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Macdonald Polynomials and Graded Characters of Generalized Demazure Modules of so(2n)[t]

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Macdonald Polynomials and Graded Characters of Generalized Demazure Modules of $\mathfrak{so}_{2n}[t]$

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

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in

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by

Maranda N. Smith

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ABSTRACT OF THE DISSERTATION

Macdonald Polynomials and Graded Characters of Generalized Demazure Modules of $\mathfrak{so}_{2n}[t]$

by

Maranda N. Smith

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2022
Dr. Vyjayanthi Chari, Chairperson

In recent work published by Biswal, Chari, Shereen, and Wand [1] the authors defined a family of symmetric polynomials indexed by pairs of dominant integral weights, $G_{\nu,\lambda}(z,q)$ where $z = (z_1, \cdots .z_{n+1}) \in \mathbf{C}^{n+1}$, and determined that $G_{0,\lambda}(z,q)$ is the graded character of a level two Demazure module for $\mathfrak{sl}_{n+1}[t]$. The aim of this thesis is to construct analogues of these polynomials for the generalized Demazure modules for $\mathfrak{so}_{2n}[t]$ as they are presented by Chari, Davis, and Moruzzi [3]. We do this by constructing modules which interpolate from the presentation provided in [3] and local Weyl modules. We then create short exact sequences between them to relate their graded characters. This allows us to identify coefficients in the corresponding graded characters with the coefficients in $G_{\nu,\lambda}(z,q)$.

Contents

Chapter 1

Introduction

For a simple Lie algebra $\mathfrak{g}$, the associated current algebra, $\mathfrak{g} \otimes \mathbb{C}[t] = \mathfrak{g}[t]$, is a maximal parabolic subalgebra of the affine lie algebra, $\hat{\mathfrak{g}}$. Therefore it is of interest to study representations of $\mathfrak{g}[t]$, as these correspond to representations of $\hat{\mathfrak{g}}$. Note that since we can grade the complex polynomial ring by degree, $\mathfrak{g}[t]$ inherits this grading. It is then sensible to consider graded modules of $\mathfrak{g}[t]$, and seek their graded character formulae.

Of particular interest are level ℓ Demazure modules, as these are highest weight modules of $\mathfrak{g}[t]$. It was shown in [4][8][10] that a level one Demazure module is isomorphic to a local Weyl modules, as introduced in [6]. It was also shown in [11][15] that for $\mathfrak{g}[t]$ which are simply laced (i.e. types A, D, E) that the graded characters of level one Demazure modules are precisely Macdonald Polynomials. It was also shown in [1] that the graded characters of a level 2 Demazure modules is a linear combination of Macdonald Polynomials, but this only applies to the type A case. It is not clear that this will also be the case for type D.

We look to generalized Demazure modules, which are fusion products of level ℓ

Demazure modules, introduced in [12]-[14]. These modules have begun appearing recently in various parts of the literature, and have been connected to graded limits of minimal affinizations [13][14] and to the graded limits prime modules over $\hat{\mathfrak{g}}$ in [2](type A) and [3](type D). Particularly, we will focus on the family of generalized Demazure modules presented in [3], which are fusion products of two level 1 modules. In [3], a graded character for these modules is given in terms of graded characters of level 2 Demazure modules. Similar work has been done in [9] for types-B, C, indicating a process similar to that of this thesis can be undertaken for these types as well.

In this thesis, we use methods inspired by [16] and [1] to construct a closed form for the graded characters of the modules from [3], proving that these graded characters are sums of Macdonald Polynomials with $\mathbb{Z}[q]$ coefficients.

We may also be able to combine this result with the character formula provided in [3] to find an explicit graded character for the level 2 Demazure modules of type D.

Chapter 2

Algebras and Modules of Interest

This chapter is designed to introduce the reader to the notation, algebras, and modules discussed throughout this thesis.

Let $\mathbb{C}$ denote the complex numbers and $\mathbb{Z}$ the integers in the usual fashion. Take $\mathbb{Z}_+$ to be the nonnegative integers. Also, take $[i, j] = \{i, i+1, \cdots, j\}$ for $i < j \in \mathbb{Z}$. Also, given two vector spaces, V and W, denote their tensor over $\mathbb{C}$ as $V \otimes W$, and define V to be **$\mathbb{Z}$-graded** if V can be written as the direct sum $V = \bigoplus_{r \in \mathbb{Z}} V[r]$, where $V[r] = \{v \in V$ such that the grade of $v = r\}$. We define a $\mathbb{Z}_+$-grading similarly.

In this thesis, we take $\mathfrak{g}$ to be a simple Lie algebra of type D_n, and we denote the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$. The Cartan subalgebra is denoted by $\mathfrak{h}$, and we denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ as R. Fixing a basis of R to be our **simple roots**, $\Delta = \{\alpha_i \mid i \in [1, n]\}$, and $(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ to be the symmetric bilinear form induced by the killing form of $\mathfrak{g}$, we define a set of **fundamental weights** $\{\omega_i \in \mathfrak{h}^* \mid (\omega_i, \alpha_j) = \delta_{i,j}\}$ with $\omega_0 = \omega_{n+1} = 0$ for convenience later.

In the usual fashion, let Q and Q^+ be the $\mathbb{Z}$-span of Δ and the $\mathbb{Z}_+$-span of Δ respectively. Take the height of a some $\gamma = \sum_{i=1}^{n} c_i\alpha_i$ to be $\mathrm{ht}_r\,\gamma = \sum_{i=1}^{n} c_i$, and with that we label the highest root θ. We define the **positive roots** to be $R^+ = R \bigcap Q^+$. Note that for type D_n, R^+ is described explicitly as

$$\alpha_{i,j} = \sum_{k=i}^{j} \alpha_k, \quad \alpha_{i,n} = (\sum_{k=i}^{n-2} \alpha_k) + \alpha_n, \quad \beta_{i,j} = \alpha_{i,n-1} + \alpha_{j,n}, \quad i < j \in [1, n-1]$$

For convenience later on, we take $\alpha_{j,i} = 0$ for $i < j$. We take $\{x_\alpha^\pm, h_i \ : \ \alpha \in R^+, \ i \in [1,n]\}$ to be a Chevalley basis of $\mathfrak{g}$, and will denote $x^\pm_{\alpha_{i,j}} = x^\pm_{i,j}$ for convenience. We take $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ to be the triangular decomposition, where $\mathfrak{n}^\pm = \{x_\alpha^\pm \mid \alpha \in R^+\}$.

We also take P and P^+ to be the $\mathbb{Z}$-span of fundamental weights (**integral weights**) and the $\mathbb{Z}_+$-span of the fundamental weights (**dominant integral weights**) respectively. We define the height of an integral weight $\lambda = \sum_{i=1}^{n} c_i\omega_i$ to be $\mathrm{ht}\,\lambda = \sum_{i=1}^{n} c_i$, and define a partial order on P^+, $\mu \prec \lambda$ if and only if $\lambda - \mu \in Q^+$. We can extend this partial order to $P^+ \times P^+$,

$$(\nu', \lambda') \prec (\nu, \lambda) \text{ if } \nu + \lambda - \nu' - \lambda' \in Q^+ \setminus \{0\} \text{ or if } \nu + \lambda - \nu' - \lambda' = 0 \text{ with } \nu - \nu' \in P^+.$$

Further, we set $P^+(1) = \{\lambda \in P^+ \mid (\lambda, \alpha_i) \leq 1; \ i \in [1,n]\}$, and note that for any weight $\lambda \in P^+$, there exists a $\lambda_0 \in P^+$ and $\lambda_1 \in P^+(1)$ such that $\lambda = 2\lambda_0 + \lambda_1$.

There are a few interesting types of pairs of weights for this simple Lie algebra, namely those that are compatible and interlacing. A pair $(\nu, \lambda) \in P^+ \times P^+$ is **compatible** if

- $\lambda_1 = 0$

- $\lambda_1 \neq 0$, $\nu_0 = \omega_i$ for $i \in [0,n]$, $\max \nu_1 < \min \lambda_1$, and if $i \neq 0$, $i < \min(\lambda_1) - 1$ with $\nu_1(h_i) = \nu_1(h_{i+1}) = 0$.

Also a pair $(\nu_1, \nu_2) \in P^+(1) \times P^+(1)$ is **interlacing** if $\nu_1 + \nu_2 \in P^+(1)$ and

- $\nu_r(h_i) = 1 = \nu_r(h_j) \implies \nu_p(h_s) = 1$ for some $s \in [i+1, j-1]$

- $\nu_1 + \nu_2(h_{n-1} + h_n) \neq 0 \implies \nu_r(h_{n-1} + h_n) = 0$

for some $r \in \{1,2\}$, $p \neq r \in \{1,2\}$. Note that for any $\nu \in P^+(1)$, there exists an interlacing pair $(\zeta_{1,\nu}, \zeta_{2,\nu})$ such that $\nu = \zeta_{1,\nu} + \zeta_{2,\nu}$. Therefore, for any weight $\lambda \in P^+$, $\lambda = 2\lambda_0 + \lambda_1 = 2\lambda_0 + \zeta_{1,\lambda} + \zeta_{2,\lambda}$ where $(\zeta_{1,\lambda}, \zeta_{2,\lambda})$ is the interlacing pair corresponding to $\lambda_1 \in P^+(1)$. This fact will also be used throughout this thesis.

Now that we have defined dominant integral weights, we can define the finite irreducible $\mathfrak{g}$-modules. For a given $\lambda \in P^+$, the **finite dimensional irreducible** $\mathfrak{g}$-module of weight λ, $V(\lambda)$, is generated by an element v_λ with the following defining relations:

$$x_i^+ v_\lambda = 0, \; h_i v_\lambda = \lambda(h_i) v_\lambda, \; (x_i^-)^{\lambda(h_i)+1} v_\lambda = 0, \; i \in [1,n].$$

These are very important to our understanding of representations, since any finite dimensional $\mathfrak{g}$-module can be decomposed into a direct sum of irreducible modules.

2.1 $\hat{\mathfrak{g}}$, $\mathfrak{g}[t]$, and their Graded modules

In this section we introduce the extensions of our Lie algebra that we are interested in. Let t be an indeterminate, $\mathbb{C}[t]$ the complex polynomials, and $\mathbb{C}[t^{\pm}]$ be the Laurent polynomials. First, the **affine Lie algebra** of $\mathfrak{g}$, $\hat{\mathfrak{g}}$, is given by

$$\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[t^{\pm}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where c is a central element and d is the derivation defined by $[d, x \otimes t^r] = r(x \otimes t^r)$ for all $x \in \mathfrak{g}$ and $r \in \mathbb{Z}$. The bracket of this algebra is given as

$$[x \otimes t^r, y \otimes t^s] = ([x, y] \otimes t^{r+s}) + tr(xy)c$$

where $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$. The Cartan subalgebra of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{h}} = \mathbb{C}\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and we describe $\hat{\mathfrak{n}}^+ = (\mathfrak{g} \otimes t\mathbb{C}[t]) \oplus \mathfrak{n}^+$ and $\hat{\mathfrak{b}} = \hat{\mathfrak{n}}^+ \oplus \hat{\mathfrak{h}}$.

As the algebra quickly becomes too cumbersome to work with practically, we move to a maximal subalgebra in $\hat{\mathfrak{b}}$, the current algebra. The **current algebra** of $\mathfrak{g}$ is defined to be $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ with the following Lie bracket,

$$[x \otimes f(t), y \otimes g(t)] = [x, y]_{\mathfrak{g}} \otimes f(t)g(t).$$

Note that the current algebra inherits a $\mathbb{Z}^+$-grading by degree, namely

$$\mathfrak{g}[t] = \bigoplus_{r \in \mathbb{Z}_+} \mathfrak{g} \otimes (\mathbb{C}[t])[r],$$

where $(\mathbb{C}[t])[r] = \{f(t) \mid \deg f(t) = r\}$. This grading also extends to $\mathbf{U}(\mathfrak{g}[t])$ by assigning the word $(x_1 \otimes t^{r_1})(x_2 \otimes t^{r_2}) \cdots (x_j \otimes t^{r_j})$ the grade $\sum_{i=1}^{j} r_i$.

As the current algebra is $\mathbb{Z}_+$-graded, we can consider $\mathbb{Z}_+$-graded modules of $\mathfrak{g}[t]$. We say V is a **graded** $\mathfrak{g}[t]$-module if it is $\mathbb{Z}_+$-graded as a vector space and $\mathfrak{g}[t]$ acts on V in the following way

$$(\mathfrak{g} \otimes \mathbb{C}t^r)V[s] \subseteq V[r + s].$$

In this thesis, it becomes necessary to shift the grade of $V[r]$, and to this end, we take τ_s^* to be the grade shift operator so that $\tau_s^* V$ is the $\mathfrak{g}[t]$-module V with each grade space shifted up uniformly s, but action of $\mathfrak{g}[t]$ remains the same.

There are two lemmas that will help us a great deal in Chapter 4. Both can be found in [8] as Lemma 1.6 and Lemma 2.3 respectively

Lemma 2.1.1 *Let V be a $\mathfrak{g}[t]$-module with $v \in V$ such that*

$$(x_i^- \otimes t^{s_i})v = 0$$

for all $i \in [1, n]$ and some $s_i \in \mathbb{Z}^+$. Set $\lambda = \sum_i s_i \omega_i$. For all $\alpha \in R^+$, we have

$$(x_\alpha^- \otimes t^{s_\alpha})v = 0, \ s_\alpha = \sum_i s_i.$$

Lemma 2.1.2 *(Garland's Formula) Given $n \in \mathbb{N}$, $r \in \mathbb{Z}^+$, and $\alpha \in R^+$ then*

$$(x_\alpha^+ \otimes t)^s (x_\alpha^- \otimes 1)^{s+r} - (-1)^s x_\alpha^-(r,s) \in \mathbf{U}(\mathfrak{g}[t])\mathfrak{n}^+ \bigoplus \mathbf{U}(\mathfrak{n}^-[t] \oplus \mathfrak{h}[t]_+)\mathfrak{h}[t]_+.$$

Here, $\mathfrak{h}[t]_+$ denotes the elements of $\mathfrak{h}$ with positive powers of t, and

$$x_\alpha^-(r,s) = \sum_{n^1 + \cdots + n^r = s} (x_\alpha^- \otimes t^{n^1}) \cdots (x_\alpha^- \otimes t^{n^r}).$$

2.2 Characters and Graded Characters

To discuss characters, it is important to define weight spaces of a given representation. Given a $\mathfrak{g}$-module V, the λ weight space of V is defined to be

$$V_\lambda = \{v \in V \mid h_i v = \lambda(h_i)v, \text{ for all } i \in [1, n]\},$$

with $\operatorname{wt} V = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}$. For a finite dimensional $\mathfrak{g}$-module, we can look at the **weight space decomposition** of V,

$$V = \bigoplus_{\lambda \in P^+} V_\lambda.$$

We can take $\mathbb{Z}[P]$ to be the polynomial ring whose indeterminates come for a basis of elements of the form e^λ for $\lambda \in P$. Then the **character** of a finite dimensional $\mathfrak{g}$-module V is

$$\operatorname{ch} V = \sum_{\lambda \in P} \dim V_\lambda e^\lambda \ \in \mathbb{Z}[P].$$

We can incorporate the notion of grading into our character by splitting each weight space into its graded components and using an indeterminate q to keep track of the grade. Doing so gives us the **graded character** of a $\mathbb{Z}^+$-graded finite-dimensional $\mathfrak{g}$-module V,

$$\operatorname{ch}_{\mathrm{gr}} V = \sum_{r \in \mathbb{Z}^+} \sum_{\lambda \in P} q^r e^\lambda \dim V_\lambda[r] \ \in \mathbb{Z}[P][q].$$

2.3 Fusion Products

There is a product of graded $\mathfrak{g}[t]$-modules that we will need for the proof of our results, the fusion product. First it is necessary to describe what the associated **graded space** of a $\mathfrak{g}[t]$-module V, $\operatorname{gr} V$. The r^{th} filtration of V,

$$F^r V = \bigoplus_{s \leq r \in \mathbb{Z}^+} V[s]$$

,and the graded space associated to V is

$$\operatorname{gr} V = \bigoplus_{r \in \mathbb{Z}^+} F^r V / F^{r-1} V$$

with $F^{-1}V$ being taken to be 0. Note that each $F^r V/F^{r-1}$ is a $\mathfrak{g}$-module, and so $\operatorname{gr} V$ is as well. We can describe $\operatorname{gr} V$ as a graded $\mathfrak{g}[t]$-module whose acts as follows

$$(x \otimes t^m)\overline{v} = \overline{(x \otimes t^m)v}$$

where $v \in V$ and $\overline{v} \in F^r V / F^{r-1} V$ for some $r \in \mathbb{Z}^+$. If V was cyclically generated by an element $v \in V$, then $\operatorname{gr} V \cong V$ as $\mathfrak{g}$-modules and $\operatorname{gr} V$ is generated by $\overline{v}$.

The following is a lemma will be useful to us in Chapter 4. It is given as Lemma 4 and proven in [16].

Lemma 2.3.1 *Let V be a cyclic $\mathfrak{g}[t]$-module generated by $v \in V$. Then for all $u \in V$, $x \in \mathfrak{g}$, $r \in \mathbb{Z}^+$, and $a_1, \cdots, a_r \in \mathbb{C}$ we have*

$$(x \otimes t^r)\overline{u} = \overline{(x \otimes t^r)u}$$

where $\overline{u}$ is the image of u in $\operatorname{gr} V$.

We can now define the fusion product. Taking $V_1, \cdots, V_m$ all to be cyclic $\mathfrak{g}[t]$-modules generated by $v_1, \cdots, v_m$ respectively and a set of distinct parameters $z_1, \cdots, z_m$ in $\mathbb{C}$, we twist the action of $\mathfrak{g}[t]$ each V_i by z_i. This precisely looks like

$$(x \otimes t^r)v_i = (x \otimes (t + z_i)^r)v_i.$$

The new twisted module we will denote as $V_i^{z_i}$, and define our **fusion product** to be the associated graded space of

$$V_1^{z_1} \otimes V_2^{z_2} \otimes \cdots \otimes V_{m-1}^{z_{m-1}} \otimes V_m^{z_m}$$

and we denote it as

$$V_1^{z_1} * V_2^{z_2} * \cdots * V_{m-1}^{z_{m-1}} * V_m^{z_m}.$$

In practice, we will often write this fusion product omitting the parameters fro a less cumbersome notation. For example the above would be written as $V_1 * V_2 * \cdots * V_{m-1} * V_m$.

With respect to characters, the fusion product is useful because

$$\dim V_1 * V_2 * \cdots * V_{m-1} * V_m = \prod_{i=1}^{m} \dim V_i.$$

This means we can utilize dimension arguments when working with fusion products, even though we don't have defining relations for these products in general.

2.4 Local Weyl, Demazure, and Kirillov-Reshetikin Modules

In this section we will introduce some modules that are relevant to our results, namely we will focus on the $\mathfrak{g}[t]$-modules. Given a weight $\lambda \in P^+$, the **local Weyl module** of weight lambda is denoted $W_{\text{loc}}(\lambda)$ and is generated by w_λ subject to the following relations

$$(x_i^+ \otimes \mathbb{C}[t])w_\lambda = 0, \quad (h_i \otimes t^r)w_\lambda = \delta_{0,r}\lambda(h_i)w_\lambda, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0$$

for each $i \in [1, n]$ and $r \in \mathbb{Z}$. These relations result in $\text{wt}\, W_{\text{loc}}(\lambda) \subset \lambda - Q^+$, $\dim W_{\text{loc}}(\lambda)_\lambda = 1$, and $W_{\text{loc}}(0) \cong \mathbb{C}$ as a $\mathfrak{g}[t]$-module.

There are many quotients of a local Weyl module, and here we define the **level ℓ Demazure module**, $D(\ell, \lambda)$ using Theorem 2 of [8]. This means the for all $\alpha \in R^+$ we add the relations

$$(x_\alpha^- \otimes t^{s_\alpha})w_\lambda = 0, \quad (x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1}w_\lambda = 0 \text{ if } m_\alpha < d_\alpha \ell$$

where $s_\alpha, m_\alpha \in \mathbb{N}$ are defined to by $\lambda(h_\alpha) = d_\alpha \ell(s_\alpha - 1) + m_\alpha$ with $m_\alpha \in [1, \ell]$ and $d_\alpha = \frac{2}{(\alpha,\alpha)}$ is the root length. For simply-laced cases, each root is of the same length, so $d_\alpha = 1$ for all alpha. In these cases, the relations for $D(1, \lambda)$ are the result of the relations of $W_{\text{loc}}(\lambda)$, so these modules are isomorphic as proven in [4], [8], [10], and [12]. It was also shown in [2] that in the type A case, the second relation is a result of the first. This can be generalized to all simply-laced cases.

The next family of $\mathfrak{g}[t]$-modules that we will be introducing are the **generalized Demazure modules** as defined in [3]. These modules are very important to the study of classical limits of quantum affine representations and so were initially introduced by [13] and [14] as modules for $\hat{\mathfrak{g}}$, but we will only present the construction for $\mathfrak{g}[t]$. We take a sequence of $\lambda_1, \lambda_2, \cdots, \lambda_r \in P^+$ and $\ell_1, \cdots, \ell_r, s_1, \cdots, s_r \in \mathbb{N}$ and define the generalized Demazure module

$$D(\lambda_1, \lambda_2, \cdots, \lambda_r) = \mathbf{U}(\mathfrak{g}[t])(w_{\lambda_1} \otimes w_{\lambda_2} \otimes \cdots \otimes w_{\lambda_r}) \subset \bigotimes_{i=1}^{r} \tau_{s_i}^* D(\ell_i, \lambda_i).$$

These representations are largely unstudied, but some work has been done to begin to understand them. In this thesis, we primarily focus on a particular family of generalized Demazure modules,

$$D(\nu, \lambda) = \mathbf{U}(\mathfrak{g}[t])(w_\nu \otimes w_\lambda) \subset D(1, \nu) \otimes D(1, \lambda).$$

Particularly, we are interested in the family [3] provides a presentation for, $D(\lambda_0 + \zeta_{1,\lambda}, \lambda_0 + \zeta_{2,\lambda})$. The presentation that is provided is as a quotient of $W_{\text{loc}}(\nu + \lambda)$ with the added relation

$$(x_\alpha^- \otimes t^{(\lambda_0,\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})w_{\nu+\lambda} = 0.$$

The last family of $\mathfrak{g}[t]$-modules that we will need are the **Kirillov-Reshetikin modules** as defined in [5]. The module $\mathbf{KR}(m\omega_j)$ is generated by $v_{j,m}$ with the following relations

$$(x_i^+ \otimes t)v_{j,m} = 0, \;\; (h \otimes t^r)v_{j,m} = \delta_{r,0} m\omega_j(h)v_{j,m},$$

$$(x_i^- \otimes 1)^{m\omega_j(h_i)+1} v_{j,m} = 0,\ (x_i^- \otimes t^{\omega_j(h_i)}) v_{j,m} = 0.$$

By observing the relations, we see that $\mathbf{KR}(\omega_j) \cong W_{\text{loc}}$. A result of [10], $\mathbf{KR}(2\omega_j) \cong D(2, 2\omega_j)$. Lastly, we define a fusion product of modules, $K_{i,m}^*$ identically to [8]. The module $K_{i,m}$ is described as follows:

$$K_{i,m} \cong \bigotimes_{j\ |\alpha_i(h_j)<0} \bigotimes_{k=0}^{-\alpha_i(h_j)-1} KR((\lceil \frac{m(\alpha_i(h_j)) - k}{\alpha_i(h_j)} \rceil)\omega_j).$$

Note that for type D, the only value for $\alpha_i(h_j) < 0$ is -1, so the above definition simplifies to

$$K_{i,m} \cong \bigotimes_{j\ |\alpha_i(h_j)=-1} KR(m\omega_j).$$

We define $K_{i,m}^*$ to be $K_{i,m}$ but replacing the tensor product with fusion product. This product will be useful to have defined for Chapter 4.

