When are there Condorcet winners despite extremist preferences

Author
Bergstrom, Ted

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When are there Condorcet winners despite "extremist" preferences?

Ted Bergstrom*

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Introduction

The beach of social decision theory has been thoroughly combed by Condorcet, de Borda, Lewis Carroll, Kenneth Arrow, Kenneth May, Duncan Black, Amartya Sen, and hundreds of other talented scholars. Chances of finding even a pretty pebble seem small, but the tides may sometimes uncover new stones.

Kenneth May [8] found that with two alternatives, if a voting method always picks a winning candidate, and if it treats voters and alternatives symmetrically and satisfies a simple Pareto criterion, then it must be majority rule voting. May also observed that if a voting method treats voters and alternatives symmetrically and also satisfies independence of irrelevant alternatives, then this method must be pairwise majority voting. He then noted that the example of Condorcet's majority voting cycle demonstrates that with three or more voters, for some configurations of preference orderings pairwise majority voting does not produce a winning outcome. Thus we have a simple proof of a variant of Arrow's impossibility theorem.

Duncan Black [2] discovered that if the domain of preference orders over candidates is restricted to those that are single peaked, then pairwise majority voting always produces a winning candidate. Inada [7] and Sen and Pattanaik [11] extended Black's result by finding other restrictions on the domain of preferences that are sufficient to imply that pairwise majority voting selects a winner. All of these conditions require that certain preference orderings are *never* found in the voting population. As they say:

> "However the restrictions we consider are those that apply only to types of permissible orderings and not on numbers holding them."

The current paper builds on a very simple idea; one that allows us to apply the results of Black, Inada, and Sen and Pattanaik to a much broader domain of

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preferences than has been previously recognized. Notice that if two voters have opposite preferences about every pair of candidates, then the results of majority voting would be unchanged if neither of them voted. Thus if one preference ordering is less common than its opposite, the majority voting results from this profile will be the same as the results in which there are no voters with the less common ordering and their number is subtracted from the number of voters with the opposite preferences. If this "reduced form" profile is single-peaked, then majority voting from the initial preference profile will produce a winning candidate.

This means, for example, that qualitative results found for a population where some preference ordering never appears will still be true in a population that has a significant number of these "deviants," so long as they are outnumbered by voters with opposite preferences. For example, even if for any left to right array of three candidates there are a significant number of voters with "extremist" preferences, majority voting will still select a winner, so long as those voters who prefer both extremes to the middle are outnumbered by voters with opposite preferences, who who prefer the center candidate to both extremes.

In a masterful survey of the voting theory literature [10], Sen acknowledges that it is possible to consider number-specific constraints on domain of preference profile but warns that

> "In order to make the exercise worthwhile, the number-specific conditions must have some intuitive meaning that helps the interpretation of the nature of the preference configurations."

I believe that that the simple "number-specific" conditions proposed here meet Sen's criterion of having "intuitive meaning" and that they can help us to better understand when it is likely that pairwise majority voting produces a winning candidate. Applying this idea also allows us to simplify and unify the conditions for transitive majority that were found by Sen and Pattanaik in a way that I believe enriches our understanding of these conditions.

1 Some Fundamentals

Preference Profiles and Voting Relations

We consider a community with n voters. Each voter i is assumed to have a reflexive, complete, and transitive preference relation $\succeq_i$ over the candidate pool. Such a relation is known as a *weak order*. The corresponding strict preference $\succ_i$ and indifference $\sim_i$ are defined in the usual way, where $A \succ_i B$ if and only if $A \succeq_i B$ and not $B \succeq_i A$, and where $A \sim_i B$ if and only $A \succeq_i B$ and $B \succeq_i A$.

Definition 1 (Preference profile)**.** *A preference profile for a community of voters is a list, showing the number of voters who have each possible preference ordering over the set of candidates.*

A preference profile for a set of voters determines the outcome of pairwise majority voting. We define the relations $\mathcal{R}$ (gets at least as many votes as) $\mathcal{P}$ (gets more votes than) and $\mathcal{I}$ (ties) in the obvious way.

Definition 2 (Pairwise majority voting relations)**.** *For any preference profile, and any two candidates, A and B, we define the induced majority relation $\mathcal{R}$, where $A \ \mathcal{R}\, B$ if and only if at least as many voters prefer A to B as prefer B to A. The corresponding strict preference relation $\mathcal{P}$ has $A \ \mathcal{P}\, B$ if and only more voters prefer A to B than prefer B to A. The corresponding "indifference relation" $\mathcal{I}$ has $A\,\mathcal{I}\,B$ if and only if exactly as many voters prefer A to B as prefer B to A.*

Definition 3 (Condorcet winner)**.** *For any preference profile, a Condorcet winner is a candidate that is not defeated in pairwise majority voting by any other candidate. Thus candidate X is a Condorcet winner if $X \mathcal{R}\, Y$ for all other candidates Y.*

A strict Condorcet winner is a candidate who is never beaten or tied.

Definition 4 (Strict Condorcet winner)**.** *For any preference profile, a strict Condorcet winner is a candidate who defeats all other candidates in pairwise majority voting. Thus candidate x is a strict Condorcet winner if $x \mathcal{P}\, y$ for all other candidates y.*

Definition 5 (Strict Condorcet Loser)**.** *For any preference profile, a strict Condorcet loser is a candidate who is defeated by all other candidates in pairwise majority voting. Thus candidate x is a strict Condorcet loser if $y \mathcal{P}\, x$ for all other candidates y.*

Definition 6 (Transitive, quasi-transitive, and acyclic majority voting)**.** *If for any three candidates x, y, and z, majority voting is:*

- *transitive if $A \mathcal{R}\, B$ and $B \mathcal{R}\, C$ implies $A \mathcal{R}\, C$.*
- *quasi-transitive $A \mathcal{P}\, B$ and $B \mathcal{P}\, C$ implies $A \mathcal{P}\, C$.*
- *has no 3-cycles if $A \mathcal{P}\, B$ and $B \mathcal{P}\, C$ implies $A \mathcal{R}\, C$.*

Sen [9] argues that, while transitivity of the majority voting relation is sufficient for the existence of a Condorcet winner, this assumption is overly restrictive. He shows that the weaker assumption of quasi-transitivity is sufficient to guarantee that there is least one Condorcet winner for any finite set of candidates. Sen [9] proved the following result, using a straightforward induction argument.

Lemma 1 (Sen)**.** *If the number of candidates is finite and the majority voting relation $\mathcal{R}$ is quasi-transitive, there must be at least one Condorcet winner.*

Although the converse of Lemma 1 is not true, we will make use of the following related result.

