some $R > 0$. If $h \in A^2(\Omega)$ and there exists $w_2 \in \Omega$ with $h(w_2) \neq 0$, then*

$$k(z) = \frac{h(z) - h(w_1)}{z - w_1} - \frac{h(w_1)}{h(w_2)}\Big(\frac{h(z) - h(w_2)}{z - w_2}\Big), \quad w_1 \in \Omega,$$

is also in $A^2(\Omega)$.

Proof. Without loss of generality suppose $|w_i| < R$ for $i = 1, 2$. When $|z| > R$,

$$\frac{1}{z - w_i} = \frac{1}{z} + \sum_{n=2}^{\infty} \frac{w_i^{n-1}}{z^n} =: \frac{1}{z} + F_i(z),$$

and $F_i \in A^2(\{z : |z| > R\})$. Let $\beta_i = h(w_i)$, and $\alpha = \frac{\beta_1}{\beta_2}$. Then

$$k(z) = \frac{h(z)}{z} + F_1(z)h(z) - \beta_1 F_1(z) - \alpha\frac{h(z)}{z} - \alpha F_2(z)h(z) + \alpha\beta_2 F_2(z), \quad |z| > R,$$

which is a sum of functions in $A^2(\{z : |z| > R\})$. The singularities at $z = w_1, w_2$ are removable and $k \in A^2(\Omega \cap D(0, R))$. Thus, the proof of the lemma is complete. $\square$

Proof of $v(\Omega) = \infty$ case of Theorem 2.42. After a translation $0 \in \Omega$, and after applying an automorphism L of $\mathbb{C}_\infty$ which fixes 0 and sends some point $z_1 \in \Omega$ to ∞,

$$K_{L(\Omega)\setminus\{\infty\}}(0, 0)|L'(0)|^2 = K_{\Omega\setminus\{z_1\}}(0, 0) = K_\Omega(0, 0) = 0.$$

It suffices to show that $L(\Omega) \setminus \{\infty\} = \mathbb{C} \setminus P$ where P is a compact polar subset of $\mathbb{C}$. For ease of notation, denote the domain $L(\Omega)$ by Ω. Suppose towards a contradiction that $A^2(\Omega) \neq \{0\}$. Then there is a function $h \in A^2(\Omega)$ and $w_1 \neq w_2 \in \Omega$ so that $h(w_1), h(w_2) \neq 0$. Necessarily,

$w_1, w_2 \neq 0$ because

$$0 = K(0,0) = \sup\{|g(0)|^2 : \|g\|_2^2 \leq 1, g \in A^2(\Omega)\}. \tag{2.33}$$

Consider

$$k(z) = \frac{h(z) - h(w_1)}{z - w_1} - \frac{h(w_1)}{h(w_2)}\Big(\frac{h(z) - h(w_2)}{z - w_2}\Big).$$

By Lemma 2.43, $k \in A^2(\Omega)$. However, $k(0) \neq 0$, which contradicts (2.33). Thus, $A^2(\Omega) = \{0\}$. Carleson proved that this is equivalent to $\Omega = \mathbb{C} \setminus P$ where P is a compact polar set. See [14, VI.Theorem 1] or [18]. □

When $v(\Omega) < \infty$, Theorem 2.42 may be proved without using the logarithmic capacity (and the inequality part of the Suita conjecture) as follows.

Proof of $v(\Omega) < \infty$ case of Theorem 2.42. By Theorems 2.1 and 2.33 and Lemmas 2.32 and 2.37,

$$\pi K(z_0) \geq c_B^2(z_0) = \gamma^2(\mathbb{C}_\infty \setminus f_{z_0}(\Omega)) \geq \frac{v(\mathbb{C}_\infty \setminus f_{z_0}(\Omega))}{\pi} \geq \frac{\pi}{v(\Omega)} = \pi K(z_0). \tag{2.34}$$

By Theorem 2.1, Ω is biholomorphic to a disk less a relatively closed polar set. Let $F : \Omega \to D(0,1)$ be biholomorphic with $F'(z_0) > 0$ and $F(z_0) = 0$. By the transformation law of the Bergman kernel,

$$F'(z)F'(z_0)K_{D(0,1)}(F(z), 0) = K_\Omega(z, z_0). \tag{2.35}$$

By Lemma 1.2, since $K_\Omega(z_0, z_0) = v(\Omega)^{-1}, K_\Omega(\cdot, z_0)$ is a constant function. Since $K_{D(0,1)}(\cdot, 0)$ is also constant, so is $F'(\cdot)$. Thus, F is affine and Ω is a disk less a closed polar set. □

As a corollary, the previous proof improves the lower bound $K(z) \geq v(\Omega)^{-1}$ for the Bergman kernel.

Corollary 2.44. *Let $f_{z_0}(z) = \frac{1}{z-z_0}$ and $\Omega \subset \mathbb{C}$ be a domain with $v(\Omega) < \infty$. Then*

$$K(z_0, z_0) \geq \frac{v(\mathbb{C}_\infty \setminus f_{z_0}(\Omega))}{\pi^2} \geq \frac{1}{v(\Omega)},$$

and the second inequality is strict if and only if Ω does not equal a disk less a relatively closed set of measure 0.

Proof. As with (2.34), this follows from Theorems 2.1 and 2.33 and Lemmas 2.32 and 2.37.

□

We may also prove Theorem 2.42 without using the analytic capacity if we use the equality part of the Suita Conjecture.

Proof of Theorem 2.42. First suppose, $v(\Omega) = \infty$. Then $c_\beta(z_0) = 0$. By Remark 2.15, $\Omega = \mathbb{C} \setminus P$ where P is a polar set. P is closed because it is the complement in $\mathbb{C}$ of an open set.

Now suppose $v(\Omega) < \infty$. Since polar sets have two-dimensional Lebesgue measure 0, by Theorem 2.11, Ω admits a Green's function. After a translation, $z_0 = 0$ and $K_\Omega(\cdot, 0) \equiv v(\Omega)^{-1}$. Let $\Omega_\tau = \{z \in \Omega : G(z, 0) < \tau\}$. Let $\lambda = \ln c_\beta(z_0)$ and

$$r_0 := e^{\tau - \lambda(0) - \epsilon}, \quad r_1 := e^{\tau - \lambda(0) + \epsilon}.$$

Then for $\tau < 0$ sufficiently negative,

$$D(0, r_0) \subset \Omega_\tau \subset D(0, r_1)$$

(cf. [9, 12].) Hence,

$$\frac{e^{-2\epsilon} e^{2\lambda(0)}}{\pi} \leq \frac{e^{2\tau}}{v(\Omega_\tau)} \leq \frac{e^{2\epsilon} e^{2\lambda(0)}}{\pi}.$$

Letting $\epsilon \to 0^+$,

$$\frac{e^{2\tau}}{v(\Omega_\tau)} \approx \frac{e^{2\lambda(0)}}{\pi}, \qquad \text{as } \tau \to -\infty.$$

By Theorem 3 of [13], $\frac{e^{2\tau}}{v(\Omega_\tau)}$ is a decreasing function on $(-\infty, 0]$; hence,

$$K(0,0) = \frac{1}{v(\Omega)} \leq \frac{e^{2\lambda(0)}}{\pi} \leq K(0,0).$$

By the equality part of the Suita Conjecture, there exists a biholomorphic map $f : D(0,1)\backslash P \to \Omega$ where P is a relatively closed polar set. After a Möbius transformation of the unit disk, we may assume $0 \notin P$, $f(0) = 0$ and $f'(0) > 0$. Since P is removable for functions in $A^2(D(0,1) \setminus P)$, $K_{D(0,1)\backslash P}(\cdot,\cdot) = K_{D(0,1)}(\cdot,\cdot)$ when both sides are well-defined. By the transformation law of the Bergman kernel, f is linear. Hence $\Omega = D(0, f'(0)) \setminus f(P)$ and $f(P)$ is a relatively closed polar set. □

2.7.1 Minimal domains

A bounded domain $\Omega \subset \mathbb{C}^n$ with $z_0 \in \Omega$ is called a minimal domain with center z_0 if for every biholomorphism $f : \Omega \to \Omega'$ with $\det(\mathrm{J}f(z_0)) = 1$,

$$v(\Omega) \leq v(\Omega').$$

Here $\mathrm{J}f$ denotes the Jacobian matrix of f. It is known that equivalently a domain Ω is minimal with center z_0 if and only if

$$K(z, z_0) = \frac{1}{v(\Omega)}, \quad z \in \Omega.$$

For more information about minimal domains, see [39, 44, 57]. Theorem 2.42 classifies the minimal domains of $\mathbb{C}$. More precisely, we have the following corollary.

Corollary 2.45. *Let $\Omega \subset \mathbb{C}$ be a domain. Suppose there exists a $z_0 \in \Omega$ such that $K(\cdot, z_0) = C$.*

Then $C = v(\Omega)^{-1}$ and

1. *If $C = 0$, then $\Omega = \mathbb{C} \setminus P$ where P is a closed polar set.*
2. *If $C > 0$, then $\Omega = D(z_0, r) \setminus P$ where P is a possibly empty, polar set closed in the relative topology of $D(z_0, r)$ with $r = \sqrt{v(\Omega)\pi^{-1}}$.*

Consequently, all minimal domains in $\mathbb{C}$ with center z_0 are disks centered at z_0 minus closed polar sets.

Proof. Since $1 = \int_\Omega K(w, z_0)dv(w)$, it follows that $C = v(\Omega)^{-1}$ and $K(z_0, z_0) = v(\Omega)^{-1}$. The result now follows from Theorem 2.42. □

2.7.2 A proof of Theorem 2.42 for smoothly bounded domains

When Ω is a bounded domain with C^∞ boundary, we provide an additional proof of Theorem 2.42, which does not use the logarithmic or analytic capacity.

Let Ω be a bounded domain with C^∞-smooth boundary and denote its boundary by $\partial\Omega$. Then Ω is n-connected with $n < \infty$ and the boundary consists of n simple closed curves parametrized by C^∞ functions $z_j : [0, 1] \to \mathbb{C}$. Without loss of generality, let z_n parametrize the boundary component which bounds the unbounded connected component of the complement and let $\partial\Omega_j$ denote the boundary component parametrized by z_j, $j = 1, ..., n$. Let $T(z)$ denote the unit tangent vector to the boundary and ds denote the arc-length measure of the boundary. Define $L^2(\partial\Omega) = \{f : \partial\Omega \to \mathbb{C} : \|f\|_{L^2(\partial\Omega)} < \infty\}$ where the norm $\|\cdot\|_{L^2(\partial\Omega)}$ is induced by the inner product

$$\langle f, g\rangle = \int_{\partial\Omega} f\bar{g}\, ds.$$

Let $A^\infty(\partial\Omega)$ denote the boundary values of functions in $\mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$. The Hardy space of $\partial\Omega$

denoted $H^2(\partial\Omega)$ is the $L^2(\partial\Omega)$ closure of $A^\infty(\partial\Omega)$. If $P : L^2(\partial\Omega) \to H^2(\partial\Omega)$ is the orthogonal projection, then the Szegö kernel for Ω, $S(z,a)$, is defined by

$$P(C_a)(z) = S(z,a), \quad a, z \in \Omega, \quad C_a(z) = \overline{\frac{1}{2\pi i}\frac{T(z)}{z-a}}$$

[5, Section 7]. It can be shown that $S(z,a) = \overline{S(a,z)}$, and from the proof of the Ahlfor's Mapping Theorem, for each a, $S(\cdot,a)$ has $n-1$ zeros counting multiplicity [5, Theorem 13.1]. We note that the proof of the Ahlfor's Mapping Theorem just cited requires C^∞-boundary regularity. Since we will need the fact about the $n-1$ zeros of $S(\cdot,a)$, we have imposed a C^∞ boundary regularity assumption on Ω in this section.