2.5 Prime Representations

In this section we introduce the family of prime representations of $\mathbf{U}_q(\hat{\mathfrak{g}})$. A representation is said to be **prime** if it can not be expressed as the tensor of two non-trivial representations. We take $\mathbb{U}_q(\hat{\mathfrak{g}})$ to be the quantized enveloping algebra of $\hat{\mathfrak{g}}$. Let $\mathcal{P}_{\mathbb{Z}}^+$ to be the free abelian monoid generated $\{\boldsymbol{\omega}_{i,q^r} \mid 1 \le i \le n,\ r \in \mathbb{Z}\}$, and let $\mathrm{wt} : \mathcal{P}^+ \to P^+$ be defined as $\mathrm{wt}\, \boldsymbol{\pi} = \mathrm{wt}(\prod_{i=1}^n \boldsymbol{\omega}_{i,\pi_i(q)}) = \sum_{i=1}^n (\deg \pi_i)\omega_i$. We define $\mathcal{P}_{\mathbb{Z}}^+(1)$ the same way as [3], as the subset of $\mathcal{P}_{\mathbb{Z}}^+$ containing the identity and elements $\prod_{i=1}^k \boldsymbol{\omega}_{i_j,a_j}$ where $1 \le i_1 < \cdots < i_k \le n$ and $a_j \in q^{\mathbb{Z}}$ such that

$$\frac{a_j}{a_{j+1}} = q^{\pm(i_{j+1}-i_j+2)},\ k \ge 2$$

$$\frac{a_j}{a_{j+1}} = q^{\pm(i_{j+1}-i_j+2)} \implies \frac{a_{j+1}}{a_{j+2}} = q^{\mp(i_{j+2}-i_{j+1}+2)},\ \forall j \le k-3$$

The last requirement will also apply to $j = k-2$ if $(i_{k-1}, i_k) \neq (n-1, n)$, but if $(i_{k-1}, i_k) = (n-1, n)$ we require $a_k = a_{k-1}$. For any element $\boldsymbol{\pi} \in \mathcal{P}^+$, we can define an irreducible finite dimensional $\mathbf{U}_q(\hat{\mathfrak{g}})$ representation, $[\boldsymbol{\pi}]$.

Lemma 2.5.1 *(from [3]) The module $[\boldsymbol{\pi}]$ is prime for all $\boldsymbol{\pi} \in \mathcal{P}^+(1)$.*

We can consider these representations for $\mathfrak{g}[t]$, and by taking a pull-back of this representation via the map taking $(x \otimes t^r) \to (x \otimes (t-1)^r) \in \mathrm{Aut}(\mathfrak{g}[t])$, we get a representation $[\boldsymbol{\pi}_{\mathbb{C}}]$. The following is Theorem 3.3 of [3] which we will need in Chapter 4

Theorem 2.5.1 *For $\boldsymbol{\pi} \in \mathcal{P}^+(1)$, there exsists an isomorphism of $\mathfrak{g}[t]$-modules*

$$[\boldsymbol{\pi}_{\mathbb{C}}] \cong D(\zeta_{1,\pi}, \zeta_{2,\pi})$$

where $\mathrm{wt}\,\boldsymbol{\pi} = \zeta_{1,\pi} + \zeta_{2,\pi}$ and $(\zeta_{1,\pi}, \zeta_{2,\pi})$ is an interlacing pair.

We define a height function, $\xi : \{1, \cdots, n\} \to \mathbb{Z}$ where

$$\xi(i) = \xi(i+1) \pm 1, \ \xi(i) = \xi(i+2), \ \xi(n-1) = \xi(n),$$

and we define

$$\mathcal{P}_\xi^+ = \{\boldsymbol{\omega}_{i,a_i} \cdots \boldsymbol{\omega}_{j,a_j} \in \mathcal{P}_{\mathbb{Z}}^+(1) \mid a_k = q^{\xi(k)\pm 1} \text{ if } \xi(k) = \xi(k-1) \pm 1\}.$$

Therefore if we take $\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega} \in \mathcal{P}_\xi^+$ with $i < j < \min \boldsymbol{\omega}$, $j \neq n-2$, and are able prove that

$$\frac{[\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]}{[\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]} \supseteq [\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}],$$

then we would have shown that

$$\dim[\boldsymbol{\omega}_{i,a}] \dim[\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}] \geq \dim[\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}] + \dim[\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}]. \qquad (2.5.1)$$

The proof of this containment is done through argument of q-characters for these quantum affine modules. Namely, the observation is made that $[\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}]$ is an irreducible quotient of $[\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]$ but not of $[\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]$. The details shall be omitted here.

When $j = n$, we use $j-1$ in place of $j+1$, and if $j = n-2$ this statement changes slightly to

$$\frac{[\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]}{[\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]} \supseteq [\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}_{n,\xi(j)}\boldsymbol{\omega}],$$

leading to

$$\dim[\boldsymbol{\omega}_{i,a}]\dim[\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}] \geq \dim[\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}] + \dim[\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}_{n,\xi(j)}\boldsymbol{\omega}]. \tag{2.5.2}$$

Chapter 3

Main Results

In this chapter, we will discuss the main results of this thesis. Our goal is to construct a graded character for our generalized Demazure modules in terms of level 1 Demazure modules. To do this we will be constructing interpolating polynomials and modules. We will prove that for compatible pairs (ν, λ) there exists short exact sequences between our modules, and use those sequences to manipulate coefficients of the graded characters and identify them with coefficients of our polynomials. This work is inspired by the recent results of [1] for level 2 Demazure modules in type A, and will follow a very similar pattern.

3.1 Macdonald Polynomials and $G_{\nu,\lambda}(z, q)$

Let $z = (z_1, \cdots, z_{n+1})$ and q be indeterminates, for weight $\lambda \in P^+$, and let $P_\lambda(z, q, 0)$ be the Macdonald polynomial associated to λ specialized to $t = 0$. The Macdonald polynomials are orthogonal and so form a basis for the ring of symmetric polynomials in $\mathbb{C}[q][z]$. Further, it has been proven in [15] for type A and in [11] for the rest of the simply laced cases $\mathrm{ch_{gr}}\, W_{\mathrm{loc}}(\lambda) = \mathrm{ch_{gr}}\, D(1,\lambda) = P_\lambda(z,q,0)$, a fact we will use freely.

Given $\lambda, \mu \in P^+$ we define p_λ^μ as in [1]:

$$p_\lambda^\mu(q) = \begin{cases} q^{\frac{1}{2}(\lambda+\mu_1,\lambda-\mu)} \prod_{j=1}^n \begin{bmatrix} (\lambda-\mu,\omega_j)+(\mu_0,\alpha_j) \\ (\lambda-\mu,\omega_j) \end{bmatrix}_q & \text{if } \lambda-\mu \in Q^+ \\ 0 & \text{else} \end{cases}$$

and notice that $p_\lambda^\lambda(q) = 1$, and just as in [1], if $\lambda - \mu \in Q^+$ then $(\lambda+\mu_1, \lambda-\mu) = (\lambda-\mu,\lambda-\mu) + 2(\mu-\mu_0,\lambda-\mu) \in 2\mathbb{Z}_+$, meaning that $p_\lambda^\mu \in \mathbb{Z}^+[q]$. For convenience of future statements, it will be helpful to extend our definition by setting $p_\lambda^\mu = 0$ if λ or μ are in $P \setminus P^+$.

Next, we will define $G_\lambda(z,q) \in \mathbb{C}[q][z]$ recursively as follows:

$$G_{\omega_i}(z,q) = P_{\omega_i}(z,q,0), \quad i \in \{0,1,n-1,n\},$$

$$P_\lambda(z,q,0) = G_\lambda(z,q) + \sum_{\mu \prec \lambda \in P^+} p_\lambda^\mu(q) G_\mu(z,q).$$

Since the $\{P_\lambda \mid \lambda \in P^+\}$ is a linearly independent set, an induction on λ shows that $\{G_\lambda(z,q) \mid \lambda \in P^+\}$ is as well, and so forms another basis of the symmetric polynomials in $\mathbb{C}[q,z]$. Hence there exists polynomials $a_\mu^\lambda(q) \in \mathbb{C}[q]$ with $a_\lambda^\lambda(q) = 1$, and $a_\lambda^\mu = 0$ if $\lambda - \mu \notin Q^+$ such that

$$G_\lambda(z,q) = \sum_{\mu \in P^+} a_\lambda^\mu(q) P_\mu(z,q,0) \text{ and,}$$

$$\sum_{\nu \in P} a_\lambda^\nu p_\nu^\mu = \delta_{\lambda,\mu} = \sum_{\nu \in P} p_\lambda^\nu a_\nu^\mu$$

Finally, given a pair of weights $\nu, \lambda \in P^+$, we set

$$G_{\nu,\lambda}(z,q) = \sum_{\mu \in P^+} q^{\lambda+\nu-\mu,\nu} a_\lambda^{\mu-\nu}(q) P_\mu(z,q,0)$$

where $a_\lambda^{\mu-\nu} = 0$ if $\mu - \nu \notin P^+$ and note that

$$G_{\nu,0} = P_\nu(z,q,0) \text{ and } G_{0,\lambda}(z,q) = G_\lambda(z,q).$$

3.2 Modules $M(\nu, \lambda)$ and their properties

In this section, we define the family of modules $M(\nu, \lambda)$. To do so, recall that for any $\lambda \in P^+$, there exists $\lambda_0 \in P^+$ and an interlacing pair $(\zeta_{1,\lambda}, \zeta_{2,\lambda})$ such that $\lambda = 2\lambda_0 + \zeta_{1,\lambda} + \zeta_{2,\lambda}$. For any $\nu, \lambda \in P^+$, define $M(\nu, \lambda)$ be the $\mathfrak{g}[t]$-module generated by some element $w_{\nu+\lambda}$ with the following relations:

$$(x_i^+ \otimes 1)w_{\nu+\lambda} = 0, \ (h \otimes t^r)w_{\nu+\lambda} = \delta_{0,r}(\lambda+\nu)(h)w_{\nu+\lambda}, \ (x_i^- \otimes 1)^{(\lambda+\nu)(h_i)+1}w_{\nu+\lambda} = 0, \quad (3.2.1)$$

$$(x_\alpha^- \otimes t^{(\nu+\lambda_0,\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})w_{\nu+\lambda} = 0, \quad (3.2.2)$$

for all $i \in [n], h \in \mathfrak{h}$ and $\alpha \in R^+$. Since the defining relations of $M(\nu, \lambda)$ are graded by their power of t, $M(\nu, \lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]$-module, setting the grade of $w_{\nu+\lambda}$ to be zero.

The inspiration for this construction comes from [16], and forces $M(\nu, 0) \cong W_{\mathrm{loc}}(\nu)$ and that $M(0, \lambda) \cong D(\lambda_0 + \zeta_{1,\lambda}, \lambda_1 + \zeta_{2,\lambda})$ as presented in [3]. Further, from [6] it is known that local Weyl modules are finite-dimensional, and since $M(\nu, \lambda)$ is a quotient of $W_{\mathrm{loc}}(\nu+\lambda)$, $M(\nu, \lambda)$ must also be finite-dimensional.

Notice that upon inspection, these relations give

$$M(\nu + \omega_i, 2\lambda_0) \cong_{\mathfrak{g}[t]} M(\nu, 2\lambda_0 + \omega_i) \text{ for } i \in [1, n]. \quad (3.2.3)$$

Also note, that because when $\lambda_1 = \omega_n + \omega_{n-1}$ the pair $(\zeta_{1,\lambda}, \zeta_{2,\lambda}) = (\omega_n + \omega_{n-1}, 0)$, an inspection of relations also provides

$$M(\nu + \omega_n + \omega_{n-1}, 2\lambda_0) \cong M(\nu, 2\lambda_0 + \omega_n + \omega_{n-1}). \quad (3.2.4)$$

We make the following observations about $M(\nu, \lambda)$ and their graded characters.

Lemma 3.2.1 *For any $\nu, \lambda \in P^+$,*

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), M(\nu,\lambda)) \neq 0 \implies \nu + \lambda - \mu \in Q^+.$$

Moreover,

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu+\lambda), M(\nu,\lambda)) = 1$$

Proof. Any non-zero element in $M(\nu,\lambda)$ is of the form $uw_{\nu+\lambda}$ with $u \in \mathbf{U}(\mathfrak{g}[t])$. Since $M(\nu,\lambda)$ is a quotient of $W_{\text{loc}}(\nu+\lambda)$ and all words in $\mathbf{U}(\mathfrak{g}[t])$ can be expressed in increasing order, we can assume that $u \in \mathbf{U}(\mathfrak{n}^-[t])$, and so $\operatorname{wt} u = \sum_{\alpha \in R^+} - c_\alpha \alpha$ where $c_\alpha \in \mathbb{N}$.

If there exists a homomorphism $\phi : V(\mu) \to M(\nu,\lambda)$ taking w_μ to $uw_{\nu+\lambda}$, then for any $h \in \mathfrak{h}$

$$\begin{aligned}
\mu(h)\phi(w_\mu) &= \phi(hw_\mu) \\
&= (h \otimes 1)uw_{\nu+\lambda} \\
&= (\nu + \lambda - \sum_{\alpha \in R^+} c_\alpha \alpha)(h)uw_{\nu+\lambda} \\
&= (\nu + \lambda - \sum_{\alpha \in R^+} c_\alpha \alpha)(h)\phi(w_\mu).
\end{aligned}$$

Thus $\mu = \nu + \lambda - \sum_{\alpha \in R^+} c_\alpha \alpha$, forcing $\nu + \lambda - \mu = \sum_{\alpha \in R^+} c_\alpha \alpha \in Q^+$.

Further, the only weight μ such that $\nu + \lambda - \mu = 0$ is $\nu + \lambda$ itself. Thus $\dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu+\lambda), M(\nu,\lambda)) \leq 1$. Notice that the map $\iota : V(\nu+\lambda) \to M(\nu,\lambda)$ which sends $v_{\nu+\lambda} \to w_{\nu+\lambda}$, $x_\alpha^\pm \to (x_\alpha^\pm \otimes 1)$, and sends $h_\alpha \to (h_\alpha \otimes 1)$ is a well defined homomorphism by the defining relations of $M(\nu+\lambda)$. Thus the $\dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu+\lambda), M(\nu,\lambda)) = 1$.

■

The above lemma shows that the highest weight spaces of the modules $M(\nu,\lambda)$ are one dimensional. Thus the set $\{\operatorname{ch}_{\text{gr}} M(\mu,0) : \mu \in P^+\}$ (resp. the set $\{\operatorname{ch}_{\text{gr}} M(0,\mu) :$

$\mu \in P^+\}$) is linearly independent. We have also shown that the $\mathbb{Z}[q, q^-1]$-span of this set contains $\mathrm{ch}_{\mathrm{gr}} V(\lambda)$, $\lambda \in P^+$.

And so, $\mathrm{ch}_{\mathrm{gr}} V(\lambda)$ is a linear combination of $\mathrm{ch}_{\mathrm{gr}} M(\mu, 0)$ (resp. $\mathrm{ch}_{\mathrm{gr}} M(\mu, 0)$), meaning that polynomials $g^{\mu}_{\nu,\lambda}$ (resp. $h^{\mu}_{\nu,\lambda}$) exist in $\mathbb{C}[q]$ such that:

$$\mathrm{ch}_{\mathrm{gr}} M(\nu, \lambda) = \sum_{\mu \in P^+} g^{\mu}_{\nu,\lambda}(q)\, \mathrm{ch}_{\mathrm{gr}} M(\mu, 0) = \sum_{\mu \in P^+} h^{\mu}_{\nu,\lambda}(q)\, \mathrm{ch}_{\mathrm{gr}} M(0, \mu). \tag{3.2.5}$$

with $g^{\nu+\lambda}_{\nu,\lambda} = 1 = h^{\nu+\lambda}_{\nu,\lambda}$ and $g^{\mu}_{\nu,\lambda} = 0 = h^{\mu}_{\nu,\lambda}$ if $\lambda + \nu - \mu \notin Q^+$. For future convenience, we will take $g^{\mu}_{\nu,\lambda} = 0$ if any of μ, ν, or λ are not dominant integral weights. As the graded characters are linearly independent, it is implied that for all $\nu, \lambda \in P^+$,

$$\delta_{\nu,\mu} = \sum_{\mu' \in P^+} g^{\mu'}_{0,\nu} h^{\mu}_{\mu',0} = \sum_{\mu' \in P^+} h^{\mu'}_{\nu,0} g^{\mu}_{0,\mu'}. \tag{3.2.6}$$

3.3 Main Theorem and key Proposition

The following is the primary result with relation to the polynomials $G_{\nu,\lambda}(z, q)$.

Theorem 3.3.1 *If $(\nu, \lambda) \in P^+ \times P^+$ is compatible, then $G_{\nu,\lambda}$ is a sum of Macdonald polynomials $P_\mu(z, q, 0)$ such that the coefficients are polynomials with positive coefficients in $\mathbb{Z}[q]$.*

Note that because $\mathrm{ch}_{\mathrm{gr}} M(\nu, 0) = P_\nu(z, q, 0)$, (3.2.5) implies that

$$\sum_{\mu \in P^+} h^{\mu}_{\nu,0}(q)\, \mathrm{ch}_{\mathrm{gr}} M(0, \mu) = \mathrm{ch}_{\mathrm{gr}} M(\nu, 0) = P_\nu(z, q, 0) = \sum_{\mu \in P^+} p^{\mu}_{\nu} G_\mu(z, q).$$

Leading us to desire this proposition:

Proposition 3.3.1 *For $\mu, \nu, \lambda \in P^+$ with (ν, λ) a compatible pair,*

$$g^{\mu}_{\nu,\lambda} = q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}, \quad h^{\mu}_{\nu,0} = p^{\mu}_{\nu}.$$

We prove this proposition in Chapter 5, and with this result have the following:

Corollary 3.3.1 *For* $\lambda \in P^+$

$$\mathrm{ch}_{gr} M(0,\lambda) = G_\lambda(z,q)$$

.

Taking this corollary, we have that

$$\sum_{\mu \in P^+} g^{\mu}_{0,\nu}(q) P_\mu(z,q,0) = \mathrm{ch}_{\mathrm{gr}} M(0,\nu) = G_\nu(z,q) = \sum_{\mu \in P^+} a^{\mu}_{\nu} P_\mu(z,q,0),$$

which implies that $a^{\mu}_{\lambda} = g^{\mu}_{0,\lambda}$. This directly implies the following, which is the primary representation theoretic result of this thesis.

Theorem 3.3.2 *For a compatible pair* $(\nu,\lambda) \in P^+ \times P^+$,

$$\mathrm{ch}_{gr} M(\nu,\lambda) = G_{\nu,\lambda}(z,q).$$

Moreover, we obtain the following isomorphism

$$M(\nu,\lambda) \cong M(\nu,0) * M(0,\lambda).$$

We then build a closed form of $g^{\mu}_{\nu,\lambda}$ and find it is an element of $\mathbb{Z}[q]$, so we also achieve Theorem 3.3.1.

The first step towards proving Proposition 3.3.1, is to find ways to find ways to manipulate these coefficients. In this thesis we do this by relating the graded characters of several $M(\nu,\lambda)$'s using a collection of short exact sequences.

Proposition 3.3.2 *Let* $\lambda,\nu \in P^+$ *and* $\lambda = 2\lambda_0 + \lambda_1 = 2\lambda_0 + \zeta_{1,\lambda} + \zeta_{2,\lambda}$ *as previously described.*

i) Let $(\nu, \lambda) \in P^+ \times P^+$ be a compatible pair.

a) If $\nu(h_j) \geq 2$ for some $1 \leq j \leq n$,then there exists an exact sequence of $\mathfrak{g}[t]$–modules,

$$0 \to \tau^*_{(\nu+\lambda_0,\alpha_j)-1} M(\nu - \alpha_j, \lambda) \xrightarrow{\varphi^-} M(\nu, \lambda) \xrightarrow{\varphi^+} M(\nu - 2\omega_j, \lambda + 2\omega_j) \to 0.$$

b) If $\nu \in P^+(1)$ with $\max \nu < \min \lambda_1 = m < n-1$, $0 < p = \min(\lambda_1 - \omega_m)$, there exists an exact sequence of $\mathfrak{g}[t]$–modules,

$$0 \to \tau^*_{(\lambda_0,\alpha_{m,p})+1} M(\nu + \omega_{m-1}, \lambda - \alpha_{m,p} - \omega_{m-1})$$

$$\xrightarrow{\varphi^-} M(\nu + \omega_m, \lambda - \omega_m) \xrightarrow{\varphi^+} M(\nu, \lambda) \to 0$$

ii) If $\lambda \in P^+(1)$ with $m < \min \lambda$ with $m \notin \{n-1, n\}$, there exists an exact sequence of $\mathfrak{g}[t]$-modules,

$$0 \to \tau^*_1 M(\omega_{m-1}, \lambda + \omega_{m+1}) \xrightarrow{\varphi^-} M(\omega_m, \lambda + \omega_m) \xrightarrow{\varphi^+} M(0, \lambda + 2\omega_m) \to 0$$

If $m \in \{n-1, n\}$, there exists an exact sequence of $\mathfrak{g}[t]$-modules,

$$0 \to \tau^*_1 M(\omega_{n-2}, \lambda) \xrightarrow{\varphi^-} M(\omega_m, \lambda + \omega_m) \xrightarrow{\varphi^+} M(0, \lambda + 2\omega_m) \to 0$$

The goal of the rest of this thesis will be proving Proposition 3.3.2 and Proposition 3.3.1.

Chapter 4

Proof of Proposition 3.3.2

First, in this chapter we focus on proving that φ^+ is a well-defined surjection and that φ^- is well-defined for each sequence in Proposition 3.3.2. For this we will focus on each sequence separately. Then we shall utilize some dimension arguments to prove injectivity of all of the φ^- maps simultaneously.

4.1 Sequence *ia)*

Here we focus on proving the right exactness of Proposition 3.3.2 *ia)*.

Lemma 4.1.1 *Let $(\nu, \lambda) \in P^+ \times P^+$ be compatible with $\nu(h_j) \geq 2$ for some $1 \leq j \leq n$. Then the map*

$$\varphi^+ : M(\nu, \lambda) \to M(\nu - 2\omega_j, \lambda + 2\omega_j)$$

which sends $w_{\nu+\lambda}$ to $w^+_{\nu+\lambda}$ is a well-defined surjection. Moreover, $\ker(\varphi^+)$ *is generated by* $(x_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$.

Proof. For φ^+ to be well-defined, it must first be shown that $w^+_{\nu+\lambda}$ satisfies the relation (3.2.2) for $M(\nu,\lambda)$.This translates to showing that

$$(x^-_\alpha \otimes t^{(\nu+\lambda_0,\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})w^+_{\nu+\lambda} = 0 \tag{4.1.1}$$

Note that for all $\alpha \in R^+$, $\omega_j(h_\alpha) \geq 0$, so

$$(\nu+\lambda_0-\omega_j,\alpha) + max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha)) \leq (\nu+\lambda_0,\alpha) + \max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha)).$$

Hence $w^+_{\nu+\lambda}$ satisfies the relation (4.1.1), and so φ^+ is proved to be well-defined.