Lemma 2. *In an election with three candidates, if there is either a strict Condorcet winner or a strict Condorcet loser, then the the majority voting relation $\mathcal{R}$ is transitive.*

Proof. If there are three candidates and if $\mathcal{R}$ is not transitive, then there must be a preference cycle of the form $\mathrm{x}\,\mathcal{R}\,\mathrm{y}$, $\mathrm{y}\,\mathcal{R}\,\mathrm{z}$, and $\mathrm{z}\,\mathcal{R}\,\mathrm{x}$. With such a cycle, we see that in pairwise contests, each candidate gets at least as many votes as some other candidate and each gets no more votes than some other candidate. A strict Condorcet winner must get more and a strict Condorcet loser must get fewer votes than either of the other candidates in pairwise voting. So with three candidates, if there is either a strict Condorcet winner or a strict Condorcet loser, there can be no cycle. Hence $\mathcal{R}$ must be transitive.

□

Opposite preferences and reduced-form preference profiles

Let us define opposite preference orderings in the following way.

Definition 7 (Opposite preference orderings)**.** *Two strict preference orderings are said to be* opposite *if these two orderings rank every pair of candidates in opposite ways.*

Where there are three candidates and strict preferences, two voters who have opposite preference orderings must have the same second choice, with the first choice of each being the third choice of the other.

Definition 8 (Dominant and dominated orderings)**.** *For any preference profile, and any pair of opposite preference orderings, if the number of voters with one ordering exceeds the number with the opposite ordering, we say that this ordering is the dominant member of the pair and the other is dominated.*

In any election between two candidates, the votes of two voters with opposite preferences will cancel each other. Removing an equal number of voters from each side of a pair of opposite preferences does not change the results from pairwise majority voting.[1] This suggests a way to find a simpler preference profile for which the results of pairwise majority voting are the same as those in the original profile.

We define the *reduced form* of a strict preference profile as a preference profile in which equal numbers of opposite preference orderings are cancelled in such a way as to reduce the number of voters holding the dominated member of each pair of opposite preference orderings to zero.

Definition 9 (Reduced form frequency)**.** *For each candidate i, where i is the second choice of a pair of opposite preference orderings, the reduced form frequency of the dominant member of this pair is the difference $m(i)$ between the numbers of voters holding the dominant and the dominated preference orderings from this pair. If the numbers of voters holding each of these opposite orderings are equal, then $m(i) = 0$.*

[1] This is also true of the Borda count, and of any other symmetric vote counting mechanism.

Definition 10 (Reduced form preference profiles)**.** *The reduced form of a preference profile is a preference profile in which the only preference orderings that appear are those that dominate the opposite ordering. In this profile, the reduced form frequency of a dominant ordering is the difference between the number of voters with this preference and the number with its opposite.*

2 Three Candidates and Strict Preferences

The theory works out especially cleanly when there are only three candidates and voters have strict, transitive preferences over the candidates. The case of three candidates is interesting in itself, since political contests with three serious contenders are quite common. In addition, it turns out that with any finite number of candidates, if preferences over every subset of three candidates are quasi-transitive, then there will necessarily be at least one Condorcet winner for the entire slate of candidates. We will show that when individual preferences are strict and transitive, a simple condition on the reduced preference profile is necessary and sufficient for the existence and uniqueness of a Condorcet winner. This condition is satisfied in much broader cirumstances than the familiar assumption of single-peaked preferencs.

Assumption 1. *There are three candidates, and each voter i has a complete, transitive, and asymmetric strict preference ordering $\succ_i$ over these candidates.*[2]

A preference profile for a community of voters lists the number of voters who have each possible preference ordering. In a preference profile with three candidates and strict preferences, a voter could have any of six possible preference orderings over the three candidates. Table 1 shows a preference profile with three candidates in its general form.

Table 1: A preference profile with 3 candidates

Ranking of candidates	Number of voters					
	n_1	n_2	n_3	n_4	n_5	n_6
1	B	C	A	C	A	B
2	A	A	B	B	C	C
3	C	B	C	A	B	A

In a reduced form preference profile with three voters and strict preferences, at most three of the six possible preference orderings appear with a positive number of voters. These are the orderings that dominate their opposites. Thus if n_1 voters order the candidates BAC and $n_2 < n_1$ voters order the candidates CAB, then the reduced preference profile has $n_1 - n_2$ voters with ordering

[2]A binary relation R on a set $\mathcal{S}$ is complete if for any two elements x and y of $\mathcal{S}$, xRy or yRx. It is transitive if for any three elements, x, y and z, if xRy and yRz, then xRz. It is asymmetric if xRy implies that *not* yRx. A relation with these three properties is sometimes called a *linear order.* Arrow [1] calls this a *strong ordering relation.*

BAC and no voters with ordering CAB. In case $n_1 = n_2$, neither ABC nor CBA appear in the reduced preference profile. With three candidates, each pair of opposite preferences shares a common second choice. Thus each candidate appears at most once as a second choice in the reduced form preference profile. In a reduced form profile, we denote the reduced form frequency of a preference ordering with Candidate i ranked second as $m(i)$.

Latin Squares and Condorcet Cycles

Table 2: A Latin Square reduced form preference profile

Ranking of candidates	Number of voters $m(B)$	$m(C)$	$m(A)$
1	A	B	C
2	B	C	A
3	C	A	B

Table 2 shows a reduced preference profile in which each candidate appears once in each row and once in each column. An array of this type is known as a "Latin square."[3]

Definition 11 (Latin Square reduced form profile)**.** *Where there are three candidates, a reduced form preference profile is a Latin square if each of the three candidates appears is the first choice in one of the three orderings, the second choice in another, and as third choice in the remaining ordering.*

It is well-known that with three candidates, if the preference profile takes the form of a Latin square, then the majority voting relation may have a cycle and thus there will be no Condorcet winner. The following result shows exactly when this is the case.

Lemma 3. *If a preference profile satisfies Assumption 1, and if the reduced form of a preference profile is a Latin square, then the majority voting relation* $\mathcal{P}$ *has a cycle if and only if each of the three preference orderings in the reduced form profile has fewer than half of the total number of voters counted in this profile.*

Proof. If each of the preference orderings in a reduced preference profile is held by fewer than half of the voters in that profile, then in pairwise voting, any candidate will be defeated by another that is preferred by two of the three preference orderings in the profile. If the reduced form preference profile is a Latin square, then for each candidate, there is another candidate that is preferred in two of the three preference orderings in the profile. Therefore the ordering $\mathcal{P}$ must have a cycle.

[3]Benjamin Ward [12] appears to be the first to have applied this term to voting theory.

