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ABSTRACT

We develop a Reggeized model for the 3-to-3 amplitude using group theoretical variables. The triple-Regge vertex is defined by the asymptotic form of the amplitude. It is defined entirely in terms of physical region quantities and therefore is, in principle, directly measurable.

I. INTRODUCTION

Since practical scattering experiments generally have only two particles in the initial state, theoretical physicists have focused most of their attention on such reactions. From a broad theoretical point of view, on the other hand, there is no reason to exclude from our consideration processes with more than two particles in the initial state. Indeed, both crossing and unitarity imply that an understanding of processes with two particles in the initial state is intimately related to more general processes.

The usefulness of Regge-pole expansions for the description of the asymptotic behavior of the 2-to-n amplitude is well known. Expressions for the amplitude in terms of group theoretical variables have been particularly convenient for the formulation of the Regge-pole hypothesis. Such variables were first introduced by Toller[1] for the 2-to-2 amplitude, and were extended to the general 2-to-n amplitude by Bali, Chew, and Pignotti.[2] The set of variables, the method of analysis, and the resulting expression for the 2-to-n amplitude can be schematically represented by a tree diagram (see Fig. 1). The variables are the magnitudes of the momentum transfers and the corresponding little group elements; there is one momentum transfer and one little group element for each internal line in the diagram. If the amplitude's dependence upon any little group element is expressed in terms of the projection of the amplitude onto the irreducible representations of the little group, a plausible physical assumption is that the leading singularity of this projection in the

ℓ-plane, where ℓ labels the irreducible representation, is a pole with a factorizable residue. Defining an asymptotic region of the variables by letting the boost parameters in all little group elements corresponding to space-like momentum transfers go to infinity, the amplitude's asymptotic behavior is described in terms of the positions of the leading poles and a set of vertex functions. There is one two-particle-one-Reggeon vertex function for each vertex with two external lines and one internal line in the diagram, and one one-particle-two-Reggeon vertex function for each vertex with one external line and two internal lines.

For the general m-to-n amplitude, more complicated tree diagrams can be drawn; in particular, diagrams containing vertices with three internal lines become possible. Toller[3] has suggested a particular set of variables for an arbitrary tree diagram, his objective being an amplitude free of kinematic singularities and constraints.

The triple-Regge vertex, i.e., a vertex function associated with a vertex with three internal lines in a tree diagram, has been studied by a number of authors. Misheloff[4] and Goddard and White[5] extended the Regge-pole hypothesis to the tree diagram with one three-internal-line-vertex for the 3-to-3 amplitude. For this reaction, it was necessary to distinguish between two parts of the physical region. In one part, the plane defined by the momentum transfers contains some time-like vectors. In the other part, this plane contains only space-like vectors. Goddard and White[6] used the analytic group variables of Toller[3] to discuss the implications of analyticity at the boundary between these two regions for the triple-Regge vertex. Landshoff and Zakrzewski[7] examined the triple-Regge vertex using several dynamical models.

The physical content of this thesis is essentially the same as that of Ref. 4. Considerably more detail is presented here and some of the mathematical calculations are presented in a more logically esthetic manner. In Sec. II we define a set of group theoretical variables for the 3-to-3 amplitude. In Sec. III we relate our variables to the invariants, and in Sec. IV we define an asymptotic region of the variables and extend the Regge-pole hypothesis to the description of the amplitude in this region. The triple-Regge vertex is defined by the asymptotic behavior. In Sec. V we study the 3-to-3 amplitude in the Veneziano model and find that it Reggeizes in the expected manner. In the Appendix we derive some properties of the irreducible representations of the three-dimensional Lorentz group.

II. DEFINITION OF VARIABLES FOR THE 3-to-3 AMPLITUDE

Let us consider the process $A_i + B_i + C_i \to A_f + B_f + C_f$. For this process there are two possible tree diagrams, which are shown in Figs. 2 and 3.

The analysis associated with Fig. 2 is very similar to the multi-Regge analysis for the 2-to-4 amplitude and is not expected to yield any essentially new information. The analysis of Fig. 3 is more complicated and contains the concept of a triple-Regge vertex. Therefore, in the following we confine our attention to the tree diagram of Fig. 3.

For simplicity we assume that all the particles are spinless and that they all have the same mass, m. We adopt the convention that incoming particles have positive energies whereas outgoing particles have negative energies.

We define the momentum transfers, Q_X, and their magnitudes, t_X, by

$$Q_X = p_{i_X} + p_{f_X} , \quad t_X = Q_X^{\,2} , \qquad (X = A,B,C) . \tag{2.1}$$

Energy-momentum conservation can be written as

$$Q_A + Q_B + Q_C = 0 . \tag{2.2}$$

Since Q_A is a space-like vector, there is a Lorentz frame in which Q_A points in the positive z direction. To specify this frame further, we require the three-vector $\underset{\sim}{p}_{i_A}$ to point in the z direction. Let this frame be called "frame a_p." Four-vectors in this frame have a superscript a_p. Equation (2.1) completely determines $p_{i_A}^{a_p}$, $p_{f_A}^{a_p}$, and $Q_A^{a_p}$:

$$\left.\begin{aligned} {}^{a_p}p_{i_A} &= [(m^2 - t_A/4)^{\frac{1}{2}},0,0,\tfrac{1}{2}(-t_A)^{\frac{1}{2}}] \\ {}^{a_p}p_{f_A} &= [-(m^2 - t_A/4)^{\frac{1}{2}},0,0,\tfrac{1}{2}(-t_A)^{\frac{1}{2}}] \\ {}^{a_p}Q_A &= [0,0,0,(-t_A)^{\frac{1}{2}}] \end{aligned}\right\} . \qquad (2.3a)$$

We define frames b_p and c_p in an analogous manner. In frame b_p

$$\left.\begin{aligned} {}^{b_p}p_{i_B} &= [(m^2 - t_B/4)^{\frac{1}{2}},0,0,\tfrac{1}{2}(-t_B)^{\frac{1}{2}}] \\ {}^{b_p}p_{f_B} &= [-(m^2 - t_B/4)^{\frac{1}{2}},0,0,\tfrac{1}{2}(-t_B)^{\frac{1}{2}}] \\ {}^{b_p}Q_B &= [0,0,0,(-t_B)^{\frac{1}{2}}] \end{aligned}\right\} . \qquad (2.3b)$$

In frame c_p

$$\left.\begin{aligned} {}^{c_p}p_{i_C} &= [(m^2 - t_C/4)^{\frac{1}{2}},0,0,\tfrac{1}{2}(-t_C)^{\frac{1}{2}}] \\ {}^{c_p}p_{f_C} &= [-(m^2 - t_C/4)^{\frac{1}{2}},0,0,\tfrac{1}{2}(-t_C)^{\frac{1}{2}}] \\ {}^{c_p}Q_C &= [0,0,0,(-t_C)^{\frac{1}{2}}] \end{aligned}\right\} . \qquad (2.3c)$$

Since Q_A is spacelike, there is a frame in which Q_A and Q_B are of the form

$$\left.\begin{aligned} Q_A &= [0,0,0,(-t_A)^{\frac{1}{2}}] \\ Q_B &= [u,v,0,w] \end{aligned}\right\} ,$$

where $u^2 - v^2 = t_B + w^2$ and $w = -(-t_A)^{-\frac{1}{2}}(Q_A \cdot Q_B)$. Using (2.2), we can write

$$
\begin{aligned}
u^2 - v^2 &= t_B - (Q_A \cdot Q_B)^2/t_A \\
&= t_B - [(Q_A + Q_B)^2 - Q_A{}^2 - Q_B{}^2]^2/(4t_A) \\
&= t_B - [(-Q_C)^2 - Q_A{}^2 - Q_B{}^2]^2/(4t_A) \\
&= t_B - [t_C - t_A - t_B]^2/(4t_A) \\
&= -\lambda(t_A,t_B,t_C)/(4t_A) \quad ,
\end{aligned}
$$

where

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \quad . \tag{2.4}$$

Since $t_A < 0$, $\lambda(t_A,t_B,t_C)$ and $u^2 - v^2$ have the same sign.

If $u^2 - v^2 > 0$, there is a frame, designated by a_r, in which Q_A points in the positive z direction and only the z and t components of Q_B are nonzero. If $u^2 - v^2 < 0$, there is a frame, designated by a_r', in which Q_A points in the positive z direction, only the x and z components of Q_B are nonzero, and the x component of Q_B is positive.

We must consider the two cases separately. The two cases are distinguished by the sign of $\lambda(t_A,t_B,t_C)$.

<u>Case I: $\lambda(t_A,t_B,t_C) > 0$</u>

We have completely determined $Q_A{}^{a_r}$, and $Q_B{}^{a_r}$ is determined by (2.2) up to the sign of the t component. We have

$$\left.\begin{aligned} Q_A{}^{a_r} &= [0,0,0,(-t_A)^{\frac{1}{2}}] \\ Q_B{}^{a_r} &= \tfrac{1}{2}(-t_A)^{-\frac{1}{2}}[\pm\lambda^{\frac{1}{2}}(t_A,t_B,t_C),0,0,t_A + t_B - t_C] \\ Q_C{}^{a_r} &= \tfrac{1}{2}(-t_A)^{-\frac{1}{2}}[\mp\lambda^{\frac{1}{2}}(t_A,t_B,t_C),0,0,t_A + t_C - t_B] \end{aligned}\right\} . \quad (2.5a)$$

By the application of a z boost of magnitude q_{ab}, where

$$\sinh q_{ab} = \mp\tfrac{1}{2}(t_A t_B)^{-\frac{1}{2}} \lambda^{\frac{1}{2}}(t_A,t_B,t_C) , \quad (2.6a)$$

to frame a_r, we arrive at a frame, called b_r, in which

$$\left.\begin{aligned} Q_A{}^{b_r} &= \tfrac{1}{2}(-t_B)^{-\frac{1}{2}}[\mp\lambda^{\frac{1}{2}}(t_A,t_B,t_C),0,0,t_A + t_B - t_C] \\ Q_B{}^{b_r} &= [0,0,0(-t_B)^{\frac{1}{2}}] \\ Q_C{}^{b_r} &= \tfrac{1}{2}(-t_B)^{-\frac{1}{2}}[\pm\lambda^{\frac{1}{2}}(t_A,t_B,t_C),0,0,t_B + t_C - t_A \end{aligned}\right\} . \quad (2.5b)$$

Similarly, by an application of a z boost of magnitude q_{bc}, where

$$\sinh q_{bc} = \mp\tfrac{1}{2}(t_B t_C)^{-\frac{1}{2}} \lambda^{\frac{1}{2}}(t_A,t_B,t_C) , \quad (2.6b)$$

to frame b_r, we arrive at a frame which we call frame c_r. In this frame

$$\left.\begin{aligned} Q_A{}^{c_r} &= \tfrac{1}{2}(-t_C)^{-\frac{1}{2}}[\pm\lambda^{\frac{1}{2}}(t_A,t_B,t_C),0,0,t_A + t_C - t_B] \\ Q_B{}^{c_r} &= \tfrac{1}{2}(-t_C)^{-\frac{1}{2}}[\mp\lambda^{\frac{1}{2}}(t_A,t_B,t_C),0,0,t_B + t_C - t_A] \\ Q_C{}^{c_r} &= [0,0,0(-t_C)^{\frac{1}{2}}] \end{aligned}\right\} . \quad (2.5c)$$

A z boost of magnitude q_{ca}, where

$$\sinh q_{ca} = \mp\frac{1}{2}(t_A t_C)^{-\frac{1}{2}} \lambda^{\frac{1}{2}}(t_A, t_B, t_C) \quad , \qquad (2.6c)$$

applied to frame c_r takes us back to frame a_r.

