Title
The Abelian-Nonabelian Correspondence for *I*-functions

Author
Webb, Rachel

Publication Date
2022-08-20

Peer reviewed

THE ABELIAN-NONABELIAN CORRESPONDENCE FOR I-FUNCTIONS

RACHEL WEBB

Abstract. We prove the abelian-nonabelian correspondence for quasimap I-functions. That is, if Z is an affine l.c.i. variety with an action by a complex reductive group G, we prove an explicit formula relating the quasimap I-functions of the GIT quotients $Z/\!\!/_\theta G$ and $Z/\!\!/_\theta T$ where T is a maximal torus of G. We apply the formula to compute the J-functions of some Grassmannian bundles on Grassmannian varieties and Calabi-Yau hypersurfaces in them.

Contents

1. Introduction

Let Z be an affine variety with at worst local complete intersection singularities and let G be a connected complex reductive algebraic group acting on Z with maximal torus T. A character θ of G defines a linearization of the trivial bundle on Z. From this data, we get two GIT quotients with a rational map between them: $Z/\!\!/T \dashrightarrow Z/\!\!/G$. The *abelian-nonabelian correspondence* is a conjectured relationship [15, Conj 3.7.1] between the genus-zero Gromov-Witten invariants of $Z/\!\!/G$ and those of $Z/\!\!/T$.

This paper studies the abelian-nonabelian correspondence via the quasimap theory of Ciocane-Fontantine–Kim–Maulik [14]. Roughly speaking, quasimap moduli spaces approximate moduli spaces of stable maps by replacing rational tails with basepoints. The quasimap invariants are packaged into a generating series called an I-function that, granting the existence of a wall-crossing theorem (as in [13] or [51]), encodes certain Gromov-Witten invariants of the target. This paper proves a correspondence of the small quasimap I-functions of $Z/\!\!/T$ and $Z/\!\!/G$, and of their big I-functions when Z is affine space. In particular, when Z is affine space we obtain an explicit formula for the big I-function. In certain cases, our I-function correspondence implies [15, Conj 3.7.1].

1.1. **Statement of the main result.** Let Z, G, T, and θ be as stated. Let $Z^s(G)$ and $Z^{ss}(G)$ denote the θ-stable and semistable points in Z. We assume that $Z^s(G) = Z^{ss}(G)$ is smooth and not empty and that G acts on $Z^s(G)$ freely. We also assume that each of these statements holds with T in place of G. Hence $V/\!\!/_\theta G$ and $V/\!\!/_\theta T$ are smooth varieties. We fix the character θ for all of this paper, and we will generally write $Z/\!\!/G := Z/\!\!/_\theta G$ and

Date: August 20, 2022.
2010 *Mathematics Subject Classification.* 14L30, 14N35, 14D23.
Key words and phrases. quasimaps, I-function, abelianization, moduli of maps from the projective line.

$Z/\!\!/T := Z/\!\!/_\theta T$. (When the targets $Z/\!\!/G$ or $Z/\!\!/T$ are considered individually, this is the same setup as used in [13, Section 2.1].)

The small quasimap I-function of $Z/\!\!/G$, defined in [14], has the form

$$I^{Z/\!\!/G}(z) = 1 + \sum_{\beta \neq 0} q^\beta I_\beta^{Z/\!\!/G}(z) \tag{1}$$

where β is in $\mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ with $\mathrm{Pic}^G(Z)$ the group of G-equivariant line bundles on Z, q^β is a formal variable, and the coefficients $I_\beta^{Z/\!\!/G}(z)$ are polynomials in z and z^{-1} with coefficients in $A^*(Z/\!\!/G)$ (see Section 3.3 for details, including a discussion of our use of $A^*(Z/\!\!/G)$ in place of $H^*(Z/\!\!/G, \mathbb{Q})$).

To state the main theorem, we make two observations. First, the rational map $Z/\!\!/T \dashrightarrow Z/\!\!/G$ may be stated more precisely via the diagram

$$\begin{tikzcd} Z^s(G)/T \arrow[r, hook, "j"] \arrow[d, "g"] & Z^s(T)/T \\ Z^s(G)/G & \end{tikzcd} \tag{2}$$

Second, there is an inclusion

$$\chi(G) \to \mathrm{Pic}^G(Z) \tag{3}$$

sending the character ξ to

$$\mathscr{L}_\xi := Z \times \mathbb{C}_\xi \tag{4}$$

where $\mathbb{C}_\xi$ is the representation with character ξ. The line bundle $\mathscr{L}_\xi$ descends to a line bundle on $Z/\!\!/G$ which we also denote by L_ξ.

Theorem 1.1.1. *The small I-functions of $Z/\!\!/G$ and $Z/\!\!/T$ satisfy*

$$g^* I_\beta^{Z/\!\!/G}(z) = \sum_{\tilde\beta \to \beta} \left(\prod_\rho \frac{\prod_{k=-\infty}^{\tilde\beta(\rho)} (c_1(\mathscr{L}_\rho) + kz)}{\prod_{k=-\infty}^{0} (c_1(\mathscr{L}_\rho) + kz)} \right) j^* I_{\tilde\beta}^{Z/\!\!/T}(z), \tag{5}$$

where the sum is over all $\tilde\beta$ mapping to β under the natural map $\mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ *and the product is over all roots ρ of G.*

Remark 1.1.2. *Our I-functions in Theorem 1.1.1 are valued in Chow. One may also define an I-function for $Z/\!\!/G$ valued in $H^*(Z/\!\!/G, \mathbb{Q})$ by first mapping the coefficients of $I^{Z/\!\!/G}(z)$ to Borel-Moore homology and then applying Poincaré duality (see* [24, Sec 19.1]*). The equality* (5) *holds when $I_\beta^{Z/\!\!/G}(z)$ and $I_{\tilde\beta}^{Z/\!\!/T}(z)$ are replaced with their cohomology-valued versions (see Remark 3.3.4).*

Since g^* is injective (see Proposition 2.4.1), the equality (5) completely determines $I^{Z/\!\!/G}$. We explain in Section 5.4 how to make sense of the right hand side of (5), in particular the denominators that appear when $\tilde\beta(\rho)$ is strictly negative. When Z is a vector space, one obtains a closed formula for $I_\beta^{Z/\!\!/G}$ by combining (5) with the formula for $I_{\tilde\beta}^{Z/\!\!/T}$ in [10, Thm 5.4] (which agrees with the formula in [27]).

Previous to this work, Theorem 1.1.1 was known for $Z/\!\!/G$ equal to a flag variety or the Hilbert scheme of n points in $\mathbb{C}^2$ [3] [4] [16]. Since the posting of this paper, a proof of this result for quiver flag varieties has also appeared [35]. Finally, a result similar to Theorem 1.1.1 was proved in the language of stable gauged maps in [28, Thm 1.4, Thm 3.9]. Gauged maps apply in a different context than quasimaps: while quasimap theory requires Z to be an affine l.c.i. variety, the theorey of gauged maps mandates that Z be smooth projective. It is not clear how the theories of gauged maps and quasimaps are related [10, Rmk 4.4].

We remark that Theorem 1.1.1 completes a provisional result in [33]. The bulk of the proof of Theorem 1.1.1 is a careful analysis of certain moduli spaces of maps from $\mathbb{P}^1$ to $[Z/G]$ and $[Z/T]$; the geometry of these moduli spaces is summarized in Proposition 4.0.1.

This geometry may be useful in other contexts. For instance, it is used in [50] to compute quasimap I-functions in K-theory.

1.2. **Extensions and applications.** This paper also proves the natural extension of Theorem 1.1.1 to the equivariant and twisted theories (Corollaries (6.1.1) and (6.2.1)). We also use the theory developed in [11] to write down a big I-function for $Z/\!\!/_\theta G$ when Z is a vector space (Corollary 6.3.1). This recovers, for example, an explicit big I-function for the Grassmannian $Gr(k,n)$.

With the reconstruction result in [15] and the mirror result in [13], the equivariant version of Theorem 1.1.1 implies the full abelian-nonabelian correspondence for projective, sufficiently Fano quotients $Z/\!\!/G$ with "nice" torus actions (Corollary 6.4.4). We use this result to explicitly compute an equivariant twisted small J-function for a Grassmannian bundles on a Grassmannian variety (Theorem 6.5.1).

1.3. **Conventions and notation.** We work over $\mathbb{C}$. If A is a $\mathbb{Z}$-module then $A_{\mathbb{Q}}$ is the tensor product with $\mathbb{Q}$. A variety is an integral separated finite type scheme. Fix an affine variety Z with a left action by G that is a complex reductive algebraic group over $\mathbb{C}$ and fix $T \subset G$ a maximal torus. Let $N_G(T)$ be the normalizer of T in G, and let $W = N_G(T)/T$ denote the Weyl group. The characters of G are $\chi(G)$.

Fix a character $\theta \in \chi(G)$. Let $Z^s(G)$ and $Z^{ss}(G)$ denote the θ-stable and semistable loci as defined in [36, Section 2]. We assume that $Z^s(G) = Z^{ss}(G)$ is smooth and not empty and that G acts on $Z^s(G)$ freely. We also assume that each of these statements holds with T in place of G. Hence $V/\!\!/_\theta G$ and $V/\!\!/_\theta T$ are smooth varieties, projective over $\mathbf{S}_G := \mathrm{Spec}(H^0(Z,\mathscr{O}_Z)^G)$ and $\mathbf{S}_T := \mathrm{Spec}(H^0(Z,\mathscr{O}_Z)^T)$, respectively.

For the convenience of the reader, we list some of the notation used in this paper.

- If $(C,\mathscr{P},\sigma,\mathbf{x})$ is a quasimap to $Z/\!\!/T$ (see Definition 3.1.1), then
  - $\sigma : C \to \mathscr{P} \times_G Z$ is a section
  - $\tilde{n} : \mathscr{P} \to Z$ is the associated morphism of principal bundles, and
  - $n : C \to [Z/G]$ is the associated morphism of stacks (see Section 3.1.1).
- τ is a transition function of a quasimap written in coordinates (Section 3.1.2).
- r_G, r_T, r_χ, and r_{Pic} are morphisms of the lattices containing quasimap classes (see (23)).
- The fixed loci used in Sections 4 and 5 are summarized in the diagram below, along with the place they are defined.

$$\begin{tikzcd}
\begin{matrix} F_{\tilde\beta}(Z/\!\!/G) \\ \text{(52)} \end{matrix} \arrow[d, hook] & & \begin{matrix} Z^0_{\tilde\beta} \\ \text{(40)} \end{matrix} \arrow[r, hook] \arrow[d] & \begin{matrix} Z_{\tilde\beta} \\ \text{(38)} \end{matrix} \arrow[d] \\
\begin{matrix} F_{\beta}(Z/\!\!/G) \\ \text{Section 3.3} \end{matrix} \arrow[d, hook] & & \begin{matrix} F^0_{\tilde\beta}(Z/\!\!/T) \\ \text{(40)} \end{matrix} \arrow[ll, "\psi_{\tilde\beta}", "\text{(42)}"'] \arrow[ull, two heads] \arrow[r, hook] \arrow[d, hook] & \begin{matrix} F_{\tilde\beta}(Z/\!\!/T) \\ \text{Section 3.3} \end{matrix} \arrow[d, hook] \\
\begin{matrix} QG(Z/\!\!/G) \\ \text{Section 3.2} \end{matrix} & & \begin{matrix} F^0_{\beta}(Z/\!\!/T) \\ \text{(46)} \end{matrix} \arrow[r, hook] & \begin{matrix} QG(Z/\!\!/T) \\ \text{Section 3.2} \end{matrix}
\end{tikzcd}$$

1.4. **Organization of the paper.** In Sections 2 and 3 we review standard facts about abelian/nonabelian correspondences and quasimaps, respectively, writing out many well-known formulae explicitly so that we can reference them in computations later. Section 4 explains the geometry behind Theorem 1.1.1 and Section 5 completes the proof of this theorem. Section 6 proves various extensions and applications of the main result.

1.5. **Acknowledgements.** Thanks to Robert Silversmith for explaining quasimaps to me in a concrete way, and to Yongbin Ruan for helpful discussions. Thank you to Jonathan Wise, Damiano Fulghesu, Matthew Satriano, and Martin Olsson for teaching me enough algebraic geometry so I could write this paper rigorously. Anonymous referees provided many helpful comments and corrections. This project was partially supported by NSF RTG grant 1045119 and an NSF Postdoctoral Research Fellowship, award number 200213.

2. Some first abelian/nonabelian correspondences

2.1. **Preliminaries on principal bundles.** A principal G-bundle on a scheme C is a scheme $\pi : \mathscr{P} \to C$ with π faithfully flat and locally finitely presented, together with an action $\mu : G \times \mathscr{P} \to \mathscr{P}$ leaving π invariant such that the map

$$G \times \mathscr{P} \xrightarrow{(\mu, pr_2)} \mathscr{P} \times_C \mathscr{P}$$

is an isomorphism. With our assumptions, a principal G-bundle is locally trivial in the étale topology (see eg [41, Rmk 4.5.7]).

If $\mathscr{P} \to C$ is a principal G-bundle on a scheme C and Z is an affine variety with a left G-action, then we define the *mixing space* to be the quotient

$$\mathscr{P} \times_G Z = (\mathscr{P} \times Z)/G \qquad \text{where } g \cdot (p, z) = (gp, gz) \text{ for } g \in G,\ (p, z) \in \mathscr{P} \times Z. \tag{6}$$

This space is an étale-locally trivial fibration on C with fiber Z. A priori it is an algebraic space, but for example if C is finite type then it is known to be a scheme (using affineness of Z, see eg [7, Sec 3]).

Our convention in (6) is necessary because we want the stack quotient $Z \to [Z/G]$ to have the property that, for any scheme S and any morphism $S \to [Z/G]$, the fiber product $Z \times_{[Z/G]} S \to S$ is a principal G-bundle. However, it has the unfortunate consequence that the transition functions of $\mathscr{P}$ and $\mathscr{P} \times_G Z$ are inverse to each other. Hence, if $T \subset G$ is a subgroup and $\mathscr{T} \to C$ is a principal T-bundle, we define the *associated G-bundle* to be

$$G \times_T \mathscr{T} = (G \times \mathscr{T})/T \qquad \text{where } t \cdot (g, s) = (gt^{-1}, ts) \text{ for } t \in T,\ (g, s) \in G \times \mathscr{T}. \tag{7}$$

The quotient $G \times_T \mathscr{T}$ is a (left) principal G-bundle with the same transition function as $\mathscr{T}$.

2.2. **Action of the Weyl group.** Heuristically, an abelian/nonabelian correspondence relates data of G to data of T and the Weyl group W. In this section we explain the action of W on several objects of interest.

The Weyl group acts on T, characters of T, dual characters and cocharacters of T in the usual ways; i.e., for $w \in N_G(T)$ and $t \in T$, $\xi \in \chi(T)$ a character, $\tilde{\alpha} \in \mathrm{Hom}(\chi(T), \mathbb{Z})$ a dual character, and $\tau : \mathbb{C}^* \to T$ a cocharacter, we have

$$\begin{aligned} w \cdot t &= wtw^{-1} & w \cdot \xi(t) &= \xi(w^{-1} \cdot t) \\ w \cdot \tilde{\alpha}(\xi) &= \tilde{\alpha}(w^{-1} \cdot \xi) & (w \cdot \tau)(t) &= w \cdot (\tau(t)). \end{aligned} \tag{8}$$

Since quasimaps are really maps to a stack quotient, we will need to understand the action of W on $[Z/T]$. As a warmup, consider the case when $[Z/T]$ is represented by a smooth scheme Z/T (e.g., replace Z by $Z^s(T)$). Then the action of the group scheme W on Z descends to Z/T as follows. For $w \in N_G(T)$ and $t \in T$, $z \in Z$ we compute

$$w(tz) = (wtw^{-1})(wz). \tag{9}$$

This shows that the action of $N_G(T)$ descends to Z/T, and clearly the action of $T \subset N_G(T)$ is trivial. Because of the computation (9), we say that the map $w : Z \to Z$ is *twisted-equivariant* for the homomorphism $a_w : T \to T$ defined by $a_w(t) = wtw^{-1}$; that is,

$$w(tz) = a_w(t)w(z).$$

This twisted equivariance manifests itself in the Weyl actions that we describe below.

2.2.1. *On maps to* $[Z/T]$. By [43, Rmk 2.4], the action of W descends to $[Z/T]$. We compute the action explicitly. According to [43, Rmk 2.4] an element $w \in N_G(T)$ defines an automorphism ϕ_w of $[Z/T]$ dependent only on the congruence class of w in $W = N_G(T)/T$ and fitting into a diagram (ignore the gray part for now)

$$\begin{tikzcd} w\mathscr{T} \arrow[r] \arrow[d] & wZ \arrow[r, "w"] \arrow[d] & Z \arrow[d] \\ S \arrow[r] & {[Z/T]} \arrow[r, "\phi_w"] & {[Z/T]} \end{tikzcd} \tag{10}$$

where wZ is determined as a scheme by the requirement that the black square be fibered, and it is determined as a T-scheme by the requirement that $w : wZ \to Z$ be T-equivariant. That is, if $\cdot$ denotes the original action of T on Z, then wZ is the scheme Z with the T-action $\cdot_w$ given by

$$t \cdot_w z := (w^{-1}tw) \cdot z \qquad \text{for } t \in T,\ z \in Z. \tag{11}$$

Now suppose we have an S-valued point of $[Z/T]$ corresponding to a principal bundle $\mathscr{T} \to S$. On the one hand, the action of w on $S \to [Z/T]$ of $[Z/G]$ is given by composition with ϕ_w. On the other hand, studying the gray part of (10) we see that the principal bundle corresponding to $S \to [Z/T] \xrightarrow{\phi_w} [Z/T]$ is $w\mathscr{T}$, where $w\mathscr{T}$ is equal to $\mathscr{T}$ as a scheme and has action $\cdot_w$ defined as in (11).

In particular, when Z is a point, we get an action on principal T-bundles given by $w \cdot \mathscr{T} = w\mathscr{T}$. The identity map $\mathscr{T} \to w\mathscr{T}$ is an a_w-equivariant isomorphism.

For use when Z is nontrivial, we also describe the action on $[Z/T]$ in terms of sections of fiber bundles. The morphism $S \to [Z/T]$ given by $\pi : \mathscr{T} \to S$ and $f : \mathscr{T} \to Z$ is equivalent to the data of $\mathscr{T}$ and the section σ of $\mathscr{T} \times_T Z$ defined by the quotient of $\mathscr{T} \xrightarrow{(id,f)} \mathscr{T} \times Z$. Then w acts by

$$w \cdot (\mathscr{T}, \sigma : S \to \mathscr{T} \times_T Z) = (w\mathscr{T}, \varpi \circ \sigma : S \to w\mathscr{T} \times_T Z) \tag{12}$$

where $\varpi : \mathscr{T} \times_T Z \to w\mathscr{T} \times_T Z$ is the isomorphism coming from the a_w-equivariant map

$$\begin{aligned} \mathscr{T} \times Z &\xrightarrow{\varpi} w\mathscr{T} \times Z \\ (x, z) &\mapsto (x, wz). \end{aligned} \tag{13}$$

2.2.2. *On* $\mathrm{Pic}^T(Z)$. Finally, the group W acts on $\mathrm{Pic}^T(Z)$, by which we mean the group of line bundles on $[Z/T]$, or equivalently T-equivariant line bundles on Z. Since W acts on $[Z/T]$ we get an action on $\mathrm{Pic}^T(Z)$ by sending $\mathscr{L} \in \mathrm{Pic}([Z/T])$ to the pullback $(w^{-1})^*\mathscr{L}$.

We translate this action to T-equivariant line bundles on Z. If $\mathscr{L} \to [Z/T]$ is a line bundle, the corresponding T-equivariant line bundle on Z is the pullback $\mathscr{L}_1 := \mathscr{L} \times_{[Z/T]} Z$. Then $w \cdot \mathscr{L}_1$ is the fiber product $(w^{-1})^*\mathscr{L} \times_{[Z/T]} Z$. We relate this to $\mathscr{L}_1$ using the following diagram.

$$\begin{tikzcd} w((w^{-1})^*\mathscr{L}_1) \arrow[r, "id"] \arrow[d] & (w^{-1})^*\mathscr{L}_1 \arrow[r] \arrow[d] & \mathscr{L}_1 \arrow[d] \\ Z \arrow[r, "id"] & w^{-1}Z \arrow[r, "w^{-1}"] & Z \end{tikzcd} \tag{14}$$

Again, $w((w^{-1})^*\mathscr{L}_1)$ is the scheme $(w^{-1})^*\mathscr{L}_1$ with action given by $\cdot_w$. The square on the right is one side of a fibered cube formed by pulling back $\mathscr{L} \to [Z/T]$ to the other corners of the black square in (10). The left square is also fibered (note that its horizontal maps are twisted-equivariant). So the desired fibered product $(w^{-1})^*\mathscr{L} \times_{[Z/T]} Z$ with its natural T-action is equal to $w((w^{-1})^*\mathscr{L}_1$ (to see this, the reader may like to pencil into (14) the fibered cube mentioned above). From this one sees that

$$w \cdot \mathscr{L}_\xi = \mathscr{L}_{w\xi}, \tag{15}$$

i.e., the map $\chi(T) \to \mathrm{Pic}^T(Z)$ defined in (3) is W-equivariant.

2.3. **Principal bundles on $\mathbb{P}^1_k$.** We will prove Theorem 1.1.1 by directly comparing certain moduli spaces of maps from $\mathbb{P}^1$ to the stack quotient $[Z/T]$ or $[Z/G]$. Hence, the correspondence between principal G-bundles and principal T-bundles on $\mathbb{P}^1$ is really the heart of our theorem, and it will be used in our proof. This corresondence follows from Grothendieck's classification of principal G-bundles on $\mathbb{P}^1_k$, where k is an algebraically closed field, and we now remind the reader of the statement.

