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Jacobians and branch points of real analytic open maps

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Summary. The main result is that the Jacobian determinant of an analytic open map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ does not change sign. A corollary of the proof is that the set of branch points of f has dimension $\leq n - 2$.

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Introduction

The main object of this paper is to prove the following result:

Theorem 1. *The Jacobian of a real analytic open map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ does not change sign.*

One of the referees kindly pointed out that the special case of polynomial maps was proved by Gamboa and Ronga [3]:

Theorem 2 (Gamboa and Ronga). *A polynomial map in $\mathbb{R}^n$ is open if and only if point inverses are finite and the Jacobian does not change sign.*

The proof of Theorem 1 is very similar to methods in [3], which are easily adapted to analytic maps; but as Theorem 1 does not seem to be known, a direct proof may be useful.

$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes a (real) analytic map in Euclidean n -space. We always assume f is *open*, that is, f maps open sets onto open sets. Denote the Jacobian matrix of f at $p \in \mathbb{R}^n$ by $df_p = \left[\frac{\partial f_i}{\partial x_j}(p) \right]$. The rank of df_p is called the *rank* of f at p , denoted by $\text{rk}_p f$; the determinant of df_p is the *Jacobian* of f at p , denoted by $Jf(p)$. When the analytic function $Jf: \mathbb{R}^n \rightarrow \mathbb{R}$ is everywhere non-negative or everywhere non-positive (in a set X), we say Jf *does not change sign* (in X).

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The following sets are defined for any C^1 map $g: M \rightarrow N$ between n -manifolds (without boundary):

- the set $R_k = \{p \in M: \operatorname{rk}_p g \leq k\}$
- the *critical set*, $C = R_{n-1}$
- the *branch set*, $B = \{p \in U: g \text{ is not a local homeomorphism at } p\}$

Note that $B \subset C$ by the inverse function theorem. When g is analytic, we also define:

- the *critical analytic hypersurface* $H \subset C$, comprising those points having a neighborhood in C that is an analytic submanifold of dimension $n - 1$
- the *constant rank analytic hypersurface* $V \subset H$, at which $g|_H$ has locally constant rank

The following results are byproducts of the proof of Theorem 1:

Theorem 3.

- (i) *The restricted map $f|_V$ has rank $n - 1$,*
- (ii) *f is a local homeomorphism at every point of V ,*
- (iii) $B \subset R_{n-2}$,
- (iv) $\dim R_{n-2} \leq n - 2$.

When $n = 2$, conclusions (iii) and (iv) imply B is a closed discrete set; thus in this case f is *light*, i.e., point inverses are 0-dimensional. From Stoilow [4], which topologically characterizes germs of light open surface maps, we obtain:

Corollary 4. *When $n = 2$, the germ of f at any point is topologically equivalent to the germ at 0 of the complex function z^d for some integer $d \neq 0$.*

A key role in our proofs is played by the following result, Theorem 1.4 of Church [2]:

Theorem 5 (Church). *If $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is C^n and open with rank $\geq n - 1$ at every point, then g is a local homeomorphism.*

Our results are close to some of those obtained by Church for C^n maps. It is interesting to compare Theorem 3 (iii) and Corollary 4 to the following results from paragraphs 1.5 to 1.8 of his paper [2]:

Theorem 6 (Church). *Let $g: M \rightarrow N$ be a C^n map between n -manifolds.*

- (i) *If $M = N = \mathbb{R}^n$ and g is light, the following conditions are equivalent:*
 - (a) *g is open,*
 - (b) *Jg does not change sign,*
 - (c) $B \subset R_{n-2}$.
- (ii) *If M is compact and g is open, then g is light.*

Proofs

Lemma 7. *Assume the critical set of f is $C = Jf^{-1}(0) = \mathbb{R}^{n-1} \times \{0\}$, and $f|_C$ has constant rank k , $0 \leq k \leq n-1$. Then f is a local homeomorphism, $k = n-1$, and Jf does not change sign in $\mathbb{R}^n$.*

Proof. It suffices to prove that the conclusion holds in some neighborhood of each point, which we may take to be the origin.

It is convenient to denote points of $\mathbb{R}^n$ as $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

By the rank theorem we assume that in some open cubical neighborhood N of the origin,

$$f_i(y, 0) \equiv 0, \quad i = k+1, \dots, n. \quad (1)$$

Identifying N with $\mathbb{R}^n$ by an analytic diffeomorphism, we assume this holds for all $y \in \mathbb{R}^n$.

Because f is analytic and open, there is a dense open set $\Lambda \subset \mathbb{R}^{n-1}$ such that for every $y \in \Lambda$, the map $t \mapsto f_n(y, t)$ is not constant on any interval. For each $y \in \Lambda$ there exists a maximal integer $\mu(y) \geq 0$ such that

$$0 < j < \mu(y) \implies \left(\frac{\partial}{\partial t} \right)^j f_n(y, 0) = 0,$$

Fix $y_* \in \Lambda$ such that the function $\mu: \Lambda \rightarrow \mathbb{N}$ takes its minimum value m at y_* . Then $\mu = m$ in a precompact open neighborhood $W \subset \Lambda$ of y_* .