Let ω_j be the (unique) solution to the Dirichlet boundary-value problem

$$\begin{cases} \Delta u(z) = 0 & z \in \Omega \\ u(z) = 1 & z \in \partial\Omega_j \\ u(z) = 0 & z \in \partial\Omega_k, \quad k \neq j \end{cases}$$

and define $F_j : \Omega \to \mathbb{C}$ by $F_j(z) = 2\partial\omega_j/\partial z$. Then the Bergman kernel and Szegö kernel are related by

$$K(z,a) = 4\pi S(z,a)^2 + \sum_{j=1}^{n-1} \lambda_j F_j(z) \tag{2.36}$$

where λ_j are constants in z and depend on a [5, Theorem 23.2]. Since $\omega_j \in C^\infty(\overline{\Omega})$ is harmonic, $F_j \in \mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega}) \subset A^2(\Omega)$. We now prove Theorem 2.42 when Ω is bounded with C^∞ boundary.

Proof. After a translation we may assume that $z_0 = 0$. Let $\{v(\Omega)^{-1/2}\} \cup \{\phi_j\}_{j=1}^\infty$ be a complete orthonormal basis for $A^2(\Omega)$. Then

$$\frac{1}{v(\Omega)} = K(0,0) = \frac{1}{v(\Omega)} + \sum_{j=1}^{\infty} \phi_j(0)\overline{\phi_j(0)},$$

which implies that $\phi_j(0) = 0$, for all j. It follows that $K(0,a) = v(\Omega)^{-1}$, and for any $f \in A^2(\Omega)$ by the reproducing property (1.1)

$$f(0) = \frac{1}{v(\Omega)} \int_\Omega f(w)dv(w).$$

In particular for $F_j, j = 1, ..., n-1$,

$$\begin{aligned} F_j(0) &= \frac{1}{2iv(\Omega)} \int_\Omega 2\frac{\partial \omega_j}{\partial w} d\bar{w} \wedge dw \\ &= \frac{-1}{iv(\Omega)} \int_{\partial\Omega} \omega_j d\bar{w} \\ &= \frac{-1}{iv(\Omega)} \int_{\partial\Omega_j} 1 d\bar{w} \\ &= 0. \end{aligned}$$

Hence setting $z = 0$ in (2.36),

$$\frac{1}{v(\Omega)} = K(0,a) = 4\pi S^2(0,a)$$

Since $S(0,\cdot) = \overline{S(\cdot,0)}$ has $n-1$ zeros counting multiplicity, $n = 1$; that is, Ω is simply-connected. Let $F : D(0,1) \to \Omega$ be the inverse of the Riemann map with $F(0) = 0, F'(0) > 0$. By the transformation law of the Bergman kernel,

$$\frac{1}{\pi} = K_{D(0,1)}(z,0) = F'(z)K_\Omega(F(z),0)\overline{F'(0)} = \frac{F'(z)\overline{F'(0)}}{v(\Omega)}. \tag{2.37}$$

So F is linear; hence $\Omega = D(0, F'(0))$.

□

2.8 Rigidity theorem of the Bergman kernel in $\mathbb{C}^n$

When $n > 1$, a much wider class of domains have Bergman kernels which satisfy $K(z_0) = v(\Omega)^{-1}$. A domain Ω is said to be circular containing its center z_0 if $z_0 \in \Omega$ and $\{z_0\} + e^{i\theta}(\Omega - \{z_0\}) \subset \Omega$. The Bergman kernel of such a domain satisfies $K(z_0) = v(\Omega)^{-1}$, see [4].

Thus, to generalize Theorem 2.42 to $\mathbb{C}^n$, $n \geq 1$, $D(z_0, r)$ cannot simply be replaced by a translation and rescaling of $\mathbb{B}^n$. However, the unit ball, in addition to being circular containing its center, is complete Reinhardt, strongly convex with algebraic boundary. So, we also consider whether Theorem 2.42 generalizes to $\mathbb{C}^n$ if Ω is required to be complete Reinhardt, strongly convex with algebraic boundary. The answer is no as the next example shows.

Definition 2.46. *A domain Ω in $\mathbb{C}^n$ is said to be complete Reinhardt if for all $z = (z_1, ..., z_n) \in \Omega$*

$$(\lambda_1 z_1, ..., \lambda_n z_n) \in \Omega, \quad |\lambda_i| \leq 1.$$

Complete Reinhardt domains are circular containing their center 0; thus, they too satisfy $K(0) = v(\Omega)^{-1}$.

Example 2.47. *Let $\Omega = \{z \in \mathbb{C}^2 : |z_1|^4 + |z_1|^2 + |z_2|^2 < 1\}$ be a domain with algebraic boundary. Then Ω is complete Reinhardt, strongly convex and not biholomorphic to $\mathbb{B}^2$.*

Proof of Example 2.47. It is easy to see that Ω is complete Reinhardt with algebraic boundary. To verify that Ω is strongly convex, one lets $\rho(z) = |z_1|^4 + |z_1|^2 + |z_2|^2 - 1$ and verifies that the real-Hessian of $\mathcal{H}(\rho)(z)$ satisfies

$$w^\tau \mathcal{H}(\rho)(z_0) w > 0, \quad z_0 \in \partial\Omega, \quad w \in \mathbb{R}^4 \setminus \{0\}.$$

Suppose towards a contradiction that there exists an $F : \mathbb{B}^2 \to \Omega$ which is biholomorphic. Since the holomorphic automorphism group of $\mathbb{B}^2$ is transitive, we may suppose that $0 \mapsto 0$. By Henri

Cartan's theorem, [51, Theorem 2.1.3.], F is linear; that is $F(z) = (a_1z_1 + a_2z_2, a_3z_1 + a_4z_2)$. Consequently, $F : \partial\mathbb{B}^2 \to \partial\Omega$. After composing with a holomorphic rotation of $\mathbb{B}^2$, we may also suppose $F((0,1)) = (0,1)$. Then, $a_2 = 0, a_4 = 1$. Since for all $\theta \in [0, 2\pi]$,

$$\partial\Omega \ni F(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{i\theta}) = (\frac{a_1}{\sqrt{2}}, \frac{a_3}{\sqrt{2}} + \frac{e^{i\theta}}{\sqrt{2}}),$$

we see that

$$\frac{|a_1|^4}{4} + \frac{|a_1|^2}{2} + \frac{|a_3|^2}{2} + Re\langle a_3, e^{i\theta}\rangle + \frac{1}{2} = 1,$$

which implies that $a_3 = 0$. Thus,

$$|a_1|^4 + 2|a_1|^2 = 2. \tag{2.38}$$

Since $(a_1, 0) = F((1,0)) \in \partial\Omega$,

$$|a_1|^4 + |a_1|^2 = 1. \tag{2.39}$$

Equations (2.38) and (2.39) do not have a simultaneous solution. Thus, F does not exist. □

2.8.1 Rigidity theorem for the Bergman kernel of ellipsoids

In general, it is difficult to calculate the Bergman kernel for a domain $\Omega \subset \mathbb{C}^n$. It is known that $K(0) = v(\Omega)^{-1}$ for any circular domain containing its center 0. We ask if there are any other domains which might satisfy that equality. In this subsection, we investigate whether that equality holds for the real ellipsoids

$$\{(x_1 + ix_2, \ldots, x_{2n-1} + ix_{2n}) \in \mathbb{C}^n : \sum_{j=1}^{n} a_j x_{2j-1}^2 + b_j x_{2j}^2 < 1\}, \quad a_j \geq b_j > 0, \quad j = 1, \ldots, n. \tag{2.40}$$

After a complex linear change of variables, the ellipsoids in (2.40) can be written in its Webster normal form [34, 55]

$$E_A = \{z \in \mathbb{C}^n : \rho(z) = |z|^2 + \sum_{j=1}^{n} A_j Re(z_j^2) < 1\}, \quad 0 \leq A_j = \frac{a_j - b_j}{a_j + b_j} < 1, \quad j = 1, \ldots, n,$$

where $A = (A_1, \ldots, A_n)$. The Bergman kernels of the ellipsoids were studied by Hirachi, [34, 55]. The ellipsoids have also been studied in pseudo-Hermitian CR geometry, see [41, 43]. The final result of the paper characterizes the ellipsoids which satisfy (2.32).

Theorem 2.48. *Let $K_A(\cdot,\cdot)$ denote the Bergman kernel of the normalized real ellipsoid E_A. Then $K_A(0,0) = v(E_A)^{-1}$ if and only if $A = (0, \ldots, 0)$.*

Proof of Theorem 2.48. If $A = (0, \ldots, 0)$, then E_A is the unit ball and (2.32) holds. The other direction will be proved by contradiction. Suppose without loss of generality that $A_n \neq 0$. If $K_A(0,0) = v(E_A)^{-1}$, then by Lemma 1.2, $K_A(0,z) = K_A(0,0)$. By the reproducing property of the Bergman kernel,

$$0 = \frac{1}{v(E_A)} \int_{E_A} z_n^2 dv(z). \tag{2.41}$$

Notice that

$$Re(\int_{E_A} z_n^2 dv(z)) = \int_{E_A} x_{2n-1}^2 \; dv(x_1, \ldots, x_{2n}) - \int_{E_A} x_{2n}^2 \; dv(x_1, \ldots, x_{2n}) =: I - II.$$

Let $X = (X_1, \ldots, X_{2n})$ where

$$X_{2i-1} = x_{2i-1}\sqrt{1 + A_i}, \quad X_{2i} = x_{2i}\sqrt{1 - A_i}, \quad i = 1, \ldots, n,$$

and $\alpha = (\prod_{i=1}^{n}(1 + A_i)(1 - A_i))^{-\frac{1}{2}}$. Using these coordinates,

$$I = \alpha \int_{\{X_1^2+\cdots+X_{2n}^2<1\}} \frac{X_{2n-1}^2}{(1+A_n)}\, dv(X) =: \frac{1}{(1+A_n)}\gamma.$$

Similarly, $II = \frac{1}{(1-A_n)}\gamma$. Notice $I - II \neq 0$, which implies that (2.41) does not hold. Thus, $K_A(0,0) \neq v(E_A)^{-1}$.

□

Chapter 3

The $\overline{\partial}$-problem on convex domains

3.1 Introduction

In several complex variables, the fundamental differential operator is the $\overline{\partial}$-operator defined on functions and differential forms. Specifically, if u is a function and $f = \sum_{j=1}^{n} f_j d\bar{z}_j$ is a (0, 1)-form, then

$$\overline{\partial}u = \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j, \quad \overline{\partial}f = \sum_{1\le i<j\le n}^{n} \frac{\partial f_j}{\partial \bar{z}_i} - \frac{\partial f_i}{\partial \bar{z}_j} d\bar{z}_i \wedge d\bar{z}_j. \tag{3.1}$$

We say that a differential-form $f = \sum_{j=1}^{n} f_j d\bar{z}_j$ is in $L^p_{(0,1)}(\Omega)$ if $f_j \in L^p(\Omega)$ for each j. When not all of the f_j's are differentiable, we interpret $\overline{\partial}f$ in (3.1) in terms of distributional derivatives. A fundamental question in several complex variables in its simplest form is the $\overline{\partial}$-problem.

