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Metric Deformations and Intermediate Ricci Curvature

A Dissertation submitted in partial satisfaction
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by

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As I write this, two small yellow parrots named Bella and Corn are perched on either of my shoulders. Diligent as they are in keeping me company, it seems only right to include them in these acknowledgments.

This dissertation is hereby dedicated to my wife, Ruby.

ABSTRACT OF THE DISSERTATION

Metric Deformations and Intermediate Ricci Curvature

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2025
Dr. Frederick Wilhelm, Chairperson

This dissertation studies two topics in Riemannian geometry.

First, we study the existence of totally geodesic submanifolds in Riemannian 3-manifolds. Murphy and Wilhelm showed that a generic closed Riemannian manifold has no totally geodesic submanifolds, provided the ambient space is at least four dimensional. We show that the set of metrics that admit totally geodesic submanifolds on a compact 3-manifold actually contains a set that is open and dense set in the C^q-topology, provided $q \geq 3$.

Second, we study the preservation of positive intermediate Ricci curvature under Riemannian submersions. Pro and Wilhelm showed that there are Riemannian submersions $\pi : M \to B$ with M a compact manifold with positive Ricci curvature, whose b-dimensional base has Ricci curvatures with both signs. We show that if $\mathrm{Ric}_k(M) > 0$, then $\mathrm{Ric}_k(B)$ must be positive if $k \in \{1, 2, \cdots, b-1\}$, yet $\mathrm{Ric}(B)$ need not be positive if $k = b$.

Contents

Chapter 1

Summary of results

1.1 Introduction

This dissertation, which features two results in Riemannian geometry, is principally concerned in the study of metric deformations of generalized curvature. Perhaps the most popular generalization of curvature is "intermediate" curvature, which interpolates between the classical three lenses of curvature (sectional, Ricci, and scalar). In particular, our results operate within what is called *intermediate Ricci curvature*, which interpolates between sectional and Ricci curvature.

Definition. *Let $n \in \mathbb{N}$ and $k \in \{1, \cdots, n-1\}$. We say that a Riemannian manifold (M, g) with dimension n has* ***positive k^{th}-intermediate Ricci curvature****, denoted by $\mathrm{Ric}_k > 0$, if for all unit vector fields $U \in TM$ and all orthonormal sets $(V_1, \cdots, V_k)$ in $U^\perp$, we have*

$$\sum_{i=1}^{k} \sec(U, V_i) > 0.$$

Additionally, given any unit vector field $U \in TM$, we define

$$\mathrm{Ric}_k(U) := \min \sum_{i=1}^{k} \sec(U, V_i) \tag{1.1.1}$$

where the minimum is over all orthonormal sets $\{V_1, \cdots, V_k\}$ in TM that are perpendicular to U.

Positive sectional curvature is equivalent to $\mathrm{Ric}_1 > 0$*, and positive Ricci curvature is equivalent to* $\mathrm{Ric}_{n-1} > 0$*. In general,* $\mathrm{Ric}_k > 0$ *is a stronger condition than* $\mathrm{Ric}_{k+1} > 0$*.*

1.2 Totally Geodesic Submanifolds

Chapter 2 presents joint work with Frederick Wilhelm, which was published in the Annals of Global Analysis and Geometry ([EW25]).

In his magnum opus, Spivak writes that it

> *"seems rather clear that if one takes a Riemannian manifold $\big(N, \langle\cdot,\cdot\rangle\big)$ 'at random,' then it will not have any totally geodesic submanifolds of dimension > 1. But I must admit that I don't know of any specific example of such a manifold."* ([Spi99], p.39)

Murphy and Wilhelm proved that this is indeed the case for compact manifolds with dimension ≥ 4:

Theorem 1.2.1 ([MW19])**.** *Let M be a compact, smooth manifold of dimension ≥ 4. For any finite $q \geq 2$, the set of Riemannian metrics on M with no nontrivial immersed totally geodesic submanifolds contains a set that is open and dense in the C^q-topology.*

Motivated by Perelman's solution of Thurston's geometrization conjecture ([Per02, Per03a, Per03b]), it is also natural to ask about the existence of totally geodesic surfaces in compact 3-manifolds that have one of Thurston's eight canonical geometries. This problem is attacked algorithmically for a special family of surfaces in cusped hyperbolic 3-manifolds by Basillo, Lee, and Malionek in [BLM24].

In the Math Overflow post [Bry15], Bryant frames the question of whether a generic manifold has a totally geodesic hypersurface in terms of the curvature tensor. In the case of dimension 3, and when the Ricci tensor has distinct eigenvalues, he identifies the only three tangent planes at a given point that can possibly be tangent to a totally geodesic surface.

Lytchak and Petrunin confirmed Spivak's intuition in dimension 3. Indeed, the following theorem is a consequence of their main theorem in [LP22], wherein the term G_δ is used to refer to a subset of a topological space that is a countable intersection of dense open sets.

Theorem 1.2.2 ([LP22])**.** *Let M be a compact, smooth 3-manifold. For any finite $q \geq 2$, let $\mathcal{M}^q(M)$ be the space of Riemannian metrics on M with the C^q-topology. Then $\mathcal{M}^q(M)$ contains a dense G_δ, $\mathcal{G}$, so that for all $g \in \mathcal{G}$, (M, g) has no immersed totally geodesic surfaces.*

Here we refine this result by showing the following:

Main Theorem A. *Let M be a compact, smooth manifold of dimension 3. For any finite $q \geq 3$, the set of Riemannian metrics on M with no immersed totally geodesic surfaces contains a set that is open and dense in the C^q-topology.*

1.3 Submersions

Chapter 3 presents joint work, as yet unpublished, with Russell Phelan and Frederick Wilhelm ([EPW25]).

It follows from the Gray–O'Neill Horizontal Curvature Equation that Riemannian submersions preserve lower bounds on sectional curvature ([Gra67, O'N66]). In particular, they preserve positive sectional curvature. On the other hand, Pro and Wilhelm showed that they need not preserve positive Ricci curvature. More specifically they showed the following:

Theorem ([PW14])**.** *For any $C > 0$, there is a Riemannian submersion $\pi : M \to B$ for which M is compact with positive Ricci curvature and B has some Ricci curvatures less than $-C$.*

As intermediate Ricci curvatures interpolate between sectional and Ricci curvatures, it is natural to ask about the extent to which the Gray–O'Neill versus the Pro–Wilhelm results extend to the setting of intermediate Ricci curvature.

Let $\pi : M^n \to B^b$ be a Riemannian submersion from a compact n-manifold to a compact b-manifold. We will show that the Gray–O'Neill result extends to $\mathrm{Ric}_k > 0$ provided $k \leq b-1$, and the Pro–Wilhelm result extends to $\mathrm{Ric}_k > 0$ provided $k \geq b$. More precisely:

Main Theorem B. *Let $\pi : M^n \to B^b$ be a Riemannian submersion from a compact n-manifold to a compact b-manifold. Suppose that $\mathrm{Ric}_k^M > 0$ for some $k \in \{1, \ldots, n-1\}$. If $k \leq b-1$, then $\mathrm{Ric}_k^B > 0$. If $k \geq b$, then B need not have positive Ricci curvature.*

Chapter 2

Totally Geodesic Submanifolds of Riemannian 3-Manifolds

2.1 Introduction

We remind the reader that this chapter presents joint work with Frederick Wilhelm, which was published in the Annals of Global Analysis and Geometry ([EW25]).

Main Theorem A is restated below:

Theorem 2.1.1. *Let M be a compact, smooth manifold of dimension 3. For any finite $q \geq 3$, the set of Riemannian metrics on M with no immersed totally geodesic surfaces contains a set that is open and dense in the C^q-topology.*

We will take an approach that is more directly connected to [MW19]. To start, we let $\mathcal{G}r(M)$ be the Grassmannian bundle of planes tangent to M. We then define an invariant $G : \mathcal{G}r(M) \to \mathbb{R}$ that vanishes on $P \subset \mathcal{G}r(M)$ whenever P is tangent to a totally geodesic surface. We will then show how to deform a given Riemannian 3-manifold so that G is nowhere 0.

Definition 2.1.2. *Given $v \in TM$, let $R_v = R(\cdot, v)v : T_pM \to T_pM$ be the corresponding Jacobi operator, and let $\nabla_{\cdot} R_v$ be the covariant derivative of R_v. Define the **generic plane** operator, $G : \mathcal{G}r(M) \to \mathbb{R}$, by*

$$G(P) := \max_{\{v,w \in P \,:\, |v|=|w|=1\}} \left\{ \left| R_v(w)^{P\perp} \right| \;,\; \left| \left(\left(\nabla_v R_v \right)(w) \right)^{P\perp} \right| \right\},$$

*where the superscript ${}^{P\perp}$ denotes the component perpendicular to P. If $G(P) \neq 0$, then we say that P is G-**generic**. If $G(P) = 0$, then we say that P is G-**rigid**. We say that (M, g) is G-**generic**, provided all $P \in \mathcal{G}r(M)$ are generic.*

Note that if P is G-generic, then P is not tangent to any totally geodesic submanifold. Thus G-rigid planes are analogous to the "Partially Geodesic" planes of [MW19], and Theorem 2.1.1 is a consequence of the following result.

Theorem 2.1.3. *Let M be a compact, smooth 3-manifold. For any finite $q \geq 3$, the set of G-generic Riemannian metrics on M is open and dense in the C^q-topology.*

Since $\mathcal{G}r(M)$ is compact and G is continuous in the C^q-topology for any finite $q \geq 3$, the set of G-generic metrics is open in the C^q-topology. Thus it suffices to show that the set of G-generic Riemannian metrics on M is dense. More specifically it suffices to show the follow assertion.

Density Assertion. *Let (M, g) be a compact Riemannian 3-manifold. Given any finite $q \geq 3$ and $\xi > 0$, there is a G-generic Riemannian metric $\tilde{g}$ on M that satisfies $\|\tilde{g} - g\|_{C^q} \leq \xi$.*

To prove the Density Assertion we will repeatedly apply the following result.

Lemma 2.1.4. *Let $\{g_s\}_{s\geq 0}$ be a smooth family of Riemannian metrics on a compact, smooth 3-manifold M with corresponding generic plane operator G^s. Let $\mathcal{U}$ be an open subset of the Grassmannian $\mathcal{G}(M)$. Let $\mathcal{R}$ be the set of G-rigid planes for g_0 in the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$. Suppose that there are c, s_0, and a neighborhood $\mathcal{V}$ of $\mathcal{R}$ so that for every $P \in \mathcal{V}$, and every $s \in (0, s_0)$,*

$$G^s(P) > cs. \tag{2.1.1}$$

Then, for all sufficiently small s, every plane of $\overline{\mathcal{U}}$ is G^s-generic.

In particular, when $\mathcal{U} = \mathcal{G}r(M)$, Lemma 2.1.4 gives a criterion for a global deformation to a metric that is G-generic.

Proof. Since $\overline{\mathcal{U} \setminus \mathcal{V}}$ is compact, there is a $\delta > 0$ such that for all $P \in \overline{\mathcal{U} \setminus \mathcal{V}}$, we have $G^0(P) > \delta$. By continuity, there is an $s_1 > 0$ so that for all $s \in (0, s_1)$ and all $P \in \overline{\mathcal{U} \setminus \mathcal{V}}$, we have $G^s(P) > \delta$. Combining this with Inequality (2.1.1), it follows that for all sufficiently small s, every plane in $\overline{\mathcal{U}}$ is G^s-generic. $\square$

With one minor modification, the proof we present here gives an alternative proof of the special case of Theorem 1.2.1, when $q \geq 3$. The reader who is familiar with [MW19] might also recognize other similarities between the proof we present here and the one in that paper. In fact, a quick summary of our proof is to use the ideas of [MW19] with the addition of $\left|(\nabla_v R_v)(w)^{P\perp}\right|$ in Definition 2.1.2.

Since we suspect that these comments alone will not be convincing to many readers, we detail the rest of the proof below. We have prioritized having a simple, clear, and self-contained exposition over attempting to avoid some textual overlap with [MW19]. Having acknowledged this debt to [MW19], we make no further mention of it.

The proof of the Density Assertion begins by establishing a local version of it, which is the main lemma of the paper (Lemma 2.3.1 below). For the idea behind the proof of the main lemma, suppose that a plane $P \subset T_pM$ is G-rigid with orthonormal basis $\{v, T, n\} \subset T_pM$, with v, T tangent to P, and $n \perp P$. We perturb $\langle n, T \rangle$ by a function f that has either a large second or third derivative in the v-direction on a neighborhood of p. Then either $R(T, v)v$ or $(\nabla_v R)(T, v)v$ will have a large component in the n-direction. Thus, P is no longer G-rigid.

Section 2.2 establishes the notations and conventions. In Section 2.3, we state and prove the main lemma, Lemma 2.3.1, which as explained above, is a local version of the Density Assertion. In Section 2.4, we explain how to combine Lemma 2.1.4 and Lemma 2.3.1 to prove the Density Assertion and hence Theorems 2.1.1 and 2.1.3.

2.2 Notation

These remarks about notation apply only within Chapter 2.

Let (M, g) be a compact Riemannian manifold of dimension 3. The Riemannian connection, curvature tensor, and Christoffel symbols are denoted by ∇, R, and Γ. Given local coordinates $\{x_i\}_{i=1}^3$, we write ∂_i to denote the partial derivative operator, $\frac{\partial}{\partial x_i}$.

Let $\mathcal{G}r(\mathcal{M})$ denote the Grassmannian of 2-planes in M, and let $\pi : \mathcal{G}r(\mathcal{M}) \to \mathcal{M}$ be the projection of $\mathcal{G}r(\mathcal{M})$ to M. As in [MW19], we fix a metric on $\mathcal{G}r(\mathcal{M})$ so that π is a Riemannian submersion with totally geodesic fibers that are isometric to the Grassmannian of 2-planes in $\mathbb{R}^3$. For a metric space X, $A \subset X$, and $r > 0$, let

$$B_r(A) = \{x \in X : \mathrm{dist}(x, A) < r\}.$$

Let $\mathcal{R}_0$ be the set of G-rigid 2-planes for g. We put a bar over a set to denote its closure. Thus $\overline{A}$ is the closure of A.

Throughout Section 2.3, we fix a coordinate neighborhood U that has the property that $\overline{U}$ has a neighborhood V which is also a coordinate neighborhood. We use the notation $O(\varepsilon)$ to denote a quantity whose absolute value is no larger than $C\varepsilon$, where C is a positive constant that only depends on U and V.

2.3 The Local Construction

In this section, we state and prove the main lemma of the paper, which gives the method to perform our local deformation.

Lemma 2.3.1 (Main Lemma)**.** *Given $K, \eta > 0$ and sufficiently small $\rho > 0$, there are $\xi, s_0 > 0$ such that if*

$$\|g - \hat{g}\|_{C^3} < \xi$$

and P is a G-rigid plane for $\hat{g}$, then there is a C^∞-family of metrics $\{g_s\}_{s\in[0,s_0]}$ so that the following hold.

1. $g_0 = \hat{g}$.

2. *For all s, $g_s = \hat{g}$ on $M \setminus B_{\rho+\eta}(\pi(P))$.*

3. *Let $\sigma(P)$ be the section of $\mathcal{G}r(B_\rho(\pi(\mathcal{P})))$ determined by P via normal coordinates at $\pi(P)$ with respect to g. For all $s \in (0, s_0)$ and $\check{P} \in \pi^{-1}\left(B_\rho(\pi(P))\right) \cap B_\rho(\sigma(P))$,*

$$G^s\left(\check{P}\right) > Ks,$$

where G^s is the generic plane operator for g_s.

The proof begins with the following construction of a function on a coordinate neighborhood.

Lemma 2.3.2. *There is a $K_0 > 1$ with the following property. Given $K > K_0$, $\varepsilon > 0$, $p \in M$, and coordinate neighborhoods U and V of p with $\overline{U} \subset V$, there is a C^∞ function $f : M \to \mathbb{R}$ so that*

1. $\|f\|_{C^1} < \varepsilon$ *on* M.

2. $\left|\partial_j \partial_k f\right| < \varepsilon$, *except when* $j = k = 1$.

3. *For* $i, j \in \{1, 2, 3\}$ *and* $k \in \{2, 3\}$, $\partial_i \partial_j \partial_k f = 0$ *on* U, *and on* V,

$$|\partial_1 \partial_1 f| < K. \tag{2.3.1}$$

4. $f \equiv 0$ *on* $M \setminus V$.

5. *On* U, *if* $|\partial_1 \partial_1 f| \leq \sqrt{K}$, *then* $|\partial_1 \partial_1 \partial_1 f| \geq K$. *In particular, on* U

$$\max\left\{|\partial_1 \partial_1 f|, |\partial_1 \partial_1 \partial_1 f|\right\} > \sqrt{K}.$$

The proof of Lemma 2.3.2 relies on the following single variable calculus result.

Lemma 2.3.3. *There is a $K_0 > 1$ and for all $K > K_0$, and $\varepsilon \in (0, 1)$, there is a C^∞ function $h : \mathbb{R} \to \mathbb{R}$ with the following properties:*

1. $\|h\|_{C^1} < \varepsilon$.

2. *If* $\left|h''(t)\right| \leq \sqrt{K}$, *then* $\left|h'''(t)\right| \geq K$. *In particular,*

$$\sqrt{K} < \max\left\{\left|h''(t)\right|, \left|h'''(t)\right|\right\}.$$

3. *For all t,*

$$\left|h''(t)\right| \leq \frac{1}{2}K. \tag{2.3.2}$$

Proof. For $\eta = \frac{\varepsilon}{K}$, let

$$h(t) = \frac{1}{2}K\eta^2 \sin\left(\frac{t}{\eta}\right).$$

Then

$$\left|h(t)\right| \;\leq\; \frac{1}{2}K\eta^2 \;=\; \frac{1}{2}K\frac{\varepsilon^2}{K^2} \;<\; \varepsilon,$$

and

$$h'(t) = \frac{1}{2}K\eta \cos\left(\frac{t}{\eta}\right),$$

therefore $\left|h'(t)\right| \leq \frac{\varepsilon}{2}$ and $\|h\|_{C^1} < \varepsilon$, proving Part 1.

Also,

$$h''(t) = -\frac{1}{2}K \sin\left(\frac{t}{\eta}\right),$$

and therefore $\left|h''(t)\right| \leq \frac{K}{2}$, proving Part 3.

