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A FADING-MEMORY THEOREM FOR MATERIALS WITH INTERNAL VARIABLES

by

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Abstract

It is shown that a material whose thermodynamical state is described by internal variables α_i which are governed by equations of evolution of the form $\dot{\alpha}_i = f_i(u_j, \alpha_k)$, where the u_j represent the "external" variables (deformation, temperature, temperature gradient) possesses the property of fading memory (as postulated by Coleman) if the f_i are differentiable and if the matrix of their partial derivatives with respect to the α_k is negative definite.

1. Introduction

The concepts of irreversible thermodynamics, in particular those of internal variables and equations of evolution, were applied to study of linear materials with memory by Biot [1-3] and Ziegler [4-6]. Recently these applications have been extended to include nonlinear materials undergoing large deformations by Valanis [7-9] and Coleman and Gurtin [10]. The last-named authors point out that this approach to continuum thermodynamics is but one of several, other approaches including those based on constitutive equations of differential type (as assumed by Coleman and Mizel [11], Schapery [12], Perzyna and Olszak [13], Mandel and Brun [14], and others) and on the axiom of fading memory (as formulated by Coleman [15] and elaborated by Coleman and Mizel [16], Mizel and Wang [17] and others).

These approaches are generally acknowledged to be independent of one another. However, some important bridges have been built: (1) Coleman and Noll [18] have proved that constitutive equations of differential type are valid approximations for materials with fading memory undergoing slow deformations; (2) linear viscoelastic materials of evolutionary type (excluding those with instantaneous irreversibility) have relaxation functions of negative exponential form and hence fading memory.

In the present paper it will be proved that, under specific but fairly broad conditions, the principle of fading memory is obeyed by nonlinear evolutionary materials. It thus follows that the generalized stress relations derived by Valanis [9] and by Coleman and Gurtin [10] are implicit in the work of Coleman [15], and that the stronger results of the last named (e.g., the independence of the internal-dissipation and heat-conduction inequalities) are valid under equivalent constitutive hypotheses.

2. Assumptions

It is assumed that the thermodynamic state at time t at a point of a body is determined by n internal or hidden variables $\alpha_1, \ldots, \alpha_n$, forming a vector $\underset{\sim}{\alpha}$, in addition to the external or observed variables (deformation or strain, temperature, temperature gradient) which will be denoted by a vector $\underset{\sim}{u}$. The internal variables are assumed to be governed by equations of evolution

$$\dot{\underset{\sim}{\alpha}} = \underset{\sim}{f}(\underset{\sim}{u},\underset{\sim}{\alpha}) \tag{1}$$

where $\cdot \equiv \frac{d}{dt}$. Functions $\underset{\sim}{u}(t)(t > -\infty)$ permitting unique, continuous solutions $\underset{\sim}{\alpha}(t)$ of eq. (1) which satisfy $\underset{\sim}{\alpha}(-\infty) = \underset{\sim}{0}$ will be considered admissible. The function $\underset{\sim}{f}(\underset{\sim}{u},\underset{\sim}{\alpha})$ is assumed to be differentiable in all $n + r$ arguments (r is the dimension of $\underset{\sim}{u}$).

In view of the existence theorem of Carathéodory (see, e.g., Coddington and Levinson [19]) and the uniqueness theorem of Diaz and Walter [20]*, $\underset{\sim}{f}(\underset{\sim}{u}(t),\underset{\sim}{\alpha})$ and hence $\underset{\sim}{u}(t)$ need not be continuous in t for $\underset{\sim}{\alpha}(t)$ to be continuous. Consequently the "present value" $\underset{\sim}{\alpha}(t)$ is determined only by the "past history" $\underset{\sim}{u}(\tau)$, $\tau < t$, and not the present value $\underset{\sim}{u}(t)$. Denoting (as in [15]) by $\underset{\sim}{u}^r(\tau)$ the restriction of $\underset{\sim}{u}(\tau)$ to $\tau < t$, the solution of (1) may be written in the form of a functional,

$$\underset{\sim}{\alpha}(t) = \hat{\underset{\sim}{\alpha}}\{\underset{\sim}{u}^r(\tau)\}_{\tau=-\infty}^{t} \tag{2}$$