$\overline{\partial}$-problem (simple version). *Given a domain Ω, $p \in [1, \infty]$, and $f \in L^p_{(0,1)}(\Omega)$ so that $\overline{\partial}f = 0$, does there exist a constant $C > 0$ depending only on Ω and $u \in L^p(\Omega)$ so that*

$$\overline{\partial}u = f, \quad \|u\|_{L^p(\Omega)} \le C\|f\|_{L^p_{(0,1)}(\Omega)}?$$

The $\overline{\partial}$-problem has been an active area of research for over half a century. Most famously, Hörmander [35] solved the $\overline{\partial}$-problem in the affirmative when $p = 2$ for pseudoconvex domains. In [33], Henkin and Leiterer asked whether the $\overline{\partial}$-problem can be solved affirmatively when $p = \infty$ and Ω is a Cartan classical bounded symmetric domain of types I-IV. Motivated by the problems raised by Henkin and Leiterer [33], in this chapter, we give the following $\overline{\partial}$-estimates.

Proposition 3.1. *Let Ω be a simple convex domain (see Definition 3.7), g be the Bergman metric and K the Bergman kernel. Then there exists a constant C such that for any $\overline{\partial}$-closed $(0,1)$-form f on Ω with $\|f\|_{g,\infty} < \infty$, if the canonical solution to $\overline{\partial}u = f$ is in $C^1(\Omega)$, then it satisfies*

$$|u(a)| \leq C\|f\|_{g,\infty} K(a,a)^{\frac{1}{2}}, \quad a \in \Omega,$$

or equivalently,

$$|u(a)| \leq C\|f\|_{g,\infty} \prod_{j=1}^{n} \tau_j^{-1}(a),$$

where in both cases C depends only on the domain.

Proposition 3.1, while applicable to a wide class of domains, is not optimal. On a narrower class of domains, the Cartan classical domains, we give a sharp estimate:

Theorem 3.2. *Let Ω be a Cartan classical domain whose Bergman kernel and metric are denoted by K and g, respectively. Then there is a constant C such that for any $\bar{\partial}$-closed $(0,1)$-form f with $\|f\|_{g,\infty} < \infty$, if the L^2-minimal solution u to $\overline{\partial}u = f$ is in $C^1(\Omega)$, then it satisfies*

$$|u(z)| \leq C\|f\|_{g,\infty} \int_{\Omega} |K(z,w)| dv(w), \quad z \in \Omega. \tag{3.2}$$

Remark 3.3. *That the above estimate* (3.2) *is sharp for the unit ball was first proved by B. Berndtsson in [8].*

Remark 3.4. *Theorem 3.2 also holds on the strictly convex domains, see the paper of Dong, Li, and myself [24].*

Remark 3.5. *If $\overline{\partial} u_2 = f$, then the canonical solution to $\overline{\partial} u = f$ is given by*

$$u = u_2 - P[u_2],$$

where P is the Bergman projection. Since $P[u_2](z)$ is holomorphic, if $u_2 \in C^1(\Omega)$, then $u \in C^1(\Omega)$.

In Section 3.7, we show that Theorem 3.2 is sharp for the Cartan classical domain $II(2)$ as $z \to \partial\Omega$ along a certain direction. I will use Remark 3.5 in that example. This chapter is based on my joint work with Dong and Li [24].

3.2 Bergman metric and estimates for convex domains

The Bergman space $A^2(\Omega)$ on a domain $\Omega \subset \mathbb{C}^n$ is the closed holomorphic subspace of $L^2(\Omega)$. The Bergman projection is the orthogonal projection $P_\Omega : L^2(\Omega) \to A^2(\Omega)$ given by

$$P_\Omega[f](z) = \int_\Omega K(z,w) f(w) dv(w), \tag{3.3}$$

where $K(z,w)$ is the Bergman kernel on Ω and dv is the Lebesgue $\mathbb{R}^{2n}$ measure. We will write $K(z)$ to denote the on-diagonal Bergman kernel $K(z,z)$. When Ω is bounded, the complex Hessian of $\log K(z)$ induces the Bergman metric $B_\Omega(z;X)$ defined by

$$B_\Omega(z;X) := \left(\sum_{j,k=1}^{n} g_{j\overline{k}} X_j \overline{X}_k \right)^{\frac{1}{2}}, \quad g_{j\overline{k}}(z) := \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K(z), \quad \text{for } z \in \Omega,\ X \in \mathbb{C}^n.$$

The Bergman distance between $z, w \in \Omega$ is

$$\beta_\Omega(z,w) := \inf \left\{ \int_0^1 B_\Omega(\gamma(t); \gamma'(t)) dt \right\},$$

where the infimum is taken over all piecewise C^1-curves $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = z, \gamma(1) = w$.

Throughout this chapter,

$$B_a(r) := \{z \in \Omega : \beta_\Omega(z, a) \leq r\} \tag{3.4}$$

will denote the hyperbolic ball in the Bergman metric centered at $a \in \Omega$ of radius r. Additionally, $K(z, w), P_\Omega$ and g will always denote the Bergman kernel, Bergman projection on Ω and the Bergman metric respectively.

Consider a convex domain Ω that contains no complex lines and $a \in \Omega$. Choose any $a^1 \in \partial\Omega$ such that $\tau_1(a) := |a - a^1| = \text{dist}(a, \partial\Omega)$ and define $V_1 = a + \text{span}(a^1 - a)^\perp$. Let $\Omega_1 = \Omega \cap V_1$ and choose any $a^2 \in \partial\Omega_1$ such that $\tau_2(a) := ||a - a^2|| = \text{dist}(a, \partial\Omega_1)$. Let $V_2 = a + \text{span}(a^1 - a, a^2 - a)^\perp$ and $\Omega_2 = \Omega \cap V_2$. Repeat this process to obtain $a^1, ..., a^n, w_j = \frac{a^j - a}{||a^j - a||}, 1 \leq j \leq n$. Define

$$D(a; w, r) = \{z \in \mathbb{C}^n : |\langle z - a, w_i \rangle| < r_i, 1 \leq i \leq n\} \tag{3.5}$$

and

$$D(a, r) = \{z \in \mathbb{C}^n : |z_i - a_i| < r_i, 1 \leq i \leq n\}. \tag{3.6}$$

By [47, Theorem 2], for convex domains that contain no complex lines, the Kobayashi metric and the Bergman metric are comparable. It follows by [48, Corollary 2] that if Ω is a convex domain with no complex lines, then for every $\epsilon > 0$ there exists constants $C_{1,\epsilon}$ and $C_{2,\epsilon}$ such that for any a,

$$D(a; w, C_{1,\epsilon}\tau(a)) \subset B_a(\epsilon) \subset D(a; w, C_{2,\epsilon}\tau(a)). \tag{3.7}$$

By [47, Theorem 1] (see also [45]),

$$\frac{1}{4^n} \leq K(a) \prod_{j=1}^{n} \pi\tau_j^2(a) \leq \frac{(2n)!}{2^n}, \tag{3.8}$$

which implies that

$$\left(\frac{C_{1,\epsilon}}{2}\right)^{2n} \leq K(a)v(B_a(\epsilon)) \leq (2n)! \left(\frac{C_{2,\epsilon}^2}{2}\right)^n. \tag{3.9}$$

For positive real-valued functions f and g on Ω, we say $f \approx g$ if there exists a constant C

$$C^{-1} \leq fg^{-1} \leq C.$$

Using this notation, (3.9) may be rewritten as for every $\epsilon > 0$, $K(a) \approx v^{-1}(B_a(\epsilon))$ where $\approx$ is independent of the choice of a.

For any open subset A of Ω, we define

$$\|\overline{\partial}u\|_{g,\infty,A} = \big\||\overline{\partial}u(z)|_g\big\|_{L^\infty(A)}. \tag{3.10}$$

In the proofs below, C will denote a numerical constant, which may be different at each appearance. The Cauchy–Pompeiu formula gives the following useful proposition.

Proposition 3.6. *Let Ω be a bounded convex domain. For any $\epsilon > 0$ sufficiently small, there exists a constant C so that for any complex-valued C^1 function u on Ω,*

$$|u(a)| \leq \oint_{B_a(\epsilon)} |u(z)| dv_z + C\|\overline{\partial}u\|_{g,\infty,B_a(\epsilon)}.$$

Proof. After a complex rotation, without loss of generality, using the notation of (3.5), we may assume the standard basis for $\mathbb{C}^n$ is $(w_k)_{k=1}^n$. Let $r_j(a) = C_{1,\epsilon}\tau_j(a)$ and consider the polydisk $D(a; w, r)$ as defined above. Define the pseudometrics

$$M_{A_1}(z; X) = \sum_{k=1}^n \frac{|X_k|}{\tau_k(z)}, \quad M_{A_2}(z; X) = \sqrt{\sum_{k=1}^n \frac{|X_k|^2}{\tau_k(z)^2}}, \quad X \in \mathbb{C}^n.$$

Notice that these pseudometrics are equivalent because

$$
\begin{aligned}
1 \le \frac{(\sum_{k=1}^n |X_k|/\tau_k(z))^2}{\sum_{k=1}^n |X_k|^2/\tau_k(z)^2} &= 1 + \frac{\sum_{1\le k<j\le n} |X_k||X_j|/\tau_k(z)\tau_j(z)}{\sum_{k=1}^n |X_k|^2/\tau_k^2(z)} \\
&\le 1 + \frac{(1/2)\sum_{1\le k<j\le n} |X_k|^2/\tau_k^2 + |X_j|^2/\tau_j^2}{\sum_{k=1}^n |X_k|^2/\tau_k^2(z)} \\
&\le 1 + \frac{(1/2)\sum_{k,j=1}^n |X_k|^2/\tau_k^2 + |X_j|^2/\tau_j^2}{\sum_{k=1}^n |X_k|^2/\tau_k^2(z)} \\
&\le C.
\end{aligned}
$$

McNeal in [46, Theorem 2.5] proved that

$$M_{A_1}(z;X) \approx B_\Omega(z;X), \quad X \in \mathbb{C}^n,$$

where $\approx$ is independent of z and X. Therefore,

$$(g^{i\bar{j}}(z))_{i,j} \approx D[\tau_1^2(z), \ldots, \tau_n^2(z)], \quad z \in \Omega,$$

where the right hand side denotes the diagonal matrix with entries $\tau_1^2(z), \ldots \tau_n^2(z)$ and $\approx$ is independent of z. Consequently, with $r = (r_1, \ldots, r_n)$ and $a = (a_1, \ldots, a_n)$,

$$
\begin{aligned}
&\fint_{D(a_1,r_1)} \cdots \fint_{D(a_{k-1},r_{k-1})} r_k \|\overline{\partial}_k u(w_1, \ldots, w_{k-1}, \cdot, a_{k+1}, \ldots, a_n)\|_{L^\infty(D(a_k,r_k))} dv(w_{k-1}, \ldots, w_1) \\
\le\ &\fint_{D(a_1,r_1)} \cdots \fint_{D(a_{k-1},r_{k-1})} \left(\sup_{\eta \in D(a;w,r)} \sum_{j=1}^n r_j |\frac{\partial u}{\partial \bar{w}_j}(\eta)| \right) dv(w_{k-1}, \ldots, w_1) \\
=\ &\sup_{\eta \in D(a;w,r)} \sum_{j=1}^n r_j |\frac{\partial u}{\partial \bar{w}_j}(\eta)| \\
\le\ &C \sup_{\eta \in D(a;w,r)} \sum_{j=1}^n \tau_j(a) |\frac{\partial u}{\partial \bar{w}_j}(\eta)|. \qquad (3.11)
\end{aligned}
$$

