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Authors
Calafiore, Giuseppe
El Ghaoui, Laurent

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Robust Dynamic Traffic Assignment under Demand and Capacity Uncertainty*

Giuseppe Calafiore† Laurent El Ghaoui‡

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Abstract

In this note we discuss the system optimum dynamic traffic assignment (SO-DTA) problem in the presence of time-dependent uncertainty on the demands and link capacities. We start from the deterministic linear programming formulation of the SO-DTA problem introduced by Ziliaskopoulos in [14], and then consider a robust version of the decision problem where the total travel time must be minimized under a worst-case scenario of demand and capacity configurations. When uncertain demands and capacities are modeled as unknown-but-bounded quantities restricted in intervals, the resulting robust decision problem can still be formulated as a linear program and solved at the same computational cost as its nominal counterpart. Worst-case solutions to assignment problems appear to be useful for establishing routing policies that are resilient to possibly large variations of network parameters and for computing lower bounds on network performance in extreme situations.

1 Introduction

Traffic assignment problems deal with management of vehicle flows in road networks and aim at determining flows in some "optimal" way so to minimize some system global or local cost functional, while satisfying practical constrains such as capacity and sustainable traffic limitations on the network links. Various models and related solution techniques have been proposed for these problems in the past, including mathematical programming models, [1]. This kind of models stem from and early discretized-time formulation of Merchant and Nemhauser [5, 6] and have been extensively studied thereafter, see for instance the recent survey in [9] and [7].

A critical issue in mathematical programming formulations of traffic assignment problems (as well as in other formulations) relates to the fact that the problem "input data,"

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†Politecnico de Torino, Italy.

‡EECS/IEOR, UC Berkeley.

such as origin-destination demand levels or effective capacities of links, are seldom known in advance with precision. If nominal or expected values of these parameters are used in an optimization model, the resulting optimal solution is likely to depend critically on these values and to yield potentially bad system performance when the actual system parameters deviate significantly from the nominal ones. Standard approaches for dealing with uncertainty include numerically-intensive scenario techniques that basically solve a large number of problem instances for randomly generated parameter patterns, see, e.g., [8]. While useful, scenario technique often lead to computationally demanding problems, that may not be easily solved already for medium scale networks.

Recently, other attempts have been proposed in the literature for accounting for data uncertainty. Notably, in [12] the authors study the impact of demand uncertainty on the evaluation of road network improvements and conclude that deterministic (nominal) traffic assignment procedures are likely to incorrectly rank improvement policies. Also, in [11], a static assignment problem with uncertain origin-destination demands is posed and solved by means of genetic programming, while the authors in [13] consider a dynamic setup and directly treat stochastic uncertainty in the mathematical problem by means of chance-constrained optimization.

In this report, we intend to explore the potentialities of robust optimization techniques (see, e.g., [2] and the references therein) for attacking traffic assignment problems in presence of uncertainty. In robust optimization, uncertain parameters in the mathematical program are assumed to be unknown-but-bounded quantities and solutions are seek such that the constraints are satisfied *for all* possible values of the parameters, which includes the worst-case scenarios. A cost objective which is optimal in this robust sense (say, for instance, the total travel time) provides in turn a performance level which is guaranteed for all the considered scenarios. To this purpose, we focus here on a specific linear programming model proposed in [14] for the system optimum dynamic traffic assignment problem (DTA), and show how this model can be posed and solved in a robust setting. Our preliminary results show that in this specific case robustness can be achieved at the same computational cost as the nominal solution. Section 6 also briefly discusses an analogous approach for static network flow optimization problems.

2 Problem setup

We start by reviewing the linear programming formulation of the DTA problem in [14], based on the cell transmission model of [3, 4]. A cell is a segment of a link of a street network having length equal to the distance traveled by a free-flowing vehicle during a fixed amount of time τ, which we shall henceforth assume to be equal to one without loss of generality. The whole network is described as the ensemble of all interconnected cells, and the state of the system at time t is given by the number of vehicles contained in each cell i at time t: x_i^t; $i \in \mathcal{C}$, $t = 1, 2, \ldots, T$, where $\mathcal{C}$ is the set of all cells, $C \doteq |\mathcal{C}|$, and T is the total time horizon over which the system is analyzed. A cell is characterized by several parameters and variables, as detailed in Table 1. Cells are classified according to the number of cells that precede or