If one of the preference orderings $\geq_i$ in the reduced preference profile has at least half of the voters in this profile, then for any three candidates, A, B, and C, if A $\mathcal{P}$ B and B $\mathcal{P}$ C, it must be that $A \geq_i B$ and $B \geq_i C$. Since $\geq_i$ is assumed to be transitive, it follows that $A \geq_i C$. Since at least half of the voters in the reduced preference profile have the preference ordering $\geq_i$, it must be that A $\mathcal{R}$ C. It follows that $\mathcal{P}$ can not have a cycle if some profile has at least half of the voters in the reduced preference profile. □

Existence of Condorcet winners

If more than half of the voters in the reduced preference profile have the same preference ordering, then in all pairwise voting contests, the preference of this majority group will prevail. Since individual preferences are assumed to be transitive, the majority voting relation is transitive. Where preferences are strict, the unique candidate that is most preferred by those with the majority preference is a strict Condorcet winner. Therefore, we have:

Lemma 4. *If Assumption 1 is satisfied and if more than half of the voters in the reduced preference profile share the same preference ordering, then the majority voting relation is transitive and there is a unique strict Condorcet winner.*

The possibility of ties in pairwise votes causes a bit of complication and leads to slightly weaker results. Therefore, we begin by considering preference profiles in which there are no troublesome tie votes.

Assumption 2 (No ties)**.** *No preference ordering in the reduced preference profile is held by exactly half of voters in this profile. For at least one pair of opposite preferences, the number of voters on one side exceeds the number on the other side.*

With three candidates, the reduced preference profile includes at most three preference orderings. Since voters with opposite preferences share the same second choice from among three candidates, if three candidates appear in the reduced preference profile, then each of them will have a different second choice. If the reduced preference profile has three candidates and is not a Latin square, it follows that one candidate is ranked first in two of the preference orderings and second in the other ordering. Another candidate must ranked first in one ordering, second in another, and third in another. The remaining candidate must never be ranked first, and must be ranked second in one ordering and third in the other two orderings.

For any reduced form preference profile with three candidates that is not a Latin square, we can assign labels as in Table 3, where candidate B is ranked first by two orderings, candidate A is ranked first by one ordering, and candidate C is never ranked first.

Lemma 5. *If Assumptions 1 and 2 are satisfied and if the reduced preference profile is not a Latin square, then there is a unique Condorcet winner and the majority voting relation is transitive.*

Table 3: Reduced Form Profile Without Latin Square

Ranking of candidates	Number of voters $m(B)$	$m(C)$	$m(A)$
1	A	B	B
2	B	C	A
3	C	A	C

Proof. If more than half of the voters in this reduced preference profile share one of the preference orderings, then in any pairwise contest, the candidate preferred by this preference ordering will prevail. Therefore, since individual preferences are assumed to be transitive, the ordering $\mathcal{P}$ will be transitive and the first choice of this preference will be a unique Condorcet winner.

If fewer than half of the voters in the reduced preference profile hold each of the preference orderings, then in any pairwise contest, the winning candidate will the one favored by two of the three preference orderings. We have shown that if a reduced preference profile with three candidates is not a Latin square, then one candidate (B in Table 3) is the first choice of two of the three orderings, and one candidate (C in Table 3) is the last choice of two of the three candidates. It follows that candidate B, the favorite of two of the three preference orderings is a unique Condorcet winner. For these three candidates, the relation $\mathcal{P}$ is also transitive, since we have $B \mathcal{P} A$, $A \mathcal{P} C$, and $B \ \mathcal{P} C$.

Assumption 2 implies that the reduced preference profile contains at least one preference ordering. The reduced preference profile would have only one or two preference orderings if for one or two of the pairs of opposite preferences, the number of voters on each side is equal. If the reduced preference profile has only one or two preference orderings, and if assumption 2 is satisfied, then more than half of all voters in the reduced preference profile share a single preference ordering. Therefore the majority voting relation $\mathcal{R}$ will be the same as the preference of ordering of these voters. Hence $\mathcal{R}$ will be transitive and the unique Condorcet winner will be the most preferred candidate of these voters. $\square$

Two kinds of ties are excluded by the No Ties assumption. One occurs where the reduced preference profile contains no preference orderings because there are equal numbers of voters on each side of all opposite pairs. The other occurs when some preference ordering is held by exactly half of all votes counted in the reduced preference profile.

In the case of equal numbers of voters on each side of all opposing pairs, there will be a tie vote in every pairwise contest. Therefore in this case, all three candidates are Condorcet winners.

If one preference ordering is held by exactly half of the voters in the reduced preference profile, then the favorite candidate of this preference ordering is a Condorcet winner, since it cannot be defeated in pairwise competition. There may, however be a tie between this candidate and another candidate who is preferred by the other voters.

Lemma 6. *If the reduced preference profile is not a Latin square and if exactly half of the voters in this profile share a single preference ordering, there exists at least one Condorcet winner, though it is not necessarily unique.*

Proof. If exactly half of the voters in the reduced preference profile share the same preference ordering, then in any pairwise contest, the candidate preferred by this group will get a least half of the votes. A candidate that is the first choice of half the voters must be Condorcet winner, since no other candidate can get more votes in pairwise competition. □

Lemma 7. *If Assumption 1 is satisfied, if the reduced preference profile is not a Latin square ,and if exactly half of the voters in the reduced preference profile share the same preference ordering, then the majority rule ordering $\mathcal{R}$ is not necessarily transitive, but it is quasi-transitive.*

Proof. To see that $\mathcal{R}$ is not necessarily transitive, consider the case where exactly half of the voters have the preference relation $B \succ C \succ A$ and the remaining voters have the ordering $A \succ B \succ C$. In this case, it must be that $C \ \mathcal{R} A$ and $A \ \mathcal{R} B$, since there is a tie vote in each case, but it is not the case that C $\mathcal{R} B$, since B is unanimously preferred to C.

To show that $\mathcal{R}$ is quasi-transitive we need to show that for any three candidates A, B, and C, if A $\mathcal{P}$ B and B $\mathcal{P}$ C, then A $\mathcal{R}$ C. Suppose that half of the voters in the reduced preference profile have the preference ordering $\succeq_i$. Then if A $\mathcal{P}$ B and B $\mathcal{P}$ C, it must be that $A \succ_i B$ and $B \succ_i C$. Since $\succ_i$ is assumed to be transitive, it follows that $A \succ_i C$. But this means that at least half of all voters prefer A to C. It follows that A $\mathcal{R}$ C. Hence $\mathcal{R}$ is quasi-transitive.

□

The results of Lemmas 4-7 can be summarized by the following propositions:

Proposition 1. *If there are three candidates and individual preferences are strict orderings, then there is at least one Condorcet winner if and only if one or both of the following are true:*

(i) One of the preference orderings in the reduced preference profile is held by at least half of the voters counted in this profile.

(ii) The reduced preference profile is not a Latin square.

Proposition 2. *If there are three candidates and individual preferences are strict orderings, then a Condorcet winner is unique if Asssumption 2, (the no-ties assumption) is satisfied.*

Single-peaked Preference Profiles

A preference profile is said to be "single-peaked" if candidates can be arrayed from left to right along a line, in such a way that every voter has a favorite candidate and for any two candidates on the same side of the favorite, the one who is closer to the favorite is strictly preferred to the other.