By Taylor's theorem there exists $\epsilon > 0$ such that for (y, t) in the open set

$$N = W \times]-\epsilon, \epsilon[\subset \mathbb{R}^{n-1} \times \mathbb{R}$$

we have

$$f_n(y, t) = t^m H(y, t), \quad H(y, t) \neq 0. \quad (2)$$

Claim. *If $k \leq n-2$ and $(y_0, t_0) \in N$ is such that $f_n(y_0, t_0) = 0$, then $f_{n-1}(y_0, t_0) = 0$. For $t_0 = 0$ by (2), and $k \leq n-2$ implies $f_{n-1}(y_0, 0) = 0$ by (1).*

Now we assume $k \leq n-2$ and reach a contradiction. Since $f(N)$ is open and contains

$$f(y_*, 0) = (a_1, \dots, a_{n-2}, 0, 0),$$

$f(N)$ also contains points $(a_1, \dots, a_{n-2}, \delta, 0)$ with $\delta > 0$. But this contradicts the claim.

As f has rank $n-1$ at every point of the critical set, f must be a local homeomorphism by Theorem 5. Therefore for every p , the induced homomorphism of homology groups

$$\mathbb{Z} = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(p)\}) = \mathbb{Z}$$

is an isomorphism, hence is multiplication by a number $\delta(p) \in \{+1, -1\}$.

Homology theory implies that each of the two level sets of $\delta: \mathbb{R}^n \rightarrow \{+1, -1\}$ is open. As $\mathbb{R}^n$ is connected, $\delta(p)$ is constant. As $\delta(p)$ is the sign of $Jf(p)$ if $Jf(p) \neq 0$, we have proved Jf does not change sign.

Proof of Theorem 1. For any set $Y \subset \mathbb{R}^n$, we say *the local theorem holds in Y* if every point of Y has a neighborhood in Y in which Jf does not change sign.

Lemma 8. *If the local theorem holds in a connected set Y , then Jf does not change sign in the closure $\overline{Y}$.*

Proof. It suffices to prove Jf does not change sign in Y , because Jf is continuous. Define Y_+ , Y_- to be the subsets of Y where Jf is respectively ≥ 0 and ≤ 0 . These sets are closed in Y by continuity of Jf , and open in Y by hypothesis. As Y is connected, either $Y = Y_+$ or $Y = Y_-$.

The local theorem obviously holds in the set $\mathbb{R}^n \setminus C$ of noncritical points. By Lemma 7, it also holds in the relatively open analytic hypersurface $V \subset C$ defined in the introduction. It remains to prove that every point of $C \setminus V$ has a neighborhood in which Jf does not change sign.

Lemma 9. *Every point $p \in C \setminus V$ has a neighborhood $X_p \subset C \setminus V$ that is an analytic variety of dimension $\leq n - 2$.*

Proof. Write

$$C \setminus V = (C \setminus H) \cup (H \setminus V)$$

Suppose $p \in C \setminus H$. In this case we take X_p to be the union of the variety C_{sing} of singular points of C and those connected components of $C \setminus C_{\text{sing}}$ having dimension $\leq n - 2$.

Suppose $p \in H \setminus V$, or equivalently: $p \in H$ and some minor determinant of df vanishes at p but not identically in any neighborhood of p in H . We take X_p to be the intersection of C with the union of the zero sets of such minors.

Now consider any point $p \in C \setminus V$. By Lemma 9, p has a neighborhood $X_p \subset C \setminus V$ that is the union of finitely many smooth submanifolds of $\mathbb{R}^n$ having dimensions $\leq n - 2$.

Choose a connected open neighborhood $N_p \subset \mathbb{R}^n$ of p such that $N_p \cap (C \setminus V) = N_p \cap X_p$. Then $N_p \setminus X_p \subset (\mathbb{R}^n \setminus C) \cup V$. Therefore the local theorem holds in $N_p \setminus X_p$.

Now $N_p \setminus X_p$ is connected, by a standard general position argument. Therefore from Lemma 8, with $Y = N_p \setminus X_p$, we infer that Jf does not change sign in $\overline{N_p \setminus X_p}$, which equals $\overline{N_p}$ because X_p is nowhere dense. This completes the proof of Theorem 1.

Proof of Theorem 3. Parts (i) and (ii) of Theorem 2 are proved by applying Lemma 7 locally. Lemma 9 implies (iii), because $B \subset C \setminus V$ by (ii). For (iv), suppose $\dim R_{n-2} = n - 1$. Then the variety R_{n-2} contains an analytic hypersurface, which must meet V . As $R_{n-2} \subset C$, this implies $R_{n-2} \cap V \neq \emptyset$, contradicting (i). $\square$

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