It remains to show that $\ker(\varphi^+)$ is generated by $(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$. Observe that as φ^+ is a well-defined homomorphism,

$$\begin{aligned}0 &= (x^-_j \otimes t^{(\nu+\lambda_0-\omega_j,\alpha_j)+\max(\zeta_{1,\lambda}(h_j),\zeta_{2,\lambda}(h_j))})w^+_{\nu+\lambda}\\ &= (x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})\varphi(w_{\nu+\lambda})\\ &= \varphi((x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}),\end{aligned}$$

and so $(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda} \in \ker(\varphi^+)$. Further, for any α such that $\omega_j(h_\alpha) \neq 0$, $(x^-_\alpha \otimes t^{(\nu+\lambda_0-\omega_j,\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})w_{\nu+\lambda} \in \ker(\varphi^+)$. Also, $\omega_j(h_\alpha) \neq 0$ implies that

$$\alpha = \begin{cases} \alpha_{i,k} & 1 \leq i \leq j \leq k \leq n \\ \beta_{i,k} & 1 \leq i \leq j < k \leq n-1 \\ \beta_{i,k} & 1 \leq i \leq k \leq j. \end{cases}$$

In each case, $\alpha = \alpha_j + \gamma_1 + \gamma_2 + \gamma_3$ where $\gamma_i \in R^+ \bigcup\{0\}$ and commute (for a more detailed explanation, please see Appendix A.1). Hence

$$\begin{aligned}&\Big(\prod_{i=1}^{3}(x^-_{\gamma_i} \otimes t^{(\nu+\lambda_0-\omega_j,\gamma_i)+\max((\zeta_{1,\lambda},\gamma_i),(\zeta_{2,\lambda},\gamma_i))})\Big)(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}\\ &\qquad = (x^-_\alpha \otimes t^{(\nu+\lambda_0-\omega_j,\alpha)+\max(\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha)})w_{\nu+\lambda}.\end{aligned}$$

Hence all elements of $\ker(\varphi^+)$ can be generated from $(x_j^- \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$. ■

Lemma 4.1.2 *Let $(\nu, \lambda) \in P^+ \times P^+$ be such that $\nu(h_j) \geq 2$ for some $1 \leq j \leq n$. Then the map*

$$\varphi^- : \tau^*_{(\nu+\lambda_0,\alpha_j)-1} M(\nu - \alpha_j, \lambda) \to M(\nu, \lambda)$$

which sends $w_{\nu-\alpha_j+\lambda}$ to $(x_j^- \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$ is well defined.

Proof. First it must be shown that $(x_j^- \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$ is a highest weight vector. This can be sufficiently shown by proving that

$$(x_i^+ \otimes 1)(x_j^- \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda} = 0.$$

for $i \in [1, n]$. For all $i \in [1, n]$, there exists a $c \in \mathbb{C}$ such that

$$\begin{aligned}&(x_i^+ \otimes 1)(x_j^- \otimes t^{(\nu(h_j)+\lambda_0,\alpha_j)-1})w_{\nu+\lambda} \\ &\qquad = \delta_{i,j} c(h_j \otimes t^{(\nu(h_j)+\lambda_0,\alpha_j)-1})w_{\nu+\lambda} + (x_j^- \otimes t^{(\nu+\lambda_0,\alpha_j)-1})(x_i^+ \otimes 1)w_{\nu+\lambda} = 0\end{aligned}$$

since $w_{\nu+\lambda}$ is highest weight. Thus this map sends highest weight vectors to highest weight vectors.

Now, it remains to be shown that the relations of $M(\nu - \alpha_j, \lambda)$ are satisfied by this mapping. This means verifying that

$$(x_\alpha^- \otimes t^{(\nu-\alpha_j+\lambda_0,\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})(x_j^- \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda} = 0, \tag{4.1.2}$$

for all $\alpha \in R^+$. To do this we examine the cases: $\alpha_j(h_\alpha) = 0$, $\alpha_j(h_\alpha) = -1$, $\alpha_j(h_\alpha) = 2$, and $\alpha_j(h_\alpha) = 1$.

In the case that $\alpha_j(h_\alpha) = 0$, $[x_\alpha^-, x_j^-] = 0$ and

$$(\nu - \alpha_j + \lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = (\nu + \lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha))$$

so commutation of the terms kills $w_{\nu+\lambda}$.

When $\alpha_j(h_\alpha) = -1$, $\alpha + \alpha_j = \alpha' \in R^+$, and

$$r = (\nu - \alpha_j + \lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = (\nu + \lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) + 1.$$

Therefore

$$s = (\nu + \lambda_0, \alpha') + \max(\zeta_{1,\lambda}(h_{\alpha'}), \zeta_{2,\lambda}(h_{\alpha'})) = r + (\nu + \lambda_0, \alpha_j) - 1)$$

and so utilizing the properties of the bracket on $\mathfrak{g}[t]$ and Lemma 2.1.1, commutation of our terms again kills $w_{\nu+\lambda}$.

In the case that $\alpha_j(h_\alpha) = 2$, $\alpha = \alpha_j$, and

$$(\nu - \alpha_j + \lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = (\lambda_0, \alpha).$$

Since the only $n_1 + n_2 = 2(\lambda_0, \alpha_j) - 1$ such that $n_i < (\nu + \lambda_0, \alpha_j)$ are $n_1 = (\lambda_0, \alpha_j)$ and $n_2 = (\lambda_0, \alpha_j) - 1$ or vice versa, Garland's Lemma 2.1.2 tells us this product of terms kills $w_{\nu+\lambda}$.

Finally for the case $\alpha_j(h_\alpha) = 1$, we have again that $[x_\alpha^-, x_j^-] = 0$, but we also have that $\alpha = \alpha_j + \gamma$ for some $\gamma \in R^+$ with $\alpha_j(h_\gamma) = -1$. This means that now

$$s = (\nu - \alpha_j + \lambda_0, \gamma) + \max(\zeta_{1,\lambda}(h_\gamma), \zeta_{2,\lambda}(h_\gamma)) = (\nu + \lambda_0, \gamma) + \max(\zeta_{1,\lambda}(h_\gamma), \zeta_{2,\lambda}(h_\gamma)) + 1$$

and $r = (\nu - \alpha_j + \lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = s + (\nu + \lambda_0, \alpha_j) - 1$. Thus utilizng the bracket of $\mathfrak{g}[t]$, and prior cases, we can commute these terms and kill $w_{\nu+\lambda}$.

We include the computation of these commutations in Appendix A.1. With these completed, we have proven (4.1.2) for all cases, and so φ^- is well-defined. ■

Together these lemmas show Sequence *ia)* is right-exact.

4.2 Sequence *ib)*

Here we focus on proving the right exactness of Proposition 3.3.2 *ib)*.

Lemma 4.2.1 *Let* $(\nu, \lambda) \in P^+ \times P^+$ *with* $\max \nu < \min \lambda_1 = m < n-1$ *and* $p = \min \lambda_1 - \omega_m > 0$. *Then the map*

$$\varphi^+ : M(\nu + \omega_m, \lambda - \omega_m) \to M(\nu, \lambda)$$

which sends $w_{\nu+\lambda}$ *to* $w^+_{\nu+\lambda}$ *is a well-defined surjection. Moreover,* $\ker(\varphi^+)$ *is generated by* $(x^-_{m,p} \otimes t^{(\lambda_0, \alpha_{m,p})+1} w_{\nu+\lambda}$.

Proof. As in the prior proof, our first step is to show that $w^+_{\nu+\lambda}$ satisfies (3.2.2) for $M(\nu + \omega_m, \lambda - \omega_m)$. We will assign our interlacing pair such that $\zeta_{1,\lambda}(h_m) = 1 = \zeta_{2,\lambda}(h_p)$, and note we want to show

$$(x^-_\alpha \otimes t^{(\nu+\omega_m+\lambda_0,\alpha)+\max((\zeta_{1,\lambda}-\omega_m,\alpha),(\zeta_{2,\lambda},\alpha))})w^+_{\nu+\lambda} = 0 \tag{4.2.1}$$

Note that for all $\alpha \in R^+$, $\omega_m(h_\alpha) \geq 0$, so

$$(\nu + \lambda_0, \alpha) + \max(\zeta_{1,\lambda}, \zeta_{2,\lambda}) \leq (\nu + \omega_m + \lambda_0, \alpha) + \max(\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda}, \alpha)).$$

Hence $w^+_{\nu+\lambda}$ satisfies (4.2.1), and so φ^+ is well-defined.

It remains to show that $\ker(\varphi^+)$ is generated by $(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda}$. Since φ^+ is a well-defined homomorphism,

$$\begin{aligned}
0 =& (x^-_{m,p} \otimes t^{\nu(h_{m,p})+\left\lceil \frac{\lambda(h_{m,p})}{2} \right\rceil})w_{\nu+\lambda} \\
=& (x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} \\
=& (x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})\varphi^+(w_{\nu+\lambda}) \\
=& \varphi^+((x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda}),
\end{aligned}$$

and so $(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} \in \ker(\varphi^+)$. For any α such that $\omega_m(h_\alpha) \neq 0$ and $\omega_p(h_\alpha) \neq 0$, $(x^-_\alpha \otimes t^{\nu(h_\alpha)+\lceil \frac{\lambda(h_\alpha)}{2} \rceil})w_{\nu+\lambda} \in \ker(\varphi^+)$. Also, $\omega_m(h_\alpha) \neq 0$ and $\omega_p(h_\alpha) \neq 0$ implies that

$$\alpha = \begin{cases} \alpha_{i,k} & 1 \le i \le m < p \le k \le n \\ \beta_{i,k} & 1 \le i \le m < p < k \le n-1 \\ \beta_{i,k} & 1 \le i < m < k \le p \\ \beta_{i,k} & 1 \le i < k \le m. \end{cases}$$

In each case, $\alpha = \alpha_{m,p} + \gamma_1 + \gamma_2 + \gamma_3$ where $\gamma_i \in R^+ \bigcup \{0\}$ and commute (again, for a more detailed explanation, please see Appendix A.2). Hence

$$\Big(\prod_{i=1}^{3} (x^-_{\gamma_i} \otimes t^{(\nu+\lambda_0,\gamma_i)+\max((\zeta_{1,\lambda},\gamma_i),(\zeta_{2,\lambda},\gamma_i))}\Big)(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda}$$
$$= (x^-_\alpha \otimes t^{(\nu+\lambda_0,\alpha)+\max(\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})w_{\nu+\lambda}.$$

Hence, $\ker(\varphi^+)$ is in fact generated by $(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda}$. ■

Lemma 4.2.2 *Let $(\nu, \lambda) \in P^+ \times P^+$ be such that $\max \nu < \min \lambda_1 = m < n-1$ and $p = \min \lambda_1 - \omega_m > 0$. Then the map*

$$\varphi^- : \tau^*_{(\lambda_0,\alpha_{m,p})+1} M(\nu + \omega_{m-1}, \lambda - \alpha_{m,p} - \omega_{m-1}) \to M(\nu + \omega_m, \lambda - \omega_m)$$

which sends $w_{\nu+\lambda-\alpha_{m,p}}$ to $(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda}$ is well-defined.

Proof. We will assign the interlacing pair associated to λ_1 so that $\zeta_{1,\lambda}(h_m) = 1 = \zeta_{2,\lambda}(h_p)$ as we did in the last proof. First we must show that $(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_m,p)+1})w_{\nu+\lambda}$ is a highest weight vector. Specifically that

$$(x^+_i \otimes 1)(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_m,p)+1})w_{\nu+\lambda} = 0$$

for $i \in [1, n]$. For all $i \in [1, n]$, there exists a $c \in \mathbb{C}$ such that

$$(x_i^+ \otimes 1)(x_{m,p}^- \otimes t^{(\lambda_0, \alpha_{m,p})+1})w_{\nu+\lambda}$$

$$= \delta_{1,\alpha_{m,p}(h_i)} c(h_i \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} + (x_{m,p}^- \otimes t^{(\lambda_0,\alpha_{m,p})+1})(x_i^+ \otimes 1)w_{\nu+\lambda} = 0.$$

Thus this map sends highest weight vectors to highest weight vectors.

It remains to show that the relations of $M(\nu+\omega_{m-1}, \lambda-\alpha_{m,p}-\omega_{m-1})$ are satisfied, which means showing that

$$(x_\alpha^- \otimes t^{(\nu+\omega_{m-1}+\lambda_0,\alpha)+\max((\zeta_{1,\lambda}-\omega_m,\alpha),(\zeta_{2,\lambda}-\omega_p+\omega_{p+1},\alpha))})(x_{m,p}^- \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} = 0, \quad (4.2.2)$$

for all $\alpha \in R^+$. To do this we split into cases, just as in the proof of Lemma 4.1.2: $\alpha_{m,p}(h_\alpha) = 0$, $\alpha_{m,p}(h_\alpha) = -1$, $\alpha_{m,p}(h_\alpha) = 2$, and $\alpha_{m,p}(h_\alpha) = 1$.

In the case that $\alpha_{m,p}(h_\alpha) = 0$, $[x_\alpha^-, x_{m,p}^-] = 0$, and

$$(\nu + \omega_{m-1} + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda} - \omega_p + \omega_{p+1} + \delta_{p,n-2}\omega_n, \alpha))$$

$$= (\nu + \omega_m + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda}, \alpha)),$$

so commutation kills $w_{\nu+\lambda}$.

When $\alpha_{m,p}(h_\alpha) = -1$, $\alpha + \alpha_{m,p} = \alpha' \in R^+$ and

$$r = (\nu + \omega_{m-1} + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda} - \omega_p + \omega_{p+1} + \delta_{p,n-2}\omega_n, \alpha))$$

$$\geq (\nu + \omega_m + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda}, \alpha)).$$

Therefore

$$s = (\nu + \omega_{m-1} + \lambda_0, \alpha') + \max((\zeta_{1,\lambda} - \omega_m, \alpha'), (\zeta_{2,\lambda} - \omega_p + \omega_{p+1} + \delta_{p,n-2}\omega_n, \alpha')) + 1$$

$$\geq (\nu + \omega_m + \lambda_0, \alpha') + \max((\zeta_{1,\lambda} - \omega_m, \alpha'), (\zeta_{2,\lambda}, \alpha')),$$

and so utilizing the bracket on $\mathfrak{g}[t]$ and Lemma 2.1.1, commutation of our terms kills $w_{\nu+\lambda}$.

In the case that $\alpha_{m,p}(h_\alpha) = 2$, $\alpha = \alpha_{m,p}$, and

$$(\nu + \omega_{m-1} + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda} - \omega_p + \omega_{p++1} + \delta_{p,n-2}\omega_n, \alpha)) = (\lambda_0, \alpha).$$

Since the only $n_1 + n_2 = 2(\lambda_0, \alpha_{m,p}) + 1$ such that $n_i < (\nu + \lambda_0, \alpha_{m,p})$ are $n_1 = (\nu + \lambda_0, \alpha_{m,p})$ and $n_2 = (\nu + \lambda_0, \alpha_{m,p}) + 1$ or vice versa, Garland's Lemma 2.1.2 tells us this product of terms kills $w_{\nu+\lambda}$.

Lastly, if α is such that $\alpha_{m,p}(h_\alpha) = 1$, then either $\alpha = \alpha_{m,p} + \gamma$ or $\alpha_{m,p} = \alpha + \gamma$. If $\alpha = \alpha_{m,p} + \gamma$ then $\alpha_{m,p}(\gamma) = -1$,

$$r = (\nu + \omega_{m-1} + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda} - \omega_p + \omega_{p+1} + \delta_{p,n-2}\omega_n, \alpha))$$

$$\geq (\nu + \omega_m + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda} -, \alpha)) - 1,$$

and

$$s = (\nu + \omega_{m-1} + \lambda_0, \gamma) + \max((\zeta_{1,\lambda} - \omega_m, \gamma), (\zeta_{2,\lambda} - \omega_p + \omega_{p+1} + \delta_{p,n-2}\omega_n, \gamma))$$

$$\geq (\nu + \omega_m + \lambda_0, \gamma) + \max((\zeta_{1,\lambda} - \omega_m, \gamma), (\zeta_{2,\lambda}, \gamma)).$$

Again by using the $\mathfrak{g}[t]$ bracket and our prior cases, we can commute our terms to kill $w_{\nu+\lambda}$.

If $\alpha_{m,p} = \alpha + \gamma$ then we have

$$r = (\nu + \omega_{m-1} + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda} - \omega_p + \omega_{p+1}, \alpha))$$

$$= (\nu + \omega_m + \lambda_0, \alpha) + \max((\zeta_{1,\lambda} - \omega_m, \alpha), (\zeta_{2,\lambda}, \alpha)) - 1,$$

and

$$s = (\nu + \omega_{m-1} + \lambda_0, \gamma) + \max((\zeta_{1,\lambda} - \omega_m, \gamma), (\zeta_{2,\lambda} - \omega_p + \omega_{p+1}, \gamma))$$

$$= (\nu + \omega_m + \lambda_0, \gamma) + \max((\zeta_{1,\lambda} - \omega_m, \gamma), (\zeta_{2,\lambda}, \gamma)) - 1.$$

Here using our bracket and Garland's Lemma 2.1.2, we can again commute our terms and kill $w_{\nu+\lambda}$.

We include the computations for these commutations in Appendix A.2. We have now proven (4.2.2), so φ^- is well-defined. ■

Together these lemmas show that Sequence *ib)* is right-exact.

4.3 Sequence *ii)*

In this Section we focus on proving the right exactness of Proposition 3.3.2*ii)*.

Lemma 4.3.1 *If $\lambda \in P^+(1)$ with $m < \min \lambda$, then the mapping*

$$\varphi^+ : M(\omega_m, \lambda + \omega_m) \to M(0, \lambda + 2\omega_m)$$

which sends $w_{\lambda+2\omega_m}$ to $w^+_{\lambda+2\omega_m}$ is surjective. Further $\ker(\varphi^+)$ is generated by $(x_m^- \otimes t)w_{\lambda+2\omega_m}$.

Proof. For sake of notation, we will assume that $\min \zeta_{1,\lambda} < \min \zeta_{2,\lambda}$. First it must be shown that $w^+_{\lambda+2\omega_m}$ satisfies (3.2.2) for $M(\omega_m, \lambda + \omega_m)$. This means showing

$$(x_\alpha^- \otimes t^{(\omega_m,\alpha)+\max(\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda}+\omega_m,\alpha)})w^+_{\lambda+2\omega_m} = 0. \tag{4.3.1}$$

Notice that for all $\alpha \in R^+$, $(\omega_m, \alpha) \geq 0$, so

$$(\nu + \omega_m + \lambda_0, \alpha) + \max(\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) \leq (\nu + \omega_m + \lambda_0, \alpha) + \max(\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha))$$

Hence by (3.2.2) for $M(0, \lambda + 2\omega_m)$, (4.3.1) is satisfied by $w^+_{\lambda+2\omega_m}$. Therefore φ^+ is well-defined.

It remains to show that $\ker(\varphi^+)$ is generated by $(x_m^- \otimes t)w_{\lambda+2\omega_m}$. Observe that since φ^+ is well-defined

$$\begin{aligned}0 &= (x_m^- \otimes t^{(\omega_m,\alpha)})w^+_{\lambda+2\omega_m} \\ &= (x_m^- \otimes t)\varphi^+(w_{\lambda+2\omega_m}) \\ &= \varphi^+((x_m^- \otimes t)w_{\lambda+2\omega_m}),\end{aligned}$$

so $(x_m^- \otimes t)w_{\lambda+2\omega_m}$ is in the kernel. Note that for all α such that $(\omega_m, \alpha) \neq 0$, $(x_\alpha^- \otimes t^{(\omega_m,\alpha)+\max(\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})w_{\lambda+2\omega_m} \in \ker(\varphi^+)$, and that if $(\omega_m, \alpha) \neq 0$, then

$$\alpha = \begin{cases} \alpha_m & \\ \alpha_{i,k} & 1 \leq i \leq m \leq k \leq n \\ \beta_{i,k} & 1 \leq i \leq m < k \leq n-1 \\ \beta_{i,k} & 1 \leq i < k \leq m. \end{cases}$$

Identically to the proof of Lemma 4.1.1, each case is such that $\alpha - \alpha_m = \gamma_0 + \gamma_1 + \gamma_2$ where $\gamma_i \in R^+ \bigcup \{0\}$ and commute (using the same details Appendix A.1, simply replacing j with m). Hence

$$\Big(\prod_{i=0}^{2}(x_{\gamma_i}^- \otimes t^{(\omega_m,\gamma_i)+\max(\zeta_{1,\lambda},\gamma_i),(\zeta_{2,\lambda},\gamma_i))})\Big)(x_m^- \otimes t)w_{\lambda+2\omega_m}$$
$$= (x_\alpha^- \otimes t^{(\omega_m,\alpha)+\max(\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))}w_{\lambda+2\omega_m}.$$

Hence $\ker(\varphi^+)$ is generated by $(x_m^- \otimes t)w_{\lambda+2\omega_m}$ as claimed. ■

Lemma 4.3.2 *If $\lambda \in P^+(1)$ with $m < \min \lambda$ and $m \notin \{n-1, n\}$, then the mapping*

$$\varphi^- : \tau_1^* M(\omega_{m-1}, \lambda + \omega_{m+1}) \to M(\omega_m, \lambda + \omega_m)$$

which sends $w_{\lambda+\omega_{m-1}+\omega_{m+1}}$ *to* $(x_m^- \otimes t)w_{\lambda+2\omega_m}$ *is well-defined.*

Proof. Again, for notation, we assume $\min \zeta_{1,\lambda} < \min \zeta_{2,\lambda}$. First, we must show that $(x_m^- \otimes t)w_{\lambda+2\omega_m}$ is a highest weight vector. This can be sufficiently shown by proving that

$$(x_i^+ \otimes 1)(x_m^- \otimes t)w_{\nu+\lambda} = 0.$$

for $i \in [1, n]$. For all $i \in [1, n]$, there exists a $c \in \mathbb{C}$ such that

$$\begin{aligned}&(x_i^+ \otimes 1)(x_m^- \otimes t)w_{\nu+\lambda}\\ &\qquad = \delta_{i,m} c(h_m \otimes t)w_{\nu+\lambda} + (x_m^- \otimes t)(x_i^+ \otimes 1)w_{\nu+\lambda} = 0\end{aligned}$$

since $w_{\nu+\lambda}$ is highest weight. Thus this map sends highest weight vectors to highest weight vectors.