We paraphrase Arrow's [1] formal definition of single-peaked preferences as follows:

Definition 12 (Single-peaked preferences-Arrow)**.** *A preference profile is single-peaked if there is some complete ordering $\mathcal{S}$ of all candidates such that for every preference ordering in this profile, and for any three candidates x, y and z such that y lies between x and z in the ordering S, if $x \succeq_i y$, it must be that $y \succ_i z$.*

Sen [10] pointed out that Arrow's definition of single-peakedness is equivalent to a definition that can be stated without explicit use of Arrow's left-right ordering, $\mathcal{S}$. Sen's version is as follows:

Definition 13 (Single-peaked preferences-Sen)**.** *A preference profile is single-peaked if and only if for any three candidates, one of these candidates is strictly preferred by all voters to at least one of the other two candidates.*

Remark 1. *Where $\geq_i$ is a complete prefernce ordering for all i, Sen's definition of a single-peaked preference profile is equivalent to Arrow's.*

Proof of Remark. Suppose that the preference profile is single-peaked by Arrow's definition. We first show that if the preference profile satisfies Arrow's definition, it also satisfies Sen's. In particular, the candidate positioned in the middle by Arrow's ordering S is strictly preferred by all voters to at least one of the other two candidates. For three candidates, x,y, and z, where $xSySz$, Arrow requires that for all voters i, if $x \succeq_i y$, then $y \succ_i z$. Since individual preference orderings are assumed to be complete, it follow that if not $x \succeq_i y$, then $y \succ_i x$. Therefore it must be that for every voter, the middle candidate y is preferrred to at least one of the other two candidates.

Conversely, suppose that a preference profile satisfies Sen's condition. Then for any three candidates, there is one candidate who is strictly preferred by all voters to at least one other candidate. Without loss of generality, name these candidates x,y, and z where y is the candidate that all voters prefer to at least one of the other two. Let the left-to-right ordering $\mathcal{S}$ rank them $xSySz$. Since y is preferred to either x or to z for every voter i, it must be that if $x \succeq_i y$, then $y \succ z$. Therefore Sen's condition implies Arrow's. □

Definition 14 (Essentially single-peaked)**.** *A preference profile over three candidates is essentially single-peaked if the reduced form of this profile is single-peaked.*

Lemma 8. *A preference profile that satisfies Assumption 1 is essentially single-peaked if and only if it is not a Latin square.*

Proof. According to Sen's definition, if a reduced preference profile is single-peaked, there is some candidate that does not appear in last place in any of the three preference orderings in this profile. In a Latin square, every candidate is the last choice of one of the profiles. Therefore if the reduced preference profile is single-peaked, it cannot be a Latin square.

In a reduced preference profile, each of the three candidates is the second choice of one of the three preference orderings included. If this profile is not a Latin square, it cannot be that each candidate is also the first choice of one of the three orderings. So it must be that one candidate is the first choice of two of the preference orderings and is not the last choice of any of the three. Therefore one of the three candidates must be the first choice of none of the three orderings and the last choice of two of them. The candidate who is the first choice of two of the orderings must then be preferred by all three preference orderings to the candidate who is the first choice of none of the orderings. (See Table 3) Therefore if the reduced preference profile is not a Latin square, it satisfies Sen's condition for single-peakedness. □

The following results follow directly from Lemma 8 and Propositions 1 and 2.

Proposition 3. *If there are three candidates and individual preferences are strict orderings, there is a Condorcet winner if the reduced preference profile is essentially single-peaked. If in addition, Assumption 2 is satisfied, there will be only one Condorcet winner.*

Proposition 4. *If there are three candidates, individual preferences are strict orderings, and each preference ordering in the reduced profile is held by fewer than half of all voters in this profile, then if there is a Condorcet winner, it must be that preferences are single-peaked.*

3 Preferences that allow indifference

The story gets a bit more complex if voters can be indifferent between two candidates. Let us consider preference profiles in which preferences of each individual i are characterized by an "at least as good" relation $\succeq_i$ that is reflexive, transitive, and complete. Such a preference relation is known as a *weak order.*

Assumption 3. *There are 3 candidates. Each voter i has a preference relation $\succeq_i$ that is a weak order defined on these candidates.*

With three candidates, when indifference is allowed, there are 13 possible preference orderings. In addition to the six orderings with strict preference only, there are six orderings with semi-strict preference orderings where there is indifference between two of the three candidates. There is also one ordering that is indifferent among all three candidates. Voters with this ranking have no effect on majority voting outcomes, since they never prefer one candidate to another. The six semi-strict orderings include three pairs of opposite preferences. Where $x(yz)$ denotes $X \succ Y$ and $Y \sim Z$, these pairs are $A(BC)$ and $(BC)A$, $B(AC)$ and $(AC)B$, and $C(AB)$ and $(AB)C$.

Any preference profile in which preferences are weak orderings will induce the same majority voting relation as its reduced preference profile. The reduced preference profile consists of a listing of six preference relations: the three strict preference orders that dominate their opposites, and the three semi-strict preference orders that dominate their opposites, along with their reduced form frequencies.

3.1 Single-peaked with weak orders

A single-peaked preference profile, as defined by Arrow and by Sen allows preferences to be a weak order. This definition requires that there is a positional ordering $\mathcal{S}$ such that a candidate who is located between the other two in the ordering $\mathcal{S}$ is strictly preferred to at least one of its neighbors.

Sen and Pattanaik [11] and Sen [10] (pp 226-231) show that if the preference profile is single peaked, then majority voting is quasi-transitive and hence selects at least one Condorcet winner. They also find less stringent qualitative restrictions on preference profiles that imply the existence of one or more Condorcet winners.

Since pairwise voting results with the reduced preference profile are the same as for the original profile, the results found by Sen and Pattanaik can be extended by noting that it is sufficient for the existence of a Condorcet winner that the reduced preference profile satisfies their restrictions on preference profiles.

Working directly with reduced preference orderings allows us to find alternative sets of assumptions with relatively simple intuitive interpretations that imply the existence and/or uniqueness of Condorcet winners.

If preferences are single peaked and there are three candidates, the middle candidate under the ordering $\mathcal{S}$ cannot be ranked last or tied for last. In the reduced form preference profile, there are at most three strict preference orderings that appear with a positive number of voters. Each of the three candidates is the second choice in one of these three orderings.

Table 4 represents all possible reduced form preference profiles with the following conventions. The number m_i is the difference between the number of voters with the preference ordering listed below and the number of voters who hold the opposite preference. If $m_i > 0$, the preference ordering listed below dominates its opposite. If $m_i < 0$, the ordering listed below is dominated by its opposite, and if $m_i = 0$,the number of voters with this ordering is equal to the number with the opposite ordering.