Symbol	Description
N_i^t	maximum number of vehicles in cell i at time t
Q_i^t	maximum number of vehicles that can flow in or out cell i during the t-th time period
δ_i^t	free-flow to backward propagation speed ratio for cell i at time t
$\Gamma(i)$	set of successor cells to cell i
$\Gamma^{-1}(i)$	set of predecessor cells to cell i
d_i^t	demand (inflow) at cell i in the t-th time interval
x_i^t	number of vehicles in cell i at the t-th time interval
y_{ij}^t	number of vehicles moving from cell i to cell j at the t-th time interval

Table 1: Cell parameters and variables

follow them. In particular, the set $\mathcal{C}_o$ of ordinary cells is such that $\Gamma(i) = \Gamma^{-1}(i) = 1$ for $i \in \mathcal{C}_o$. Diverging cells have one predecessor and more than one successor, that is the set $\mathcal{C}_d$ of diverging cells is such that $\Gamma(i) > 1, \Gamma^{-1}(i) = 1$ for $i \in \mathcal{C}_d$. The set $\mathcal{C}_m$ of merging cells is such that $\Gamma(i) = 1, \Gamma^{-1}(i) > 1$ for $i \in \mathcal{C}_m$. Finally, source cells in $\mathcal{C}_r$ are such that $\Gamma(i) = 1, \Gamma^{-1}(i) = 0$ for $i \in \mathcal{C}_r$, and sink cells in $\mathcal{C}_s$ are such that $\Gamma(i) = 0, \Gamma^{-1}(i) = 1$ for $i \in \mathcal{C}_s$. Initial conditions on the system are set by x_i^0, $i \in \mathcal{C}$. Sink cells are assumed to have infinite capacity and allow infinite input flows ($N_i^t = \infty$, $Q_i^t = \infty$, for all t and all $i \in \mathcal{C}_s$). Source cells are assumed to have infinite capacity ($N_i^t = \infty$, for all t and all $i \in \mathcal{C}_r$).

In this setup, the objective of the system optimal dynamic traffic assignment (SO DTA) problem is to determine network states x_i^t and flows y_{ij}^t so to minimize the total travel time in the network, that is the travel time experienced by all vehicles in the network. Since x_i^t vehicles stay in the i-th cell for τ units of time, the cost objective to be minimized is (sink cells do not contribute to total system time)

$$J = \sum_{t=1}^{T} \sum_{i \in \mathcal{C} \setminus \mathcal{C}_s} \tau x_i^t. \tag{1}$$

The overall LP model for the single destination SO DTA problem has been developed in [14], and can be simplified reported next:

$$\begin{aligned}
\min \quad & J && (2)\\
\text{subject to,} \quad & \forall i, j \in \mathcal{C}, t = 1, \ldots, T: \\
& x_i^t = x_i^{t-1} + \textstyle\sum_{k \in \Gamma^{-1}(i)} y_{ki}^{t-1} - \sum_{j \in \Gamma(i)} y_{ij}^{t-1} + d_i^{t-1}, && (3)\\
& \textstyle\sum_{j \in \Gamma(i)} y_{ij}^t \leq x_i^t && (4)\\
& \textstyle\sum_{j \in \Gamma(i)} y_{ij}^t \leq Q_i^t && (5)\\
& \textstyle\sum_{i \in \Gamma^{-1}(j)} y_{ij}^t \leq Q_j^t && (6)\\
& \textstyle\sum_{i \in \Gamma^{-1}(j)} y_{ij}^t \leq \delta_j^t (N_j^t - x_j^t) && (7)\\
& x_i^t \geq 0 && (8)\\
& y_{ij}^t \geq 0, && (9)
\end{aligned}$$

with initial conditions $x_i^0 = \xi_i$, $y_{ij}^0 = 0$, $\forall i, j \in \mathcal{C}$, and where $d_i^t = 0$ for all non-source nodes. The meaning of constraints in the above LP is summarized as follows: equation (3) represents the flow balance at cells, equations (4), (5) impose that the output flow from cell i should be smaller than $\min(x_i^t, Q_i^t)$, whereas equations (6), (7) impose that the input flow to cell j should not exceed the maximum input flow capacity Q_j^t and the remaining capacity at the target node. The last two constraints impose that all states and flows be non negative.