Let $\mathrm{Bun}_G(\mathbb{P}^1_k)$ denote the set of isomorphism classes of principal G-bundles on $\mathbb{P}^1_k$. There is a natural map

$$\psi : \mathrm{Bun}_T(\mathbb{P}^1) \to \mathrm{Bun}_G(\mathbb{P}^1)$$

defined by sending a principal T-bundle $\mathscr{T}$ to $G \times_T \mathscr{T}$ (see (7)). The group W acts on $\mathrm{Bun}_T(\mathbb{P}^1_k)$ as in (11). Moreover, for $w \in N_G(T)$ and $\mathscr{T} \in \mathrm{Bun}_T(\mathbb{P}^1_k)$, the principal G-bundles $G \times_T \mathscr{T}$ and $G \times_T w\mathscr{T}$ are isomorphic via the map $G \times \mathscr{T} \to G \times w\mathscr{T}$ given by $(g, x) \mapsto (gw^{-1}, x)$ (compare with (13)). The following theorem is due to Grothendieck; see also [40, 393].

Theorem 2.3.1 ([29])**.** *The map ψ induces a bijection $\mathrm{Bun}_T(\mathbb{P}^1_k)/W \to \mathrm{Bun}_G(\mathbb{P}^1_k)$.*

Remark 2.3.2. *The isomorphism class of $\mathscr{T}$ is determined by the homomorphism $\tilde{\alpha} \in \mathrm{Hom}(\chi(T), \mathbb{Z})$ defined by*

$$\tilde{\alpha}(\xi) = \deg_{\mathbb{P}^1}(\mathscr{T} \times_T \mathbb{C}_\xi) \qquad \xi \in \chi(T).$$

Hence Theorem 2.3.1 says that elements of $\mathrm{Bun}_G(\mathbb{P}^1_k)$ biject with Weyl-orbits on $\mathrm{Hom}(\chi(T), \mathbb{Z})$.

2.4. **Chow groups of $Z/\!\!/G$ and $Z/\!\!/T$.** We recall a weak abelian-nonabelian correspondence for Chow groups of $Z/\!\!/G$ and $Z/\!\!/T$—in fact, the proposition below only compares the Chow group of $Z/\!\!/G$ and that of an open subset of $Z/\!\!/T$.

Proposition 2.4.1. *The pullback g^* in diagram (2) induces an isomorphism*

$$g^* : A_*(Z^s(G)/G) \otimes \mathbb{Q} \xrightarrow{\sim} (A_*(Z^s(G)/T) \otimes \mathbb{Q})^W. \tag{16}$$

An analogous statement holds for cohomology rings with coefficients in $\mathbb{Q}$.

Remark 2.4.2. *The inclusion $j : Z^s(G)/T \hookrightarrow Z/\!\!/T$ induces a morphism j^* of Chow groups or cohomology groups. The composition $(g^*)^{-1} \circ j^*$ defines a morphism $(A_*(Z/\!\!/T) \otimes \mathbb{Q})^W \to A_*(Z/\!\!/G) \otimes \mathbb{Q}$, but one should expect this morphism to have kernel (in one setting, this is* [39, Thm A]*). We will not need to understand the kernel in this paper.*

Remark 2.4.3. *Since G acts with trivial stabilizers on $Z^s(G)$, one can identify $A_*(Z^s(G)/G)$ with $A^G_*(Z^s(G)/T)^W$ by* [21, Prop 8] *and then Proposition 2.4.1 follows from* [21, Prop 6]*. Likewise, the statement for cohomology rings follows from* [6, Prop 1]*. Because we need ideas from the proof later (in Section 5.4), we include here a direct proof of Proposition 2.4.1 in the case of Chow groups.*

Proof of Proposition 2.4.1. We prove the statement for Chow groups. Let $B \subset G$ be a Borel subgroup containing T. Then there is a geometric quotient scheme $Z^s(G)/B$ ([22, Sec 2.5]) and g factors as the composition

$$Z^s(G)/T \xrightarrow{f} Z^s(G)/B \to Z^s(G)/G.$$

Since f is an affine bundle, f^* is an isomorphism of Chow groups by [46, Lem 2.2].

Let $S = Sym(\chi(T)_\mathbb{Q})$ and let S^W_+ be the ideal generated by W-invariants of positive degree. The *characteristic homomorphism* is an isomorphism

$$c : S/S^W_+ \xrightarrow{\sim} A^*(G/B)_\mathbb{Q}$$

sending a character ξ to the first Chern class of the line bundle on G/B associated to ξ ([22, Sec 1.3]). Here, A^* is the operational Chow group.

Similarly, sending ξ to $c_1(\mathscr{L}_\xi)$ defines a W-equivariant map $c_T : S \to A^*(Z^s(G)/T)_\mathbb{Q}$ that factors as

$$S \xrightarrow{c_B} A^*(Z^s(G)/B)_\mathbb{Q} \underset{\sim}{\xrightarrow{f^*}} A^*(Z^s(G)/T)_\mathbb{Q} \tag{17}$$

where the map c_B restricts to the characteristic homomorphism of any fiber $G/B \hookrightarrow Z^s(G)/B$. Hence, if $\chi_1, \ldots, \chi_k$ are elements of S that map to a basis of S/S_+^W, then $c_B(\chi_i)$ are elements of $A^*(Z^s(G)/B)_{\mathbb{Q}}$ that restrict to a basis of the Chow group of any fiber. Now by [22, Lem 2.8] and (17) the map

$$\begin{aligned} S/S_+^W \otimes_{\mathbb{Q}} A_*(Z^s(G)/G)_{\mathbb{Q}} &\to A_*(Z^s(G)/T)_{\mathbb{Q}} \\ \sum_i [\chi_i] \otimes b_i &\mapsto \sum_i c_T(\chi_i) \cap g^*(b_i) \end{aligned} \tag{18}$$

is an isomorphism. To prove the proposition, take W-invariants of both sides of (18), noting that $(S/S_T^W)^W = \mathbb{Q}$.

□

3. The quasimap I-function

The I-function of $Z/\!\!/G$ can be defined using the quasimap theory of Ciocan-Fontanine and Kim, developed in [13], [14], and [12]. To compute the I-function, we only need to study genus-zero quasimaps to $Z/\!\!/G$ with no markings and one parameterized component, so we will restrict our review of quasimap theory to this situation.

3.1. **Quasimaps.** Fix for the duration of this paper a copy of the projective line with projective coordinates $[u : v]$ and denote it $\mathbb{P}^1$. For a scheme S we write $\mathbb{P}^1_S := \mathbb{P}^1 \times S$. Stable graph quasimaps are defined in [14, Def 7.2.1]; one easily sees that when $g = 0$ and there are no marked points, that definition is equivalent to the following.

Definition 3.1.1. *A* stable graph quasimap *to $Z/\!\!/G$ over a base scheme S is a tuple $(C, \mathscr{P}, \sigma, \mathbf{x})$ where*

- *$C \to S$ is a nodal genus-0 projective curve*
- *$\mathscr{P} \to C$ is a principal G-bundle*
- *σ is a section of $\mathscr{P} \times_G Z$*
- *$\mathbf{x} : C \to \mathbb{P}^1_S$ is a morphism that restricts to an isomorphism on every geometric fiber over S.*

Moreover, for every geometric point $s \in S$, the set of points $p \in C_s$ such that $\sigma(p) \notin Z^s_\theta$ must be finite.

An isomorphism of stable graph quasimaps $(C, \mathscr{P}, \sigma, \mathbf{x})$ and $(C, \mathscr{P}', \sigma', \mathbf{x}')$ on S is a commuting diagram

$$\begin{tikzcd} & \mathscr{P}' \arrow[r, "\sim"] \arrow[d] & \mathscr{P} \arrow[d] \\ \mathbb{P}^1_S & C' \arrow[l, "\mathbf{x}'"'] \arrow[r, "\sim", "f"'] & C \arrow[ll, bend left, "\mathbf{x}"] \end{tikzcd} \tag{19}$$

such that the square is fibered and $f^*\sigma = \sigma'$. Note that stability depends on the character θ; since we have fixed θ once and for all in this paper, we will generally omit it from the notation. The set $\{p \in C \mid \sigma(p) \notin Z^s\}$ is called the *base locus* of the quasimap (it has finite fibers over the topological space of S).

Definition 3.1.2. *A* general quasimap *is a quasimap as defined in* [14, Sec 3.1]. *That is, the data of a general quasimap over a base scheme S is a tuple $(C, \mathscr{P}, \sigma, p_i)$ where $C \to S$ is a nodal curve of genus g, the principal bundle $\mathscr{P}$ and section σ are as in Definition 3.1.1, and $p_i : S \to C$ are markings for $i = 1, \ldots, n$. These data are required to satisfy the same nondegeneracy condition as in Definition 3.1.1; namely, for every geometric point $s \in S$, the set of points $p \in C_s$ such that $\sigma(p) \notin Z^s_\theta$ must be finite.*

Remark 3.1.3. *Because we work almost exclusively with the stable graph quasimaps in Definition 3.1.1, we call them simply "quasimaps." We will refer explicitly to Definition 3.1.2 when the more general notion arises.*

Lemma 3.1.4. *If $(C, \mathscr{P}, \sigma, \mathbf{x})$ is a quasimap, then $\mathbf{x}$ is an isomorphism (so $C = \mathbb{P}^1_S$).*

Proof. The condition that the geometric fibers of $\mathbf{x}$ are isomorphisms forces $\mathbf{x}$ to be an isomorphism. One can show this directly, or one can note that $(C, \mathbf{x})$ defines a family in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 1)$. This latter moduli space is isomorphic to the Grassmannian $Gr(\mathbb{P}^1, \mathbb{P}^1)$ (according to [26, 5]), which is represented by a point. So its universal curve is trivial. □

3.1.1. *Quasimaps as maps to* $[Z/G]$. We will make extensive use of an equivalent definition of quasimaps. Given a principal G-bundle $\mathscr{P} \to C$ on a curve C over S, giving a section $\sigma : C \to \mathscr{P} \times_G Z$ is equivalent to giving a G-equivariant morphism

$$\tilde{n} : \mathscr{P} \to Z.$$

which in turn defines a map $n : C \to [Z/G]$. Indeed, a morphism $\tilde{n} : \mathscr{P} \to Z$ is equivalent to a section of

$$\mathscr{P} \times Z \to \mathscr{P},$$

and the quotient of this section by G recovers σ. The morphism $\tilde{n}$ defines a stable quasimap if for every geometric fiber C_s of $C \to S$, the set of points $p \in C_s$ such that $\tilde{\sigma}(\mathscr{P}_p) \not\subset Z^s_\theta$ is finite. The correspondence between σ and $\tilde{n}$ is made more precise using the moduli of sections in Section 3.2.

3.1.2. *Quasimaps in local coordinates.* We will often work with a certain class of quasimaps which we now describe. Let

$$U_S := S \times \mathbb{A}^1 \xrightarrow{(s,u)\mapsto(s,[u:1])} S \times \mathbb{P}^1 \qquad\qquad V_S := S \times \mathbb{A}^1 \xrightarrow{(s,v)\mapsto(s,[1:v])} S \times \mathbb{P}^1$$

be the distinguished open subsets of $\mathbb{P}^1_S = S \times \mathbb{P}^1$, with gluing morphism $\kappa : U_S \setminus (S \times \{0\}) \to V_S \setminus (S \times \{0\})$ given by $\kappa(s, u) = (s, u^{-1})$. Then any morphism $\tau : U_S \setminus \{S \times 0\} \to G$ defines an isomorphism of the restrictions of the trivial G-bundles $U_S \times G$ and $V_S \times G$ by sending

$$(u, g) \mapsto (\kappa(u), g\tau^{-1}(u)) \quad \text{for} \quad u \in U_S \setminus \{S \times 0\}. \tag{20}$$

We denote the resulting principal G-bundle on $\mathbb{P}^1_S$ by $\mathscr{P}_\tau$. In particular, any cocharacter τ of G defines a principal G-bundle $\mathscr{P}_\tau$ on $\mathbb{P}^1$, or more generally on any $\mathbb{P}^1_S$ by pullback. If $\mathscr{T}$ is a principal T-bundle, then the map $\tau \mapsto \mathscr{T}_\tau$ is W-equivariant with respect to the actions defined in (8) and (11).

A quasimap $(\mathbb{P}^1_S, \mathscr{P}_\tau, \sigma, id)$ may be described as follows. Due to the convention in (6), the fiber bundle $\mathscr{P}_\tau \times_G Z$ is given by gluing the trivial bundles $U_S \times Z$ and $V_S \times Z$ via

$$(u, z) \mapsto (\kappa(u), \tau(u)z) \quad \text{for} \quad u \in U_S \setminus \{S \times 0\}.$$

So σ is determined by the maps $\sigma_U : U_S \to Z$ and $\sigma_V : V_S \to Z$, where σ_U is the composition

$$U_S \xrightarrow{\sigma|_{U_S}} (\mathscr{P}_\tau \times_G Z)|_{U_S} = U_S \times Z \xrightarrow{pr_2} Z$$

and σ_V is defined similarly. Hence we have

$$\tau \cdot \sigma_U = \sigma_V \circ \kappa \qquad \text{on } U_S \setminus \{S \times 0\}, \tag{21}$$

and conversely a pair of morphisms $\sigma_U : U_S \to Z$ and $\sigma_V : V_S \to Z$ satisfying (21) define a section of $\mathscr{P}_\tau \times_G Z$. Two quasimaps $(\mathbb{P}^1_S, \mathscr{P}_\tau, \sigma, id)$ and $(\mathbb{P}^1_S, \mathscr{P}_\omega, \rho, id)$ are isomorphic if and only if there are functions $\phi_U : U_S \to G$ and $\phi_V : V_S \to G$ such that

$$\begin{aligned} (\phi_V \circ \kappa)\tau = \omega\phi_U & \qquad \text{as maps } U_S \setminus (S \times \{0\}) \to G \\ \phi_U \cdot \sigma_U = \rho_U & \qquad \text{as maps } U_S \to Z \\ \phi_V \cdot \sigma_V = \rho_V & \qquad \text{as maps } V_S \to Z. \end{aligned} \tag{22}$$

Remark 3.1.5. *If k is an algebraically closed field, then $U = V = \mathbb{A}^1_k$, and since every principal G-bundle is trivial on $\mathbb{A}^1_k$ we see that any k-quasimap is isomorphic to one of the form $(\mathbb{P}^1_k, \mathscr{P}_\tau, \sigma, id)$ for some transition function τ.*

3.1.3. *Class.* Let k be an algebraically closed field. The *class* of a quasimap $(\mathbb{P}^1_k, \mathscr{P}, \sigma, \mathbf{x})$ is the homomorphism $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ given by

$$\beta(\mathscr{L}) = \deg_C(\sigma^*(\mathscr{P} \times_G \mathscr{L})) \qquad \mathscr{L} \in \mathrm{Pic}^G(Z).$$

A family of quasimaps $(\mathbb{P}^1_S, \mathscr{P}, \sigma, \mathbf{x})$ has class β if each of its geometric fibers over S has class β.

Let $T \subset G$ be a maximal torus. From the morphisms $\chi(G) \to \chi(T)$ and $\mathrm{Pic}^G(Z) \to \mathrm{Pic}^T(Z)$ and the inclusion (3), we have the following diagram, crucial for understanding how class works in the abelian-nonabelian correspondence:

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z}) \arrow[r, "r_{\mathrm{Pic}}"] \arrow[d, "r_T"] & \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z}) \arrow[d, "r_G"] \\ \mathrm{Hom}(\chi(T), \mathbb{Z}) \arrow[r, "r_\chi"] & \mathrm{Hom}(\chi(G), \mathbb{Z})) \end{tikzcd} \tag{23}$$

The maps are all given by restriction of homomorphisms. The maps r_G and r_T are isomorphisms when Z is a vector space, but not in general.

Definition 3.1.6. *Recall the general quasimaps as defined in Definition 3.1.2, with nodal, marked, possibly disconnected source curves. The classes of all general quasimaps form a semigroup, called the θ-*effective classes *of (Z, G) (see* [14, Def 3.2.2]*).*

Because this paper focuses on computing small I-functions, it will be convenient to have a name for the subset of θ-effective classes that show up in the expansion of that power series. These are precisely the I-effective classes defined below. In contrast with the θ-effective classes, one can often compute the I-effective classes directly (see Section 6.5).

Definition 3.1.7. *The classes $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ that are realized as the class of some stable (graph) quasimap to $Z /\!\!/_\theta G$ are called the I-*effective classes *of (Z, G, θ).*

Remark 3.1.8. *The I-effective classes generate the sub-semigroup of θ-effective classes defined by source curves with genus 0. Indeed, let $(C, \mathscr{P}, \sigma, p_i)$ be a general quasimap of class β as in Definition 3.1.2, such that the genus of C is zero. Let $C_1, \ldots, C_N$ denote the irreducible components of C. Observe that by choosing any parametrization $\mathbf{x}_i$ of C_i and forgetting the markings, we get stable (graph) quasimaps $(C_i, \mathscr{P}|_{C_i}, \sigma_{C_i}, \mathbf{x}_i)$ to $Z /\!\!/_\theta G$, and the class β_i of $(C_i, \mathscr{P}|_{C_i}, \sigma_{C_i}, \mathbf{x}_i)$ does not depend on the choice of parametrization $\mathbf{x}_i$. Moreover, for $\mathscr{L} \in \mathrm{Pic}^G(Z)$, we have*

$$\beta(\mathscr{L}) = \deg_C(\sigma^*(\mathscr{P} \times_G \mathscr{L})) = \sum_{i=1}^N \deg_{C_i}(\sigma|_{C_i}^*(\mathscr{P}|_{C_i} \times_G \mathscr{L})) = \sum_{i=1}^N \beta_i(\mathscr{L}).$$

Hence the θ-effective class β is a sum of the I-effective classes β_i.

Let $QG_\beta(Z/\!\!/G)$ denote the groupoid of stable class-β quasimaps to $Z/\!\!/G$. We will denote it simply QG_β when the target $Z/\!\!/G$ is understood. The space QG_β is called a *quasimap graph space* in analogy with Gromov-Witten theory, and it is equal to the space $\mathrm{Qmap}_{0,0}(Z/\!\!/G, \beta; \mathbb{P}^1)$ from [14].

Theorem 3.1.9 ([14, Theorem 7.2.2])**.** *The moduli space $QG_\beta(Z/\!\!/G)$ is a separated Deligne-Mumford stack of finite type, proper over $\mathbf{S}_G$.*

The following lemma is essentially proved in the proof of [14, Theorem 7.2.2]. Since we will use the statement, we explicitly extract its proof.

Lemma 3.1.10. *When r_{Pic} is restricted to I-effective classes in both the source and target, it has finite fibers.*

Proof. By [14, Prop 2.5.2], for any Z there is a G-equivariant embedding $Z \subset V$ into a vector space V. The vector space V may not satisfy the assumption $V^s = V^{ss}$, but for $\tilde{\alpha} \in \mathrm{Hom}(\mathrm{Pic}^G(V), \mathbb{Z})$ one can still define the stack $QG_{\tilde{\alpha}}(V/\!\!/G)$, and it is shown in [14, Thm 3.2.5] that this stack is finite type. In fact, it is argued there that for classes that are effective for (V, G, θ), the map r_χ has finite fibers. Since we may also assume $Z^s \subset V^s$, it follows that r_χ has finite fibers for general Z.

Now, for $\tilde{\alpha} \in \mathrm{Hom}(\chi(T), \mathbb{Z})$ let $\mathfrak{X} = \bigsqcup_{\tilde{\beta} \in r_T{}^{-1}(\tilde{\alpha})} QG_{\tilde{\beta}}(Z/\!\!/T)$. Then $\mathfrak{X}$ is a closed substack of $QG_{\tilde{\alpha}}(V/\!\!/T)$, hence finite type. On the other hand, the components $QG_{\tilde{\beta}}(Z/\!\!/T)$ are open in $\mathfrak{X}$ because they are loci where (flat) families of line bundles have specified degrees. Since $\mathfrak{X}$ has finite type, this shows that r_T also has finite fibers in effective classes. $\square$

3.2. **Perfect obstruction theories on $QG_\beta(Z/\!\!/G)$.** To catalog our options for a perfect obstruction theory on $QG_\beta(Z/\!\!/G)$, we recall some general results about moduli of secions, using the language and notation of [9, Appendix A]. Let $\mathfrak{U}$ and Z be algebraic stacks with $C \to \mathfrak{U}$ a family of prestable curves and $Z \to C$ a morphism. The *moduli of sections* $\underline{\mathrm{Sec}}_{\mathfrak{U}}(Z/C)$ is the groupoid whose fiber over $S \to \mathfrak{U}$ is $\mathrm{Hom}_C(C \times_{\mathfrak{U}} S, Z)$. There is a universal curve $\pi : \mathfrak{C} \to \underline{\mathrm{Sec}}_{\mathfrak{U}}(Z/C)$ and a universal section $n : \mathfrak{C} \to Z$. If $\omega^\bullet_{\mathfrak{C}}$ is the relative dualizing complex of $\mathfrak{C} \to \underline{\mathrm{Sec}}_{\mathfrak{U}}(Z/C)$, there is a canonical morphism

$$\phi : R\pi_*(Ln^*\mathbb{L}_{Z/C} \otimes \omega^\bullet_{\mathfrak{C}}) \to \mathbb{L}_{\underline{\mathrm{Sec}}_{\mathfrak{U}}(Z/C)/\mathfrak{U}} \tag{24}$$

These objects have the following properties:

- If $Z \to \mathfrak{U}$ is locally of finite presentation, quasi-separated, and has affine stabilizers, then $\underline{\mathrm{Sec}}_{\mathfrak{U}}(Z/C)$ is an algebraic stack [31, Thm 1.3].
- If $\mathfrak{U}$ is quasi-separated and locally Noetherian, the morphism (24) is an obstruction theory [48, Thm 2.1.2].
- The morphism ϕ satisfies the same functoriality properties as the cotangent complex [9, Lem A.2.3, A.2.4].
- If a group acts on the input data $Z \to C \to \mathfrak{U}$, then $\underline{\mathrm{Sec}}_{\mathfrak{U}}(Z/C)$ has a group action and ϕ is equivariant [9, Sec A.3].