It remains to show that the relation (3.2.2) of $M(\omega_{m-1}, \lambda+\omega_{m+1})$ are satisfied by $(x_m^- \otimes t)w_{\lambda+2\omega_m}$. If $\lambda(h_{m+1}) = 0$, this means showing

$$(x_\alpha^- \otimes t^{(\omega_{m-1},\alpha)+\max(\zeta_{1,\lambda},\alpha)(\zeta_{2,\lambda}+\omega_{m+1},\alpha))})(x_m^- \otimes t)w_{\lambda+2\omega_m} = 0, \tag{4.3.2}$$

and if $\lambda(h_{m+1}) = 1$, this means showing

$$(x_\alpha^- \otimes t^{(\omega_{m-1}+\omega_{m+1},\alpha)+\max((\zeta_{1,\lambda}-\omega_{m+1},\alpha),(\zeta_{2,\lambda},\alpha))})(x_m^- \otimes t)w_{\lambda+2\omega_m} = 0. \tag{4.3.3}$$

To do this, we'll examine the cases

- $\omega_m(h_\alpha) = \omega_{m-1}(h_\alpha) = \omega(h_\alpha)$
- $\omega_m(h_\alpha) = \omega_{m-1}(h_\alpha) = \omega_{m+1}(h_\alpha) - 1$ or $\omega_m(h_\alpha) = \omega_{m+1}(h_\alpha) = \omega_{m-1}(h_\alpha) - 1$
- $\omega_m(h_\alpha) - 1 = \omega_{m-1}(h_\alpha) = \omega_{m+1}(h_\alpha) = 0$

- $\omega_m(h_\alpha) = \omega_{m-1}(h_\alpha) = \omega_{m+1}(h_\alpha) + 1$ or $\omega_m(h_\alpha) = \omega_{m+1}(h_\alpha) = \omega_{m-1}(h_\alpha) + 1$

If $\omega_m(h_\alpha) = \omega_{m-1}(h_\alpha) = \omega(h_\alpha)$, then

$$(\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha))$$
$$= r = \begin{cases} (\omega_{m-1}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_{m+1}, \alpha)) & \lambda(h_{m+1}) = 0 \\ (\omega_{m-1} + \omega_{m+1}, \alpha) + \max((\zeta_{1,\lambda} - \omega_{m+1}, \alpha), (\zeta_{2,\lambda}, \alpha)) & \lambda(h_{m+1}) = 1 \end{cases}$$

, and if $\alpha + \alpha_m = \alpha' \in R^+$, then

$$(\omega_m, \alpha') + \max((\zeta_{1,\lambda}, \alpha'), (\zeta_{2,\lambda} + \omega_m, \alpha')$$
$$= s = \begin{cases} (\omega_{m-1}, \alpha') + \max((\zeta_{1,\lambda}, \alpha'), (\zeta_{2,\lambda} + \omega_{m+1}, \alpha')) + 1 & \lambda(h_{n-1}) = 0 \\ (\omega_{m-1} + \omega_{m+1}, \alpha') + \max((\zeta_{1,\lambda} - \omega_{m+1}, \alpha'), (\zeta_{2,\lambda}, \alpha')) + 1 & \lambda(h_{n-1}) = 1, \end{cases}$$

Hence (taking $x_{\alpha'}$ to be 0 if $\alpha + \alpha_m \notin R^+$) we can use the bracket of $\mathfrak{g}[t]$ allows us to commute our terms and kill $w_{\lambda+2\omega_m}$.

Next, if $\omega_m(h_\alpha) = \omega_{m-1}(h_\alpha) = \omega_{m+1}(h_\alpha) - 1$ or $\omega_m(h_\alpha) = \omega_{m+1}(h_\alpha) = \omega_{m-1}(h_\alpha) - 1$, then

$$(\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha))$$
$$\leq r = \begin{cases} (\omega_{m-1}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_{m+1}, \alpha)) & \lambda(h_{m+1}) = 0 \\ (\omega_{m-1} + \omega_{m+1}, \alpha) + \max((\zeta_{1,\lambda} - \omega_{m+1}, \alpha), (\zeta_{2,\lambda}, \alpha)) & \lambda(h_{m+1}) = 1 \end{cases}$$

, and if $\alpha + \alpha_m = \alpha' \in R^+$, then

$$(\omega_m, \alpha') + \max((\zeta_{1,\lambda}, \alpha'), (\zeta_{2,\lambda} + \omega_m, \alpha')$$
$$\leq s = \begin{cases} (\omega_{m-1}, \alpha') + \max((\zeta_{1,\lambda}, \alpha'), (\zeta_{2,\lambda} + \omega_{m+1}, \alpha')) + 1 & \lambda(h_{n-1}) = 0 \\ (\omega_{m-1} + \omega_{m+1}, \alpha') + \max((\zeta_{1,\lambda} - \omega_{m+1}, \alpha'), (\zeta_{2,\lambda}, \alpha')) + 1 & \lambda(h_{n-1}) = 1, \end{cases}$$

so again taking $x_{\alpha'}$ to be 0 if $\alpha + \alpha_m \notin R^+$, we can use the bracket of $\mathfrak{g}[t]$ allows us to commute our terms and kill $w_{\lambda+2\omega_m}$.

When $\omega_m(h_\alpha) - 1 = \omega_{m-1}(h_\alpha) = \omega_{m+1}(h_\alpha) = 0$, then $\alpha = \alpha_m$ and

$$0 = (\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha)) - 2$$
$$= \begin{cases} (\omega_{m-1}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_{m+1}, \alpha)) & \lambda(h_{m+1}) = 0 \\ (\omega_{m-1} + \omega_{m+1}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) & \lambda(h_{m+1}) = 1. \end{cases}$$

Since the only $n_1 + n_2 = 1$ are $n_1 = 0$ and $n_2 = 1$ or vice versa, we can use Garland's Lemma 2.1.2 to see that this product kills $w_{\lambda+2\omega_m}$.

Lastly, when $\omega_m(h_\alpha) = \omega_{m-1}(h_\alpha) = \omega_{m+1}(h_\alpha) + 1$ or $\omega_m(h_\alpha) = \omega_{m+1}(h_\alpha) = \omega_{m-1}(h_\alpha) + 1$, we have that for some $\gamma \in R^+$, $\alpha = \alpha_m + \gamma$ with

$$(\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha)$$
$$\leq r = \begin{cases} (\omega_{m-1}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_{m+1}, \alpha)) & \lambda(h_{n-1}) = 0 \\ (\omega_{m-1} + \omega_{m+1}, \alpha) + \max((\zeta_{1,\lambda} - \omega_{m+1}, \alpha), (\zeta_{2,\lambda}, \alpha)) & \lambda(h_{n-1}) = 1, \end{cases}$$

and

$$(\omega_m, \gamma) + \max((\zeta_{1,\lambda}, \gamma), (\zeta_{2,\lambda} + \omega_m, \gamma))$$
$$\leq s = \begin{cases} (\omega_{m-1}, \alpha) + \max((\zeta_{1,\lambda}, \gamma), (\zeta_{2,\lambda} + \omega_{m+1}, \gamma)) & \lambda(h_{m+1}) = 0 \\ (\omega_{m-1} + \omega_{m+1}, \gamma) + \max((\zeta_{1,\lambda} - \omega_{m+1}, \gamma), (\zeta_{2,\lambda}, \gamma)) & \lambda(h_{m+1}) = 1. \end{cases}$$

Therefore using the $\mathfrak{g}[t]$ bracket and our prior cases, we can commute our terms and kill $w_{\lambda+2\omega_m}$.

As with the prior sections, we inculde the full commutations in Appendix A.3, and thus (4.3.2) and (4.3.3) are satisfied, and φ^- is well-defined. ■

Lemma 4.3.3 *If $\lambda \in P^+(1)$ with $m < \min\lambda$ and $m \in \{n-1, n\}$, then the mapping*

$$\varphi^- : \tau_1^* M(\omega_{n-2}, \lambda) \to M(\omega_m, \lambda + \omega_m)$$

which sends $w_{\lambda+\omega_{n-2}}$ to $(x_m^- \otimes t)w_{\lambda+2\omega_m}$ is well defined.

Proof. For ease of notation, we will take $(\zeta_{1,\lambda}, \zeta_{2,\lambda}+\omega_m)$ to be the interlacing pair associated to $\lambda+\omega_m$. First, we must show that $(x_m^- \otimes t)w_{\lambda+2\omega_m}$ is a highest weight vector. The proof is identical to the previous lemma, the mapping sending highest weight vectors to highest weight vectors.

It remains to show that the relation (3.2.2) of $M(\omega_{n-2}, \lambda)$ are satisfied by $(x_m^- \otimes t)w_{\lambda+2\omega_m}$. This means showing that

$$(x_\alpha^- \otimes t^{(\omega_{n-2},\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})(x_m^- \otimes t)w_{\lambda+2\omega_m} = 0. \tag{4.3.4}$$

To do this we examine the cases $\alpha_m(h_\alpha) = 0, -1, 2$, and 1.

If $\alpha_m(h_\alpha) = 0$, $[x_\alpha^-, x_m^-] = 0$, and

$$r = (\omega_{n-2}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = (\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha)),$$

so we can commute our terms and kill $w_{\lambda+2\omega_m}$.

Next, if $\alpha_m(h_\alpha) = -1$, then $\alpha + \alpha_m = \alpha' \in R^+$ and

$$r = (\omega_{n-2}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = (\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha)) + 1,$$

so

$$s = (\omega_{n-2}, \alpha') + \max((\zeta_{1,\lambda}, \alpha'), (\zeta_{2,\lambda}, \alpha')) + 1 = (\omega_m, \alpha') + \max((\zeta_{1,\lambda}, \alpha'), (\zeta_{2,\lambda} + \omega_m, \alpha')).$$

Using the bracket of $\mathfrak{g}[t]$ and Lemma 2.1.1, the commutation of our terms kills $w_{\lambda+2\omega_m}$.

If $\alpha_m(h_\alpha) = 2$, then $\alpha = \alpha_m$. In this case

$$r = (\omega_{n-2}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = (\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha)) - 2,$$

so by using Garland's Lemma 2.1.2 we see that this product will kill $w_{\lambda+2\omega_m}$.

Lastly, if $\alpha_m(h_\alpha) = 1$, then $\alpha = \alpha_m + \alpha_{i,n-2}$ and

$$r = (\omega_{n-2}, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = (\omega_m, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda} + \omega_m, \alpha)) - 1,$$

with

$$s = (\omega_{n-2}, \alpha_{i,n-2}) + \max((\zeta_{1,\lambda}, \alpha_{i,n-2}), (\zeta_{2,\lambda}, \alpha_{i,n-2}))$$

$$= (\omega_m, \alpha_{i,n-2}) + \max((\zeta_{1,\lambda}, \alpha_{i,n-2}), (\zeta_{2,\lambda} + \omega_m, \alpha_{i,n-2})) - 1.$$

Using the bracket of $\mathfrak{g}[t]$ and prior cases, we can commute our terms to kill $w_{\lambda+2\omega_m}$.

We provide the full commutations in Appendix A.3 again, and thus (4.3.4) is satisfied for all $\alpha \in R$ and φ^- is well-defined. ■

Together these lemmas prove the Sequences in Proposition 3.3.2*ii)* are right-exact.

4.4 Dimension Arguments

In this section, we will prove 2 propositions regarding the dimensions of certain $M(\nu, \lambda)$ in order to prove that the sequences in Proposition 3.3.2 are exact. In order to prove these two propositions, we will need the following lemma.

Lemma 4.4.1 *For $\lambda \in P^+$,*

$$\dim M(0, \lambda) = \dim D(2, 2\lambda_0) \dim M(0, \lambda_1).$$

Proof. Firstly note that if $\lambda_1(h_{n-1}+h_n)=1$ or if $\lambda=\omega_{i-1}+\omega_i+\delta_{i,n-1}\omega_n$, by Proposition 1.10 of [3], $M(0,\lambda)\cong D(2,\lambda)$, and from [7] and a second application of Proposition 1.10 that

$$M(0,\lambda)\cong D(2,2\lambda_0)*D(2,\lambda_1)\cong D(2,2\lambda_0)*M(0,\lambda_1).$$

If $\lambda_1(h_{n-1}+h_n)\in\{0,\ 2\}$ and $\lambda_1\neq\omega_{i-1}+\omega_i+\delta_{i,n-1}\omega_n$, then we must begin by defining β_{λ_1}. Take $p\leq n-2$ to be maximal such that $\lambda(h_{p+1})=1$, and $1\leq p'\leq p$ to be maximal such that $(\zeta_{1,\lambda}-\zeta_{2,\lambda})(h_{p',p})=0$. Then we set $\beta_{\lambda_1}=\beta_{p',p+1}$. By Lemma 1.11 of [3] there exists a $\mu\in P^+$ such that $\lambda_1-\beta_{\lambda_1}-2\mu\in P^+(1)$. This in turn means that $\lambda-\beta_{\lambda_1}=2(\lambda_0+\mu)+(\lambda_1-\beta_{\lambda_1}-2\mu)$. From this point we will use $\beta_\lambda=\beta_{\lambda_1}$ for notational ease.

Next we define a sequence of weights recursively, beginning with

$$\lambda^0=\lambda,\ \lambda^1=\lambda^0-\beta_{\lambda^0},\ \cdots\ ,\lambda^s=\lambda^{s-1}-\beta_{\lambda^{s-1}}.$$

We take s to be minimal such that $\beta_{\lambda^s}=0$, and we define

$$r_k=(\lambda_0^k,\beta_{\lambda^k})+\max((\zeta_{1,\lambda}^k,\beta_{\lambda^k}),(\zeta_{2,\lambda}^k,\beta_{\lambda^k}))$$

where $(\zeta_{1,\lambda}^k,\zeta_{2,\lambda}^k)$ is the interlacing pair corresponding to λ_1^k. Note that from the above discussion, there exists a $\mu^k\in P^+$ such that $\lambda^{k+1}=2(\lambda_0^k+\mu^k)+(\lambda_1^k-\beta_{\lambda^k}-2\mu^k)$. We can therefore describe $\lambda^{k+1}=2(\lambda_0+\sum_{i=0}^k\mu^i)+(\lambda_1-(\sum_{i=0}^k\beta_{\lambda^i}+2\mu^i))$.

Therefore the graded character formula given in Section 1.15 of [3] translates to

$$\begin{aligned}\dim M(0,\lambda)&=\sum_{k=0}^{s}q^{r_k}\dim D(2,\lambda^k)\\&=\sum_{k=0}^{s}q^{r_k}\dim D(2,2(\lambda_0+\sum_{i=0}^{k-1}\mu^i)+(\lambda_1-(\sum_{i=0}^{k-1}\beta_{\lambda^i}+2\mu^i)))\end{aligned}$$

Through application of [7] and the formula once again.

$$\dim M(0,\lambda) = \dim D(2,2\lambda_0)\sum_{k=0}^{s} q^{r_k} \dim D(2, 2(\sum_{i=0}^{k-1}\mu^i) + (\lambda_1 - (\sum_{i=0}^{k-1}\beta_{\lambda^i} + 2\mu^i)))$$

$$= \dim D(2,2\lambda_0)\dim M(0,\lambda_1)$$

■

Proposition 4.4.1 *For all weights satisfying the given conditions:*

i) $\nu_1, \nu_2 \in P^+$ *where* $\nu = \nu_1 + \nu_2$*, then* $\dim M(\nu, 0) = \dim M(\nu_1, 0) \dim M(\nu_2, 0)$.

ii) $\lambda, \mu \in P^+$*, where* $\lambda_0 - \mu \in P^+$*, then* $\dim M(0,\lambda) = \dim M(0, \lambda - 2\mu) \dim M(0, 2\mu)$

iii) $\lambda, \nu \in P^+$*, then* $\dim M(\nu,\lambda) \geq \dim M(\nu, 0) \dim M(0,\lambda)$

iv) $\lambda \in P^+(1)$*,* $m < \min \lambda = p$*, then* $\dim M(\omega_m, \lambda) = \dim M(\omega_m, 0) \dim M(0,\lambda)$ *and there exists a short exact sequence*

$$0 \to \tau_1^* M(\omega_{m-1}, \lambda - \alpha_{m,p} + \omega_m - \omega_{m-1}) \to M(\omega_m, \lambda) \to M(0, \lambda + \omega_m) \to 0$$

Proof. Firstly, the proof of *i)* is in [10], however the authors use the notation $W(\nu)$ in place of $M(\nu, 0)$. For the proof of *ii)*, note the by Lemma 4.4.1 and [7] for a $\mu \in P^+$ such that $\lambda_0 - \mu \in P^+$

$$\dim M(0,\lambda) = \dim D(2, 2\lambda_0) \dim M(0,\lambda_1) = \dim D(2, 2\mu) \dim D(2, 2(\lambda_0 - \mu)) \dim M(0, \lambda_1)$$

$$= \dim M(0, 2\mu) \dim M(0, \lambda - 2\mu)$$

Next, for *iii)*, we will take z_1, z_2 to be the parameters of our the fusion product $M(\nu, 0) * M(0,\lambda)$, and examine the map $\psi : M(\nu,\lambda) \to M(\nu, 0) * M(0,\lambda)$ which sends

$w_{\nu+\lambda} \to w_\nu * w_\lambda$. To show that this map is a well-defined surjection we need to show that $w_\nu * w_\lambda$ satisfies the relations (3.2.1) and (3.2.2) for $M(\nu, \lambda)$. Note that because both $M(\nu, 0)$ and $M(0, \lambda)$ are quotients of $W_{\text{loc}}(\nu)$ and $W_{\text{loc}}(\lambda)$ respectively, (3.2.1) is clearly satisfied. To show that (3.2.2) is satisfied, take $(\nu + \lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha)) = r$, and note that for any $r_1 + r_2 = r$, by Lemma 2.3.1

$$\begin{aligned}(x_\alpha^- \otimes (t+z_1)^{r_1}(t+z_2)^{r_2})&\overline{(w_\nu \otimes w_\lambda)} \\ &= \overline{(x_\alpha^- \otimes t^r)(w_\nu \otimes w_\lambda)} = 0,\end{aligned}$$

since $r \geq (\nu, \alpha)$ and $r \geq (\lambda_0, \alpha) + \max((\zeta_{1,\lambda}, \alpha), (\zeta_{2,\lambda}, \alpha))$. Hence ψ is a well defined surjection, and $\dim M(\nu, \lambda) \geq \dim M(\nu, 0) \dim M(0, \lambda)$.

Lastly, for *iv)*, we will induct on m. Note that for $m = 0$, $\dim M(\omega_0, 0) = \dim M(0, 0) = \dim \mathbb{C} = 1$ so our claim is obvious. For $m > 0$, it is worth noting that (3.2.3) gives us that $\dim M(\omega_m, 0) = \dim M(0, \omega_m)$. Specifically for $m = 1$, Proposition 3.3.2*ib)* and Proposition 4.4.1*ii)* gives us

$$\dim M(\omega_1, 0) \dim M(0, \lambda) \leq \dim M(0, \lambda + \omega_1) + \dim M(0, \lambda - \alpha_{1,p} + \omega_1).$$

Note that because Theorem 2.5.1 and $\lambda \in P^+(1)$, $M(0, \lambda) = [\boldsymbol{\lambda}_{\mathbb{C}}]$ and the above becomes

$$\dim[\boldsymbol{\omega}_{1,a}] \dim[\boldsymbol{\lambda}] \geq \dim[\boldsymbol{\omega}_{1,a} + \boldsymbol{\lambda}] + \dim[\boldsymbol{\omega}_{p+1,\xi p} + \delta_{p,n-2}\boldsymbol{\omega}_{n,b} + \boldsymbol{\lambda}']$$

where $\text{wt}\,\boldsymbol{\lambda}' = \lambda - \omega_p$, which is precisely (2.5.1) and (2.5.2). Assuming our hypothesis for $M < m$, we again use Proposition 3.3.2*ib)*, Proposition 4.4.1*ii)*, and our inductive hypothesis to arrive at

$$\begin{aligned}&\dim M(\omega_m, 0) \dim M(0, \lambda) \\ &\qquad \leq \dim M(0, \lambda + \omega_m) + \dim M(\omega_{m-1}, 0) \dim M(0, \lambda - \alpha_{m,p} + \omega_m - \omega_{m-1}),\end{aligned}$$

which is shown to be an equality by (2.5.1) and (2.5.2). Hence our claim hold for m, and the induction complete. ■

Proposition 4.4.2 *Let (ν, λ) be a compatible pair.*

i) For all (ν, λ), $\dim M(\nu, \lambda) = \dim M(\nu, 0) \dim M(0, \lambda)$

ii) if $\nu(h_j) = 2$, $\dim M(\nu, \lambda) = \dim M(\nu - 2\omega_j, \lambda + 2\omega_j) + \dim M(\nu - \alpha_j, \lambda)$

iii) if $\nu \in P^+(1)$, $m = \max \nu < \min \lambda_1 = p$,

$$\dim M(\nu, \lambda) = \dim M(\nu - \omega_m, \lambda + \omega_m) + \dim M(\nu - \omega_m + \omega_{m-1}, \lambda - \alpha_{m,p} + \omega_m - \omega_{m-1})$$

iv) if $\lambda \in P^+(1)$, with $m < \min \lambda$ with $m \notin \{n-1, n\}$,

$$\begin{aligned}\dim M(\omega_m, \lambda + \omega_m) =& \dim M(\omega_m, 0) \dim M(0, \lambda + \omega_m)\\ =& \dim M(0, \lambda + 2\omega_m) + \dim M(\omega_{m-1}, \lambda + \omega_{m+1})\end{aligned}$$

v) if $\lambda \in P^+(1)$, with $m < \min \lambda$ with $m \in \{n-1, n\}$,

$$\begin{aligned}\dim M(\omega_m, \lambda + \omega_m) =& \dim M(\omega_m, 0) \dim M(0, \lambda + \omega_m)\\ =& \dim M(0, \lambda + 2\omega_m) + \dim M(\omega_{n-2}, \lambda)\end{aligned}$$

Proof. We prove *i)-iii)* by inducting on compatible pairs. Observe this is true for $(0, \omega_i)$'s by (3.2.3). Assume this is true for all $(\nu', \lambda') \prec (\nu, \lambda)$.