We also have

$$h'''(t) \;=\; -\frac{1}{2}\frac{K}{\eta}\cos\left(\frac{t}{\eta}\right) \;=\; -\frac{1}{2}\frac{K^2}{\varepsilon}\cos\left(\frac{t}{\eta}\right).$$

Thus, if $\left|h''(t)\right| \leq \sqrt{K}$, then

$$\frac{1}{4}K^2 \sin^2\left(\frac{t}{\eta}\right) \leq K,$$

so

$$1 - \cos^2\left(\frac{t}{\eta}\right) = \sin^2\left(\frac{t}{\eta}\right) \leq \frac{4}{K}$$

and

$$\begin{aligned}
\left|h'''(t)\right|^2 &= \frac{1}{4}\frac{K^4}{\varepsilon^2}\cos^2\left(\frac{t}{\eta}\right) \\
&\geq \frac{1}{4}\frac{K^4}{\varepsilon^2}\left(1-\frac{4}{K}\right) \\
&> K^2,
\end{aligned}$$

so long as K is sufficiently large and ε is sufficiently small, proving Part 2. $\square$

Proof of Lemma 2.3.2. Let $\Phi : V \to \mathbb{R}^3$ be the coordinate chart. Let $\pi_1 : \mathbb{R}^3 \to \mathbb{R}$ be the orthogonal projection onto the first factor. Let $\chi : \mathbb{R}^3 \to [0,1]$ be C^∞ and satisfy

$$\chi|_{\Phi(U)} \equiv 1 \qquad \text{and} \qquad \chi|_{\mathbb{R}^3 \setminus \left\{\Phi(V)\right\}}.$$

Let $m_3 > 1$ satisfy $\|\chi\|_{C^3} \leq m_3$. Choose a C^∞ function $h : \mathbb{R} \to \mathbb{R}$ that satisfies the conclusion of Lemma 2.3.3 with ε replaced by $\frac{\varepsilon}{3m_3}$. Thus

$$\|h\|_{C^1} \leq \frac{\varepsilon}{3m_3},$$

$$\max\left\{\left|h''(t)\right|, \left|h'''(t)\right|\right\} > \sqrt{K}, \quad \text{and}$$

$$\left|h''(t)\right| < \frac{1}{2}K,$$

and if $\left|h''(t)\right| \leq \sqrt{K}$, then $\left|h'''(t)\right| \geq 2K$.

Let $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\tilde{f}(p) = \chi(p) \cdot (h \circ \pi_1)(p)$$

and let $f : M \to \mathbb{R}$ be

$$f = \tilde{f} \circ \Phi.$$

Since $\chi|_{\mathbb{R}^3 \setminus \{\Phi(V)\}} \equiv 0$, we have $f \equiv 0$ on $M \setminus V$, and so Part 4 holds.

Noting that $\|h\|_{C^1} \leq \frac{\varepsilon}{3m_3}$, $\|\chi\|_{C^3} \leq m_3$, and

$$\partial_k \tilde{f}(p) = \partial_k \chi(p) \cdot (h \circ \pi_1)(p) + \chi(p) \cdot \partial_k (h_i \circ \pi_1)(p), \tag{2.3.3}$$

we conclude that $\|f\|_{C^1} < \varepsilon$, so Part 1 holds.

From (2.3.3) we get

$$\begin{aligned} \partial_j \partial_k \tilde{f}(p) = & \ \partial_j \partial_k \chi(p) \cdot (h \circ \pi_1)(p) + \partial_j \chi(p) \cdot \partial_k (h \circ \pi_1)(p) \\ & + \partial_k \chi(p) \cdot \partial_j (h \circ \pi_1)(p) + \chi(p) \cdot \partial_j \partial_k (h \circ \pi_1)(p) \end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned} \partial_i \partial_j \partial_k \tilde{f}(p) = & \ \partial_i \partial_j \partial_k \chi(p) \cdot (h \circ \pi_1)(p) + \partial_j \partial_k \chi(p) \cdot \partial_i (h \circ \pi_1)(p) \\ & + \partial_i \partial_j \chi(p) \cdot \partial_k (h \circ \pi_1)(p) + \partial_j \chi(p) \cdot \partial_i \partial_k (h \circ \pi_1)(p) \\ & + \partial_i \partial_k \chi(p) \cdot \partial_j (h \circ \pi_1)(p) + \partial_k \chi(p) \cdot \partial_i \partial_j (h \circ \pi_1)(p) \\ & + \partial_i \chi(p) \cdot \partial_j \partial_k (h \circ \pi_1)(p) + \chi(p) \cdot \partial_i \partial_j \partial_k (h \circ \pi_1)(p). \end{aligned}$$

In particular, since $\chi|_{\Phi(U)} \equiv 1$, on $\Phi(U)$,

$$\partial_j \partial_k \tilde{f}(p) = \partial_j \partial_k (h \circ \pi_1)(p) \text{ and } \partial_i \partial_j \partial_k \tilde{f}(p) = \partial_i \partial_j \partial_k (h \circ \pi_1)(p).$$

Part 5 follows from this and Lemma 2.3.3. Since $\|h\|_{C^1} \leq \frac{\varepsilon}{3m_3}$ and $\|\chi\|_{C^3} \leq m_3$, we conclude from (2.3.4) that

$$\left|\chi(p) \cdot \partial_j \partial_k \left(h \circ \pi_1\right)(p) - \partial_j \partial_k \tilde{f}(p)\right| < \varepsilon.$$

Inequality (2.3.1) follows from this and Inequality (2.3.2), provided ε is sufficiently small and K is sufficiently large. Similarly, if either i or j is different from 1, then $\partial_j \partial_k \left(h \circ \pi_1\right) = 0$, so

$$\left|\partial_j \partial_k \tilde{f}(p)\right| < \varepsilon,$$

proving Part 2. $\square$

To continue the proof of Lemma 2.3.1 let $P \in \mathcal{R}_0$ and $\hat{g}$ be as in Lemma 2.3.1 and have foot point p. Let $\{v, T, n\}$ be an ordered $\hat{g}$-orthonormal triplet at p with $v, T \in P$ and n normal to P. Let $\{E_i\}_{i=1}^3$ be an ordered coordinate frame that is defined on a coordinate neighborhood U of p with

$$E_1(p) = v, E_2(p) = T, \text{ and } E_3(p) = n. \tag{2.3.5}$$

We further assume, as in Lemma 2.3.2, that the coordinate chart of U is defined on a neighborhood V of the closure of U.

Choose g_s so that with respect to $\{E_i\}_{i=1}^3$, the matrix of $g_s - \hat{g}$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & sf \\ 0 & sf & 0 \end{pmatrix} \tag{2.3.6}$$

where f was constructed via Lemma 2.3.2 with $U = B_\rho\left(\pi(P)\right)$.

We write $\tilde{g}$ for g_s and use $\tilde{}$ for objects associated with $\tilde{g}$. With respect to $\{E_i\}_{i=1}^{3}$, the Christoffel symbols of the first kind are

$$\tilde{\Gamma}_{ij,k} = \tilde{g}\left(\tilde{\nabla}_{E_i} E_j, E_k\right).$$

To emphasize the special role of our first coordinate vector field we will write "v" for the index "1" that corresponds to E_1.

Letting $\tilde{R}_{ijkl} := \tilde{R}\left(E_i, E_j, E_k, E_l\right)$, we then have

$$\tilde{R}_{ijkl} = \partial_i \tilde{\Gamma}_{jk,l} - \partial_j \tilde{\Gamma}_{ik,l} + \tilde{g}^{\sigma\tau}\left(\tilde{\Gamma}_{ik,\sigma}\tilde{\Gamma}_{jl,\tau} - \tilde{\Gamma}_{jk,\sigma}\tilde{\Gamma}_{il,\tau}\right), \tag{2.3.7}$$

where the Einstein summation convention is used on indices σ and τ (see, for example, page 89 of [Pet16]).

We calculate $\tilde{\nabla}\tilde{R}$ in terms of Christoffel symbols of the second kind which are

$$\tilde{\nabla}_{E_i} E_j = \Gamma_{ij}^{k} E_k, \tag{2.3.8}$$

where the Einstein summation convention is used. We then have that

$$\begin{aligned}\left(\tilde{\nabla}_v \tilde{R}\right)(E_2, v, v, E_3) \quad := \quad & \tilde{\nabla}_v\left(\tilde{R}(E_2, v, v, E_3)\right) - \tilde{R}\left(\tilde{\nabla}_v E_2, v, v, E_3\right) \\ & -\tilde{R}\left(E_2, \tilde{\nabla}_v v, v, E_3\right) - \tilde{R}\left(E_2, v, \tilde{\nabla}_v v, E_3\right) \\ & -\tilde{R}\left(E_2, v, v, \tilde{\nabla}_v E_3\right).\end{aligned}$$

Together with (2.3.8), this gives us

$$2\left(\tilde{\nabla}_v \tilde{R}\right)_{2vv3} = 2\left(\partial_v \tilde{R}_{2vv3} - \tilde{\Gamma}^\tau_{v2}\tilde{R}_{\tau vv3} - \tilde{\Gamma}^\tau_{vv}\tilde{R}_{2\tau v3} - \tilde{\Gamma}^\tau_{vv}\tilde{R}_{2v\tau 3} - \tilde{\Gamma}^\tau_{v3}\tilde{R}_{2vv\tau}\right) \tag{2.3.9}$$

$$= 2\partial_v \tilde{R}_{2vv3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{v2,\mu}\tilde{R}_{\tau vv3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{vv,\mu}\tilde{R}_{2\tau v3}$$

$$- \tilde{g}^{\tau\mu}\tilde{\Gamma}_{vv,\mu}\tilde{R}_{2v\tau 3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{v3,\mu}\tilde{R}_{2vv\tau},$$

where we use the Einstein convention and the formula

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\Gamma_{ij,l}$$

on the bottom of page 66 of [Pet16] to convert between the two Christoffel types.

Before stating the next three results, we remind the reader that we have suppressed the role of s by writing $\tilde{g}$ for g_s and use $\tilde{}$ for objects associated to $\tilde{g}$. For example, this is important to keep in mind when reading Part 3 of the following result.

Proposition 2.3.4. *Let U be a coordinate neighborhood as in (2.3.5), let i, j, k, l be arbitrary elements of $\{1, 2, 3\}$, and let $\varepsilon > 0$ be as in Lemma 2.3.2.*

1. *Writing $\left(\tilde{\Gamma} - \hat{\Gamma}\right)_{ij,k}$ for $\tilde{\Gamma}_{ij,k} - \hat{\Gamma}_{ij,k}$, we have*

$$\left(\left|\tilde{\Gamma} - \hat{\Gamma}\right|\right)_{ij,k} \leq O(\varepsilon s).$$

2. *We have*

$$\begin{aligned} \partial_v(\tilde{\Gamma} - \hat{\Gamma})_{2v,3} &= \partial_v(\tilde{\Gamma} - \hat{\Gamma})_{v3,2} = -\partial_v(\tilde{\Gamma} - \hat{\Gamma})_{23,v} = \partial_v(\tilde{\Gamma} - \hat{\Gamma})_{v2,3} \\ &= \partial_v(\tilde{\Gamma} - \hat{\Gamma})_{3v,2} = -\partial_v(\tilde{\Gamma} - \hat{\Gamma})_{32,v} \\ &= \frac{s}{2}\partial_v\partial_v f, \end{aligned} \tag{2.3.10}$$

and all of the other expressions $\partial_i(\tilde{\Gamma}-\hat{\Gamma})_{jk,l}$ *with* $i,j,k\in\{1,2,3\}$ *have absolute value* $< O(\varepsilon s)$.

3. *There is a* $C>0$ *so that for all* $s\in[0,1]$,

$$\left|\tilde{\Gamma}_{ij,k}\right|\le C \text{ and } \left|\hat{\Gamma}_{ij,k}\right|\le C$$

throughout U.

4. *We have*

$$\begin{aligned}\partial_v\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{2v,3} &= \partial_v\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{v3,2}=-\partial_v\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{23,v}=\partial_v\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{v2,3}\\ &= \partial_v\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{3v,2}=-\partial_v\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{32,v}\\ &= \frac{s}{2}\partial_v\partial_v\partial_v f.\end{aligned}$$

Thus by Part 5 of Lemma 2.3.2, if $\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{2v,3}\le 2\sqrt{K}s$, *then*

$$\partial_v\partial_v(\tilde{\Gamma}-\hat{\Gamma})_{2v,3}\ge 2Ks \quad \text{on } U.$$

Proof. The proofs of Parts 1 and 2 are identical to the proofs of Propositions 2.5 and 2.6 in [MW19].

To prove Part 3, first recall that our coordinates and hence our Christoffel symbols for all of our metrics are defined on a neighborhood V of closure of U. Part 3 follows from this and the fact that $\overline{U}\times[0,1]$ is compact.

To prove Part 4, we note that by the Koszul formula, each of the expressions is equal to

$$
\begin{aligned}
&\frac{1}{2}\partial_v\partial_v\left[\partial_v(\tilde{g}-\hat{g})(E_2,E_3)+\partial_2(\tilde{g}-\hat{g})(E_3,E_v)-\partial_3(\tilde{g}-\hat{g})(E_v,E_2)\right] \\
&\quad=\frac{1}{2}\partial_v\partial_v\partial_v(\tilde{g}-\hat{g})(E_2,E_3) \qquad \text{by (2.3.6)} \\
&\quad=\frac{s}{2}\partial_v\partial_v\partial_v(f).
\end{aligned}
$$

□

Additionally, we need the following proposition.

Proposition 2.3.5. *Let U be a coordinate neighborhood as in (2.3.5). There is a $C>0$ so that on U the coefficients $\tilde{g}^{ij}$ and $\hat{g}^{ij}$ of the inverses of $\{\tilde{g}\}_{ij}$ and $\{\hat{g}\}_{ij}$ satisfy*

$$\left\|\tilde{g}^{ij}-\hat{g}^{ij}\right\|_{C^1}<C(\varepsilon s), \tag{2.3.11}$$

and for all $s\in[0,1]$,

$$\max\left\{\tilde{g}^{ij},\hat{g}^{ij}\right\}<C \tag{2.3.12}$$

throughout U.

Proof. It follows from Lemma 2.3.2 and the definition of $(\tilde{g}-\hat{g})_{ij}$ in (2.3.6) that

$$\left\|(\tilde{g}-\hat{g})_{ij}\right\|_{C^1}<O\left(\varepsilon s\right). \tag{2.3.13}$$

Since $\tilde{g}_{ij}$ and $\hat{g}_{ij}$ are defined on a neighborhood V of $\overline{U}$ they define maps

$$\hat{G} : \overline{U} \to Gl\,(n) \quad \text{and} \quad \tilde{G} : \overline{U} \times [-\varepsilon, \varepsilon] \to Gl\,(n)$$

$$\hat{G}\,(u) = \left(\hat{g}_{ij}\,(u)\right)_{ij} \quad \text{and} \quad \tilde{G}\,(u,s) = \left(\tilde{g}^{s}_{ij}\,(u)\right)_{ij}$$

that take values in a compact subset of $Gl\,(n)$. This, together with (2.3.13) and the fact that the inversion map $\mathcal{I} : Gl\,(n) \to Gl\,(n)$ is C^{∞}, gives us that there is a $C > 0$ so that

$$\left\|\tilde{g}^{ij} - \hat{g}^{ij}\right\|_{C^1} < C(\varepsilon s)$$

as claimed in (2.3.11). The proof of (2.3.12) is similar. □

We continue to write v for the first element in our frame.

Proposition 2.3.6. *Let i, j, k, l be arbitrary elements of $\{1, 2, 3\}$.*

1. *We have*
$$\left|\left(\tilde{R} - \hat{R}\right)_{ijkl}\right| \leq O(\varepsilon s), \tag{2.3.14}$$
except for up to a symmetry of $\tilde{R} - \hat{R}$, the case of $(\tilde{R} - \hat{R})_{2vv3}$. In that event, we have
$$(\tilde{R} - \hat{R})_{2vv3} = \partial_v \left(\hat{\Gamma} - \tilde{\Gamma}\right)_{2v,3} + O(\varepsilon s) = -\frac{s}{2}\partial_v\partial_v f + O(\varepsilon s). \tag{2.3.15}$$

2. *There is a $C > 0$ so that for all $s \in [0,1]$,*
$$\left|\tilde{R}_{ijkl}\right| \leq C \text{ and } \left|\hat{R}_{ijkl}\right| \leq C$$
throughout U.

3. *For K sufficiently large and as in Lemma 2.3.2, if* $\left|(\tilde{R}-\hat{R})_{2vv3}\right| \leq s\sqrt{K}$, *then*

$$\left(\tilde{\nabla}_v \tilde{R} - \hat{\nabla}_v \hat{R}\right)_{2vv3} \geq K.$$

In particular,

$$\max\left\{\left|(\tilde{R}-\hat{R})_{2vv3}\right|, \left|(\tilde{\nabla}_v \tilde{R} - \hat{\nabla}_v \hat{R})_{2vv3}\right|\right\} \geq s\sqrt{K}.$$

The proof uses two calculus estimates that we make explicit in the next two lemmas.

Lemma 2.3.7. *Let $f, g, h, \tilde{f}, \tilde{g}, \tilde{h} : \mathbb{R} \to \mathbb{R}$ be C^1-functions. Suppose that all six functions are bounded by C, and*

$$\left|f-\tilde{f}\right| < D, \quad \left|g-\tilde{g}\right| < D, \text{ and } \quad \left|h-\tilde{h}\right| < D.$$

Then $\left|fgh - \tilde{f}\tilde{g}\tilde{h}\right| \leq 3DC^2$.

Proof. Notice that

$$fgh = (f-\tilde{f})gh + \tilde{f}(g-\tilde{g})h + \tilde{f}\tilde{g}(h-\tilde{h}) + \tilde{f}\tilde{g}\tilde{h}.$$

Due to the bounds on the values of $f, g, h, \tilde{f}, \tilde{g}, \tilde{h}$ and $(f-\tilde{f})$, $(g-\tilde{g})$, $(h-\tilde{h})$ from our hypothesis, we conclude that $\left|fgh - \tilde{f}\tilde{g}\tilde{h}\right| < 3DC^2$. □

Lemma 2.3.8. *Suppose that all six functions from 2.3.7 are C^1-bounded by C, and*

$$\left\|f-\tilde{f}\right\|_{C^1} < D, \quad \left\|g-\tilde{g}\right\|_{C^1} < D, \text{ and } \quad \left\|h-\tilde{h}\right\|_{C^1} < D.$$

Then $\left\|fgh - \tilde{f}\tilde{g}\tilde{h}\right\|_{C^1} < 9DC^2$.