Furthermore, the Fréchet derivative of $\hat{\underset{\sim}{\alpha}}$ with respect to a variation $\underset{\sim}{u} \to \underset{\sim}{u} + \underset{\sim}{v}$ ($\underset{\sim}{u} + \underset{\sim}{v}$ assumed admissible),

$$\delta_{\underset{\sim}{u}^r} \hat{\underset{\sim}{\alpha}}\{\underset{\sim}{u}^r(\tau) | \underset{\sim}{v}^r(\tau)\}_{\tau=-\infty}^{t}$$

is equal to the solution $\delta\underset{\sim}{\alpha}$ of the variational equation of (1),

*These theorems are valid for finite intervals $t_0 \leq t \leq t_1$; the existence of limits of the solutions as $t_0 \to -\infty$, uniform in t, is assumed.

$$\delta\dot{\alpha} = f_{\alpha}\delta\alpha + f_{u}v \tag{3}$$

satisfying $\delta\alpha(-\infty) = 0$. In eq. (3) f_{α} and f_{u} denote the respective matrices $[\partial f_i/\partial \alpha_j]$ and $[\partial f_i/\partial u_k]$ evaluated at $u(t)$, $\alpha(t)$.

If the Helmholtz free energy ψ is assumed to be a function of (the present values of) u and α , then eq. (2) also defines it as a functional of $u(\tau)$, $\tau \leq t$:

$$\begin{aligned}\psi(u,\alpha) &= \psi(u,\hat{\alpha}\{\overset{t}{\underset{\tau=-\infty}{u^r}}(\tau)\})\\ &= \hat{\psi}(\overset{t}{\underset{\tau=-\infty}{u^r}}(\tau);u)\end{aligned} \tag{4}$$

Furthermore,

$$D_{u}\hat{\psi} = \frac{\partial\psi}{\partial u} \tag{5}$$

and, by the chain rule,

$$\delta_{u^r}\hat{\psi} = \partial\psi/\partial\alpha \cdot \delta\alpha . \tag{6}$$

To establish the fading-memory property for $\hat{\psi}$ it clearly suffices to do so for $\hat{\alpha}$. Once this is done, eq. (5) shows the aforementioned equivalence of the generalized stress relations.

3. Fading Memory

A sufficient condition for the possession of fading memory by $\hat{\alpha}$ is, according to Mizel and Wang [17], the existence of a decreasing non-negative function $h(s)$ (the influence function), $0 \leq s < \infty$, such that

$$\int_0^\infty h^2(s)ds < \infty \tag{7}$$

and

$$\hat{\alpha}\{\underset{\sim}{u}^r(\tau) + \underset{\sim}{v}^r(\tau)\} - \hat{\alpha}\{\underset{\sim}{u}^r(\tau)\}$$
$$= \delta_{\underset{\sim}{u}^r} \hat{\alpha}\{\underset{\sim}{u}^r(\tau) | \underset{\sim}{v}^r(\tau)\} + o(||\underset{\sim}{v}||) \tag{8}$$

where

$$||\underset{\sim}{v}||^2 = \int_0^\infty |\underset{\sim}{v}(t - s)|^2 \, h^2(s) ds < \infty , \tag{9}$$

$|\underset{\sim}{v}|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$, and $o(\ \ )$ is interpreted in the sense "as the argument goes to zero". The following theorem will now be proved:

If $\underset{\sim}{f}_{\underset{\sim}{u}}$ is bounded and $\underset{\sim}{f}_{\underset{\sim}{\alpha}}$ is negative definite for any admissible $\underset{\sim}{u}(t)$ and the corresponding $\underset{\sim}{\alpha}(t)$ then the functional $\hat{\underset{\sim}{\alpha}}$ has the fading-memory property with an influence function $h(s)$ given by

$$h^2(s) = \begin{cases} e^{-ks} , & 0 \leq s \leq 1/k \\ kse^{-ks} & 1/k \leq s \end{cases} \tag{10}$$

where k is a positive number such that $\underset{\sim}{y} \cdot \underset{\sim}{f}_{\underset{\sim}{\alpha}} \underset{\sim}{y} \leq - k \underset{\sim}{y} \cdot \underset{\sim}{y}$ for any n-vector $\underset{\sim}{y}$.