Similarly, when $k = 1$,

$$r_1 \|\overline{\partial}_1 u(\cdot, a_2, \dots a_n)\|_{L^\infty(D(a_1,r_1))} \leq C \sup_{\eta \in D(a;w,r)} \sum_{j=1}^{n} \tau_j(a) \left|\frac{\partial u}{\partial \bar{w}_j}(\eta)\right|. \tag{3.12}$$

Thus, (3.11) and (3.12) are less than or equal to

$$\begin{aligned} C \sup_{z \in D(a;w,r)} \sqrt{\sum_{j=1}^{n} \tau_j^2(z) \left|\frac{\partial u}{\partial \bar{w}_j}(z)\right|^2} &\leq C \sup_{z \in D(a;w,r)} \sqrt{\sum_{i,j=1}^{n} g^{i\bar{j}}(z) \frac{\partial u}{\partial \bar{w}_i}(z) \overline{\frac{\partial u}{\partial \bar{w}_j}}(z)} \\ &= C \|\overline{\partial} u\|_{g,\infty,D(a;w,r)} \\ &\leq C \|\overline{\partial} u\|_{g,\infty,B_a(\epsilon)}. \end{aligned}$$

By the Cauchy-Pompeiu Theorem,

$$\begin{aligned} u(a) &= \frac{1}{2\pi i} \int_{|w_1 - a_1| = s_1} \frac{u(w_1, a_2, \dots, a_n)}{w_1 - a_1} dw_1 + \frac{1}{2\pi i} \int_{|w_1 - a_1| < s_1} \frac{\partial u}{\partial \bar{w}_1} \frac{1}{w_1 - a_1} dw_1 \wedge d\bar{w}_1 \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(a_1 + s_1 e^{i\theta}, a_2, \dots, a_n) d\theta + \frac{1}{2\pi i} \int_{|w_1 - a_1| < s_1} \frac{\partial u}{\partial \bar{w}_1} \frac{1}{w_1 - a_1} dw_1 \wedge d\bar{w}_1. \end{aligned}$$

Multiplying by s_1 and integrating with respect to s_1 from 0 to r_1,

$$\frac{r_1^2}{2} u(a) = \frac{1}{2\pi} \int_{D(0,r_1)} u(a_1 + \omega, a_2, \dots, a_n) dv(\omega) + \frac{1}{2\pi i} \int_0^{r_1} s_1 \int_{|w_1 - a_1| < s_1} \frac{\partial u}{\partial \bar{w}_1} \frac{1}{w_1 - a_1} dw_1 \wedge d\bar{w}_1 \wedge ds_1.$$

Thus,

$$
\begin{aligned}
&|u(a)| \\
&\leq \oint_{D(0,r_1)} |u(a_1+\omega, a_2, \dots a_n)| dv_\omega + \frac{1}{\pi r_1^2}\left|\int_0^{r_1}\int_{|w_1-a_1|<s_1} \frac{\partial u}{\partial \bar{w}_1}\frac{s_1}{w_1-a_1} dw_1 \wedge d\bar{w}_1 \wedge ds_1\right| \\
&\leq \oint_{D(a_1,r_1)} |u(w_1, a_2, \dots, a_n)| dv_{w_1} + \frac{1}{\pi r_1^2}\left|\int_0^{r_1}\int_{|w_1|<s_1} \frac{\partial u}{\partial \bar{w}_1}(a_1+w_1, a')\frac{s_1}{w_1} dw_1 \wedge d\bar{w}_1 \wedge ds_1\right| \\
&= \oint_{D(a_1,r_1)} |u(w_1, a_2, \dots, a_n)| dv_{w_1} + \frac{1}{\pi r_1^2}\left|\int_0^{r_1}\int_0^{2\pi}\int_0^{s_1} \frac{\partial u}{\partial \bar{w}_1}(a_1+te^{i\theta}, a')\frac{2s_1}{te^{i\theta}} t\ dt \wedge d\theta \wedge ds_1\right| \\
&\leq \oint_{D(a_1,r_1)} |u(w_1, a_2, \dots, a_n)| dv_{w_1} + \frac{1}{\pi r_1^2}\int_0^{r_1}\int_0^{2\pi}\int_0^{s_1} \left|\frac{\partial u}{\partial \bar{w}_1}(a_1+te^{i\theta}, a') 2s_1\right|\ dt \wedge d\theta \wedge ds_1 \\
&\leq \oint_{D(a_1,r_1)} |u(w_1, a_2, \dots, a_n)| dv_{w_1} + \frac{1}{\pi r_1^2}\int_0^{r_1} \|\frac{\partial u}{\partial \bar{w}_1}\|_{L^\infty(D(a_1,r_1)\times\{(a_2,\dots,a_n)\})} 4\pi s_1^2 ds_1 \\
&= \oint_{D(a_1,r_1)} |u(w_1, a_2, \dots, a_n)| dv_{w_1} + \frac{4\pi}{\pi r_1^2}\frac{r_1^3}{3}\|\frac{\partial u}{\partial \bar{w}_1}\|_{L^\infty(D(a_1,r_1)\times\{(a_2,\dots,a_n)\})} \\
&= \oint_{D(a_1,r_1)} |u(w_1, a_2, \dots, a_n)| dv_{w_1} + \frac{4}{3} r_1\|\frac{\partial u}{\partial \bar{w}_1}\|_{L^\infty(D(a_1,r_1)\times\{(a_2,\dots,a_n)\})} \\
&\leq \oint_{D(a_1,r_1)} |u(w_1, a_2, \dots, a_n)| dv_{w_1} + C\|\overline{\partial} u\|_{g,\infty,B_a(\epsilon)}.
\end{aligned}
$$

Repeating similar steps for $a_2, \dots a_n$, one gets that

$$
\begin{aligned}
&|u(a)| \\
&\leq \oint_{D(a,C_{1,\epsilon}\tau(a))} |u(w_1, ..., w_n)| dv_w + C\|\overline{\partial} u\|_{g,\infty,B_a(\epsilon)} \\
&\qquad + \sum_{k=2}^{n}\oint_{D(a_1,r_1)}\cdots\oint_{D(a_{k-1},r_{k-1})} C r_k \|\overline{\partial}_k u(w_1, \dots, w_{k-1}, \cdot, a_{k+1}, \dots, a_n)\|_{L^\infty(D(a_k,r_k))} dv(w_{k-1}, \dots, w_1) \\
&\leq \oint_{D(a,C_{1,\epsilon}\tau(a))} |u(w_1, ..., w_n)| dv_w + C\|\overline{\partial} u\|_{g,\infty,B_a(\epsilon)} \\
&\leq \oint_{B_a(\epsilon)} |u(w)| dv_w + C\|\overline{\partial} u\|_{g,\infty,B_a(\epsilon)}.
\end{aligned}
$$

Therefore the proof is complete. □

3.3 Pointwise estimates on the simple convex domains

McNeal proved an L^2-estimate using the Bergman metric for the simple domains, see [46]

Definition 3.7. [46] *A domain $\Omega \subset \mathbb{C}^n$ is a simple convex domain if it is a smoothly bounded convex domain of finite type.*

Theorem 3.8. [46, Proposition 3.3] *Let $\Omega \subset \mathbb{C}^n$ be a simple convex domain. There exists a constant $C > 0$ such that if f is a $\overline{\partial}$-closed (0, 1)-form on Ω, then there exists a solution to $\overline{\partial} u = f$ which satisfies*

$$\int_\Omega |u(z)|^2 dv(z) \leq C \int_\Omega |f(z)|_g^2 dv(z).$$

Proposition 3.6 and Theorem 3.8 yields a pointwise estimate for the canonical solution to the $\overline{\partial}$-problem on the simple convex domains.

Proposition 3.1. *Let Ω be a simple convex domain. Then there exists a constant C such that for any $\overline{\partial}$-closed $(0,1)$-form f on Ω with $\|f\|_{g,\infty} < \infty$, if the canonical solution to $\overline{\partial} u = f$ is in $C^1(\Omega)$, then it satisfies*

$$|u(a)| \leq C \|f\|_{g,\infty} K(a,a)^{\frac{1}{2}}, \quad a \in \Omega,$$

or equivalently,

$$|u(a)| \leq C \|f\|_{g,\infty} \prod_{j=1}^{n} \tau_j^{-1}(a)$$

where in both cases C depends only on the domain.

Proof. By Theorem 3.8 and Hölder's inequality,

$$\int_{B_a(\epsilon)} |u| dv \leq \|u\|_{L^2(\Omega)} v^{\frac{1}{2}}(B_a(\epsilon)) \leq C \|f\|_{g,\infty} v^{\frac{1}{2}}(B_a(\epsilon)).$$

By Proposition 3.6 and (3.8),

$$\begin{aligned} |u(a)| &\leq C\|f\|_{g,\infty} v^{\frac{-1}{2}}(B_a(\epsilon)) + C\|f\|_{g,\infty,B_a(\epsilon)} \\ &\approx C\|f\|_{g,\infty} K(a,a)^{\frac{1}{2}} + C\|f\|_{g,\infty,B_a(\epsilon)} \\ &\approx C\|f\|_{g,\infty} K(a,a)^{\frac{1}{2}}, \end{aligned} \tag{3.13}$$

where $\approx$ is independent of $a \in \Omega$.

Equivalently, by (3.7) and (3.8), (3.13) is equivalent to

$$|u(a)| \leq C\|f\|_{g,\infty} \prod_{j=1}^{n} \tau_j^{-1}(a).$$

□

In Section 3.6, on a different class of domains, we will derive a better estimate for $|u(a)|$ using more complicated L^2-estimates due to Donnelly and Fefferman. We will additionally show that these estimates are sharp.

3.4 Bergman kernel on the Cartan classical domains

A domain Ω is homogeneous if it has a transitive (holomorphic) automorphism group. A domain Ω is symmetric if for all $a \in \Omega$, there is an involutive automorphism G such that a is isolated in the set of fixed points of G. All bounded symmetric domains are convex and homogeneous. E. Cartan proved that all bounded symmetric domains in $\mathbb{C}^N$ up to biholomorphism are the Cartesian product(s) of the following four types of Cartan classical domains and two domains of exceptional types.