Proof. We have

$$\begin{aligned}(fgh)' - (\tilde{f}\tilde{g}\tilde{h})' = (f'gh - \tilde{f}'\tilde{g}\tilde{h})& \\ + (fg'h - \tilde{f}\tilde{g}'\tilde{h})& \\ + (fgh' - \tilde{f}\tilde{g}\tilde{h}').&\end{aligned}$$

We apply the following expansion to the term $f'gh$ located in the right hand side of the previous display:

$$f'gh = (f-\tilde{f})'gh + \tilde{f}'(g-\tilde{g})h + \tilde{f}'\tilde{g}(h-\tilde{h}) + \tilde{f}'\tilde{g}\tilde{h}.$$

Doing the analogous substitution for $fg'h$ and fgh' yields

$$\begin{aligned}(fgh)' - (\tilde{f}\tilde{g}\tilde{h})' = (f-\tilde{f})'gh + \tilde{f}'(g-\tilde{g})h + \tilde{f}'\tilde{g}(h-\tilde{h})& \\ + (f-\tilde{f})g'h + \tilde{f}(g-\tilde{g})'h + \tilde{f}\tilde{g}'(h-\tilde{h})& \\ + (f-\tilde{f})gh' + \tilde{f}(g-\tilde{g})h' + \tilde{f}\tilde{g}(h-\tilde{h})'.&\end{aligned}$$

Combining this with our hypothesis, we get $\left\|fgh - \tilde{f}\tilde{g}\tilde{h}\right\|_{C^1} < 9DC^2$. □

Proof of Proposition 2.3.6. By Equation 2.3.7, we have

$$\tilde{R}_{ijkl} = \partial_i\tilde{\Gamma}_{jk,l} - \partial_j\tilde{\Gamma}_{ik,l} + \tilde{g}^{\sigma\tau}\left(\tilde{\Gamma}_{ik,\sigma}\tilde{\Gamma}_{jl,\tau} - \tilde{\Gamma}_{jk,\sigma}\tilde{\Gamma}_{il,\tau}\right) \text{ and} \tag{2.3.16}$$

$$\hat{R}_{ijkl} = \partial_i\hat{\Gamma}_{jk,l} - \partial_j\hat{\Gamma}_{ik,l} + (\hat{g})^{\sigma\tau}\left(\hat{\Gamma}_{ik,\sigma}\hat{\Gamma}_{jl,\tau} - \hat{\Gamma}_{jk,\sigma}\hat{\Gamma}_{il,\tau}\right). \tag{2.3.17}$$

If $i = j = v$, then $\tilde{R}_{ijkl} = \hat{R}_{ijkl} = 0$. Otherwise, at most one of $\partial_i(\tilde{\Gamma} - \hat{\Gamma})_{jk,l}$ or $\partial_j(\tilde{\Gamma} - \hat{\Gamma})_{ik,l}$ can correspond to the indices in (2.3.10). In the event that it is $\partial_i(\tilde{\Gamma} - \hat{\Gamma})_{jk,l}$, then $i = v$ and j, k, l are distinct elements of $\{v, 2, 3\}$, so the curvature difference $(\tilde{R} - \hat{R})_{ijkl}$ is up to a sign the exceptional one in (2.3.15). For similar reasons, if $\partial_j(\tilde{\Gamma} - \hat{\Gamma})_{ik,l}$ corresponds to the indices in (2.3.10), then curvature difference $(\tilde{R} - \hat{R})_{ijkl}$ is up to a sign the exceptional one in (2.3.15). If neither $\partial_i(\tilde{\Gamma} - \hat{\Gamma})_{jk,l}$ nor $\partial_j(\tilde{\Gamma} - \hat{\Gamma})_{ik,l}$ corresponds to the indices in (2.3.10), then by combining Propositions 2.3.4 and 2.3.5 with Equations (2.3.16) and (2.3.17) and the two calculus lemmas above, we conclude that

$$\left|\left(\tilde{R} - \hat{R}\right)_{ijkl}\right| \leq O(\varepsilon s), \text{ proving (2.3.14).}$$

To prove (2.3.15), note that in this special case, Equations (2.3.16) and (2.3.17) become

$$\tilde{R}_{2vv3} = \partial_2 \tilde{\Gamma}_{vv,3} - \partial_v \tilde{\Gamma}_{2v,3} + \tilde{g}^{\sigma\tau}\left(\tilde{\Gamma}_{2v,\sigma}\tilde{\Gamma}_{v3,\tau} - \tilde{\Gamma}_{vv,\sigma}\tilde{\Gamma}_{23,\tau}\right)$$

and

$$\hat{R}_{2vv3} = \partial_2 \hat{\Gamma}_{vv,3} - \partial_v \hat{\Gamma}_{2v,3} + \tilde{g}^{\sigma\tau}\left(\hat{\Gamma}_{2v,\sigma}\hat{\Gamma}_{v3,\tau} - \hat{\Gamma}_{vv,\sigma}\hat{\Gamma}_{23,\tau}\right)$$

It follows from the Koszul formula that

$$\begin{aligned} 2\tilde{\Gamma}_{vv,3} &= 2\tilde{g}\left(\tilde{\nabla}_{E_V} E_V, E_3\right) \\ &= 2D_{E_V}\tilde{g}\left(E_V, E_3\right) - D_{E_3}\tilde{g}\left(E_V, E_V\right) \end{aligned}$$

and similarly

$$2\hat{\Gamma}_{vv,3} = 2D_{E_V}\hat{g}\left(E_V, E_3\right) - D_{E_3}\hat{g}\left(E_V, E_V\right).$$

However, from the definition of $\tilde{g}-\hat{g}$, in (2.3.6) we see that

$$\tilde{g}\left(E_V, E_3\right)-\hat{g}\left(E_V, E_3\right)=\tilde{g}\left(E_V, E_V\right)-\hat{g}\left(E_V, E_V\right) \equiv 0.$$

Hence

$$\partial_2 \tilde{\Gamma}_{vv,3}-\partial_2 \hat{\Gamma}_{vv,3} \equiv 0. \tag{2.3.18}$$

Thus

$$\begin{aligned}\left(\tilde{R}-\hat{R}\right)_{2vv3}-\partial_v\left(\hat{\Gamma}-\tilde{\Gamma}\right)_{2v,3}=\tilde{g}^{\sigma\tau}&\left(\tilde{\Gamma}_{2v,\sigma}\tilde{\Gamma}_{v3,\tau}-\tilde{\Gamma}_{vv,\sigma}\tilde{\Gamma}_{23,\tau}\right)\\ &-\tilde{g}^{\sigma\tau}\left(\hat{\Gamma}_{2v,\sigma}\hat{\Gamma}_{v3,\tau}-\hat{\Gamma}_{vv,\sigma}\hat{\Gamma}_{23,\tau}\right).\end{aligned}$$

By combining Propositions 2.3.4 and 2.3.5 with Lemma 2.3.7, we see that the right hand side of the previous display is $\leq O\left(\varepsilon s\right)$. Thus

$$\left|(\tilde{R}-\hat{R})_{2vv3}-\partial_v\left(\hat{\Gamma}-\tilde{\Gamma}\right)_{2v,3}\right| \leq O\left(\varepsilon s\right),$$

and by Part 2 of Proposition 2.3.4, $\partial_v\left(\hat{\Gamma}-\tilde{\Gamma}\right)_{2v,3}=-\frac{s}{2}\partial_v\partial_v f$, so

$$(\tilde{R}-\hat{R})_{2vv3}=\partial_v\left(\hat{\Gamma}-\tilde{\Gamma}\right)_{2v,3}+O(\varepsilon s)=-\frac{s}{2}\partial_v\partial_v f+O(\varepsilon s),$$

as claimed.

Part 2 follows from a compactness argument as in the proof of Part 3 of Proposition 2.3.4.

To prove Part 3, we recall that via Equation (2.3.9) we have

$$
\begin{aligned}
2\left(\tilde{\nabla}_v \tilde{R}\right)_{2vv3} &= 2\partial_v \tilde{R}_{2vv3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{v2,\mu}\tilde{R}_{\tau vv3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{vv,\mu}\tilde{R}_{2\tau v3} \\
&\quad - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{vv,\mu}\tilde{R}_{2v\tau 3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{v3,\mu}\tilde{R}_{2vv\tau},
\end{aligned}
$$

and

$$
\begin{aligned}
2\left(\hat{\nabla}_v \hat{R}\right)_{2vv3} &= 2\partial_v \hat{R}_{2vv3} - \hat{g}^{\tau\mu}\hat{\Gamma}_{v2,\mu}\hat{R}_{\tau vv3} - \hat{g}^{\tau\mu}\hat{\Gamma}_{vv,\mu}\hat{R}_{2\tau v3} \\
&\quad - \hat{g}^{\tau\mu}\hat{\Gamma}_{vv,\mu}\hat{R}_{2v\tau 3} - \hat{g}^{\tau\mu}\hat{\Gamma}_{v3,\mu}\hat{R}_{2vv\tau}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2\left(\left(\tilde{\nabla}_v \tilde{R}\right)_{2vv3} - \partial_v \tilde{R}_{2vv3}\right) - 2\left(\left(\hat{\nabla}_v \hat{R}\right)_{2vv3} - 2\partial_v \hat{R}_{2vv3}\right) \\
&= \left(\hat{g}^{\tau\mu}\hat{\Gamma}_{v2,\mu}\hat{R}_{\tau vv3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{v2,\mu}\tilde{R}_{\tau vv3}\right) + \left(\hat{g}^{\tau\mu}\hat{\Gamma}_{vv,\mu}\hat{R}_{2\tau v3} - \hat{g}^{\tau\mu}\hat{\Gamma}_{vv,\mu}\hat{R}_{2\tau v3}\right) \\
&\quad + \left(\hat{g}^{\tau\mu}\hat{\Gamma}_{vv,\mu}\hat{R}_{2v\tau 3} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{vv,\mu}\tilde{R}_{2v\tau 3}\right) + \left(\hat{g}^{\tau\mu}\hat{\Gamma}_{v3,\mu}\hat{R}_{2vv\tau} - \tilde{g}^{\tau\mu}\tilde{\Gamma}_{v3,\mu}\tilde{R}_{2vv\tau}\right).
\end{aligned}
\tag{2.3.19}
$$

By combining our hypothesis that $\left|(\tilde{R}-\hat{R})_{2vv3}\right| \leq s\sqrt{K}$ with Parts 1 and 2 of this result, Proposition 2.3.4, Proposition 2.3.5, and Lemma 2.3.7, we see that each of the four terms in the parentheses on the right hand side of (2.3.19) is $\leq O\left(s\sqrt{K}\right)$. So

$$
\left|2\left(\left(\tilde{\nabla}_v \tilde{R}\right)_{2vv3} - \partial_v \tilde{R}_{2vv3}\right) - 2\left(\left(\hat{\nabla}_v \hat{R}\right)_{2vv3} - 2\partial_v \hat{R}_{2vv3}\right)\right| \leq O\left(s\sqrt{K}\right). \tag{2.3.20}
$$

By Equation 2.3.7, we have

$$
\partial_v \tilde{R}_{2vv3} = \partial_v \partial_2 \tilde{\Gamma}_{vv,3} - \partial_v \partial_v \tilde{\Gamma}_{2v,3} + \partial_v\left(\tilde{g}^{\sigma\tau}\left(\tilde{\Gamma}_{2v,\sigma}\tilde{\Gamma}_{v3,\tau} - \tilde{\Gamma}_{vv,\sigma}\tilde{\Gamma}_{23,\tau}\right)\right)
$$

and

$$\partial_v \hat{R}_{2vv3} = \partial_v \partial_2 \hat{\Gamma}_{vv,3} - \partial_v \partial_v \hat{\Gamma}_{2v,3} + \partial_v \left(\hat{g}^{\sigma\tau} \left(\hat{\Gamma}_{2v,\sigma} \hat{\Gamma}_{v3,\tau} - \hat{\Gamma}_{vv,\sigma} \hat{\Gamma}_{23,\tau} \right) \right).$$

It follows from Equation (2.3.18) that

$$\partial_v \partial_2 \tilde{\Gamma}_{vv,3} - \partial_v \partial_2 \hat{\Gamma}_{vv,3} \equiv 0.$$

Thus

$$\begin{aligned} &\left| \left(\partial_v \left(\tilde{R}_{2vv3} - \hat{R}_{2vv3} \right) \right) - \partial_v \partial_v \left(\hat{\Gamma}_{2v,3} - \tilde{\Gamma}_{2v,3} \right) \right| \\ &= \partial_v \left(\left(\tilde{g}^{\sigma\tau} \tilde{\Gamma}_{2v,\sigma} \tilde{\Gamma}_{v3,\tau} - \hat{g}^{\sigma\tau} \hat{\Gamma}_{2v,\sigma} \hat{\Gamma}_{v3,\tau} \right) \right) \\ &\quad + \partial_v \left(\hat{g}^{\sigma\tau} \hat{\Gamma}_{vv,\sigma} \hat{\Gamma}_{23,\tau} - \tilde{g}^{\sigma\tau} \tilde{\Gamma}_{vv,\sigma} \tilde{\Gamma}_{23,\tau} \right). \end{aligned} \tag{2.3.21}$$

Equation (2.3.15), together with our hypothesis that $\left| (\tilde{R} - \hat{R})_{2vv3} \right| \leq s\sqrt{K}$, and Part 2 of Proposition 2.3.4 together give us that

$$\left| \partial_v \left(\tilde{\Gamma} - \hat{\Gamma} \right)_{ij,k} \right| \leq 2s\sqrt{K}. \tag{2.3.22}$$

Combining this with Proposition 2.3.4, Proposition 2.3.5, and Lemma 2.3.8, we see that each term on the right hand side of (2.3.21) is $\leq O\left(s\sqrt{K} \right)$. So

$$\left| \left(\partial_v \tilde{R}_{2vv3} - \partial_v \hat{R}_{2vv3} \right) - \left(\partial_v \partial_v \left(\hat{\Gamma}_{2v,3} - \tilde{\Gamma}_{2v,3} \right) \right) \right| \leq O\left(s\sqrt{K} \right).$$

From Part 4 of Proposition 2.3.4, we have $\partial_v \partial_v \left(\hat{\Gamma}_{2v,3} - \tilde{\Gamma}_{2v,3}\right) = \frac{s}{2}\partial_v \partial_v \partial_v f$. This, the previous display, and (2.3.20) combine to give us that

$$\left|\left(\tilde{\nabla}_v \tilde{R}\right)_{2vv3} - \left(\hat{\nabla}_v \hat{R}\right)_{2vv3} - \frac{s}{2}\partial_v \partial_v \partial_v f\right| \leq O\left(s\sqrt{K}\right).$$

Inequality (2.3.22) together with (2.3.10) gives us $|\partial_v \partial_v f| \leq 4\sqrt{K}$. Thus by Part 5 of Lemma 2.3.2, with $4\sqrt{K}$ playing the role of $\sqrt{K}$, $|\partial_v \partial_v \partial_v f| \geq 16K$. Therefore

$$\left|\left(\tilde{\nabla}_v \tilde{R}\right)_{2vv3} - \left(\hat{\nabla}_v \hat{R}\right)_{2vv3}\right| \geq 16K - O\left(s\sqrt{K}\right) > K,$$

if K is sufficiently large. □

Part 1 of Lemma 2.3.1 follows from our definition of $g_s - \hat{g}$ in (2.3.6). Part 2 of Lemma 2.3.1 follows from our choice of f; see Part 4 of Lemma 2.3.2. Part 3 of Lemma 2.3.1 follows from Part 3 of Proposition 2.3.6 and the definition of the generic plane operator.

2.4 The Global Argument

We prove the Density Assertion by repeatedly applying Lemma 2.3.1 and Lemma 2.1.4. Except for changes of notation and terminology, the argument is the same as the proof of the "Modified l^{th}-Partially Geodesic Assertion" in [MW19], so we only give a brief description here.

First construct an open cover $\{\mathcal{U}_i\}_{i=1}^K$ of the set $\mathcal{R}_0$ of G-rigid planes. We will apply the Lemma 2.3.1 and Lemma 2.1.4 successively to $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_K$. So we assume that each $\mathcal{U}_i$ has the form $B_\rho\left(\pi(P)\right)$, where the notation is as in Lemma 2.3.1. Prior to the first step, notice that since $\{\mathcal{U}_i\}_{i=1}^K$ covers $\mathcal{R}_0$, the generic plane operator

$G : \mathcal{G}r(M) \to \mathbb{R}$ is positive on $\mathcal{G}r(M) \setminus \cup_{i=1}^K \mathcal{U}_i$. Since $\mathcal{G}r(M) \setminus \cup_{i=1}^K \mathcal{U}_i$ is compact, there is a $\delta_0 > 0$ so that

$$G|_{\mathcal{G}r(M)\setminus\cup_{i=1}^K \mathcal{U}_i} > \delta_0. \tag{2.4.1}$$

With δ_0 in hand, we perform our first deformation by applying Lemma 2.3.1 to $\mathcal{U}_1$. It follows from Lemma 2.1.4 that if the deformation is small enough, then the resulting metric g^1 is G-generic on $\mathcal{U}_1$. By combining (2.4.1) with a possible further restriction of the deformation size, we can also conclude that the set $\mathcal{R}_1$ of G-rigid planes of g^1 is contained in $\cup_{i=2}^K \mathcal{U}_i$. As above, it follows from compactness that there is a δ_1 so that

$$G^{g^1}|_{\mathcal{G}r(M)\setminus\cup_{i=2}^K \mathcal{U}_i} > \delta_1, \tag{2.4.2}$$

where G^{g^1} is the generic plane operator for g^1.

For the second step, apply Lemma 2.3.1 to g^1 on $\mathcal{U}_2$ to obtain a deformation of metrics that, by Lemma 2.1.4, are G-generic on $\mathcal{U}_2$. By (2.4.2), if the deformation is small enough, then the set $\mathcal{R}_2$ of G-rigid planes of g^2 is contained in $\cup_{i=3}^K \mathcal{U}_i$. Proceeding inductively in this manner, we obtain a G-generic metric that is as close as we please to g.

Chapter 3

Riemannian Submersions and Intermediate Ricci Curvature

3.1 Introduction

We remind the reader that this chapter presents joint work, as yet unpublished, with Russell Phelan and Frederick Wilhelm ([EPW25]).