If the reduced form preference profile is single-peaked, there is some candidate that is not the last choice in any of the three strict preference orderings. Let us label this candidate B. Since for the three strict preference orderings, Candidate B is never the third choice and only once the second choice, it follows that Candidate B is the first choice in two of the strict orderings and the second choice in one of them. Let us apply the label, A, to the candidate that is the first choice when Candidate B is the second choice. It follows that the third candidate, labelled C, must be the second choice in one of the strict orderings and the third choice in two of them. Then if preferences are single-peaked, it must bet that $m_1 \geq 0$, $m_2 \geq 0$ and $m_3 \geq 0$ in Table 4. If preferences are single-peaked, then it must be that in the semi-strict orderings, Candidate B is never ranked last or tied for last. This implies that in Table 4, $m_4 \geq 0$, $m_5 \geq 0$ and $m_6 \geq 0$.

Table 4: Essential Single-peaked profile with weak orders

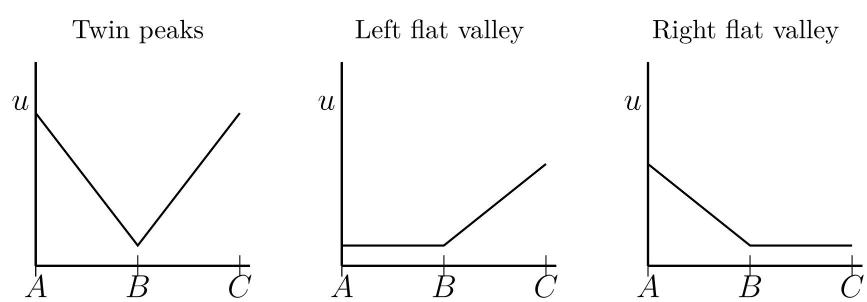
Ranking of candidates	Number of voters m_1	m_2	m_3	m_4	m_5	m_6
1	A	B	B	B	(AB)	(BC)
2	B	C	A	(AC)	C	A
3	C	A	C			

3.2 Beyond the single peaks

The assumption that preferences are essentially single-peaked is stronger than necessary to ensure the existence of a Condorcet winner. With three candidates and single-peaked preferences, if Candidate B is the middle candidate in a positional order **S** for which the preference profile is single-peaked, then in the reduced preference profile, the three semi-strict preference relations, B(AC), (AB)C, and (BC)A must dominate their opposites, (AC)B, C(AB) and A(BC). Thus, if preferences are essentially single-peaked, it must be that where the preference profile is represented as in Table 4, $m_i \geq 0$ for $i = 1, 2, \ldots 6$.

The three semi-strict preference relations that must be dominated if the reduced preference profile is single-peaked are shown in Figure 3.2, where the horizontal axes show the candidates in a left-to-right positional order, where Candidate A is on the left, Candidate B is in the middle and Candidate C is on the right, and where the vertical axis represents utility for each candidate.

Figure 1: Examples violating single-peakedness



As it turns out, in order to ensure that there is a Condorcet winner, we do not need to exclude the right flat valley preference relation A(BC) be dominated by its opposite.

Lemma 9. *Where the preference profile is represented as in Table 4, there will be at least one Condorcet winner and the pairwise majority relation $\mathcal{R}$ will be quasi-transitive if $m_i \geq 0$ for $i = 1, 2, \ldots 5$.*

Proof. Suppose that $A \mathcal{P} B$. From Table 4, we see that if $m_i \geq 0$ for $i = 1, \ldots, 5$, then the only dominant preference order for which $A \succ B$ also has $A \succ C$. Therefore $A \mathcal{P} B$ implies $A \mathcal{P} C$, Thus if $A \mathcal{P} B$, A is a strict Condorcet winner, and according to Lemma 2 $\mathcal{R}$ is transitive.

Suppose that $B \mathcal{P} A$. We see from Table 4 that i if $m_i \geq 0$ for $i = 1, 2, \ldots 5$, then $B \succeq C$ for all voters in the reduced preference profile. Therefore $B \mathcal{R} C$. Since $B \mathcal{P} A$ and $B \mathcal{R} C$, it must be that B is a Condorcet winner and the relation $\mathcal{R}$ is quasi-transitive.

Suppose that $A \mathcal{I} B$. From Table 4, we see that

$$m_1 + m_6 = m_2 + m_3 + m_4 \tag{1}$$

Since $m_3 \geq 0$ and $m_5 \geq 0$, this implies that

$$m_1 + m_3 + m_5 + m_6 \geq m_2. \tag{2}$$

Equation 2 implies that $A \mathcal{R} C$. Since $m_i \geq 0$ for $i = 1, 2 \ldots 5$, it must be that voters who prefer C to B never outnumber their opposites who prefer B to C. Hence $B \mathcal{R} C$. We now have $A \mathcal{I} B$, $A \mathcal{R} C$, and $B \mathcal{R} C$. It follows that both A and B are Condorcet winners. In this case there are no three alternatives x, y, and z such that $x \mathcal{P} y$ and $y \mathcal{P} z$. Therefore quasi-transitivity is trivially satisfied. □

3.3 The remaining troublemakers

Where there are three candidates and the reduced preference profile over strict preferences is single-peaked, there remain two semi-strict preference orderings that can result in the absence of a Condorcet winner. With the labeling conventions used in Tables 4, these are the semi-strict preference orderings $A \sim C \succ B$ and $C \succ A \sim B$, which we will denote as (AC)B and C(AB).

Examples 1 and 2 show reduced preference profiles that include positive numbers of voters with preference orderings with these preference orderings and for which the majority voting relation is cyclic and hence there is no Condorcet winner. In the appendix to this paper, we show that in a preference profile where either of these "trouble-makers" appear, there will be no Condorcet winner for a large set of distributions of voter preferences.

Table 5: Example 1: No Condorcet winner with C(BA)

Ranking of candidates	Number of voters					
	5	2	1	1	6	1
1	A	B	B	B	C	A
2	B	C	A	(AC)	(AB)	(BC)
3	C	A	C			

In Example 1, Candidate A beats B by 6 votes to 4. B beats C by 9 votes to 6. C beats A by 8 votes to 7. Thus, we have $A \mathcal{P} B$, $B \mathcal{P} C$ and $C \mathcal{P} A$. There is a Condorcet cycle and there is no Condorcet winner.

Table 6: Example 2: No Condorcet winner with (AC)B

Ranking of candidates	Number of voters					
	1	3	1	5	1	1
1	A	B	B	(AC)	(AB)	(BC)
2	B	C	A	B	C	A
3	C	A	C			

In Example 2, A beats B by 6 votes to 5. B beats C by 6 votes to 5, and C beats A by 4 votes to 3. Since we have $\mathrm{A}\,\mathcal{P}\,\mathrm{B}$, $\mathrm{B}\,\mathcal{P}\,\mathrm{C}$ and $\mathrm{C}\,\mathcal{P}\,\mathrm{A}$, there is a Condorcet cycle and there is no Condorcet winner.