Example 1. *Let BG denote the global quotient $[pt/G]$. Then the stack $\underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times BG/\mathbb{P}^1)$ is the moduli of principal G-bundles on $\mathbb{P}^1$, which we will denote $\mathfrak{Bun}_G$.*

Let $QG(Z/\!\!/G)$ denote the stack of stable quasimaps to $Z/\!\!/G$ of any class. Following [14, Sec 7.2], we let $\mu : QG(Z/\!\!/G) \to \mathfrak{Bun}_G$ and $\nu : QG_\beta(Z/\!\!/G) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 1) = pt$ be the forgetful maps (note that in [14], the codomain of ν is not always a point). Let $\mathscr{P}$ denote the universal principal bundle on $\mathbb{P}^1_{\mathfrak{Bun}_G}$. The definition of quasimaps in Definition 3.1.1 realizes $QG(Z/\!\!/G)$ as an open substack of $\underline{\mathrm{Sec}}_{\mathfrak{Bun}_G}(\mathscr{P} \times_G Z/\mathbb{P}^1_{\mathfrak{Bun}_G})$, and hence defines a canonical μ-relative obstruction theory on $QG(Z/\!\!/G)$. As in [14, Sec 7.2] there is an associated ν-relative theory, which in our case is an absolute obstruction theory (since ν maps to a point), and it is defined via a mapping cone construction to have the same virtual cycle as the canonical μ-relative theory. It is proved in [14, Thm 7.1.6] that the μ- (and hence ν-) relative theories are perfect.

Remark 3.2.1. *In this paper we need an absolute obstruction theory on QG_β in order to compute the virtual fundamental classes of the fixed loci in the definition of the I-function (Section 3.3). The ν-relative theory is absolute, but because it is defined as a non-canonical mapping cone, we were not able to show that it satisfies the abelianization diagram* (54) *needed later.*

In light of Remark 3.2.1, we need to construct an absolute obstruction theory on $QG(Z/\!\!/G)$ that is a canonical obstruction theory on a moduli of sections, so that it will enjoy all the functoriality properties of those theories. To do so we use the tower

$$\mathbb{P}^1 \times [Z/G] \to \mathbb{P}^1 \times BG \to \mathbb{P}^1 \to pt. \tag{25}$$

From this we construct the stack $\underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1)$, which contains $QG(Z/\!\!/G)$ as an open substack by Lemma 3.1.4 and the discussion in Section 3.1.1. We take the induced absolute obstruction theory on $QG(Z/\!\!/G)$ as our chosen one in this paper; that is, we define

$$\phi : \mathbb{E}_{QG_\beta} := R\pi_*(n^*\mathbb{L}_{[Z/G]} \otimes \omega^\bullet) \to \mathbb{L}_{QG_\beta} \tag{26}$$

to be the chosen obstruction theory on QG_β, where ϕ is defined as in [9, (18)]. Note that $\pi : \mathbb{P}^1 \times QG_\beta(Z/\!\!/G) \to QG_\beta(Z/\!\!/G)$ is the trivial family. Also, since the cotangent complex $\mathbb{L}_{[Z/G]}$ is perfect, there is a canonical isomorphism

$$\mathbb{E}_{QG_\beta} \simeq (R\pi_* n^* \mathbb{T}_{[Z/G]})^\vee$$

given by [23, (4.1)] (it is an isomorphism by [23, Thm 4.4] and [48, Prop 2.2.6]).

A priori it is not clear that (26) defines a *perfect* obstruction theory, or that the resulting virtual cycle agrees with the one used in [14] (which is defined with the μ-relative theory). To compare the μ-relative theory and our chosen ϕ, first note that the projection $\mathbb{P}^1 \times [Z/G] \to \mathbb{P}^1 \times BG$ induces a morphism

$$\underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1) \to \mathfrak{B}un_G, \tag{27}$$

and by [9, Lem A.1.2] there is a canonical isomorphism

$$\underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1) \simeq \underline{\mathrm{Sec}}_{\mathfrak{B}un_G}(\mathscr{P} \times_G Z/\mathbb{P}^1_{\mathfrak{B}un_G})$$

compatible with the open embeddings of $QG(Z/\!\!/G)$. Hence we may compare (26) and the μ-relative theory on the same stack.

Lemma 3.2.2. *The arrow* (26) *is an absolute perfect obstruction theory on QG_β inducing the same virtual cycle as the one used in* [14].

Proof. The argument in [9, Sec A.2.3] shows that (26) is in fact compatible with the μ-relative theory in [14], implying that ϕ is a *perfect* obstruction theory and that it induces the same virtual cycle as the μ- and ν-relative theories in [14]. □

3.3. **I-function.** Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by

$$\lambda \cdot [u : v] = [\lambda u : v], \qquad \lambda \in \mathbb{C}^*. \tag{28}$$

This induces an action on QG_β via

$$\lambda \cdot (C, \mathscr{P}, \sigma, \mathbf{x}) = (C, \mathscr{P}, \sigma, \lambda \circ \mathbf{x}). \tag{29}$$

Every quasimap is isomorphic to one with $\mathbf{x} = id$. For these quasimaps, the action in (29) is equivalent to setting

$$\lambda \cdot (\mathbb{P}^1_S, \mathscr{P}, \sigma, id) = (\mathbb{P}^1_S, (\lambda^{-1})^* \mathscr{P}, \sigma \circ \lambda^{-1}, id). \tag{30}$$

In terms of the moduli of sections, the action described on QG_β comes from the $\mathbb{C}^*$-equivariant structure on the tower of morphisms (25) given by letting $\mathbb{C}^*$ act on $\mathbb{P}^1$ via (28). By [9, Sec A.3] this equivariant tower induces $\mathbb{C}^*$-actions on QG_β and C_{QG_β} making π and n equivariant. It also induces a canonical $\mathbb{C}^*$-equivariant structure on the perfect obstruction theory (26).

We define the *fixed locus* of QG_β under the $\mathbb{C}^*$-action as in [8, Sec 3]. Its closed points are geometric quasimaps $(\mathbb{P}^1_k, \mathscr{P}, \sigma, \mathbf{x})$ such that $\lambda \cdot (\mathbb{P}^1_k, \mathscr{P}, \sigma, \mathbf{x})$ is isomorphic to $(\mathbb{P}^1_k, \mathscr{P}, \sigma, \mathbf{x})$ for every $\lambda \in \mathbb{C}^*$ (see eg [1, Prop 5.23]). The I-function of $Z/\!\!/G$ is defined in terms of localization residues at certain fixed loci (see [14, Sec 7.3]).

Lemma 3.3.1. *A fixed quasimap has its base locus supported on $\{[1 : 0] \cup [0 : 1]\} \times S$, and the resulting map*

$$\mathbb{P}^1_S \setminus (\{[1 : 0] \cup [0 : 1]\} \times S) \to Z/\!\!/G \tag{31}$$

factors through the projection $\mathbb{P}^1_S \to S$.

Proof. Let $(\mathbb{P}^1_S, \mathscr{P}, \sigma, \mathbf{x})$ be a fixed quasimap. Its base locus may be computed on geometric fibers, and is contained in the set specified by [49, Stack Example 7.2]. On the other hand, because it is $\mathbb{C}^*$-fixed, the map $\mathbb{P}^1_S \to [Z/G]$ factors through $\mathbb{P}^1_S \to [\mathbb{P}^1_S/\mathbb{C}^*]$. Removing the set $\{[1 : 0] \cup [0 : 1]\} \times S$ we get the desired factorization. □

Let $F_\beta(Z/\!\!/G)$ denote the component of the fixed locus of $QG_\beta(Z/\!\!/G)$ corresponding to quasimaps that have a unique basepoint at $[0:1]$ (it may not be connected). We will omit the space $Z/\!\!/G$ from the notation when there is no danger of confusion. Let $ev_\bullet : F_\beta \to Z/\!\!/G$ send a quasimap over S to the morphism $S \to Z/\!\!/G$ through which (31) factors. Then we can define the I-function of $Z/\!\!/_\theta G$ as a formal power series in the q-adic completion of the semigroup ring generated by the semigroup of θ-effective classes.

Definition 3.3.2. *The (small) I-function of $Z/\!\!/_\theta G$ is*

$$I^{Z/\!\!/G}(z) = 1 + \sum_{\beta\neq 0} q^\beta I_\beta^{Z/\!\!/G}(z) \qquad \textit{where} \qquad I_\beta^{Z/\!\!/G}(z) = (ev_\bullet)_* \left(\frac{[F_\beta(Z/\!\!/G)]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{F_\beta(Z/\!\!/G)})} \right), \tag{32}$$

and the sum is over all I-effective classes of (Z, G, θ). Here, $ev_\bullet$ is proper pushforward for Chow groups and $I_\beta^{Z/\!\!/G}(z)$ is an element of $A_*(Z/\!\!/G) \otimes_{\mathbb{Z}} \mathbb{Q}[z, z^{-1}]$.

Remark 3.3.3. *In* (32) *it is equivalent to sum over all θ-effective classes.*

Remark 3.3.4 (Comparison with the cohomology-valued I-function)**.** *As explained in Remark 1.1.2, we use the cycle map (see* [24, Sec 19.1]

$$cl : A_*(Z/\!\!/G) \to H_*(Z/\!\!/G)$$

and the Poincaré duality map for the smooth $2n$-dimensional manifold $Z/\!\!/G$ (see [24, Sec 19.1(3)]*)*

$$P : H^{2n-i}(Z/\!\!/G) \xrightarrow[\sim]{\cap[Z/\!\!/G]} H_i(Z/\!\!/G)$$

to define a cohomology-valued I-function $P^{-1}cl(I^{Z/\!\!/G}(z))$ where the morphisms P and cl are extended $\mathbb{Q}$-linearly to series in q and z.

To prove the main result (5) *for the cohomology-valued I-functions, we apply $P^{-1}cl$ to* (5)*, using the fact that for any smooth morphism $f : X \to Y$ of schemes we have $P^{-1}cl f^* = f^*P^{-1}cl$. To see that this equality holds, first note that the Gysin map $f^* : H_*(Y) \to H_*(X)$ for Borel-Moore homology is defined and commutes with cl by* [24, Example 19.2.1]*. In fact, the singular cohomology of schemes forms a bivariant theory with Borel-Moore homology for the singular classes. Hence, the final formula $f^*P = Pf^*$ is* [25, 2.5(G_4.iii)]*, noting that f has even (real) relative dimension as a map of complex schemes so the sign is $+1$.*

Finally, we note that it would be equivalent to define the cohomology-valued I-function to have coefficients $P^{-1}[(ev_\bullet)_*(cl([F_\beta(Z/\!\!/G)]^{vir})/e_{\mathbb{C}^*}(N^{vir}_{F_\beta(Z/\!\!/G)}))]$, where $(ev_\bullet)_*$ is now proper pushforward for Borel-Moore homology. Equivalence holds because cl commutes with proper pushforward ([24, 372]*) and Euler classes (*[24, Prop 19.1.2]*). This definition is the one we must take if we want to include cohomology-valued insertions in the I-function, as we will in Section 6.3.*

4. Relate the fixed loci and evaluation maps

The goal of this section is to "pull back" diagram (2) to the $\mathbb{C}^*$-fixed loci in the quasimap moduli spaces. That is, we prove the following.

Proposition 4.0.1. *Let $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ be I-effective. For every $\tilde\beta \in r_{\mathrm{Pic}}^{-1}(\beta)$, the map $ev_\bullet : F_{\tilde\beta}(Z/\!\!/T) \to Z^s(T)/T$ is a closed embedding, and there is*

- *a parabolic subgroup $P_{r_T(\tilde\beta)} \subset G$, and*
- *a morphism $\psi_{\tilde\beta} : F_{\tilde\beta}(Z/\!\!/T) \cap Z^s(G) \to F_\beta(Z/\!\!/G)$ whose image we denote $F_{\tilde\beta}(Z/\!\!/G)$,*

fitting into the following commutative diagram:

$$
\begin{tikzcd}
F_{\tilde\beta}(Z/\!\!/G) \arrow[d, hook, "i"] & F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G) \arrow[l, "\psi_{\tilde\beta}"'] \arrow[r, hook, "h"] \arrow[d, hook] & F_{\tilde\beta}(Z/\!\!/T) \arrow[d, hook, "ev_\bullet"] \\
Z^s(G)/P_{r_T(\tilde\beta)} \arrow[dr, "f"] & Z^s(G)/T \arrow[l, "p"] \arrow[r, hook, "j"] \arrow[d, "g"] & Z^s(T)/T \\
 & Z^s(G)/G &
\end{tikzcd}
\tag{33}
$$

Here, the two squares are fibered and the composition $f\circ i$ is the evaluation map $ev_\bullet$.

4.1. **Preliminaries, including definition of $P_{r_T(\tilde\beta)}$.** We may identify cocharacters with dual characters of T as follows. A dual character $\tilde\alpha\in\mathrm{Hom}(\chi(T),\mathbb{Z})$ determines a cocharacter $\tau_{\tilde\alpha}$ via the rule

$$\xi(\tau_{\tilde\alpha}(t)) = t^{-\tilde\alpha(\xi)} \qquad \text{for any} \qquad \xi\in\chi(T). \tag{34}$$

The negative sign in the exponential appears so that for $\xi\in\chi(T)$ we have

$$\deg_{\mathbb{P}^1}(\mathscr{T}_{\tau_{\tilde\alpha}}\times_T\mathbb{C}_\xi)=\tilde\alpha(\xi)$$

(so in particular, if Z is a vector space, a quasimap to $Z/\!\!/T$ with principal bundle $\mathscr{T}_{\tau_{\tilde\alpha}}$ has class $\tilde\alpha$). One can check that this identification of cocharacters and dual characters is W-equivariant under the actions defined in (8). To lighten the notation we will write $\mathscr{T}_{\tilde\alpha}$ for $\mathscr{T}_{\tau_{\tilde\alpha}}$ and $\mathscr{P}_{\tilde\alpha}$ for the associated principal G-bundle (which is equal to $\mathscr{P}_{\tau_{\tilde\alpha}}$).

Remark 4.1.1. *Let k be an algebraically closed field. By the classification of principal bundles on $\mathbb{P}^1_k$, every k-point of QG_β is represented by a quasimap of the form $(\mathbb{P}^1_k,\mathscr{P}_{\tilde\alpha},\sigma,id)$ where $r_\chi(\tilde\alpha)=r_G(\beta)$ (see Theorem 2.3.1 and Remark 2.3.2).*

The construction of the parabolic subgroup $P_{r_T(\tilde\beta)}$ uses the "dynamic method" (see e.g. [18, Sec 2.1]). If $\tilde\alpha=r_T(\tilde\beta)$ is a dual character and $\tau_{\tilde\alpha}$ the cocharacter defined in (34), then the dynamic method defines a parabolic subgroup whose points over a G-scheme S are

$$P_{\tilde\alpha}(S)=\{g\in G(S)\mid \lim_{t\to0}\tau_{\tilde\alpha}(t)^{-1}g\tau_{\tilde\alpha}(t)\text{ exists in }G\}. \tag{35}$$

By "the limit exists in G" we mean that there is a dotted arrow making the following diagram of S-schemes commute:

$$
\begin{tikzcd}
\mathbb{C}^*_S \arrow[r, "\tau_{\tilde\alpha}(t)^{-1}g\tau_{\tilde\alpha}(t)"] \arrow[d, hook] & G_S \\
\mathbb{A}^1_S \arrow[ur, dashed] &
\end{tikzcd}
\tag{36}
$$

By considering the case when S is affine (so all schemes in the diagram are also affine), one sees that the dotted arrow is unique if it exists. The subgroup $P_{\tilde\alpha}$ clearly contains T.

Remark 4.1.2. *The group $P_{\tilde\alpha}$ has a natural inclusion into $\mathrm{Aut}(\mathscr{P}_{\tilde\alpha})\subset\mathrm{Aut}(\mathscr{P}_{\tilde\alpha}\times_G Z)$, given by sending $g\in P_{\tilde\alpha}$ to the automorphism defined as in (22) by setting $\phi_V(v)=g$ and setting ϕ_U to be the unique dotted arrow in (36).*

The dynamic method also produces a canonical Levi subgroup $L_{\tilde\alpha}\subset P_{\tilde\alpha}$, equal to the centralizer of $\tau_{\tilde\alpha}$:

$$L_{\tilde\alpha}=\{g\in G\mid \tau_{\tilde\alpha}(t)^{-1}g\tau_{\tilde\alpha}(t)=g\}$$

In fact this is the unique Levi sugbroup of $P_{\tilde\alpha}$ containing T (see [17, Prop 12.3.1]).

We close this section with some properties of $F_\beta(Z/\!\!/G)$.

Lemma 4.1.3. *The stack $F_\beta(Z/\!\!/G)$ is represented by a separated algebraic space, and it is proper over $\mathbf{S}_G$.*

Proof. From the definition of torus fixed loci in [8, Sec 3] we see that $F_\beta(Z/\!\!/G)$ is a closed substack of QG_β, hence proper and separated by Theorem 3.1.9.

To see that its automorphism groups are trivial, let $(\mathbb{P}^1_S, \mathscr{P}, \sigma, id)$ be a quasimap in F_β over a scheme S, and let ϕ be an automorphism of it, i.e., ϕ is an automorphism of $\mathscr{P}$ such that the induced automorphism of $\mathscr{P} \times_G Z$ fixes σ. If $U \to \mathbb{P}^1_S$ is an étale chart where $\mathscr{P}$ is trivial, then σ is given by a map $\sigma_U : U \to Z$ and ϕ is given by $\phi_U : U \to G$, and these data satisfy

$$\phi_U(u)\sigma_U(u) = \sigma_U(u)$$

for each $u \in U$. This means $\phi_U(u)$ is in the stabilizer $G_{\sigma_U(u)}$. Because the quasimap is stable, the group $G_{\sigma_U(u)}$ is trivial on an open subset of U. Hence ϕ_U is the identity, and ϕ is trivial. □

4.2. **Abelian case.** The goal of this section is to prove the following.

Lemma 4.2.1. *The map $ev_\bullet : F_{\tilde\beta}(Z/\!\!/T) \to Z^s(T)/T$ is a closed embedding.*

At the end of the section, we use Lemma 4.2.1 to describe the universal family on $F_{\tilde\beta}(Z/\!\!/T)$ (Proposition 4.2.6). The results in this section are related to those in [10, Sec 5.2]. We begin with three lemmas, the first two of which are probably standard.

Lemma 4.2.2. *Let $S \hookrightarrow S'$ be a square-zero extension of schemes. If $\mathscr{P}_1$ and $\mathscr{P}_2$ are two principal G-bundles on $\mathbb{P}^1_{S'}$ such that $\mathscr{P}_1|_{\mathbb{P}^1_S} \simeq \mathscr{P}_2|_{\mathbb{P}^1_S}$, then $\mathscr{P}_1 \simeq \mathscr{P}_2$.*

Proof. This follows from [42, Thm 1.5] and the fact that $\mathbb{L}_{[\bullet/G]}$ is represented by a vector bundle in degree 1. □

Lemma 4.2.3. *Let X, Y be algebraic spaces over $\mathbb{C}$, locally of finite type. Let $\pi : X \to Y$ be a separated morphism that is injective on $\mathbb{C}$-points. Then π is universally injective.*

Proof. We show that the diagonal $\Delta_\pi : X \to X \times_Y X$ is surjective. Because π is separated, Δ_π is closed, so the complement of the image $|\Delta_\pi|^C$ is an open subset of $|X \times_Y X|$ and by [44, Tag 06G2] it contains a $\mathbb{C}$-point if it is nonempty. So it suffices to show that Δ_π is surjective on $\mathbb{C}$-points. But if $(x_1, x_2) \in (X \times_Y X)(\mathbb{C})$, then $\pi(X_1) = \pi(x_2)$ so $x_1 = x_2$. So $(x_1, x_2) = \Delta_\pi(x_1, x_1)$ as desired. □

We will use the description of a quasimap as a tuple $(\mathbb{P}^1_S, \mathscr{T}, \tilde n, id)$ where $\tilde n : \mathscr{T} \to Z$ is a T-equivariant map. Let $\star = [1 : 1]$ in $U \cap V \subset \mathbb{P}^1_{\mathbb{C}}$ and let $\iota_\star : S \to \mathbb{P}^1_S$ be the section with constant image $\star$, with $\mathscr{T}_S := \iota_\star^* \mathscr{T}$. Then $ev_\bullet$ is represented by the map $ev_\bullet : F_{\tilde\beta}(Z/\!\!/T) \to [Z^s(T)/T]$ that sends q_i to the map $S \to [Z^s(T)/T]$ given by

$$\mathscr{T}_S \to \mathscr{T} \xrightarrow{\tilde n} Z. \tag{37}$$

Lemma 4.2.4. *If $q = (\mathbb{P}^1_S, \mathscr{T}, \sigma, id)$ is an S-point of $F_{\tilde\beta}(Z/\!\!/T)$, then we have a commuting diagram as below, with the square fibered:*

$$\begin{tikzcd} \mathscr{T}|_{V\times S} \arrow[r] \arrow[d] \arrow[rr, bend left, "\tilde n"] & \mathscr{T}_S \arrow[r, "(37)"'] \arrow[d] & Z \\ V \times S \arrow[r] & S & \end{tikzcd}$$

Proof. If we replace V with $\mathbb{C}^* \subset V$, this is just a restatement of the fact that the morphism $n : \mathbb{P}^1_S \to [Z/T]$ defined by q factors through the quotient $\mathbb{P}^1_S \to [\mathbb{P}^1_S/\mathbb{C}^*]$, and hence $n|_{\mathbb{C}^*_S}$ is equivalent (2-isomorphic) to $\mathbb{C}^*_S \to S \xrightarrow{n|_\star} [Z/T]$. Hence n and $V_S \to S \xrightarrow{n|_\star} [Z/T]$ agree on the open subset $\mathbb{C}^*_S \subset V_S$, so since $n|_{V_S}$ factors through the separated substack $Z/\!\!/T \subset [Z/T]$, they agree on all of V_S. This translates to the desired diagram. □

Lemma 4.2.5. *Let $q_i = (\mathbb{P}^1_S, \mathscr{T}, \sigma_i, id)$ for $i = 1, 2$ be two quasimaps in $F_{\tilde\beta}(Z/\!\!/T)$ with the same base S and principal T-bundle $\mathscr{T}$. If $ev_\bullet(q_1) = ev_\bullet(q_2)$, then $q_1 \simeq q_2$.*

Proof. Suppose we have two quasimaps $q_i = (\mathbb{P}^1_S, \mathscr{T}, \tilde{n}_i, id)$ for $i = 1, 2$ with $ev_\bullet(q_1) = ev_\bullet(q_2)$. Then we have an automorphism of $\mathscr{T}_S$ sending $\tilde{n}_1|_{\mathscr{T}_S}$ to $\tilde{n}_2|_{\mathscr{T}_S}$. Since T is abelian, this automorphism is given by a morphism $\phi : S \to T$ (for example, because the adjoint bundle in [2, Prop 2.11] is trivial). Then the composition $\mathbb{P}^1_S \to S \to T$ defines an element of $T(\mathbb{P}^1_S)$, hence an automorphism Φ of $\mathscr{T}$. It remains to check that $\tilde{n}_1 \circ \Phi$ and $\tilde{n}_2$ define the same map from $\mathscr{T}$ to Z, given that their restrictions to $\mathscr{T}_S$ agree. Lemma 4.2.4 implies that $\tilde{n}_1 \circ \Phi$ and $\tilde{n}_2$ agree on the open subset $\mathscr{T}|_{V \times S} \subset \mathscr{T}$. Since Z is separated and the complement of $\mathscr{T}|_{V\times S}$ in $\mathscr{T}$ is an effective Cartier divisor, they agree on all of $\mathscr{T}$ (see [47, 10.2.G]). $\square$

Proof of Lemma 4.2.1. By Lemma 4.1.3, $F_{\tilde{\beta}}(Z/\!\!/T)$ is an algebraic space, so we can use [44, Tag 05W8]: it is enough to show that $ev_\bullet$ is proper, formally unramified, and universally injective.