Definition 3.9. *A Cartan classical domain is a domain of one of the following types:*

(i) $I(m,n) := \{z \in M_{(m,n)}(\mathbb{C}) : I_m - zz^* > 0\}$*,* $m \leq n$*;*

(ii) $II(n) := \{z \in I(n,n) : z^\tau = z\}$*;*

(iii) $III(n) := \{z \in I(n,n) : z^\tau = -z\}$*;*

(iv) $IV(n) := \{z \in \mathbb{C}^n : 1 - 2|z|^2 + |s(z)|^2 > 0$ *and* $|s(z)| < 1\}$*, where* $s(z) := \sum_{j=1}^n z_j^2$ *and* $n > 2$*.*

Here $z^* := \bar{z}^\tau$ *is the conjugate transpose of* z*.*

Hua [37] obtained explicit formulas for the Bergman kernels on the Cartan classical domains. For a domain Ω of type I, II or III,

$$K(z,w) = C_\Omega \left[\det(I - zw^*)\right]^{-p}, \tag{3.14}$$

for some constant p depending on Ω, and for a domain of type IV,

$$K(z,w) = C_n[1 - 2\sum_{j=1}^n z_j \overline{w}_j + s(z)\overline{s(w)}]^{-n}. \tag{3.15}$$

We can give further estimates for the Bergman kernel beyond (3.8) for these types of domains, (see [38]).

Proposition 3.10. *Let* Ω *be a bounded homogeneous convex domain. Then,*

$$|K(z,a)| \approx K(a) \approx \frac{1}{v(B_a(\epsilon))}, \quad z \in B_a(\epsilon),$$

where $\approx$ *is independent of the choice of* a*. If* Ω *is furthermore a Cartan classical domain, then for any* $\epsilon > 0$*, there is a* C_ϵ *such that for any* $a \in \Omega$*,*

$$\max_{w \in B_a(\epsilon)} \left|\frac{K(z,w)}{K(z,a)}\right| \leq C_\epsilon, \quad z \in \Omega.$$

Proof. The first $\approx$ is Theorem A of [38], which only requires the domain to be bounded and homogeneous. The second equivalence is (3.8).

Let ϕ_a be an involutive automorphism such that $\phi_a(a) = 0$. By the transformation law of the Bergman kernel

$$\max_{w \in \overline{B_a(\epsilon)}} \left| \frac{K(z,w)}{K(z,a)} \right| = \max_{w \in \overline{B_a(\epsilon)}} \left| \frac{det J_{\mathbb{C}}\phi_a(w)}{det J_{\mathbb{C}}\phi_a(a)} \right| \left| \frac{K(\phi_a(z), \phi_a(w))}{K(\phi_a(z), 0)} \right|. \tag{3.16}$$

Since biholomorphisms are isometries of the Bergman metric,

$$\beta_\Omega(\phi_a(w), 0) = \beta_\Omega(\phi_a(w), \phi_a(a)) = \beta_\Omega(w, a) \le \epsilon.$$

Since Ω is bounded and symmetric, $K(\cdot,\cdot)$ is nonzero and continuous on $\overline{B_0(\epsilon)} \times \overline{\Omega}$ and $K(\cdot, 0) \equiv$ *const*. (cf. [6], [38], [42]). This can also be deduced from the explicit formulas (3.14), (3.15). Hence (3.16) is less than or equal to

$$C \max_{w \in \overline{B_a(\epsilon)}} \left| \frac{det J_{\mathbb{C}}\phi_a(w)}{det J_{\mathbb{C}}\phi_a(a)} \right|.$$

By the transformation law of the Bergman kernel

$$\left| \frac{K(w,a)}{K(a,a)} \right| = \left| \frac{det J_{\mathbb{C}}\phi_a(w) \overline{det J_{\mathbb{C}}\phi_a(a)} K(\phi_a(w), 0)}{|det J_{\mathbb{C}}\phi_a(a)|^2 K(0,0)} \right| = \left| \frac{det J_{\mathbb{C}}\phi_a(w)}{det J_{\mathbb{C}}\phi_a(a)} \right|.$$

Since $|K(w,a)| \approx K(a)$ for $w \in B_a(\epsilon)$ independent of the choice of a,

$$\left| \frac{K(w,a)}{K(a,a)} \right| \le C_\epsilon,$$

which completes the proof of the Proposition. $\square$

Lemma 3.11. *Let Ω be a Cartan classical domain. Let $\phi(z) := \gamma \log K(z)$, $\gamma > 0$. Then, for γ sufficiently small, $\|\overline{\partial}\phi\|^2_{i\partial\overline{\partial}\phi} \le \frac{1}{2}$.*

Proof. Notice $\|\overline{\partial}\phi\|^2_{i\partial\overline{\partial}\phi} = \gamma\|\overline{\partial}\log K\|^2_g$. In Theorem 3.16, we compute

$$|\overline{\partial}\log K|^2_g = cTr(zz^*).$$

Thus, the inequality follows by taking γ to be sufficiently small. □

3.5 L^2-estimates of solutions to the $\overline{\partial}$-problem

An upper semicontinuous function ϕ defined on a domain $\Omega \subset \mathbb{C}^n$ with values in $\mathbb{R}\bigcup\{-\infty\}$ is called plurisubharmonic if its restriction to every complex line is subharmonic. Let $L^2(\Omega,\phi)$ denote the set of measurable functions h satisfying $\int_\Omega |h(z)|^2 e^{-\phi(z)} dv_z < \infty$. A C^2 function ϕ is called strongly plurisubharmonic if $i\partial\overline{\partial}\phi$ is strictly positive definite. Now, let Ω be a bounded pseudoconvex domain and ϕ be strongly plurisubharmonic on Ω. Then, for any $(0,1)$-form $f = \sum_{k=1}^n f_k(z)d\overline{z}_k$, define the norm of f induced by $i\partial\overline{\partial}\phi$ as

$$|f|^2_{i\partial\bar{\partial}\phi}(z) := \sum_{j,k=1}^{n} \phi^{j\bar{k}}(z)\overline{f_j(z)}f_k(z), \tag{3.17}$$

where $(\phi^{j\bar{k}})^\tau$ equals the inverse of the complex Hessian matrix $H(\phi)$. Demailly's reformulation [19, 20] of Hörmander's theorem [35] says that *for any* $\bar{\partial}$*-closed* $(0,1)$*-form* f*, the canonical solution in* $L^2(\Omega,\phi)$ *(i.e. the* $L^2(\Omega,\phi)$*-minimal solution) of* $\bar{\partial}u = f$ *satisfies*

$$\int_\Omega |u|^2 e^{-\phi}\, dv \le \int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\phi}\, dv. \tag{3.18}$$

From this we see that when the $(0,1)$-form f is bounded in the Bergman metric g, then the canonical solution u to $\bar{\partial}u = f$ exists and satisfies the estimate (3.18).

Donnelly and Fefferman [27] (see also the papers by Berndtsson [7, 8]) modified Hörmander's theorem further as follows.

Theorem 3.12 (Donnelly-Fefferman type estimate)**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let ψ and ϕ be plurisubharmonic functions on Ω such that $i\partial\bar{\partial}\phi > 0$ and $|\partial\phi|^2_{i\partial\bar{\partial}\phi} \le \frac{1}{2}$. Then, the canonical solution $u_0 \in L^2(\Omega, \psi + \frac{\phi}{2})$ to $\bar{\partial}u = f$ satisfies*

$$\int_\Omega |u_0|^2 e^{-\psi} dv \le 4 \int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi} dv. \tag{3.19}$$

Next, we prove the following lemma, using the estimates (3.18) and (3.19).

Lemma 3.13. *Let Ω be a bounded pseudoconvex domain and f be a $\overline{\partial}$-closed $(0,1)$-form on Ω. Let ψ and ϕ be plurisubharmonic on Ω and u_0 and u_1 be the L^2-minimal solutions to $\overline{\partial}u = f$ in $L^2(\Omega, \psi + \frac{\phi}{2})$ and $L^2(\Omega, \phi)$, respectively. Suppose B is a compact subset of Ω and $h \in L^\infty(\Omega)$ with support in B.*

(i) If $i\partial\bar{\partial}\phi > 0$ and $\|\partial\phi\|^2_{i\partial\bar{\partial}\phi} \le \frac{1}{2}$ on Ω, then

$$\int_B |u_0| dv \le 2\Big(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi} dv\Big)^{\frac{1}{2}} \Big(\int_B e^{\psi} dv\Big)^{\frac{1}{2}} \tag{3.20}$$

and

$$\Big|\int_\Omega u_0 \overline{P(h)} dv\Big| \le 2v(B)\|h\|_\infty \Big(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi} dv\Big)^{\frac{1}{2}} \Big(\int_\Omega \max_{w\in B} |K(z,w)|^2 e^{\psi(z)} dv_z\Big)^{\frac{1}{2}}. \tag{3.21}$$

(ii)

$$\int_B |u_1| dv \le 2\Big(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\phi} dv\Big)^{\frac{1}{2}} \Big(\int_B e^{\phi} dv\Big)^{\frac{1}{2}}$$

and

$$\Big|\int_\Omega u_1 \overline{P(h)} dv\Big| \le 2v(B)\|h\|_\infty \Big(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\phi} dv\Big)^{\frac{1}{2}} \Big(\int_\Omega \max_{w\in B} |K(z,w)|^2 e^{\phi(z)} dv_z\Big)^{\frac{1}{2}}.$$

Proof. Let χ_B denote the characteristic function on B, and let $\beta := \chi_B \frac{u_0(z)}{|u_0(z)|}$. By (3.19),

$$\left(\int_B |u_0| dv\right)^2 = \left|\int_\Omega u_0\, \bar{\beta}\, dv\right|^2 \le \int\limits_\Omega |u_0|^2 e^{-\psi} dv \cdot \int\limits_B |\beta|^2 e^{\psi} dv \le 4 \int\limits_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi} dv \int\limits_B e^{\psi} dv,$$

which proves (3.20). Notice that

$$\begin{aligned}\left|\int_\Omega u_0 \overline{P(h)} dv\right|^2 &\le \int_\Omega |u_0|^2 e^{-\psi} dv \cdot \int_\Omega |P(h)|^2 e^{\psi} dv \\ &\le 4 \int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi} dv \cdot v^2(B) \cdot \|h\|^2_\infty \cdot \int_\Omega \max_{w\in B} |K(z,w)|^2 e^{\psi(z)} dv_z,\end{aligned}$$

which proves (3.21). Part (ii) can be proved similarly using Hörmander's estimate (3.18) in place of Donnelly-Fefferman's estimate (3.19). □