Main Theorem B is restated below:

Theorem 3.1.1. *Let $\pi : M^n \to B^b$ be a Riemannian submersion from a compact n-manifold to a compact b-manifold. Suppose that $\mathrm{Ric}_k^M > 0$ for some $k \in \{1, \ldots, n-1\}$.*

1. *If $k \leq b-1$, then $\mathrm{Ric}_k^B > 0$.*

2. *If $k \geq b$, then B need not have positive Ricci curvature.*

Part 1 of Theorem 3.1.1 is an easy consequence of the Gray–O'Neill Horizontal Curvature Equation. Suppose that $\mathrm{Ric}_k^M > 0$ for some $k \in \{1, 2, \ldots b-1\}$. Given any unit vector U tangent to B, by compactness, there is an orthonormal set $\{U, V_1, \ldots, V_k\}$

that realizes $\min(\mathrm{Ric}_k^B)$. Let $\{\tilde{U}, \tilde{V}_1, \ldots, \tilde{V}_k\}$ be any horizontal lift of $\{U, V_1, \ldots, V_k\}$ to TM. Then

$$\begin{aligned}
\min(\mathrm{Ric}_k^B) &= \sum_{i=1}^{k} \sec^B(U, V_i) \\
&= \sum_{i=1}^{k} \left(\sec^M \left(\tilde{U}, \tilde{V}_i \right) + 3 \left| A_{\tilde{U}} \tilde{v}_i \right|^2 \right) \\
&\geq \sum_{i=1}^{k} \sec^M(\tilde{U}, \tilde{V}_i) \\
&> 0,
\end{aligned}$$

where A is the Gray–O'Neill A-tensor. Thus $\mathrm{Ric}_k^B > 0$ as claimed in Part 1 of Theorem 3.1.1.

In the remainder of the paper we prove the following generalization of Part 2 of Theorem 3.1.1.

Theorem 3.1.2. *Let $\pi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion from a compact n-manifold M. Suppose that $\dim(B) = b$ and for some $k \geq b$, $\mathrm{Ric}_k^M > 0$.*

Then for every $\varepsilon > 0$ there are Riemannian metrics $\tilde{g}_M$ and $\tilde{g}_B$ on M and B so that

1. *With respect to the new metrics $\tilde{g}_M$ and $\tilde{g}_B$, the function $\pi : (M, \tilde{g}_M) \to (B, \tilde{g}_B)$ is a Riemannian submersion.*

2. *With respect to $\tilde{g}_M$, $\widetilde{\mathrm{Ric}}_k^M > 0$.*

3. *$(B, \tilde{g}_B)$ has Ricci curvatures of both signs.*

4. *The C^1 distances of $\tilde{g_M}$ and $\tilde{g_B}$ to the original metrics satisfy,*

$$\|g_M - \tilde{g_M}\|_{C^1} < \varepsilon \qquad \textit{and} \qquad \|g_B - \tilde{g_B}\|_{C^1} < \varepsilon.$$

In fact, given any $K > 0$ and $p \in B$, we can choose $\tilde{g}_M$ to have all of the properties above and, additionally, have

$$\begin{aligned} \mathrm{Ric}(B, \tilde{g}_B)|_p &< -K(b-1), \quad \textit{and} \\ \mathrm{Ric}_k(M, \tilde{g}_M) &\geq \min \mathrm{Ric}_k(M, g_M) - \varepsilon \end{aligned} \tag{3.1.1}$$

In Theorem 3.1.2, when $k = n-1$, M has positive Ricci curvature. So Theorem 3.1.2 implies that among metrics with positive Ricci curvature for which a given submersion is Riemannian, those that do not project to positive Ricci curvature always exist, and in fact are dense in the C^1-topology.

Section 3.2 establishes the notations and conventions of this paper. In particular, because it is convenient for the proof of Theorem 3.1.2, we define the C^1-norm of a $(0, 2)$-tensor relative to background metric. We then compare this definition to the standard one that is given in terms of a fixed atlas. Section 3.3 details the construction of the functions that will be used to deform g_M and g_B to create $\tilde{g}_M$ and $\tilde{g}_B$, respectively, both of which will be rigorously defined in Section 3.4. In Section 3.5, we verify Inequality (3.1.1). This together with Theorem 2 of [PW14] implies that $\tilde{g}_B$ has Ricci curvature of both signs. Section 3.6 establishes a method to verify $\mathrm{Ric}_k > 0$, which is based, in part, on a lemma of Reiser and Wraith from [RW23]. Finally, Section 3.7 completes the proof of Theorem 3.1.2 by verifying positive intermediate Ricci curvature on the total space.

3.2 Notation

We denote the Riemannian connection, curvature tensor, and curvature operator by ∇, R, and $\mathcal{R}$ and the vertical and horizontal distributions of a submersion π :

$(M^m, g_M) \to (B^b, g_B)$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. The horizontal or vertical component of a vector E are denoted by $E^{\mathbf{h}}$ and $E^{\mathbf{v}}$, respectively. We often write X, Y, and Z to represent horizontal vector fields, and T, U, and V to represent vertical vector fields.

We will rely on several curvature calculations from Gromoll and Walschap's book [GW09], so to facilitate the verification of our assertions, we adopt the notation system in [GW09] for the fundamental tensors of a Riemannian submersion, rather than those of Gray and O'Neill ([Gra67], [O'N66]). For the reader's convenience we review these here.

We define the tensor $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ by

$$A_X Y = \frac{1}{2}[X, Y]^{\mathbf{v}}, \tag{3.2.1}$$

and its adjoint $A^* : \mathcal{H} \times \mathcal{V} \to \mathcal{H}$ is denoted $A^*_X V$. We also define the tensor $S : \mathcal{V} \times \mathcal{H} \to \mathcal{V}$ by

$$S_X U = -(\nabla_U X)^{\mathbf{v}}. \tag{3.2.2}$$

The second fundamental form of the fibers is denoted by σ, thus

$$\sigma(U, V) = (\nabla_U V)^{\mathbf{h}}. \tag{3.2.3}$$

The operation $\circ$ denotes the Kulkarni–Nomizu product of two $(0, 2)$-tensors, given by

$$\begin{aligned} \alpha \circ \beta(v_1, v_2, v_3, v_4) = &\frac{1}{2}\left(\alpha(v_1, v_4)\beta(v_2, v_3) + \alpha(v_2, v_3)\beta(v_1, v_4)\right) \\ &+ \frac{1}{2}\left(\alpha(v_1, v_3)\beta(v_2, v_4) + \alpha(v_2, v_4)\beta(v_1, v_3)\right). \end{aligned} \tag{3.2.4}$$

(See Exercise 3.4.23 in [Pet16].)

C^1-Topology on the space of Riemannian metrics

Fix a compact smooth manifold M. The usual definitions of the C^0 and C^1-norms on the space of $(0, 2)$-tensors on M is the maximum one obtains by evaluating a metric h on all of the coordinated fields of some fixed finite atlas, and in the case of the C^1-topology also considering the derivatives of the components of h in all possible coordinate directions. More precisely,

Definition 3.2.1. *Let M be a compact, smooth n-manifold with fixed finite atlas $\mathcal{A}$. Given $(U, \varphi) \in \mathcal{A}$, write*

$$h_{i,j} := h(E_i, E_j),$$

where E_i, E_j, are coordinate fields of (U, φ). The C^0-norm of a $(0, 2)$-tensor h with respect to $\mathcal{A}$ is then

$$\|h\|_{C^0,\mathcal{A}} := \max h(E_i, E_j),$$

where the maximum taken is over all of the coordinate fields E_i, E_j of all of the coordinate charts of $\mathcal{A}$. Similarly, the C^1-norm of a $(0, 2)$-tensor h with respect to $\mathcal{A}$ is

$$\|h\|_{C^1,\mathcal{A}} := \max\{\partial_k h_{i,j}, \|h\|_{C^0,\mathcal{A}}\},$$

where the maximum taken is over all of the coordinate fields E_i, E_j E_k of all of the coordinate charts of $\mathcal{A}$.

For us, it will be simpler and more convenient to define these notions in terms of a fixed background Riemannian metric g_0.

Definition 3.2.2. *Fix a compact Riemannian manifold (M, g_0). Given $\varepsilon > 0$ and a $(0, 2)$-tensor h we say that*

$$\|h\|_{C^0,g_0} < \varepsilon,$$

provided for all $u \in TM$,

$$\left|h(u,u)\right| \leq \varepsilon g_0(u,u).$$

Definition 3.2.3. *Let (M, g_0) be a compact Riemannian manifold. Given $\varepsilon > 0$ and a $(0,2)$-tensor h, we say that*

$$\|h\|_{C^1, g_0} < \varepsilon,$$

provided

$$\|h\|_{C^0, g_0} < \varepsilon,$$

and for all $u, w \in TM$,

$$\left|\left(\nabla^{g_0}_w h\right)(u,u)\right| \leq \varepsilon g_0(u,u) \|w\|_{g_0},$$

where ∇^{g_0} denotes covariant derivative with respect to g_0.

All four notions of norms then yield a metric space structure in the usual way, that is,

$$\operatorname{dist}(h, \tilde{h}) := \left\|h - \tilde{h}\right\|,$$

where $\left\|h - \tilde{h}\right\|$ is any of the four notions of norms defined above.

Finally, we record the fact for $p \in \{0, 1\}$, the two C^p-norms yield the same topology on the set of $(0,2)$-tensors.

Proposition 3.2.4. *Let M be a compact, smooth n-manifold with fixed finite atlas $\mathcal{A} := \{(U, \varphi_U)\}$ and Riemannian metric g_0. Given $\varepsilon > 0$ and any $(0,2)$-tensor h there are $\delta, \tilde{\delta} > 0$ with the following property. Given any $p \in \{0, 1\}$ and any $(0,2)$-tensor $\tilde{h}$ on M, if*

$$\left\|h - \tilde{h}\right\|_{C^p, \mathcal{A}} < \delta, \text{ then } \left\|h - \tilde{h}\right\|_{C^p, g_0} < \varepsilon. \tag{3.2.5}$$

and if

$$\left\|h-\tilde{h}\right\|_{C^p,g_0}<\tilde{\delta},\ \textit{then}\ \left\|h-\tilde{h}\right\|_{C^p,\mathcal{A}}<\varepsilon. \tag{3.2.6}$$

Proof. Since there are only a finite number of possibilities for coordinate fields u and w for $(U,\varphi_U)\in\mathcal{A}$, there is a $K>1$ so that for all coordinate fields u,w of $\mathcal{A}$,

$$\frac{1}{K}\le\max\left\{\|u\|_{g_0}\ ,\ \|\nabla^{g_0}_w u\|_{g_0}\right\}\le K. \tag{3.2.7}$$

The versions of Statements (3.2.5) and (3.2.6) for the C^0-topology follows from

$$\frac{1}{K}\le\|u\|_{g_0}\le K.$$

The bounds in (3.2.7) also give us

$$\begin{aligned}\left|\nabla^{g_0}_w(h-\tilde{h})(u,u)\right| &\le \left|D_wh(u,u)-D_w\tilde{h}(u,u)\right|+2\left|(h-\tilde{h})(\nabla^{g_0}_w u,u)\right|\\ &\le \left\|h-\tilde{h}\right\|_{C^1,\mathcal{A}}+2K\left\|h-\tilde{h}\right\|_{C^0,\mathcal{A}}\\ &\le (1+2K)\left\|h-\tilde{h}\right\|_{C^1,\mathcal{A}}.\end{aligned}$$

The version of (3.2.5) where $p=1$ follows by combining this with the version of (3.2.5) for $p=0$.

In the event that $\left\|h-\tilde{h}\right\|_{C^1,\mathcal{A}}=\left\|h-\tilde{h}\right\|_{C^0,\mathcal{A}}$, the version of (3.2.6) for $p=1$ follows from the version for $p=0$. Otherwise, we let u,v, and w be coordinate fields so that at a point p,

$$\left|D_wh(u,v)-D_w\tilde{h}(u,v)\right|=\left\|h-\tilde{h}\right\|_{C^1,\mathcal{A}}$$

then

$$\begin{aligned}
\left\| h - \tilde{h} \right\|_{C^1,\mathcal{A}} &= \left| D_w(h - \tilde{h})(u, v)) \right| \\
&\leq \left| \left(\nabla_w^{g_0}(h - \tilde{h}) \right) (u, v) \right| + \left| (h - \tilde{h})(\nabla_w^{g_0} u, v) \right| \\
&\leq \left\| h - \tilde{h} \right\|_{C^1,g_0} + K^2 \left\| h - \tilde{h} \right\|_{C^0,g_0} \\
&\leq (1 + K^2) \left\| h - \tilde{h} \right\|_{C^1,g_0}.
\end{aligned}$$

The version of (3.2.6) where $p = 1$ follows by combining this with the version of (3.2.6) for $p = 0$. $\square$

For the remainder of the paper, the notation $\|\cdot\|$ (with no indication of which norm is being taken) refers to the (C^0, g_0)-norm from Definition 3.2.2.

We use the notation $O(t)$ to denote a number, vector, or tensor whose norm is no larger than Ct, where C is a positive constant that only depends on g_0.

3.3 Construction of Functions

To create the metric $\tilde{g}_M$ that proves Theorem 3.1.2, we will construct two functions using the following lemma.

Lemma 3.3.1. *Let M be a compact Riemannian n-manifold. Given any $p \in M$, $C \in \mathbb{R} \setminus [-1, 1]$, and $\varepsilon, \eta \in \left(0, \frac{1}{2} \min\{1, \operatorname{inj}_M\}\right)$, and any $\tau \in (0, \frac{\varepsilon}{2C})$ there is a smooth function $\omega : M \to \mathbb{R}$ so that*

1. $\operatorname{grad} \omega|_p = 0$,

2. $\|\omega\|_{C^1} < \varepsilon$,

3. $\mathrm{supp}(\omega) \subset B(p, 2\eta)$.

Moreover, if $C > 0$, we can choose ω so that for all unit vector fields $Z \in TM$,

$$-\varepsilon \leq \mathrm{Hess}_\omega(Z, Z) \leq 3C, \tag{3.3.1}$$

and for all unit vector fields $Z \in TM|_{B(p,\tau)}$,

$$C \leq \mathrm{Hess}_\omega(Z, Z). \tag{3.3.2}$$

If $C < 0$, we can choose ω so that for all unit vector fields $Z \in TM$,

$$3C \leq \mathrm{Hess}_\omega(Z, Z) \leq \varepsilon. \tag{3.3.3}$$

and for all unit vector fields $Z \in TM|_{B(p,\tau)}$,

$$\mathrm{Hess}_\omega(Z, Z) \leq C. \tag{3.3.4}$$

Before proving Lemma 3.3.1 we establish two preliminary results.

Lemma 3.3.2. *For every $\varepsilon > 0$ and $K > 1$, there is a $\delta > 0$ with the following property. Let $\eta > 0$ and let $\lambda : \mathbb{R} \to [0, 1]$ be a C^2-smooth function satisfying*

$$\begin{aligned} \lambda|_{\mathbb{R}\setminus[-2\eta,2\eta]} &\equiv 0, \\ \lambda|_{[-\eta,\eta]} &\equiv 1, \\ \|\lambda\|_{C^2} &< K. \end{aligned} \tag{3.3.5}$$

Let $f, g : [-3\eta, 3\eta] \to \mathbb{R}$ *be* C^2 *and set*

$$\varphi := \lambda f + (1-\lambda)g. \tag{3.3.6}$$

If

$$\|f - g\|_{C^1} < \delta, \tag{3.3.7}$$

then

$$\begin{gathered} \left\|(f-\varphi)\right\|_{C^1} < \varepsilon, \qquad \left\|(g-\varphi)\right\|_{C^1} < \varepsilon, \quad \textit{and} \\ \left\|(\varphi'' - (\lambda f'' + (1-\lambda)g''))\right\|_{C^0} < \varepsilon. \end{gathered} \tag{3.3.8}$$

Proof. Choose $\delta < \min\{\frac{\varepsilon}{3K}, \varepsilon\}$. Then (3.3.5), (3.3.6), and (3.3.7) together give us

$$\begin{aligned} \left|\varphi' - (\lambda f' + (1-\lambda)g')\right| &= \left|\lambda'(f-g)\right| \\ &< K\delta \\ &< \varepsilon \end{aligned}$$

and

$$\begin{aligned} \left|\varphi'' - (\lambda f'' + (1-\lambda)g'')\right| &= \left|2\lambda'(f'-g') + \lambda''(f-g)\right| \\ &< 3K\delta \\ &< \varepsilon. \end{aligned}$$

Equation (3.3.6) together with the previous two displays tell us that the values of φ, φ', and φ'' never deviate further than ε from a weighted average of f and g, f' and

g', and f'' and g'', respectively. This together with $\|(f-g)\|_{C^1} < \delta < \varepsilon$, yields the inequalities in (3.3.8). $\square$

Lemma 3.3.3. *Given $\varepsilon, \eta \in (0,1)$, $C \in \mathbb{R} \setminus \{0\}$, and any $\tau \in (0, \frac{\varepsilon}{2C})$ there is a function $\varphi : \mathbb{R} \to \mathbb{R}$ with the following properties:*

$$\varphi(t) = \frac{C}{2} t^2 \ \textit{ for } \ t \in [-\tau, \tau],$$

$$\varphi \equiv 0 \ \textit{ on } \mathbb{R} \setminus [-2\eta, 2\eta],$$

$$\|\varphi\|_{C^1} < \varepsilon \ \textit{ on } \mathbb{R}, \tag{3.3.9}$$

and

$$\|\varphi\|_{C^2} < \varepsilon \ \textit{ on } \mathbb{R} \setminus [-\eta, \eta]. \tag{3.3.10}$$

Moreover if $C > 0$, we can choose φ so that

$$\varphi'|_{[0,\eta]} \geq 0, \ \ \textit{and} \ \ \varphi'' \in (-\varepsilon, C], \tag{3.3.11}$$

and if $C < 0$, we can choose φ so that

$$\varphi'|_{[0,\eta]} \leq 0 \ \ \textit{and} \ \ \varphi'' \in [C, \varepsilon).$$

Proof. For simplicity, we just consider the case where $C > 0$. For some $\nu \in (0, \frac{\varepsilon}{2C})$ and $\tau \in (0, \nu)$ let $h_\nu : [-2\eta, 2\eta] \to [0, \infty)$ satisfy

$$h_\nu(t) = C \quad \text{for} \quad t \in [-\tau, \tau] \tag{3.3.12}$$

$$h_\nu|_{(-\nu,\nu)} \geq 0, \ \ h'_\nu|_{(-\nu,0)} > 0 \quad \text{and} \quad h'_\nu|_{(0,\nu)} < 0, \tag{3.3.13}$$

and

$$h_\nu|_{[-2\eta,2\eta]\setminus[-\nu,\nu]} \equiv 0.$$

We set

$$f(x) := \int_0^x \int_0^t h_\nu(s)\,\mathrm{d}s\,\mathrm{d}t,$$

and it follows that

$$f'(t) := \int_0^t h_\nu(s)\,\mathrm{d}s \;\text{ and }\; f''(s) = h_\nu(s).$$

In particular,

$$f'(t) \geq 0 \quad \text{for} \quad t \geq 0, \tag{3.3.14}$$

$$f(t) = \frac{C}{2}t^2 \quad \text{for} \quad t \in [-\tau,\tau], \tag{3.3.15}$$

and

$$\begin{aligned} \left\| f|_{[-2\eta,2\eta]} \right\|_{C^1} \leq \; & \sup_{x\in[-2\eta,2\eta]} \left| \int_0^x \int_0^t h_\nu(s)\,\mathrm{d}s\,\mathrm{d}t \right| + \sup_{t\in[-2\eta,2\eta]} \left| \int_0^t h_\nu(s)\,\mathrm{d}s \right| \\ \leq \; & C\nu\eta + C\nu. \end{aligned}$$

Since

$$f''|_{[-2\eta,2\eta]\setminus[-\eta,\eta]} = h_\nu|_{[-2\eta,2\eta]\setminus[-\eta,\eta]} \equiv 0$$

we also have that $\|f\|_{C^2} = \|f\|_{C^1}$ on $[-2\eta,2\eta]\setminus[-\eta,\eta]$, so

$$\left\| f|_{[-2\eta,2\eta]\setminus[-\eta,\eta]} \right\|_{C^2} \leq C\nu\eta + C\nu.$$

To construct φ, apply Lemma 3.3.2 with f as above and $g \equiv 0$. Note that since $\varphi|_{[0,\eta]} = f|_{[0,\eta]}$ it follows from (3.3.15) that

$$\varphi(t) = \frac{C}{2}t^2 \text{ for } t \in [-\tau, \tau].$$

Similarly, $\varphi|_{[0,\eta]} = f|_{[0,\eta]}$ together with (3.3.14) gives us that $\varphi'|_{[0,\eta]} \geq 0$, as claimed in (3.3.11).