The map $ev_\bullet$ is proper because $F_{\tilde{\beta}}(Z/\!\!/T)$ is proper over $\mathbf{S}_T$ and $Z/\!\!/T$ is separated. It is universally injective by Remark 4.1.1, Lemma 4.2.5, and Lemma 4.2.3. Finally, to check that it is formally unramified, let $S \hookrightarrow S'$ be a square-zero extension of schemes fitting into a solid diagram

$$\begin{tikzcd} S \arrow[r, "q"] \arrow[d, hook] & F_{\tilde{\beta}}(Z/\!\!/T) \arrow[d, "ev_\bullet"] \\ S' \arrow[r] \arrow[ur, dashed] & Z/\!\!/T \end{tikzcd}$$

Suppose we have two dotted arrows q_1, q_2 making the diagram above commute. The arrows q_1, and q_2 define principal T-bundles $\mathbb{P}^1_{S'}$ that agree after restriction to $\mathbb{P}^1_S$, so by Lemma 4.2.2 the two principal bundles on $\mathbb{P}^1_{S'}$ are isomorphic. Now Lemma 4.2.5 shows $q_1 = q_2$. $\square$

Define

$$Z_{\tilde{\beta}} = Z^s(T) \times_{Z/\!\!/T} F_{\tilde{\beta}}(Z/\!\!/T). \tag{38}$$

Observe that $Z_{\tilde{\beta}} \to F_{\tilde{\beta}}(Z/\!\!/T)$ is a principal T-bundle (meaning it is represented by such) with a T-equivariant map to $Z^s(T)$ (this map is a closed embedding by Lemma 4.2.1), and in fact this data defines the evaluation morphism $ev_\bullet : F_{\tilde{\beta}}(Z/\!\!/T) \to [Z^s(T)/T]$ (see (37)). Let

$$\tilde{\alpha} = r_T(\tilde{\beta}).$$

Proposition 4.2.6. *The universal family on $F_{\tilde{\beta}}(Z/\!\!/T)$ has fiber bundle $\mathcal{Z}$ on $\mathbb{P}^1_{F_{\tilde{\beta}}(Z/\!\!/T)}$ and section $\mathcal{S}$ defined as follows:*

$$\mathcal{Z} = \frac{Z_{\tilde{\beta}} \times (\mathbb{C}^2 \setminus \{0\}) \times Z}{(x, \mathbf{u}, y) \sim (tx, s\mathbf{u}, \tau_{\tilde{\alpha}}(s)^{-1}ty)} \qquad \mathcal{S}(x, \mathbf{u}) = (x, \mathbf{u}, \tau_{\tilde{\alpha}}(u)^{-1}x) \tag{39}$$

where $(x, \mathbf{u}, y) \in Z_{\tilde{\beta}} \times (\mathbb{C}^2 \setminus \{0\}) \times Z$ with $\mathbf{u} = (u, v)$ and $(t, s) \in T \times \mathbb{C}^*$.

Remark 4.2.7. *The section $\mathcal{S}$ is a priori defined only for $u \neq 0$, but we will see in the proof of Proposition 4.2.6 that it has a unique extension over all of $\mathbb{P}^1_{F_{\tilde{\beta}}(Z/\!\!/T)}$.*

Remark 4.2.8. *We will often use the tautological family on $Z_{\tilde{\beta}}$ that is the pullback of* (39)*. It is given by the same formulas as in* (39) *but without dividing by the T-action. The benefit of studying this family on $Z_{\tilde{\beta}}$ is that it is of the form in Section 3.1.2: its underlying principal bundle is $\mathscr{T}_{\tilde{\alpha}}$, as can be shown by computing its transition function, and we have*

$$\begin{aligned} \mathcal{S}_U : \mathbb{A}^1 \times Z_{\tilde{\beta}} &\to Z \\ (u, z) &\mapsto \tau_{\tilde{\alpha}}(u)^{-1}z \end{aligned} \qquad\qquad \begin{aligned} \mathcal{S}_V : \mathbb{A}^1 \times Z_{\tilde{\beta}} &\to Z \\ (v, z) &\mapsto z \end{aligned}$$

where $\mathcal{S}_U$ is defined as explained in Remark 4.2.7.

Proof of Proposition 4.2.6. We show that the tautological family on $Z_{\tilde{\beta}}$ is as described in Remark 4.2.8. Then (39) defines a family of fixed quasimaps on $F_{\tilde{\beta}}(Z/\!\!/T)$ that is sent by $ev_\bullet$ to the inclusion $[Z_{\tilde{\beta}}/T] \subset [Z^s(T)/T]$. Since $ev_\bullet$ is a closed embedding, (39) must be the universal family.

Let $(\mathbb{P}^1_{Z_{\tilde{\beta}}}, \mathscr{T}, \mathcal{S}, id)$ be the tautological family. We first show that $\mathscr{T}$ is isomorphic to $\mathscr{T}_{\tilde{\alpha}}$. Using the description of the evaluation map $ev_\bullet$ in the proof of Lemma 4.2.5, we see that $\mathscr{T}_{Z_{\tilde{\beta}}} = \iota_\star^* \mathscr{T}$ is canonically isomorphic to $Z_{\tilde{\beta}} \times_{F_{\tilde{\beta}}(Z/\!\!/T)} Z_{\tilde{\beta}}$, with the map to the base $Z_{\tilde{\beta}}$ equal to one of the projections. This principal T-bundle is trivial (it has the diagonal section). From Lemma 4.2.4 we see that $\mathscr{T}|_{\mathbb{C}^* \times Z_{\tilde{\beta}}}$ is also trivial.

We show that $\mathscr{T}|_{U \times Z_{\tilde{\beta}}}$ and $\mathscr{T}|_{V \times Z_{\tilde{\beta}}}$ are trivial. Let $\{S_i\} \to Z_{\tilde{\beta}}$ be an affine open cover. If S_i^{red} is the reduced subscheme of S_i, then since S_i is Noetherian, the containment $S_i^{red} \subset S_i$ may be factored as a finite sequence of square-zero extensions. So by Lemma 4.2.2 the restriction map $\operatorname{Pic}(S_i \times U) \to \operatorname{Pic}(S_i^{red} \times U)$ is injectve. Since $S_i^{red} \times U$ is a reduced Noetherian affine scheme, by [34] the restriction map $\operatorname{Pic}(S_i^{red} \times U) \to \operatorname{Pic}(S_i^{red} \times \mathbb{C}^*)$ is also injective. This implies that the restriction $\operatorname{Pic}(S_i \times U) \to \operatorname{Pic}(S_i \times \mathbb{C}^*)$ is injective. Since $\mathscr{T}|_{\mathbb{C}^* \times S_i}$ is trivial, this implies $\mathscr{T}|_{U \times S_i}$ is trivial. On the other hand, the transition function for $\mathscr{T}$ on $\mathbb{C}^* \times (S_i \cap S_j)$ is constant and equal to the identity, since $\mathscr{T}_{\mathbb{C}^* \times Z_{\tilde{\beta}}}$ is trivial. So the transition function for $\mathscr{T}$ on each $U \times (S_i \cap S_j)$ is trivial, and we conclude that $\mathscr{T}|_{U \times Z_{\tilde{\beta}}}$ is trivial. Likewise $\mathscr{T}|_{V \times Z_{\tilde{\beta}}}$ is trivial.

Finally we compute the transition function $\tau : \mathbb{C}^*_{Z_{\tilde{\beta}}} \to T$ satisfying (20) for $\mathscr{T}$. The morphism τ is given by a ring map $\Gamma(T, \mathscr{O}_T) \to \Gamma(\mathbb{C}^*_{Z_{\tilde{\beta}}}, \mathscr{O}_{\mathbb{C}^* \times Z_{\tilde{\beta}}})$. If we choose a basis $\xi_1, \dots, \xi_N$ of characters of T, then τ is determined by a collection $p_1, \dots, p_N$ of invertible elements of $\Gamma(Z_{\tilde{\beta}}, \mathscr{O}_{Z_{\tilde{\beta}}})[u, u^{-1}]$ (p_j is the image of ξ_j). Since geometric fibers of $\mathscr{T}$ have class $\tilde{\alpha}$, the restriction of p_j to every geometric fiber is a monomial of degree $-\tilde{\alpha}(\xi_j)$. So the restriction of p_j to $Z_{\tilde{\beta}}^{red}$ is also a monomial of degree $-\tilde{\alpha}(\xi_j)$. Changing the trivialization on $V_{Z_{\tilde{\beta}}}$ by the appropriate element of T and recalling the relationship (34), we can assume $\tau|_{Z_{\tilde{\beta}}^{red}} = \tau_{\tilde{\alpha}}$. By Lemma 4.2.2 the bundle $\mathscr{T}|_{\mathbb{P}^1_{Z_{\tilde{\beta}}}}$ is also isomorphic to $\mathscr{T}_{\tilde{\alpha}}$ as claimed.

The second step is to show that the tautological section $\mathcal{S}$ is given by the formulae for $\mathcal{S}_U$ and $\mathcal{S}_V$ in Remark 4.2.8. Because evaluation is tautological, $\mathcal{S}_V$ sends $(1, z)$ to z. By Lemma 4.2.4, the function σ_V is pulled back from this fiber. A priori the pullback map is not unique, and hence σ_V may not be completely determined; however, any two choices for σ_V would differ by an element of $T(V_{Z_{\tilde{\beta}}})$, which is given by a collection of invertible elements in $\Gamma(Z_{\tilde{\beta}}, \mathscr{O}_{Z_{\tilde{\beta}}})[v]$. Since these are constant with respect to v, we see that the only option for $\mathcal{S}_V$ is the map $\mathcal{S}_V(v, z) = z$. Then by (21) we see that $\sigma_U(u, z) = \tau_{\tilde{\alpha}}(u)^{-1} z$ for $u \neq 0$, and in particular this map has an extension to all of $U_{Z_{\tilde{\beta}}}$. Uniqueness of the extension may be checked affine locally on $Z_{\tilde{\beta}}$, since the ring map $\Gamma(U \times \operatorname{Spec}(A), \mathscr{O}_{U \times \operatorname{Spec}(A)}) \to \Gamma(\mathbb{C}^* \times \operatorname{Spec}(A), \mathscr{O}_{\mathbb{C}^* \times \operatorname{Spec}(A)})$ is injective. □

4.3. **Proof of Proposition 4.0.1.** Let

$$F^0_{\tilde{\beta}}(Z/\!\!/T) := \Big(F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G)/T\Big) \subset Z^s(T)/T, \tag{40}$$
$$Z^0_{\tilde{\beta}} := \Big(Z_{\tilde{\beta}} \cap Z^s(G)\Big) \subset Z^s(T)$$

so $F^0_{\tilde{\beta}}(Z/\!\!/T)$ is the open substack of $F_{\tilde{\beta}}(Z/\!\!/T)$ where $ev_\bullet$ lands in $Z^s(G)/T$ and $Z^0_{\tilde{\beta}}$ is the natural T-torsor on it. As a guide to keeping track of the spaces in (40), the reader may like to consult the diagram in Section 1.3.

Lemma 4.3.1. *The subscheme $Z^0_{\tilde{\beta}} \subset Z^s(G)$ is invariant under the action of $P_{\tilde{\alpha}} \subset G$ on $Z^s(G)$.*

Proof. By the definition of the semi-stable locus, there is a cover of $Z^s(G)$ by G-invariant affine subschemes. Passing to this cover, we may assume $Z^s(G)$ is affine, and hence it suffices to check that for $p : Z^0_{\tilde\beta} \to P_{\tilde\alpha}$ we have $pZ^0_{\tilde\beta} \subset Z^0_{\tilde\beta}$ (i.e., global sections are invariant). Since the entire set $P_{\tilde\alpha} Z^0_{\tilde\beta}$ is T-invariant, if we let $(pZ^0_{\tilde\beta})/T$ denote the quotient of the T-orbits meeting $pZ^0_{\tilde\beta}$, it is in fact sufficient to show that $(pZ^0_{\tilde\beta})/T \subset Z^0_{\tilde\beta}/T = F^0_{\tilde\beta}(Z/\!\!/T)$.

To do this let $(\mathbb{P}^1_{Z^0_{\tilde\beta}}, \mathscr{T}_{\tilde\alpha}, \mathcal{S}, id)$ be the tautological family on $Z^0_{\tilde\beta}$ defined by (39), and observe that $\mathscr{T}_{\tilde\alpha} \times_T Z = \mathscr{P}_{\tilde\alpha} \times_G Z$. Let $\wp \in \mathrm{Aut}(\mathscr{T}_{\tilde\alpha} \times_T Z)$ be the automorphism defined by p as in Remark 4.1.2. We claim that

$$\mathcal{F} = (\mathbb{P}^1_{Z^0_{\tilde\beta}}, \mathscr{T}_{\tilde\alpha}, \wp \circ \mathcal{S}, id) \tag{41}$$

is another family of $\mathbb{C}^*$-fixed quasimaps on $Z^0_{\tilde\beta}$ of class $\tilde\beta$.[1] Granting this, its evaluation map $ev_{\bullet,\mathcal{F}} : Z^0_{\tilde\beta} \to Z/\!\!/T$ must factor through the universal one, namely the inclusion $F_{\tilde\beta}(Z/\!\!/T) \subset Z/\!\!/T$. But by construction the image of $ev_{\bullet,\mathcal{F}}$ is precisely $pZ^0_{\tilde\beta}/T$ (note that $\mathcal{S}_V : Z^0_{\tilde\beta} \times \{u \neq 0\} \to Z^0_{\tilde\beta}$ is the projection). So we have $(pZ^0_{\tilde\beta})/T \subset Z_{\tilde\beta}/T$. Since the image of $ev_{\bullet,\mathcal{F}}$ is also contained in $Z^s(G)/T$, we have $(pZ^0_{\tilde\beta})/T \subset Z^0_{\tilde\beta}/T$ as desired.

That the family $\mathcal{F}$ is $\mathbb{C}^*$-fixed follows from the definition of the $\mathbb{C}^*$-action (29). To see that geometric fibers have class $\tilde\beta$, let k be an algebraically closed field and let $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha}, \sigma, id)$ be a fiber of the tautological family over a k-point of $Z^0_{\tilde\beta}$. The fiber of (41) over the same point is $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha}, \wp \circ \sigma, id)$. Then as a quasimap to $Z/\!\!/T$, the class of $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha}, \wp\circ\sigma, id)$ is the homomorphism that sends $\mathscr{L} \in \mathrm{Pic}^T(Z)$ to

$$\deg_{\mathbb{P}^1}((\wp \circ \sigma)^*(\mathscr{T} \times_T \mathscr{L})).$$

Because $P_{\tilde\alpha}$ is a connected subgroup of $\mathrm{Aut}(\mathscr{T}_{\tilde\alpha} \times_T Z)$, there is a (piecewise linear) homotopy from the automorphism $\wp$ to the identity on $\mathscr{T}_{\tilde\alpha} \times_T Z$. In particular the images of σ and $\wp \circ \sigma$ are rationally equivalent, hence the degree of $\mathscr{T}_{\tilde\alpha} \times_T \mathscr{L}$ along these two rational curves is the same.

□

Let $\beta = r_{\mathrm{Pic}}(\tilde\beta)$. We define

$$\begin{aligned} \psi_{\tilde\beta} : F^0_{\tilde\beta}(Z/\!\!/T) &\to F_\beta(Z/\!\!/G) \\ (C, \mathscr{T}, \sigma, \mathbf{x}) &\mapsto (C, G \times_T \mathscr{T}, \sigma, \mathbf{x}), \end{aligned} \tag{42}$$

noting that σ is a section of $\mathscr{T} \times_T Z = (G \times_T \mathscr{T}) \times_G Z$. A priori $\psi_{\tilde\beta}$ is a map to $QG_\beta(Z/\!\!/G)$; it is straightforward to check that it factors through $F_\beta(Z/\!\!/G)$. One uses the fact that isomorphisms of principal T-bundles induce isomorphisms of associated G-bundles.

Lemma 4.3.2. *The composition*

$$Z^0_{\tilde\beta} \to F^0_{\tilde\beta}(Z/\!\!/T) \xrightarrow{\psi} F_\beta(Z/\!\!/G) \tag{43}$$

is invariant under the action of $P_{\tilde\alpha}$ on $Z^0_{\tilde\beta}$.

Proof. As in the proof of Lemma 4.3.1, we may replace $Z^0_{\tilde\beta}$ with a G-equivariant affine cover, and hence it is enough to show that (43) is unchanged by precomposition with an arbitrary automorphism of $Z^0_{\tilde\beta}$ induced by $p : Z^0_{\tilde\beta} \to P_{\tilde\alpha}$. The morphism (43) is given by the family $(\mathbb{P}^1_{Z^0_{\tilde\beta}}, \mathscr{T}_{\tilde\alpha} \times_T G, \mathcal{S}, id)$ where $\mathcal{S}$ is the tautological section defined in (39). The composition

$$Z^0_{\tilde\beta} \xrightarrow{p} Z^0_{\tilde\beta} \xrightarrow{(43)} F_\beta(Z/\!\!/G)$$

is induced by the pullback of this family, which is precisely $(\mathbb{P}^1_{Z^0_{\tilde\beta}}, \mathscr{T}_{\tilde\alpha} \times_T G, \wp \circ \mathcal{S}, id)$ where $\wp \in \mathrm{Aut}(\mathscr{T}_{\tilde\alpha} \times_T G) \subset \mathrm{Aut}(\mathscr{T}_{\tilde\alpha} \times_T Z)$ is the automorphism defined by p in Remark 4.1.2.

[1]Using the presentation for $\mathscr{T}_{\tilde\alpha} \times_T Z$ in (39), the automorphism $\wp$ is given by $(x, \mathbf{u}, z) \to (x, \mathbf{u}, \tau_{\tilde\alpha}(u)^{-1} p \tau_{\tilde\alpha}(u) z)$ and $\wp \circ \mathcal{S}(x, \mathbf{u}) = (x, \mathbf{u}, \tau_{\tilde\alpha}(u)^{-1} p x)$.

This can be seen by checking that the square

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "{(x,\mathbf{u},y)\mapsto(px,\mathbf{u},y)}"] \arrow[d] & \mathcal{Z} \arrow[d] \\ \mathbb{P}^1_{Z^0_{\tilde\beta}} \arrow[r, "{(x,\mathbf{u})\mapsto(px,\mathbf{u})}"] \arrow[u, bend left, "\wp\circ\mathcal{S}"] & \mathbb{P}^1_{Z^0_{\tilde\beta}} \arrow[u, bend right, "\mathcal{S}"'] \end{tikzcd}$$

is a fibered square of fiber bundles with sections, where we interpret $\mathcal{Z}$ and $\mathbb{P}^1_{Z^0_{\tilde\beta}}$ using the GIT presentation in (39), and we observe that in those coordinates we have $\wp\circ\mathcal{S}(x,\mathbf{u}) = (x,\mathbf{u},\tau_{\tilde\alpha}(u)^{-1}px)$. Since $\wp\in\mathrm{Aut}(\mathscr{P}_{\tilde\alpha})$, the families $(\mathbb{P}^1_{Z^0_{\tilde\beta}}, G\times_T\mathscr{T}_{\tilde\alpha}, \mathcal{S}, id)$ and $(\mathbb{P}^1_{Z^0_{\tilde\beta}}, G\times_T\mathscr{T}_{\tilde\alpha}, \wp\circ\mathcal{S}, id)$ are isomorphic quasimaps to $Z/\!\!/G$. $\square$

By Lemma 4.3.2 we have an induced morphism

$$Z^0_{\tilde\beta}/P_{\tilde\alpha}\to F_\beta(Z/\!\!/G). \tag{44}$$

Lemma 4.3.3. *The morphism* (44) *is a closed embedding.*

Proof. There is a closed embedding $[Z^0_{\tilde\beta}/P_{\tilde\alpha}]\to[Z^s(G)/P_{\tilde\alpha}]$, but $[Z^s(G)/P_{\tilde\alpha}]$ is represented by a flag bundle on the variety $Z/\!\!/G$ that is proper over $\mathbf{S}_G$, so $Z^0_{\tilde\beta}/P_{\tilde\alpha}$ is represented by a scheme that is proper over $\mathbf{S}_G$. Since $F_\beta(Z/\!\!/G)$ is separated, the morphism (44) is proper, so to prove the lemma we need only show that it is a monomorphism; i.e., fully faithful.