4. Proof of the Theorem

The negative definiteness of $\underset{\sim}{f}_{\underset{\sim}{\alpha}}$ implies the existence of k . The boundedness of $\underset{\sim}{f}_{\underset{\sim}{u}}$ implies the existence of a positive number m such that

$$|\underset{\sim}{f}_{\underset{\sim}{u}} \underset{\sim}{z}| \leq m |\underset{\sim}{z}| \tag{11}$$

for any r-vector $\underset{\sim}{z}$.

Let us denote the left-hand side of eq. (8), viewed as a function of t , by $\underset{\sim}{\beta}(t)$. Then

$$\dot{\underset{\sim}{\beta}} = \underset{\sim}{f}(\underset{\sim}{u} + \underset{\sim}{v}, \underset{\sim}{\alpha} + \underset{\sim}{\beta}) - \underset{\sim}{f}(\underset{\sim}{u}, \underset{\sim}{\alpha}) . \tag{12}$$

Furthermore, let $\underset{\sim}{\gamma}$ denote $\underset{\sim}{\beta} - \delta \underset{\sim}{\alpha}$, and let us subtract (2) from (10), writing the result as

$$\dot{\gamma} = f_{\alpha}\gamma + \omega \tag{13}$$

where

$$\omega = f(u + v\ ,\ \alpha + \beta) - f(u,\alpha) - f_u v - f_{\alpha}\beta\ .$$

The differentiability of f implies

$$\omega = o\left(\sqrt{|v|^2 + |\beta|^2}\right).$$

The Schwartz and triangle inequalities yield

$$\sqrt{|v|^2 + |\beta|^2} \le |v| + |\beta| \le |v| + |\delta\alpha| + |\gamma|\ .$$

Hence, using the distributive property of the o symbol (for this and other properties, see Erdelyi [21]) we have

$$\omega = o(|v|) + o(|\delta\alpha|) + o(|\gamma|)\ . \tag{14}$$

Let $\Phi(t;\tau)$ be the $n \times n$ matrix solution of

$$\Phi = f_{\alpha}\Phi \tag{15}$$

satisfying $\Phi(\tau;\tau) = I$, the unit matrix. Then the solution of (13) may be written as

$$\gamma(t) = \int_{-\infty}^{t} \Phi(t;\tau)\omega(\tau)d\tau\ . \tag{16}$$

A result due to Ważewski [22] states*

*In [22] continuity of all functions is assumed. The proof in Zadeh and Desoer [23] dispenses with this assumption. See also a similar result attributed to Wintner by Cesari [24].

$$|\underset{\sim}{\Phi}(t;\tau)\underset{\sim}{\omega}(\tau)| \leq e^{\int_\tau^t s(\lambda)d\lambda} |\underset{\sim}{\omega}(\tau)| \tag{17}$$

where $s(t)$ is the largest eigenvalue of 1/2 $(\underset{\sim}{f}_{\underset{\sim}{\alpha}} + \underset{\sim}{f}_{\underset{\sim}{\alpha}}^T)$. Since, by hypothesis,

$$s(t) \leq - k$$

we have

$$|\underset{\sim}{\gamma}(t)| \leq \int_{-\infty}^{t} e^{-k(t-\tau)} |\underset{\sim}{\omega}(\tau)| d\tau . \tag{18}$$

Combining the inequality (18) with the asymptotic expression (14) and using the commutativity of the o symbol with integration [21], we obtain

$$|\underset{\sim}{\gamma}(t)| = o\left(\int_{\infty}^{t} e^{-k(t-\tau)} |\underset{\sim}{v}(\tau)| d\tau\right)$$
$$+ o\left(\int_{-\infty}^{t} e^{-k(t-\tau)} |\delta\underset{\sim}{\alpha}(\tau)| d\tau\right) + o\left(\int_{-\infty}^{t} e^{-k(t-\tau)} |\underset{\sim}{\gamma}(\tau)| d\tau\right) . \tag{19}$$

Considering, first, the last term of (19), we see that its contribution to $|\underset{\sim}{\gamma}|$ will vanish faster than the approximation which neglects this term. Hence we need only investigate the contributions of the other two terms.