If $\nu(h_j) = 2$, Proposition 3.3.2*ia)*, Proposition 4.4.1*i)-iii)*, and our inductive hypothesis achieve

$$\begin{aligned}&\dim M(\nu, 0) \dim M(0, \lambda)\\ &\qquad \le (\dim M(0, 2\omega_j) + \dim M(2\omega_j - \alpha_j, 0)) \dim M(\nu - 2\omega_j, 0) \dim M(0, \lambda).\end{aligned}$$

Note that $\dim M(2\omega_j, 0) = \dim M(\omega_j, 0) \dim M(\omega_j, 0) = (\dim M(\omega_j, 0))^2$ by (3.2.3) and Proposition 4.4.1*i*, $\dim M(0, 2\omega_j) = \dim D(2, 2\omega_j)$ by Proposition 4.4.1, and $\dim M(2\omega_j - \alpha_j, 0) = \prod_{\alpha_j(h_k)=-1} \dim M(0, \omega_k)$ again by (3.2.3) and Proposition 4.4.1*i*. Observe that here $\mathbf{KR}(\omega_j) \cong M(\omega_j, 0)$, $\mathbf{KR}(2\omega_j) \cong M(0, 2\omega_j)$, and $\dim K^*_{j,1} = \dim M(2\omega_j - \alpha_j, 0)$. Therefore by Theorem 4 of [8],

$$\dim M(2\omega_j - \alpha_j, 0) + \dim M(0, 2\omega_j) = \dim M(2\omega_j, 0).$$

Hence, using the above along with Proposition 4.4.1*i)* we have

$$\begin{aligned}\dim M(\nu, 0) \dim M(0, \lambda) &\leq \dim M(2\omega_j, 0) \dim M(\nu - 2\omega_j, 0) \dim M(0, \lambda) \\ &= \dim M(\nu, 0) \dim M(0, \lambda).\end{aligned}$$

If $\nu \in P^+(1)$ with $\max \nu = m$ and $\lambda_1 = 0$, then using (3.2.3), the inductive hypothesis, and Proposition 4.4.1*i)*,

$$\begin{aligned}\dim M(\nu, \lambda) = \dim M(\nu - \omega_m, \lambda + \omega_m) &= \dim M(\nu - \omega_m, 0) \dim M(0, \lambda + \omega_m) \\ &= \dim M(\nu - \omega_m, 0) \dim M(\omega_m, 0) \dim M(0, \lambda) \\ &= \dim M(\nu, 0) \dim M(0, \lambda)\end{aligned}$$

If $\nu \in P^+(1)$, such that $m = \max \nu < \min \lambda_1 = p$, then by Proposition 3.3.2*ib)*, Proposition 4.4.1, and our inductive hypothesis

$$\begin{aligned}&\dim M(\nu, 0) \dim M(0, \lambda) \\ &\quad \leq \dim M(\nu - \omega_m, \lambda + \omega_m) + \dim M(\nu - \omega_m + \omega_{m-1}, \lambda - \alpha_{m,p} + \omega_m - \omega_{m-1}) \\ &\quad = \dim M(\nu, 0 \dim M(0, \lambda).\end{aligned}$$

We prove *iv)* and *v)* using a downward induction on m. If $m = n$, $\lambda = 0$, so using (3.2.3), Proposition 3.3.2*ii)*, and Proposition 4.4.2*ii)* we see

$$\dim M(\omega_n, 0) \dim M(0, \omega_n) = \dim M(2\omega_n, 0) \leq \dim M(\omega_{n-2}, 0) + \dim M(0, 2\omega_n)$$

$$= \dim M(\omega_{n-2}, 0) + \big(\dim M(2\omega_n, 0) - \dim M(\omega_{n-2}, 0)\big) = \dim M(2\omega_n, 0).$$

When $m = n - 1$, $\lambda \in \{0,\ \omega_n\}$, so using either (3.2.3) or (3.2.4), Proposition 3.3.2*ii)*, and Proposition 4.4.2*ii)*

$$\dim M(\omega_{n-1}, 0) \dim M(0, \lambda + \omega_{n-1}) = \dim M(2\omega_{n-1}, \lambda)$$

$$\leq \dim M(\omega_{n-2}, \lambda) + \dim M(0, \lambda + 2\omega_{n-1})$$

$$= \dim M(\omega_{n-2}, \lambda) + \big(\dim M(2\omega_{n-1}, \lambda) - \dim M(\omega_{n-2}, \lambda)\big)$$

$$= \dim M(2\omega_{n-1}, \lambda).$$

Now, for our induction, assume our claim holds for $M > m$ and take $\min \lambda = p$. We can use Proposition 4.4.1*i)*, Proposition 3.3.2*ii)*, and Proposition 4.4.2*i)*-*iii)* to see

$$\dim M(\omega_m, 0) \dim M(0, \lambda + \omega_m)$$

$$\leq \big(\dim M(\omega_{m-1} + \omega_{m+1}, \lambda) - \dim M(\omega_{m-1} + \omega_m, \lambda - \alpha_{m+1,p} + \omega_{m+1} - \omega_m)\big)$$

$$+ \big(\dim M(2\omega_m, \lambda) - \dim M(\omega_{m-1} + \omega_{m+1}, \lambda)\big)$$

$$= \dim M(\omega_m, 0)\big(\dim M(\omega_m, \lambda) - \dim M(\omega_{m-1}, \lambda - \alpha_{m,p} + \omega_m - \omega_{m-1})\big)$$

$$= \dim M(\omega_m, 0) \dim M(0, \lambda + \omega_m)$$

since $\alpha_{m+1,p} - \omega_{m+1} + \omega_m = \alpha_{m,p} - \omega_m + \omega_{m-1}$. This completes our induction. ■

Proposition 4.4.2*ii)*-*v)* prove exactness of all sequences of Proposition 3.3.2, so we complete this chapter.

Chapter 5

Proof of Proposition 3.3.1

In this chapter, we will be creating a closed form of $g^{\mu}_{\nu,\lambda}(q)$ and proving that $h^{\mu}_{\nu,0} = p^{\mu}_{\nu}$. To do this, we prove some properties of $g^{\mu}_{\nu,\lambda}$ to use in our construction of its closed form, take note of some recursions of p^{μ}_{ν}, and then prove some properties of $h^{\mu}_{\nu,\lambda}$ and their relationship to $g^{\mu}_{\nu,\lambda}$ and show that these produce recursions identical to those of p^{μ}_{ν}.

5.1 Properties of $g^{\mu}_{\nu,\lambda}$

Proposition 5.1.1 *Assume $(\nu, \lambda) \in P^+ \times P^+$ is a compatible pair, then for $b \in \{g, h\}$*

ia) If $\nu(h_j) \geq 2$ for $1 \leq j \leq n$ then

$$b^{\mu}_{\nu,\lambda} = b^{\mu}_{\nu-2\omega_j,\lambda+2\omega_j} + q^{(\nu+\lambda_0,\alpha_j)-1} b^{\mu}_{\nu-\alpha_j,\lambda}$$

ib) If $\nu \in P^+(1)$ with $\max \nu < \min \lambda_1 = m < n-1$, $0 < p = \min \lambda_1 - \omega_m$, then

$$b^{\mu}_{\nu+\omega_m,\lambda-\omega_m} = b^{\mu}_{\nu,\lambda} + q^{(\lambda_0,\alpha_{m,p})+1} b^{\mu}_{\nu+\omega_{m-1},\lambda-\alpha_{m,p}-\omega_{m-1}}$$

ii) If $\lambda \in P^+(1)$ with $m < \min \lambda$ with $m \notin \{n-1, n\}$,

$$b^{\mu}_{\omega_m,\lambda+\omega_m} = b^{\mu}_{0,\lambda+2\omega_m} + qb^{\mu}_{\omega_{m-1},\lambda+\omega_{m+1}},$$

and if $m \in \{n-1, n\}$,

$$b^{\mu}_{\omega_m,\lambda+\omega_m} = b^{\mu}_{0,\lambda+2\omega_m} + qb^{\mu}_{\omega_{n-2},\lambda}$$

Proof. Given the the exact sequences of Proposition 3.3.2, then these equalities come form identifying elements and equating their coefficients in the graded character formulas. For an example of the first item, see Appendix B.1 ■

Lemma 5.1.1 *For all compatible pairs (ν, λ) and dominant integral weights μ, $g^{\mu}_{\nu,\lambda} = q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}$.*

Proof. This proof will be done by induction on the height of λ_1. We will assume throughout that (ν, λ) is a compatible pair.

First, we examine the base case of $\operatorname{ht} \lambda = 0$. If $\operatorname{ht} \lambda = 0$, then $\lambda = 0$ and

$$g^{\mu}_{\nu,0} = \delta_{\nu,\mu} = g^{\mu-\nu}_{0,0}.$$

Thus we assume inductively $g^{\mu}_{\nu,\lambda'} = q^{(\nu+\lambda'-\mu,\nu)} g^{\mu-\nu}_{0,\lambda'}$ for $\operatorname{ht} \lambda' < \operatorname{ht} \lambda$ and move to cases of $\lambda \neq 0$. For brevity, the full computations have been moved to Appendix B.2.

When $\operatorname{ht} \lambda_1 = 0$, then $\lambda = 2\lambda_0$ and $\lambda(h_j) \neq 0$ for some $1 \leq j \leq n$. Since $(\nu + 2\omega_j, \lambda - 2\omega_j)$ and $(\nu - \alpha_j + 2\omega_m, \lambda - 2\omega_j)$ are also a compatible pairs with $\operatorname{ht} \lambda >$

$\mathrm{ht}(\lambda - 2\omega_j)$, we can use Proposition 5.1.1*ia)* and the inductive hypothesis to compute

$$\begin{aligned} g^{\mu}_{\nu,\lambda} &= g^{\mu}_{\nu+2\omega_j,\lambda-2\omega_j} - q^{(\nu+\lambda_0+\omega_j)(h_j)-1} g^{\mu}_{\nu+2\omega_j-\alpha_j,\lambda-2\omega_j} \\ &= q^{(\nu+\lambda-\mu,\nu)}(q^{(\nu+\lambda-\mu,2\omega_j)} g^{\mu-\nu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(\nu+\lambda-\mu,2\omega_j-\alpha_j)} g^{\mu+\alpha_j-\nu-2\omega_j}_{0,\lambda-2\omega_j}) \\ &= q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}. \end{aligned}$$

Next we explore the case $\mathrm{ht}\,\lambda_1 = 1$. Thus $\lambda_1 = \omega_m$ for some $1 \le m \le n$ and $\lambda = 2\lambda_0 + \omega_m$. Then by (3.2.3) and the fact that $(\nu + \omega_m, 2\lambda_0)$ is a compatible pair with $\mathrm{ht}\,\lambda > \mathrm{ht}(2\lambda_0)$ out inductive hypothesis provides

$$g^{\mu}_{\nu,2\lambda_0+\omega_m} = q^{(\nu+\lambda-\mu,\nu+\omega_m)} g^{\mu-\nu-\omega_m}_{0,2\lambda_0} = q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}$$

Then we move to the case of $\mathrm{ht}\,\lambda_1 \ge 2$. Here, $\lambda_1(h_{m,p}) = 2$ for $1 \le m < p \le n$. Note that the case when $\mathrm{ht}\,\lambda_1 = 2$ with $m = n-1$ is immediate from (3.2.4) and the inductive hypothesis. For all other instances, since $(\nu + \omega_m, \lambda - \omega_m)$ and $(\nu + \omega_{m-1}, \lambda - \alpha_{m,p} - \omega_{m-1})$ are compatible pairs with $\mathrm{ht}\,\lambda > \mathrm{ht}(\lambda - \omega_m) = \mathrm{ht}(\lambda - \alpha_{m,p} - \omega_{m-1})$, we can use Proposition 5.1.1*ib)* and the inductive hypothesis to compute

$$\begin{aligned} g^{\mu}_{\nu,\lambda} &= g^{\mu}_{\nu+\omega_m,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1} g^{\mu}_{\nu+\omega_{m-1},\lambda-\alpha_{m,p}-\omega_{m-1}} \\ &= q^{(\nu+\lambda-\mu,\nu)}(q^{(\nu+\lambda-\mu,\omega_m)} g^{\mu-\nu-\omega_m}_{0,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1+(\nu+\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g^{\mu-\nu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}) \\ &= q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}. \end{aligned}$$

Hence our induction is complete. ■

Once we have this proposition and lemma, we can prove even more properties of $g^{\mu}_{\nu,\lambda}$.

Proposition 5.1.2 *For* $\lambda \in P^+$ *with* $\lambda = 2\lambda_0 + \lambda_1$*,* $m = \min(\lambda_1)$*, and* $p = \min(\lambda - \omega_m)$*;*

a) $\sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda} = 0$

b) $g^{\mu}_{0,\lambda} \neq 0$ *implies* $(\lambda-\mu,\omega_s) \leq (\lambda,\alpha_s)$ *for all* $1 \leq s \leq m$.

c) When $\lambda_0 = 0$, $m < n-1$, $p \neq 0$

$$g^{\mu}_{0,\lambda} = g^{\mu-\omega_m}_{0,\lambda-\omega_m} - q g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}$$

d) When $\lambda_0 = \omega_j$ *with* $j+1 < m$

$$g^{\mu}_{0,\lambda} = g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q g^{\mu-2\omega_j+\alpha_j}_{0,\lambda-2\omega_j}$$

Proof. From the proof of Lemma 5.1.1, we know that for compatible pairs (ν,λ) we have

$$g^{\mu}_{0,\lambda} = q^{(\lambda-\mu,\omega_m)} g^{\mu-\omega_m}_{0,\lambda-\omega_m} - (1-\delta_{p,0}) q^{(\lambda_0,\alpha_{m,p})+1+(\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}} \tag{5.1.1}$$

$$g^{\mu}_{0,\lambda} = q^{(\lambda-\mu,2\omega_j)} g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(\lambda-\mu-\alpha_j,2\omega_j-\alpha_j)} g^{\mu+\alpha_j-2\omega_j}_{0,\lambda-2\omega_j} \tag{5.1.2}$$

and so we can prove each piece of Proposition 5.1.2 with an induction on $\operatorname{ht}\lambda$. As with our other proofs, we give the steps of our proof, and list the full computations in Appendix B.3.

For *a)*, if $\lambda = 0$, then $\sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,0} = 0$. We will suppose that $\sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda} = 0$ for $\operatorname{ht}\lambda < M$. Taking $\lambda \in P^+(1)$ with $\operatorname{ht}\lambda = M$, there exists $m = \min(\lambda)$, so

$$\sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda} = \sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} (q^{(\lambda-\mu,\omega_m)} g^{\mu-\omega_m}_{0,\lambda-\omega_m} \\ - (1-\delta_{p,0}) q^{(\lambda_0,\alpha_{m,p})+1+(\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}} = 0$$

by (5.1.1) and the inductive hypothesis. Taking $\lambda \notin P^+(1)$ with $\operatorname{ht}\lambda = M$. Then there exists a j such that $\lambda_0(h_{\alpha_j}) > 0$ the

$$\sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda}$$

$$= \sum_{\mu\in P} q^{\frac{1}{2}(\mu-2\omega_j,\mu-2\omega_j)}(q^{(\lambda,2\omega_j)} g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(\lambda-\alpha_j,2\omega_j-\alpha_j)} g^{\mu+\alpha_j-2\omega_j}_{0,\lambda-2\omega_j} = 0$$

by (5.1.2) and the inductive hypothesis. Thus for all $\operatorname{ht}\lambda = M$ *a)* holds.

For *b)* if $\lambda = 0$, then $g^{\mu}_{0,0} = \delta_{0,\mu}$, so $g^{\mu}_{0,0} \neq 0$ unless $\mu = 0$, and $(0-0,\omega_s) = (0,\alpha_s)$. Assume for $\operatorname{ht}\lambda < M$ that $g^{\mu}_{0,\lambda} \neq 0$ implies that $(\lambda-\mu,\omega_s) \leq (\lambda,\alpha_s)$ for all $1 \leq s \leq m$. Taking a $\lambda \in P^+(1)$ with $\operatorname{ht}\lambda = M$, then there exists $m = \min\lambda$ and $p = \min(\lambda-\omega_m)$. Firstly if $p = 0$, then (3.2.3) and the inductive hypothesis give

$$0 \neq g^{\mu}_{0,\lambda} = q^{(\lambda-\mu,\omega_m)} g^{\mu-\omega_m}_{0,0} \text{ and } (\omega_m-\mu,\omega_s) \leq (0,\alpha_s) = 0$$

for all $s \in [1,m]$. If $p \neq 0$, then 5.1.1 and the inductive hypothesis give us that

$$0 \neq g^{\mu}_{0,\lambda} = q^{(\lambda-\mu,\omega_m)} g^{\mu-\omega_m}_{0,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1+(\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}, \text{ so}$$

$$\begin{aligned}(\lambda-\alpha_{m,p}-\omega_{m-1}-(\mu-\omega_{m-1}),\omega_s) &\leq (\lambda-\omega_m-(\mu-\omega_m),\omega_s)\\ &\leq (\lambda-\omega_m,\alpha_s) \leq (\lambda,\alpha_s)\end{aligned}$$

for all $s \in [1,m]$. Taking a $\lambda \notin P^+(1)$, then $\lambda_0(h_{\alpha_j}) > 0$ for some $j \in [1,n]$, and (5.1.2) and the inductive hypothesis lead to

$$0 \neq g^{\mu}_{0,\lambda} = q^{(\lambda-\mu,2\omega_j)} g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(\lambda-\mu-\alpha_j,2\omega_j-\alpha_j)} g^{\mu+\alpha_j-2\omega_j}_{0,\lambda-2\omega_j}, \text{ so}$$

$$\begin{aligned}(\lambda-\alpha_j-\mu,\omega_s) &\leq (\lambda-2\omega_j-(\mu-2\omega_j),\omega_s)\\ &\leq (\lambda-2\omega_j,\alpha_s) \leq (\lambda,\alpha_s)\end{aligned}$$

for all $s \in [1,m]$. Thus, *b)* holds for all $\mathrm{ht}\,\lambda > M$.

For *c)* take $\lambda_0 = 0$ and $m < n-1$ with $p \neq 0$, notice that using *b)*, we have that $(\lambda-\mu, \omega_m) = (\lambda - \alpha_{m,p} - \mu, \omega_{m-1}) = 0$, and applying this to (5.1.1) we achieve our desired result.

Finally for *d)* take $\lambda_0 = \omega_j$ with $j+1 < m$, using *b)*, we have that $(\lambda-\mu, \omega_j) = (\lambda - \mu - \alpha_j, \omega_j) = 0$, and applying this to (5.1.2) we achieve the desired result. ■

There is one final lemma we need to construct the closed form of $g^{\mu}_{\nu,\lambda}$.

Lemma 5.1.2 *For $m \in [1,n]$ and $\lambda \in P^+(1)$ with $m < p = min\lambda$ we have*

$$g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q g^{\mu-(1-\delta_{m,n})\omega_{m-1}-\delta_{m,n}\omega_{n-2}}_{0,\lambda+(1-\delta_{m,n-1})\omega_{m+1}} = q^{(\lambda+2\omega_m-\mu,\omega_m)} g^{\mu-\omega_m}_{0,\lambda+\omega_m}.$$

Proof. To prove the first equality, note that from Proposition 5.1.1*ii)*, we have for $m \notin \{n-1,n\}$ $g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q g^{\mu}_{\omega_{m-1},\lambda+\omega_{m+1}}$. By Lemma 5.1.1, the compatibility of the pair $(\omega_{m-1}, \lambda+\omega_{m+1})$, and Proposition 5.1.2*b)*, $(\lambda+\omega_{m+1}+\omega_{m-1}-\mu, \omega_{m-1}) = 0$ and

$$g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q g^{\mu-\omega_{m-1}}_{0,\lambda+\omega_{m+1}}.$$

Similarly, if $m \in \{n-1,n\}$, Proposition 5.1.1*ii)* gives us $g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q g^{\mu}_{\omega_{n-2},\lambda}$. Again using Lemma 5.1.1, the fact that the pair (ω_{n-2}, λ) is compatible, and Proposition 5.1.2*b)* implying that $(\lambda+\omega_{n-2}-\mu, \omega_{n-2}) = 0$, we see that

$$g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q g^{\mu-\omega_{n-2}}_{0,\lambda}.$$

Hence the first equality holds.

To prove the second equality, we shall employ a downward induction on m. If $m = n$, then $\lambda = 0$, so

$$g^{\mu-\omega_n}_{0,\omega_n} = \delta_{\mu-\omega_n,\omega_n}, \qquad g^{\mu-\omega_{n-2}}_{0,0} = \delta_{\mu-\omega_{n-2},0}$$

therefore the equality holds for $m = n$ if $\delta_{\mu,2\omega_n} = g^{\mu}_{0,\lambda+2\omega_n} + q\delta_{\mu,\omega_{n-2}}$. This is precisely the application of (5.1.2) to $\lambda = 2\omega_n$.

If $m = n-1$, then $\lambda \in \{0, \omega_n\}$. If $\lambda = 0$, we follow the same steps discuss when $m = n$. If $\lambda = \omega_n$, note that using (5.1.2)

$$g^{\mu}_{0,2\omega_{n-1}+\omega_n} = q^{(\omega_n+2\omega_{n-1}-\mu,2\omega_{n-1})} g^{\mu-2\omega_{n-1}}_{0,\omega_n} - q g^{\mu-\omega_{n-2}}_{0,\omega_n}.$$

Note that by Proposition 5.1.2*b)* $g^{\mu-2\omega_{n-1}}_{0,\omega_n} \neq 0 \implies (\omega_n + 2\omega_{n-1} - \mu, \omega_{n-1}) \leq (\omega_n, \alpha_{n-1}) = 0$, so when we apply Lemma 5.1.1 and (3.2.3) we see that

$$\begin{aligned} g^{\mu}_{0,2\omega_{n-1}+\omega_n} &= q^{(\omega_n+2\omega_{n-1}-\mu,\omega_{n-1})} g^{\mu-2\omega_{n-1}}_{0,\omega_n} - q g^{\mu-\omega_{n-2}}_{0,\omega_n} \\ &= g^{\mu-\omega_{n-1}}_{\omega_{n-1},\omega_n} - q g^{\mu-\omega_{n-2}}_{0,\omega_n} = g^{\mu-\omega_{n-1}}_{0,\omega_n+\omega_{n-1}} - q g^{\mu-\omega_{n-2}}_{0,\omega_n} \end{aligned}$$

so our equality holds.

For the next step of this induction, when $m < n-1$ and $p \neq n$, we apply Proposition 5.1.2*c)* or inductive hypothesis to $g^{\mu-\omega_m}_{0,\lambda+\omega_m}$ and $g^{\mu-\omega_{m-1}}_{\lambda+\omega_{m+1}}$ and have

$$g^{\mu-\omega_m}_{0,\lambda+\omega_m} = g^{\mu-2\omega_m}_{0,\lambda} - q g^{\mu-\omega_m-\omega_{m-1}}_{0,\lambda-\omega_p+\omega_{p+1}} \quad g^{\mu-\omega_{m-1}}_{0,\lambda+\omega_{m+1}} = g^{\mu-2\omega_m+\alpha_m}_{0,\lambda} - q g^{\mu-\omega_{m-1}-\omega_m}_{0,\lambda-\omega_p+\omega_{p+1}}.$$

Using Proposition 5.1.2*b)*, we have that if $g^{\mu-\omega_{m-1}-\omega_m}_{0,\lambda-\omega_p-\omega_{p+1}} \neq$ then $(\lambda + 2\omega_m - \alpha_{m,p} - \mu, \omega_m) = 0$ and if $g^{\mu-2\omega_m}_{0,\lambda}$ then $(\lambda + 2\omega_m - \mu, \omega_m) = 0$. This means what we are attempting to show is

$$\begin{aligned} g^{\mu}_{0,\lambda+2\omega_m} =& q^{(\lambda+2\omega_m-\mu,\omega_m)}(g^{\mu-2\omega_m}_{0,\lambda} - q g^{\mu-\omega_m-\omega_{m-1}}_{0,\lambda-\omega_p+\omega_{p+1}}) - q(g^{\mu-2\omega_m+\alpha_m}_{0,\lambda} - q g^{\mu-\omega_{m-1}-\omega_m}_{0,\lambda-\omega_p+\omega_{p+1}}) \\ =&(q^{(\lambda+2\omega_m-\mu,\omega_m)} g^{\mu-2\omega_m}_{0,\lambda} - q g^{\mu-2\omega_m+\alpha_m}_{0,\lambda}) - q(q^{(\lambda+2\omega_m-\mu,\omega_m)} - q) g^{\mu-\omega_{m-1}-\omega_m}_{0,\lambda-\omega_p+\omega_{p+1}} \\ =& g^{\mu-2\omega_m}_{0,\lambda} - q g^{\mu-2\omega_m+\alpha_m}_{0,\lambda}. \end{aligned}$$

When $p = n$, we get an identical statement using identical steps, with $p-1$ in place of $p+1$. To prove this simplified statement, we split into three cases: $p > m+1$, $\lambda = \omega_{m+1}$, and

$p = m+1$ with $\min(\lambda - \omega_p) = r > 0$. Firstly, if $p > m+1$, then $(2\omega_m, \lambda)$ is a compatible pair, so our expression is exactly the result of (5.1.2). If $\lambda = \omega_{m+1}$ then by (3.2.3) in combination with Proposition 5.1.1*ib)* to rewrite the desired statement as

$$\begin{aligned} g_{0,\lambda}^{\mu-2\omega_m} =& g_{0,\omega_{m+1}+2\omega_m}^{\mu} + q g_{0,\lambda}^{\mu-2\omega_m+\alpha_m} \\ =& (g_{2\omega_m,0}^{\mu-\omega_{m+1}} - q g_{2\omega_m-\alpha_m,0}^{\mu-\omega_{m+1}}) + q g_{0,\lambda}^{\mu-2\omega_m+\alpha_m}, \end{aligned}$$

which is true if $\delta_{\mu-2\omega_m,\omega_{m+1}} = \delta_{\mu-\omega_{m+1},2\omega_m} - q\delta_{\mu-\omega_{m+1},2\omega_m-\alpha_m} + q\delta_{\mu-2\omega_m+\alpha_m,\omega_{m+1}}$, and this is clear.