By Lemma 3.3.2, on $[-2\eta, 2\eta] \backslash [-\eta, \eta]$, the C^2-distances of φ to each of f and 0 satisfy

$$\|\varphi - f\|_{C^2} < \varepsilon \text{ and } \|\varphi\|_{C^2} < \varepsilon.$$

Thus (3.3.10) holds, and we have that $\varphi'' > -\varepsilon$ on $[-2\eta, 2\eta] \backslash [-\eta, \eta]$. Since $\varphi = g \equiv 0$ on $\mathbb{R} \setminus [-2\eta, 2\eta]$, we have

$$\varphi \equiv 0 \text{ on } \mathbb{R} \setminus [-2\eta, 2\eta].$$

Since $\varphi = f$ on $[-\eta, \eta]$, we have that $\varphi''|_{[-\tau,\tau]} = f''|_{[-\tau,\tau]} = C$, and since $f'' = h$ is positive on that interval and has a maximum at 0 (see Equations (3.3.12) and (3.3.13)), we have that

$$\varphi'' \in (-\varepsilon, C],$$

which completes the proof of (3.3.11).

Finally,

$$\begin{aligned} \left\|\varphi|_{[-\eta,\eta]}\right\|_{C^1} &= \left\|f_{[-\eta,\eta]}\right\|_{C^1} \\ &< C\nu\eta + C\nu \\ &< 2C\nu, \text{ since } \eta < 1 \\ &< \varepsilon, \end{aligned}$$

since $\nu < \frac{\varepsilon}{2C}$, which together with (3.3.10) completes the proof of (3.3.9). $\square$

Proof of Lemma 3.3.1. Set

$$X = \nabla \operatorname{dist}_p .$$

Given any $\varepsilon, \eta > 0$, notice that if the conclusion holds for η and some $\tilde{\varepsilon} \in (0, \varepsilon)$, then the conclusion holds for η and our given ε. Now apply Lemma 3.3.3 with ε, C, η, and τ transformed as indicated by the following table:

Lemma 3.3.1 Quantity	$\longrightarrow$	Lemma 3.3.3 Quantity
ε	$\longmapsto$	$\varepsilon\eta^3$
C	$\longmapsto$	$2C$
η	$\longmapsto$	η
τ	$\longmapsto$	τ

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the resulting function, and set

$$\omega(x) := \varphi \circ \left(\operatorname{dist}_p(x)\right) .$$

Since $\varphi(t) = \frac{C}{2}t^2$ on $[-\tau, \tau]$, we have that the restriction of ω to $B(x, \tau)$ is

$$\omega(\cdot) = \frac{C}{2}\left(\operatorname{dist}_p(\cdot)\right)^2 , \tag{3.3.16}$$

which is smooth. Since dist_p is smooth on $B(p, \operatorname{inj}_M)\backslash\{0\}$, and $\operatorname{supp}(\varphi) \subset [-2\eta, 2\eta]$, and $2\eta \in (0, \operatorname{inj}_M)$, it follows that ω is smooth and supported in $B(p, 2\eta)$. Since $\|\varphi\|_{C^0} < \varepsilon\eta^3$, we have that $\|\omega\|_{C^0} < \varepsilon\eta^3 < \varepsilon$. Moreover,

$$(\operatorname{grad}\omega)|_x = \varphi'\left(\operatorname{dist}_p(x)\right) X,$$

and since $\left|\varphi'\left(\mathrm{dist}_p(x)\right)\right| < \varepsilon\eta^3$, it follows that $\left|(\mathrm{grad}\,\omega)|_x\right| < \varepsilon\eta^3$, and therefore $\|\omega\|_{C^1} < \varepsilon\eta^3 < \varepsilon$. Since $\varphi'(0) = 0$, it follows that $(\mathrm{grad}\,\omega)|_p = 0$.

It remains to prove Inequalities (3.3.1)–(3.3.4). Besides mechanical changes the proofs when $C > 0$ and $C < 0$ are identical, so we only detail the proofs when $C > 0$, that is of (3.3.1) and (3.3.2).

Since

$$\mathrm{Hess}_\omega(X, X)_x = \varphi''\left(\mathrm{dist}_p(x)\right) \quad \text{and} \quad \left|\varphi''\right| \in (-\varepsilon\eta^3, 2C],$$

we have that

$$-\varepsilon\eta^3 \leq \mathrm{Hess}_\omega(X, X) \leq 2C.$$

On the other hand, if $Y \in T_xM$ is a unit vector perpendicular to X, then $\nabla_Y\left(\varphi' \circ \mathrm{dist}_p(\cdot)\right) = 0$, and so

$$\nabla_Y \mathrm{grad}\ \omega = \varphi'\left(\mathrm{dist}_p(x)\right)\nabla_Y X.$$

Thus,

$$\begin{aligned}
\mathrm{Hess}_\omega(Y, Y) &= g\left(\nabla_Y\,\mathrm{grad}\,\omega, Y\right) \\
&= \varphi'\left(\mathrm{dist}_p(x)\right) g\left(\nabla_Y X, Y\right) \\
&= \varphi'\left(\mathrm{dist}_p(x)\right) \mathrm{Hess}_{\mathrm{dist}_p}(Y, Y) \\
&= \varphi'\left(\mathrm{dist}_p(x)\right)\left(\frac{1}{\mathrm{dist}_p(x)} + O\left(\mathrm{dist}_p(x)\right)\right), \qquad (3.3.17)
\end{aligned}$$

where for the last equality we are using Equation 2.7.1 in [SW15].

Since $\varphi'(0) = 0$ and $\varphi''(t) \leq 2C$, we have $\varphi'(t) \leq 2Ct$ for $t > 0$. This together with (3.3.17) and $\|\varphi\|_{C^1} < \varepsilon\eta^3$ gives us,

$$\begin{aligned} \mathrm{Hess}_\omega(Y,Y) &\leq 2C\,\mathrm{dist}_p(x)\frac{1}{\mathrm{dist}_p(x)} + \varepsilon\eta^3 \cdot O\left(\mathrm{dist}_p(x)\right) \\ &\leq 3C. \end{aligned}$$

Since $\varphi'|_{(0,\eta)} > 0$, (3.3.17) gives us that $0 \leq \mathrm{Hess}_\omega(Y,Y)|_{B(p,\eta)}$, and since $\|\varphi'\|_{C^0} \leq \varepsilon\eta^3$, (3.3.17) also gives us that

$$\mathrm{Hess}_\omega(Y,Y)|_{M\setminus B(p,\eta)} \geq -2\frac{\varepsilon\eta^3}{\eta} \geq -\varepsilon.$$

Hence $-\varepsilon \leq \mathrm{Hess}_\omega(Z,Z) \leq 3C$, as claimed in (3.3.1).

To show that (3.3.2) holds, for $Z \in TM|_{B(p,\tau)}$ we note that since for $t \in [0,\tau]$, $\varphi''(t) = 2C$,

$$\mathrm{Hess}_\omega(X,X)|_{B(p,\tau)} \equiv 2C,$$

and $\varphi'(t) = 2Ct$, so (3.3.17) gives us that for any unit vector $Y \perp X$ and $Y \in TM|_{B(p,\tau)}$

$$\begin{aligned} \mathrm{Hess}_\omega(Y,Y) &= \varphi'\left(\mathrm{dist}_p(x)\right)\left(\frac{1}{\mathrm{dist}_p(x)} + O\left(\mathrm{dist}_p(x)\right)\right) \\ &\geq 2C\,\mathrm{dist}_p(x)\left(\frac{1}{\mathrm{dist}_p(x)} + O\left(\mathrm{dist}_p(x)\right)\right) \\ &= 2C + 2CO\left(\mathrm{dist}_p(x)\right)^2 \\ &> C, \end{aligned}$$

since we are at a point where $\mathrm{dist}_p(x) < \tau$. □

3.4 Definition of the Metric

For some $k \geq b$, let $\pi : (M^n, g_M) \to (B^b, g_B)$ be a Riemannian submersion with $\operatorname{Ric}_k M \geq k$. For $p \in B$, $\varepsilon_h, \eta_h, \varepsilon_v, \eta_v \in (0, \frac{\mathrm{inj}_M}{2})$, $C_h > 1$, and $C_v < -1$, let

$$\omega_h : B \to \mathbb{R} \text{ and } \omega_v : B \to \mathbb{R}$$

be as in Lemma 3.3.1 with ε_h, η_h, and C_h playing the role of ε, η, and C in the construction of ω_h and ε_v, η_v, and C_v playing the role of ε, η, and C in the construction of ω_v. Now set

$$\tilde{\omega}_h : M \to \mathbb{R} \quad \text{and} \quad \tilde{\omega}_v : M \to \mathbb{R},$$

$$\tilde{\omega}_h = \omega_h \circ \pi \quad \text{and} \quad \tilde{\omega}_v = \omega_v \circ \pi,$$

$$\tilde{g}_B = e^{2\omega_h} g_B, \tag{3.4.1}$$

and

$$\tilde{g}_M = e^{2\tilde{\omega}_h} g_M|_{\mathcal{H}\oplus\mathcal{H}} + e^{2\tilde{\omega}_v} g_M|_{\mathcal{V}\oplus\mathcal{V}}. \tag{3.4.2}$$

It is immediate that $\pi : (M^n, \tilde{g}_M) \to (B^b, \tilde{g}_B)$ continues to be a Riemannian submersion. Our goal is to show that there is a choice of ω_h and ω_v so that

$$\widetilde{\operatorname{Ric}}_k^M := \operatorname{Ric}_k(M, \tilde{g}_M) > 0$$

is positive and

$$\widetilde{\operatorname{Ric}}^B := \operatorname{Ric}(B, \tilde{g}_B)$$

obtains both positive and negative values.

Theorem 2 of [PW14] says that if $\pi : M \to B$ is a Riemannian submersion from a compact manifold M with $\mathrm{Ric}^M > 0$, then B can not have nonpositive Ricci curvature. Thus to prove that $\widetilde{\mathrm{Ric}}^B$ has both signs, we only need show that $\widetilde{\mathrm{Ric}}^M_k > 0$ and that $\widetilde{\mathrm{Ric}}^B$ obtains a negative value.

3.5 Verification on the Base

In this section, we estimate the impact on the curvature of the base of the conformal change

$$\tilde{g}_B = e^{2\omega_h} g_B,$$

defined in (3.4.1). We write R^b and Ric^B for the curvature and Ricci tensors of (B, g_B), $\widetilde{\mathrm{Ric}}^B$ and $\tilde{R}_B$ for the curvature and Ricci tensors of $(B, \tilde{g}_B)$. We set

$$\mathfrak{D} R_B := \tilde{R}_B - R_B,$$

and ultimately we will choose ω_h so that $\tilde{g}_B$ satisfies the conclusion of the following result.

Proposition 3.5.1. *Given any $\varepsilon, K > 0$, and $\eta_h \in \left(0, \frac{\mathrm{inj}_M}{2}\right)$ there is an $\varepsilon_h^0 > 0$ so that if $\varepsilon_h \in (0, \varepsilon_h^0)$ and*

$$C_h > \tilde{K} := K + \max\left\{ \left| \mathrm{Ric}^B |_p \right| \right\} + 1,$$

then

$$\widetilde{\mathrm{Ric}}^B(\cdot,\cdot)|_p < -K(b-1)g(\cdot,\cdot)|_p$$

and

$$\left\| \mathfrak{D} R_B - (-2\,\mathrm{Hess}_{\omega_h} \circ g_B) \right\| < \varepsilon, \tag{3.5.1}$$

where $\circ$ is the Kulkarni–Nomizu product defined in (3.2.4).

Proof. By Part 4 of Exercise 4.7.14 in [Pet16]

$$\widetilde{\mathrm{Ric}}^B = \mathrm{Ric}^B - (b-2)\left(\mathrm{Hess}^B_{\omega_h} - (\mathrm{d}\omega_h \otimes \mathrm{d}\omega_h)\right) - \left(\Delta\omega_h + (b-2)\,|\mathrm{d}\omega_h|^2\right) g_B. \quad (3.5.2)$$

Since $|\omega_h|_{C^1} < \varepsilon_h$, all first order terms in Equation (3.5.2) are ε_h-small, so

$$\widetilde{\mathrm{Ric}}^B = \mathrm{Ric}^B - (b-2)(\mathrm{Hess}^B_{\omega_h}) - \Delta\omega_h \; g_B + O(\varepsilon_h).$$

Noting that $C_h > \tilde{K}$, Inequality (3.3.2) gives us that for all unit vectors $Z \in T_pM$,

$$\mathrm{Hess}^B_{\omega_h}(Z,Z)|_p \geq \tilde{K}$$

and hence that $\Delta\omega_h|_p \geq b\tilde{K}$. Thus if ε_h is small enough, then for all g_B-unit vectors $Z \in T_pM$,

$$\begin{aligned}
\widetilde{\mathrm{Ric}}^B(Z,Z)|_p &\leq \mathrm{Ric}^B(Z,Z)|_p - (b-2)\tilde{K} - b\tilde{K} + O(\varepsilon_h) \\
&= \mathrm{Ric}^B(Z,Z)|_p - \tilde{K}(2b-2) + O(\varepsilon_h) \\
&\leq -K(2b-2) \\
&< -K(b-1),
\end{aligned}$$

since $\tilde{K} := K + \max\left\{\left|\mathrm{Ric}^B|_p\right|\right\} + 1$ and $b \geq 2$.

To prove (3.5.1) we first observe that by Part 2 of Exercise 4.7.14 in [Pet16]

$$e^{-2\omega_h}\tilde{R}_B = R_B - \left(2\,\mathrm{Hess}^B_{\omega_h} - 2(\mathrm{d}\omega_h \otimes \mathrm{d}\omega_h) + |d\omega_h|^2 g_B\right) \circ g_B \qquad (3.5.3)$$

Since $\|\omega_h\|_{C^1} < \varepsilon_h$, this becomes

$$\left\|\mathfrak{D} R_B - (-2\operatorname{Hess}_{\omega_h} \circ g_B)\right\| < O(\varepsilon_h) < \varepsilon, \tag{3.5.4}$$

as claimed. □

3.6 Method to Verify Positive Intermediate Ricci Curvature

In this section we prove a lemma to estimate changes in intermediate Ricci curvature, caused by changes in the curvature operator. In large part, our result relies on a result of Reiser and Wraith from [RW23]. Before stating either result we introduce some notation.

Let g and $\tilde{g}$ be two Riemannian metrics on a smooth n-manifold M with sectional curvatures $\sec^M$ and $\widetilde{\sec}^M$. To understand the change in the intermediate Ricci curvature lower bound when we transform (M, g_M) to $(M, \tilde{g}_M)$ we will say that

$$\mathfrak{D}\operatorname{Ric}_k^M \geq r,$$

provided for all g_M-orthonormal sets $\{u, v_1, \cdots, v_k\}$ we have

$$\sum_{i=1}^{k} \widetilde{\sec}^M(u, v_i) - \sec^M(u, v_i) > r.$$

The following lemma is the main result of this section.

Lemma 3.6.1. *Given $\varepsilon > 0$ and $b \in \mathbb{N}$, there is a $\delta > 0$ with the following property. Let g and $\tilde{g}$ be two complete Riemannian metrics on a smooth n-manifold M with* *curvature operators $\mathcal{R}_M$ and $\widetilde{\mathcal{R}}_M$, respectively. Let $\mathcal{H}$ and $\mathcal{V}$ be orthogonal distributions so that $TM = \mathcal{H} \oplus \mathcal{V}$ with $\dim \mathcal{H} = b$. Set*

$$\mathfrak{D}\mathcal{R}_M := \widetilde{\mathcal{R}}_M - \mathcal{R}_M.$$

Suppose for some $\lambda_{h,h}, \lambda_{h,v} \in \mathbb{R}$ and all orthonormal pairs $\{X, Y\} \subset \mathcal{H}$, and $\{V, W\} \subset \mathcal{V}$,

$$\begin{aligned}
\left| \left| g\left(\mathfrak{D}\mathcal{R}_M\left(X \wedge Y\right), X \wedge Y\right) \right| - \left\| \mathfrak{D}\mathcal{R}_M\left(X \wedge Y\right) \right\| \right| &< \delta, && (3.6.1) \\
\left| \left| g\left(\mathfrak{D}\mathcal{R}_M\left(X \wedge V\right), X \wedge V\right) \right| - \left\| \mathfrak{D}\mathcal{R}_M\left(X \wedge V\right) \right\| \right| &< \delta, && (3.6.2) \\
g(\mathfrak{D}\mathcal{R}_M\left(X \wedge Y\right), X \wedge Y) &\geq \lambda_{h,h}, && (3.6.3) \\
g(\mathfrak{D}\mathcal{R}_M\left(X \wedge V\right), X \wedge V) &\geq \lambda_{h,v}, \quad \textit{and} && (3.6.4) \\
\left\| \mathfrak{D}\mathcal{R}(V \wedge W) \right\| &< \delta && (3.6.5)
\end{aligned}$$

Suppose further that either

$$\lambda_{h,v} > 0 \quad \textit{and} \quad (k-1)\lambda_{h,h} + \lambda_{h,v} > 0 \tag{3.6.6}$$

or

$$\left|\lambda_{h,v}\right| < \delta \quad \textit{and} \quad \left|\lambda_{v,v}\right| < \delta \tag{3.6.7}$$

Then for any $k \in \{b, b+1, \ldots, n-1\}$, we have $\mathfrak{D}\operatorname{Ric}_k^M > -\varepsilon$.