Let $a_i : S\to Z^0_{\tilde\beta}$ be two morphisms from a scheme S. We show that if they induce isomorphic maps to $F_\beta(Z/\!\!/G)$, then they differ by an element of $P_{\tilde\alpha}$. The map to $F_\beta(Z/\!\!/G)$ induced by a_i is given by the family $(\mathbb{P}^1_S, \mathscr{T}_{\tilde\alpha}\times_T G, \sigma_i, id)$ with

$$\sigma_i(x,\mathbf{u}) = (x,\mathbf{u},\tau_{\tilde\alpha}(u)^{-1}a_i(x))$$

using the GIT notation of (39). Observe that $\sigma_{i,V} = a_i\circ pr_1$ where $pr_1 : V_S = S\times V\to S$ is the projection. If these define isomorphic quasimaps to $Z/\!\!/G$ then by (22) there are maps $\phi_U : U_S\to G$ and $\phi_V : V_S\to G$ such that

$$\begin{aligned} \phi_V\cdot(a_1\circ pr_1) &= a_2\circ pr_1 \qquad \text{as maps } V_S\to Z\\ (\phi_V\circ\kappa)\tau_{\tilde\alpha} &= \tau_{\tilde\alpha}\phi_U \qquad \text{as maps } U_S\setminus(S\times\{0\})\to G. \end{aligned} \tag{45}$$

The first equation shows that ϕ_V is given by a composition $S\times V\xrightarrow{pr_1} S\xrightarrow{g} G$ for some $g\in G(S)$ sending a_1 to a_2. The second equation shows that $\tau_{\tilde\alpha}^{-1}g\tau_{\tilde\alpha} : S\times\mathbb{C}^*\to G$ has an extension to $S\times\mathbb{A}^1$ (the extension is $\phi_U : S\times U\to G$), so that $g\in P_{\tilde\alpha}(S)$. Hence the isomorphism defined by ϕ_U and ϕ_V is precisely the element of $\mathrm{Aut}(G\times_T\mathscr{T}_{\tilde\alpha})$ defined by $g\in P_{\tilde\alpha}(S)$ as in Remark 4.1.2. $\square$

5. Compute the I-function

5.1. **Weyl group action.** It is now our goal to show that the images of the closed embeddings (44) are always disjoint or equal, and to write $F_\beta(Z/\!\!/G)$ as a disjoint union of a certain collection of these images. Define

$$F^0_\beta(Z/\!\!/T) = \bigsqcup_{\tilde\beta\to\beta} F^0_{\tilde\beta}(Z/\!\!/T) \tag{46}$$

and let $\psi : F^0_\beta(Z/\!\!/T)\to F_\beta(Z/\!\!/G)$ be defined to equal $\psi_{\tilde\beta}$ on $F^0_{\tilde\beta}(Z/\!\!/T)$. As a guide to keeping track of the spaces in (46), the reader may like to consult the diagram in Section 1.3.

Lemma 5.1.1. *The map $\psi : F^0_\beta(Z/\!\!/T)\to F_\beta(Z/\!\!/G)$ is surjective.*

Proof. By Chevalley's Theorem [44, Tag 0ECX] and [44, Tag 06G2], it is enough to show that ψ is surjective on $\mathbb{C}$-points. First we observe that if $(\mathbb{P}^1, \mathscr{T}_{\tilde\alpha}, \sigma, id)$ is a quasimap to

$Z/\!\!/T$ with σ_V constant, then this quasimap is fixed (necessarily with a unique basepoint at $[0:1]$). For, by (21) we have

$$\sigma_U(u) = \tau_{\tilde{\alpha}}(u)^{-1}\sigma_V \tag{47}$$

on $U\setminus\{0\}$. In fact (47) determines $\sigma_U(u)$ on all of U, and the underlying map $q:\mathbb{P}^1\to[Z/T]$ is given in homogeneous coordinates by $[u:v]\mapsto\tau_{\tilde{\alpha}}(u)^{-1}\sigma_V$. One may check directly that this map is invariant under the $\mathbb{C}^*$-action on $\mathbb{P}^1$.

Now let $(\mathbb{P}^1,\mathscr{P}_{\tilde{\alpha}},\sigma,id)$ be a $\mathbb{C}$-point of $F_\beta(Z/\!\!/G)$, for some $\tilde{\alpha}$ with $r_\chi(\tilde{\alpha})=r_G(\beta)$ (see Remark 4.1.1). We will find an automorphism ϕ^0 of $\mathscr{P}_{\tilde{\alpha}}$ sending σ to a section ρ with ρ_V a constant function. By the above discussion the quasimap $(\mathbb{P}^1,\mathscr{T}_{\tilde{\alpha}},\rho,id)$ is an element of $F^0_{\tilde{\beta}}(Z/\!\!/T)$, and $(\mathbb{P}^1,\mathscr{P}_{\tilde{\alpha}},\sigma,id)$ is in its essential image.

We construct ϕ^0 as a limit in the following way. Since $(\mathbb{P}^1,\mathscr{P}_{\tilde{\alpha}},\sigma,id)$ is fixed by $\mathbb{C}^*$, from (30) and (22) we see that for each $\lambda\in\mathbb{C}^*$ we have morphisms $\psi_V^\lambda:V\to G$ and $\psi_U^\lambda:U\to G$ such that

$$\psi_V^\lambda(u^{-1})\tau_{\tilde{\alpha}}(u)=\tau_{\tilde{\alpha}}(\lambda^{-1}u)\psi_U^\lambda(u)\qquad u\in U\setminus\{0\} \tag{48}$$

$$\psi_U^\lambda(u)=\sigma_U(u)=\sigma_U(\lambda^{-1}u)\qquad u\in U$$

$$\psi_V^\lambda(v)\sigma_V(v)=\sigma_V(\lambda v)\qquad v\in V. \tag{49}$$

The morphisms ψ_V^λ and ψ_U^λ are in fact fibers of morphisms of schemes $\Psi_V:\mathbb{C}^*\times V\to G$ and $\Psi_U:\mathbb{C}^*\times U\to G$, respectively, and the equations above hold as equalities of morphisms of schemes when ψ_V^λ and ψ_U^λ are replaced by Ψ_V and Ψ_U, respectively. We will construct Ψ_V below and also show that Ψ_V extends to a morphism $\Phi_V:\mathbb{A}^1\times V\to G$. The construction of Ψ_U is similar.

To construct Φ_V, let $\iota:G\times V\to Z^s(G)\times V$ be defined by $\iota(g)=(g\sigma_V(v),v)$. This is a closed embedding as follows. Since $Z^s(G)\to Z/\!\!/G$ is a principal G-bundle, the map $G\times Z^s(G)\to Z^s(G)\times_{Z/\!\!/G}Z^s(G)$ is an isomorphism. On the other hand by [44, Tag 02XE] there is a fiber square

$$\begin{array}{ccc} Z^s(G)\times_{Z/\!\!/G}Z^s(G) & \longrightarrow & Z/\!\!/G \\ \downarrow & & \downarrow \\ Z^s(G)\times Z^s(G) & \longrightarrow & Z/\!\!/G\times Z/\!\!/G \end{array}$$

so since $Z/\!\!/G$ is a separated scheme, the composition $G\times Z^s(G)\to Z^s(G)\times Z^s(G)$ sending (g,z) to $(g\cdot z,z)$ is a closed embedding. Pulling back along $\sigma_V:V\to Z^s(G)$ yields ι.

We claim that $\sigma_V^\lambda:\mathbb{A}^1\times V\to Z^s(G)\times V$ given by $(\lambda,v)\mapsto(\sigma_V(\lambda v),v)$ factors through the embedding ι, or equivalently $\sigma_V(\lambda v)$ and v are in the same G-orbit for arbitrary $\lambda\in\mathbb{A}^1$. For $\lambda\neq0$ this follows from (49). In particular, fixing v, we see that $\sigma_V(\lambda v)$ are contained in a single G-orbit on $Z^s(G)$ for all $\lambda\neq0$. Since G-orbits on $Z^s(G)$ are closed, $\sigma_V(0)=\sigma_V(0\cdot v)$ is in this same orbit. Now define $\Phi_V=\iota^{-1}\circ\sigma_V^\lambda$, so we have

$$\Phi_V(\lambda,v)\sigma_V(v)=\sigma_V(\lambda v)\qquad(\lambda,v)\in\mathbb{A}^1\times V. \tag{50}$$

We also use Ψ_U to define a map $\Phi_U:\mathbb{A}^1\times U\to G$, but the strategy here is different. First define $\Phi'_U:(\mathbb{A}^1\times U)\setminus\{(0,0)\}\to G$ piecewise via

$$\Phi'_U=\begin{cases}\tau_{\tilde{\alpha}}(\lambda^{-1})\Psi_U(\lambda u) & \lambda\neq0\\ \tau_{\tilde{\alpha}}(u)^{-1}\Phi_V(\lambda,u^{-1})\tau_{\tilde{\alpha}}(u) & u\neq0.\end{cases}$$

These pieces agree on the overlap $(\mathbb{A}^1\setminus\{0\})\cap(U\setminus\{0\})$ by (48). By Hartog's theorem, Φ'_U extends to a function $\Phi_U:\mathbb{A}^1\times U\to G$.

We have constructed $\Phi_U:\mathbb{A}^1\times U\to G$ and $\Phi_V:\mathbb{A}^1\times V\to G$ such that (by definition)

$$\Phi_V(\lambda,u^{-1})\tau_{\tilde{\alpha}}(u)=\tau_{\tilde{\alpha}}(u)\Phi_U(u)\qquad u\in U\setminus\{0\},\lambda\in\mathbb{A}^1. \tag{51}$$

We set $\phi_U^0=\Phi_U(0,u)$ and $\phi_V^0=\Phi_V(0,v)$. The restriction of (51) says that this defines an automorphism of $\mathscr{P}_{\tilde{\alpha}}$. Let $\rho=\phi^0\circ\sigma$; by (50) we have $\rho_V=\sigma_V(0\cdot v)=\sigma_V(0)$ a constant function, as desired. □

Define $ev_\bullet : F^0_\beta(Z/\!\!/T) \to Z^s(G)/T$ to equal $ev_\bullet$ on each component $F^0_{\tilde\beta}(Z/\!\!/T)$. Notice that $F^0_\beta(Z/\!\!/T)$ is a stack of maps to $[Z/T]$ and hence carries an action by the group scheme W as in (12). Under this action, $ev_\bullet$ is equivariant and ψ is invariant.

For $\tilde\alpha \in \mathrm{Hom}(\chi(T), \mathbb{Z})$, let $W_{\tilde\alpha} = N_{L_{\tilde\alpha}}(T)/T$ be the Weyl group of $L_{\tilde\alpha}$, the unique Levi subgroup of $P_{\tilde\alpha}$ containing T. Recall that the group W acts on $\mathrm{Pic}^T(Z)$ as in Section 2.2.2; this defines an action on $\mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$ analogous to (8). Observe also that since W is finite, if S is connected then there is a bijection between elements of $W(S)$ and $W(\mathrm{Spec}(k))$ for k an algebraically closed field.

Lemma 5.1.2. *The action of the W on $F^0_\beta(Z/\!\!/T)$ and $r^{-1}_{\mathrm{Pic}}(\beta) \subset \mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$ has the following properties.*

(1) If $(C, \mathscr{T}, \sigma, \mathbf{x})$ is an S-quasimap of class $\tilde\beta$ and $w \in W(S)$, then $w \cdot (C, \mathscr{T}, \sigma, \mathbf{x})$ has class $w \cdot \tilde\beta$. In particular the action of W permutes the components $F^0_{\tilde\beta}(Z/\!\!/T)$ of $F^0_\beta(Z/\!\!/T)$.

(2) If $\tilde\beta_1, \tilde\beta_2 \in \mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$ are not in the same W-orbit, then the intersection of $\psi(F^0_{\tilde\beta_1}(Z/\!\!/T))$ and $\psi(F^0_{\tilde\beta_2}(Z/\!\!/T))$ is empty.

(3) If $\tilde\alpha = r_T(\tilde\beta)$, then the stabilizer of $\tilde\beta \in \mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$ is $W_{\tilde\alpha}$.

Proof. To prove (1), let k be algebraically closed and let $(\mathbb{P}^1_k, \mathscr{T}, \sigma, id)$ be a k-quasimap in $F^0_{\tilde\beta}(Z/\!\!/T)$ and choose an equivariant line bundle $\mathscr{L} \in \mathrm{Pic}^T(Z)$. We use the description of $w \cdot (\mathbb{P}^1_k, \mathscr{T}, \sigma, id)$ in (12). Then from (14) we have a fiber square

$$\begin{array}{ccc} \mathscr{T} \times w^{-1}(w^*\mathscr{L}) & \longrightarrow & w\mathscr{T} \times \mathscr{L} \\ \downarrow & & \downarrow \\ \mathscr{T} \times Z & \xrightarrow{(id, w\cdot)} & w\mathscr{T} \times Z \end{array}$$

where the horizontal maps are twisted-equivariant isomorphisms and vertical maps are T-equivariant. Hence we can quotient the square by T to obtain a fiber square over ϖ defined in (13). From this it follows that

$$\deg_{\mathbb{P}^1}(\varpi \circ \sigma)^*(w\mathscr{T} \times_T \mathscr{L}) = \deg_{\mathbb{P}^1} \sigma^*(\mathscr{T} \times_T w^{-1}(w^*\mathscr{L})) = \deg_{\mathbb{P}^1} \sigma^*(\mathscr{T} \times_T (w^{-1} \cdot \mathscr{L})),$$

or in other words, the class of the quasimap $(\mathbb{P}^1_k, w\mathscr{T}, \varpi \circ \sigma, id)$ applied to $\mathscr{L}$ is $(w \cdot \tilde\beta)(\mathscr{L})$.

For (2), for $i = 1, 2$ let $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_i}, \sigma_i, id)$ be a k-quasimap of class $\tilde\beta_i$, and suppose these two quasimaps have the same image in $F_\beta(Z/\!\!/G)$. Then in particular the associated G-bundles $\mathscr{P}_{\tilde\alpha_1}$ and $\mathscr{P}_{\tilde\alpha_2}$ are isomorphic, so by Theorem 2.3.1 there is some $w \in N_G(T)$ such that $w\tilde\alpha_1 = \tilde\alpha_2$. Then the proof of Lemma 4.3.3 shows that there exists $p \in P_{\tilde\alpha_2}$ such that

$$w \cdot (\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_1}, \sigma_1, id) = (\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_2}, \varpi \circ \sigma_1, id) = (\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_2}, \wp \circ \sigma_2, id)$$

(in particular, this argument did not require the two quasimaps to have the same class, just the same bundle type). Finally, as argued in the proof of Lemma 4.3.1, the quasimaps $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_2}, \wp\circ\sigma_2, id)$ and $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_2}, \sigma_2, id)$ have the same class. So the class of $w\cdot(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_1}, \sigma_1, id)$ equals the class of $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha_2}, \sigma_2, id)$, which by part (1) implies $w \cdot \tilde\beta_1 = \tilde\beta_2$.

To prove (3), first note that by definition, $L_{\tilde\alpha}$ is the G-stabilizer of the cocharacter $\tau_{\tilde\alpha}$. Since the identification (34) of $\tilde\alpha$ and $\tau_{\tilde\alpha}$ is W-equivariant, we see that $W_{\tilde\alpha}$ is the stabilizer of $\tilde\alpha$. So the stabilizer of $\tilde\beta$ is a subgroup of $W_{\tilde\alpha}$. Conversely, if $w \in N_{L_{\tilde\alpha}}(T)$ and $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha}, \sigma, id)$ is a quasimap of class $\tilde\beta$, we want to show that $(\mathbb{P}^1_k, \mathscr{T}_{w\tilde\alpha}, \varpi \circ \sigma, id)$ also has class $\tilde\beta$ (for then part (1) implies $w \cdot \tilde\beta = \tilde\beta$). Because w is in the stabilizer of $\tilde\alpha$, the bundles $\mathscr{T}_{\tilde\alpha}$ and $\mathscr{T}_{w\tilde\alpha}$ are identically the same. In fact the morphism

$$\varpi : \mathscr{T}_{\tilde\alpha} \times_T G \to \mathscr{T}_{w\tilde\alpha} \times_T G$$

defined in (13) is the same as the automorphism $\wp \in \mathrm{Aut}(\mathscr{T}_{\tilde\alpha} \times_T G)$ determined by w as an element of $P_{\tilde\alpha}$. It was argued in the proof of Lemma 4.3.1 that the class of $(\mathbb{P}^1_k, \mathscr{T}_{\tilde\alpha}, \wp \circ \sigma, id)$ is $\tilde\beta$. □

Lemma 5.1.2 shows that the images

$$F_{\tilde{\beta}}(Z/\!\!/G) := \psi(F^0_{\tilde{\beta}}(Z/\!\!/T)) \tag{52}$$

are either disjoint or identical subspaces of $F_\beta(Z/\!\!/G)$. Moreover, from Lemma 5.1.1 we know ψ is surjective. Let $\tilde{\beta}_i$ be elements of $\mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$ such that

$$F_\beta(Z/\!\!/G) = \bigsqcup_i F_{\tilde{\beta}_i}(Z/\!\!/G), \tag{53}$$

i.e., the $\tilde{\beta}_i$ are elements of distinct Weyl orbits on $\mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$. It follows from Lemmas 4.3.3 and 3.1.10 that (53) is a decomposition of $F_\beta(Z/\!\!/G)$ as a disjoint union of open and closed subschemes.

5.2. **Relate the perfect obstruction theories.** The main goal of this section is to relate the perfect obstruction theory of $F^0_{\tilde{\beta}}(Z/\!\!/T)$ to the pullback of the perfect obstruction theory of $F_{\tilde{\beta}}(Z/\!\!/G)$ under $\psi_{\tilde{\beta}}$. Let

$$\mathbb{E}_G := \mathbb{E}_{QG_\beta(Z/\!\!/G)} \qquad \mathbb{E}_T := \mathbb{E}_{QG_{\tilde{\beta}}(Z/\!\!/T)}$$

denote the absolute perfect obstruction theories defined in (26). In what follows, if A (resp. B) is a complex on $QG_\beta(Z/\!\!/G)$ (resp. $QG_{\tilde{\beta}}(Z/\!\!/T)$), we will use the notation

$$A|_F := A|_{F_{\tilde{\beta}}(Z/\!\!/G)} \qquad (\text{resp. } B|_F := B|_{F^0_{\tilde{\beta}}(Z/\!\!/T)})$$

for the restricted complex whenever the intended class $\tilde{\beta}$ is clear. In particular, we have

$$\mathbb{E}_G|_F := \mathbb{E}_G|_{F_{\tilde{\beta}}(Z/\!\!/G)} \qquad \mathbb{E}_T|_F := \mathbb{E}_T|_{F^0_{\tilde{\beta}}(Z/\!\!/T)}.$$

To relate the obstruction theories, we use the tower of morphisms

$$\mathbb{P}^1 \times [X/T] \xrightarrow{\psi} \mathbb{P}^1 \times [X/G] \to \mathbb{P}^1 \to pt$$

which leads to a morphism of moduli of sections

$$\psi : \underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times [Z/T]/\mathbb{P}^1) \to \underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1).$$

Define $QG^0_\beta(Z/\!\!/T)$ and the map ψ^0 to be the fiber product

$$\begin{tikzcd} QG^0_\beta(Z/\!\!/T) \arrow[r] \arrow[d, "\psi^0"] & \underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times [Z/T]/\mathbb{P}^1) \arrow[d, "\psi"] \\ QG_\beta(Z/\!\!/G) \arrow[r] & \underline{\mathrm{Sec}}_{pt}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1) \end{tikzcd}$$

Notice that $QG^0_\beta(Z/\!\!/T)$ is an open substack of the (finite) disjoint union of moduli spaces $QG_{\tilde{\beta}}(Z/\!\!/T)$ with $\tilde{\beta}$ mapping to β.

Lemma 5.2.1. *In the derived category of $F^0_{\tilde{\beta}}(Z/\!\!/T)$, there is a morphism of distinguished triangles*

$$\begin{tikzcd} \psi^*_{\tilde{\beta}}(\mathbb{E}_G|_F) \arrow[r] \arrow[d] & \mathbb{E}_T|_F \arrow[r] \arrow[d] & (R\pi_* n_F^* \mathbb{T}_\psi)^\vee \arrow[r] \arrow[d] & {} \\ \psi^*_{\tilde{\beta}}(\mathbb{L}_{QG_\beta(Z/\!\!/G)}|_F) \arrow[r] & \mathbb{L}_{QG_{\tilde{\beta}}(Z/\!\!/T)}|_F \arrow[r] & \mathbb{L}_{\psi^0}|_F \arrow[r] & {} \end{tikzcd} \tag{54}$$

where ψ is the canonical map $[Z/T] \to [Z/G]$ and n_F is the restriction to $\mathbb{P}^1 \times F^0_{\tilde{\beta}}(Z/\!\!/T)$ of the universal map defined in Section 3.2.

Proof. On $QG^0_{\tilde\beta}(Z/\!\!/T)$ we have the following morphism of distinguished triangles, where the middle and left vertical arrows are the absolute perfect obstruction theories of (26) (see [9, Lem A.2.3]).

$$\begin{tikzcd}(\psi^0)^*\mathbb{E}_G \arrow[r] \arrow[d, "(\psi^0)^*\phi_G"] & \mathbb{E}_T \arrow[r] \arrow[d, "\phi_T"] & (R\pi_*n^*\mathbb{T}_\psi)^\vee \arrow[r] \arrow[d] & {} \\ (\psi^0)^*\mathbb{L}_{QG_{\tilde\beta}(Z/\!\!/G)} \arrow[r] & \mathbb{L}_{QG_{\tilde\beta}(Z/\!\!/T)} \arrow[r] & \mathbb{L}_{\psi^0} \arrow[r] & {}\end{tikzcd} \tag{55}$$

Now restrict this diagram to $F^0_{\tilde\beta}(Z/\!\!/T)$ and use that $R\pi_*$ commutes with restriction to the fixed locus by [30, Cor 4.13].