For the argument of the first term we readily see

$$\int_{-\infty}^{t} e^{-k(t-\tau)} |\underset{\sim}{v}(\tau)| d\tau = \int_{0}^{\infty} e^{-ks} |\underset{\sim}{v}(t-s)| ds \leq \int_{0}^{\infty} h^2(s) |\underset{\sim}{v}(t-s)| ds \tag{20}$$

if $h^2(s)$ is given by (10). Furthermore, by the Cauchy-Schwartz inequality,

$$\left[\int_0^\infty h^2(s)|\underset{\sim}{v}(t-s)|\,ds\right]^2 \leq \int_0^\infty h^2(s)ds \cdot \int_0^\infty h^2(s)|\underset{\sim}{v}(t-s)|^2 ds = \frac{1+e^{-1}}{k}||\underset{\sim}{v}||^2 , \quad (21)$$

with $||v||$ given by (9). Hence the first term is $o(||v||)$.

To evaluate the second term we first solve eq. (3):

$$\delta\underset{\sim}{\alpha} = \int_{-\infty}^t \Phi(t;\tau)B(\tau)\underset{\sim}{v}(\tau)d\tau \quad (22)$$

where $B(t) = \underset{\sim}{f}_{\underset{\sim}{u}}(\underset{\sim}{u}(\tau), \underset{\sim}{\alpha}(\tau))$. Using, again, Ważewski's inequality, as well as the boundedness of $\underset{\sim}{f}_{\underset{\sim}{u}}$, we obtain

$$|\delta\underset{\sim}{\alpha}(t)| \leq m \int_0^\infty e^{-ks}|\underset{\sim}{v}(t-s)|ds \quad (23)$$

Noting that

$$\int_{-\infty}^t e^{-k(t-\tau)} \int_0^\infty e^{-ks}|\underset{\sim}{v}(\tau - s)|ds\, d\tau$$

$$= \int_0^\infty se^{-ks}|\underset{\sim}{v}(t - s)|ds$$

$$\leq \frac{1}{k}\int_0^\infty h^2(s)|\underset{\sim}{v}(t - s)|ds ,$$

with $h^2(s)$ given by (10), we find, through (21), that the second term of (19) is also $o(||v||)$. The theorem is thus proved.

5. <u>Remarks</u>

a. The functionals (2) have been proved to possess fading memory in the sense of Coleman [15]. The materials which they characterize are, however, somewhat more general than those considered by Coleman, who explicitly limited himself to the simplest hypothesis compatible with equipresence. In particular, Coleman assumed the constitutive functionals to be independent of the past history of the temperature gradient and depend only on its present value. It is this latter dependence of the free-energy functional and hence the stress and entropy which is then eliminated by the Clausius-Duhem inequality. In view of eq. (4), an equivalent assumption for evolutionary materials would be that for each i $(i = 1, \ldots, n)$ either $\partial\psi/\partial\alpha_i = 0$, or f_i is independent of temperature gradient.

b. Let $\underset{\sim}{\tau}$ be a generalized stress vector conjugate to $\underset{\sim}{u}$ such that

$$\underset{\sim}{\tau} = \psi_{\underset{\sim}{u}} \; . \tag{24}$$

Differentiation of (24) with respect to time, together with elimination of $\dot{\underset{\sim}{\alpha}}$ by means of (1), yields

$$\dot{\tau}_i = \frac{\partial^2\psi}{\partial u_i \partial u_j} \dot{u}_j + \frac{\partial^2\psi}{\partial u_i \partial \alpha_k} f_k \, (\underset{\sim}{u},\underset{\sim}{\alpha}) \; , \tag{25}$$

and, more generally,

$$\frac{d^s}{dt^s} \tau_i = \frac{\partial^2\psi}{\partial u_i \partial u_j} \frac{d^s u_j}{dt^s} + \phi_{is} \, (\underset{\sim}{\alpha},\underset{\sim}{u},\dot{\underset{\sim}{u}}, \ldots, \frac{d^{s-1}\underset{\sim}{u}}{dt^{s-1}}) \; . \tag{26}$$

$s = 1,2, \ldots$. It is possible, in principle, to take n independent equations of the form (26) and solve then for $\underset{\sim}{\alpha}$ in terms of time derivatives of $\underset{\sim}{\tau}$ and $\underset{\sim}{u}$, the result being substituted in (24) which then becomes a constitutive equation of rate type. Evolutionary materials are therefore both of fading-memory and of rate type.

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