2.1 Formulation in matrix notation

Let $Y(t)$ be a matrix with non-negative entries and such that $[Y(t)]_{ij} = y_{ij}^t = 0$ whenever $j \notin \Gamma(i)$, and denote by $\mathcal{Y}$ the set of non-negative matrices having this sparsity pattern. Then, the total flow out cell i at t is $Y_{i,:}(t)\mathbf{1}$, where $Y_{i,:}(t)$ denotes the i-th row of $Y(t)$ and $\mathbf{1}$ is a vector of ones, while the total flow in i at t is $Y_{i,:}^T(t)\mathbf{1}$, being $Y_{i,:}^T(t)$ the i-th row of $Y^T(t)$. Let also $x(t) = [x_1^t \cdots x_C^t]^T$ denote the state vector, $d(t) = [d_1^t \cdots d_C^t]^T$ denote the input (demand) vector, $Q(t) = [Q_1^t \cdots Q_C^t]^T$, $N(t) = [N_1^t \cdots N_C^t]^T$, $\Delta(t) = \mathrm{diag}(\delta_1^t, \ldots, \delta_C^t)$, and let e be a vector such that $e_i = 1$ if $i \notin \mathcal{C}_s$ and $e_i = 0$ otherwise. Then the SO DTA LP can be written in the equivalent matrix format

$$\min_{x(t)\geq 0, Y(t)\in\mathcal{Y}} \quad \sum_{t=1}^T e^T x(t) \tag{10}$$

subject to

$$\text{for } t = 1, \ldots, T:$$

$$x(t) = x(t-1) + [Y^T(t-1) - Y(t-1)]\mathbf{1} + d(t-1), \tag{11}$$

$$Y(t)\mathbf{1} \leq x(t) \tag{12}$$

$$Y(t)\mathbf{1} \leq Q(t) \tag{13}$$

$$Y^T(t)\mathbf{1} \leq Q(t) \tag{14}$$

$$Y^T(t)\mathbf{1} \leq \Delta(t)[N(t) - x(t)]. \tag{15}$$

2.1.1 Reduction of variables

Let now $f_{\text{in}} \doteq Y^T(t)\mathbf{1}$, $f_{\text{out}} \doteq Y(t)\mathbf{1}$ denote the vector of total inflows and total outflows at cells, respectively. Then, applying equation (11) recursively, we obtain

$$x(t) = x(0) + \sum_{\nu=0}^{t-1} (f_{\text{in}}(\nu) - f_{\text{out}}(\nu)) + \sum_{\nu=0}^{t-1} d(\nu), \quad t = 1, \ldots, T, \tag{16}$$

with $x(0) = \xi$, $f_{\text{in}}(0) = f_{\text{out}}(0) = 0$. Notice that, since under the standard assumption in [14] that diverging and merging cells cannot be directly connected, the knowledge of the total inflows and outflows at cells is sufficient for reconstructing all the individual flows in Y. Indeed, sources, sinks and ordinary cells have only one inflow or outflow link, that would obviously coincide to the corresponding entry in f_{in}, f_{out}. Diverging cells have multiple output flows, but each of them is uniquely determined by the corresponding inflow at the successor cells. Similarly, merging cells have multiple inflows, each of them corresponding to the single outflow of a predecessor cell.

Therefore, substituting (16) into (10)–(15) we obtain a linear program where the x variables have been eliminated and only the total flows $f_{\text{in}}(t)$, $f_{\text{out}}(t)$ appear. The individual flows $Y(t)$ can be reconstructed a posteriori on the basis of the network topology. If we let the accumulated demand up to time t be

$$\psi(t) \doteq \sum_{\nu=0}^{t} d(\nu), \tag{17}$$

then our LP takes the following form:

$$\min_{f_{\text{in}} \in \mathcal{F}_{\text{in}}, f_{\text{out}} \in \mathcal{F}_{\text{in}}, \gamma} \quad \gamma \tag{18}$$

subject to:

$$\textstyle\sum_{t=1}^{T} e^T \xi + \sum_{t=1}^{T} \sum_{\nu=0}^{t-1} e^T \left(f_{\text{in}}(\nu) - f_{\text{out}}(\nu) \right) + \sum_{t=1}^{T} e^T \psi(t-1) \leq \gamma \tag{19}$$

and, for $t = 1, \ldots, T$:

$$f_{\text{out}}(t) \leq \xi + \textstyle\sum_{\nu=0}^{t-1} \left(f_{\text{in}}(\nu) - f_{\text{out}}(\nu) \right) + \psi(t-1) \tag{20}$$

$$f_{\text{out}}(t) \leq Q(t) \tag{21}$$

$$f_{\text{in}}(t) \leq Q(t) \tag{22}$$

$$f_{\text{in}}(t) \leq \Delta(t) \left(N(t) - \xi + \textstyle\sum_{\nu=0}^{t-1} \left(f_{\text{in}}(\nu) - f_{\text{out}}(\nu) \right) + \psi(t-1) \right). \tag{23}$$