□

We use Lemma 5.2.1 to relate the virtual and Euler classes appearing in the definition (32) of the I-function. We recall the definitions of these classes. According to [8, Sec 3], the composition

$$\mathbb{E}_G|_F^{\mathrm{fix}} \xrightarrow{\phi|_F^{\mathrm{fix}}} \mathbb{L}_{QG_\beta(Z/\!\!/G)}|_F^{\mathrm{fix}} \to \mathbb{L}_{F_{\tilde\beta}(Z/\!\!/G)}$$

is a perfect obstruction theory on $F_{\tilde\beta}(Z/\!\!/G)$. The virtual class $[F_{\tilde\beta}(Z/\!\!/G)]^{\mathrm{vir}}$ in (32) is the one defined by this perfect obstruction theory. By definition we have

$$N^{\mathrm{vir}}_{F_{\tilde\beta}(Z/\!\!/G)} := (\mathbb{E}_G|_F^{\mathrm{mov}})^\vee. \tag{56}$$

We note that the complex (56) has a global resolution by vector bundles. (This follows from [44, Tag 0F8E] and the fact that $F_{\tilde\beta}(Z/\!\!/G)$ has the resolution property, see [45, Thm 2.1].) Thus we may define its equivariant Euler class as in [8, Def 3.3]; i.e., if $N^\bullet$ is a finite complex of vector bundles with $e_{\mathbb{C}^*}(N^i)$ invertible for i odd, then

$$e_{\mathbb{C}^*}(N) = \prod_i e_{\mathbb{C}^*}(N^i)^{(-1)^i}. \tag{57}$$

One may show as in [44, Tag 0ESZ] that the definition (57) depends only on the complex represented by N and not on the choice of resolution. We make a simple observation about this definition that is useful in our computation.

Lemma 5.2.2. *If moreover the cohomology sheaves $H^i(N)$ are locally free, then $e(N) = \prod_i e(H^i(E))^{(-1)^i}$.*

Proof. Let N be the complex $\ldots \to N^n \xrightarrow{d_n} N^{n+1} \to \ldots$ with locally free cohomology sheaves. For each n, we have short exact sequences

$$0 \to \ker(d_n) \to N^n \to \mathrm{im}(d_n) \to 0$$
$$0 \to \mathrm{im}(d_{n-1}) \to \ker(d_n) \to H^n(N) \to 0.$$

The result now follows from a routine computation using additivity of the Euler class and the fact that $\ker(d_n)$ and $\mathrm{im}(d_n)$ are locally free by [44, Tag 0F8J].

□

Corollary 5.2.3. *We have the following relationships on $F^0_{\tilde\beta}(X/\!\!/T)$:*

$$\psi^*_{\tilde\beta}[F_{\tilde\beta}(Z/\!\!/G)]^{\mathrm{vir}} = [F^0_{\tilde\beta}(Z/\!\!/T)]^{\mathrm{vir}} \tag{58}$$

$$\psi^*_{\tilde\beta} e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{\tilde\beta(Z/\!\!/G)}}) = e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^0_{\tilde\beta(Z/\!\!/T)}})\frac{\prod_{\tilde\beta(\rho)<0}\prod_{k=\tilde\beta(\rho)+1}^{-1}(c_1(\mathscr{L}_\rho)+kz)}{\prod_{\tilde\beta(\rho)\geq 0}\prod_{k=1}^{\tilde\beta(\rho)}(c_1(\mathscr{L}_\rho)+kz)}. \tag{59}$$

Here, ρ ranges over roots of G with respect to T.

Proof. For (58), modify (54) by applying the "fix" functor, and then use the commuting square

$$\begin{tikzcd} F^0_{\tilde\beta}(Z/\!\!/T) \arrow[r] \arrow[d, "\psi_{\tilde\beta}"] & QG^0_\beta(Z/\!\!/T) \arrow[d, "\psi^0"] \\ F_{\tilde\beta}(Z/\!\!/G) \arrow[r] & QG_\beta(Z/\!\!/G) \end{tikzcd}$$

and functoriality of the cotangent complex to map the bottom row of the fixed part of (54) to the canonical distinguished triangle for $\psi_{\tilde\beta}$. The resulting morphism of distinguished triangles

$$\begin{tikzcd} \psi^*_{\tilde\beta}(\mathbb{E}_G|^{\mathrm{fix}}_F) \arrow[r] \arrow[d] & \mathbb{E}_T|^{\mathrm{fix}}_F \arrow[r] \arrow[d] & ((R\pi_* n_F^*\mathbb{T}_\psi)^{\mathrm{fix}})^\vee \arrow[r] \arrow[d, "\phi_\psi"] & {} \\ \psi^*_{\tilde\beta}(\mathbb{L}_{F_{\tilde\beta}(Z/\!\!/G)}) \arrow[r] & \mathbb{L}_{F^0_{\tilde\beta}(Z/\!\!/T)} \arrow[r] & \mathbb{L}_{\psi_{\tilde\beta}} \arrow[r] & {} \end{tikzcd}$$

is exactly the diagram for checking that we can define virtual pullback along $\psi_{\tilde\beta}$ as in [38]. Observe that $\psi_{\tilde\beta}$ is smooth (being the projection morphism of a flag bundle). We claim that the arrow ϕ_ψ is a quasi-isomorphism; granting this, the diagram above implies that virtual pullback along $\psi_{\tilde\beta}$ is defined and agrees with the usual flat pullback [38, Rmk 3.10]. By [38, Cor 4.9], we get (58).

To show that ϕ_ψ is a quasi-isomorphism, it suffices to show that it induces an isomorphism of cohomology sheaves of degree -1 (since a standard diagram chase shows that ϕ_ψ is an obstruction theory). Because $\mathbb{L}_{\psi_{\tilde\beta}}$ is represented by a vector bundle in degree 0, it suffices to show that $(R^1\pi_* n_F^*\mathbb{T}_\psi)^{\mathrm{fix}} = 0$. By Nakayama's lemma it suffices to check that fibers at closed points vanish. If $q = (\mathbb{P}^1 \xrightarrow{n} [Z/T])$ is a closed point of $F^0_{\tilde\beta}(X/\!\!/T)$ then $(R^1\pi_* n_F^*\mathbb{T}_\psi)|_q = H^1(\mathbb{P}^1, n^*\mathbb{T}_\psi)$. Since the $\mathbb{C}^*$-linearization on $n^*\mathbb{T}_\psi$ is trivial, and $H^1(\mathbb{P}^1, n^*\mathbb{T}_\psi)$ has a basis of monomials in u, v where each variable has degree at most -1, we see that this representation has no fixed part.

To compute (59), observe from the universal family (39) that we may write the universal curve $F^0_{\tilde\beta}(Z/\!\!/T)\times\mathbb{P}^1$ as the quotient $(Z^0_{\tilde\beta}\times(\mathbb{C}^2\setminus\{0\})/(T\times\mathbb{C}^*)$ and that with this presentation, the vector bundle $n_F^*\mathbb{T}_\psi$ on $F^0_{\tilde\beta}(Z/\!\!/T)\times\mathbb{P}^1$ is induced from a topologically trivial bundle on $Z^0_{\tilde\beta}\times(\mathbb{C}^2\setminus\{0\})$. This trivial bundle has fiber equal to the subspace of the Lie algebra $\mathfrak{g}$ of G with nontrivial weights, viewed as a $T\times\mathbb{C}^*$ representation via the homomorphism

$$T\times\mathbb{C}^* \xrightarrow{(t,s)\to t\tau_{\tilde\alpha}(s)^{-1}} T$$

and the adjoint representation of T on $\mathfrak{g}$. In other words, $n_F^*\mathbb{T}_\psi$ splits as a direct sum of line bundles

$$n_F^*\mathbb{T}_\psi = \bigoplus_\rho \left(\pi^*\mathscr{L}_\rho \otimes \mathscr{O}_{\mathbb{P}^1\times F_{\tilde\beta}(Z/\!\!/T)}(\tilde\beta(\rho))\right)$$

where the sum ranges over the roots of $\mathfrak{g}$ relative to T. By the projection formula and flat base change, we have

$$R^i\pi_*(n_F^*\mathbb{T}_\psi) = \bigoplus_\rho \left(\mathscr{L}_\rho \otimes H^i(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(\tilde\beta(\rho))\right),$$

and in particular these sheaves are locally free. Now we apply [44, Tag 0F8G, 0F9F] and Lemma 5.2.2 to the dual of the top row of (54), recalling the definition (56). We get

$$\psi^*_{\tilde\beta} e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde\beta}(Z/\!\!/G)}) = e_{\mathbb{C}^*}(N^{vir}_{F^0_{\tilde\beta}(Z/\!\!/T)})\frac{e_{\mathbb{C}^*}((R^1\pi_* n_F^*\mathbb{T}_\psi)^{\mathrm{mov}})}{e_{\mathbb{C}^*}((R^0\pi_* n_F^*\mathbb{T}_\psi)^{\mathrm{mov}})}. \tag{60}$$

If $\tilde\beta(\rho)$ is nonnegative, then $R^1\pi_*(\mathscr{O}_{\mathbb{P}^1}(\tilde\beta(\rho)))$ vanishes, but $R^0\pi_*(\mathscr{O}_{\mathbb{P}^1}(\tilde\beta(\rho)))$ is nonzero on a closed fiber of π, and a basis is given by the monomials $u^{\tilde\beta(\rho)}, u^{\tilde\beta(\rho)-1}v, u^{\tilde\beta(\rho)-2}v^2, \ldots, v^{\tilde\beta(\rho)}$

which have $\mathbb{C}^*$-weights $0, 1, 2, \ldots, \tilde{\beta}(\rho)$, respectively. Hence the Euler class of the moving part of the corresponding summand of $R^0\pi_* n_F^* \mathbb{T}_\psi$ is $\prod_{k=1}^{\tilde{\beta}(\rho)}(c_1(\mathscr{L}_\rho) + kz)$.

If $\tilde{\beta}(\rho)$ is less than -1, then $R^0\pi_*(\mathscr{O}_{\mathbb{P}^1}(\tilde{\beta}(\rho))$ vanishes but $R^1\pi_*(\mathscr{O}_{\mathbb{P}^1}(\tilde{\beta}(\rho))$ is nonzero on a closed fiber of π, and a basis is given by monomials $u^{-1}v^{\tilde{\beta}(\rho)+1}, u^{-2}v^{\tilde{\beta}(\rho)+2}, \ldots, u^{\tilde{\beta}(\rho)+1}v^{-1}$. Hence the Euler class of the moving part of the corresponding summand of $R^1\pi_* n_F^* \mathbb{T}_\psi$ is $\prod_{k=\tilde{\beta}(\rho)+1}^{-1}(c_1(\mathscr{L}_\rho) + kz)$. □

5.3. **Proof of the main theorem.** The following lemma, a restatement of [5, Prop 2.1], lets us navigate around the bottom left triangle of (33).

Lemma 5.3.1. *For any $\delta \in A_*(Z^s(G)/P_{\tilde{\alpha}})$, we have*

$$g^* f_* \delta = \sum_{w \in W/W_{\tilde{\alpha}}} (w^{-1})^* \left[\frac{p^*\delta}{\prod_{\rho \in R_{\tilde{\alpha}}^-} c_1(\mathscr{L}_\rho)} \right] \tag{61}$$

where $R_{\tilde{\alpha}}^-$ is the set of roots of G whose inner product with the dual character $\tilde{\alpha}$ is negative.

Proof. We reduce this statement to the one in [5, Prop 2.1]. Using the dynamic method, one may obtain a Borel subgroup B of G, contained in $P_{\tilde{\alpha}}$, equal to P_μ for some cocharacter μ that is positive on any root where $\tau_{\tilde{\alpha}}^{-1}$ is positive (see e.g. [17, 45], noting that our definition of $P_{\tilde{\alpha}}$ is dual to the one in that reference). So the opposite roots of this Borel, minus the roots of $L_{\tilde{\alpha}}$, are precisely those roots where $\tau_{\tilde{\alpha}}^{-1}$ is negative. Recalling the relationship (34), we see that this is the set $R_{\tilde{\alpha}}^-$.

The statement of [5, Prop 2.1] is for classes in $A_*(Z^s(G)/P_{\tilde{\alpha}})_{\mathbb{Q}}$ that are in the image of

$$c^{W_{\tilde{\alpha}}} : Sym(\chi(T)_{\mathbb{Q}})^{W_{\tilde{\alpha}}} \to A_*(Z^s(G)/P_{\tilde{\alpha}})_{\mathbb{Q}},$$

a morphism defined in [5, 47]. We claim that when the classes in the image of $c^{W_{\tilde{\alpha}}}$ are restricted to any fiber of $Z^s(G)/P_{\tilde{\alpha}} \to Z^s(G)/G$, they generate the Chow group of that fiber, so that by the Leray-Hirsch theorem [20, Lem 6] it suffices to show (61) for δ in the image of $c^{W_{\tilde{\alpha}}}$. To prove the claim, let $z \in Z^s(G)/G$. We have a commuting diagram

$$\begin{array}{ccccc}
Sym(\chi(T)_{\mathbb{Q}}) & \xrightarrow{c} & A_*(Z^s(G)/B) & \xrightarrow{\iota_z^*} & A_*(G/T) \\
\uparrow & & \uparrow & & \uparrow F \\
Sym(\chi(T)_{\mathbb{Q}})^{W_{\tilde{\alpha}}} & \xrightarrow{c^{W_{\tilde{\alpha}}}} & A_*(Z^s(G)/P_{\tilde{\alpha}}) & \xrightarrow{\iota_z^*} & A_*(G/P_{\tilde{\alpha}})
\end{array}$$

where the maps labeled ι_z^* are restrictions to fibers over z, vertical maps of Chow groups are pullbacks, the left square commutes by definition of $c^{W_{\tilde{\alpha}}}$, and we have used that the pullback $A_*(G/B) \to A_*(G/T)$ is an isomorphism [22, (2.6)]. The morphism c is the characteristic homomorphism, and by [22, (1.3)] the composition of the top arrows is surjective. By Proposition 2.4.1, the map F is injective. Now a diagram chase shows that the composition of the bottom arrows is surjective, as desired.

Now for $\delta = c^{W_{\tilde{\alpha}}}(\xi)$, the result [5, Prop 2.1] tells us

$$g^* f_* \delta = p^* f^* f_* c^{W_{\tilde{\alpha}}}(\xi) = p^* c^{W_{\tilde{\alpha}}} \left(\sum_{w \in W/W_{\tilde{\alpha}}} w \cdot (\xi / \prod_{\rho \in R_{\tilde{\alpha}}^-} \rho) \right).$$

It follows from the definition of $c^{W_{\tilde{\alpha}}}$ that the composition $p^* c^{W_{\tilde{\alpha}}} : Sym(\chi(T)_{\mathbb{Q}})^{W_{\tilde{\alpha}}} \to A_*(Z^s(G)/T)_{\mathbb{Q}}$ sends a character ξ to $c_1(\mathscr{L}_\xi)$. By (15) this composition is W-equivariant, so (61) follows.

□

Let $\tilde{\alpha}_i = r_T(\tilde{\beta}_i)$. Turning to formula (32) for $I_\beta^{Z/\!\!/G}(z)$, we first write it as a sum of pushforwards from $F_{\tilde{\beta}_i}(Z/\!\!/G)$ using (53). We use Proposition 4.0.1 to identify the evaluation map on each component, and then apply Lemma 5.3.1, obtaining

$$g^* I_\beta^{Z/\!\!/G} = \sum_{\tilde{\beta}_i} \sum_{w \in W/W_{\tilde{\alpha}_i}} (w^{-1})^* \left[\frac{p^* i_*([F_{\tilde{\beta}_i}(Z/\!\!/G)]^{vir} e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}_i}(Z/\!\!/G)})^{-1})}{\prod_{\rho \in R^-_{\tilde{\alpha}_i}} c_1(\mathscr{L}_\rho)} \right]. \tag{62}$$

Let us simplify the numerator of a summand of (62). From Lemma 5.1.2 and the equivariance of $ev_\bullet$, we have a commuting diagram

$$\begin{tikzcd} F^0_{w\tilde{\beta}_i}(Z/\!\!/T) \arrow[r, "w^{-1}"] \arrow[d, hook, "ev_\bullet"] & F^0_{\tilde{\beta}_i}(Z/\!\!/T) \arrow[r, "\psi_{\tilde{\beta}_i}"] \arrow[d, hook, "ev_\bullet"] & F_{\tilde{\beta}_i}(Z/\!\!/G) \arrow[d, hook, "i"] \\ Z^s(G)/T \arrow[r, "w^{-1}"] & Z^s(G)/T \arrow[r, "p_{\tilde{\alpha}_i}"] & Z^s(G)/P_{\tilde{\alpha}_i} \end{tikzcd} \tag{63}$$

The square on the left is fibered because w^{-1} is an isomorphism and the square on the right is fibered by Proposition 4.0.1, so the outer square is fibered. Because w^{-1} and $p_{\tilde{\alpha}_i}$ are flat, by [24, Prop 1.7] we have $(w^{-1})^* p^*_{\tilde{\alpha}_i} i_* = (ev_\bullet)_* (w^{-1})^* \psi^*_{\tilde{\beta}_i}$, so that the numerator of a summand in (62) is

$$(ev_\bullet)_* \psi^*_{w\tilde{\beta}_i}\big([F_{\tilde{\beta}_i}(Z/\!\!/G)]^{vir} e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}_i}(Z/\!\!/G)})^{-1}\big) \tag{64}$$

where we have also used that ψ (defined on $F^0_\beta(Z/\!\!/T)$) is equivariant.

Now let us compute the denominator of a summand of (62). We get

$$(w^{-1})^* \prod_{\rho \in R^-_{\tilde{\alpha}_i}} c_1(\mathscr{L}_\rho) = \prod_{\rho \in R^-_{\tilde{\alpha}_i}} c_1(\mathscr{L}_{w\cdot\rho}) = \prod_{\rho \in R^-_{w\cdot\tilde{\alpha}_i}} c_1(\mathscr{L}_\rho). \tag{65}$$

The first equality uses (15) and the second follows from the fact that the natural pairing between $\chi(T)$ and $\mathrm{Hom}(\chi(T), \mathbb{Z})$ is invariant.

Finally we apply equations (64) and (65) and use Lemma 5.1.2 to combine the double sum in (62) into a single sum, obtaining

$$g^* I_\beta^{Z/\!\!/G} = \sum_{\tilde{\beta} \mapsto \beta} \frac{(ev_\bullet)_* \psi^*_{\tilde{\beta}}([F_{\tilde{\beta}}(Z/\!\!/G)]^{vir} e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}}(Z/\!\!/G)})^{-1}))}{\prod_{\rho \in R^-_{r_T(\tilde{\beta})}} c_1(\mathscr{L}_\rho)}. \tag{66}$$

We can compute the pullbacks in the numerator with Corollary 5.2.3. Finally, applying the projection formula and recalling that $R^-_{r_T(\tilde{\beta})}$ is just the set of roots with $r_T(\tilde{\beta})(\rho) = \tilde{\beta}(\rho) < 0$, we recover Theorem 1.1.1.

5.4. **Denominators in the main theorem.** Consider the symbolic expression

$$B(z) := \sum_{\tilde{\beta} \to \beta} \left(\prod_\rho \frac{\prod_{k=-\infty}^{\tilde{\beta}(\rho)} (c_1(\mathscr{L}_\rho) + kz)}{\prod_{k=-\infty}^{0} (c_1(\mathscr{L}_\rho) + kz)} \right) j^* I_{\tilde{\beta}}^{Z/\!\!/T}(z) \tag{67}$$

appearing on the right hand side of (5). A priori it is not clear how this expression defines an element of $A^*(Z^s(G)/T) \otimes \mathbb{Q}[z, z^{-1}]$ because it contains denominators of the form $\prod_{\tilde{\beta}(\rho)<0} c_1(\mathscr{L}_{\tilde{\beta}(\rho)})$. We claim that in explicit examples, these denominators will always cancel with some factor in a numerator (though one may have to pass through painful algebraic manipulations to achieve this).

One reason to believe this claim is our proof of the equality (5). The denominators were introduced when we used Lemma 5.3.1, which is in turn a restatement of the formula in [5, Prop 2.1]. In particular the formula in [5, Prop 2.1] contains the same denominators $\prod_{\tilde{\beta}(\rho)<0} c_1(\mathscr{L}_{\tilde{\beta}(\rho)})$. If one believes that this is a sensible formula then our claim follows.

Nevertheless, for the skeptical reader, we make sense of (67) directly. Let R be the set of roots of G with respect to T and let R^+ be any system of positive roots. We recall (see e.g. [22, Sec 1]) the sign function $\mathrm{sgn} : W \to \{\pm 1\}$, and that an element α of some W-module is said to be W-anti-invariant if $w \cdot \alpha = (-1)^{\mathrm{sgn}(w)}\alpha$ for every $w \in W$. If for $\rho \in R$ we let L_ρ denote the corresponding linear element of $Sym(\chi(T)_\mathbb{Q})$, then the product

$$\Delta := \prod_{\rho \in R^+} L_\rho \in Sym(\chi(T)_\mathbb{Q})$$

is the *fundamental W-anti-invariant*. As reviewed in [22, Sec 1], the element Δ plays a role analogous to the Vandermonde determinant for symmetric functions, namely, if $\alpha \in Sym(\chi(T)_\mathbb{Q})$ is a W-anti-invariant, then there exists a unique β such that $\Delta\beta = \alpha$, and β is necessarily W-invariant. If X is a variety where T acts with trivial stabilizers, we will also use Δ to denote $\prod_{\rho \in R^+} c_1(\mathscr{L}_\rho) \in A^*(X/T)_\mathbb{Q}$.