3.4 Consistent with strict preference

Where all voters have strict preferences over three candidates, we found that that essentially single-peaked preferences are not only sufficient for there to be a Condorcet winner, but also necessary in the following sense. If, the reduced preference profile is not single-peaked and if no preference ordering in this profile is shared by more than half of the voters, then the majority voting relation will be cyclic and there will be no Condorcet winner.

When preferences are a weak order, single-peakedness remains a sufficient condition but is no longer necessary for existence of a Condorcet winner. According to Lemma 9, there is a Condorcet winner if the preference profile takes the form found in Table **??**, although this preference profile is not single-peaked. How can we interpret this weaker sufficient condition? When all preferences are strict and there are three candidates, the assumption of single-peaked preferences requires that there is some candidate whom all voters agree is "moderate," and any voter who prefers one of the extreme candidates to the moderate will prefer the moderate to the other extreme. When extending this notion to preferences with weak orders, the definition of single-peaked preferences used by Arrow and by Sen requires, not only that no voters rank the moderate candidate as strictly worse than the other two, but never rank this candidate in a tie for worst. This assumption is stronger than one might hope, since it excludes the possibility that some voters may find one of the extremes to be a clear first choice, and do not bother to distinguish between the two less desired candidates.

Lemma 9 shows that a Condorcet winner can be guaranteed even if some voters rank the moderate candidate in a tie for third place. As we see from Table **??** there will be a Condorcet winner when the reduced preference profile includes an ordering that ranks the extreme candidate, A as first choice and is indifferent between the moderate, B, and the other extreme, C. The profile in Table **??** does not, however allow the semi-strict ordering C(BA), in which the first choice is Candidate C and where A and B are tied for last. Indeed, as we will show, in some orderings where the semi-strict ordering C(BA) dominates its opposite there will be no Condorcet winner.

Recall that with our naming convention, voters with strict preference orderings who rank C as their first choice, are outnumbered by their opposites, who rank A their first choice and C as their last choice. Thus the assumption that voters with the semi-strict ordering C(BA) are outnumbered by their opposites imposes a kind of consistency between relative numbers of semi-strict orderings and corresponding strict orderings. This motivates the following definition.

Definition 15 (Consistent with strict preferences)**.** *A preference profile is consistent with strict preferences if: (i) a candidate who is not ranked first by any strict preference ordering is not strictly preferred to the other two candidates in the profile's semi-strict orderings. (ii) a candidate who is not ranked last by any strict preference ordering is not ranked as worse than the other two candidates in any of the profile's semi-strict ordering.*

Definition 16 (Consistently single-peaked)**.** *Where preferences are a weak order with three candidates, the reduced preference profile is consistently single-peaked if the preference profile is consistent with strict preferences and the dominant strict preference relations do not make a Latin square.*

From Proposition **??** and Lemma 9, we have the following proposition, which extends the domain of preference profiles for which there is a Condorcet winner by excluding only two of the three preference orderings excluded by single-peakedness.

Proposition 5. *Where there are three candidates and individual preferences are weak orders, if the reduced form preference profile is consistently single-peaked, there is at least one Condorcet winner and the majority voting relation $\mathcal{R}$ is quasi-transitive. Except in the case where there is a voting tie, the Condorcet winner is unique and between $\mathcal{R}$ is transitive.*

4 Ties, cycles, and strategic Voting

Eric Maskin and Partha Dasgupta [4] compare pairwise majority voting with other voting mechanisms. They show that where the number of voters is large and voter preferences are strict, and the preference profile has no Condorcet cycles, pairwise majority voting is the only voting mechanism that satisfies a set of appealing assumptions, crucially including the assumption that it is strategy-proof in the very strong sense that no coalition of voters, acting cooperatively can benefit some and harm nocoalition member.

Maskin and Dasgupta avoid the complications that arise from tie votes by assuming that there there is a continuum of voters and that voters are never indifferent between two candidates. They justify the assumption of having an infinite number of voters as a good approximation to there being a very large number of voters, in which case, with any reasonable assumption about the distribution of preferences, the probability of a tie vote is negligible. The assumption of a large number of voters and the near impossibility of ties is probably appropriate for most governmental elections, but the theory of voting also has important applications for decisions of small organizations and of committees only a few voters. In these cases, tie votes become quite likely and the prospect of tie votes can not be ignored. Like Maskin and Dasgupta, we find that in the absence of tie votes, if there are no tie votes and preferences are single-peaked, pairwise majority voting produces a unique Condorcet winner and it is in the interest of voters to vote their true preferences. [4]

Lemma 10. *If pairwise majority voting always produces a unique Condorcet winner, then for any voter, it is a weakly dominant strategy to vote ones true preference.*

Proof. Suppose Candidate X is a unique Condorcet winner. The only way that a voter could cause another candidate Y to be a Condorcet winner would be to rank Y above X in his stated ordering, although he prefers X to Y. But if the voter prefers X to Y, then such a deception would make him worse off, not better off.

□

4.1 Multiple Condorcet winners and strategic voting

If, however, sincere voting produces more than one Condorcet winner, it is possible that a voter could benefit by claiming a preference ordering different from his actual ordering. If there is more than one Condorcet winner, a fully specified voting process must determine what happens when there are ties. For example, the winner might be selected at random, or by applying some other criterion based on voting results.

If there are at least two Condorcet winners and at least three candidates, it may be that a voter who prefers one of the Condorcet winners to another can gain by strategically casting her vote in such a way as to eliminate a less preferred outcome from the set of Condorcet winners.

Here is an example: The reduced preference profile is shown in Table 7. We assume that there is a positive number of voters number of voters with preference order $A \succ B \succ C$, but that this number is equal to the number with ordering $C \succ B \succ A$ so that in the reduced preference profile there are no voters of either type. We also assume that the number of voters with order $A \succ C \succ B$ is equal to the number with $B \succ A \succ C$. With this preference profile, the majority rule ordering has A $\mathcal{I}$ B, A $\mathcal{P}$ C and B $\mathcal{I}$ C. In this case, A and B are undefeated in pairwise voting and hence are both Condorcet winners. If the winner is randomly chosen from among the Condorcet winners, the outcome has a positive probability of being either A or B.

A voter whose preferences are $A \succ B \succ C$ would like to eliminate B from the Condorcet set. In the contest between B and C, this voter prefers B, and if she votes sincerely, B and C receive equally many votes. If this voter submits the ranking $A \succ C \succ B$ instead of her actual preference order, the reduced profile of submitted votes becomes that shown in Table 8.

[4] Maskin and Dasgupta use a stronger concept of strategy-proofness. Their version requires that no coalition, acting cooperatively can improve on sincere voting.