3.6 Pointwise estimates on the Cartan classical domains

Theorem 3.2. *Let Ω be a Cartan classical domain. Then, there is a constant C such that for any $\bar{\partial}$-closed $(0,1)$-form f on Ω with $\|f\|_{g,\infty} < \infty$, the canonical solution to $\overline{\partial} u = f$ satisfies*

$$|u(z)| \le C \|f\|_{g,\infty} \int_\Omega |K(z,w)| dv_w, \quad z \in \Omega.$$

Proof. By (3.18), the canonical solution to $\bar{\partial} u = f$ exists. For an arbitrary $a \in \Omega$ and any sufficiently small $\epsilon > 0$, let $\beta := \chi_{B_a(\epsilon)} \frac{u(z)}{|u(z)|}$, where $\chi_{B_a(\epsilon)}$ is the characteristic function of the hyperbolic ball $B_a(\epsilon)$. Let $\phi := \gamma \log K(z)$ be a plurisubharmonic function on Ω for some chosen γ that satisfies the condition in Lemma 3.11. Define $\psi_a(z) := -\log |K(z,a)|$. Since $K(\cdot, a) \neq 0$ and continuous on $\overline{\Omega}$, ψ_a is pluriharmonic and bounded on Ω. Also define the function

$$\phi_0 := \psi_a + \frac{\phi}{2},$$

and let u_0 be the $L^2(\Omega,\phi_0)$ minimal solution to the equation $\bar\partial v = f$. Let $m_a = \min_{z\in\overline{\Omega}} |K(z,a)| > 0$. By Theorem 3.12,

$$m_a \int_\Omega |u_0|^2 dv \le \int_\Omega |u_0|^2 e^{-\psi_a} dv \le C \int_\Omega |f|_g^2 e^{-\psi_a} dv \le C\|f\|_{g,\infty}^2 \int_\Omega e^{-\psi_a} dv < \infty,$$

where the last inequality follows because $K(\cdot,a)$ is nonzero and continuous on $\overline{\Omega}$. This implies that $u_0 \in L^2(\Omega)$. So $u-u_0 \in A^2(\Omega)$ and

$$\int_{B_a(\epsilon)} |u| dv = \int_\Omega u\bar\beta dv = \int_\Omega u(\overline{\beta - P(\beta)})dv = \int_\Omega u_0(\overline{\beta - P(\beta)})dv = \int_\Omega u_0\overline{\beta}dv - \int_\Omega u_0\overline{P(\beta)}dv.$$

By Lemma 3.11 and (3.20) in Lemma 3.13,

$$\begin{aligned}\left|\int_\Omega u_0\,\bar\beta\,dv\right|^2 &\le 4\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi_a} dv \cdot \int_{B_a(\epsilon)} e^{\psi_a} dv \\ &\le C\|f\|_{g,\infty}^2 \int_\Omega |K(z,a)| dv_z \cdot \int_{B_a(\epsilon)} |K(z,a)|^{-1} dv_z \\ &\le C_\epsilon \|f\|_{g,\infty}^2 \int_\Omega |K(z,a)| dv_z \cdot v(B_a(\epsilon)) \cdot K(a)^{-1} \\ &\le C_\epsilon \|f\|_{g,\infty}^2 v^2(B_a(\epsilon)) \int_\Omega |K(z,a)| dv_z,\end{aligned}$$

where the last two inequalities hold due to Proposition 3.10, and C_ϵ is a constant depending on ϵ. On the other hand, by (3.21) in Lemma 3.13 and Proposition 3.10 again,

$$\begin{aligned}\left|\int_\Omega u_0\overline{P(\beta)}dv\right|^2 &\le Cv^2(B_a(\epsilon)) \int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi_a} dv \cdot \int_\Omega \max_{w\in\overline{B_a(\epsilon)}} |K(z,w)|^2 e^{\psi_a(z)} dv_z \\ &\le C_\epsilon v^2(B_a(\epsilon)) \int_\Omega |f|^2_{i\partial\bar\partial\phi}(z) |K(z,a)| dv_z \cdot \int_\Omega |K(z,a)|^{2-1} dv_z \\ &\le C_\epsilon \|f\|_{g,\infty}^2 v^2(B_a(\epsilon)) \left(\int_\Omega |K(z,a)| dv_z\right)^2.\end{aligned}$$

Combining the above estimates, one can see easily

$$\frac{1}{v(B_a(\epsilon))}\int\limits_{B_a(\epsilon)}|u|dv \le C_\epsilon\|f\|_{g,\infty}\int_\Omega |K(z,a)|dv_z.$$

Fix $\epsilon > 0$. By Proposition 3.6, there exists a constant C depending only on Ω such that

$$|u(a)| \le C\|f\|_{g,\infty}\int_\Omega |K(z,a)|dv_z.$$

□

Remark 3.14. *These estimates also hold on the strictly convex domain. See the paper of Dong, Li and myself [24].*

3.7 Example on $II(2)$

Lemma 3.15. *Let Ω be a Cartan classical domain and $u(z) = \log K(z)$. Then $P[u](z)$ is a constant function on Ω.*

Proof. Notice that for all $z \in \Omega$,

$$\begin{aligned}
P[u](z) &= \int_\Omega u(w)K(z,w)dv_w \\
&= \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}w)K(z,e^{i\theta}w)d\theta dv_w \\
&= \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} u(w)K(z,e^{i\theta}w)d\theta dv_w \\
&= \int_\Omega u(w)K(z,0)dv_w \\
&= \frac{1}{v(\Omega)}\int_\Omega u(w)dv_w,
\end{aligned}$$

where the second equality uses that Ω is circular, the third equality follows from the transformation rule of the Bergman kernel, the fourth equality follows from the mean-value property of (anti-)holomorphic functions, and the fifth because Ω is a circular domain containing its center. $\square$

Theorem 3.16. *Let Ω be a Cartan classical domain and $u(z) = \log K(z)$. Then there exists a constant c so that $|\overline{\partial} u|_g^2 = c\,\mathrm{Tr}(zz^*)$.*

Proof. By [15, Proposition 2.1] and [37], using the notation of [15], for $z \in M_{(m,n)}(\mathbb{C})$, if $W(z,w) = I_m - zw^*$, $V(z) = W(z,z)$ and V_{uv} denotes the (u,v) entry of V, then

$$g^{j\alpha,\overline{k\beta}}(z) = \begin{cases} V_{jk}(\delta_{\alpha\beta} - \sum_{l=1}^{m} z_{l\alpha}\overline{z}_{l\beta}) = \overline{g^{k\beta,\overline{j\alpha}}} & z \in I(m,n) \\ \frac{1}{4}V_{jk}\frac{V_{\alpha\beta}}{(1-\frac{\delta_{j\alpha}}{2})(1-\frac{\delta_{k\beta}}{2})} & z \in II(n) \\ \frac{1}{4}V_{jk}V_{\alpha\beta}(1-\delta_{j\alpha})(1-\delta_{k\beta}) & z \in III(n) \end{cases}$$

and for a type IV domain,

$$g^{j,\overline{k}}(z) = r(z)(\delta_{jk} - 2z_j\overline{z}_k) + 2(\overline{z_j} - \overline{s(z)}z_j)(z_k - s(z)\overline{z_k}) \tag{3.22}$$

where $z \in \mathbb{C}^n$ and

$$s(z) = \sum z_j^2, \quad \text{and} \quad r(z) = 1 - 2|z|^2 + |s(z)|^2.$$

For a domain of types $I - III$, by (3.14) up to a constant C, depending on the type of domain,

$$\log K(z) = C \log\det V(z) = C\log\det W(z,z).$$

Hence for those domains, it suffices to show that $|\overline{\partial}\log\det W(z,z)|_g^2 = Tr(zz^*)$.

In [15], the authors proved, for $z \in II(n)$ and $w \in \overline{II(n)}$,

$$c_{j\alpha}(z,w) := \frac{\partial \log\det W(z,w)}{\partial z_{j\alpha}} = -(2-\delta_{j\alpha})[w^* W^{-1}(z,w)]_{j\alpha}$$

and

$$\overline{c_{k\beta}}(z,w) = -(2-\delta_{k\beta})[W^{-1}(w,z)w]_{k\beta}$$

Similar elementary proofs show

$$\frac{\partial \log\det W(z,w)}{\partial z_{j\alpha}} = \begin{cases} [-w^* W^{-1}]_{\alpha j} & z,w \in I(m,n) \\ -2(1-\delta_{j\alpha})[w^* W^{-1}]_{\alpha j} & z,w \in III(n). \end{cases}$$

Thus, for $I(m,n)$,

$$\begin{aligned}
&\sum_{j,\beta}\sum_{k,\alpha} g^{j\alpha,\overline{k\beta}} \frac{\partial \log\det W(z,z)}{\partial z_{j\alpha}} \overline{\frac{\partial \log\det W(z,z)}{\partial z_{k\beta}}} \\
&= \sum_{j,\beta,k,\alpha} V_{jk}(\delta_{\alpha\beta} - \sum_{l=1}^{m} z_{l\alpha}\bar{z}_{l\beta})(-[z^* V^{-1}]_{\alpha j}(-\overline{[z^* V^{-1}]}_{\beta k}) \\
&= \sum_{j,\beta,k,\alpha} V_{jk}[I - z^\tau \bar{z}]_{\alpha\beta}[z^* V^{-1}]_{\alpha j}[z^\tau \overline{V^{-1}}]_{\beta k} \\
&= \sum_{j,\beta,k,\alpha} [z^* V^{-1}]_{\alpha j} V_{jk}[I - z^\tau \bar{z}]_{\alpha\beta}[z^\tau \overline{V^{-1}}]_{\beta k} \\
&= \sum_{\alpha k} [z^* V^{-1} V]_{\alpha k}[(I - z^\tau \bar{z}) z^\tau \overline{V^{-1}}]_{\alpha k} \\
&= \sum_{\alpha,k} [z^*]_{\alpha k}[z^\tau (I - \bar{z} z^\tau)\overline{V^{-1}}]_{\alpha k} \\
&= \sum_{\alpha,k} [z^*]_{\alpha k}[z^\tau (I - \bar{z} z^\tau)\overline{(I - zz^*)^{-1}}]_{\alpha k} \\
&= \sum_{\alpha,k} [z^*]_{\alpha k}[z^\tau]_{\alpha k} \\
&= \mathrm{Tr}(zz^*).
\end{aligned}$$

For $z \in II(n)$,

$$\begin{aligned}
\sum_{k,\alpha} g^{j\alpha,\overline{k\beta}} \frac{\partial \log \det W(z,z)}{\partial z_{j\alpha}} \overline{\frac{\partial \log \det W(z,z)}{\partial z_{k\beta}}} &= \sum_{k,\alpha} V_{jk} V_{\alpha\beta} [z^* V(z)^{-1}]_{j\alpha} \overline{[V(z)^{-1} z]_{k\beta}} \\
&= \sum_{\alpha} [z^* V(z)^{-1}]_{j\alpha} V_{\alpha\beta} \sum_{k} V_{jk} \overline{[V(z)^{-1} z]_{k\beta}} \\
&= [z^*]_{j\beta} [z]_{j\beta}
\end{aligned}$$

Thus, for $z \in II(n)$,

$$\sum_{j,\beta} \sum_{k,\alpha} g^{j\alpha,\overline{k\beta}} \frac{\partial \log \det W(z,z)}{\partial z_{j\alpha}} \overline{\frac{\partial \log \det W(z,z)}{\partial z_{k\beta}}} = \sum_{j,\beta} [z^*]_{j\beta} [z]_{j\beta} = \sum_{j,\beta} [z^\tau]_{\beta j} [z^*]_{j\beta} = \text{Tr}(zz^*).$$