Stacking all variables in vector

$$X = \left[\begin{array}{c} f_{\text{in}}(1) \\ \vdots \\ f_{\text{in}}(T) \\ \hline f_{\text{out}}(1) \\ \vdots \\ f_{\text{out}}(T) \\ \hline \gamma \end{array} \right]$$

it can be observed that this LP can be written in the compact form

$$\min_{X \in \mathcal{X}} \quad c^T X \tag{24}$$

subject to:

$$AX \leq B + E\Theta, \tag{25}$$

where $c^T \doteq [0\, 0\, \cdots\, 0\, 1]$, and A, B, E are appropriate coefficient matrices that may be easily inferred from (18)–(23). Here, Θ is given by

$$\Theta \doteq \left[\begin{array}{c} \Psi \\ \Gamma \end{array} \right],$$

where Ψ represents the overall vector of accumulated demands and Γ the overall vector of cell capacities:

$$\Psi \doteq \left[\begin{array}{c} \psi(0) \\ \vdots \\ \psi(T-1) \end{array} \right], \quad \Gamma \doteq \left[\begin{array}{c} N(1) \\ \vdots \\ N(T) \end{array} \right].$$

If the demand history and the cell capacities are exactly known in advance, the SO DTA problem then reduces to solving the above LP in the total flux variables. In the next section we shall instead consider the situation when these input data are imprecisely known, and seek for solutions that are guaranteed against the worst-case scenario.

3 Traffic assignment under uncertainty

We consider the situation when accumulated demand and capacities in Θ are only known to belong to given intervals. Specifically, we assume that

$$\Theta_{\text{low}} \leq \Theta \leq \Theta_{\text{up}}$$

where Θ_{low}, Θ_{up} are vectors containing the lower and upper bounds on the entries of Θ, respectively. We are interested in determining a solution of the SO DTA problem which will be guaranteed to work in the worst-case scenario. That is, we seek a solution to the following *interval* LP problem:

$$\min_{X \in \mathcal{X}} \quad c^T X \tag{26}$$

subject to:

$$AX \leq B + E\Theta, \quad \forall \Theta : \ \Theta_{\text{low}} \leq \Theta \leq \Theta_{\text{up}}. \tag{27}$$

It turns out that this interval LP can be readily cast and hence solved as a standard LP with no additional computational effort, as specified in the next proposition.

Proposition 1 *A solution for the robust SO DTA in (28) can be obtained by solving the following LP:*

$$\min_{X \in \mathcal{X}} \quad c^T X \tag{28}$$

subject to:

$$AX \leq B + E_+\Theta_{\text{low}} - E_-\Theta_{\text{up}}, \tag{29}$$

where

$$[E_+]_{ij} = \begin{cases} [E]_{ij} & \text{if } [E]_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}, \quad [E_-]_{ij} = \begin{cases} -[E]_{ij} & \text{if } [E]_{ij} < 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Considering the i-th row of constraints (27), we have

$$\begin{aligned} A_{i,:}X - B_i &\leq \sum_j E_{ij}\Theta_j, \quad \forall\ \Theta_{\text{low},j} \leq \Theta_j \leq \Theta_{\text{up},j} \\ &\Updownarrow \\ A_{i,:}X - B_i &\leq \min_{\Theta_{\text{low}} \leq \Theta \leq \Theta_{\text{up}}} \sum_j E_{ij}\Theta_j \\ &= \sum_j \min_{\Theta_{\text{low},j} \leq \Theta_j \leq \Theta_{\text{up},j}} E_{ij}\Theta_j \\ &= \sum_j \begin{cases} E_{ij}\Theta_{\text{low},j} & \text{if } E_{ij} > 0 \\ E_{ij}\Theta_{\text{up},j} & \text{if } E_{ij} \leq 0 \end{cases} \end{aligned}$$

from which the statement easily follows.

4 Numerical examples

tbd.
The numerical example (six nodes, seven links) reported in [14] as well as the example (22 nodes, 25 links) in [13] might be reformulated and solved in the robust framework and results compared.

5 Further work

- Prove that constraint (11) can be replaced by

$$x(t) \geq x(t-1) + [Y^T(t-1) - Y(t-1)]\mathbf{1} + d(t-1)$$

  without changing the solution at optimum.

- Impose robustness in probabilistic sense.

- Static Network Assignment Problems (NAP). See next section.