Frame X_r is related to frame X_p by a Lorentz transformation g_X which preserves $Q_X^{X_p} = Q_X^{X_r}$, i.e., an element of the three-dimensional Lorentz group, O(2,1). We may parameterize g_X by a rotation through an angle μ_X around the z axis, a boost of magnitude ζ_X in the x direction, and a final rotation around the z axis through an angle ν_X[2]:

$$g_X = R_z(\nu_X)\, B_x(\zeta_X)\, R_z(\mu_X) \quad , \qquad (X = a,b,c) \quad . \qquad (2.7)$$

The set $\{t_A, g_a, t_B, g_b, t_C, g_c\}$ is our set of variables for the case in which $\lambda(t_A, t_B, t_C) > 0$. Of course, the amplitude can depend upon only eight independent variables. We show below how to eliminate four of the above variables; however, it will be convenient in Sec. IV to express the amplitude as a function of all twelve variables.

Frame a_p has been specified only up to an arbitrary rotation about the z axis. A redefinition of frame a_p by an arbitrary angle ϕ is equivalent to replacing μ_a by $\mu_a + \phi$. Therefore the amplitude must be left invariant by the transformation $\mu_a \to \mu_a + \phi$, i.e., it is independent of μ_a.[8] Similarly, the amplitude can not depend upon μ_b or μ_c.

Frame a_r is also specified only up to an arbitrary z rotation, and redefinition of this frame by an arbitrary angle ϕ is equivalent to the following change of variables: $\nu_a \rightarrow \nu_a + \phi$, $\nu_b \rightarrow \nu_b + \phi$, and $\nu_c \rightarrow \nu_c + \phi$. This implies that the amplitude can depend upon ν_a, ν_b, and ν_c only in the combinations ω_{ab}, ω_{bc}, and ω_{ca}, where

$$\omega_{ij} = \nu_i - \nu_j \quad . \tag{2.8}$$

Clearly $\omega_{ab} + \omega_{bc} + \omega_{ca} = 0$. Therefore, the amplitude depends upon only eight independent variables.

Case II: $\lambda(t_A,t_B,t_C) < 0$

Equation (2.2) completely determines $Q_A^{a'_r}$, $Q_B^{a'_r}$, and $Q_C^{a'_r}$. We have

$$\left.\begin{aligned} Q_A^{a'_r} &= [0,0,0,(-t_A)^{\frac{1}{2}}] \\ Q_B^{a'_r} &= \tfrac{1}{2}(-t_A)^{-\frac{1}{2}}\{0,[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}},0,t_A + t_B - t_C\} \\ Q_C^{a'_r} &= \tfrac{1}{2}(-t_A)^{-\frac{1}{2}}\{0,-[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}},0,t_A + t_C - t_B\} \end{aligned}\right\} . \tag{2.9a}$$

A rotation about the y axis through an angle Θ_{ab}, where

$$\left.\begin{aligned} \sin\Theta_{ab} &= -\tfrac{1}{2}(t_At_B)^{-\frac{1}{2}}[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}} \\ \cos\Theta_{ab} &= \tfrac{1}{2}(t_At_B)^{-\frac{1}{2}}(t_A + t_B - t_C) \end{aligned}\right\} , \tag{2.10a}$$

carries us from frame a'_r to frame b'_r. A rotation about the y axis through an angle Θ_{bc}, where

$$\left.\begin{aligned} \sin\Theta_{bc} &= -\tfrac{1}{2}(t_Bt_C)^{-\frac{1}{2}}[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}} \\ \cos\Theta_{bc} &= \tfrac{1}{2}(t_Bt_C)^{-\frac{1}{2}}(t_B + t_C - t_A) \end{aligned}\right\} , \tag{2.10b}$$

carries us from frame b_r' to frame c_r'. Finally, a rotation about the y axis through an angle θ_{ca}, where

$$\left.\begin{aligned} \sin\theta_{ca} &= -\tfrac{1}{2}(t_A t_C)^{-\frac{1}{2}}[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}} \\ \cos\theta_{ca} &= \tfrac{1}{2}(t_A t_C)^{-\frac{1}{2}}(t_A + t_C - t_B) \end{aligned}\right\} , \qquad (2.10c)$$

carries us from c_r' back to frame a_r'.

In frame b_r'

$$\left.\begin{aligned} Q_A^{b_r'} &= \tfrac{1}{2}(-t_B)^{-\frac{1}{2}}\{0,-[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}},0,t_A + t_B - t_C\} \\ Q_B^{b_r'} &= [0,0,0(-t_B)^{\frac{1}{2}}] \\ Q_C^{b_r'} &= \tfrac{1}{2}(-t_B)^{-\frac{1}{2}}\{0,[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}},0,t_B + t_C - t_A\} \end{aligned}\right\} . \qquad (2.9b)$$

In frame c_r'

$$\left.\begin{aligned} Q_A^{c_r'} &= \tfrac{1}{2}(-t_C)^{-\frac{1}{2}}\{0,[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}},0,t_A + t_C - t_B\} \\ Q_B^{c_r'} &= \tfrac{1}{2}(-t_C)^{-\frac{1}{2}}\{0,-[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}},0,t_B + t_C - t_A\} \\ Q_C^{c_r'} &= [0,0,0,(-t_C)^{\frac{1}{2}}] \end{aligned}\right\} . \qquad (2.9c)$$

Frame X_r' is related to frame X_p by an element of $O(2,1)$, denoted by g_X':

$$g_X' = R_z(\nu_X')\, B_x(\zeta_X')\, R_z(\mu_X'), \qquad (X = a,b,c) . \qquad (2.10)$$

As in Case I, the amplitude can not depend upon μ'_a, μ'_b, or μ'_c. The removal of the fourth dependent variable is more complicated, however. It arises from the fact that frame a'_r is defined up to an arbitrary y boost. A redefinition of frame a'_r by a boost of magnitude χ is equivalent to the following transformation of variables:

$$g'_X \to g''_X = B_y(\chi)\, g'_X \ , \qquad (X = a,b,c) \ . \tag{2.11}$$

Expressed in terms of ν'_X, ζ'_X, and μ'_X, Eq. (2.11) takes the form

$$\left.\begin{aligned}
\cosh \zeta'_X \to \cosh \zeta''_X &= \cosh \zeta'_X \cosh \chi + \sinh \zeta'_X \sinh \chi \sin \nu'_X \\
\cos \nu'_X \to \cos \nu''_X &= \sinh \zeta'_X \cos \nu'_X / \sinh \zeta''_X \\
\sin \nu'_X \to \sin \nu''_X & \\
&= (\cosh \zeta'_X \sinh \chi + \sinh \zeta'_X \cosh \chi \sin \nu'_X)/\sinh \zeta''_X \\
&(X = a,b,c)
\end{aligned}\right\} \ . \tag{2.12}$$

Therefore, the amplitude must be left invariant by the transformation (2.12).

Alternatively, we may parameterize g'_X in the following way:

$$g'_X = B_y(\eta_X)\, B_x(\gamma_X)\, R_z(\phi_X) \ , \qquad (X = a,b,c) \ . \tag{2.13}$$

As before, the amplitude can not depend upon ϕ_a, ϕ_b, or ϕ_c. However, the covariance condition (2.12) is replaced by the simpler statement that the amplitude depends upon η_a, η_b, and η_c only in the combinations δ_{ab}, δ_{bc}, and δ_{ca} where

$$\delta_{ij} = \eta_i - \eta_j \quad . \tag{2.14}$$

Clearly $\delta_{ab} + \delta_{bc} + \delta_{ca} = 0$.

The relationship between (2.10) and (2.13) is given by

$$\left.\begin{aligned}
\sinh \gamma_X &= \sinh \zeta'_X \cos \nu'_X \\
\sinh \eta_X &= \sinh \zeta'_X \sin \nu'_X / \cosh \gamma_X \\
\cos \phi_X &= (\cosh \zeta'_X \cos \mu'_X \cos \nu'_X - \sin \mu'_X \sin \nu'_X)/\cosh \gamma_X \\
\sin \phi_X &= (\cosh \zeta'_X \sin \mu'_X \cos \nu'_X + \cos \mu'_X \sin \nu'_X)/\cosh \gamma_X
\end{aligned}\right\} . \tag{2.15}$$

We note that as $\zeta'_X \to \infty$

$$\left.\begin{aligned}
\sinh \gamma_X &\sim \sinh \zeta'_X \cos \nu'_X \\
\sinh \eta_X &\sim \mathrm{sgn}(\cos \nu'_X) \tan \nu'_X \\
\cos \phi_X &\sim \mathrm{sgn}(\cos \nu'_X) \cos \mu'_X \\
\sin \phi_X &\sim \mathrm{sgn}(\cos \nu'_X) \sin \mu'_X
\end{aligned}\right\} , \tag{2.16}$$

except at the isolated points $\nu'_X = \pm\pi/2$.

III. EXPRESSION OF THE INVARIANTS IN TERMS OF OUR VARIABLES

We define the following invariants:

$$\left.\begin{aligned} s_a &\equiv (p_{i_A} + Q_B)^2 \\ s_b &\equiv (p_{i_B} + Q_C)^2 \\ s_c &\equiv (p_{i_C} + Q_A)^2 \end{aligned}\right\} , \tag{3.1}$$

and

$$\left.\begin{aligned} s_{ab} &\equiv (p_{i_A} + p_{f_B})^2 \\ s_{bc} &\equiv (p_{i_B} + p_{f_C})^2 \\ s_{ca} &\equiv (p_{i_C} + p_{f_A})^2 \end{aligned}\right\} . \tag{3.2}$$

The calculation of these invariants in terms of our variables is straightforward but tedious. We indicate below how the calculation proceeds and quote the results.

Case I: $\lambda(t_A,t_B,t_C) > 0$

In frame a_p, Q_B is given by ${}^{a}Q_B{}^{p} = L(g_a{}^{-1})\ {}^{a}Q_B{}^{r}$. Using (2.3a), (2.5a), and (2.7), we can calculate s_a. The result is

$$s_a = m^2 + \tfrac{1}{2}(t_B + t_C - t_A) \pm (\tfrac{1}{4} - m^2/t_A)^{\frac{1}{2}} \lambda^{\frac{1}{2}}(t_A,t_B,t_C) \cosh \zeta_a . \tag{3.3}$$

We can express s_b and s_c by cyclic permutations of (A,B,C) in Eq. (3.3).