The proof involves several preliminary results, one of which is the following linear algebra lemma of Reiser–Wraith from [RW23].

Lemma 3.6.2 (Reiser-Wraith, [RW23])**.** *Let V be a finite-dimensional, real vector space with a fixed inner product $\langle \cdot, \cdot \rangle$. Let $A : \Lambda^2 V \to \Lambda^2 V$ be a self-adjoint linear map and let $k \in \{1, \cdots, \dim(V) - 1\}$. Suppose that V splits orthogonally as $V = V_1 \oplus V_2 \oplus V_3$, so that the spaces $V_i \wedge V_j$ with $i, j \in \{1, 2, 3\}$ are eigenspaces of A with eigenvalues λ_{ij}. If for every $i \in \{1, 2, 3\}$ and every $n_{i1}, n_{i2}, n_{i3} \in \mathbb{N}_0$ with $n_{ij} \leq \dim\big(V_j\big)$ $(j \neq i)$, $n_{ii} \leq \dim(V_i) - 1$, and*

$$n_{i1} + n_{i2} + n_{i3} = k \tag{3.6.8}$$

we have

$$n_{i1}\lambda_{i1} + n_{i2}\lambda_{i2} + n_{i3}\lambda_{i3} > c \tag{3.6.9}$$

for some $c \in \mathbb{R}$, then for every orthonormal set $\{u_0, \cdots, u_k\}$ in V,

$$\sum_{i=1}^{k} \big\langle A(u_0 \wedge u_i), u_0 \wedge u_i \big\rangle > c.$$

The following lemma is an easy consequence of Lemma 3.6.2.

Lemma 3.6.3. *Let $A : \Lambda^2 V \to \Lambda^2 V$ be as in Lemma 3.6.2, with $V_3 = \{0\}$, and $\dim(V_1) = b$. Given $\varepsilon > 0$, there is a $\delta > 0$ with the following property. Suppose that*

$$\lambda_{12} > 0, \quad (b-1)\lambda_{11} + \lambda_{12} > 0, \quad \textit{and} \ \ |\lambda_{22}| < \delta, \tag{3.6.10}$$

then for every $k \geq b$ and every orthonormal set $\{u_0, u_1 \cdots, u_k\}$ in V,

$$\sum_{i=1}^{k} \big\langle A(u_0 \wedge u_i), u_0 \wedge u_i \big\rangle > -\varepsilon. \tag{3.6.11}$$

Proof. Let n_{ij} be as in Lemma 3.6.2. Since $V_3 = \{0\}$, $n_{13} = n_{23} = n_{33} = 0$ and we only need consider the cases when $i = \{1, 2\}$. When $i = 1$, (3.6.8) becomes

$$n_{11} + n_{12} = k.$$

If $\lambda_{11} > 0$, then since $\lambda_{12} > 0$,

$$n_{11}\lambda_{11} + n_{12}\lambda_{12} > 0, \tag{3.6.12}$$

a fortiori. If $\lambda_{11} \leq 0$, then since $k \geq b$ and $n_{11} \leq b - 1, n_{12} \geq 1$. This together with $\lambda_{12} > 0$ gives us

$$n_{11}\lambda_{11} + n_{12}\lambda_{12} \geq (b - 1)\,\lambda_{11} + \lambda_{12} > 0. \tag{3.6.13}$$

When $i = 2$, since $\lambda_{21} > 0$ we have

$$n_{21}\lambda_{21} + n_{22}\lambda_{22} > n_{22}\lambda_{22} > -|\delta n_{22}| > -\varepsilon, \tag{3.6.14}$$

provided we choose $\delta \in \left(0, \frac{\varepsilon}{\dim(V)}\right)$. Together (3.6.12), (3.6.13), and (3.6.14) give us that (3.6.9) holds in all cases with $c = -\varepsilon$, hence we get that (3.6.11) holds, as desired. $\square$

Lemma 3.6.4. *Let $A : \Lambda^2 V \to \Lambda^2 V$ be as in Lemma 3.6.2, with $V_3 = \{0\}$, and $\dim(V_1) = b$. If*

$$|\lambda_{11}| < \delta, \quad |\lambda_{12}| < \delta, \quad \text{and } |\lambda_{22}| < \delta, \tag{3.6.15}$$

then $\|A\| \leq \delta$.

Proof. A unit vector $\mathfrak{u} \in \wedge^2 V$ has the form

$$\mathfrak{u} = \mu_{11}e_{11} + \mu_{12}e_{12} + \mu_{22}e_{22},$$

where $e_{11} \in V_1 \wedge V_1$, $e_{12} \in V_1 \wedge V_2$, $e_{22} \in V_2 \wedge V_2$, and $(\mu_{11})^2 + (\mu_{12})^2 + (\mu_{22})^2 = 1$. Since $V_1 \wedge V_1$, $V_1 \wedge V_2$, and $V_2 \wedge V_2$, are orthogonal,

$$\begin{aligned}
\left|\left\langle A(\mathfrak{u}), A(\mathfrak{u})\right\rangle\right| &= \|\mu_{11}\lambda_{11}e_{11} + \mu_{12}\lambda_{12}e_{12} + \mu_{22}\lambda_{22}e_{22}\|^2 \\
&= \lambda_{11}^2\mu_{11}^2 + \lambda_{12}^2\mu_{12}^2 + \lambda_{22}^2\mu_{22}^2 \\
&\leq \delta^2\left(\mu_{11}^2 + \mu_{12}^2 + \mu_{22}^2\right) \\
&= \delta^2,
\end{aligned}$$

as claimed. □

Proof of Lemma 3.6.1. Let $\lambda_{h,h}, \lambda_{h,v}$ be as in Lemma 3.6.1, and define

$$\begin{aligned}
\mathcal{A} : TM \wedge TM &\longrightarrow TM \wedge TM \text{ by setting} \\
\mathcal{A}(X \wedge Y) &= \lambda_{h,h}(X \wedge Y), \text{ for all } X, Y \in \mathcal{H} \\
\mathcal{A}(X \wedge V) &= \lambda_{h,v}(X \wedge V), \text{ for all } X \in \mathcal{H}, Y \in \mathcal{V} \\
\mathcal{A}(V \wedge W) &= 0 \text{ for all } V, W \in \mathcal{V}
\end{aligned}$$

Informally, Inequalities (3.6.1)–(3.6.5) mean that the operator $\mathcal{A}$ is almost $\geq \mathfrak{D}\mathcal{R}_M$. More precisely, given any $\varepsilon > 0$, there is a $\delta > 0$ so that if (3.6.1)–(3.6.5) hold for our choice of $\delta > 0$, then for all $p \in M$ and all orthonormal $(k+1)$-frames $\{u, v_1, \ldots, v_k\} \subset T_pM$,

$$\sum_{i=1}^{k} g(\mathfrak{D}\mathcal{R}_M(u \wedge v_i), u \wedge v_i) > \sum_{i=1}^{k} g(\mathcal{A}(u \wedge v_i), u \wedge v_i) - \frac{\varepsilon}{2} \tag{3.6.16}$$

Thus if (3.6.6) holds, then by combining (3.6.16) with Lemma 3.6.3 and possibly choosing an even smaller $\delta > 0$, we get that

$$\begin{aligned}\sum_{i=1}^{k} g(\mathfrak{D}\mathcal{R}_M(u\wedge v_i), u\wedge v_i) &\geq \sum_{i=1}^{k} g(\mathcal{A}(u\wedge v_i), u\wedge v_i) - \frac{\varepsilon}{2} \\ &\geq -\frac{\varepsilon}{2} - \frac{\varepsilon}{2},\end{aligned} \tag{3.6.17}$$

as desired. Similarly, if (3.6.7) holds, then we combine (3.6.16) with Lemma 3.6.4 and also get that (3.6.17) holds, after possibly making δ even smaller. □

3.7 Analysis on the Total Space

We will compare the curvatures of $\tilde{g}_M$ with those of g_M, by comparing both to those of the following auxiliary metric:

$$\begin{aligned}\hat{g}_M &= \pi^*(\tilde{g}_B)|_{\mathcal{H}\oplus\mathcal{H}} + g_M|_{\mathcal{V}\oplus\mathcal{V}} \\ &= e^{2\tilde{\omega}_h} g_M|_{\mathcal{H}\oplus\mathcal{H}} + g_M|_{\mathcal{V}\oplus\mathcal{V}}\end{aligned} \tag{3.7.1}$$

Noting that submersion $\pi : M \to B$ is Riemannian with respect to both metrics $\hat{g}_M$ and $\tilde{g}_B$, and that in either case the induced metric on B is $\tilde{g}_B$, to ease the exposition we set

$$\hat{g}_B := \tilde{g}_B.$$

The comparison between the curvatures of $\hat{g}_M$ and $\tilde{g}_M$ will be based on the Gromoll–Walschap formulas for the changes in the infinitesimal geometry of a Riemannian submersion when the metric on the fibers is changed by a warping function of the base (cf. (3.4.2) and (3.7.1)).

We denote the covariant derivative of g_M by ∇. We add a tilde or a hat above a covariant derivative or a curvature to denote its association to the warped metrics $\tilde{g}_M$ and $\hat{g}_M$, respectively.

We start by recording the following general proposition on the impact of a C^1-small change to a fixed metric on the metric's covariant derivative and the gradient and Hessian of a fixed real valued function.

Proposition 3.7.1. *Let (M, g) be a compact Riemannian manifold M. Let $\mathcal{U}$ be a fixed finite atlas for M, and let $f : M \to \mathbb{R}$ be a fixed C^∞ function on M.*

For every $\varepsilon > 0$, there is a $\delta > 0$, that also depends on g, $\mathcal{U}$ and f, with the following property. If $\hat{g}$ is a Riemannian metric on M that satisfies

$$\|g - \hat{g}\|_{C^1, g_0} < \delta, \tag{3.7.2}$$

then

$$\left\|\nabla f - \hat{\nabla} f\right\|_g < \varepsilon, \tag{3.7.3}$$

$$\left\|\mathrm{Hess}_f - \widehat{\mathrm{Hess}}_f\right\|_g < \varepsilon, \quad \textit{and} \tag{3.7.4}$$

$$\left\|\nabla . E - \hat{\nabla} . E\right\|_g < \varepsilon, \tag{3.7.5}$$

for any coordinate field E of any chart of $\mathcal{U}$.

Proof. For all $y \in TM$

$$g(\nabla f, y) = df(y) = \hat{g}(\hat{\nabla} f, y). \tag{3.7.6}$$

This gives us

$$\begin{aligned}\left|g(\nabla f,y)-g(\hat{\nabla} f,y)\right| &\leq \left|g(\nabla f,y)-\hat{g}(\hat{\nabla} f,y)\right| + \left|\hat{g}(\hat{\nabla} f,y)-g(\hat{\nabla} f,y)\right| \\ &\leq 0 + \left|(g-\hat{g})\left(\hat{\nabla} f,y\right)\right|, \quad \text{by (3.7.6)} \qquad (3.7.7)\end{aligned}$$

Since M is compact, there is a $K>0$ so that

$$\max\left\|\hat{\nabla} f\right\|_{\hat{g}} \leq K.$$

This together with $\|g-\hat{g}\|_{C^0,g_0} < \delta$ implies that

$$\max\left\|\hat{\nabla} f\right\|_{g} \leq K\left(1+\delta\right),$$

and for $\|y\|_g = 1$, we have $\|y\|_{\hat{g}} \leq 1+\delta$. Thus for $\|y\|_g = 1$,

$$\left|(g-\hat{g})\left(\hat{\nabla} f,y\right)\right| \leq 4K\delta.$$

So

$$\begin{aligned}\left|g(\nabla f,y)-g(\hat{\nabla} f,y)\right| &< 4K\delta \\ &< \varepsilon,\end{aligned}$$

provided we choose $\delta < \frac{\varepsilon}{4K}$, which completes the proof of (3.7.3).

To prove (3.7.5) we write

$$\left|\hat{g}(\hat{\nabla}_X E,Y)-\hat{g}(\nabla_X E,Y)\right| \leq \left|\hat{g}(\hat{\nabla}_X E,Y)-g(\nabla_X E,Y)\right| + \left|(g-\hat{g})(\nabla_X E,Y)\right|.$$

Now suppose that X and Y are coordinate fields for $\mathcal{U}$. Since the there are only a finite number of possible X and Y, the Kozul formula together with (3.7.2) and Proposition 3.2.4 gives us that there is a $C > 0$ so that

$$\left|\hat{g}(\hat{\nabla}_X E, Y) - g(\nabla_X E, Y)\right| < C\delta.$$

Similarly, (3.7.2) gives us

$$\|g - \hat{g}\| < \|g - \hat{g}\|_{C^1} < \delta.$$

Since the there are only a finite number of possible coordinate fields, the previous display gives us that there is a $C > 0$ so that

$$\left|(\hat{g} - g)(\nabla_X E, Y)\right| < C\delta.$$

Setting $\delta = \frac{\varepsilon}{C}$ and combining the previous three displays gives us (3.7.5).

We have

$$\begin{aligned}
\mathrm{Hess}_f(X, Y) &= g(\nabla_X \nabla f, Y) \\
&= Xg(\nabla f, Y) - g(\nabla f, \nabla_X Y) \\
&= Xdf(Y) - g(\nabla f, \nabla_X Y),
\end{aligned}$$

and similarly,

$$\widehat{\mathrm{Hess}}_f(X, Y) = Xdf(Y) - \hat{g}(\hat{\nabla} f, \hat{\nabla}_X Y).$$

In particular,

$$\mathrm{Hess}_f(X,Y) - \widehat{\mathrm{Hess}}_f(X,Y) = \hat{g}(\hat{\nabla} f, \hat{\nabla}_X Y) - g(\nabla f, \nabla_X Y).$$

Thus, for coordinate fields X and Y,

$$\begin{aligned}
\left|(\mathrm{Hess}_f - \widehat{\mathrm{Hess}}_f)(X,Y)\right| \leq\ & \left|\hat{g}(\hat{\nabla} f, \hat{\nabla}_X Y) - g(\hat{\nabla} f, \hat{\nabla}_X Y)\right| \\
& + \left|g(\hat{\nabla} f, \hat{\nabla}_X Y) - g(\nabla f, \hat{\nabla}_X Y)\right| \\
& + \left|g(\nabla f, \hat{\nabla}_X Y) - g(\nabla f, \nabla_X Y)\right| \\
=\ & \left|(\hat{g} - g)(\hat{\nabla} f, \hat{\nabla}_X Y)\right| \\
& + \left|g(\hat{\nabla} f - \nabla f, \hat{\nabla}_X Y)\right| \\
& + \left|g(\nabla f, \hat{\nabla}_X Y - \nabla_X Y)\right|.
\end{aligned}$$

Together with (3.7.2), (3.7.3), and (3.7.5), this gives us that

$$\left|(\mathrm{Hess}_f - \widehat{\mathrm{Hess}}_f)(X,Y)\right| < \varepsilon,$$

provided δ is small enough. □

Proposition 3.7.2. *Let g_M and $\hat{g}_M$ be as above. Then for any $X, Y \in \mathcal{H}$ and $U, V \in \mathcal{V}$,*

$$A_X Y = \hat{A}_X Y, \tag{3.7.8}$$

$$S_X U = \hat{S}_X U, \tag{3.7.9}$$

$$A_X^* U = e^{2\omega_h} \hat{A}_X^* U, \tag{3.7.10}$$

and

$$\sigma(U,V) = e^{2\omega_h}\hat{\sigma}(U,V), \tag{3.7.11}$$

where the tensors A, S, and σ are defined in (3.2.1), (3.2.2), and (3.2.3), respectively. In particular, since ω_h is C^1-small, the tensors $\hat{\sigma}$ and $\hat{A}^*$ are C^1-close to their g_M-counterparts.

Proof. Equation (3.7.8) follows from the fact that

$$A_X Y = \frac{1}{2}[X,Y]^{\mathbf{v}} = \hat{A}_X Y.$$

Dualizing this we find that

$$\begin{aligned} g_M(A_X^* U, Y) &= g_M(U, A_X Y) \\ &= g_M(U, \hat{A}_X Y) \\ &= \hat{g}_M(U, \hat{A}_X Y) \\ &= \hat{g}_M(\hat{A}_X^* U, Y) \\ &= e^{-2\omega_h} g_M(\hat{A}_X^* U, Y). \end{aligned}$$

As this holds for all horizontal vectors Y,

$$A_X^* T = e^{-2\omega_h}\hat{A}_X^* U, \text{ proving (3.7.10).}$$

For vertical vectors $V \in \mathcal{V}$, $g_M(V,\cdot) = \hat{g}_M(V,\cdot)$, so by the Kozul formula,

$$g_M(S_X V, U) = \hat{g}_M(\hat{S}_X V, U), \text{ for all } U, V \in \mathcal{V}. \tag{3.7.12}$$

Since $g_M(\cdot, U) = \hat{g}_M(\cdot, U)$ for all $U \in \mathcal{V}$, this proves, Equation (3.7.9).

Dualizing, we get,

$$\begin{aligned} g_M(\sigma(U,V), X) &= g_M(S_X V, U) \\ &= \hat{g}_M(\hat{S}_X V, U) \text{ by (3.7.12)} \\ &= \hat{g}_M(\hat{\sigma}(V,U), X) \\ &= e^{2\omega_h} g_M(\hat{\sigma}(V,U), X). \end{aligned}$$

As the previous display is valid for all horizontal vectors X,

$$\sigma(U,V) = e^{2\omega_h}\hat{\sigma}(U,V),$$

proving (3.7.11). □

Combining the previous result with the Grey-O'Neill formulas for the curvatures of the total space of a Riemannian submersion immediately gives us the following comparison between the curvatures of R and $\hat{R}$ (see [Gra67], [O'N66], or Theorem 1.5.1 in [GW09]).