Table 7: Room for strategic voting

Ranking of candidates	Number of voters 0	k	k
1	A	A	B
2	B	C	A
3	C	B	C

Table 8: Manipulated profile

Ranking of candidates	Number of voters 1	k+1	k
1	C	A	B
2	B	C	A
3	A	B	C

In this case, A $\mathcal{I}$ B, C $\mathcal{P}$ B, and A $\mathcal{P}$ C. The unique Condorcet winner is now A. This outcome is preferred by the voter with preference ordering $A \succ B \succ C$ to the outcome from sincere voting. So it is not in her interest to vote sincerely.

4.2 Cycles and random choice

One way to choose a winner when majority voting yields cycles is to construct the "transitive closure" $\mathcal{R}^*$ of the majority preference ordering. This amounts to declaring any set of candidates in a voting cycle to be indifferent to each other. Since $\mathcal{R}^*$ is transitive, there will be a non-empty "Condorcet set" of candidates such that no member of this set can be defeated by a candidate not in the set. The mechanism could then declare the winner to be a randomly selected member of the Condorcet set.

Table 9: Room for manipulation

Ranking of candidates	Number of voters $2k-1$	k	k
1	A	B	C
2	B	C	A
3	C	A	B

A problem with this mechanism is that when it is used, voting one's true preference is not in general optimal. To see why, consider the voter profile in Table 9. This profile is a Latin square. No preference order has more than half of the voters and if voters vote their true preferences, the majority voting relation is cyclic, with no Condorcet winner. Then if all vote their true preferences, the outcome will be a lottery in which each of the three candidates is chosen with probability 1/3. Suppose that a voter with preference ordering $A \succ B \succ C$ strongly dislikes Candidate C and would rather have Candidate B than this lottery. If the other voters vote their true preferences, the voter who prefers A to B and B to C could change the outcome by voting for B in the pairwise contest between A and B. When he does so, Candidate B will defeat A by $k+1$ votes to is shared by more then half of the voters, Candidate A becomes the Condorcet winner and the majority voting relation ranks Candidate A first, Candidate B second and Candidate C third.

4.3 More thoughts on ties

As Maskin and Dasgupta [4] say, with a large number of voters, ties are extremely unlikely. But even if ties are very unlikely, is not obvious that this means that voters will ignore the possibility of ties. Timothy Fedderson and Wolfgang Pesendoerfer [5] remind us that a rational voter who cares mainly about the outcome of the election would use his vote to maximize his payoff conditional on his vote making a difference. But a voter's vote will matter only if it makes or breaks a tie. Of course a voter who believes that the probability is nearly zero that his vote will be pivotal will have a negligible incentive to think carefully about how his vote could affect the outcome and could hardly be expected to determine his best strategy in the event that his vote is pivotal.

In general, determining how to cast ones vote strategically, a voter would need to make a guess about what the others are going to do. Consider the preference profile shown in Table 7. Suppose, for example, that a voter with preference ordering $A \succ B \succ C$ does not know whether in the reduced preference profile, the number of voters with ordering $A \succ C \succ B$ is k or $k+1$, but knows that the number of voters with the other two orderings are as in the table. We have seen that if the number is k, this voter can gain by pretending to have preference relation $A \succ C \succ B$. But if, instead, the number of voters with the order $A \succ C \succ B$ is $k+1$, then if this voter votes his true preference, the outcome will be A, his first choice. If this voter pretends to have preference $A \succ C \succ B$, the result will be a tie between A and B, which leads to a random choice between A and B. Thus the voter will have harmed himself by not voting his true preference.

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This paper needs some more work. Among other things: The last 5 pages need to be edited. It also needs a concluding section and an Appendix that details the relation between this paper's results and those of Sen and Pattanaik. Some comments on plurality voting, and alternative forms of ranked voting, with attention to [3] and to [6] would probably be worthwhile.

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Appendix 1: Relation to Sen-Pattanaik results

4.4 The Sen-Pattanaik conditions

Sen and Pattanaik [11] presented conditions on preference profiles that guarantee the existence of a Condorcet winner when there are three candidates and preferences are a weak order. The Sen-Pattanaik conditions most closely related to this discussion are the following.

Definition 17 (The Not Worst condition)**.** *For some candidate x, candidate x is strictly preferred to at least one of the other two candidates by every voter.*

As Sen and Pattanaik point out:

Remark 2. *The Not Worst assumption is equivalent to the assumption of single-peaked preferences.*

Definition 18 (The Not Best condition:)**.** *There is some candidate x such that every voter prefers at least one other candidate to x.*

Remark 3. *The assumption that the reduced form profile satisfies the Not Best condition is equivalent to the assumption that this profile can be written in the form of Table* **??***.*

For each candidate x, every voter strictly prefers some other candidate to x. The Not Best assumption requires that some candidate is not the first choice of any voters. Suppose that the reduced form preference profile satisfies Not Best. Where we attach label C to the candidate that is not any voter's first choice, this requires that the reduced form profile is as shown in Table **??**. Therefore, according to Lemma 9, there is a Condorcet winner if the reduced form preference profile satisfies Not Best.

Definition 19 (The Limited Agreement condition)**.** *For some pair of candidates x and y, all voters agree that $x \mathcal{R} y$.*

Remark 4. *The Limited Agreement assumption is equivalent to the assumption that either Not Worst or Not Best applies.*

For an election with three candidates, Sen and Pattanaik [11] find three conditions on preference profiles, each of which is sufficient for pairwise majority voting to find a Condorcet winner.

These conditions are as follows:

Value restriction For any three candidates x,y,z, and for all voters i, one of the three candidates (call it x) satisfies one of these conditions.

- **Not worst** Candidate x is strictly preferred to at least one other candidate. That is: $x \succ_i y$ or $x \succ_i z$.
- **Not best** At least one other candidate is strictly preferred to Candidate x. That is: $y \succ_i x$ or $z \succ_i x$.

Not middle Candidate x is either preferred to both of the other candidates or both of the other candidates are preferred to x. That is: ($x \succ y$ and $x \succ z$) or ($y \succ x$ and ($z \succ x$)).

Extremal restriction If for at least one voter j, $x \succ_j y \succ_j z$, then for any voter i, if $z \succ_i x$. then $z \succ_i y \succ_i x$.

Limited agreement there is some pair of candidates (call them x and y) such that $x \succeq_i y$ for all voters i.

According to Sen and Pattanaik, when individual preferences are weak orders (indifference is allowed), these conditions are completely logically independent. That is, a preference profile could satisfy any of these conditions or any pair of them while not satisfying the remaining conditions.

4.5 When preferences are strict

Sen and Pattanaik show that if individual preference orderings $\succ_i$ are strong orders (without indifference), then the value restriction assumption is implied by either the extremal restriction assumption or by the limited agreement assumption. Hence the value restriction assumption is not only sufficient for there to be a Condorcet winner, but also necessary in the sense that there is no other ban on preference orderings that implies that this is the case.