In frame ${}^{a}p$, p_{f_B} is given by $p_{f_B}{}^{a_p} = L(g_a{}^{-1}\, q_{ab}{}^{-1}\, g_b)\, p_{f_B}{}^{b_p}$. Using (2.3a), (2.3b), (2.6a), and (2.7), we can calculate s_{ab}. The result is

$$
\begin{aligned}
s_{ab} = {} & 2m^2 - (t_A + t_B - t_C)/4 \\
& \pm \tfrac{1}{2} \cosh \zeta_a (-t_A)^{-\frac{1}{2}} (m^2 - t_A/4)^{\frac{1}{2}} \lambda^{\frac{1}{2}}(t_A, t_B, t_C) \\
& \pm \tfrac{1}{2} \cosh \zeta_b (-t_B)^{-\frac{1}{2}} (m^2 - t_B/4)^{\frac{1}{2}} \lambda^{\frac{1}{2}}(t_A, t_B, t_C) \\
& - (t_A t_B)^{-\frac{1}{2}} \cosh \zeta_a \cosh \zeta_b (m^2 - t_A/4)^{\frac{1}{2}} (m^2 - t_B/4)^{\frac{1}{2}} \\
& \qquad \times (t_A + t_B - t_C) \\
& + 2 \sinh \zeta_a \sinh \zeta_b \cos \omega_{ab} (m^2 - t_A/4)^{\frac{1}{2}} (m^2 - t_B/4)^{\frac{1}{2}} . \qquad (3.4)
\end{aligned}
$$

We can express $(p_{i_A} + p_{i_B})^2$ by changing the sign of $(m^2 - t_B/4)^{\frac{1}{2}}$ in (3.4); $(p_{f_A} + p_{f_B})^2$ by changing the sign of $(m^2 - t_A/4)^{\frac{1}{2}}$; and $(p_{f_A} + p_{i_B})^2$, by changing the signs of both $(m^2 - t_A/4)^{\frac{1}{2}}$ and $(m^2 - t_B/4)^{\frac{1}{2}}$. Expressions for the other two particle invariants can be obtained by cyclic permutations of (A,B,C). All the other invariants can easily be expressed in terms of the two-particle invariants.

We note that the invariants depend upon ν_a, ν_b, and ν_c only in the combinations ω_{ab}, ω_{bc}, and ω_{ca}, and that no invariant depends upon μ_a, μ_b, or μ_c.

Case II: $\lambda(t_A,t_B,t_C) < 0$

The calculation of the invariants proceeds in the same manner as in Case I. The results expressed in terms of the parameterization (2.10) are

$$s_a = m^2 + \tfrac{1}{2}(t_B + t_C - t_A)$$
$$- \sinh \zeta_a' \cos \nu_a' (\tfrac{1}{4} - m^2/t_A)^{\frac{1}{2}} [-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}} \quad , \qquad (3.5)$$

and

$$s_{ab} = 2m^2 - (t_A + t_B - t_C)/4$$
$$-2(m^2 - t_A/4)^{\frac{1}{2}} (m^2 - t_B/4)^{\frac{1}{2}} \{\cosh \zeta_a' \cosh \zeta_b'$$
$$- \sinh \zeta_a' \sinh \zeta_b' [\sin \nu_a' \sin \nu_b'$$
$$+ \cos \nu_a' \cos \nu_b' \tfrac{1}{2}(t_A t_B)^{-\frac{1}{2}} (t_A + t_B - t_C)]\}$$
$$- \tfrac{1}{2}[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}} [(-t_B)^{-\frac{1}{2}} (m^2 - t_B/4)^{\frac{1}{2}} \sinh \zeta_b' \cos \nu_b'$$
$$+ (-t_A)^{-\frac{1}{2}} (m^2 - t_A/4)^{\frac{1}{2}} \sinh \zeta_a' \cos \nu_a'] \quad . \qquad (3.6)$$

Other invariants can be expressed by appropriate sign changes and permutations of (A,B,C) in (3.5) and (3.6).

We note that the invariants do not depend upon μ_a', μ_b', and μ_c' and that they are left unchanged by transformation (2.12).

Expressed in terms of the parameterization (2.13), Eqs. (3.5) and (3.6) assume the form

$$s_a = m^2 + \frac{1}{2}(t_B + t_C - t_A)$$

$$- (\tfrac{1}{4} - m^2/t_A)^{\frac{1}{2}} \, [-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}} \sinh \gamma_a \quad , \tag{3.7}$$

and

$$s_{ab} = 2m^2 - (t_A + t_B - t_C)/4$$

$$- \frac{1}{2}[-\lambda(t_A,t_B,t_C)]^{\frac{1}{2}} \, [(\tfrac{1}{4} - m^2/t_B)^{\frac{1}{2}} \sinh \gamma_b$$

$$+ (\tfrac{1}{4} - m^2/t_A)^{\frac{1}{2}} \sinh \gamma_a]$$

$$+ (t_A + t_B - t_C)(\tfrac{1}{4} - m^2/t_A)^{\frac{1}{2}}(\tfrac{1}{4} - m^2/t_B)^{\frac{1}{2}} \sinh \gamma_a \sinh \gamma_b$$

$$- (m^2 - t_A/4)^{\frac{1}{2}} \, (m^2 - t_B/4)^{\frac{1}{2}} \cosh \gamma_a \cosh \gamma_b \cosh \delta_{ab} \quad . \tag{3.8}$$

IV. ASYMPTOTIC BEHAVIOR OF THE AMPLITUDE AND DEFINITION OF THE TRIPLE-REGGE VERTEX

In this section we use the method of Toller[1] to extend the Regge-pole hypothesis to the 3-to-3 amplitude. The concept of the triple-Regge vertex will arise naturally in this section.

Case I: $\lambda(t_A, t_B, t_C) > 0$

Let the amplitude be written as $f(t_A, g_a, t_B, g_b, t_C, g_c; \tau)$. The index τ refers to the sign of the t component of $Q_B{}^{a}{}_{r}$.[9] We can expand the amplitude's dependence on g_a in terms of its projection onto the unitary irreducible representations of $O(2,1)$.[10] We write this projection as[11]

$$f_{mn}{}^{\ell} = \int dg_a \, e^{-im\nu_a} \, d_{mn}{}^{\ell}(\zeta_a) \, e^{-in\mu_a} \, f(g_a) \quad , \tag{4.1}$$

where dg_a is the invariant measure on the group,

$$dg_a = \frac{1}{32\pi^2} \sinh \zeta_a \, d\mu_a \, d\nu_a \, d\zeta_a$$

and $d_{mn}{}^{\ell}(\zeta_a)$ is defined in the Appendix. Since the external particles are assumed to be spinless, $f_{mn}{}^{\ell}$ vanishes unless $n = 0$:

$$f_{mn}{}^{\ell} \equiv f_m{}^{\ell} \, \delta_{n0} \; . \tag{4.2}$$

In particular, this implies that the projections of the amplitude onto representations of the discrete class vanish. From Eq. (A.19) of the Appendix, we have

$$f_m{}^{-\ell-1} = \eta_m^{-\ell-1} \, f_m{}^{\ell} \, \eta_0{}^{\ell} \quad , \tag{4.3}$$

where η_m^{ℓ} is defined by Eq. (A.20). The inverse formula for the amplitude is given by[1]

$$f(g_a) = \sum_{m=-\infty}^{\infty} e^{im\nu_a} f_m(\zeta_a) \quad , \tag{4.4}$$

where

$$f_m(\zeta_a) = -\frac{1}{2} i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \, \frac{2\ell + 1}{\tan \pi\ell} \, [d_{m0}^{\ell}(\zeta_a)]^* \, f_m^{\ell} \quad . \tag{4.5}$$

Using Eqs. (A.15) and (A.17), we can write (4.5) as

$$f_m(\zeta_a) = -\frac{1}{2} i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \, \frac{2\ell + 1}{\tan \pi\ell} \, d_{m0}^{-\ell-1}(\zeta_a) \, f_m^{\ell} \quad . \tag{4.6}$$

Equation (A.21) enables us to write (4.6) in the form

$$f_m(\zeta_a) = -\frac{1}{2} i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \, \frac{2\ell + 1}{\tan \pi\ell} \, a_{m0}^{-\ell-1}(\zeta_a) \, f_m^{\ell}$$

$$- \frac{1}{2} i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \, \frac{2\ell + 1}{\tan \pi\ell} \, \eta_m^{-\ell-1} \, a_{m0}^{\ell}(\zeta_a) \, \eta_0^{\ell} \, f_m^{\ell} \quad .$$

Using Eq. (4.3), we obtain

$$f_m(\zeta_a) = -\frac{1}{2} i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \, \frac{2\ell + 1}{\tan \pi\ell} \, a_{m0}^{-\ell-1}(\zeta_a) \, f_m^{\ell}$$

$$- \frac{1}{2} i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \, \frac{2\ell + 1}{\tan \pi\ell} \, a_{m0}^{\ell}(\zeta_a) \, f_m^{-\ell-1} \quad .$$

Letting $\ell = -\ell' - 1$ in the second integral, we obtain

$$f_m(\zeta_a) = -i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \, \frac{2\ell + 1}{\tan \pi\ell} \, a_{m0}^{-\ell-1}(\zeta_a) \, f_m^{\ell} \, . \qquad (4.7)$$

Because of the asymptotic behavior of $a_{m0}^{-\ell-1}(\zeta_a)$ [see Eq. (A.23)], in order to obtain an asymptotic expansion for $f_m(\zeta_a)$, we shift the contour of integration to the line $\text{Re}\,\ell = L$ where $L < \frac{1}{2}$. The contours at infinity give no contribution.[1] If we assume that f_m^{ℓ} is meromorphic in the ℓ plane, we pick up the contributions of all poles with $L < \text{Re}\,\alpha_i < -\frac{1}{2}$, where α_i is the position of the $\underline{i}$th pole. There is no contribution from the vanishing of $\tan \pi\ell$ at $\ell = -N$ $(N = 1,2,\cdots)$ since, from (A.22),

$$a_{m0}^{N-1}(\zeta) = 0$$

for $|m| < N$, and from (4.1) and (A.16),

$$f_m^{-N} = 0$$

for $|m| \geq N$. We then have

$$f_m(\zeta_a)$$

$$= -i \int_{L-i\infty}^{L+i\infty} a_{m0}^{-\ell-1}(\zeta_a) \, f_m^{\ell} \, \frac{2\ell + 1}{\tan \pi\ell} \, d\ell + \sum_j 2\pi \, \beta_m^{\,j} \, \frac{2\alpha_j + 1}{\tan \pi \alpha_j} \, a_{m0}^{-\alpha_j-1}(\zeta_a), \qquad (4.8)$$

where $\beta_m^{\,j}$ is the residue of the pole at $\ell = \alpha_j$. From Eq. (4.3), we note that if there is a pole at $\ell = \alpha$, there is also a pole at $\ell = -\alpha-1$.