Corollary 3.7.3. *For every $\varepsilon > 0$ there is a $\delta > 0$ so that if $\|\tilde{\omega}_h\|_{C^1} < \delta$, then the curvature tensors of g_M and $\hat{g}_M$ evaluated on unit horizontal vectors $X, Y, Z \in \mathcal{H}$ satisfy*

$$\left\| \left(\hat{R}_M^{\mathbf{h}} - \hat{R}_B\right)(X,Y)Z - \left(R_M^{\mathbf{h}} - R_B\right)(X,Y)Z \right\| < \varepsilon$$

and

$$\left\| \left(\hat{R}_M^{\mathbf{v}} - R_M^{\mathbf{v}}\right)(X,Y)Z \right\| < \varepsilon,$$

and when we evaluate $\left\|\hat{R}-R\right\|$ *on any other triple of unit vectors in* $\mathcal{H}\cup\mathcal{V}$*, the result has length* $<\varepsilon$*.*

The following result from Gromoll and Walschap's book [GW09] describes the change in curvature tensor of the total space of a Riemannian submersion when the metric on the vertical distribution is warped by a function on the base. When the warping function is arbitrary, these equations (3.7.13–3.7.18, below) are rather complicated; however, in our planned application the function $\tilde{\varphi}: M \to (0,\infty)$ is C^1-small and hence nearly every term on the right side of Equations (3.7.13)–(3.7.18) will be very small.

Proposition 3.7.4. *Let* $\pi:(M^n,\hat{g}_M)\to(B^b,\hat{g}_B)$ *be any Riemannian submersion, let* $\varphi: B\to(0,\infty)$ *be* C^∞*, and let* $\tilde{\varphi}:=\varphi\circ\pi$ *be its pullback on* M*. Set*

$$\tilde{g}_M=\pi^*(\hat{g}_B)|_{\mathcal{H}\oplus\mathcal{H}}+e^{2\tilde{\varphi}}\hat{g}_M|_{\mathcal{V}\oplus\mathcal{V}}.$$

Let $X,Y,Z\in\mathcal{H}$ *and* $T,T_1,T_2,T_3\in\mathcal{V}$*, and let* $\langle\cdot,\cdot\rangle$ *denote the inner product associated with the metric* $\hat{g}_M$*. The components of the curvature tensor associated with* $\tilde{g}_M$ *are as follows.*

$$\begin{aligned}\tilde{R}^{\mathbf{v}}_M(X,Y)Z &= \hat{R}^{\mathbf{v}}_M(X,Y)Z+\left\langle\hat{\nabla}\tilde{\varphi},X\right\rangle\hat{A}_YZ\\ &\quad-\left\langle\hat{\nabla}\tilde{\varphi},Y\right\rangle\hat{A}_XZ-\left\langle\hat{\nabla}\tilde{\varphi},Z\right\rangle\hat{A}_XY\end{aligned}\tag{3.7.13}$$

$$\tilde{R}^{\mathbf{h}}_M(X,Y)Z=e^{2\tilde{\varphi}}\hat{R}^{\mathbf{h}}_M(X,Y)Z+(1-e^{2\tilde{\varphi}})\hat{R}_B(X,Y)Z\tag{3.7.14}$$

$$\begin{aligned}\tilde{R}^{\mathbf{v}}_M(X,T)Y &= \hat{R}^{\mathbf{v}}_M(X,T)Y + (1-e^{2\tilde{\varphi}})\hat{A}_X\hat{A}^*_Y T \\ &\quad + \left(\widehat{\mathrm{Hess}}^M_{\tilde{\varphi}}(X,Y) + \left\langle \hat{\nabla}\tilde{\varphi}, X\right\rangle \left\langle \hat{\nabla}\tilde{\varphi}, Y\right\rangle\right) T \\ &\quad - \left(\left\langle \hat{\nabla}\tilde{\varphi}, X\right\rangle \hat{S}_Y T + \left\langle \hat{\nabla}\tilde{\varphi}, Y\right\rangle \hat{S}_X T\right)\end{aligned} \tag{3.7.15}$$

$$\begin{aligned}e^{-2\tilde{\varphi}}\tilde{R}^{\mathbf{h}}_M(X,T)Y &= \hat{R}^{\mathbf{h}}_M(X,T)Y - \left\langle \hat{\nabla}\tilde{\varphi}, Y\right\rangle \hat{A}^*_X T \\ &\quad - 2\left\langle \hat{\nabla}\tilde{\varphi}, X\right\rangle \hat{A}^*_Y T + \left\langle \hat{A}_X Y, T\right\rangle \hat{\nabla}\tilde{\varphi}\end{aligned} \tag{3.7.16}$$

$$e^{-2\tilde{\varphi}}\tilde{R}^{\mathbf{h}}_M(T_1,T_2)X = \hat{R}^{\mathbf{h}}_M(T_1,T_2)X + (1-e^{2\tilde{\varphi}})(\hat{A}^*_{\hat{A}^*_X T_1}T_2 - \hat{A}^*_{\hat{A}^*_X T_2}T_1), \tag{3.7.17}$$

$$\tilde{R}^{\mathbf{h}}_M(T_1,T_2)T_3 = \hat{R}^{\mathbf{h}}_M(T_1,T_2)T_3, \tag{3.7.18}$$

$$\begin{aligned}\tilde{R}^{\mathbf{v}}_M(T_1,T_2)T_3 &= \hat{R}^{\mathbf{v}}_M(T_1,T_2)T_3 + (1-e^{2\tilde{\varphi}})\left\{\hat{S}_{\hat{\sigma}(T_2,T_3)}T_1 - \hat{S}_{\hat{\sigma}(T_1,T_3)}T_2\right\} \\ &\quad + e^{2\tilde{\varphi}}\left\{\left(\hat{S}_{\hat{\nabla}\tilde{\varphi}} - |\hat{\nabla}\tilde{\varphi}|^2 I\right)\left(\langle T_2,T_3\rangle T_1 - \langle T_1,T_3\rangle T_2\right)\right. \\ &\quad \left. + \left\langle \hat{\nabla}\tilde{\varphi}, \hat{\sigma}(T_2,T_3)\right\rangle T_1 - \left\langle \hat{\nabla}\tilde{\varphi}, \hat{\sigma}(T_1,T_3)\right\rangle T_2\right\},\end{aligned} \tag{3.7.19}$$

where I denotes the identity map.

In our usage of Proposition 3.7.4, our function $\tilde{\omega}_v$ plays the role of $\tilde{\varphi}$. To ease the notation, we will henceforth remove the hat from atop the Hessian. As mentioned before, the functions $\tilde{\omega}_h$ and $\tilde{\omega}_v$ that define our deformation will be chosen to be C^1-small. This together with Propositions 3.7.1, 3.7.2, and 3.7.4 imply that most components of the difference of the curvature operators $\mathfrak{D}\mathcal{R}_M := \widetilde{\mathcal{R}}_M - \mathcal{R}_M$ are small. We make this precise in the following proposition.

Proposition 3.7.5. *Let M be a compact smooth manifold, and let $\pi : (M^n, g_M) \to (B^b, g_B)$ be a Riemannian submersion. Then for every $\varepsilon > 0$ there is $\delta > 0$ that depends on g_M and has the following properties.*

Let $\tilde{g}_M$ be as in (3.4.2). Let $\mathcal{R}_M$ and $\widetilde{\mathcal{R}}_M$ be the curvature operators of g_M and $\tilde{g}_M$, respectively, and set $\mathfrak{D}\mathcal{R}_M := \widetilde{\mathcal{R}}_M - \mathcal{R}_M$. If

$$\|\tilde{\omega}_h\|_{C^1} < \delta \ \ \textit{and} \ \ \|\tilde{\omega}_v\|_{C^1} < \delta, \tag{3.7.20}$$

then for all horizontal unit vectors Y, Z and vertical unit vectors U, V,

$$\left\| \mathfrak{D}\mathcal{R}_M(Y \wedge Z) + 2\,\mathrm{Hess}^M_{\tilde{\omega}_h}(Y,Y)(Y \wedge Z) \right\| \ \leq \ \varepsilon. \tag{3.7.21}$$

$$\left\| \mathfrak{D}\mathcal{R}_M(Y \wedge U) + \mathrm{Hess}^M_{\tilde{\omega}_v}(Y,Y)(Y \wedge U) \right\| \ \leq \ \varepsilon, \ \ \textit{and} \tag{3.7.22}$$

$$\left\| \mathfrak{D}\mathcal{R}_M(U \wedge V) \right\| \ \leq \ \varepsilon, \tag{3.7.23}$$

Proof. Since M is compact each of the fundamental tensors, A, A^*, σ, and S, for the submersion $\pi : (M, g_M) \to (B^b, g_B)$ is uniformly bounded. This together with the hypothesis that $\|\tilde{\omega}_h\|_{C^1}$, $\|\tilde{\omega}_v\|_{C^1} < \delta$ gives us that each of these tensors is within $O(\delta)$ of the corresponding tensors $\hat{A}$, $\hat{A}^*$, $\hat{S}$, and $\hat{\sigma}$ for $(M, \hat{g}_M) \to (B, \tilde{g}_B)$. This together with Corollary 3.7.3 gives us that, with one exception, in each of the equations (3.7.13)–(3.7.19), the difference between the two curvature tensors satisfies

$$\left\| \tilde{R} - \hat{R} \right\| < O(\delta). \tag{3.7.24}$$

The exception is Inequality (3.7.15) which after substituting Y for X, U for T, and $\tilde{\omega}_v$ for $\tilde{\varphi}$ becomes

$$\tilde{R}^{\mathbf{v}}_M(U,Y)Y = \hat{R}^{\mathbf{v}}_M(U,Y)Y - \mathrm{Hess}^M_{\tilde{\omega}_v}(Y,Y)U + O(\delta).$$

Re-writing this in terms of the curvature operator gives us

$$\left\| \mathfrak{D}\mathcal{R}_M(Y \wedge U) + \mathrm{Hess}^M_{\tilde{\omega}_v}(Y,Y)(Y \wedge U) \right\| \leq O(\delta),$$

which proves (3.7.22).

To obtain (3.7.21), we note that Inequality (3.7.14) together with the fact that $\|\omega_h\|_{C^1} < \delta$ gives us

$$\begin{aligned} \tilde{R}^{\mathbf{h}}_M(X,Y)Z &= e^{2\tilde{\omega}_h}\hat{R}^{\mathbf{h}}_M(X,Y)Z + (1-e^{2\tilde{\omega}_h})\tilde{R}_B(X,Y)Z \\ &= \hat{R}^{\mathbf{h}}_M(X,Y)Z + O(\delta). \end{aligned} \tag{3.7.25}$$

Equation (3.7.8) together with the Grey-O'Neill Horizontal Curvature Equation and the fact that the restrictions of $\hat{g}_M$ and g_M to the vertical distributions coincide give us that

$$\begin{aligned} \|(\hat{R}^{\mathbf{h}}_M - R^h_M)(X,Y)Z\| &= \|(\hat{R}_B - R_B)(X,Y)Z\| \\ &= \|\mathfrak{D}R_B(X,Y)Z\|, \end{aligned} \tag{3.7.26}$$

by the definition of $\mathfrak{D}R_B$. Combining (3.7.26) with Inequality (3.5.1) we get

$$\|\hat{R}^{\mathbf{h}}_M - R^h_M + 2\,\mathrm{Hess}_{\omega_h} \circ g_B)\| < O(\delta). \tag{3.7.27}$$

Thus

$$\begin{aligned}
&\left\|\mathfrak{D}\mathcal{R}_M(Y\wedge Z)+2\,\mathrm{Hess}^M_{\tilde{\omega}_h}(Y,Y)(Y\wedge Z)\right\| \\
&\quad\le\ \left\|(\tilde{\mathcal{R}}_M-\hat{\mathcal{R}}_M)(Y\wedge Z)\right\| \\
&\qquad +\ \left\|(\hat{\mathcal{R}}_M-\mathcal{R}_M)(Y\wedge Z)+2\,\mathrm{Hess}^M_{\tilde{\omega}_h}(Y,Y)(Y\wedge Z)\right\| \\
&\quad\le\ O(\delta) \qquad \text{by (3.7.25) and (3.7.27)},
\end{aligned}$$

proving (3.7.21). □

To finish the proof, we fix $p\in B$ and remind the reader that (3.4.2) defines the metric $\tilde{g}_M$ that eventually proves Theorem 3.1.2 as follows:

$$\tilde{g}_M := e^{2\tilde{\omega}_h} g_M|_{\mathcal{H}\oplus\mathcal{H}} + e^{2\tilde{\omega}_v} g_M|_{\mathcal{V}\oplus\mathcal{V}}. \tag{3.7.28}$$

Thus $(\tilde{g}_M - g_M)$ is determined by the choice of functions $\tilde{\omega}_h$ and $\tilde{\omega}_v$, which in turn we will construct via Lemma 3.3.1. The functions $\tilde{\omega}_h$ and $\tilde{\omega}_v$ are each constrained by a choice of the four constants τ,η,ε,C mentioned in Lemma 3.3.1. Thus to prove Theorem 3.1.2 we show that are two quadruples of constants

$$(\tau_h,\eta_h,\varepsilon_h,C_h)\ \text{ and }\ (\tau_v,\eta_v,\varepsilon_v,C_v)$$

so that when we use the corresponding functions $\tilde{\omega}_h$ and $\tilde{\omega}_v$ to define $\tilde{g}_M$ via (3.7.28) the resulting metric satisfies the conclusion of Theorem 3.1.2. We explain this more formally via the following two definitions.

Definition 3.7.6. *Let $\tilde{\varepsilon}>0$. A quadruple $(\tau_h,\eta_h,\varepsilon_h,C_h)$ is called **$\tilde{\varepsilon}$-Horizontal Admissible** if there exists a corresponding function $\omega_h : B\to\mathbb{R}$ constructed via*

Lemma 3.3.1 so that $(B, \tilde{g}_B)$ satisfies the conclusion of Proposition 3.5.1, with $\tilde{\varepsilon}$ playing the role of ε.

Definition 3.7.7. *Let $\varepsilon, \tilde{\varepsilon} > 0$, and suppose $(\tau_h, \eta_h, \varepsilon_h, C_h)$ is a $\tilde{\varepsilon}$-Horizontal Admissible quadruple. A quadruple $(\tau_v, \eta_v, \varepsilon_v, C_v)$ is called* ***$(\varepsilon, \tau_h, \eta_h, \varepsilon_h, C_h)$-Vertical Admissible*** *if there exists a corresponding function $\omega_v : B \to \mathbb{R}$ constructed via Lemma 3.3.1 so that $(M, \tilde{g}_M)$, defined via (3.4.2), satisfies*

$$\widetilde{\mathrm{Ric}}_b^M \geq \min \mathrm{Ric}_b^M - \varepsilon.$$

In particular, Theorem 3.1.2 follows from the following proposition.

Proposition 3.7.8. *Given $\varepsilon > 0$, there are an $\eta_h^0, \tilde{\varepsilon}_0 > 0$ with the following property. For all $\eta_h \in \left(0, \eta_h^0\right)$ and $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$ there is an $\tilde{\varepsilon}$-Horizontal Admissible $(\tau_h,\ \eta_h, \varepsilon_h, C_h)$ and an $(\varepsilon, \tau_h, \eta_h, \varepsilon_h, C_h)$-Vertical Admissible quadruple $(\tau_v,\ \eta_v, \varepsilon_v, C_v)$.*

Proof. From Proposition 3.5.1 we know that there is an $\tilde{\varepsilon}_0 > 0$ so that for all $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$ there are $\tilde{\varepsilon}$ -Horizontal Admissible quadruples, $(\tau_h,\ \eta_h, \varepsilon_h, C_h)$. In particular, the constant C_h is positive so from (3.3.1) we have that for all unit vector fields $Z \in TM$

$$-\varepsilon_h \leq \mathrm{Hess}_{\omega_h}(Z, Z) \leq 3C_h. \tag{3.7.29}$$

We will choose $C_v < 0$. This together with (3.3.3) and gives us that the Hessian of $\tilde{\omega}_v$ satisfies

$$3C_v \leq \mathrm{Hess}_{\tilde{\omega}_v}(Z, Z) \leq \varepsilon_v \tag{3.7.30}$$

for all unit vector fields $Z \in TM$. Additionally we have from (3.3.4) that for all unit vector fields $Z \in TM|_{B(p,\tau_v)}$

$$\mathrm{Hess}_{\tilde{\omega}_v}(Z, Z) \leq C_v < 0. \tag{3.7.31}$$

In addition to choosing $C_h > 0$ and $C_v < 0$ we will further require that

$$3\eta_h < \tau_v, \text{ and} \tag{3.7.32}$$

$$6(b-1)C_h + C_v < 0. \tag{3.7.33}$$

Note in particular, that (3.7.32) implies that the support of $\tilde{\omega}_h$ is contained in $B(p, \tau_v)$. Thus throughout the support of ω_h we have that (3.7.31) holds. Combining this with Proposition 3.7.5 and Inequality (3.7.33) we see that there is a $\delta > 0$ so that if $\varepsilon_h, \varepsilon_v \in (0, \delta)$, then $\mathfrak{D}\mathcal{R}_M$ satisfies the hypotheses of Lemma 3.6.1 on $B(p, \tau_v)$ with

$$\lambda_{h,h} = -2C_h \text{ and } \lambda_{v,v} = -C_v.$$

Note, in particular, the version of Lemma 3.6.1 that holds on $B(p, \tau_v)$ is the version where hypothesis (3.6.6) holds, and that this follows from our choice (3.7.33) together with (3.7.29) and (3.7.31).

On the other hand, on $M \setminus B(p, \tau_v)$, we see that $\omega_h \equiv 0$, so we still have that the (3.6.6) version of Lemma 3.6.1 holds wherever, $\mathrm{Hess}_{\tilde{\omega}_v} < 0$.

Finally notice that the upper bound in (3.7.30) gives us that the (3.6.7) version of Lemma 3.6.1 holds wherever, $\mathrm{Hess}_{\tilde{\omega}_v} \geq 0$.

$\square$

Appendix A

The Connection for a Warped Metric

Recall that in Chapter 3, we warp the vertical and horizontal distributions of a submersion by vertically warping an auxiliary metric $\hat{g}_M$ that is already conformally warped, as defined in Equation (3.7.1). This was done to expedite our reasoning, since we took advantage of the fact that most of the terms in each of the equations in Proposition 3.7.4 are ε-small.

However, if the reader is curious about how to proceed with such calculations in an especially unexpedited manner, this appendix details the equations for the Levi-Civita connection of a metric whose horizontal and vertical distributions are (for lack of a better word) *simultaneously* warped.