Lemma 5.4.1. *If $\alpha \in A_*(Z^s(G)/T)_\mathbb{Q}$ is W-anti-invariant, then there exists a unique $\beta \in A_*(Z^s(G)/T)_\mathbb{Q}^W$ such that $\Delta \cap \beta = \alpha$.*

Proof. We recall the notation S and χ_i in the proof of Proposition 2.4.1. Let $S^a \subset S$ denote the subspace of W-anti-invariants. In fact S^a is a direct summand of S with splitting $\pi : S \to S^a$ given by

$$\pi(\chi) = |W|^{-1} \sum_{w \in W} \mathrm{sgn}(w) w(\chi). \tag{68}$$

Hence we may choose the elements χ_i to have a partition $\{\chi_\ell\}_{\ell=1}^k = \{\chi_i^a\}_{i=1}^{k_1} \cup \{\chi_j^b\}_{j=k_1+1}^k$ such that $\pi(\chi_i^a) = \chi_i^a$ and $\pi(\chi_j^b) = 0$. Since χ_i^a is anti-invariant, there is a unique (W-invariant) element $\bar{\chi}_i^a \in S^W$ such that $\chi_i^a = \Delta\bar{\chi}_i^a$.

Using (18) we may write $\alpha = \sum_\ell c_T(\chi_\ell) \cap g^*(b_\ell)$. If we define an endomorphism π_A of $A_*(Z^s(G)/T)_\mathbb{Q}$ by the same formula (68), then

$$\alpha = \pi_A(\alpha) = \sum_i \pi_A(c_T(\chi_i^a)) \cap g^*(b_i) + \sum_j \pi_A(c_T(\chi_j^b)) \cap g^*(b_j) = \sum_i c_T(\chi_i^a) \cap g^*(b_i),$$

where the first equality is because α is anti-invariant and the third is because c_T is W-equivariant. Hence we may write $\alpha = \Delta \cap \beta$ with $\beta = \sum_i c_T(\bar{\chi}_i^a) \cap g^*(b_i) \in A_*(Z^s(G)/T)_\mathbb{Q}^W$. Uniqueness follows from Proposition 2.4.1. Here's why: if $\beta, \beta' \in A_*(Z^s(G)/T)_\mathbb{Q}^W$, then $\beta = g^*(b)$ and $\beta' = g^*(b')$. So $\Delta \cap \beta = \Delta \cap \beta'$ implies $\Delta \cap g^*(b) = \Delta \cap g^*(b')$. Since we can choose the basis χ_i so that $\Delta = \chi_1^a$, we conclude that $b = b'$. $\square$

Finally we can explain the right hand side of (5).

Lemma 5.4.2. *The expression $\Delta B(z)$ defines a W-anti-invariant element of $A_*(Z^s(G)/T)$. Hence, we may define $B(z) \in A_*(Z^s(G)/T)^W$ to be the unique element β such that $\Delta \cap \beta = \Delta B(z)$ of Lemma 5.4.1.*

Combined with Proposition 2.4.1, this explains how the equality (5) completely determines $I^{Z/\!\!/G}(z)$.

Proof of Lemma 5.4.2. For $\square \in \{=, <, >, \leq, \geq\}$, let $S_{\tilde\beta}^{\square} = \{\rho \in R \mid \tilde\beta(\rho) \,\square\, 0\}$. Symbolically, we compute $\Delta B(z) = \sum_{\tilde\beta \to \beta} B_{\tilde\beta}$, where

$$B_{\tilde\beta} = (-1)^{\#(R^+ \cap S_{\tilde\beta}^{<})} \left(\prod_{\rho \in R^+ \cap S_{\tilde\beta}^{=}} c_1(\mathscr{L}_\rho) \prod_{\rho \in S_{\tilde\beta}^{>}} (-1)^{\tilde\beta(\rho)-1}(c_1(\mathscr{L}_\rho) + \tilde\beta(\rho)z) \right) j^* I_{\tilde\beta}^{Z/\!\!/T}(z)$$

We will show that when $w \in W$ is the reflection along a root, $w \cdot B_{\tilde\beta} = -B_{w\tilde\beta}$. Since W acts on the set of $\tilde\beta$ mapping to β it follows that $\Delta B(z)$ is W-anti-invariant as desired.

We first claim that for $w \in W$, we have $(w^{-1})^* I_{\tilde\beta}^{Z/\!\!/T}(z) = I_{w\tilde\beta}^{Z/\!\!/T}(z)$. To see this, let $QG(Z/\!\!/T)$ denote the moduli of quasimaps to $Z/\!\!/T$ (of any class). The W-action on $[Z/T]$

induces an action on $QG(Z/\!\!/T)$ that is compatible with the action on classes and makes $ev_\bullet$ W-equivariant (see Lemma 5.1.2). Hence we have

$$(w^{-1})^*(ev_\bullet)_* = (ev_\bullet)_*(w^{-1})^*$$

where the evaluation map on the left hand side is for $F_{\tilde{\beta}}(Z/\!\!/T)$ and on the right hand side it is for $F_{w\tilde{\beta}}(Z/\!\!/T)$ (see (63)). Moreover, by [9, Sec A.3], the obstruction theory on $QG(Z/\!\!/T)$ is W-equivariant. This means that $(w^{-1})^*\mathbb{E}_{QG}$ is canonically isomorphic to $\mathbb{E}_{QG}$, and in particular they have the same localization residue on $F_{w\tilde{\beta}}(Z/\!\!/T)$.

Next, by (15) we have $w \cdot c_1(\mathscr{L}_\rho) = c_1(\mathscr{L}_{w\rho})$. After reindexing and using (8), we see that we want to show

$$(-1)^{\#(R^+\cap S^{\leq}_{\tilde{\beta}})} \prod_{\delta \in wR^+\cap S^{=}_{w\tilde{\beta}}} c_1(\mathscr{L}_\delta) = \alpha(-1)^{\#(R^+\cap S^{<}_{w\tilde{\beta}})} \prod_{\delta \in R^+\cap S^{=}_{w\tilde{\beta}}} c_1(\mathscr{L}_\delta). \tag{69}$$

with $\alpha = -1$. A priori the sign α in (69) is -1 raised to the quantity

$$\#\left(R^+ \cap \left[(S^{\geq}_{\tilde{\beta}} \cap S^{<}_{w\tilde{\beta}}) \cup (S^{\leq}_{\tilde{\beta}} \cap S^{\geq}_{w\tilde{\beta}})\right]\right) + \#\left(R^- \cap wR^+ \cap S^{=}_{w\tilde{\beta}}\right). \tag{70}$$

We rewrite the second quantity using the map $R \to R$ sending ρ to $-\rho$, and we partition the first quantity by intersecting with the partition $wR^+ \sqcup wR^-$ of R. We see that the parity of (70) is equal to the parity of

$$\begin{aligned}&\#\left(R^+ \cap wR^+ \cap \left[(S^{\geq}_{\tilde{\beta}} \cap S^{<}_{w\tilde{\beta}}) \cup (S^{\leq}_{\tilde{\beta}} \cap S^{\geq}_{w\tilde{\beta}})\right]\right)\\ &\quad + \#\left(R^+ \cap wR^- \cap \left[(S^{\geq}_{\tilde{\beta}} \cap S^{<}_{w\tilde{\beta}}) \cup (S^{\leq}_{\tilde{\beta}} \cap S^{>}_{w\tilde{\beta}}) \cup (S^{\geq}_{\tilde{\beta}} \cap S^{=}_{w\tilde{\beta}})\right]\right)\end{aligned}$$

On the first set, w is an involution with no fixed point. The second set is equal to $R^+ \cap wR^- \cap [(S^{\geq}_{\tilde{\beta}} \cap S^{\leq}_{w\tilde{\beta}}) \cup (S^{\leq}_{\tilde{\beta}} \cap S^{>}_{w\tilde{\beta}})]$, and here $-w$ is an involution with a unique fixed point (the root along which w is a reflection), so this parity is odd. $\square$

6. Extensions and applications

6.1. **Equivariant I-functions.** Let S be a torus and suppose that we have an action of $S \times G$ on Z extending the action of $G = \{1\} \times G$ on Z. In other words, S acts on Z and this action commutes with the action of G. Then S acts on $[Z/G]$ and $[Z/T]$ (see [43, Rmk 2.4]) and this defines actions on $QG_\beta(X/\!\!/G)$ and $QG_{\tilde{\beta}}(X/\!\!/T)$ and their universal families by [9, Sec A.3], viewing them as substacks of the moduli of sections as in Section 3.2. Moreover the perfect obstruction theories E_{QG} in (26) are canonically S-equivariant as in [9, Sec A.3].

Because the actions of S and $\mathbb{C}^*$ on $\mathbb{P}^1 \times [Z/G]$ commute, the $\mathbb{C}^*$-fixed locus $F_\beta(Z/\!\!/G)$ is invariant under the action of S and the $\mathbb{C}^*$-fixed and moving parts of the perfect obstruction theory E_{QG_β} are also S-equivariant. Finally the map $ev_\bullet$ is S-equivariant since the universal family on $QG_\beta(X/\!\!/G)$ is. These statements also hold for T in place of G. Since the spaces $F_\beta(Z/\!\!/G)$ are schemes, we can use the equivariant intersection theory of [21] to define $[F_\beta(Z/\!\!/G)]^{S,\mathrm{vir}}$ in $A^S_*(F)$. The class $e_{S\times\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta(Z/\!\!/G)})$ is defined as in [44, Tag 0F9E] but with the Euler classes replaced by their $S \times \mathbb{C}^*$-equivariant counterparts. Hence we can define the S-equivariant I-function via the same formulas (32), but with all objects replaced by their S-equivariant counterparts.

For $\rho \in \chi(T)$, let L_ρ be the S-equivariant line bundle on $X^s(T)/T$ given by

$$L_\rho = X^s(T) \times_T \mathbb{C}_\rho \tag{71}$$

where $\mathbb{C}_\rho$ is the $S \times T$-equivariant representation where S acts trivially and T acts with character ρ.

Corollary 6.1.1. *The S-equivariant I-functions of $Z/\!\!/G$ and $Z/\!\!/T$ satisfy the equation* (5), *with $I^{S,Z/\!\!/G}_\beta(z)$ and $I^{S,Z/\!\!/T}_\beta(z)$ in place of $I^{Z/\!\!/G}_\beta(z)$ and $I^{Z/\!\!/T}_\beta(z)$.*

Proof. First note that Proposition 2.4.1 and Lemma 5.3.1 also hold S-equivariantly (in Lemma 5.3.1, the line bundles $c_1(\mathscr{L}_\rho)$ are S-equivariant as in (71) and we take the S-equivariant first Chern class). The same proofs work after replacing $Z^s(G)$ with $Z^s(G)\times_S U$, where $U \to U/S$ is an appropriate approximation of the universal S-bundle (definition as in [21, Sec 2.2]).

Now the computation in Section 5 proceeds as follows. The diagram (54) is equivariant; i.e., it is isomorphic to the pullback of a morphism of distinguished triangles on $[F^0_{\tilde{\beta}}(Z/\!\!/T)/S]$. This is true because the diagram (55) is equivariant by [9, Lem A.3.3]. Next, to compute the equivariant Euler class in Corollary 5.2.3, note that since S commutes with G its action on the Lie algebra $\mathfrak{g}$ is trivial. The remainder of the proof is the same as in the non-equivariant case. □

6.2. **Twisted I-functions.** Let S be a torus and suppose we have an action of $S \times G$ on Z as in Section 6.1. Furthermore, let $R = \mathbb{C}^*$ act trivially on Z with equivariant parameter μ; note this induces the trivial action on $F_\beta(Z/\!\!/G)$ as a moduli space of maps. Let E be a $S \times G$-equivariant vector bundle on Z, and let $\mathbb{C}_\mu$ be the R-equivariant vector bundle on Z that is topologically trivial and has its R-action given by scaling fibers. Let E_G denote the $S\times R$-equivariant vector bundle on $[Z/G]$ corresponding to $E\otimes\mathbb{C}_\mu$. Recall that $\mathbb{C}^*$ acts on $\mathbb{P}^1$ via (28) and hence on $F_\beta(Z/\!\!/G)\times\mathbb{P}^1$, and that the universal map $n : F_\beta(Z/\!\!/G)\times\mathbb{P}^1 \to [Z/G]$ is invariant with respect to this action. So n^*E_G is naturally $S \times R \times \mathbb{C}^*$-equivariant. We assume that the complex $R\pi_*n^*E_G$ has a $S\times R\times\mathbb{C}^*$-equivariant global resolution by vector bundles; i.e., it is an element of the rational Grothendieck group

$$K^\circ_{S\times R\times\mathbb{C}^*}(F_\beta(Z/\!\!/G) = K^\circ_{S\times\mathbb{C}^*}(F_\beta) \otimes \mathbb{Q}[\mu, \mu^{-1}]$$

of $S\times R\times\mathbb{C}^*$-equivariant vector bundles on $F_\beta(Z/\!\!/G)$. This assumption holds, for example, if $R^1\pi_*n^*E_G$ is zero and $R^0\pi_*n^*E_G$ is a vector bundle (see also [14, Sec 6.2]).

Fix an invertible multiplicative characteristic class $\mathbf{c}$ defining a group homomorphism

$$\mathbf{c} : K^\circ_{S\times R\times\mathbb{C}^*}(F_\beta(Z/\!\!/G)) \to (A^*_{S\times R\times\mathbb{C}^*}(F_\beta(Z/\!\!/G), \mathbb{Q}))^\times$$

to the group of units in $A^*_{S\times R\times\mathbb{C}^*}(F_\beta(Z/\!\!/G), \mathbb{Q})$. A priori, $\mathbf{c}$ may be defined only for vector bundles; its invertibility means its definition extends to elements of K°. Let $\underline{E}_G$ denote the $S \times R$-equivariant vector bundle on $Z/\!\!/G$ induced by $E \otimes \mathbb{C}_\mu$. Now we define the S-equivariant, $\mathbf{c}(E)$-twisted I-function to be

$$I^{Z/\!\!/G,\,S,\,\mathbf{c}(E)}(z) = 1 + \sum_{\beta\neq 0} q^\beta I_\beta^{Z/\!\!/G,\,S,\,\mathbf{c}(E)}(z)$$

where

$$I_\beta^{Z/\!\!/G,\,S,\,\mathbf{c}(E)}(z) = \mathbf{c}(\underline{E}_G)^{-1}(ev_\bullet)_*\left(\frac{[F_\beta(Z/\!\!/G)]^{S\times R,\mathrm{vir}} \cap \mathbf{c}(R\pi_*n^*E_G)}{e_{S\times R\times\mathbb{C}^*}(N^{vir}_{F_\beta(Z/\!\!/G)})}\right). \tag{72}$$

(see [13, (7.2.3)]). Note that the torus R is omitted from the superscripts in the I-function notation.

For the abelianization theorem, observe that E is naturally a T-equivariant vector bundle on Z, so we can also define the $\mathbf{c}(E)$-twisted I-function of $Z/\!\!/T$.

Corollary 6.2.1. *If the class* $\mathbf{c}$ *is functorial with respect to pullback, then Theorem* (1.1.1) *holds with* $I_\beta^{Z/\!\!/G,\,S,\,\mathbf{c}(E)}(z)$ *and* $I_\beta^{Z/\!\!/T,\,S,\,\mathbf{c}(E)}(z)$ *in place of* $I_\beta^{Z/\!\!/G}(z)$ *and* $I_\beta^{Z/\!\!/T}(z)$.

Proof. To complete the computation in Section 5.3, first note that

$$g^*\mathbf{c}(\underline{E}_G)^{-1} = \mathbf{c}(g^*\underline{E}_G)^{-1} = \mathbf{c}(\underline{E}_T)^{-1}.$$

The remainder of the computation is the same until the last line when we replace the numerator in the right-hand side of (66) with

$$(ev_\bullet)_*\psi^*_{\tilde{\beta}}([F_{\tilde{\beta}}(Z/\!\!/G)]^{vir} \cap e_{S\times R\times\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}}(Z/\!\!/G)})^{-1} \cap \mathbf{c}(R\pi_*n^*E_G)).$$

By functoriality of $\mathbf{c}$, the term $\psi_{\tilde{\beta}}^*(\mathbf{c}(R\pi_* n^* E_G))$ is equal to

$$\mathbf{c}(\psi_{\tilde{\beta}}^* R\pi_* n^* E_G) = \mathbf{c}(R\pi_* n^* \psi^* E_G)$$

where ψ is the natural map from $[Z/T]$ to $[Z/G]$. The bundle $\psi^* E_G$ is just E_T. $\square$

Remark 6.2.2. *The standard application of twisted invariants is to choose E that satisfies $R^1\pi_* n^* E_G = 0$ and set $\mathbf{c}$ to be the R-equivariant Euler class e_S. Then the non-equivariant limit of* (72) *exists—i.e., one can set $\mu = 0$. This non-equivariant limit is the definition of the twisted I-function in* [13, Sec 7.2]. *Taking the non-equivariant limit of Corollary 6.2.1, we see that abelianization holds for these twisted I-functions as well. We will denote these non-equivariant, Euler-twisted I-functions by $I^{Z/\!\!/G,E}$.*

6.3. **Big I-functions.** The I-function we have been discussing in this paper is most directly related to Gromov-Witten invariants with only one insertion, from which one can easily obtain information about invariants with insertions in $H^2(Z/\!\!/G, \mathbb{Q})$ by using the divisor equation. The *big I-function* is related to Gromov-Witten invariants with arbitrary insertions, and it is defined in [11] to be the generating series
(73)

$$\mathbb{I}^{Z/\!\!/G}(z) = \sum_\beta q^\beta \mathbb{I}_\beta^{Z/\!\!/G}(z) \qquad \text{where} \qquad \mathbb{I}_\beta^{Z/\!\!/G}(z) = P^{-1}(ev_\bullet)_* \left(\exp(\hat{ev}_\beta^*(\mathbf{t})/z) \frac{cl([F_\beta]^{vir})}{e_{\mathbb{C}^*}(N_{F_\beta}^{vir})} \right).$$

The space $F_\beta := F_\beta(Z/\!\!/G)$ and $ev_\bullet$ are defined as in (32), the maps P and cl were defined in Remark 3.3.4, the sum is over all I-effective classes β of (Z, G, θ), the quantity $\hat{ev}_\beta^*(\mathbf{t})$ is a $H^*(Z/\!\!/G)$-valued polynomial in z and formal variables t_i (to be defined momentarily), and $(ev_\bullet)_*$ is proper pushforward in Borel-Moore homology (see Remark 3.3.4). When S is a torus acting on Z as in Section 6.1, the series $\mathbb{I}^{Z/\!\!/G}(z)$ may be defined S-equivariantly. The goal of this section is to prove an abelianization formula for $\mathbb{I}^{Z/\!\!/G}(z)$ when Z is a vector space, yielding a closed formula for $\mathbb{I}^{Z/\!\!/G}(z)$ in this situation.

We define $\hat{ev}_\beta^*(\mathbf{t})$. If $Z^s \subset Z^s(G)$ is an open subset, recall the Kirwan map

$$\kappa_G : H^*_{S\times G}(Z, \mathbb{Q}) \to H^*_S(Z^s/G, \mathbb{Q}).$$

We will write out the definition of the Kirwan map when S is trivial. Let EG be the universal principal G-bundle. Then we have maps

$$EG \times_G Z \xleftarrow{a} EG \times_G Z^s \xrightarrow{b} Z^s/G$$

where a is an open embedding and b is projection to the second factor. Then b^* induces an isomorphism on cohomology, and the Kirwan map is defined by $\kappa_G = (b^*)^{-1} \circ a^*$. This map is surjective by [37].

In similar spirit we define

$$\hat{ev}_\beta^* : H^*_{G\times S}(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z] \to H^*_{S\times\mathbb{C}^*}(F_\beta, \mathbb{Q}).$$

Let $\mathscr{P} \to F_\beta \times \mathbb{P}^1$ be the universal principal bundle and let $\sigma : F_\beta \times \mathbb{P}^1 \to \mathscr{P} \times_G Z$ be the universal section. When S is trivial, the map $\hat{ev}_\beta^*$ is simply the pullback in cohomology along the composition of maps

$$F_\beta \xrightarrow{(id,0)} F_\beta \times \mathbb{P}^1 \xrightarrow{\sigma} \mathscr{P} \times_G Z \to EG \times_G Z.$$

At last we define $\hat{ev}_\beta^*(\mathbf{t})$. Fix a homogeneous basis γ_i of $H^*_S(Z/\!\!/G, \mathbb{Q})$. Let $\tilde{\gamma}_i \in H^*_{S\times G}(Z, \mathbb{Q})$ be classes such that $\kappa_G(\tilde{\gamma}_i) = \gamma_i$, and set

$$\mathbf{t} = \sum_i \tilde{\gamma}_i t_i$$

for t_i some formal variables. The term $\exp(\hat{ev}_\beta^*(\mathbf{t})/z)$ is interpreted as a polynomial in the t_i with coefficients in $H^*_{S\times\mathbb{C}^*}(F_\beta, \mathbb{Q})$ via the power series expansion of the exponential.

When Z is a vector space, we can explicitly compute (73) as follows. By Proposition 2.4.1, the classes γ_i are uniquely determined by their pullbacks $g^*\gamma_i \in H^*_S(Z^s(G)/T, \mathbb{Q})$, and these pullbacks may be expressed as W-invariant polynomials in the classes $c_1(\mathscr{L}_{\xi_j})$, where ξ_j are the characters of the T-action on Z. Write

$$g^*\gamma_i = q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}}))$$

for these polynomials, where $q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}}))$ is shorthand for $q_i(c_1(\mathscr{L}_{\xi_1}), \dots, c_1(\mathscr{L}_{\xi_r}))$.

Corollary 6.3.1. *The big I-function of $Z/\!\!/_\theta G$ satisfies*

$$g^*\mathbb{I}_\beta^{Z/\!\!/G}(z) = \sum_{\tilde{\beta}\to\beta} \exp\left(\sum_i t_i q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}}) + \tilde{\beta}(\boldsymbol{\xi})z)/z\right)\left(\prod_\rho \frac{\prod_{k=-\infty}^{\tilde{\beta}(\rho)}(c_1(\mathscr{L}_\rho)+kz)}{\prod_{k=-\infty}^{0}(c_1(\mathscr{L}_\rho)+kz)}\right) j^* I_{\tilde{\beta}}^{Z/\!\!/T}(z), \tag{74}$$

where

$$q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}}) + \tilde{\beta}(\boldsymbol{\xi})z) := q_i(c_1(\mathscr{L}_{\xi_1}) + \tilde{\beta}(\xi_1)z, \dots, c_1(\mathscr{L}_{\xi_r}) + \tilde{\beta}(\xi_r)z)$$

and the sum is over all $\tilde{\beta}$ mapping to β under the natural map $r_{\mathrm{Pic}} : \mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ and the product is over all roots ρ of G.