If $p = m+1$ and $r = \min(\lambda - \omega_p) > 0$, we can use the fact that $(\omega_{m+1}, \lambda - \omega_{m+1})$, $(2\omega_m, \lambda - \omega_{m+1})$, and $(2\omega_m, \lambda - \omega_{m+1} - \omega_r + \omega_{r+1})$ are all compatible pairs, (5.1.1), Lemma 5.1.1, and (5.1.2) establish that,

$$\begin{aligned} g_{0,\lambda}^{\mu-2\omega_m} =& (g_{0,\lambda-\omega_{m+1}+2\omega_m}^{\mu-\omega_{m+1}} + q g_{2\omega_m-\alpha_m,\lambda-\omega_{m+1}}^{\mu-\omega_{m+1}}) - q(g_{0,\lambda-\alpha_{m+1,r}+\omega_m}^{\mu-\omega_m} + g_{2\omega_m-\alpha_m,\lambda-\alpha_{m+1,r}-\omega_m}^{\mu-\omega_m}) \\ =& (g_{0,\lambda-\omega_{m+1}+2\omega_m}^{\mu-\omega_{m+1}} - q g_{0,\lambda-\alpha_{m+1,r}+\omega_m}^{\mu-\omega_m}) + q(g_{0,\lambda-\omega_{m+1}}^{\mu-2\omega_{m+1}-\omega_{m-1}} - q g_{0,\lambda-\alpha_{m+1,r}-\omega_m}^{\mu-\omega_{m+1}-\omega_{m-1}-\omega_m}) \\ =& g_{0,\lambda+2\omega_m}^{\mu} + q g_{0,\lambda}^{\mu-\omega_{m+1}-\omega_{m-1}}. \end{aligned}$$

Hence, we complete our induction and the second equality is true. ■

5.2 Closed from of $g_{\nu,\lambda}^{\mu}$

In this section, we will build a closed form for $g_{\nu,\lambda}^{\mu}$. Thus far we have a recursive definition of $g_{0,\lambda}^{\mu}$ using (5.1.1), (5.1.2), beginning with $g_{0,0}^{\mu} = \delta_{\mu,0}$. To create a closed form, we need to take $\rho = \sum_{i\in[1,n]} \omega_i$ and note that for $\eta = \sum_{i\in[1,n]} c_i\alpha_i \in Q^+$, we set $\eta^\vee = \sum_{i\in[1,n]} c_i\omega_i$. Observe that when $\lambda \neq 0$ and $\operatorname{ht}\lambda_1 = 0$, (5.1.2) leads us to understand

$$g_{0,2\lambda_0}^{\mu} = q^{(2\lambda_0-\mu,2\omega_j)} g_{0,2(\lambda_0-\omega_j)}^{\mu-2\omega_j} - q^{(\lambda_0,\alpha_j)+(2\lambda_0-\mu-\alpha_j,2\omega_j-\alpha_j)} g_{0,2(\lambda_0-\omega_j)}^{\mu+\alpha_j-2\omega_j}.$$

So long as $\lambda_0 - \omega_j \neq 0$, allowing $s_j = (\lambda_0, \alpha_j)$, this iterates to

$$g^{\mu}_{0,2\lambda_0} = \sum_{i=0}^{s_j} -1^i q^{\frac{1}{2}(2\lambda_0-\mu, 2\lambda_0-(i+1)\alpha_j+(i+2)\omega_j)} \begin{bmatrix} s_j \\ i \end{bmatrix}_q g^{\mu-2s_j\omega_j+i\alpha_j}_{0,2(\lambda_0-s_j\omega_j)}.$$

Notice that at this point, all terms are $g^{\mu-2s_j\omega_j+\eta}_{0,2(\lambda_0-s_j)}$ where $\eta \in \mathbb{Z}_+\alpha_j$, and $g^{\mu-2s_j\omega_j+\eta}_{0,2(\lambda_0-s_j)} \neq 0$ implies $(2\lambda_0 - \mu - \eta, \omega_j) \leq (2(\lambda_0 - s_j\omega_j), \alpha_j) = 0$. Hence, only one term will be non-zero, namely for $(2\lambda_0 - \mu, \omega_j) = m$,

$$g^{\mu}_{0,2\lambda_0} = -1^m q^{\frac{1}{2}(2\lambda_0-\mu, 2\lambda_0-(m+1)\alpha_j+(m+2)\omega_j)} \begin{bmatrix} s_j \\ m \end{bmatrix}_q g^{\mu-2s_j\omega_j+m\alpha_j}_{0,2(\lambda_0-s_j\omega_j)}.$$

We can further iterate, utilizing the same process for any $k \neq j$ such that $(\lambda_0 - s_j\omega_j, \alpha_k) = s_k > 0$, until we have exhausted λ_0. Note that the only term that remains non-zero by the end of this process is $g^{\mu-2\lambda_0+(2\lambda_0-\mu)}_{0,0} = 1$. Hence we arrive at the equation

$$g^{\mu}_{0,2\lambda_0} = -1^{(\lambda-\mu,\rho)} q^{\frac{1}{2}(2\lambda_0-\mu, \mu+(2\lambda_0-\mu)^\vee+\rho)} \prod_{i=1}^{n} \begin{bmatrix} (\lambda_0, \alpha_i) \\ (2\lambda_0-\mu, \omega_i) \end{bmatrix}_q. \tag{5.2.1}$$

Now we increase the height of λ_1. When $\operatorname{ht}\lambda_1 = 1$, then $\lambda_1(h_m) = 1$ for some $1 \leq m \leq n$, so by (3.2.3) $g^{\mu}_{0,\lambda} = g^{\mu-\lambda_1}_{0,2\lambda_0}$. Together with (5.2.1), we have a closed form. Moving to $\operatorname{ht}\lambda_1 \geq 2$ becomes significantly more complicated.

To begin working with these weights, we need to introduce some new sets. We will define $\Sigma_s(\lambda)$ for $\lambda \in P^+(1)$ recursively. Firstly, $\Sigma_s(\lambda) = \Sigma^0_s(\lambda) \bigcup \Sigma^1_s(\lambda)$, $\Sigma^0_0(\lambda) = \{\lambda\}$, and $\Sigma^1_0(\lambda) = \emptyset$. For our fundamental weights, $\Sigma^1_s(\omega_m) = \Sigma^0_s(\omega_m) = \emptyset$ for all $m \in [0, n]$ and $s > 0$. Also, $\Sigma^1_s(\omega_n + \omega_{n-1}) = \Sigma^0_s(\omega_n + \omega_{n-1}) = \emptyset$ for all $s > 0$ For $\operatorname{ht}\lambda \geq 2$ and $s > 0$, where $m = \min\lambda$ and $p = \min(\lambda - \omega_m)$,

$$\Sigma^0_s(\lambda) = \{\omega_m\} + \Sigma_s(\lambda - \omega_m)$$

If $\lambda(h_{p+1}) = 0$, for $s > 0$,

$$\Sigma^1_s(\lambda) = \Sigma^0_{s-1}(\lambda - \alpha_{m,p}).$$

If $\lambda_1(h_{p+1}) = 1$, for $s > 0$,

$$\Sigma_1^1(\lambda) = \{\lambda - \alpha_{m,p}\}$$

$$\Sigma_s^1(\lambda) = \{2\omega_{p+1} + \Sigma_{s-1}^0(\lambda - 2\omega_{p+1} - \alpha_{m,p})\} \bigcup \{2\omega_{p+1} - \alpha_{p+1} + \Sigma_{s-2}^0(\lambda - 2\omega_{p+1} - \alpha_{m,p})\}$$

when $p \neq n-2$ or $\lambda(h_n + h_{n-1}) \neq 2$, and

$$\Sigma_s^1(\lambda) = \{2\omega_{n-1} + 2\omega_n + \Sigma_{s-1}^0(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-2})\}$$

$$\bigcup \{2\omega_{n-1} + 2\omega_n - \alpha_{n-1} + \Sigma_{s-2}^0(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-2})\}$$

when $p = n-2$ and $\lambda(h_n + h_{n-1}) = 2$.

We recall that if $\Sigma_s^r(\lambda) = \emptyset$ then set addition means $\{\nu\} + \Sigma_s^r(\lambda) = \emptyset$. Examples for the type A weights can be found in [1], but we will also provide a few more for the type D weights in Appendix B.4. We introduce these sets to provide conditions for μ that force the following properties:

Lemma 5.2.1 *Take $\lambda \in P^+(1)$ and $\mu \in \Sigma_s(\lambda)$ for $s \geq 0$. Then*

$$\mu(h_k) = 0 \quad \forall k < \min(\lambda) - 1; \qquad \mu(h_{\min(\lambda)-1}) \leq 1; \qquad \text{and } \lambda - \mu \in \sum_{k \geq \min(\lambda)-1} \mathbb{Z}_+ \alpha_k.$$

Moreover, $\Sigma_s^r(\lambda)$ and $\Sigma_{s'}^{r'}(\lambda)$ are disjoint if $(s,r) \neq (s',r')$.

Proof. We prove the above using an induction on the height of λ, beginning with the $\operatorname{ht}\lambda = 0$. In this case,

$$\Sigma_s(0) = \Sigma_s^0(0) \bigcup \Sigma_s^1(0) = \emptyset,$$

so our statements are vacuously true.

Observing the case of $\operatorname{ht}\lambda = 1$, $\lambda = \omega_m$ for some $m \in [1, n]$. Here

$$\Sigma_s^0(\omega_m) \bigcup \Sigma_s^1(\omega_m) = \begin{cases} \omega_m & s = 0 \\ \emptyset & \text{else.} \end{cases}$$

Here, $\mu(h_k) = 0$ for all $k < m-1$, $\lambda - \mu = 0 = \sum_{k \geq i} 0\alpha_k$ when $s = 0$, and $\Sigma_s^r(\lambda) = \emptyset$ otherwise, so clearly these sets are disjoint.

Examining the case of $\operatorname{ht}\lambda = 2$, $\lambda = \omega_m + \omega_p$, so

$$\Sigma_s^0(\omega_m + \omega_p) \bigcup \Sigma_s^1(\omega_m + \omega_p) = \begin{cases} \{\omega_m + \omega_p\} & s = 0 \\ \{\lambda - \alpha_{m,p}\} & s = 1, m \neq n-1 \\ \emptyset & \text{else} \end{cases}$$

In all of the above cases, $\mu(h_k) = 0$ for all $k < m-1$, $\mu(h_{m-1}) \leq 1$, $\lambda - \mu = \delta_{s,1}(1 - \delta_{m,n-1})\alpha_{m,p}$ for $\mu \in \Sigma_s^r$, and $\Sigma_s^r(\lambda) = \emptyset$ expect for $(s, r) \in \{(0,0), (1,1)\}$, which are disjoint. Therefore all claims of our lemma are satisfied.

For sake of our induction, we shall assume that all claims hold for $\operatorname{ht}\lambda < M$. Taking $\operatorname{ht}\lambda = M$ with $\min(\lambda) = m$ and $\min(\lambda - \omega_m) = p$,

$$\Sigma_s^0(\lambda) \bigcup \Sigma_s^1(\lambda) = \{\omega_m + \Sigma_s(\lambda - \omega_m)\} \bigcup \Sigma_s^1(\lambda).$$

Depending on λ, $\Sigma_s^1(\lambda)$ is defined one of three ways as described previously. We will observe that in all three of these cases, our claims will still hold true.

Beginning with the case when $\lambda(h_{p+1}) = 0$,

$$\Sigma_s^1(\lambda) = \Sigma_{s-1}^0(\lambda - \alpha_m, p) = \omega_{m-1} + \Sigma_{s-1}(\lambda - \alpha_{m,p} - \omega_{m-1}), \text{ so}$$

$$\Sigma_s(\lambda) = \{\omega_m + \Sigma_s(\lambda - \omega_m)\} \bigcup \{\omega_{m-1} + \Sigma_{s-1}(\lambda - \alpha_{m,p} - \omega_{m-1})\}.$$

Hence for all $\mu \in \Sigma_s(\lambda)$, either $\mu = \omega_m + \mu'$ where $\mu' \in \Sigma_s(\lambda - \omega_m)$ or $\mu = \omega_{m-1} + \mu''$ where $\mu'' \in \Sigma_{s-1}(\lambda - \alpha_{m,p} - \omega_{m-1})$. Note both $\mathrm{ht}(\lambda - \omega_m)$ and $\mathrm{ht}(\lambda - \alpha_{m,p} - \omega_{m-1})$ are less than M with $m < \min(\lambda - \omega_m) < \min(\lambda - \alpha_{m,p} - \omega_{m-1})$, so inductive hypothesis provides that $\omega_m + \mu'$ and $\omega_{m-1} + \mu''$ satisfy all of the claims of our lemma, forcing μ to as well.

Next we examine the case when $\lambda(h_{p+1}) = 1$ with $p \neq n-2$, here when $s = 1$

$$\Sigma_1^1(\lambda) = \{\lambda - \alpha_{m,p}\}$$

which makes

$$\Sigma_1(\lambda) = \{\omega_m + \Sigma_1(\lambda - \omega_m)\} \bigcup \{\lambda - \alpha_{m,p}\},$$

so for all $\mu \in \Sigma_1(\lambda)$, $\mu(h_k) = 0$ for all $k < m-1$, $\mu(h_k) \leq 1$, and $\lambda - \mu = \delta_{r,1}\alpha_{m,p}$ for $\mu \in \Sigma_s^r(\lambda)$. When $s > 1$ however,

$$\begin{aligned}\Sigma_s^1(\lambda) =& \{2\omega_{p+1} + \Sigma_{s-1}^0(\lambda - 2\omega_{p+1} - \alpha_{m,p})\} \bigcup \{2\omega_{p+1} - \alpha_{p+1} + \Sigma_{s-2}^0(\lambda - 2\omega_{p+1} - \alpha_{m,p})\} \\ =& \{2\omega_{p+1} + \omega_{m-1} + \Sigma_{s-1}(\lambda - 2\omega_{p+1} - \alpha_{m,p} - \omega_{m-1})\} \\ & \bigcup \{2\omega_{p+1} - \alpha_{p+1} + \omega_{m-1} + \Sigma_{s-2}(\lambda - 2\omega_{p+1} - \alpha_{m,p} - \omega_{m-1})\}.\end{aligned}$$

Hence any $\mu \in \Sigma_s(\lambda)$ is of one of three forms:

$$\mu = \begin{cases} \omega_m + \mu' & \mu' \in \Sigma_s(\lambda - \omega_m) \\ \omega_{m-1} + 2\omega_{p+1} + \mu' & \mu' \in \Sigma_{s-1}(\lambda - 2\omega_{p+1} - \alpha_{m,p} - \omega_{m-1}) \\ \omega_{m-1} + 2\omega_{p+1} - \alpha_{p+1} + \mu' & \mu' \in \Sigma_{s-2}(\lambda - 2\omega_{p+1} - \alpha_{m,p} - \omega_{m-1}). \end{cases}$$

Note that $\mathrm{ht}(\lambda - \omega_m)$ and $\mathrm{ht}(\lambda - 2\omega_{p+1} - \alpha_{m,p} - \omega_{m-1})$ are both less than M, and $m < \min(\lambda - \omega_m) < \min(\lambda - 2\omega_{p+1} - \alpha_{m,p} - \omega_{m-1})$. Thus the inductive hypothesis provides that each form of μ satisfies the claims of our lemma.

Lastly, we examine the case when $\lambda(h_{p+1}) = 1$ with $p = n-2$. Here when $s = 1$

$$\Sigma_1^1(\lambda) = \{\lambda - \alpha_{m,n-1}\}.$$

This makes

$$\Sigma_1(\lambda) = \{\omega_m + \Sigma_1(\lambda - \omega_m)\} \bigcup \{\lambda - \alpha_{m,n-1}\},$$

so for all $\mu \in \Sigma_1(\lambda)$, $\mu(h_k) = 0$ for all $k < m-1$, $\mu(h_{m-1}) \leq 1$, and $\lambda - \mu = \delta_{r,1}\alpha_{m,n-1}$ for $\mu \in \Sigma_1^r(\lambda)$. When $s > 1$ however,

$$\begin{aligned}\Sigma_s^1(\lambda) =& \{2\omega_{n-1} + 2\omega_n + \Sigma_{s-1}^0(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1})\}\\ &\bigcup \{\omega_{n-2} + 2\omega_n + \Sigma_{s-2}^0(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1})\}\\ =& \{2\omega_{n-1} + 2\omega_n + \omega_{m-1} + \Sigma_{s-1}(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1} - \omega_{m-1})\}\\ &\bigcup \{\omega_{n-2} + 2\omega_n + \omega_{m-1} + \Sigma_{s-2}^0(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1} - \omega_{m-1})\}.\end{aligned}$$

Hence any $\mu \in \Sigma_s(\lambda)$ is of one of three forms:

$$\mu = \begin{cases} \omega_m + \mu' & \mu' \in \Sigma_s(\lambda - \omega_m) \\ \omega_{m-1} + 2\omega_{n-1} + 2\omega_n + \mu' & \mu' \in \Sigma_{s-1}(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1} - \omega_{m-1}) \\ \omega_{m-1} + \omega_{n-2} + 2\omega_n + \mu' & \mu' \in \Sigma_{s-2}(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1} - \omega_{m-1}). \end{cases}$$

Note that $\mathrm{ht}(\lambda - \omega_m)$ and $\mathrm{ht}(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1} - \omega_{m-1})$ are both less than M, and $m < \min(\lambda - \omega_m) < \min(\lambda - 2\omega_{n-1} - 2\omega_n - \alpha_{m,n-1} - \omega_{m-1})$. Thus the inductive hypothesis provides that each form of μ satisfies the claims of our lemma. ■

Having the properties of these sets better understood, we make a few observations. Firstly, that for $\mathrm{ht}\,\lambda_1 \leq 2$, $g_{0,\lambda}^{\mu}$ can be rewritten using (5.1.1), and the result is equal to

$$\sum_{\nu \in \Sigma_0(\lambda_1)} (q^{(\lambda - \mu - (\lambda_1 - \nu), \nu)} g_{0,2\lambda_0}^{\mu - \nu}) - \sum_{\nu \in \Sigma_1(\lambda_1)} (q^{(\lambda_0, \lambda_1 - \nu) + (\lambda - \mu - (\lambda_1 - \nu), \nu)} g_{0,2\lambda_0}^{\mu - \nu}).$$

Working with our exponent on q,

$$(\lambda_0, \lambda_1 - \nu) + (\lambda - \mu - (\lambda_1 - \nu), \nu) = \frac{1}{2}(\lambda, \lambda_1) + \frac{1}{2}(2\lambda_0 - 2\mu + \nu, \nu),$$

and so we arrive at the understanding that

$$g^{\mu}_{0,\lambda} = q^{\frac{1}{2}(\lambda,\lambda_1)} \sum_{s\geq 0}(-1)^s \sum_{\nu\in\Sigma_s(\lambda_1)} q^{\frac{1}{2}(2\lambda_0-2\mu+\nu,\nu)} g^{\mu-\nu}_{0,2\lambda_0} \tag{5.2.2}$$

where the second sum is taken to be 0 if $\Sigma_s(\lambda_1) = \emptyset$. With (5.2.1), this gives the closed form.

For $\operatorname{ht}\lambda_1 > 2$, we utilize an induction. Assuming that (5.2.2) holds for $\operatorname{ht}\lambda_1 < M$, we apply (5.1.1) to $\operatorname{ht}\lambda_1 = M$:

$$g^{\mu}_{0,\lambda} = q^{(\lambda-\mu,\omega_m)} g^{\mu-\omega_m}_{0,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1+(\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}$$

Note that $\operatorname{ht}(\lambda_1 - \omega_m) = \operatorname{ht}(\lambda_1 - \alpha_{m,p} - \omega_{m-1}) = M - 1$. Therefore, we apply the inductive hypothesis to each term and the exponents of our q's can be simplified in the following ways:

$$\begin{aligned}(\lambda - \mu, \omega_m) + \frac{1}{2}(\lambda - \omega_m, \lambda_1 - \omega_m) + \frac{1}{2}(2\lambda_0 - 2\mu + \nu, \nu)& \\ =&\frac{1}{2}(\lambda, \lambda_1) + \frac{1}{2}(2\lambda_0 - 2\mu + \nu + \omega_m, \nu + \omega_m)\end{aligned}$$

$$\begin{aligned}(\lambda_0, \alpha_{m,p}) + 1 + (\lambda - \mu - \alpha_{m,p}, \omega_{m-1})& \\ +\frac{1}{2}(\lambda - \alpha_{m,p} - \omega_{m-1}, \lambda_1 - \alpha_{m,p} - \omega_{m-1}) + \frac{1}{2}(2\lambda_0 - 2\mu + \nu, \nu)& \\ =\frac{1}{2}(\lambda, \lambda_1) + \frac{1}{2}(2\lambda_0 - 2\mu + \nu + \omega_{m-1}, \nu + \omega_{m-1}).&\end{aligned}$$

Hence our equation simplifies to

$$g^{\mu}_{0,\lambda} = \sum_{s\geq 0}(-1)^s(q^{\frac{1}{2}(\lambda\lambda_1)}\big[\sum_{\nu\in\Sigma_s(\lambda_1-\omega_m)} q^{\frac{1}{2}(2\lambda_0-2\mu+\nu+\omega_m,\nu+\omega_m)}g^{\mu-\nu-\omega_m}_{0,2\lambda_0}$$

$$-\sum_{\nu\in\Sigma_s(\lambda_1-\alpha_{m,p}-\omega_{m-1})} q^{\frac{1}{2}(2\lambda_0-2\mu+\nu+\omega_{m-1},\nu+\omega_{m-1})}g^{\mu-\nu-\omega_{m-1}}_{0,2\lambda_0}\big].$$

Using the definitions of our $\Sigma^r_s(\lambda)$ sets, this equation is equivalent to

$$g^{\mu}_{0,\lambda} = q^{\frac{1}{2}(\lambda,\lambda_1)}\sum_{s\geq 0}(-1)^s\sum_{\nu\in\Sigma_s(\lambda_1)} q^{\frac{1}{2}(2\lambda_0-2\mu+\nu,\nu)}g^{\mu-\nu}_{0,2\lambda_0}.$$

This concludes our induction, and we combine (5.2.2) with (5.2.1) to arrive at a closed form for $g^{\mu}_{0,\lambda}$.