For $z \in III(n)$,

$$\begin{aligned}
& \sum_{j,\beta} \sum_{k,\alpha} g^{j\alpha,\overline{k\beta}} \frac{\partial \log \det W(z,z)}{\partial z_{j\alpha}} \overline{\frac{\partial \log \det W(z,z)}{\partial z_{k\beta}}} \\
= \;& \frac{1}{4} \sum V_{jk} V_{\alpha\beta} ((1-\delta_{j\alpha})(1-\delta_{k\beta})(-2)(1-\delta_{j\alpha})(-2)(1-\delta_{k\beta}) [z^* V^{-1}]_{\alpha j} \overline{[z^* V^{-1}]}_{\beta k} \\
= \;& \sum V_{jk} V_{\alpha\beta} (1-\delta_{j\alpha})(1-\delta_{k\beta}) [z^* V^{-1}]_{\alpha j} \overline{[z^* V^{-1}]}_{\beta k}. \\
= \;& \sum_{\alpha,\beta,j,k} [z^* V^{-1}]_{\alpha j} V_{jk} V_{\alpha\beta} [-V^{-1} z]_{\beta k} \\
= \;& -\sum_{\alpha,k} [z^*]_{\alpha k} [z]_{\alpha k} \\
= \;& -\text{Tr}(z^\tau z^*) \\
= \;& \text{Tr}(zz^*).
\end{aligned}$$

For the $IV(n)$ case, recall that its Bergman kernel is

$$K(z) = \frac{1}{Vol(\mathcal{R}_{IV})} (1 + |z \otimes z^\tau|^2 - 2\overline{z} \otimes z^\tau)^{-n}.$$

Notice that

$$
\begin{aligned}
& n^2 \log(1+|z\otimes z^\tau|^2 - 2\overline{z}\otimes z^\tau)_j \log(1+|z\otimes z^\tau|^2 - 2\overline{z}\otimes z^\tau)_{\bar{k}} \\
= \;& \frac{n^2}{r(z)^2}[2z_j\overline{z\otimes z^\tau} - 2\overline{z}_j][2\overline{z}_k(z\otimes z^\tau) - 2z_k] \\
= \;& \frac{4n^2}{r(z)^2}[z_j s(\overline{z}) - \overline{z}_j][\overline{z}_k s(z) - z_k].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S(z) := \quad & \sum g^{j,\bar{k}}(z)(\log K(z))_j(\log K(z))_{\bar{k}} = 4n^2\sum_{j,k=1}^{n}[r(z)(\delta_{jk} - 2z_j\overline{z}_k) \\
& +2(\overline{z}_j - s(\overline{z})z_j)(z_k - s(z)\overline{z}_k)][(\frac{1}{r(z)^2})[\overline{z}_j - s(\overline{z})z_j)(z_k - s(z)\overline{z}_k)]
\end{aligned}
$$

Let

$$
F(z) = \frac{4n^2}{r(z)}\sum_{j,k=1}^{n}(\delta_{jk} - 2z_j\overline{z_k})[\overline{z}_j - \overline{s(z)}z_j][z_k - s(z)\overline{z}_k]
$$

and

$$
G(z) = \frac{8n^2}{r(z)^2}\sum_{k,j=1}^{n}(\overline{z}_j - \overline{s(z)}z_j)^2(z_k - s(z)\overline{z_k})^2
$$

Then $S(z) = F(z) + G(z)$. We calculate F and G separately:

$$\begin{aligned}
F(z) &= \frac{4n^2}{r(z)} \sum_{j,k=1}^{n} [\delta_{jk} - 2z_j\overline{z}_k][\overline{z}_j z_k - s(z)\overline{z_j z_k} - z_j z_k \overline{s(z)} + |s(z)|^2 z_j \overline{z_k}] \\
&= \frac{4n^2}{r(z)}[\sum_{j=1}^{n} |z_j|^2 - s(z)\overline{z_j}^2 - z_j^2\overline{s(z)} + |s(z)|^2|z_j|^2 \\
&\qquad -2\sum_{j,k=1}^{n} |z_j|^2|z_k|^2 - s(z)|z_j|^2\overline{z_k}^2 - z_j^2|z_k|^2\overline{s(z)} + |s(z)|^2 z_j^2\overline{z_k}^2)] \\
&= \frac{4n^2}{r(z)}[|z|^2 - |s(z)|^2 - |s(z)|^2 + |s(z)|^2|z|^2 \\
&\qquad -2(|z|^4 - s(z)|z|^2\overline{s(z)} - s(z)|z|^2\overline{s(z)} + |s(z)|^2 s(z)\overline{s(z)})] \\
&= \frac{4n^2}{r(z)}[|z|^2 - 2|s(z)|^2 + |s(z)|^2|z|^2 - 2|z|^4 + 4|s(z)|^2|z|^2 - 2|s(z)|^4] \\
&= \frac{4n^2}{r(z)}[-2|z|^4 + 5|s(z)|^2|z|^2 - 2|s(z)|^2 + |z|^2 - 2|s(z)|^4].
\end{aligned}$$

Now, we calculate $G(z)$:

$$\begin{aligned}
G(z) &= \frac{8n^2}{r(z)^2}|\sum_{j=1}^{n}(z_j - s(z)\overline{z_j})^2|^2 \\
&= \frac{8n^2}{r(z)^2}|\sum_{j=1}^{n} z_j^2 - 2s(z)|z_j|^2 + s(z)^2\overline{z_j}^2|^2 \\
&= \frac{8n^2}{r(z)^2}|s(z) - 2s(z)|z|^2 + s(z)^2\overline{s(z)}|^2 \\
&= \frac{8n^2}{r(z)^2}|s(z)|^2|1 - 2|z|^2 + |s(z)|^2|^2 \\
&= \frac{8n^2}{r(z)^2}(|s(z)|^2 r(z)^2 \\
&= 8n^2|s(z)|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
F(z)+G(z) &= \frac{4n^2}{r(z)}[-2|z|^4+5|s(z)|^2|z|^2-2|s(z)|^2+|z|^2-2|s(z)|^4]+8n^2|s(z)|^2 \\
&= \frac{4n^2}{r(z)}[-2|z|^4+5|s(z)|^2|z|^2-2|s(z)|^2+|z|^2-2|s(z)|^4]+\frac{4n^2}{r(z)}2|s(z)|^2r(z) \\
&= \frac{4n^2}{r(z)}[-2|z|^4+5|s(z)|^2|z|^2-2|s(z)|^2+|z|^2-2|s(z)|^4 \\
&\qquad +2|s(z)|^2(1-2|z|^2+|s(z)|^2)] \\
&= \frac{4n^2}{r(z)}[-2|z|^4+|z|^2|s(z)|^2+|z|^2] \\
&= \frac{4n^2|z|^2}{r(z)}[1+|s(z)|^2-2|z|^2] \\
&= 4n^2|z|^2,
\end{aligned}$$

which concludes the IV(n) case.

□

Theorem 3.17. *If $\Omega = II(2)$, $u(z) = \log K(z)$, and $a = D[r, 0]$, the diagonal matrix with diagonal entries r and 0, then*

$$c\|\overline{\partial}u\|_{g,\infty}\int_\Omega |K(z,a)|dv(z) \le u(a) \le C\|\overline{\partial}u\|_{g,\infty}\int_\Omega |K(z,a)|dv(z),$$

for all r sufficiently close to 1, which shows that Theorem 3.2 is sharp on $\Omega = II(2)$ in this direction.

Proof. Since $u(z) \in C^1(\Omega)$, the second inequality follows from Theorem 3.2 and Remark 3.5. Since the canonical solution of $\overline{\partial}u = f$ is $u(z) - P[u](z)$, by Lemma 3.15, the canonical solution is $u(z) - C$ for some constant C. By Theorem 3.16, $\|\overline{\partial}u\|_{g,\infty}$ is finite. From the formula for the Bergman kernel of $II(2)$, (3.14),

$$\log K(z) = C\log(1-r^2).$$

Thus, it remains to verify that

$$\int_{\Omega} |K(z,a)| dv(z) \approx \log(1-r), \quad r \to 1^-.$$

This was verified by Englis and Zhang [28, Theorem 1].

□

Chapter 4

Quasi-analytic solutions to the $\overline{\partial}$-problem

4.1 Introduction

In 1997, Christ and Li [17] proved the following.

Theorem 4.1. *[17] Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with real-analytic boundary, and suppose that $0 \leq p \leq n$ and $0 < q \leq n$. Let $f \in C^{\omega}_{(p,q)}(\overline{\Omega})$ be $\overline{\partial}$-closed. Then there is a $u \in C^{\omega}_{(p,q-1)}(\overline{\Omega})$ so that $\overline{\partial}u = f$ on $\overline{\Omega}$.*

In this chapter, we extend their result from forms f with real-analytic coefficients to those that belong to a quasi-analytic class, see Theorem 4.23 below.

The chapter is organized as follows. In Section 4.2, we define quasi-analytic classes of forms and review Hörmander's elliptic regularity theorems for certain partial differential equations with sources from quasi-analytic classes. In Section 4.3, we define the $\overline{\partial}$-Neumann operator, and in Section 4.4, we prove the main theorem, Theorem 4.23.

4.2 Quasi-analytic classes and elliptic regularity

Let $X \subset \mathbb{R}^n$, $\mathcal{D}'(X)$ denote the set of distributions on X and $\mathcal{E}'(X)$ denote the set of distributions with compact support in X.

Definition 4.2. *Let $\{L_k\}_{k=0}^{\infty}$ be an increasing sequence of positive numbers such that $L_0 = 1$ and*

$$k \leq L_k, \quad L_{k+1} \leq CL_k,$$

for some constant C. Let X be an open subset of $\mathbb{R}^n$. We define a class of functions $C^L(X)$ to be the set of all $u \in C^\infty(X)$ so that for every compact set $K \subset X$, there is a constant C_K with

$$|D^\alpha u(x)| \leq C_K (C_K L_{|\alpha|})^{|\alpha|}, \quad x \in K, \quad \alpha \in \mathbb{N}^n.$$

Example 4.3. *Let $L_k = k + 1$. Then $C^L(X)$ is the set of real-analytic functions in X.*

Example 4.4. *More generally, let $L_k = (k+1)^a$ where $a \geq 1$. Then $C^L(X)$ is the Gevrey class of order a.*

We extend this definition to forms.

Definition 4.5. *Let Ω be a bounded domain in $\mathbb{C}^n$. For $0 \leq p, q \leq n$, let $C^L_{(p,q)}(\bar{\Omega})$ denote the (p,q) forms*

$$\omega = \sum_{I,J}{}' f_{I,J} dz^I \wedge d\bar{z}^J,$$

where the primed-summation denotes that the summation is over the strictly increasing p-tuples and q-tuples $I = (i_1, \ldots i_p)$ and $J = (j_1, \ldots j_q)$ respectively,

$$dz^I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}, \quad d\bar{z}^J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q},$$

and $f_{I,J} \in C^L(X)$ for some neighborhood X of $\overline{\Omega}$.

In this chapter, we restrict our attention to the classes $C^L(X)$ and $C^L_{(p,q)}(X)$ that are quasi-analytic.

Definition 4.6. *Let* $(L_k)_{k=1}^{\infty}$ *satisfy the conditions of Definition 4.2.* $C^L_{(p,q)}(\Omega)$ *is said to be a quasi-analytic class if and only if*

$$\{f_{I,J} \in C^L(\Omega) : \exists x_0 \in \Omega \ s.t. \ \forall \alpha \in \mathbb{N}^n, D^\alpha(f)(x_0) = 0\} = \{0\}, \quad |I| = p, \quad |J| = q,$$

where 0 denotes the function which is identically 0.