The leading term in the asymptotic expansion of f_m as $\zeta_a \to \infty$ is given by the term arising from the pole farthest to the right. If we assume that the residues are factorizable, we obtain from (A.23)

$$f_m(t_A,\zeta_a,t_B,g_b,t_C,g_c;\ \tau) \underset{\zeta_a\to\infty}{\sim} \rho(t_A)[\cosh\ \zeta_a]^{\alpha(t_A)}$$

$$\times\ \rho_m(t_A,t_B,g_b,t_C,g_c;\ \tau)\quad . \tag{4.9}$$

The expression for the full amplitude is

$$f(t_A,g_a,t_B,g_b,t_C,g_c;\ \tau) \underset{\zeta_a\to\infty}{\sim} \rho(t_A)[\cosh\ \zeta_a]^{\alpha(t_A)}$$

$$\times\ \rho(t_A,\nu_a,t_B,g_b,t_C,g_c;\ \tau)\quad , \tag{4.10}$$

where

$$\rho(t_A,\nu_a,t_B,g_b,t_C,g_c;\ \tau)$$

$$= \sum_{m=-\infty}^{\infty} e^{im\nu_a}\ \rho_m(t_A,t_B,g_b,t_C,g_c;\ \tau)\quad . \tag{4.11}$$

We repeat the above analysis for the dependence of $\rho(t_A,\nu_a,t_B,g_b,t_C,g_c;\ \tau)$ on g_b and g_c. The final result is

$$f(t_A, g_a, t_B, g_b, t_C, g_c; \tau) \underset{\zeta_a, \zeta_b, \zeta_c \to \infty}{\sim} \rho(t_A)\, \rho(t_B)\, \rho(t_C)$$

$$\times\ [\cosh \zeta_a]^{\alpha(t_A)} [\cosh \zeta_b]^{\alpha(t_B)} [\cosh \zeta_c]^{\alpha(t_C)}$$

$$\times\ \rho(t_A, \nu_a, t_B, \nu_b, t_C, \nu_c; \tau) \quad . \tag{4.12}$$

The triple-Regge vertex, $\rho(t_A, \nu_a, t_B, \nu_b, t_C, \nu_c; \tau)$, for the case $\lambda(t_A, t_B, t_C) > 0$ is defined by Eq. (4.12). It is defined entirely in terms of physical region quantities; therefore it is, in principle, directly measurable. Remembering the dependence of the amplitude on ν_a, ν_b, ν_c, we can write

$$\rho(t_A, \nu_a, t_B, \nu_b, t_C, \nu_c; \tau) \quad = \quad V(t_A, t_B, t_c, \omega_{ab}, \omega_{bc}; \tau) \quad . \tag{4.13}$$

By using the results of Sec. III, we can express Eq. (4.12) in terms of the invariants. The result is[9]

$$f(t_A, t_B, t_C, s_a, s_b, s_c, s_{ab}, s_{bc}, s_{ca}) \underset{\substack{s_a, s_b, s_c \to \pm\infty \\ \kappa_{ab}, \kappa_{bc}, \kappa_{ca}\ \text{fixed}}}{\sim}$$

$$g(t_A)\, g(t_B)\, g(t_C) |s_a|^{\alpha(t_A)} |s_b|^{\alpha(t_B)} |s_c|^{\alpha(t_C)}$$

$$\times\ V(t_A, t_B, t_C, \kappa_{ab}, \kappa_{bc}, \kappa_{ca}; \tau) \ , \tag{4.14}$$

where

$$\kappa_{ij} \quad = \quad s_i s_j / s_{ij} \quad . \tag{4.15}$$

In terms of invariants, $\tau = \mathrm{sgn}(s_a) = \mathrm{sgn}(s_b) = \mathrm{sgn}(s_c)$. The quantities $t_A, t_B, t_C, \kappa_{ab}, \kappa_{bc}$, and κ_{ca} are, of course, not all independent. There is a complicated nonlinear relationship among them. This relationship can be derived from the fact that $\omega_{ab} + \omega_{bc} + \omega_{ca} = 0$ and from the results of Sec. III.

Case II: $\lambda(t_A, t_B, t_C) < 0$

There are two mathematically equivalent methods in which we can proceed. In the first method, g'_X is parameterized by (2.10). The second method employs the parameterization (2.13).

The details of the mathematics in the first method are exactly the same as in Case I. The final result for the asymptotic behavior of the amplitude is

$$f(t_A, g'_a, t_B, g'_b, t_C, g'_c) \underset{\zeta'_a, \zeta'_b, \zeta'_c \to \infty}{\sim} \rho(t_A)\, \rho(t_B)\, \rho(t_C)$$

$$\times [\cosh \zeta'_a]^{\alpha(t_A)} [\cosh \zeta'_b]^{\alpha(t_B)} [\cosh \zeta'_c]^{\alpha(t_C)}$$

$$\times \rho'(t_A, \nu'_a, t_B, \nu'_b, t_C, \nu'_c) \quad . \tag{4.16}$$

Equation (4.16) defines the triple-Regge vertex, $\rho'(t_A, \nu'_a, t_B, \nu'_b, t_C, \nu'_c)$, for Case II. As in Case I, the triple-Regge vertex is, in principle, directly measurable.

Equation (4.16) must be left invariant by the transformation (2.12). Asymptotically, (2.12) becomes

$$\left.\begin{aligned}
\cosh \zeta'_X \to \cosh \zeta''_X &\sim \cosh \zeta'_X(\cosh \chi + \sinh \chi \sin \nu'_X) \\
\cos \nu'_X \to \cos \nu''_X &\sim \frac{\cos \nu'_X}{\cosh \chi + \sinh \chi \sin \nu'_X} \\
\sin \nu'_X \to \sin \nu''_X &\sim \frac{\sinh \chi + \cosh \chi \sin \nu'_X}{\cosh \chi + \sinh \chi \sin \nu'_X}
\end{aligned}\right\} . \quad (4.17)$$

This implies that the triple-Regge vertex must satisfy the condition

$$\begin{aligned}
\rho'(t_A,\nu'_a,t_B,\nu'_b,t_C,\nu'_c) &= (\cosh \chi + \sinh \chi \sin \nu'_a)^{\alpha(t_A)} \\
&\times (\cosh \chi + \sinh \chi \sin \nu'_b)^{\alpha(t_B)} (\cosh \chi + \sinh \chi \sin \nu'_c)^{\alpha(t_C)} \\
&\times \rho'(t_A,\nu''_a,t_B,\nu''_b,t_C,\nu''_c)
\end{aligned} \quad (4.18)$$

for arbitrary χ.

The complexity of the covariance condition (4.18) satisfied by the triple-Regge vertex is a reflection of the complexity of the covariance condition (2.12) satisfied by the complete amplitude. An inspection of Eqs. (2.13) and (2.14) indicates that the triple-Regge vertex arising in the second method will satisfy a simpler covariance condition.

In the second method, we again expand the amplitude's dependence on g'_a in terms of its projections onto the unitary irreducible representations of $O(2,1)$. Since the parameterization (2.13) is used, it is convenient to use the representations expressed in the mixed basis. We write this projection as[12]

$$f^{\ell}_{\mu\sigma,m} = \int dg'_a \, e^{-\mu\eta_a} \, d^{\ell}_{\mu\sigma,m}(\gamma_a) \, e^{-im\phi_a} \, f(g'_a) \, , \qquad (4.19)$$

where $d^{\ell}_{\mu\sigma,m}(\zeta_a)$ is defined in the Appendix. As in Case I, the projections of the amplitude onto the representations of the discrete class vanish, and $f^{\ell}_{\mu\sigma,m}$ vanishes unless $m = 0$:

$$f^{\ell}_{\mu\sigma,m} = f_{\mu\sigma}{}^{\ell} \, \delta_{m0} \, . \qquad (4.20)$$

The inverse formula for the amplitude is given by

$$f(g'_a) = -i \sum_{\sigma} \int_{-i\infty}^{+i\infty} d\mu \, e^{\mu\eta_a} \, f_{\mu\sigma}(\gamma_a) \, ,$$

where

$$f_{\mu\sigma}(\gamma_a) = -\frac{i}{2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} [d^{\ell}_{\mu\sigma,0}(\gamma_a)]^* \, f_{\mu\sigma}{}^{\ell} \, \frac{2\ell + 1}{\tan \pi\ell} \, d\ell \, . \qquad (4.21)$$

Using Eq. (A.45) of the Appendix, we can write (4.21) in the form

$$f_{\mu\sigma}(\gamma_a) = -\frac{1}{2} \, i \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d^{-\ell-1}_{\mu\sigma,0}(\gamma_a) \, f_{\mu\sigma}{}^{\ell} \, \frac{2\ell + 1}{\tan \pi\ell} \, d\ell \, . \qquad (4.22)$$

Because of the asymptotic behavior of $d^{-\ell-1}_{\mu+,0}(\gamma_a)$ as $\gamma_a \to -\infty$ [see Eq. (A.47)], for $\gamma_a < 0$ and $\sigma = +$ we shift the contour of integration to the line $\mathrm{Re}\,\ell = L$, where $L < -\frac{1}{2}$. If we assume that $f_{\mu\sigma}{}^{\ell}$ is meromorphic in the ℓ plane, we pick up the contributions of the poles at $\ell = \alpha_j$, where $L < \mathrm{Re}\,\alpha_j < -\frac{1}{2}$. We obtain

$$f_{\mu+}(\Upsilon_a) = -\frac{1}{2} i \int_{L-i\infty}^{L+i\infty} d^{-\ell-1}_{\mu+,0}(\Upsilon_a) \; f_{\mu+}{}^{\ell} \; \frac{2\ell+1}{\tan \pi\ell} \, d\ell$$

$$+ \pi \sum_j \beta_{\mu+}{}^{j} \frac{2\alpha_j + 1}{\tan \pi \alpha_j} \, d^{-\alpha_j-1}_{\mu+,0}(\Upsilon_a) \; . \qquad (4.23)$$

If we assume that the residues are factorizable, we obtain from (A.47)

$$f_{\mu+}(t_A, \Upsilon_a, t_B, g_b', t_C, g_c') \underset{\Upsilon_a \to -\infty}{\sim} \rho(t_A) |\sinh \Upsilon_a|^{\alpha(t_A)}$$

$$\times \; \rho_{\mu+}(t_A, t_B, g_b', t_C, g_c') \quad , \qquad (4.24)$$

where $\alpha(t_A)$ is the position of the leading pole.