5.3 Properties of $h^{\mu}_{\nu,0}$ and p^{μ}_{ν}

In this section, we discuss some recursions of p^{μ}_{ν} and $h^{\mu}_{\nu,0}$. First we shall refresh that so long as $\nu-\mu\in Q^+$,

$$p^{\mu}_{\nu}(q) = q^{\frac{1}{2}(\nu+\mu_1,\nu-\mu)}\prod_{j=1}^{n}\begin{bmatrix}(\nu-\mu,\omega_j)+(\mu_0,\alpha_j)\\ (\nu-\mu,\omega_j)\end{bmatrix}_q.$$

We also remind the reader that $\begin{bmatrix}m\\ n\end{bmatrix}_q = \begin{bmatrix}m-1\\ n\end{bmatrix}_q + q^{m-n}\begin{bmatrix}m-1\\ n-1\end{bmatrix}_q$. Together these mean that

$$\text{if } \nu(h_j)\geq 2,\qquad p^{\mu}_{\nu} = q^{(\omega_j,\nu-\mu)}p^{\mu-\omega_j}_{\nu-\omega_j} + q^{(\nu,\alpha_j)-1}p^{\mu}_{\nu-\alpha_j}$$

$$\text{, and if } \mu_1(h_m)=0,\qquad p^{\mu+\omega_m}_{\nu+\omega_m} = q^{(\omega_m,\nu-\mu)}p^{\mu}_{\nu}.$$

Our aim is to show that $h^{\mu}_{\nu,0}$ adheres to the same recursions, identifying $h^{\mu}_{\nu,0}$ and p^{μ}_{ν}. To do this we must first discuss some properties of $h^{\mu}_{\nu,0}$.

Lemma 5.3.1 *Let $\nu \in P^+$ and $0 \le r \le n$ be the index of the fundamental weight such that $\nu - \omega_r \in Q^+$. Then*

$$h^{\omega_k}_{\nu,0} = q^{\frac{1}{2}(\nu+\omega_r,\nu-\omega_r)}\delta_{k,r} \qquad 0 \le k \le n.$$

Proof. We will prove the above lemma inductively, progressing ν upward through the partial order on P^+. Firstly, if $\nu = \omega_r$ for some $0 \le r \le n$, then $\omega_r - \omega_k \in Q^+$ only if $r = k$, so $h^{\omega_k}_{\nu,0} = \delta_{k,r}$.

Let us assume our claim for $\omega_m \prec \nu' \prec \nu$ with $\operatorname{ht}\nu \ge 2$. Recall that $\delta_{\nu,\omega_k} = \sum_{\mu\in P^+} g^{\mu}_{0,\nu} h^{\omega_k}_{\mu,0}$ and from Proposition 5.1.2 *a)*, we see that,

$$q^{\frac{1}{2}(\nu+\omega_r,\nu-\omega_r)} + \sum_{\mu\prec\nu} q^{\frac{1}{2}(\mu+\omega_r,\mu-\omega_r)} g^{\mu}_{0,\nu} = 0 = h^{\omega_k}_{\nu,0} + \delta_{k,r}\sum_{\mu\prec\nu} q^{\frac{1}{2}(\nu+\omega_r,\nu-\omega_r)} g^{\mu}_{0,\nu}$$

hence $h^{\omega_k}_{\nu,0} = q^{\frac{1}{2}(\nu+\omega_r,\nu-\omega_r)}\delta_{r,k}$. ■

Lemma 5.3.2 *If $(\nu,\lambda) \in P^+$ are compatible or $(\nu,\lambda) = (\omega_m,\lambda)$ with $\min\lambda = m$, then*

$$h^{\mu}_{\nu,\lambda} = \sum_{\mu'\in P^+} q^{(\lambda+\nu-\mu',\nu)} g^{\mu'-\nu}_{0,\lambda} h^{\mu}_{\mu',0}.$$

Proof. Recall (3.2.5) and note that if we use (3.2.5) to substitute for $\operatorname{ch}_{\mathrm{gr}} M(\mu,0)$ and use Lemma 5.1.1 we have

$$\operatorname{ch}_{\mathrm{gr}} M(\nu,\lambda) = \sum_{\mu,\mu'\in P^+} q^{(\lambda+\nu-\mu',\nu)} g^{\mu'-\nu}_{0,\lambda} h^{\mu}_{\mu',0} \operatorname{ch}_{\mathrm{gr}} M(0,\mu).$$

Taking this summation along with the second sum of (3.2.5) we see that

$$\sum_{\mu\in P^+} h^{\mu}_{\nu,\lambda} \operatorname{ch}_{\mathrm{gr}} M(0,\mu) = \sum_{\mu,\mu'\in P^+} q^{(\lambda+\nu-\mu',\nu)} g^{\mu'-\nu}_{0,\lambda} h^{\mu}_{\mu',0} \operatorname{ch}_{\mathrm{gr}} M(0,\mu)$$

so the claim is achieved by identifying the coefficients of $\operatorname{ch}_{\mathrm{gr}} M(0,\mu)$ in both sums. ■

Before stating the next lemma, recall Proposition 5.1.1. It provides us with many equations for $h^{\mu}_{\nu,\lambda}$ dependant upon the form of a compatible pair (ν,λ).

Lemma 5.3.3 *For $(\nu, \lambda) \in P^+ \times P^+$ which are compatible, $1 \leq k \leq n$,*

$$h^{\mu+2\omega_k}_{\nu,\lambda+2\omega_k} = q^{(\lambda+\nu-\mu,\omega_k)} h^{\mu}_{\nu,\lambda},$$

hence when $\nu(h_j) \geq 2$ for $1 \leq j \leq n$,

$$h^{\mu}_{\nu,\lambda} = q^{(\lambda+\nu-\mu,\omega_j)} h^{\mu-2\omega_j}_{\nu-2\omega_j,\lambda} + q^{(\nu+\lambda_0,\alpha_j)-1} h^{\mu}_{\nu-\alpha_j,\lambda}.$$

Proof. We will prove this lemma using an induction on the partial order on compatible pairs, beginning with the minimal elements $(0, \omega_i)$.

$$h^{\mu+2\omega_k}_{0,\omega_i+2\omega_k} = \delta_{\mu+2\omega_k,\omega_i+2\omega_k} = \delta_{\mu,\omega_i} = h^{\mu}_{0,\omega_i} = q^{(\omega_i-\mu,\omega_k)} h^{\mu}_{0,\omega_i}.$$

Assuming our inductive hypothesis for all $(\nu', \lambda') \prec (\nu, \lambda)$, we must examine each possible form of the pair (ν, λ).

Firstly, is $\nu(h_j) \geq 2$, we utilize the first equation of Proposition 5.1.1 and our inductive hypothesis to see

$$h^{\mu}_{\nu,\lambda} = h^{\mu}_{\nu-2\omega_j,\lambda+2\omega_j} + q^{(\nu+\lambda_0,\alpha_j)-1} h^{\mu}_{\nu-\alpha_j,\lambda} = q^{-(\lambda+\nu-\mu,\omega_k)} h^{\mu+2\omega_k}_{\nu,\lambda+2\omega_k}.$$

In a similar fashion, we will utilize the second equation of Proposition 5.1.1 and the inductive hypothesis when $\nu \in P^+(1)$, $m = \max \nu < \min \lambda = p$. Here

$$h^{\mu}_{\nu,\lambda} = h^{\mu}_{\nu-\omega_m,\lambda+\omega_m} + q^{(\lambda_0,\alpha_m,p)+1} h^{\mu}_{\nu-\omega_m+\omega_{m-1},\lambda-\omega_p+\omega_{p+1}} = q^{-(\lambda+\nu-\mu,\omega_k)} h^{\mu+2\omega_k}_{\nu,\lambda+2\omega_k}.$$

The second claim of this lemma can be seen by applying the first claim to the first term in the equation for $\nu(h_j) \geq 2$ like so

$$h^{\mu}_{\nu,\lambda} = h^{\mu}_{\nu-2\omega_j,\lambda+2\omega_j} + q^{(\nu+\lambda_0,\alpha_j)-1} h^{\mu}_{\nu-\alpha_j,\lambda} = q^{(\lambda+\nu-\mu,\omega_j)} h^{\mu-2\omega_j}_{\nu-2\omega_j,\lambda} + q^{(\nu+\lambda_0,\alpha_j)-1} h^{\mu}_{\nu-\alpha_j,\lambda}.$$

■

Lemma 5.3.4 *Take* $\lambda = 2\lambda_0 + \lambda_1 \in P^+$ *and* $m < \min \lambda_1$ *if* $\lambda_0 \neq 0$ *or* $m \leq \min \lambda_1$ *if* $\lambda_0 = 0$. *Then* $h^{\mu}_{\omega_m,\lambda} = 0$ *for* $\mu \in P^+$ *with* $\mu(h_m) \in 2\mathbb{Z}_+ + 1$ *and* $\lambda + \omega_m \neq \mu$.

Proof. In the case that $m < \min \lambda_1 = p$, we can examine the contrapositive of this statement, namely that if $h^{\mu}_{\omega_m,\lambda} \neq 0$ then $\mu(h_m) \in 2\mathbb{Z}_+$ or $\lambda + \omega_m = \mu$. It can be shown that a stronger statement is true,

$$h^{\mu}_{\omega_m,\lambda} \neq 0 \implies \mu(h_s) \in 2\mathbb{Z}_+ \text{ for all } m \leq s \leq p, \text{ or } \lambda + \omega_m = \mu$$

We will prove this statement by inducting on m.

When $m = 0$, observe that $h^{\mu}_{0,\lambda} = \delta_{\mu,\lambda}$, hence when $h^{\mu}_{0,\lambda} \neq 0$, $\lambda = \mu$. Now moving to $1 \leq m \leq n-2$, we assume our statement inductively for all $s < m$, and we know from Proposition 5.1.1 that

$$h^{\mu}_{\omega_m,\lambda} = h^{\mu}_{0,\omega_m+\lambda} + q^{\frac{1}{2}(\lambda+\omega_m,\alpha_{m,p})} h^{\mu}_{\omega_{m-1},\lambda-\omega_p+\omega_{p+1}},$$

so $h^{\mu}_{\omega_m,\lambda} \neq 0$ implies that either $\omega_m + \lambda = \mu$ or $h^{\mu}_{\omega_{m-1},\lambda-\omega_p+\omega_{p+1}} \neq 0$. If $h^{\mu}_{\omega_{m-1},\lambda-\omega_p+\omega_{p+1}} \neq 0$, our inductive hypothesis gives that either $\lambda + \omega_m - \alpha_{m,p} = \mu$ or $\mu(h_s) \in 2\mathbb{Z}_+$ for all $m-1 \leq s \leq p+1$. Note that as $(\lambda + \omega_m - \alpha_{m,p})(h_m) \in 2\mathbf{Z}_+$, we have achieved the desired result.

We still must consider when $m = \min \lambda_1$ with $\lambda_0 = 0$. We again work inductively to prove our lemma, increasing on m. Here we use Proposition 5.1.1 to see that

$$h^{\mu}_{\omega_m,\lambda} = h^{\mu}_{0,\lambda+\omega_m} + q h^{\mu}_{\omega_{m-1},\lambda-\omega_m+\omega_{m+1}},$$

and so $h^{\mu}_{\omega_m,\lambda} \neq 0$ implies that $\lambda + \omega_m = \mu$ or that $h^{\mu}_{\omega_{m-1},\lambda-\omega_m+\omega_{m+1}} \neq 0$. Our induction shows that if $h^{\mu}_{\omega_{m-1},\lambda-\omega_m+\omega_{m+1}} \neq 0$ then $\lambda - \alpha_m + \omega_m = \mu$ or $\mu(h_s) \in 2\mathbb{Z}_+$ for all $m-1 \leq s \leq m+1$. Note that $(\lambda - \alpha_m + \omega_m)(h_m) = 0$, so our statement and lemma are proven. ∎

5.4 Identifying $h^\mu_{\nu,0}$ and p^μ_ν

In this section we will prove that for all $\nu, \mu \in P^+$, $h^\mu_{\nu,0} = p^\mu_\nu$ by comparing the recursive properties discussed in the last section. Firstly, note that if $\nu - \mu \notin Q^+$, then $h^\mu_{\nu,0} = 0 = p^\mu_\nu$ by definition. From this point, $\nu - \mu \in Q^+$ and we will induct on the $\mathrm{ht}_r(\nu - \mu)$.

If $\mathrm{ht}_r(\nu - \mu) = 0$, then $\nu = \mu$, so by definition $h^\mu_{\nu,0} = 1 = p^\mu_\nu$. For sake of our induction, we assume that our statement holds for all $\mathrm{ht}_r(\nu - \mu) < N$.

To examine the case when $\mathrm{ht}_r(\nu - \mu) = N$, we will induct on $\mathrm{ht}(\mu)$. Note that if $\mathrm{ht}(\mu) \le 1$, Lemma 5.3.1 states that $h^\mu_{\nu,0} = q^{\frac{1}{2}(\nu+\mu,\nu-\mu)}\delta_{\mu,\omega_r} = p^\mu_\nu$. We assume our statement holds for $1 \le \mathrm{ht}\,\mu \le s-1$, and examine $\mathrm{ht}\,\mu = s$. Here our proof splits into different cases based on the form of ν.

First, if $\nu(h_j) \ge 2$ for some j, then we utilize Lemma 5.3.3 and both induction assumptions to see that

$$h^\mu_{\nu,0} = q^{(\nu-\mu,\omega_j)} h^{\mu-2\omega_j}_{\nu-2\omega_j,0} + q^{(\nu,\alpha_j)-1} h^\mu_{\nu-\alpha_j,0} = q^{(\nu-\mu,\omega_j)} p^{\mu-2\omega_j}_{\nu-2\omega_j} + q^{(\nu,\alpha_j)-1} p^\mu_{\nu-\alpha_j} = p^\mu_\nu.$$

Next, if $\nu \in P^+(1)$, we take $m = \min\nu$ and examine the parity of $\mu(h_m)$. If $\mu(h_m) \in 2\mathbf{Z}_+ + 1$, then by Lemma 5.3.4 and Lemma 5.3.2, we can re-index to the sum,

$$0 = h^\mu_{\omega_m,\nu-\omega_m} = \sum_{\mu' \le \nu-\omega_m} q^{(\nu-\mu'-\omega_m,\omega_m)} g^{\mu'}_{0,\nu-\omega_m} h^\mu_{\mu'+\omega_m,0}.$$

When $\mu' \prec \nu - \omega_m$, $\mathrm{ht}_r(\mu' + \omega_m - \mu) < \mathrm{ht}_r(\nu - \mu)$, so our inductive hypothesis implies $h^\mu_{\mu'+\omega_m,0} = q^{(\omega_m,\mu'-\mu+\omega_m)} h^{\mu-\omega_m}_{\mu',0}$. Thus we can rewrite the prior sum and utilize (3.2.6) on the pair $(\nu - \omega_m, \mu - \omega_m)$ to see that $0 = \delta_{\nu-\omega_m,\mu-\omega_m} = \sum_{\mu' \in P^+} g^{\mu'}_{0,\nu-\omega_m} h^{\mu-\omega_m}_{\mu',0}$, so

$$q^{(\nu-\mu,\omega_m)} \sum_{\mu' \in P^+} g^{\mu'}_{0,\nu-\omega_m} h^{\mu-\omega_m}_{\mu',0} = h^\mu_{\nu,0} + q^{(\nu-\mu,\omega_m)} \sum_{\mu' \prec \nu-\omega_m} g^{\mu'}_{0,\nu-\omega_m} h^{\mu-\omega_m}_{\mu',0}.$$

Thus

$$h^{\mu}_{\nu,0} = q^{(\nu-\mu,\omega_m)} h^{\mu-\omega_m}_{\nu-\omega_m,0} = q^{(\nu-\mu,\omega_m)} p^{\mu-\omega_m}_{\nu-\omega_m} = p^{\mu}_{\nu}.$$

Lastly, we consider if $\mu(h_m) \in 2\mathbb{Z}_+$. Since we know that $(\nu + \omega_m)(h_m) = 2$, we can utilize our work in that case. Namely we see $h^{\mu+\omega_m}_{\nu+\omega_m,0} = p^{\mu+\omega_m}_{\nu+\omega_m} = q^{(\nu-\mu,\omega_m)} p^{\mu}_{\nu}$. Also, since $\mu + \omega_m(h_m) \in 2\mathbb{Z}_+ + 1$, we can follow the same steps as the prior case to arrive at

$$q^{(\nu-\mu,\omega_m)} h^{\mu}_{\nu,0} = h^{\mu+\omega_m}_{\nu+\omega_m,0} = q^{(\nu-\mu,\omega_m)} p^{\mu}_{\nu}$$

Hence our inductions conclude, and $h^{\mu}_{\nu,0} = p^{\mu}_{\nu}$.

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Appendix A

Sequence Computations

A.1 Proposition 3.3.2*ia)*

For Lemma 4.1.1, we need to show that $\alpha = \alpha_j + \gamma_1 + \gamma_2 + \gamma_3$ where $\gamma_i \in R^+ \bigcup \{0\}$ and commute for all $\alpha \in R^+$ of the form:

$$\alpha = \begin{cases} \alpha_j & \\ \alpha_{i,k} & 1 \leq i \leq j \leq k \leq n \\ \beta_{i,k} & 1 \leq i \leq j < k \leq n-1 \\ \beta_{i,k} & 1 \leq i \leq k \leq j. \end{cases}$$

If $\alpha = \alpha_j$, $\gamma_i = 0$ and we are done. For the other cases:

- If $\alpha = \alpha_{i,k}$, then $\alpha - \alpha_j = \alpha_{i,j-1} + \alpha_{j+1,k}$ if $j \neq n-2$ or $k \neq n$. In the case of $j = n-2$ and $k = n$, then this is instead $\alpha - \alpha_{n-2} = \alpha_{i,n-3} + \alpha_n$.
- If $\alpha = \beta_{i,k}$ where $1 \leq i \leq j < k \leq n-1$ and $k \neq j+1$, then $\alpha - \alpha_j = \alpha_{i,j-1} + \beta_{j+1,k}$. When $k = j+1$, $\alpha - \alpha_j = \alpha_{i,j-1} + \beta_{j+1,j+2} + \alpha_{j+1}$ instead.

- If $\alpha = \beta_{i,k}$ where $1 \leq i \leq j = k$ and $j \neq n-1$, then $\alpha - \alpha_j = \beta_{i,j+1} \in R^+$. In the case $j = n-1$, $\alpha - \alpha_j = \alpha_{i,n} \in R^+$.

- Lastly, if $\alpha = \beta_{i,k}$ where $1 \leq i < k < j$ and $j \neq n-1$, then $\alpha - \alpha_j = \beta_{i,j+1} + \alpha_{k,j-1}$. When $j = n-1$, then $\alpha - \alpha_j = \alpha_{i,n-2} + \alpha_{k,n}$.

In all of the above cases, we see that $[x^-_{\gamma_i}, x^-_{\gamma'_i}] = 0$, and so the elements commute with each other.

For Lemma 4.1.2, we need the show (4.1.2) is satisfied by $(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$. We include computations for each case of $\alpha \in R^+$ here for the reader's convenience. For $\alpha_j(h_\alpha) = 0$, the commutation behaves as follows:

$$(x^-_\alpha \otimes t^{(\nu-\alpha_j+\lambda_0,\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda} =$$
$$(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})(x^-_\alpha \otimes t^{(\nu+\lambda_0,\alpha)+\max((\zeta_{1,\lambda},\alpha),(\zeta_{2,\lambda},\alpha))})w_{\nu+\lambda} = 0.$$

For $\alpha_j(h_\alpha) = -1$, the commutation behaves like this for some $c \in \mathbb{C}$:

$$\begin{aligned}&(x^-_\alpha \otimes t^r)(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}\\ &= c(x^-_{\alpha'} \otimes t^s)w_{\nu+\lambda} +\\ &\quad (x^-_j \otimes t^{(\nu(h_j)+\lceil \frac{\lambda(h_j)}{2} \rceil -1})(x^-_\alpha \otimes t^r)w_{\nu+\lambda} = 0.\end{aligned}$$

For $\alpha_j(h_\alpha) = 2$, using Garland's Lemma 2.1.2 we see that

$$\begin{aligned}&(x^-_j \otimes t^{(\lambda_0,\alpha)})(x^-_j \otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu+\lambda}\\ &= \frac{1}{2}\sum_{n_1+n_2=2(\lambda_0,\alpha)-1}(x^-_j \otimes t^{n_1})(x_j \otimes t^{n_2})w_{\nu+\lambda}\\ &= \frac{1}{2}x^-_j(2, 2(\lambda_0,\alpha)-1)w_{\nu+\lambda} = 0.\end{aligned}$$

For $\alpha_j(h_\alpha) = 1$, the commutation behaves like this for some $c \in \mathbb{C}$:

$$c(x_\alpha^- \otimes t^r)(x_j^- \otimes t^{(\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$$
$$= (x_j^- \otimes t^{(\lambda_0,\alpha_j)})(x_\gamma^- \otimes t^s)(x_j^- \otimes t^{(\lambda_0,\alpha_j)-1})w_{\nu+\lambda}$$
$$- (x_\gamma^- \otimes t^s)(x_j^- \otimes t^{(\lambda_0,\alpha_j)})(x_j^- \otimes t^{(\lambda_0,\alpha_j)-1})w_{\nu+\lambda} = 0.$$

A.2 Proposition 3.3.2*ib)*

For Lemma 4.2.1, we need to show that $\alpha = \alpha_{m,p}+\gamma_1+\gamma_2+\gamma_3$ where $\gamma_i \in R^+ \bigcup \{0\}$ and commute for all $\alpha \in R^+$ of the form:

$$\alpha = \begin{cases} \alpha_{i,k} & 1 \le i \le m < p \le k \le n \\ \beta_{i,k} & 1 \le i \le m < p < k \le n-1 \\ \beta_{i,k} & 1 \le i < m < k \le p \\ \beta_{i,k} & 1 \le i < k \le m. \end{cases}$$

Similar to the previous sequence, if $\alpha = \alpha_{m,p}$, then $\gamma_i = 0$ and we are done. For the rest of the cases:

- If $\alpha = \alpha_{i,k}$ with $i \le m < p \le k$, $p \ne n-2$ or $k \ne n$ then $\alpha - \alpha_{m,p} = \alpha_{i,m-1} + \alpha_{p+1,k}$. When $p = n-2$ and $k = n$, the above is instead $\alpha - \alpha_{m,n-2} = \alpha_{i,m-1} + \alpha_n$.

- If $\alpha = \beta_{i,k}$ with $1 \le i \le m < p < k \le n-1$ with $k \ne p+1$, then $\alpha - \alpha_{m,p} = \alpha_{i,m-1} + \beta_{p+1,k}$. When $k = p+1$, $\alpha - \alpha_{m,p} = \alpha_{i,m-1} + \beta_{p+1,p+2} + \alpha_{p+1}$.

- If $\alpha = \beta_{i,k}$ with $1 \le i < m \le k \le p \ne n-1$, then $\alpha - \alpha_{m,p} = \alpha_{i,m-1} + \beta_{k,p+1}$. If $\alpha = \beta_{i,n-1}$ and $p = n-1$, then $\alpha - \alpha_{m,n-1} = \alpha_{i,m-1} + \alpha_n$.

- Lastly, if $\alpha = \beta_{i,k}$ with $1 \le i < k \le m$ and $p \ne n$, then $\alpha - \alpha_{m,p} = \alpha_{i,m-1} + \alpha_{k,n-2} + \beta_{p+1,n-1}$. When $p = n$, this changes slightly to $\alpha - \alpha_{m,n} = \alpha_{i,m-1} + \alpha_{k,n-1}$.

In all of these cases, we see that $[x^-_{\gamma_i}, x^-_{\gamma'_i}] = 0$, and so the elements commute with each other.