6 Static network assignment problems with demand uncertainty

The static network flow assignment problem is defined as follows. Consider a graph $\mathcal{G}(V, E)$, where V is the set of nodes and E is the set of directed edges (links), and let y_k, q_k denote respectively the flow and the capacity on link $k \in E$. The travel time for a vehicle on a link bearing flow y_k is typically expressed by means of the so-called "Bureau of Public Roads function":

$$c_k(y_k) = c_{0k}\left(1 + \alpha_k\left(\frac{y_k}{q_k}\right)^{\beta_k}\right),$$

where c_{0k} is the free-flow travel time on the link, and $\alpha_k \geq 0$, $\beta_k \geq 1$ are link-specific parameters. The system-optimum assignment problem then aims at minimizing the overall travel time while satisfying a given set of origin-destination demand flows. For each node pair $(i, j) \in V \times V$, denote with $\mathcal{P}_{(i,j)}$ the set of possible network paths that lead from i to j. Let $h^k_{(i,j)}$ be the flow on the k-th of such paths, $k \in \mathcal{P}_{(i,j)}$, and $d_{(i,j)}$ be the input demand between origin i and destination j. Then, according to [10, 12] the system-optimum assignment problem can be formulated as follows.

$$\min_{h^k_{(i,j)}} \qquad \sum_{k\in E} y_k c_k(y_k)$$

subject to:

$$\begin{aligned} \textstyle\sum_{k\in\mathcal{P}_{(i,j)}} h^k_{(i,j)} &= d_{(i,j)}, \quad \forall\,(i,j)\in V\times V \\ h^k_{(i,j)} &\geq 0, \qquad \forall\,(i,j)\in V\times V; \forall k\in\mathcal{P}_{(i,j)} \\ y_k &= \textstyle\sum_{(i,j)\in E\times E}\sum_{\nu\in\mathcal{P}_{(i,j)}} h^\nu_{(i,j)}\delta^{k,\nu}_{(i,j)}, \quad \forall k\in E. \end{aligned}$$

where $\delta^{\nu,k}_{(i,j)} = 1$ if link k is present on the path ν from i to j. However, we shall next use an alternative and equivalent representation which takes origin-destination class flows on each link as independent variables. Let $M \subseteq V\times V$ denote the set of origin-destination pairs and, for $(i,j)\in M$ denote with $E^{(i,j)}$ the set of all edges involved in paths form i to j. Denote with $y^{(i,j)}_k$ the flow of vehicles with origin i and destination j that are engaging the k-th link, $k\in E^{(i,j)}$. Then, the total flow on link k is $y_k \doteq \sum_{(i,j)\in M} y^{(i,j)}_k$, and the total flow from i to j is $y^{(i,j)} = \sum_{k\in E} y^{(i,j)}_k$. Then, the system-optimum problem can be written as

$$\begin{aligned} \min \quad & \textstyle\sum_{k\in E} y_k c_k(y_k) \\ \text{subject to:} \quad & \\ & \textstyle\sum_{k\in E^{(i,j)}} y^{(i,j)}_k = d_{(i,j)}, \quad \forall\,(i,j)\in M \\ & y^{(i,j)}_k \geq 0, \quad \forall k\in E, \forall (i,j)\in M. \end{aligned}$$

Here, the demand fulfillment constraint can be substituted by an inequality without changing the optimal solution. Also, a capacity bound can be directly introduced in the constraints:

$$\begin{aligned} \min \quad & \textstyle\sum_{k\in E} y_k c_k(y_k) && (30) \\ \text{subject to:} \quad & && (31) \\ & \textstyle\sum_{k\in E^{(i,j)}} y^{(i,j)}_k \geq d_{(i,j)}, \quad \forall\,(i,j)\in M && (32) \\ & y^{(i,j)}_k \geq 0, \quad \forall k\in E, \forall (i,j)\in M && (33) \\ & \textstyle\sum_{(i,j)\in M} y^{(i,j)}_k \leq q_k, \quad \forall k\in E. && (34) \end{aligned}$$

Note that the objective function is convex for $\beta_k \geq 1$ and that the constrains are linear, hence (30) is a convex optimization problem. In particular, it is a convex quadratic programming problem if $\beta_k = 1\ \forall k$.

This problem is easily "robustifiable" both in case of deterministic interval uncertainty and stochastic uncertainty in demand. In case of interval bounds on demand

$$d^{\text{lb}}_{(i,j)} \leq d_{(i,j)} \leq d^{\text{ub}}_{(i,j)}$$

it clearly suffices to substitute (32) by

$$\sum_{k\in E^{(i,j)}} y^{(i,j)}_k \geq d^{\text{ub}}_{(i,j)}.$$

If $d_{(i,j)}$ is instead assumed to be random, then (32) can be substituted by a probability constraint of the form

$$\text{Prob}\left\{\sum_{k\in E^{(i,j)}} y_k^{(i,j)} \geq d_{(i,j)}\right\} \geq 1-\epsilon$$

where ϵ is the acceptable risk level. This form can be converted into an explicit convex constraint if $\epsilon < 0.5$ and if either the distribution or its first two moments are given.

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