For $\Upsilon_a < 0$ and $\sigma = -$, we use (A.50) to write (4.22) in the form

$$f_{\mu-}(\Upsilon_a) = -\frac{i}{2} \cos \pi\mu \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d^{-\ell-1}_{\mu+,0}(\Upsilon_a) \; f_{\mu-}{}^{\ell} \; \frac{2\ell+1}{\sin \pi\ell} \, d\ell$$

$$+ \frac{i}{2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(\ell+1)}{\Gamma(-\ell)\,\Gamma(\ell+1+\mu)\,\Gamma(\ell+1-\mu)} \, d^{\ell}_{\mu+,0}(\Upsilon_a)$$

$$\times \; f_{\mu-}{}^{\ell} \; \frac{2\ell+1}{\sin \pi\ell} \, d\ell \quad . \qquad (4.25)$$

For the first integral, we shift the contour of integration to the left picking up the contributions of the poles at $\ell = \alpha_j$. For the second integral, we shift the contour to the right picking up the contributions of the poles at $\ell = -\alpha_j - 1$. The pole contribution is given by

$$f_{\mu-}(\gamma_a)_{\text{poles}} = \pi \sum_j \frac{2\alpha_j + 1}{\sin \pi \alpha_j} d_{\mu+,0}^{-\alpha_j-1}(\gamma_a)$$

$$\times \left[\cos \pi\mu\, \beta_{\mu-}^{j} + \frac{\Gamma(-\alpha_j)}{\Gamma(\alpha_j + 1)\, \Gamma(-\alpha_j + \mu)\, \Gamma(-\alpha_j - \mu)} \beta_{\mu-}^{j'}\right] , \quad (4.26)$$

where $\beta_{\mu-}^{j}$ and $\beta_{\mu-}^{j'}$ are the residues of the poles at $\ell = \alpha_j$ and $\ell = -\alpha_j - 1$, respectively. If we assume that the residues are factorizable in such a way that the factors coming from the external particle vertex in $\beta_{\mu-}^{j}$ and $\beta_{\mu-}^{j'}$ are the same, we can write for the asymptotic behavior

$$f_{\mu-}(t_A,\gamma_a,t_B,g_b',t_C,g_c') \underset{\gamma_a \to -\infty}{\sim} \rho(t_A)|\sinh \gamma_a|^{\alpha(t_A)}$$

$$\times \rho_{\mu-}(t_A,t_B,g_b',t_C,g_c') . \quad (4.27)$$

The asymptotic behavior for the full amplitude is given by

$$f(t_A,g_a',t_B,g_b',t_C,g_c') \underset{\gamma_a \to -\infty}{\sim} \rho(t_A)|\sinh \gamma_a|^{\alpha(t_A)}$$

$$\times \rho(t_A,\eta_a,t_B,g_b',t_C,g_c') , \quad (4.28)$$

where

$$\rho(t_A,\eta_a,t_B,g_b',t_C,g_c')$$

$$= -i \sum_\sigma \int_{-i\infty}^{+i\infty} d\mu\, e^{\mu\eta_a} \rho_{\mu\sigma}(t_A,t_B,g_b',t_C,g_c') . \quad (4.29)$$

For $\gamma_a > 0$, we use Eqs. (A.48) and (A.50) which relate the representation functions for $\gamma_a > 0$ to those for $\gamma_a < 0$. The details of the calculation are similar to the details for $\gamma_a < 0$, the only difference being a different residue function. The final result for either sign of γ_a can be written as

$$f(t_A,g'_a,t_B,g'_b,t_C,g'_c) \underset{|\gamma_a|\to\infty}{\sim} \rho(t_A)|\sinh\gamma_a|^{\alpha(t_A)}$$

$$\times \rho(t_A,\eta_a,t_B,g'_b,t_C,g'_c;\tau_a) \quad , \tag{4.30}$$

where

$$\tau_a = -\text{sgn}\,\gamma_a \; . \tag{4.31}$$

We repeat the above analysis for the dependence of $\rho(t_A,\eta_a,t_B,g'_b,t_C,g'_c;\tau_a)$ on g'_b and g'_c. The final result for the amplitude is[13]

$$f(t_A,g'_a,t_B,g'_b,t_C,g'_c) \underset{|\gamma_a|,|\gamma_b|,|\gamma_c|\to\infty}{\sim} \rho(t_A)\,\rho(t_B)\,\rho(t_C)$$

$$\times |\sinh\gamma_a|^{\alpha(t_A)}|\sinh\gamma_b|^{\alpha(t_B)}|\sinh\gamma_c|^{\alpha(t_C)}$$

$$\times \rho(t_A,\eta_a,t_B,\eta_b,t_C,\eta_c;\tau_a,\tau_b,\tau_c) \; . \tag{4.32}$$

Remembering the dependence of the amplitude on η_a, η_b, η_c, we can write the following covariance condition on the triple-Regge vertex:

$$\rho(t_A, \eta_a, t_B, \eta_b, t_C, \eta_c; \tau_a, \tau_b, \tau_c) = V(t_A, t_B, t_C: \delta_{ab}, \delta_{bc}; \tau_a, \tau_b, \tau_c) . \tag{4.33}$$

Using the results of Sec. III, we can express Eq. (4.16) [or (4.32)] in terms of the invariants. The result is

$$f(t_A, t_B, t_C, s_a, s_b, s_c, s_{ab}, s_{bc}, s_{ca}) \underset{\substack{|s_a|, |s_b|, |s_c| \to \infty \\ \kappa_{ab}, \kappa_{bc}, \kappa_{ca} \text{ fixed}}}{\sim} g(t_A)\, g(t_B)\, g(t_C)\, |s_a|^{\alpha(t_A)} |s_b|^{\alpha(t_B)} |s_c|^{\alpha(t_C)}$$

$$\times\ V(t_A, t_B, t_C, \kappa_{ab}, \kappa_{bc}, \kappa_{ca}; \tau_a, \tau_b, \tau_c) , \tag{4.34}$$

where now

$$\tau_i = \operatorname{sgn} s_i . \tag{4.35}$$

Equation (4.34) is valid at all points excluding the isolated points $\cos \nu_a' = 0$, $\cos \nu_b' = 0$, or $\cos \nu_c' = 0$.

V. THE ASYMPTOTIC FORM OF THE AMPLITUDE IN THE VENEZIANO MODEL

An explicit expression for the Veneziano six-point function has been given by Chan.[14] In our notation, this expression is

$$f = \int_0^1 du_1\, du_2\, du_3\, u_1^{-1-\alpha(t_A)} u_2^{-1-\alpha(t_B)} u_3^{-1-\alpha(t_C)}$$

$$\times (1 - u_1)^{-1-\alpha(s_{ab})} (1 - u_2)^{-1-\alpha(s_b)} (1 - u_3)^{-1-\alpha(s_{bc})}$$

$$\times [1 - u_1(1 - u_2)]^{\alpha(s_{ab})+\alpha(t_B)-\alpha(s_a)}$$

$$\times [1 - u_3(1 - u_2)]^{\alpha(s_{bc})+\alpha(t_B)-\alpha(s_c)}$$

$$\times [1 - u_1 u_3(1 - u_2)]^{\alpha(s_a)+\alpha(s_c)-\alpha(s_{ca})-\alpha(t_B)} , \qquad (5.1)$$

where $\alpha(s) = a + bs$.

After making the following change of variables in the integral:

$$\left.\begin{aligned} u_1 &= 1 - \exp[v_1/\alpha(s_a)] \\ u_2 &= 1 - \exp[v_2/\alpha(s_b)] \\ u_3 &= 1 - \exp[v_3/\alpha(s_c)] \end{aligned}\right\} , \qquad (5.2)$$

the asymptotic form of the amplitude as $s_a \to -\infty$, $s_b \to -\infty$, and $s_c \to -\infty$ can easily be obtained. The result is

$$f \sim |s_a|^{\alpha(t_A)} |s_b|^{\alpha(t_B)} |s_c|^{\alpha(t_C)}$$

$$\times G(t_A, t_B, t_C, \kappa_{ab}, \kappa_{bc}, \kappa_{ca}) , \qquad (5.3)$$

where

$$G(t_A, t_B, t_C, \kappa_{ab}, \kappa_{bc}, \kappa_{ca}) = b^{\alpha(t_A)+\alpha(t_B)+\alpha(t_C)}$$

$$\times \int_0^\infty dv_1\, dv_2\, dv_3\, v_1^{-1-\alpha(t_A)}\, v_2^{-1-\alpha(t_B)}\, v_3^{-1-\alpha(t_C)}$$

$$\times \exp\left\{-v_1 - v_2 - v_3 + \frac{1}{b}\left(\frac{v_1 v_2}{\kappa_{ab}} + \frac{v_2 v_3}{\kappa_{bc}} + \frac{v_3 v_1}{\kappa_{ca}}\right)\right\} \quad . \tag{5.4}$$

Comparing (5.3) with (4.14) or (4.34), we see that the asymptotic behavior of the Veneziano amplitude is correctly predicted by the group theoretical arguments of the preceding sections.

VI. CONCLUSION

We have extended the Regge-pole hypothesis to the 3-to-3 amplitude. The assumptions that we made are the same as those made in previous Regge-pole hypotheses; therefore from the point of view discussed here, our hypothesis is as plausible as previous Regge-pole hypotheses. The concept of a triple-Regge vertex arises naturally in our considerations. The triple-Regge vertex is defined entirely in terms of physical region quantities; therefore it can, in principle, be directly measured. The considerations discussed here can evidently be extended to an arbitrary process.

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APPENDIX: THE UNITARY IRREDUCIBLE REPRESENTATIONS OF O(2,1)

This appendix contains a derivation of the properties of the irreducible representations of O(2,1) needed in the main text. The material presented here is essentially the same as that which appears in Toller,[1] Mukunda,[15] and Ciafaloni, DeTar, and Misheloff;[16] however, our conventions differ from those used by any of the above references.[17]

1. The Group SU(1,1)

The spinor group corresponding to O(2,1) is SU(1,1). Although only the single-valued unitary irreducible representations of O(2,1) are needed in the main text, it is easier to deal with the matrices of SU(1,1) than those of O(2,1).