If we apply the Sen-Pattanaik conditions to the reduced form of the preference profile, the equivalence of these conditions can be taken a step further. The "not best" and "not worst" assumptions imply each other and the extremal restriction assumption and the limited agreement assumption both imply and are implied by either not best or not worst. To see this, note that in the reduced form preference profile, each candidate is the second choice of at most one preference ordering, the ordering that dominates its opposite. Table **??** shows that each of the three possible profiles that are not Latin Squares satisfies each of the Sen-Pattanaik conditions, while the Latin square profile violates all of them.

In the reduced form profile, the condition "Not middle" holds only if some candidate never appears as a second choice or if the number of voters with one ordering is exactly equal to the number with its opposite. In this case, the reduced preference profile has have only two different orderings. If the number of voters with one of these orderings exceeds the number with the other, then the majority voting relation $\mathcal{P}$ is transitive and is the same as the ordering of the voters with the more common ordering. If there are equal numbers of voters with each of these two preference orderings, then $\mathcal{P}$ is quasi-transitive and the two candidates who are the first choice of one or the other of these preference orderings will be Condorcet winners.

Remark 5. *If the reduced profile satisfies any of the Sen-Pattanaik conditions, then it is not a Latin square. If the reduced profile is not a Latin square, it satisfies each of the Sen-Pattanaik conditions. Therefore if the reduced profile satisfies any of the Sen-Pattanaik conditions, it satisfies all of them.*

As Sen and Pattanaik, remark, their not worst condition is equivalent to the assumption that preferences are single-peaked, with respect to an ordering that places the candidate who is not worst between the other two candidates. Thus, when preferences are strict, all of the Sen-Pattanaik conditions as applied to the reduced preference profile are equivalent to essential single-peakedness.

4.6 When indifference is allowed

Each of the Sen-Pattanaik conditions requires that the strict preference orderings in the preference profile must not be a Latin square. In the reduced form preference profile, the three strict preference orderings that appear are those that dominate their opposites. Each candidate in the reduced form preference profile appears exactly once as second choice. When the reduced form profile is not a Latin square, one candidate must be the first choice in two of the three orderings, one candidate must appear as first choice in exactly one ordering, and one candidate must not appear as first choice in any of the orderings. We label these three candidates A, B,and C, respectively. With this naming convention, if the preference profile of strict preferences is not a Latin square, it must take the form shown in Figure 10.

Table 10: Reduced strict preference profile

Ranking of candidates	Number of voters m_1	m_2	m_3
1	A	B	B
2	B	C	A
3	C	A	C

When indifference is allowed, in addition to the three strict preference orderings that appear in the reduced form preference profile, there are three pairs of opposite preference relations in which there is indifference between two of the three candidates.[5] Where we use the same naming convention as in Table 10, these three pairs of opposite preference are (i) $A \succ B \sim C$ and $B \sim C \succ A$, (ii) $B \succ A \sim C$ and $A \sim C \succ B$ and (iii) $C \succ B \sim A$ and $B \sim A \succ C$. The reduced preference profile will include one ordering from each pair, where this ordering is held by more voters than its opposite.

The assumption of single-peaked preferences and the other Sen-Pattanaik conditions assume that some of these orderings are taboo. The reduced form construction assumes only that the "excluded" orderings are those whose adherents are outnumbered by voters with opposite preferences.

[5]There is one relation where a voter is indifferent between all three candidates. Such voters have no influence in majority voting and will be ignored here.

4.6.1 The not worst condition

The only candidate that is not worst for voters with strict preferences is Candidate B. If Candidate B is not worst for any of the preference orders that include ties, then the preference orders $A \sim C \succ B$, $C \succ B \sim A$, and $A \succ B \sim C$, cannot appear in the reduced form preference profile. Therefore the full preference profile must be as in Table 11. This can be seen to be the same reduced preference profile that we under the assumption of essential single-peaked preferences, shown in Table 4.

Table 11: Reduced form with "not worst" condition

Ranking of candidates	Number of voters m_1	m_2	m_3	m_4	m_5	m_6
1	A	B	B	B	(AB)	(BC)
2	B	C	A	(AC)	C	A
3	C	A	C			

4.6.2 The not best condition

The only candidate that is not best for any voters who have strict preferences is Candidate C. The assumption that Candidate C is not best requires that every consumer prefers some other candidate to C. To assume that the not best assumption applies to the reduced preference profile is to assume that the preference orders $A \sim C \succ B$, $C \succ B \sim A$, and $B \sim C \succ A$ are held by fewer voters than their opposites. This means that the preference profile must take the form shown in Table 12. This is the same preference profile that we found for the reduced form under the assumptions of weak single-peakedness and consistency with strict preferences.

Table 12: Reduced form with "not best" condition

Ranking of candidates	Number of voters m_1	m_2	m_3	m_4	m_5	m_6
1	A	B	B	B	(AB)	A
2	B	C	A	(AC)	C	(BC)
3	C	A	C			

4.6.3 The extremal restriction

We have labeled candidates A, B, and C, in such a way that if the extremal restriction applies, then for some voters $A \succ B \succ C$ and for some voters, $B \succ C \succ A$. The extremal condition requires that if some voters have the ordering $A \succ B \succ C$, which ranks Candidate B in the middle, then if any voter strictly prefers C to A, this voter must strictly prefer C to B and strictly prefer B to A. This requirement excludes the ordering $A \sim C \succ P$ and the ordering $C \succ A \sim B$. Since there are also some voters with the preference order $B \succ C \succ A$, it must be that any one who strictly prefers A to C must have the ranking $A \succ B \succ C$. This excludes the ordering $A \succ B \sim C$. With these three exclusions, it follows that the preference profile must be as in Table 11.

4.6.4 Limited Agreement

The limited agreement condition requires that for some pair of candidates, all voters agree that one of them is at least as good as the other. The only two candidates about which all of the strict preference orderings are Candidates B and C, with B $\mathcal{R}$ C. From the preference relations with indifference, this excludes $A \sim C \succ B$ and $C \succ B \sim A$. The remaining four preference orderings with indifference are not excluded. Preference profiles that satisfy limited agreement therefore must be one of the two profiles shown in Tables 11 and 12.

observe that with three candidates single-peakedness of the original preference profile is sufficient, but not necessary for majority voting to be transitive. They offer additional conditions, which they call value

To identify a left-right ordering for single-peakedness (if one exists) for three candidates, one can simply examine the reduced form profile. There will be such a single peaked ordering if the reduced form is not a Latin square. If the reduced profile is not a Latin square, one of the candidates must not be the third choice of any of the three preference orderings. Preferences will be single-peaked with respect to either of the left-right orderings in which that candidate is the second choice.

Notice that a sufficient condition for a preference profile to have a Condorcet winner is that some preference ordering is common to at least half of the voters.

The assumption that in the reduced form of a preference profile, some candidate has at least half of the votes is a much weaker assumption and also implies that there is a Condorcet winner.