Let $\pi : (M^n, g_M) \to (B^b, g_B)$ be a Riemannian submersion whose horizontal and vertical distributions are denoted by $\mathcal{H}$ and $\mathcal{V}$, respectively. Suppose there are smooth functions $\psi : B \to \mathbb{R}$ and $\varphi : B \to \mathbb{R}$ and define their pullbacks on π by

$$\tilde{\psi} : M \to \mathbb{R} \quad \text{and} \quad \tilde{\varphi} : M \to \mathbb{R},$$

$$\tilde{\psi} = \psi \circ \pi \quad \text{and} \quad \tilde{\varphi} = \varphi \circ \pi.$$

We then warp the metrics g_B and g_M in the following manner, respectively:

$$\tilde{g}_B = e^{2\psi} g_B, \qquad \text{and} \qquad \tilde{g}_M = e^{2\tilde{\psi}} g_M|_{\mathcal{H}\oplus\mathcal{H}} + e^{2\tilde{\varphi}} g_M|_{\mathcal{V}\oplus\mathcal{V}}.$$

In particular, the new map $\pi : (M^n, \tilde{g}_M) \to (B^b, \tilde{g}_B)$ is still a submersion.

Continuing with the notation from Chapter 3, the superscripts $^{\mathbf{h}}$ and $^{\mathbf{v}}$ denote the horizontal or vertical component of a vector, respectively. Furthermore, we let $\langle \cdot, \cdot \rangle_{\sim} := \tilde{g}_M$ and $\langle \cdot, \cdot \rangle := g_M$. Let $X, Y, Z \in \mathcal{H}$ and $T, U, V \in \mathcal{V}$. Let ∇ and $\widetilde{\nabla}$ be the connections for g_M and $\tilde{g}_M$, respectively.

Proposition A.0.1. *The connection $\widetilde{\nabla}$ is described below.*

$$\widetilde{\nabla}^{\mathbf{h}}_X Y = \nabla^{\mathbf{h}}_X Y \tag{A.0.1}$$

$$\widetilde{\nabla}^{\mathbf{v}}_X Y = \nabla^{\mathbf{v}}_X Y - e^{2\psi} e^{-2\varphi} \langle X, Y \rangle \nabla\psi \tag{A.0.2}$$

$$\widetilde{\nabla}^{\mathbf{h}}_X T = e^{-2\psi} e^{2\varphi} \nabla^{\mathbf{h}}_X T + \langle \nabla\psi, T \rangle X \tag{A.0.3}$$

$$\widetilde{\nabla}^{\mathbf{v}}_X T = \nabla^{\mathbf{v}}_X T + \langle \nabla\varphi, X \rangle T \tag{A.0.4}$$

$$\widetilde{\nabla}^{\mathbf{h}}_T X = e^{-2\psi} e^{2\varphi} \nabla^{\mathbf{h}}_T X + \langle \nabla\psi, T \rangle X \tag{A.0.5}$$

$$\widetilde{\nabla}^{\mathbf{v}}_T X = \nabla^{\mathbf{v}}_T X + \langle \nabla\varphi, X \rangle T \tag{A.0.6}$$

$$\widetilde{\nabla}^{\mathbf{h}}_{T}U = e^{-2\psi}e^{2\varphi}\left(\nabla^{\mathbf{h}}_{T}U + \langle T, U\rangle \nabla\varphi\right) \tag{A.0.7}$$

$$\widetilde{\nabla}^{\mathbf{v}}_{T}U = \nabla^{\mathbf{v}}_{T}U \tag{A.0.8}$$

The calculations that give us Equations (A.0.1) to (A.0.8) are listed below.

Equations (A.0.1) and (A.0.8)

These equations follow directly from the Kozul formula.

Equation (A.0.2)

$$\begin{aligned}
\left\langle \widetilde{\nabla}_X Y, T\right\rangle_{\sim} &= \frac{1}{2}\left\langle [X,Y], T\right\rangle_{\sim} - \frac{1}{2}T\langle X, Y\rangle_{\sim} \\
&= \frac{1}{2}e^{2\varphi}\left\langle [X,Y], T\right\rangle - T(\psi)e^{2\psi}\langle X, Y\rangle - \cancel{\frac{1}{2}e^{2\psi}T\langle X, Y\rangle} \\
&= e^{2\varphi}\langle \nabla_X Y, T\rangle - T(\psi)e^{2\psi}\langle X, Y\rangle \\
&= \langle \nabla_X Y, T\rangle_{\sim} - e^{2\psi}e^{-2\varphi}\langle X, Y\rangle \langle \nabla\psi, T\rangle_{\sim}
\end{aligned}$$

Therefore,

$$\widetilde{\nabla}^{\mathbf{v}}_X Y = \nabla^{\mathbf{v}}_X Y - e^{2\psi}e^{-2\varphi}\langle X, Y\rangle \nabla\psi$$

Equation (A.0.3)

By compatibility:

$$\left\langle \widetilde{\nabla}_X T, Y\right\rangle_{\sim} = \cancel{X\langle T, Y\rangle_{\sim}} - \left\langle \widetilde{\nabla}_X Y, T\right\rangle_{\sim}$$

The same formula holds in the original metric as well:

$$\left\langle \widetilde{\nabla}_X T, Y \right\rangle = -\left\langle \widetilde{\nabla}_X Y, T \right\rangle$$

By (A.0.2),

$$\begin{aligned}\left\langle \widetilde{\nabla}_X T, Y \right\rangle_{\sim} &= e^{2\varphi} \left\langle \nabla_X T, Y \right\rangle - e^{2\psi} \left\langle \nabla \psi, T \right\rangle \left\langle X, Y \right\rangle \\ &= e^{-2\psi} e^{2\varphi} \left\langle \nabla_X T, Y \right\rangle_{\sim} - \left\langle \nabla \psi, T \right\rangle \left\langle X, Y \right\rangle_{\sim}\end{aligned}$$

Therefore:

$$\widetilde{\nabla}^{\mathbf{h}}_X T = e^{-2\psi} e^{2\varphi} \nabla^{\mathbf{h}}_X T + \left\langle \nabla \psi, T \right\rangle X$$

Equation (A.0.4)

$$\begin{aligned}\left\langle \widetilde{\nabla}_X T, U \right\rangle_{\sim} &= \frac{1}{2} \left(X \left\langle T, U \right\rangle_{\sim} + \left\langle U, [X, T] \right\rangle_{\sim} + \left\langle T, [U, X] \right\rangle_{\sim} \right) \\ &= X(\varphi) e^{2\varphi} \left\langle T, U \right\rangle + \frac{1}{2} e^{2\varphi} X \left\langle T, U \right\rangle + \frac{1}{2} e^{2\varphi} \left\langle U, [X, T] \right\rangle + \frac{1}{2} e^{2\varphi} \left\langle T, [U, X] \right\rangle \\ &= e^{2\varphi} \left\langle X, \nabla \varphi \right\rangle \left\langle T, U \right\rangle + e^{2\varphi} \left\langle \nabla_X T, U \right\rangle \\ &= \left\langle X, \nabla \varphi \right\rangle \left\langle T, U \right\rangle_{\sim} + \left\langle \nabla_X T, U \right\rangle_{\sim}\end{aligned}$$

Therefore:

$$\widetilde{\nabla}^{\mathbf{v}}_X T = \nabla^{\mathbf{v}}_X T + \left\langle \nabla \varphi, X \right\rangle T$$

Equation (A.0.5)

By symmetry:

$$\left\langle \widetilde{\nabla}_T X, Y \right\rangle_\sim = \left\langle \cancel{[X,T]} + \widetilde{\nabla}_X T, Y \right\rangle_\sim = \left\langle \widetilde{\nabla}_X T, Y \right\rangle_\sim$$

The same formula holds in the original metric as well:

$$\left\langle \widetilde{\nabla}_T X, Y \right\rangle = \left\langle \widetilde{\nabla}_X T, Y \right\rangle$$

By (A.0.3),

$$\left\langle \widetilde{\nabla}_T X, Y \right\rangle_\sim = e^{-2\psi} e^{2\varphi} \left\langle \nabla_T X, Y \right\rangle_\sim + \left\langle \nabla \psi, T \right\rangle \left\langle X, Y \right\rangle_\sim$$

Therefore:

$$\widetilde{\nabla}^{\mathbf{h}}_T X = e^{-2\psi} e^{2\varphi} \nabla^{\mathbf{h}}_T X + \left\langle \nabla \psi, T \right\rangle X$$

Equation (A.0.6)

This equation follows from applying the same method used to calculate Equation (1.4).

Therefore:

$$\widetilde{\nabla}^{\mathbf{v}}_T X = \nabla^{\mathbf{v}}_T X + \left\langle \nabla \varphi, X \right\rangle T$$

Equation (A.0.7)

By compatibility:

$$\left\langle \widetilde{\nabla}_T X, U \right\rangle_\sim = \cancel{T \left\langle X, U \right\rangle_\sim} - \left\langle \widetilde{\nabla}_T U, X \right\rangle_\sim$$

The same formula holds in the original metric as well:

$$\langle \nabla_T X, U \rangle_\sim = - \langle \nabla_T U, X \rangle_\sim$$

By (A.0.4),

$$\begin{aligned}
\left\langle \widetilde{\nabla}_T X, U \right\rangle_\sim &= e^{2\varphi} \langle \nabla_T U, X \rangle - e^{2\varphi} \langle T, U \rangle \langle \nabla \varphi, X \rangle \\
&= e^{-2\psi} e^{2\varphi} \left(\langle \nabla_T U, X \rangle_\sim - \langle T, U \rangle \langle \nabla \varphi, X \rangle_\sim \right)
\end{aligned}$$

Therefore:

$$\widetilde{\nabla}^{\mathbf{h}}_T U = e^{-2\psi} e^{2\varphi} \left(\nabla^{\mathbf{h}}_T U + \langle T, U \rangle \nabla \varphi \right)$$

Appendix B

Eigenspace Approximations

In a previous version of Section 3.6, our implementation of Lemma 3.6.2 (from [RW23]) relied on our ability to verify the closeness of an eigenspace to a given approximate eigenspace. After some deliberation, it was decided that this strategy needed to be reworked into what is now the current version of Section 3.6. However, we believe that some aspects of the old strategy may be of independent interest, and so its key lemma is recorded in this appendix. The author reminds the reader that, along with Chapter 3, the following lemma was proved in collaboration with Russell Phelan and Frederick Wilhelm (cf. [EPW25]).

Lemma B.0.1. *Let $L : \mathcal{V} \to \mathcal{V}$ be a self adjoint linear map of an n-dimensional inner product space $\mathcal{V}$. For some $\lambda \in \mathbb{R}$ and $\delta > 0$, let $\mathcal{U} \subset \mathcal{V}$ be a subspace so that for all unit vectors $U \in \mathcal{U}$,*

$$\left|\lambda - \left\langle L(U), U \right\rangle\right| < \delta. \tag{B.0.1}$$

1. *Given any* $\varepsilon > 0$, *there is a* $\delta_0 > 0$ *so that if* $\delta \in (0, \delta_0)$, *then there is an* L*-invariant subspace* $\mathcal{I} \subset \mathcal{V}$ *so that*

$$\left\|\lambda \operatorname{id} - L|_{\mathcal{I}}\right\|^2 < \varepsilon \tag{B.0.2}$$

   and

$$\left\|\operatorname{id}_{\mathcal{U}} - \operatorname{pr}_{\mathcal{I}}\right\|^2 < \varepsilon, \tag{B.0.3}$$

   where $\operatorname{pr}_{\mathcal{I}}$ *is the orthogonal projection onto* $\mathcal{I}$. *In particular, if* $\dim \mathcal{I} = \dim \mathcal{U}$, *and* ε *is sufficiently close to* 0, *then* $\operatorname{pr}_{\mathcal{I}}$ *is an isomorphism.*

2. *Given any* $c, R > 0$, *there are* $\varepsilon_1, \delta_1 > 0$ *with the following properties. Let* $\mathcal{I}^{\perp}$ *be the orthogonal complement of* $\mathcal{I}$. *Suppose that* $\dim \mathcal{I} = \dim \mathcal{U}$. *Suppose for some* $\delta \in (0, \delta_1)$ *and* $\varepsilon \in (0, \varepsilon_1)$, L *and* $\mathcal{I}$ *satisfy (B.0.1), (B.0.2), (B.0.3), that for any eigenvalue* $\lambda_{\perp}$ *of* $L|_{\mathcal{I}^{\perp}}$,

$$|\lambda - \lambda_{\perp}| > c \;\text{ and }\; \|L\| \leq R. \tag{B.0.4}$$

   Then $\mathcal{I}$ *is the unique* L*-invariant subspace of* $\mathcal{V}$ *satisfying (B.0.3).*

Proof. Combining the Spectral Theorem with the assumption that L is self-adjoint, we decompose $\mathcal{V}$ into eigenspaces $\mathcal{V} = \bigoplus_{i=1}^{n} \mathcal{E}^i$, with corresponding eigenvalues λ_i. For $\eta > 0$, let

$$\Lambda_\eta = \{\lambda_i : |\lambda - \lambda_i| < \eta\},$$

and define

$$\mathcal{I} = \bigoplus_{\{i \,:\, \lambda_i \in \Lambda_\eta\}} \mathcal{E}^i.$$

Thus (B.0.2) is satisfied if η is sufficiently small. If Λ_η contains all the eigenvalues of L, then $\mathcal{I} = \mathcal{V}$, and the proof of (B.0.3) is trivial.

Otherwise, there is a unique decomposition

$$U = \sum_{j=1}^{n} U_i, \qquad U_i \in \mathcal{E}^i$$

of any unit vector $U \in \mathcal{U}$, and it follows that

$$\begin{aligned} L(U) &= \sum_{i=1}^{n} L(U_i) \\ &= \sum_{i=1}^{n} \lambda_i U_i. \end{aligned}$$

Hence

$$\big\langle L(U), U \big\rangle = \sum_{i=1}^{n} \lambda_i \left\| U_i \right\|^2.$$

For some $\delta_0 > 0$, suppose that

$$\delta_0 > \Big| \lambda - \big\langle L(U), U \big\rangle \Big|.$$

Since U is a unit vector, $\lambda = \sum_{i=1}^{n} \lambda \left\| U_i \right\|^2$. Thus,

$$\begin{aligned} \delta_0 &> \Big| \lambda - \big\langle L(U), U \big\rangle \Big| \\ &> \sum_{i=1}^{n} \left| \lambda - \lambda_i \right| \left\| U_i \right\|^2 \\ &= \sum_{\{i \,:\, \lambda_i \in \Lambda_\eta\}} \left| \lambda - \lambda_i \right| \left\| U_i \right\|^2 + \sum_{\{i \,:\, \lambda_i \notin \Lambda_\eta\}} \left| \lambda - \lambda_i \right| \left\| U_i \right\|^2 \end{aligned}$$

In particular,

$$\begin{aligned}
\delta_0 &> \sum_{\{i\,:\,\lambda_i \notin \Lambda_\eta\}} |\lambda - \lambda_i| \, \|U_i\|^2 \\
&\geq \eta \sum_{\{i\,:\,\lambda_i \notin \Lambda_\eta\}} \|U_i\|^2 .
\end{aligned}$$

So if $\delta_0 = \varepsilon\eta$, then for any unit vector $U \in \mathcal{U}$,

$$\begin{aligned}
\left\| U - \mathrm{pr}_{\mathcal{I}}(U) \right\|^2 &= \sum_{\{i\,:\,\lambda_i \notin \Lambda_\eta\}} \|U_i\|^2 \\
&< \frac{\delta_0}{\eta} \\
&= \varepsilon,
\end{aligned}$$

proving (B.0.3). If ε is close enough to 0, then (B.0.3) implies that that $\mathrm{pr}_{\mathcal{I}}$ is an injection. Thus the additional hypothesis, $\dim \mathcal{I} = \dim \mathcal{U}$, implies that $\mathrm{pr}_{\mathcal{I}}$ is an isomorphism.

For part 2, suppose $\mathcal{J}$ is another L-invariant subspace so that for all unit vectors $U \in \mathcal{U}$,

$$\left\| U - \mathrm{pr}_{\mathcal{J}}(U) \right\|^2 < \varepsilon. \tag{B.0.5}$$

Since $\mathcal{J}$ is L-invariant, it decomposes into a direct sum of eigenspaces,

$$\mathcal{J} = \bigoplus_{\{i\,:\,\lambda_i \in S\}} \mathcal{E}^i,$$

where $S \subset \{\lambda_1, \cdots, \lambda_n\}$. If $\mathcal{J} \neq \mathcal{I}$, then for some $\lambda_b \in S$,

$$|\lambda_b - \lambda| > c. \tag{B.0.6}$$

Since (B.0.3) also holds for $\mathcal{J}$ and $\dim \mathcal{J} = \dim \mathcal{U}$, we have that $\mathrm{pr}_{\mathcal{J}} : \mathcal{U} \to \mathcal{J}$ is an isomorphism, provided ε is sufficiently small. Now let E_b be a unit vector in the intersection $\mathcal{E}^b \cap \mathcal{J}$, of the eigen space $\mathcal{E}^b$ with $\mathcal{J}$, and let $\tilde{E}_b := \mathrm{pr}_{\mathcal{J}}^{-1}(E_b)$. Then

$$\begin{aligned}
c &< |\lambda_b - \lambda| \quad \text{by (B.0.5)} \\
&\leq \left| \left\langle L(E_b), E_b \right\rangle - \left\langle L(\tilde{E}_b), \tilde{E}_b \right\rangle \right| + O(\varepsilon), \quad \text{by (B.0.3)} \\
&\leq \left| \left\langle L(E_b), E_b \right\rangle - \left\langle L(E_b), \tilde{E}_b \right\rangle \right| + \left| \left\langle L(E_b), \tilde{E}_b \right\rangle - \left\langle L(\tilde{E}_b), \tilde{E}_b \right\rangle \right| + O(\varepsilon) \\
&\leq \|L\| \, |E_b| \left| E_b - \tilde{E}_b \right| + \|L\| \left| \tilde{E}_b \right| \left| E_b - \tilde{E}_b \right| + O(\varepsilon) \\
&\leq \|L\| \left| E_b - \tilde{E}_b \right| + 2 \|L\| \left| E_b - \tilde{E}_b \right| + O(\varepsilon), \quad \text{by (B.0.5)} \\
&\leq 3R\varepsilon,
\end{aligned}$$

by (B.0.5) and the fact that we have assumed that $\|L\| < R$. Thus $c < 3R\varepsilon$, which is a contradiction, provided we choose that $\varepsilon_1 < \frac{c}{3R}$. $\square$

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