The group SU(1,1) is isomorphic to the group of matrices of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} , \tag{A.1}$$

with

$$|\alpha|^2 - |\beta|^2 = 1 . \tag{A.2}$$

The Lie algebra of SU(1,1) contains three linearly independent elements, K_1, K_2, and J_3, with the following commutation relations:

$$[K_1,K_2] = -iJ_3 , \tag{A.3a}$$

$$[J_3,K_1] = iK_2 , \tag{A.3b}$$

$$[J_3,K_2] = -iK_1 . \tag{A.3c}$$

The Casimir operator of the Lie algebra is the operator Q defined by

$$Q = K_1^2 + K_2^2 - J_3^2 \quad . \tag{A.4}$$

For the defining nonunitary representation given by (A.1) and (A.2), we may choose

$$K_1 = \frac{1}{2} i \sigma_2 = \frac{1}{2} i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tag{A.5a}$$

$$K_2 = -\frac{1}{2} i \sigma_1 = -\frac{1}{2} i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{A.5b}$$

$$J_3 = \frac{1}{2} \sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.5c}$$

Let the elements of the one-parameter subgroups generated by J_3, K_1 and K_2 be denoted by $R_z(\mu)$, $B_x(\gamma)$, and $B_y(\eta)$, respectively. In the representation given by (A.1) and (A.2), these elements assume the form

$$R_z(\mu) \equiv \exp(-i\mu J_3) = \begin{pmatrix} e^{-i\mu/2} & 0 \\ 0 & e^{i\mu/2} \end{pmatrix} , \tag{A.6a}$$

$$B_x(\gamma) \equiv \exp(-i\gamma K_1) = \begin{pmatrix} \cosh \gamma/2 & -i \sinh \gamma/2 \\ i \sinh \gamma/2 & \cosh \gamma/2 \end{pmatrix} , \tag{A.6b}$$

$$B_y(\eta) \equiv \exp(-i\eta K_2) = \begin{pmatrix} \cosh \eta/2 & -\sinh \eta/2 \\ -\sinh \eta/2 & \cosh \eta/2 \end{pmatrix} . \tag{A.6c}$$

2. Irreducible Representations of SU(1,1)

Bargmann[18] was the first person to determine the unitary irreducible representations of SU(1,1). They fall into several distinct classes. Each unitary irreducible representation can be characterized by the value of the Casimir operator Q, and the spectrum of eigenvalues of the generators J_3. If the eigenvalues of Q are denoted by q, and those of J_3 by m, the different classes of unitary irreducible representations are the following:

a. Continuous class, integral case, nonexceptional interval:

$$1/4 \leq q < \infty; \quad m = 0, \pm 1, \pm 2, \cdots;$$

b. Continuous class, exceptional interval:

$$0 < q < 1/4; \quad m = 0, \pm 1, \pm 2, \cdots;$$

c. Continuous class, half-integral case:

$$1/4 < q < \infty; \quad m = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots;$$

d. Discrete class, positive m:

$$k = \frac{1}{2}, 1, \frac{3}{2}, \cdots; \quad q = k(1 - k); \quad m = k, k+1, k+2, \cdots;$$

e. Discrete class, negative m:

$$k = \frac{1}{2}, 1, \frac{3}{2}, \cdots; \quad q = k(1 - k); \quad m = -k, -k-1, -k-2, \cdots;$$

3. Representations of the Continuous Class of O(2,1)

The single-valued unitary irreducible representations of O(2,1) are those of SU(1,1) with integral m. For unitary irreducible representations of the continuous class (nonexceptional interval), we may write

$$q = -\ell(\ell + 1) , \qquad \ell = -\frac{1}{2} + is \quad . \tag{A.7}$$

These unitary irreducible representations may be realized by unitary transformations in the Hilbert space of square-integrable functions on the unit circle. In this realization, the inner product of an element f with an element h is given by

$$(f,h) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \; f^*(\phi)\, h(\phi) \quad . \tag{A.8}$$

Let g be an element of SU(1,1) specified by the parameters α and β. Let U(g) be the unitary operator corresponding to g. In the ϕ realization the vector U(g)f is given in terms of f by

$$[U(g)f](\phi) = |\alpha^* - \beta\, e^{i\phi}|^{-2\ell-2}\, f(\phi') , \tag{A.9}$$

where

$$e^{i\phi'} = \frac{\alpha\, e^{i\phi} - \beta^*}{\alpha^* - \beta\, e^{i\phi}} \quad . \tag{A.10}$$

4. Representations of the Continuous Class in the O(2) Basis

The "Euler angle" parameterization of g is given by

$$g = \exp(-i\mu J_3)\, \exp(-i\zeta K_1)\, \exp(-i\nu J_3) \quad . \tag{A.11}$$

The range of parameters is

$$0 \leq \zeta < \infty, \qquad 0 \leq \mu < 4\pi, \qquad 0 \leq \nu < 4\pi \quad ,$$

but the parameters μ, ζ, ν, and $\mu \pm 2\pi$, ζ, $\nu \pm 2\pi$ correspond to the same group element.

For this parameterization, it is convenient to express the representation in terms of the orthonormal basis states $|\ell,m\rangle$ given by

$$|\ell,m\rangle = e^{im(\phi-\pi/2)} \quad . \tag{A.12}$$

In this basis the representation for $\exp(-i\mu J_3)$ is diagonal:

$$D_{mn}{}^{\ell}[\exp(-i\mu J_3)] \equiv \langle \ell,m|U[\exp(-i\mu J_3)]|\ell,n\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi\ e^{-im(\phi-\pi/2)}\ [e^{i(\phi-\frac{\pi}{2}-\mu)}]^n$$

$$= e^{-im\mu}\ \delta_{mn} \quad . \tag{A.13}$$

The representation for g is given by

$$D_{mn}{}^{\ell}(g) = e^{-im\mu}\ d_{mn}{}^{\ell}(\zeta)\ e^{-in\nu} \ , \tag{A.14}$$

where the function $d_{mn}{}^{\ell}(\zeta)$ is given by

$$d_{mn}^{\ell}(\zeta) \equiv \langle \ell,m|U[\exp(-i\zeta K_1)]|\ell,n\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-im\phi}(\cosh\tfrac{\zeta}{2} - \sinh\tfrac{\zeta}{2}\, e^{i\phi})^{-\ell-1}$$

$$\times \left[\frac{\cosh\frac{\zeta}{2}\, e^{i\phi} - \sinh\frac{\zeta}{2}}{e^{i\phi}}\right]^{-\ell-1} \left[\frac{\cosh\frac{\zeta}{2}\, e^{i\phi} - \sinh\frac{\zeta}{2}}{\cosh\frac{\zeta}{2} - \sinh\frac{\zeta}{2}\, e^{i\phi}}\right]^{n}$$

$$= -\frac{i}{2\pi}\oint_c dz\, z^{-1-m}(\cosh\tfrac{\zeta}{2} - z\sinh\tfrac{\zeta}{2})^{-\ell-1}$$

$$\times \left[\frac{z\cosh\frac{\zeta}{2} - \sinh\frac{\zeta}{2}}{z}\right]^{-\ell-1} \left[\frac{z\cosh\frac{\zeta}{2} - \sinh\frac{\zeta}{2}}{\cosh\frac{\zeta}{2} - z\sinh\frac{\zeta}{2}}\right]^{n}, \quad \text{(A.15)}$$

where the contour of integration follows the unit circle in a counter-clockwise sense. In the integrand, the factor $f(z) = (\cosh\frac{\zeta}{2} - z\sinh\frac{\zeta}{2})^{-\ell-1}$ is defined with the cut between $z = \coth\frac{\zeta}{2}$ and $z = +\infty$ such that $f(x + i\epsilon) = \exp[-(\ell + 1)\log|\cosh\frac{\zeta}{2} - x\sinh\frac{\zeta}{2}|]$ for $x > \coth\frac{\zeta}{2}$. The factor $g(z) = [(z\cosh\frac{\zeta}{2} - \sinh\frac{\zeta}{2})/z]^{-\ell-1}$ is defined with the cut between $z = 0$ and $z = \tanh\frac{\zeta}{2}$ with $g(x + i\epsilon) = \exp[-(\ell + 1)\log|(x\cosh\frac{\zeta}{2} - \sinh\frac{\zeta}{2})/x| + i\pi(\ell + 1)]$ for $0 < x < \tanh\frac{\zeta}{2}$. Since $\tanh\frac{\zeta}{2} < 1 < \coth\frac{\zeta}{2}$, the contour of integration never passes through any cut. Equation (A.15) can be written as

$$d_{mn}^{\ell}(\zeta) = -\frac{i}{2\pi}\oint_c dz\, z^{\ell-m}\left(z \cosh\frac{\zeta}{2} - \sinh\frac{\zeta}{2}\right)^{-\ell-1+n}$$

$$\times \left(\cosh\frac{\zeta}{2} - z \sinh\frac{\zeta}{2}\right)^{-\ell-1-n}$$

$$= \left(\sinh\frac{\zeta}{2}\right)^{-\ell-1+n}\left(\cosh\frac{\zeta}{2}\right)^{-\ell-1-n}\frac{i}{2\pi i}\oint_c dz\, z^{\ell-m}$$

$$\times \left(z \coth\frac{\zeta}{2} - 1\right)^{-\ell-1+n}\left(1 - z \tanh\frac{\zeta}{2}\right)^{-\ell-1-n} .$$

We let $z = t \tanh\frac{\zeta}{2}$ in the integral and obtain

$$d_{mn}^{\ell}(\zeta) = \left(\sinh\frac{\zeta}{2}\right)^{-\ell-1+n}\left(\cosh\frac{\zeta}{2}\right)^{-\ell-1-n}\left(\tanh\frac{\zeta}{2}\right)^{\ell+1-m}$$

$$\times \frac{1}{2\pi i}\oint dt\, t^{\ell-m}(t-1)^{-\ell-1+n}\left(1 - t \tanh^2\frac{\zeta}{2}\right)^{-\ell-1-n} .$$

For $n \geq m$, the integral is an integral representation of the hypergeometric function:[19]

$$d_{mn}^{\ell}(\zeta) = \left(\sinh\frac{\zeta}{2}\right)^{n-m}\left(\cosh\frac{\zeta}{2}\right)^{-2\ell-2+m-n}\frac{\Gamma(\ell+1-m)}{\Gamma(\ell+1-n)}$$

$$\times \frac{1}{(n-m)!} F\left(\ell+1+n, \ell+1-m; n-m+1; \tanh^2\frac{\zeta}{2}\right) . \quad (A.15a)$$

Using a standard relation between hypergeometric functions,[20] we can transform (A.15a) into the form

$$d_{mn}^{\ell}(\zeta) = \frac{1}{(n-m)!}\frac{\Gamma(\ell-m+1)}{\Gamma(\ell-n+1)}\left(\cosh\frac{\zeta}{2}\right)^{-m-n}\left(\sinh\frac{\zeta}{2}\right)^{-m+n}$$

$$\times F\left(-\ell-m, \ell-m+1; n-m+1; -\sinh^2\frac{\zeta}{2}\right) . \quad (A.16)$$

By letting $\phi = -\phi'$ in Eq. (A.15), we obtain for $m \geq n$

$$d_{mn}^{\ell}(\zeta) = d^{\ell}_{-m,-n}(\zeta) \quad . \tag{A.17}$$

For $m \geq n$, we obtain from (A.15a), (A.16), and (A.17)

$$\begin{aligned}
d_{mn}^{\ell}(\zeta) &= d^{\ell}_{-m,-n}(\zeta) \\
&= (\sinh \tfrac{\zeta}{2})^{m-n} (\cosh \tfrac{\zeta}{2})^{-2\ell-2+n-m} \frac{\Gamma(\ell+1+m)}{\Gamma(\ell+1+n)} \\
&\times \frac{1}{(m-n)!} F(\ell+1-n, \ell+1+m; m-n+1; \tanh^2 \tfrac{\zeta}{2}) \\
&= \frac{\Gamma(\ell+1+m)}{\Gamma(\ell+1+n)} \frac{\Gamma(\ell+1-m)}{\Gamma(\ell+1-n)} d_{nm}^{\ell}(\zeta) \\
&= \frac{\sin \pi(\ell+1+n)}{\sin \pi(\ell+1+m)} \frac{\Gamma(-\ell-n)}{\Gamma(-\ell-m)} \frac{\Gamma(\ell+1-m)}{\Gamma(\ell+1-n)} d_{nm}^{\ell}(\zeta) \\
&= (-1)^{m-n} \frac{1}{(m-n)!} \frac{\Gamma(-\ell-n)}{\Gamma(-\ell-m)} (\cosh \tfrac{\zeta}{2})^{-n-m} (\sinh \tfrac{\zeta}{2})^{-n+m} \\
&\times F(-\ell-n, \ell-n+1; m-n+1; -\sinh^2 \tfrac{\zeta}{2}) \\
&= (-1)^{m-n} d_{nm}^{-\ell-1}(\zeta) \quad .
\end{aligned} \tag{A.18}$$