Theorem 4.7. ***(Denjoy-Carleman theorem)*** *Let* $L = (L_k)_{k=1}^{\infty}$ *satisfy the conditions in Definition 4.2.* $C^L(\Omega)$ *is a quasi-analytic class if and only if*

$$\sum_{k=0}^{\infty} \frac{1}{L_k} = \infty.$$

From the Denjoy-Carleman theorem, one can conclude that the Gevrey classes of order $a > 1$ are not quasi-analytic. We will be interested in partial differential equations on forms where the source forms belong to a quasi-analytic class $C^L_{(p,q)}(\Omega)$. We begin by recalling Hörmander's work on linear partial differential operators [36, Chapter 8].

Proposition 4.8. $C^L(X)$ *is a ring (with binary operations function addition and multiplication) and it is closed under differentiation.*

Definition 4.9. *Let* X *be an open set in* $\mathbb{R}^n$*. For any distribution* $u \in \mathcal{D}'(X)$ *we define* sing supp$_L$ u *to be the smallest closed subset of* X *such that* u *is* C^L *in the complement of the closed set.*

Definition 4.10. *If* $X \subset \mathbb{R}^n$ *and* $u \in \mathcal{D}'(X)$*, then we denote by* $WF_L(u)$ *the complement in* $X \times (\mathbb{R}^n \setminus \{0\})$ *of the set of* (x_0, ξ_0) *such that there is a neighborhood* $U \subset X$ *of* x_0*, a conic neighborhood* Γ *of* ξ_0 *and a bounded sequence* $u_N \in \mathcal{E}'(X)$ *that satisfies*

1. $u_N|_U = u$

2.

$$|\hat{u}_N(\xi)| \leq C\Big(\frac{CL_N}{|\xi|}\Big)^N, \quad \xi \in \Gamma, \quad N = 1, 2, ...$$

For a distribution u, $WF_L(u)$ and *sing supp*$_L(u)$ are related by

Theorem 4.11. *If $u \in \mathcal{D}'(X)$, then the projection of $WF_L(u)$ in X is equal to sing supp$_L(u)$.*

Definition 4.12. *Let P be a linear partial differential operator of order m in X with C^∞ coefficients. That is,*

$$P = P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad a_\alpha(x) \in C^\infty(X).$$

The principal symbol P_m of P is defined to be

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

Recall the definition of an elliptic operator:

Definition 4.13. *Let $T^*(X)$ denote the cotangent bundle of X. The characteristic set of P, $Char P$ is defined by*

$$Char P = \{(x, \xi) \in T^*(X) \setminus \{0\} : P_m(x, \xi) = 0\}.$$

P is called elliptic if

$$Char P = \emptyset.$$

Example 4.14. *The Laplacian $\triangle$ is an elliptic operator.*

With these definitions, Hörmander proved

Theorem 4.15. *If $P(x, D)$ is a linear partial differential operator of order m with real analytic* *coefficients in* X*, then*

$$WF_L(u) \subset Char P \cup WF_L(Pu), \quad u \in \mathcal{D}'(X).$$

Corollary 4.16. *If* $P(x, D)$ *is elliptic of order* m *with real analytic coefficients in* X*, then*

$$WF_L(u) \subset WF_L(Pu), \quad u \in \mathcal{D}'(X).$$

Proof. If P is an elliptic operator then, $WF_L(u) \cap Char P = \emptyset$. □

Corollary 4.17. *Let* $x_0 \in X$*. If* $P(x, D)$ *is elliptic of order* m *with real analytic coefficients in* X *and* $Pu \in C^L(U)$ *where* U *is a neighborhood of* x_0*, then there exists a neighborhood* V *of* x_0 *so that* $u \in C^L(V)$*.*

Proof. By Theorem 4.11 and the previous corollary, if $x \notin sing\ supp_L(Pu)$, then $x \notin sing\ supp_L(u)$. Thus, if Pu is a C^L function in a neighborhood of a point $x \in X$, then u is also a C^L function in some neighborhood of $x \in X$. □

4.3 $\overline{\partial}$-Neumann operator

We now restrict our attention from domains in $\mathbb{R}^n$ to domains $\Omega \subset \mathbb{C}^n$. The $\overline{\partial}$-operator may be extended to an unbounded operator on forms in $L^2_{(p,q)}(\Omega)$ with

$$Dom(\overline{\partial}) = \{f \in L^2_{(p,q)}(\Omega) : \overline{\partial} f \in L^2_{(p,q+1)}(\Omega)\}, \quad q < n,$$

and

$$Dom(\overline{\partial}) = L^2_{(p,q)}(\Omega), \quad q = n.$$

By integration by parts, the formal adjoint ϑ of $\overline{\partial}$ is given by

$$\vartheta f = (-1)^{(p-1)} \sum_{I,K}{}' \sum_{j=1}^{n} \frac{\partial f_{I,jK}}{\partial z_j} dz^I \wedge d\bar{z}^K, \quad |I| = p, |K| = q-1. \tag{4.1}$$

As an unbounded operator of a Hilbert space, $\overline{\partial}$ has an adjoint operator denoted $\overline{\partial}^*$. When $f \in Dom(\overline{\partial}^*)$, we have $\overline{\partial}^* f = \vartheta f$.

The $\overline{\partial}$-Neumann problem is closely related to the $\overline{\partial}$-problem. One of the fundamental operators in the problem is the complex Laplacian.

Definition 4.18. *The complex Laplacian is the operator* $\Box_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ *defined by*

$$\Box_{(p,q)} = \overline{\partial}_{(p,q-1)} \overline{\partial}^*_{(p,q)} + \overline{\partial}^*_{(p,q+1)} \overline{\partial}_{(p,q)},$$

where

$$\begin{aligned} Dom(\Box) = \{ f \in L^2_{(p,q)}(\Omega) : \quad & f \in Dom(\overline{\partial}_{(p,q)}) \cap Dom(\overline{\partial}^*_{(p,q)}), \overline{\partial}_{(p,q)} f \in Dom(\overline{\partial}^*_{(p,q+1)}), \\ & \overline{\partial}^*_{(p,q)} f \in Dom(\overline{\partial}_{p,q-1}) \}. \end{aligned}$$

The complex Laplacian acts coordinate-wise as the standard Laplacian.

Proposition 4.19. [16, Theorem 4.2.4] *If* $f = \sum'_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J \in C^2_{(p,q)}(\overline{\Omega}) \cap Dom(\Box_{(p,q)})$, *then*

$$\Box_{(p,q)} f = \frac{-1}{4} \sum_{I,J}{}' \Delta f_{I,J} dz^I \wedge d\bar{z}^J,$$

where Δ *is the Laplacian.*

Corollary 4.20. *If* $f \in C^2_{(p,q)}(\overline{\Omega}) \cap Dom(\Box_{(p,q)})$ *and* $\Box_{(p,q)} f \in C^L_{(p,q)}(\overline{\Omega})$, *then* $f \in C^L_{(p,q)}(\Omega)$.

Proof. This follows by Corollary 4.17 because Δ is elliptic with constant coefficients. □

The inverse of the complex Laplacian is the $\overline{\partial}$-Neumann operator.

Theorem 4.21. [16, Theorem 4.4.1 and Theorem 5.3.9] *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq p \leq n, 1 \leq q \leq n$, there exists a bounded operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ such that*

1. *$Ran(N_{(p,q)})$ is contained in the domain of $\Box_{(p,q)}$*
2. *$\Box_{(p,q)} N_{(p,q)} = I$*
3. *$\overline{\partial} N_{(p,q)} = N_{(p,q+1)} \overline{\partial}$ on $Dom(\overline{\partial})$, $\quad 1 \leq q \leq n-1$*

Additionally, if Ω is strongly pseudoconvex with C^∞ boundary, then

4. *$N_{(p,q)}(C^\infty_{(p,q)}(\overline{\Omega})) \subset C^\infty_{(p,q)}(\overline{\Omega})$, $\quad q \geq 0$.*

4.4 $\overline{\partial}$-problem in the quasi-analytic class

Definition 4.22. *Let Ω be a bounded domain. $\overline{\Omega}$ is said to have a* **Stein neighborhood basis** *if there exists a sequence of strictly pseudoconvex domains with C^∞ boundaries $\{\Omega_j\}_{j=1}^\infty$ such that $\overline{\Omega} = \cap_{j=1}^\infty \Omega_j$ and $\overline{\Omega}_{j+1} \subset \Omega_j$.*

The closures of all bounded strictly pseudoconvex domains with C^∞ boundaries and all weakly pseudoconvex domains with real-analytic boundary have a Stein neighborhood basis. The worm domain is an example of a bounded pseudoconvex domain with C^∞ boundary regularity whose closure does not have a Stein neighborhood basis. See [22, 21].

We now prove the main theorem. As stated in the introduction, the case where Ω has real-analytic boundary was solved by Christ and Li, [17].

Theorem 4.23. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, whose closure has a Stein neighborhood basis. Let $C^L_{(p,q)}(\overline{\Omega})$ be quasi-analytic with $0 \leq p \leq n$, $1 \leq q \leq n$. If $f \in C^L_{(p,q)}(\bar{\Omega})$ is $\bar{\partial}$-closed, then there is a $u \in C^L_{(p,q-1)}(\bar{\Omega})$ such that $\bar{\partial} u = f$ in $\bar{\Omega}$.*

Proof. Let $\{\Omega_j\}_{j=1}^{\infty}$ be the Stein neighborhood basis, and without loss of generality, let $f = \sum_{I,J} f_{IJ}\, dz^I \wedge d\bar{z}^J$. Select Ω_k so that each component of f is in $C^L(\Omega_k)$. Since $C^L(\Omega_k)$ is quasi-analytic and, by Proposition 4.8, closed under differentiation, each component of $\bar{\partial} f$ is identically 0 in Ω_k. Thus, f is $\bar{\partial}$-closed in Ω_k.

Since Ω_{k+1} is a bounded strictly pseudoconvex domain with C^∞ boundary, by Theorem 4.21, there exists a $g \in C^\infty_{(p,q)}(\overline{\Omega_{k+1}})$ that satisfies $\Box g = f$ on Ω_{k+1}. Since $\Box g \in C^L_{(p,q)}(\Omega_{k+1})$, by Corollary 4.20, $g \in C^L_{(p,q)}(\Omega_{k+1})$. Since $g \in Dom(\overline{\partial}^*_{(p,q)})$, by (4.1) and Proposition 4.8, $\overline{\partial}^* g \in C^L_{(p,q-1)}(\Omega_{k+1})$. Let $u = \overline{\partial}^* g = \overline{\partial}^* N f$. Then on Ω_{k+1},

$$\begin{aligned}
\overline{\partial} u &= \overline{\partial}\overline{\partial}^* N f \\
&= (\Box - \overline{\partial}^*\overline{\partial}) N f \\
&= \Box N f - \overline{\partial}^*\overline{\partial}) N f \\
&= f - \overline{\partial}^* N \overline{\partial} f \\
&= f
\end{aligned}$$

where the second to last equality follows Theorem 4.21, and the last equality follows because $\overline{\partial} f = 0$. The proof of the theorem is complete. □

Remark 4.24. *Since the closure of a pseudoconvex domain with real-analytic boundary admits a Stein neighborhood basis, in the case where Ω has real-analytic boundary, the Stein neighborhood basis hypothesis in the above theorem is redundant.*

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