For Lemma 4.2.2, we need the show (4.2.2) is satisfied by $(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_j)+1})w_{\nu+\lambda}$. We include computations for each case of $\alpha \in R^+$ here for the reader's convenience. For $\alpha_{m,p}(h_\alpha) = 0$, the commutation behaves as follows:

$$(x^-_\alpha \otimes t^{(\nu+\omega_m+\lambda_0,\alpha)+\max((\zeta_{1,\lambda}-\omega_m,\alpha),(\zeta_{2,\lambda},\alpha))})(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_m,p)+1})w_{\nu+\lambda} =$$

$$(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_m,p)+1})(x^-_\alpha \otimes t^{(\nu+\omega_m+\lambda_0,\alpha)+\max((\zeta_{1,\lambda}-\omega_m,\alpha),(\zeta_{2,\lambda},\alpha))})w_{\nu+\lambda} = 0.$$

For $\alpha_{m,p}(h_\alpha) = -1$, the commutation behaves like this for some $c \in \mathbb{C}$:

$$(x^-_\alpha \otimes t^r)(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_m,p)+1})w_{\nu+\lambda} =$$

$$c(x^-_{\alpha+\alpha_{m,p}} \otimes t^s)w_{\nu+\lambda} + (x^-_{m,p} \otimes t^{(\lambda_0,\alpha_m,p)+1})(x^-_\alpha \otimes t^r)w_{\nu+\lambda} = 0.$$

For $\alpha_{m,p}(h_\alpha) = 2$, using Garland's Lemma 2.1.2 we see that

$$\begin{aligned}
&(x^-_{m,p} \otimes t^{(\lambda_0,\alpha)})(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} \\
&= \frac{1}{2} \sum_{n_1+n_2=2(\lambda_0,\alpha)+1} (x^-_{m,p} \otimes t^{n_1})(x^-_{m,p} \otimes t^{n_2})w_{\nu+\lambda} \\
&= \frac{1}{2} x^-_{m,p}(2, 2(\lambda_0,\alpha)+1)w_{\nu+\lambda} = 0.
\end{aligned}$$

For $\alpha_{m,p}(h_\alpha) = 1$ with $\alpha = \alpha_{m,p} + \gamma$, the commutation behaves like this for some $c \in \mathbb{C}$:

$$\begin{aligned}
&c(x^-_\alpha \otimes t^r)(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} \\
&= (x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})})(x^-_\gamma \otimes t^s)(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} \\
&\quad - (x^-_\gamma \otimes t^s)(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})})(x^-_{m,p} \otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda} = 0.
\end{aligned}$$

For $\alpha_{m,p}(h_\alpha)=1$ with $\alpha_{m,p}=\alpha+\gamma$, the commutation behaves like this for some $c\in\mathbb{C}$:

$$
\begin{aligned}
&(x_\alpha^-\otimes t^r)(x_{m,p}^-\otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda}\\
&=\frac{1}{2}((x_\alpha^-\otimes t^r)(x_{m,p}^-\otimes t^{(\lambda_0,\alpha_{m,p})+1})w_{\nu+\lambda}+(x_{m,p}^-\otimes t^{(\lambda_0,\alpha_{m,p})+1})(x_\alpha^-\otimes t^r)w_{\nu+\lambda})\\
&=\frac{1}{2}((x_\alpha^-\otimes t^r)^2(x_\gamma^-\otimes t^{s+1})w_{\nu+\lambda}+(x_\gamma^-\otimes t^{s+1})(x_\alpha^-\otimes t^r)^2w_{\nu+\lambda})=0.
\end{aligned}
$$

A.3 Proposition 3.3.2*ii)*

For Lemma 4.3.2 and Lemma 4.3.4 we need the show (4.3.2), (4.3.3), and (4.3.4) are satisfied by $(x_m^-\otimes t)w_{\nu+\lambda}$. We include computations for each case of $\alpha\in R^+$ here for the reader's convenience.

First, if $m\notin\{n-1,n\}$ the cases are as follows: If $\omega_m(h_\alpha)=\omega_{m-1}(h_\alpha)=\omega(h_\alpha)$, there exists a $c\in\mathbb{C}$ so that we commute as follows

$$
\begin{aligned}
&(x_\alpha^-\otimes t^r)(x_m^-\otimes t)w_{\lambda+2\omega_m}=\\
&c(x_{\alpha'}^-\otimes t^s)w_{\lambda+2\omega_m}+(x_m^-\otimes t)(x_\alpha^-\otimes t^r)w_{\lambda+2\omega_m}=0.
\end{aligned}
$$

If $\omega_m(h_\alpha)=\omega_{m-1}(h_\alpha)=\omega_{m+1}(h_\alpha)-1$ or $\omega_m(h_\alpha)=\omega_{m+1}(h_\alpha)=\omega_{m-1}(h_\alpha)-1$, the commutation behaves like this for some $c\in\mathbb{C}$

$$
\begin{aligned}
&(x_\alpha^-\otimes t^r)(x_m^-\otimes t)w_{\lambda+2\omega_m}\\
&=c(x_{\alpha'}^-\otimes t^s)w_{\lambda+2\omega_m}+(x_m^-\otimes t)(x_\alpha^-\otimes t^r)w_{\lambda+2\omega_m}=0.
\end{aligned}
$$

If $\omega_m(h_\alpha)-1=\omega_{m-1}(h_\alpha)=\omega_{m+1}(h_\alpha)=0$, using Garland's Lemma 2.1.2 we see that

$$
\begin{aligned}
(x_m^-\otimes 1)(x_m^-\otimes t)w_{\lambda+2\omega_m}&=\frac{1}{2}\sum_{n_1+n_2=1}(x_m\otimes t^{n_1})(x_m^-\otimes t^{n_2})w_{\lambda+2\omega_m}\\
&=\frac{1}{2}x_m^-(2,1)w_{\lambda+2\omega_m}=0.
\end{aligned}
$$

If $\omega_m(h_\alpha) = \omega_{m-1}(h_\alpha) = \omega_{m+1}(h_\alpha) + 1$ or $\omega_m(h_\alpha) = \omega_{m+1}(h_\alpha) = \omega_{m-1}(h_\alpha) + 1$, the commutation behaves like this for some $c \in \mathbb{C}$

$$\begin{aligned} &c(x_\alpha^- \otimes t^r)(x_m^- \otimes t)w_{\lambda+2\omega_m} \\ &\qquad =(x_m^- \otimes 1)(x_\gamma^- \otimes t^s)(x_m^- \otimes t)w_{\lambda+2\omega_m} - (x_\gamma^- \otimes t^s)(x_m^- \otimes 1)(x_m^- \otimes t)w_{\lambda+2\omega_m} = 0. \end{aligned}$$

If $m \in \{n-1, n\}$ the cases are as follows: For $\alpha_m(h_\alpha) = 0$, the commutation behaves as follows:

$$(x_\alpha^- \otimes t^r)(x_m^- \otimes t)w_{\lambda+2\omega_m} = (x_m^- \otimes t)(x_\alpha^- \otimes t^r)w_{\nu+\lambda} = 0.$$

For $\alpha_m(h_\alpha) = -1$, the commutation behaves like this for some $c \in \mathbb{C}$:

$$\begin{aligned} &(x_\alpha^- \otimes t^r)(x_m^- \otimes t)w_{\lambda+2\omega_m} \\ &\qquad =c(x_{\alpha'}^- \otimes t^s)w_{\lambda+2\omega_m} + (x_m^- \otimes t)(x_\alpha^- \otimes t^r)w_{\lambda+2\omega_m} = 0. \end{aligned}$$

For $\alpha_m(h_\alpha) = 2$, using Garland's Lemma 2.1.2 we see that

$$\begin{aligned} (x_m^- \otimes t^r)(x_m^- \otimes t)w_{\lambda+2\omega_m}0 =& \frac{1}{2}\sum_{n_1+n_2=2r+1}(x_m^- \otimes t^{n_1})(x_m^- \otimes t^{n_2})w_{\lambda+2\omega_m} \\ =& \frac{1}{2}x_m^-(2, r+1)w_{\lambda+2\omega_m} = 0. \end{aligned}$$

For $\alpha_j(h_\alpha) = 1$, the commutation behaves like this for some $c \in \mathbb{C}$:

$$\begin{aligned} &(x_\alpha^- \otimes t^r)(x_m^- \otimes t)w_{\lambda+2\omega_m} \\ &\quad = (x_m^- \otimes 1)(x_{i,n-2}^- \otimes t^s)(x_m^- \otimes t)w_{\lambda+2\omega_m} - (x_{i,n-2}^- \otimes t^s)(x_m^- \otimes 1)(x_m^- \otimes t)w_{\lambda+2\omega_m} = 0. \end{aligned}$$

Appendix B

Polynomial Computations

B.1 Proposition 5.1.1

An example of $\nu(h_j) \geq 2$ for $j \in [1, n]$, $b^{\mu}_{\nu,\lambda} = b^{\mu}_{\nu-2\omega_j,\lambda+2\omega_j} + q^{(\nu+\lambda_0,\alpha_j)-1} b^{\mu}_{\nu-\alpha_j,\lambda}$:

From Proposition 3.3.2 *ia)*, $M(\nu, \lambda) \cong_{\mathfrak{g}[t]} M(\nu - 2\omega_j, \lambda + 2\omega_j) \oplus \tau^*_{(\nu+\lambda_0,\alpha_j)-1} M(\nu - \alpha_j, \lambda)$. Therefore $\mathrm{ch}_{\mathrm{gr}}\, M(\nu, \lambda) = \mathrm{ch}_{\mathrm{gr}}\, M(\nu - 2\omega_j, \lambda + 2\omega_j) + q^{(\nu+\lambda_0,\alpha_j)-1} \mathrm{ch}_{\mathrm{gr}}\, M(\nu - \alpha_j, \lambda)$, and since $\mathrm{ch}_{\mathrm{gr}}\, M(\nu, \lambda) = \sum g^{\mu}_{\nu,\lambda} M(\mu, 0) = \sum h^{\mu}_{\nu,\lambda} M(0, \mu)$, we can identify the coefficients of $M(\mu, 0)$(resp. $M(0, \mu)$) on either side of this equality. Doing so, we see that

$$b^{\mu}_{\nu,\lambda} = b^{\mu}_{\nu-2\omega_j,\lambda+2\omega_j} + q^{(\nu+\lambda_0,\alpha_j)-1} b^{\mu}_{\nu-\alpha_j,\lambda}$$

where $b \in \{g, h\}$. The proof of the other three equalities is identical.

B.2 Lemma 5.1.1

For Proposition 5.1.1 we need to show that $g^{\mu}_{\nu,\lambda} = q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}$ for compatible pairs (ν, λ), and we are doing this by inducting on the height of λ_1. We include the full computations here for the reader.

When $\operatorname{ht}\lambda_1 = 0$, we can use Proposition 5.1.1*ia)* and the inductive hypothesis to see

$$\begin{aligned}
g^{\mu}_{\nu,\lambda} &= g^{\mu}_{\nu+2\omega_j,\lambda-2\omega_j} - q^{(\nu+\lambda_0+\omega_j)(h_j)-1} g^{\mu}_{\nu+2\omega_j-\alpha_j,\lambda-2\omega_j} \\
&= q^{(\nu+\lambda-\mu,\nu+2\omega_j)} g^{\mu-\nu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\nu+\lambda_0,\alpha_j)+(\nu+\lambda-\mu-\alpha_j,\nu+2\omega_j-\alpha_j)} g^{\mu+\alpha_j-\nu-2\omega_j}_{0,\lambda-2\omega_j} \\
&= q^{(\nu+\lambda-\mu,\nu)}(q^{(\nu+\lambda-\mu,2\omega_j)} g^{\mu-\nu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(\nu+\lambda-\mu,2\omega_j-\alpha_j)} g^{\mu+\alpha_j-\nu-2\omega_j}_{0,\lambda-2\omega_j}) \\
&= q^{(\nu+\lambda-\mu,\nu)}(g^{\mu-\nu}_{2\omega_j,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)} g^{\mu-\nu}_{2\omega_j-\alpha_j,\lambda-2\omega_j}) \\
&= q^{(\nu+\lambda-\mu,\nu)}(g^{\mu-\nu}_{2\omega_j,\lambda-2\omega_j} - q^{(2\omega_j+\lambda_0,\alpha_j)-1} g^{\mu-\nu}_{2\omega_j-\alpha_j,\lambda-2\omega_j}) \\
&= q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}.
\end{aligned}$$

When $\operatorname{ht}\lambda_1 = 1$, then $\lambda = 2\lambda_0 + \omega_m$ for some $m \in [1, n]$, so by (3.2.3) $M(\nu, 2\lambda_0 + \omega_m) \cong M(\nu + \omega_m, 2\lambda_0)$, $(\nu + \omega_m, 2\lambda_0)$ is a compatible pair, and $\operatorname{ht}\lambda > \operatorname{ht}(2\lambda_0)$ so

$$g^{\mu}_{\nu,2\lambda_0+\omega_m} = g^{\mu}_{\nu+\omega_m,2\lambda_0} = q^{(\nu+\lambda-\mu,\nu+\omega_m)} g^{\mu-\nu-\omega_m}_{0,2\lambda_0} = q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{\omega_m,2\lambda_0} = q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}$$

using the inductive hypothesis.

When $\operatorname{ht}\lambda_1 = 2$, then $\lambda_1(h_{m,p}) = 2$ for $1 \le m < p \le n$. The case when $\operatorname{ht}\lambda_1 = 2$ with $m = n-1$ is immediate from (3.2.4) and the inductive hypothesis, so we show the computation for the other instances. In these instances, $(\nu+\omega_m, \lambda-\omega_m)$ and $(\nu+\omega_{m-1}, \lambda-\alpha_{m,p}-\omega_{m-1})$ are compatible pairs with $\operatorname{ht}\lambda > \operatorname{ht}(\lambda-\omega_m) = \operatorname{ht}(\lambda-\alpha_{m,p}-\omega_{m-1})$, so we can use Proposition 5.1.1*ib)* and the inductive hypothesis to see that

$$\begin{aligned} g^{\mu}_{\nu,\lambda} &= g^{\mu}_{\nu+\omega_m,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1} g^{\mu}_{\nu+\omega_{m-1},\lambda-\alpha_{m,p}-\omega_{m-1}} \\ &= q^{(\nu+\lambda-\mu,\nu+\omega_m)} g^{\mu-\nu-\omega_m}_{0,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1+(\nu+\lambda-\mu-\alpha_{m,p},\nu+\omega_{m-1})} g^{\mu-\nu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}} \\ &= q^{(\nu+\lambda-\mu,\nu)} (q^{(\nu+\lambda-\mu,\omega_m)} g^{\mu-\nu-\omega_m}_{0,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1+(\nu+\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g^{\mu-\nu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}) \\ &= q^{(\nu+\lambda-\mu,\nu)} (g^{\mu-\nu}_{\omega_m,\lambda-\omega_m} - q^{(\lambda_0,\alpha_{m,p})+1} g^{\mu-\nu}_{\omega_{m-1},\lambda-\alpha_{m,p}-\omega_{m-1}}) \\ &= q^{(\nu+\lambda-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}. \end{aligned}$$

B.3 Proposition 5.1.2

We need to compute the steps of our induction for each piece of Proposition 5.1.2. For part *a)*, we suppose that $\sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda} = 0$ for $\operatorname{ht}\lambda < M$. Taking $\lambda \in P^+(1)$ with $\operatorname{ht}\lambda = M$,

$$\begin{aligned} \sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda} = \sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} (q^{(\lambda-\mu,\omega_m)} g^{\mu-\omega_m}_{0,\lambda-\omega_m} \\ - (1-\delta_{p,0}) q^{(\lambda_0,\alpha_{m,p})+1+(\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}} \\ = \sum_{\mu\in P} q^{\frac{1}{2}(\mu-\omega_m,\mu-\omega_m)+(\lambda,\omega_m)} g^{\mu-\omega_m}_{0,\lambda-\omega_m} \\ - (1-\delta_{p,0}) q^{\frac{1}{2}(\mu-\omega_{m-1},\mu-\omega_{m-1})+(\lambda_0,\alpha_{m,p})+1+(\lambda-\alpha_{m,p},\omega_{m-1})} g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}} = 0 \end{aligned}$$

by (5.1.1) and the inductive hypothesis. Taking $\lambda \notin P^+(1)$ with $\operatorname{ht}\lambda = M$,

$$\begin{aligned} \sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda} &= \sum_{\mu\in P} q^{\frac{1}{2}(\mu,\mu)} (q^{(\lambda-\mu,2\omega_j)} g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(\lambda-\mu-\alpha_j,2\omega_j-\alpha_j)} g^{\mu+\alpha_j-2\omega_j}_{0,\lambda-2\omega_j}) \\ &= \sum_{\mu\in P} q^{\frac{1}{2}(\mu-2\omega_j,\mu-2\omega_j)} (q^{(\lambda,2\omega_j)} g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(\lambda-\alpha_j,2\omega_j-\alpha_j)} g^{\mu+\alpha_j-2\omega_j}_{0,\lambda-2\omega_j} = 0 \end{aligned}$$

by (5.1.2) and the inductive hypothesis.

For part *c)* we are attempting to simplify (5.1.1), so we make note that by using part *b)*,

$$g_{0,\lambda-\omega_m}^{\mu-\omega_m} \neq 0 \implies (\lambda-\mu,\omega_m) \leq (\lambda-\omega_m,\alpha_m) = 0,$$

$$g_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}^{\mu-\omega_{m-1}} \neq 0 \implies (\lambda-\alpha_{m,p}-\mu,\omega_{m-1}) \leq (\lambda-\alpha_{m,p}-\omega_{m-1},\alpha_{m-1}) = 0.$$

, and so (5.1.1) simplifies to

$$g_{0,\lambda}^{\mu} = q^{(\lambda-\mu,\omega_m)} g_{0,\lambda-\omega_m}^{\mu-\omega_m} - q^{1+(\lambda-\mu-\alpha_{m,p},\omega_{m-1})} g_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}^{\mu-\omega_{m-1}}$$
$$= g_{0,\lambda-\omega_m}^{\mu-\omega_m} - q g_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}^{\mu-\omega_{m-1}}.$$

Similarly for part *d)* we are attempting to simplify (5.1.2), so we make note that by using *b)*,

$$g_{0,\lambda-2\omega_j}^{\mu-2\omega_j} \neq 0 \implies (\lambda-\mu,\omega_j) \leq (\lambda-2\omega_j,\alpha_j) = 0,$$

$$g_{0,\lambda-2\omega_j}^{\mu+\alpha_j-2\omega_j} \neq 0 \implies (\lambda-\mu-\alpha_j,\omega_j) \leq (\lambda-2\omega_j,\alpha_j) = 0.$$

we have that $(\lambda-\mu,\omega_j) = (\lambda-\mu-\alpha_j,\omega_j) = 0$

B.4 Examples of $\Sigma_s^r(\lambda)$ for type-D weights

Here we provide a few type-D examples of our sets $\Sigma_s^r(\lambda)$. Take $\lambda = \omega_{n-1}+\omega_n$, then

$$\Sigma_s^r(\omega_{n-1}+\omega_n) = \begin{cases} \{\omega_{n-1}+\omega_n\} & ; r=0, s=0 \\ \emptyset & ; \text{else.} \end{cases}$$

If $\lambda = \omega_m + \omega_{n-1} + \omega_n$, then

$$\Sigma_s^0(\omega_m + \omega_{n-1} + \omega_n) = \{\omega_m + \omega_{n-1} + \omega_n\} \quad ; s = 0$$

$$\Sigma_s^1(\omega_m + \omega_{n-1} + \omega_n) = \begin{cases} \{\omega_{m-1} + 2\omega_n\} & ; s = 1 \\ \{\omega_{m-1} + \omega_{n-2}\} & ; s = 2 \\ \emptyset & ; \text{else} \end{cases}$$

If $\lambda = \omega_m + \omega_p + \omega_{n-1} + \omega_n$, then

$$\Sigma_s^0(\omega_m + \omega_p + \omega_{n-1} + \omega_n) = \begin{cases} \{\omega_m + \omega_p + \omega_{n-1} + \omega_n\} & ; s = 0 \\ \{\omega_m + \omega_{p-1} + 2\omega_n\} & ; s = 1 \\ \{\omega_m + \omega_{p-1} + \omega_{n-2}\} & ; s = 2 \\ \emptyset & ; \text{else} \end{cases}$$

$$\Sigma_s^1(\omega_m + \omega_p + \omega_{n-1} + \omega_n) = \begin{cases} \{\omega_{m-1} + \omega_{p+1} + \omega_{n-1} + \omega_n\} & ; s = 1, p \neq n-2 \\ \{\omega_{m-1} + 2\omega_{n-1} + 2\omega_n\} & ; s = 1, p = n-2 \\ \{\omega_{m-1} + \omega_p + 2\omega_n\} & ; s = 2 \\ \{\omega_{m-1} + \omega_p + \omega_{n-2}\} & ; s = 3, p \neq n-2 \\ \emptyset & ; \text{else} \end{cases}$$

Lastly, if $\lambda = \omega_m + \omega_p + \omega_\ell + \omega_{n-1} + \omega_n$, then

$$\Sigma_s^0(\omega_m + \omega_p + \omega_\ell + \omega_{n-1} + \omega_n)$$
$$= \begin{cases} \{\omega_m + \omega_p + \omega_\ell + \omega_{n-1} + \omega_n\} & ; s = 0 \\ \{\omega_m + \omega_p + \omega_{\ell-1} + 2\omega_n, \omega_m + \omega_{p-1} + \omega_{\ell+1} + \omega_{n-1} + \omega_n\} & ; s = 1, \ell \neq n-2 \\ \{\omega_m + \omega_p + \omega_{\ell-1} + 2\omega_n, \omega_m + \omega_{p-1} + 2\omega_{n-1} + 2\omega_n\} & ; s = 1, \ell = n-2 \\ \{\omega_m + \omega_p + \omega_{\ell-1} + \omega_{n-2}, \omega_m + \omega_{p-1} + \omega_\ell + 2\omega_n\} & ; s = 2 \\ \{\omega_m + \omega_{p-1} + \omega_\ell + \omega_{n-2}\} & ; s = 3, \ell \neq n-2 \\ \emptyset & ; \text{else} \end{cases}$$

$$\Sigma_s^1(\omega_m + \omega_p + \omega_\ell + \omega_{n-1} + \omega_n)$$
$$= \begin{cases} \{\omega_{m-1} + \omega_{p+1} + \omega_\ell + \omega_{n-1} + \omega_n\} & ; s = 1 \\ \{\omega_{m-1} + \omega_{p+1} + \omega_{\ell-1} + 2\omega_n, & ; s = 2, \ell \neq p+1, \ell \neq n-2 \\ \quad \omega_{m-1} + \omega_p + \omega_{\ell+1} + \omega_{n-1} + \omega_n\} & \\ \{\omega_{m-1} + \omega_{p+1} + \omega_{\ell-1} + 2\omega_n, & ; s = 2, \ell \neq p+1, \ell = n-2 \\ \quad \omega_{m-1} + \omega_p + 2\omega_{n-1} + 2\omega_n\} & \\ \{\omega_{m-1} + 2\omega_{p+1} - \alpha_{p+1} + \omega_{n-1} + \omega_n\} & ; s = 2, \ell = p+1 \\ \{\omega_{m-1} + \omega_{p+1} + \omega_{\ell-1} + \omega_{n-2}, \omega_{m-1} + \omega_p + \omega_\ell + 2\omega_n\} & ; s = 3 \\ \{\omega_{m-1} + \omega_p + \omega_\ell + \omega_{n-2}\} & ; s = 4, \ell \neq p+1, \ell \neq n-2 \\ \emptyset & ; \text{else} \end{cases}$$