From Eq. (A.16) we obtain the equivalence relation

$$d_{mn}^{-\ell-1}(\zeta) = \eta_m^{-\ell-1} d_{mn}^{\ell}(\zeta) \eta_n^{\ell} , \tag{A.19}$$

where

$$\eta_m^{\ell} = \frac{\Gamma(\ell+m+1)}{\Gamma(m-\ell)} , \qquad \eta_m^{-\ell-1} \eta_m^{\ell} = 1. \tag{A.20}$$

It is convenient to express the matrix elements $d_{mn}^{\ell}(\zeta)$ in terms of functions which have a simple asymptotic behavior as $\zeta \to \infty$. This expression can be obtained by using Eq. (2.1.4.17) of Ref. 20, and can be written as follows:

$$d_{mn}^{\ell}(\zeta) = a_{mn}^{\ell}(\zeta) + \eta_m^{\ell}\, a_{mn}^{-\ell-1}(\zeta)\, \eta_n^{-\ell-1} , \tag{A.21}$$

where

$$a_{mn}^{\ell}(\zeta) = \frac{(-1)^{m-n}\,\Gamma(-2\ell-1)}{\Gamma(-\ell-m)\,\Gamma(-\ell+m)} \left(\sinh\frac{\zeta}{2}\right)^{-2\ell-2} \left(\tanh\frac{\zeta}{2}\right)^{m+n}$$

$$\times\; F\left[\ell+1+n,\ \ell+1+m;\ 2\ell+2;\ -\left(\sinh\frac{\zeta}{2}\right)^{-2}\right] . \tag{A.22}$$

The asymptotic behavior of $a_{mn}^{\ell}(\zeta)$ is given by

$$a_{mn}^{\ell}(\zeta) \underset{\zeta\to\infty}{\sim} \frac{(-1)^{m-n}\,\Gamma(-2\ell-1)}{\Gamma(-\ell-m)\,\Gamma(-\ell+m)} (\cosh \zeta)^{-\ell-1} . \tag{A.23}$$

5. The O(1,1) Basis

It is useful for some purposes to express the representation functions using a basis in which $\exp(-i\eta K_2)$ is diagonal. For this purpose, it is convenient to map the unit circle onto two real lines. We define a real variable q as a function of ϕ as follows:

$$e^{q} = \tan \phi/2 \qquad \text{for } 0 \le \phi \le \pi , \tag{A.24a}$$

$$e^{-q} = \tan \tfrac{1}{2}(\phi - \pi) \qquad \text{for } \pi \le \phi \le 2\pi . \tag{A.24b}$$

If a vector f is specified by the function $f(\phi)$ in the ϕ realization, it is given in the q realization by two functions $f_1(q)$ and $f_2(q)$ defined in terms of $f(\phi)$ by

$$f_1(q) = [\cosh q]^{-\ell-1} f(\phi) \qquad \text{for} \quad 0 \leq \phi \leq \pi \ , \qquad \text{(A.25a)}$$

$$f_2(q) = [\cosh q]^{-\ell-1} f(\phi) \qquad \text{for} \quad \pi \leq \phi \leq 2\pi \ , \qquad \text{(A.25b)}$$

Writing f as a two-component column vector, the Hilbert space in the q realization is given by all vectors of the form

$$f = \begin{bmatrix} f_1(q) \\ \\ f_2(q) \end{bmatrix} \ , \qquad -\infty < q < \infty \ , \qquad \text{(A.26)}$$

where f_1 and f_2 are independently chosen square-integrable functions. The scalar product of two elements, f and h, can be expressed in the q realization by combining (A.8), (A.24), and (A.25). The result is

$$(f,h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq [f_1^*(q)\, h_1(q) + f_2^*(q)\, h_2(q)] \ . \qquad \text{(A.27)}$$

By combining Eqs. (A.6), (A.9), and (A.10), (A.24), and (A.25), we can determine the result of letting $U(g)$ act on f in the q basis. We find for $g = \exp(i\mu J_3)$ and $0 \leq \mu \leq \pi$,

$$[U(g)f]_1(q)$$

$$= (\cos \mu - \sinh q \sin \mu)^{-\ell-1} f_1(q_1)\, \Theta(\cos \mu - \sinh q \sin \mu)$$

$$+ (\sinh q \sin \mu - \cos \mu)^{-\ell-1} f_2(q_2)\, \Theta(\sinh q \sin \mu - \cos \mu) \ , \qquad \text{(A.28)}$$

where

$$e^{q_1} = \frac{e^q + \tan\frac{\mu}{2}}{1 - e^q \tan\frac{\mu}{2}} \quad , \tag{A.29a}$$

and

$$e^{q_2} = \frac{e^q + \tan\frac{\mu}{2}}{e^q \tan\frac{\mu}{2} - 1} \quad ; \tag{A.29b}$$

and

$$[U(g)f]_2(q)$$

$$= (\cos\mu + \sinh q \sin\mu)^{-\ell-1} f_2(q_3)\, \Theta(\cos\mu + \sinh q \sin\mu)$$

$$+ (-\cos\mu - \sinh q \sin\mu)^{-\ell-1} f_1(q_4)\, \Theta(-\cos\mu - \sinh q \sin\mu) \ , \tag{A.30}$$

where

$$e^{q_3} = \frac{e^q - \tan\frac{\mu}{2}}{1 + e^q \tan\frac{\mu}{2}} \quad , \tag{A.31a}$$

and

$$e^{q_4} = \frac{\tan\frac{\mu}{2} - e^q}{1 + e^q \tan\frac{\mu}{2}} \quad . \tag{A.31b}$$

For $g = \exp(i\gamma K_1)$ and $\gamma > 0$, we have

$$[U(g)f]_1(q) = (\cosh\gamma + \cosh q \sinh\gamma)^{-\ell-1} f_1(q') \ , \tag{A.32}$$

where

$$e^{q'} = \frac{e^q + \tanh\frac{\gamma}{2}}{1 + e^q \tanh\frac{\gamma}{2}} \quad ; \tag{A.33}$$

and

$$[U(g)f]_2(q)$$

$$= (\cosh q \sinh \gamma - \cosh \gamma)^{-\ell-1} f_1(q_1)\, \theta(\cosh q \sinh \gamma - \cosh \gamma)$$

$$+ (\cosh \gamma - \cosh q \sinh \gamma)^{-\ell-1} f_2(q_2)\, \theta(\cosh \gamma - \cosh q \sinh \gamma), \tag{A.34}$$

where

$$e^{q_1} = \frac{e^q - \tanh\frac{\gamma}{2}}{e^q \tanh\frac{\gamma}{2} - 1} \quad , \tag{A.35a}$$

and

$$e^{q_2} = \frac{e^q - \tanh\frac{\gamma}{2}}{1 - e^q \tanh\frac{\gamma}{2}} \quad . \tag{A.35b}$$

For $g = \exp(i\eta K_2)$, we have

$$[U(g)f]_r(q) = f_r(q + \eta) \quad , \qquad r = 1,2. \tag{A.36}$$

Using these equations, one can obtain equations that are valid for the other ranges of μ and γ.

We choose for a basis the states $|\ell,\rho\pm\rangle$ defined by

$$|\ell,\rho+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\rho q} \quad , \tag{A.37a}$$

and

$$|\ell,\rho-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\rho q} \quad . \qquad \text{(A.37b)}$$

In this basis, the representation for $\exp(-i\eta K_2)$ is diagonal:

$$\langle \ell,\rho\sigma|U[\exp(-i\eta K_2)]|\ell,\rho'\sigma'\rangle = e^{-i\rho\eta}\, \delta_{\sigma\sigma'}\, \delta(\rho - \rho') \quad . \qquad \text{(A.38)}$$

We note the relation

$$U[\exp(i\pi J_3)]|\ell,\rho+\rangle = |\ell,-\rho-\rangle \quad . \qquad \text{(A.39)}$$

6. Representations of the Continuous Class in the Mixed Basis

The mixed parameterization of g is given by

$$g = \exp(-i\eta K_2)\, \exp(-i\gamma K_1)\, \exp(-i\theta J_3) \quad . \qquad \text{(A.40)}$$

The whole group is covered if the parameters are allowed to vary over the range

$$0 \le \theta < 2\pi, \qquad -\infty < \gamma < \infty, \qquad -\infty < \eta < \infty \; .$$

For this parameterization, it is convenient to express the representation in terms of a mixed basis. Letting $\mu = i\rho$, we define

$$D^{\ell}_{\mu\sigma,m}(g) \equiv \langle \ell,\rho\sigma|U(g)|\ell,m\rangle = e^{-i\rho\eta}\, d^{\ell}_{\mu\sigma,m}(\gamma)\, e^{-im\theta} \quad , \qquad \text{(A.41)}$$

where

$$d^{\ell}_{\mu\sigma,m}(\gamma) = \langle \ell,\rho\sigma|U[\exp(-i\gamma K_1)]|\ell,m\rangle \quad . \qquad \text{(A.42)}$$

To determine the functions $d^{\ell}_{\mu\sigma,m}(\gamma)$, we must express the states $|\ell,m\rangle$ in the q realization. Using (A.24) and (A.25), we find

$$|\ell,m\rangle = e^{-im\pi/2}\begin{pmatrix} f_m(q) \\ f_{-m}(q)\end{pmatrix} , \tag{A.43}$$

where

$$f_m(q) = (\cosh q)^{-\ell-1}\left(\frac{1+ie^q}{1-ie^q}\right)^m . \tag{A.44}$$

From Eqs. (A.32), (A.33), (A.34), and (A.35), we obtain for $\gamma > 0$

$$d^{\ell}_{\mu\pm,m}(-\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, e^{-\mu q}(\cosh q \cosh\gamma \pm \sinh\gamma)^{-\ell-1}\, e^{im\phi_{\pm}(q)} \tag{A.45}$$

where

$$\tan\frac{1}{2}\phi_{\pm}(q) = \frac{e^q \pm \tanh\frac{\gamma}{2}}{e^q \tanh\frac{\gamma}{2} \pm 1} . \tag{A.46}$$

In Eq. (A.45), the principal value of the function $(\cosh q \cosh\gamma \pm \sinh\gamma)^{-\ell-1}$ is to be taken. Since $\mu = i\rho$ is purely imaginary, the integral on the right-hand side of (A.45) converges. For $m = 0$ and $\sigma = +$, Eq. (A.45) can be written as

$$d^{\ell}_{\mu+,0}(-\gamma) = \frac{1}{2}\int_{-\infty}^{\infty} dq\; e^{-\mu q}(\cosh q \cosh\gamma + \sinh\gamma)^{-\ell-1}$$

$$= \frac{1}{2\pi}\, e^{i\pi(\ell+1)} \int_{-\infty}^{\infty} dq\; e^{-\mu q}[\cosh q(i \cosh\gamma) + i \sinh\gamma]^{-\ell-1}$$

$$= \frac{1}{2\pi}\, i^{\ell+1} \int_{0}^{\infty} dq\; e^{-\mu q}[\cosh q(i \cosh\gamma) + i \sinh\gamma]^{-\ell-1}$$

$$+ \frac{1}{2\pi}\, i^{\ell+1} \int_{-\infty}^{0} dq\; e^{-\mu q}[\cosh q(i \cosh\gamma) + i \sinh\gamma]^{-\ell-1}$$

$$= \frac{1}{\pi}\, i^{\ell+1} \int_{0}^{\infty} dq\; \cosh(\mu q)[\cosh q(i \cosh\gamma) + i \sinh\gamma]^{-\ell-1}$$

$$= \frac{1}{\pi}\, i^{\ell+1} \int_{0}^{\infty} dq\; \cosh(\mu q)[z + (z^2 - 1)^{\frac{1}{2}} \cosh q]^{-\ell-1} \quad \text{(A.45a)}$$

where $z = i \sinh\gamma$ and the phase of $(z^2 - 1)^{\frac{1}{2}}$ is the same as that of z. The right-hand side of (A.45a) is a standard integral representation of a Q_ℓ function:[21]

$$d^{\ell}_{\mu+,0}(-\gamma) = \frac{1}{\pi}\frac{\Gamma(\ell + 1 - \mu)}{\Gamma(\ell + 1)}\, i^{\ell+1}\, Q_\ell^{\;\mu}(i \sinh\gamma) \quad . \qquad \text{(A.47)}$$

Using Eq. (A.39) and the fact that $d^{\ell}_{\mu+,0}$ is even in μ, we obtain

$$d^{\ell}_{\mu-,0}(\gamma) = d^{\ell}_{\mu+,0}(-\gamma) \quad . \qquad \text{(A.48)}$$

For $m = 0$ and $\sigma = -$, the right-hand side of (A.45) is proportional to the analytic continuation of $Q_\ell^{\mu}(i \sinh \gamma)$ from $\gamma > 0$ to $\gamma < 0$ onto the Riemann sheet reached through the cut $-1 < \mathrm{Re}(z) < 1$. Therefore, by making use of the discontinuity formula[22]

$$Q_\ell^{\mu}(x + i\epsilon) = e^{i\mu\pi} Q_\ell^{\mu}(x - i\epsilon) - i\pi P_\ell^{\mu}(x - i\epsilon) \quad , \tag{A.49}$$

we obtain

$$d^{\ell}_{\mu-,0}(-\gamma) = d^{\ell}_{\mu+,0}(\gamma) = -\frac{\cos \pi\mu}{\cos \pi\ell} d^{\ell}_{\mu+,0}(-\gamma)$$

$$+ \frac{\pi\Gamma(-\ell)}{\Gamma(\ell+1)\cos\pi\ell} \frac{d^{-\ell-1}_{\mu+,0}(-\gamma)}{\Gamma(\mu-\ell)\,\Gamma(-\mu-\ell)} \quad . \tag{A.50}$$

FOOTNOTES AND REFERENCES

1. M. Toller, Nuovo Cimento 37, 631 (1965).

2. N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967).

3. M. Toller, Nuovo Cimento 62A, 341 (1969).

4. M. N. Misheloff, Phys. Rev. 184, 1732 (1969).

5. P. Goddard and A. R. White, Nucl. Phys. B17, 45 (1970).

6. P. Goddard and A. R. White, Nucl. Phys. B17, 88 (1970).

7. P. V. Landshoff and W. J. Zakrzewski, Nucl. Phys. B12, 216 (1969).

8. A kinematical dependence on μ_a would appear if particle A_i or particle A_f had spin.

9. The two values of τ refer to two disjoint sections of the physical region. The amplitude's dependence on τ was neglected in Ref. 4.

10. We are assuming that t_A is sufficiently negative so that f is a square-integrable function of g_a. The final result may be analytically continued to larger values of t_A.

11. We suppress the variables t_A, t_B, t_C, g_b, and g_c and the index τ for the moment.

12. We temporarily suppress the variables t_A, t_B, t_C, g_b', and g_c'.

13. The amplitude's dependence on τ_a, τ_b, and τ_c was neglected in Ref. 4.

14. Chan Hong-Mo, Phys. Letters 28B, 425 (1969).

15. N. Mukunda, J. Math. Phys. 8, 2210 (1967).

16. M. Ciafaloni, C. DeTar, and M. N. Misheloff, Phys. Rev. 188, 2522 (1969).

17. Our representation of the Lie algebra of SU(1,1) is the same as Mukunda's. Our $|\ell,m\rangle$ basis states differ from those of Refs. 1, 15, and 16 by a factor of $e^{-im\pi/2}$. Our $|\ell,\rho\sigma\rangle$ basis states are the same as Ciafaloni, DeTar, and Misheloff's. Our conventions are chosen in such a way that our $d_{mm'}^{\ell}$ functions are the same as Toller's, and our $d^{\ell}_{\mu\sigma,0}$ functions are the same as Ciafaloni, DeTar, and Misheloff's.

18. V. Bargmann, Ann. Math. 48, 568 (1947).

19. J. Mathews and R. L. Walker, Mathematical Methods of Physics (W. A. Benjamin, Inc., New York, 1964), Eq. (7-94).

20. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, Higher Transcendental Functions (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, Sec. (2.9).

21. Reference 20, Sec. (3.7). In our Q_{ℓ}^{μ} functions the factor $e^{i\mu\pi}$ of the definition of Ref. 19 has been dropped.

22. Reference 20, Sec. (3.4).

FIGURE CAPTIONS

Fig. 1. Tree diagram for the multi-Regge amplitude.

Fig. 2. Tree diagram with no three-internal-line vertex for the 3-to-3 amplitude.

Fig. 3. Tree diagram with one three-internal-line vertex for the 3-to-3 amplitude.

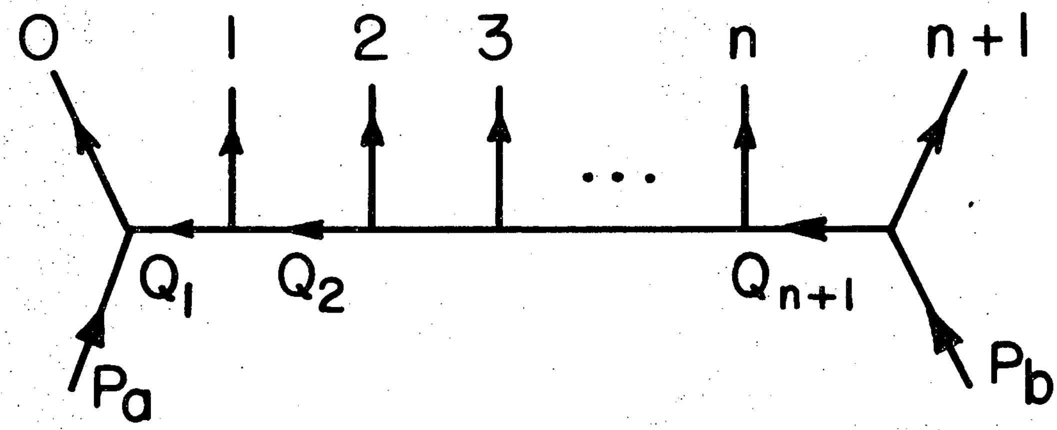

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Fig. 1

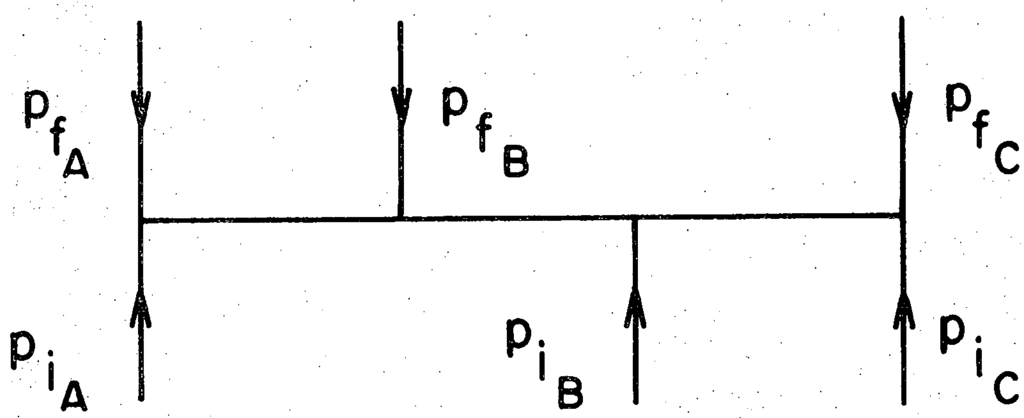

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Fig. 2

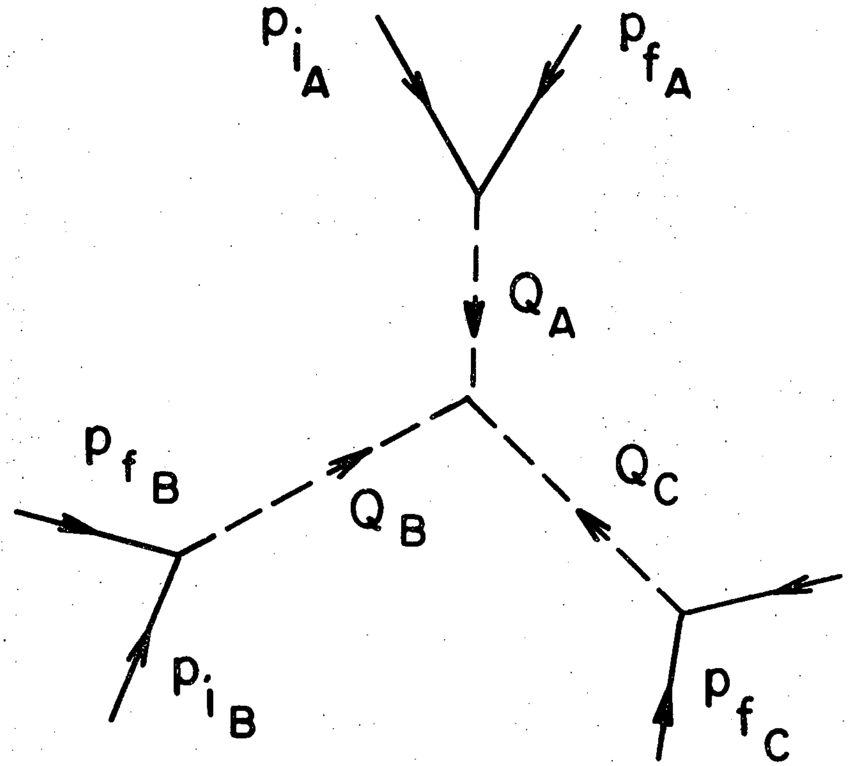

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Fig. 3

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