Remark 6.3.2. *Unlike* (5)*, the equality* (74) *holds only in cohomology.*

Proof of Corollary 6.3.1. The first step is to carefully choose the lifts $\tilde{\gamma}_i$. We have a commuting diagram of topological spaces

$$\begin{tikzcd} ET\times_T Z \arrow[d, "\psi"] & ET\times_Z Z^s(G) \arrow[l, "a"'] \arrow[r, "b", "\sim"'] \arrow[d] & Z^s(G)/T \arrow[d, "g"] \\ EG\times_G Z & EG\times_G Z^s(G) \arrow[l, "a"'] \arrow[r, "b", "\sim"'] & Z^s(G)/G \end{tikzcd}$$

which leads to a commuting diagram of cohomology maps

$$\begin{tikzcd} H^*_{S\times T}(Z,\mathbb{Q})^W \arrow[r, "\kappa_T"] & H^*_S(Z^s(G)/T,\mathbb{Q})^W \\ H^*_{S\times G}(Z,\mathbb{Q}) \arrow[u, "\psi^*", "\sim"'] \arrow[r, "\kappa_G"] & H^*_S(Z/\!\!/G,\mathbb{Q}) \arrow[u, "\sim", "g^*"'] \end{tikzcd} \tag{75}$$

The right vertical arrow is an isomorphism by Proposition 2.4.1, and the left vertical arrow is an isomorphism by [6, Prop 1]. Let

$$\tilde{\delta}_i = q_i(c_1(L_{\boldsymbol{\xi}})) \in H^*_{S\times T}(Z,\mathbb{Q})^W,$$

so $\kappa_T(\tilde{\delta}_i) = g^*\gamma_i$. Then set $\tilde{\gamma}_i = \psi^*\tilde{\delta}_i$. Commutativity of (75) implies that $\kappa_G(\tilde{\gamma}_i) = \gamma_i$ as desired.

To prove (74) we note that it is enough to prove (74) for the homology-valued functions $P\mathbb{I}^{Z/\!\!/G}(z)$ and $P\mathbb{I}^{Z/\!\!/T}(z)$, where P is the Poincaré duality map. This is because P commutes with flat pullback (see Remark 3.3.4). We may directly apply the computation in Section 5.3 to the homology-valued series $g^*(P\mathbb{I}^{Z/\!\!/G}(z))$ since Borel-Moore homology has the same functoriality properties as Chow: in a little more detail, Lemma 5.3.1 holds in $H_*(Z^s(G)/P_{\tilde{\alpha}})$ by applying cl to both sides of (61), and flat pullback commutes with proper pushforward by [25, 2.5(G$_2$.i)]. In place of (66) we arrive at the formula

$$g^*P\mathbb{I}_\beta^{Z/\!\!/G} = \sum_{\tilde{\beta}\mapsto\beta} \frac{(ev_\bullet)_*\psi_{\tilde{\beta}}^*(\exp(\hat{ev}_{\tilde{\beta}}^*(\mathbf{t})/z)[F_{\tilde{\beta}}(Z/\!\!/G)]^{vir} e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}}(Z/\!\!/G)})^{-1})}{\prod_{\rho\in R^-_{r_T(\tilde{\beta})}} c_1(\mathscr{L}_\rho)}$$

and we may replace the numerator by

$$(ev_\bullet)_*\big[\psi_{\tilde{\beta}}^*(\exp(\hat{ev}_{\tilde{\beta}}^*(\mathbf{t})/z)) \cap cl\,\psi_{\tilde{\beta}}^*([F_{\tilde{\beta}}(Z/\!\!/G)]^{vir} e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}}(Z/\!\!/G)})^{-1})\big] \tag{76}$$

using [25, 2.5(G$_4$.iii)], [24, Prop 19.1.2], and the fact that cl commutes with flat pullback. Here $\hat{ev}_{\tilde{\beta}}^*$ denotes the composition

$$H^*_{G\times S}(Z,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{Q}[z] \xrightarrow{\hat{ev}_{\tilde{\beta}}^*} H^*_{S\times\mathbb{C}^*}(F_\beta(Z/\!\!/G),\mathbb{Q}) \to H^*_{S\times\mathbb{C}^*}(F_{\tilde{\beta}}(Z/\!\!/G),\mathbb{Q})$$

where the second map is restriction to an open and closed subspace of $F_\beta(Z/\!\!/G)$.

To compute $\psi^*_{\tilde\beta}\hat{ev}^*_{\tilde\beta}$ we use the commuting diagram

$$\begin{array}{ccc} H^*_{S\times T}(Z,\mathbb{Q}) & \xrightarrow{\hat{ev}^*_{\tilde\beta}} & H^*_S(F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G),\mathbb{Q}) \\ \psi^*\big\uparrow & & \big\uparrow\psi^*_{\tilde\beta} \\ H^*_{S\times G}(Z,\mathbb{Q}) & \xrightarrow{\hat{ev}^*_{\tilde\beta}} & H^*_S(F_{\tilde\beta}(Z/\!\!/G),\mathbb{Q}) \end{array}$$

which follows from the commuting diagram of topological spaces (in the case when S is trivial)

$$\begin{array}{ccccccc} ET\times_T Z & \longleftarrow & \mathscr{T}\times_T Z & \xleftarrow[\sigma]{} & F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G)\times\mathbb{P}^1 & \xleftarrow[(id,0)]{} & F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G) \\ \big\downarrow\psi & & \big\downarrow & & \big\downarrow & & \big\downarrow\psi_{\tilde\beta} \\ EG\times_G Z & \longleftarrow & \mathscr{P}\times_G Z & \xleftarrow[\sigma]{} & F_{\tilde\beta}(Z/\!\!/G)\times\mathbb{P}^1 & \xleftarrow[(id,0)]{} & F_{\tilde\beta}(Z/\!\!/G) \end{array}$$

We see that

$$\psi^*_{\tilde\beta}\hat{ev}^*_{\tilde\beta}(\tilde\gamma_i)=\hat{ev}^*_{\tilde\beta}\psi^*(\tilde\gamma_i)=\hat{ev}^*_{\tilde\beta}(\tilde\delta_i)$$

by the definition of $\tilde\gamma_i$. Finally, we have

$$\hat{ev}^*_{\tilde\beta}(\tilde\delta_i)=ev^*_\bullet q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}})+\tilde\beta(\boldsymbol{\xi})z). \tag{77}$$

This follows from [11, Lem 5.2, Rmk 5.3] when the GIT chamber of θ has full dimension, but (as pointed out by a referee) it also follows without any additional hypothesis from our description of the universal family on $F_{\tilde\beta}(Z/\!\!/T)$ in Proposition 4.2.6 (for details, see the computation of $n_F^*\mathbb{T}_\psi$ in the proof of Corollary 5.2.3).

After replacing $\psi^*_{\tilde\beta}(\exp(\hat{ev}^*_{\tilde\beta}(\mathbf{t})/z))$ in (76) with the quantity in (77), we may use [24, Prop 19.1.2] to move the cycle map all the way to the left. Now we finish the computation using operations in Chow groups as in Section 5.3. ☐

6.4. **Applications to Gromov-Witten theory.** The mirror theorems of [13] (resp. [11, Thm 3.3]) state that $I^{Z/\!\!/G}$ (resp. $\mathbb{I}^{Z/\!\!/G}$) is on the Lagrangian cone when $Z/\!\!/G$ has a torus action with good properties (but [51] proves results without a torus action). This means that Theorem 1.1.1 and Corollary 6.3.1 can be translated to statements about J-functions, though the translation is simplest when $Z/\!\!/G$ is sufficiently positive. As an example we will use Theorem 1.1.1 to show that the abelianization conjectures [4, Conj 4.2] and [15, Conj 3.7.1] hold in good circumstances.

In order to have a clear statement to use in our application, we summarize [13, Cor 7.3.2] here. If E is a vector space with a linear G-action, we say that the resulting vector bundle on $Z/\!\!/G$ is *convex* if, when $(C,\mathscr{P},\sigma,p_i)$ is a general quasimap to $Z/\!\!/G$ as in Definition 3.1.2, then $H^1(C,\mathscr{P}\times_G E)=0$. (See [14, Prop 6.2.3] for some sufficient conditions for E to be convex.) The following theorem applies to the twisted theory described in Remark 6.2.2.

Theorem 6.4.1 ([13, Cor 7.3.2])**.** *Assume $Z/\!\!/_\theta G$ has an S-action with isolated fixed points. Let E be a convex representation satisfying*

$$\beta(\det(T_Z))-\beta(\det(Z\times E)\geq 2 \tag{78}$$

for all I-effective classes $\beta\neq 0$, where T_Z is the (G-equivariant) tangent bundle of Z. Then $J^{X/\!\!/G,E}=I^{X/\!\!/G,E}$, both S-equivariantly and nonequivariantly.

Proof. The cited result [13, Cor 7.3.2] requires us to check (78) for all θ-effective classes β, whereas we assume that it holds a priori only for I-effective classes. However, the proof of [13, Cor 7.3.2] only uses (78) for θ-effective classes that are realized as the class of some general quasimap (as in Definition 3.1.2) of *genus zero*. Since we have assumed that (78) holds for all I-effective classes, it follows from Remark 3.1.8 that (78) holds for each β that is a class of a general genus-zero quasimap . ☐

Before we can use [13, Cor 7.3.2] and Theorem 1.1.1 to say something about [4, Conj 4.2] and [15, Conj 3.7.1], we must address a difference in setup between the current paper and the cited conjectures: our Z is affine, but in [4, 15] Z is projective. The translation is accomplished (at least in many cases) by Lemmas 6.4.2 and 6.4.3.

Lemma 6.4.2. *Let Z be a vector space with an action by a reductive group G and character θ satisfying the assumptions in Section 1.1. Fix a torus $H \simeq \mathbb{C}^*$. If $\tau : H \hookrightarrow Z(G)$ is a 1-parameter subgroup of the center of G such that the resulting weights of H on Z are all positive and the character $\theta \circ \tau$ also has a positive exponent, then*

1. *$\mathbb{P}_\tau(Z) := (Z \setminus \{0\})/H$ is a weighted projective space with an action by a reductive group $\overline{G} = G/H$,*
2. *$\mathscr{O}_{\mathbb{P}_\tau(Z)}(\theta) := (Z \times \mathbb{C}_{\theta\circ\tau})/H$ is a $\overline{G}$-linearized ample line bundle on $\mathbb{P}_\tau(Z)$,*
3. *Every semi-stable point for $\mathscr{O}_{\mathbb{P}_\tau(Z)}(\theta)$ is stable, and*
4. *$Z/\!\!/_\theta G = \mathbb{P}_\tau(Z)/\!\!/_{\mathscr{O}_{\mathbb{P}_\tau(Z)}} \overline{G}$*

Proof. Statement (1) is an immediate consequence of the hypotheses. The $\overline{G}$-action in statement (2) comes from the diagonal G-action on $V \times \mathbb{C}_\theta$ (here, it is important that τ lands in the center of G). For $n \in \mathbb{Z}_{>0}$, we have a fiber square

$$\begin{array}{ccc} (Z \setminus \{0\}) \times \mathbb{C}_{n\theta} & \longrightarrow & \mathscr{O}_{\mathbb{P}_\tau(Z)}(n\theta) \\ \downarrow & & \downarrow \\ Z \setminus \{0\} & \xrightarrow{\pi} & \mathbb{P}_\tau(Z) \end{array}$$

where the horizontal maps are H-torsors and are also equivariant with respect to the projection homomorphism $G \to \overline{G}$. Hence pullback defines a 1-to-1 correspondence between G-equivariant sections of $(Z \setminus \{0\}) \times \mathbb{C}_{n\theta} \to Z \setminus \{0\}$ and $\overline{G}$-equivariant sections of $\mathscr{O}_{\mathbb{P}_\tau(Z)}(n\theta) \to \mathbb{P}_\tau(Z)$. The zero locus of a section of $(Z \setminus \{0\}) \times \mathbb{C}_{n\theta}$ is the inverse image under π of the zero locus of the corresponding section of $\mathscr{O}_{\mathbb{P}_\tau(Z)}(n\theta) \to \mathbb{P}_\tau(Z)$. This shows (3). Finally, (4) follows from descent for closed subschemes. $\square$

Example 2. *Lemma 6.4.2 applies if Z is a vector space with a linear G-action that contains the dilations, and if the composition of θ with a dilation yields a character $\mathbb{C}^* \to \mathbb{C}^*$ with a positive exponent.*

As pointed out by a helpful referee, Lemma 6.4.2 always applies in the following situation.

Lemma 6.4.3. *Let (Z, G, θ) be a triple satisfying the assumptions in Section 1.1 with Z a vector space. If $Z/\!\!/T$ is projective, then the hypotheses of Lemma 6.4.2 are satisfied.*

Proof. Let $\Sigma \subset \chi(T)$ be the cone in the character lattice of T generated by the weights of V with respect to T. Since these weights come from a G-representation, Σ is sent to itself by the action of W on $\chi(T)$. Since $Z/\!\!/T$ is projective, by [19, Prop 14.3.10], Σ is strongly convex, hence by [19, Prop 1.1.12] the dual $\Sigma^\vee$ in the lattice of cocharacters has nonempty interior. In other words, we have a 1-parameter subgroup $\tau' : H \to T$ such that the resulting weights of H on Z are all positive.

Now set $\tau = \sum_{w\in W} w\cdot\tau'$. Since W sends Σ and hence $\Sigma^\vee$ to itself, τ is also in the interior of $\Sigma^\vee$. By definition of τ we have $w \cdot \tau = \tau$; plugging in $h \in H$ we see that $w \cdot \tau(h) = \tau(h)$, so that τ is actually a 1-parameter subgroup of (the identity component of) T^W. Since the inclusion $Z(G) \hookrightarrow T^W$ is an isomorphism on identity components (see e.g. [32]), we see that τ is central.

Finally, since $Z/\!\!/_\theta T$ is projective, θ is not trivial. Since $Z^{ss}_\theta(T)$ is not empty we see that for some $k \geq 0$ the character $k\theta$ is a nonnegative linear combination of the weights, and some coefficient is strictly positive. Hence $\theta \circ \tau$ also has positive exponent. $\square$

Now we are ready to study the conjecture [15, Conj 3.7.1], which is a priori a statement about the Frobenius manifolds defined by the Gromov-Witten theories of $Z/\!\!/T$ and $Z/\!\!/G$.

Corollary 6.4.4. *Let (Z, G) be a pair satisfying the hypotheses of Theorem 6.4.1 and Lemma 6.4.2. If Z^{us} has codimension at least 2, then the conjecture* [15, Conj 3.7.1] *holds.*

We make the (mild) assumption on codimension in Corollary 6.4.4 to be in agreement with the context of the cited conjecture.

Proof. Our strategy is to first apply the reconstruction result [15, Thm 4.3.6] to reduce the Frobenius manifold correspondence to the relationship of "small" J-functions in [4, Conj 4.2]. The latter statement essentially Theorem 1.1.1 combined with [13, Cor 7.3.2], but in executing this strategy we have to be careful in a few places.

We first resolve the above-mentioned difference in setup: our Z is affine but in [4, 15] Z is projective. By Lemma 6.4.2, the quotient $Z/\!\!/_\theta G$ is identified with the projective quotient $\mathbb{P}_\tau(Z)/\!\!/_{\mathscr{O}_{\mathbb{P}_\tau(Z}}\overline{G}$. Since $\tau : H \to T \subset G$ is injective, it is also split, so $\overline{T} = T/H$ is a torus. We use this as a maximal torus of $\overline{G}$.

Now the reconstruction theorem in [15, Thm 4.3.6] applies because the localized equivariant cohomology ring

$$H^*_S(Z/\!\!/G, \mathbb{Q}) \otimes \mathrm{Frac}(H^*_S(pt, \mathbb{Q}))$$

is generated by divisors (this follows from the torus localization theorem). Also, since $Z/\!\!/G$ satisfies (78), it follows from Remark 3.1.8 that the intersection of the anticanonical divisor of $Z/\!\!/G$ with any curve in $Z/\!\!/G$ is at least 2. This is the "Fano of index ≥ 2" hypothesis needed to apply [15, Thm 4.3.6].

We must be careful one more time: the "small J-function" in [4, 15] differs from the small I-function considered in this paper; in fact, it corresponds to the *big* I-function in (73) with the classes γ_i in $\mathbf{t}$ restricted to a basis of $H^2(Z/\!\!/G, \mathbb{Q})$. However, Theorems 1.1.1 and 6.4.1 yield the expected formulas for these "middle-sized" I and J-functions as explained in [13, p. 404]. Now the Corollary follows from Theorem 1.1.1 and the mirror theorem of [13, Cor 7.3.2] (restated in Theorem 6.4.1 above). $\square$

6.5. **Example: A Grassmannian bundle on a Grassmannian variety.** To illustrate the geometry in Proposition 4.0.1 and to give an example of applying Theorem 1.1.1, in this section we investigate a family of Fano hypersurfaces. The ambient space in this example is a quiver flag variety, as studied in [35].

Theorem 6.5.1. *Let $Q := Gr_{Gr(k,n)}(\ell, U^{\oplus m})$ be the Grassmannian bundle of ℓ-planes in m copies of the tautological bundle U on $Gr(k, n)$, and assume $n - \ell m \geq 2$ and $km \geq 3$. Let $\mathscr{D}$ be the dual of the determinant of the tautological bundle on Q. Let $S = (\mathbb{C}^*)^{n+m}$ act on Q and $\mathscr{D}$ as defined in Section 6.5.3. Then the $\mathscr{D}$-twisted equivariant small J-function of Q equals $1 + \sum_{d,e>0} q_1^d q_2^e J_{(d,e)}(z)$, where $J_{(d,e)}(z)$ equals*

(79)
$$\sum_{\substack{d_1+\ldots+d_k=d,\, d_i\geq 0\\ e_1+\ldots+e_\ell=e,\, e_i\geq 0}} \prod_{\substack{i,j=1\\ i\neq j}}^{k} \left(\frac{\prod_{h=-\infty}^{d_i-d_j}(x_i - x_j + hz)}{\prod_{h=-\infty}^{0}(x_i - x_j + hz)} \right) \prod_{\substack{i,j=1\\ i\neq j}}^{\ell} \left(\frac{\prod_{h=-\infty}^{e_i-e_j}(y_i - y_j + hz)}{\prod_{h=-\infty}^{0}(y_i - y_j + hz)} \right)$$
$$\cdot \prod_{i=1}^{k}\prod_{\alpha=1}^{n} \left(\frac{\prod_{h=-\infty}^{0}(x_i + \lambda^1_\alpha + hz)}{\prod_{h=-\infty}^{d_i}(x_i + \lambda^1_\alpha + hz)} \right) \prod_{i=1}^{k}\prod_{j=1}^{\ell}\prod_{\beta=1}^{m} \left(\frac{\prod_{h=-\infty}^{0}(y_j - x_i + \lambda^2_\beta + hz)}{\prod_{h=-\infty}^{e_j-d_i}(y_j - x_i + \lambda^2_\beta + hz)} \right)$$
$$\cdot \left(\frac{\prod_{h=-\infty}^{e}(\sum_{j=1}^{\ell} y_j + hz)}{\prod_{h=-\infty}^{0}(\sum_{j=1}^{\ell} y_j + hz)} \right),$$

where the x_i are the Chern roots of the dual of U, the y_j are the Chern roots of the dual of the tautological bundle on Q, and the λ's are the equivariant parameters.

In the formula (79), the first line is the factor coming from the roots of G, the second line is the J-function of the abelian quotient, and the last line is the $\mathscr{D}$-twisting factor.

6.5.1. *Defining the target.* To define the GIT target, choose integers k, n, ℓ, and m with $k < n$ and $\ell < km$. Let $M_{k\times n}$ denote the space of $k \times n$ matrices with complex entries, and set

- the vector space $Z = M_{k\times n} \times M_{\ell\times km}$
- the group $G = GL_k \times GL_\ell$
- the action $(g, h) \cdot (X, Y) = (gX, hY \operatorname{diag}(g^{-1}))$ for $(g, h) \in G$ and $(X, Y) \in X$ where $\operatorname{diag}(g^{-1})$ is the block diagonal $km \times km$ matrix with g^{-1} repeated m times
- the character $\theta(g, h) = \det(g) \det(h)$

For the maximal torus $T \subset G$ choose the group of diagonal matrices. We can check stability of points using the numerical criterion [36, Prop 2.5]. It is straightforward to compute that

$$Z^{ss}_\theta(G) = Z^s_\theta(G) = (M_{k\times n} \setminus \Delta) \times (M_{\ell\times km} \setminus \Delta),$$

where Δ denotes matrices of less than full rank. Thus,

$$Z /\!\!/_\theta G = Gr_{Gr(k,n)}(\ell, U^{\oplus m}) =: Q$$

is the Grassmannian bundle of ℓ-planes in m copies of the tautological bundle U on $Gr(k, n)$. Observe that $\det(T_Z)$ is the G-equivariant line bundle on Z corresponding to the character

$$(g, h) \mapsto \det(g)^{n-\ell m} \det(h)^{km}.$$

The line bundle $\mathscr{D}$ is the G-equivariant line bundle on Z corresponding to the character

$$(g, h) \mapsto \det(h). \tag{80}$$

6.5.2. *Quasimaps and I-function.* To make (5) explicit for our chosen target we must compute the I-effective classes (Definition 3.1.7). As an illustration of Proposition 4.0.1 we will also describe $F_{\tilde\beta}(Z/\!\!/G)$. A stable quasimap to Q is equivalent to the following data:

- a rank-k vector bundle $\oplus_{i=1}^k \mathscr{O}_{\mathbb{P}^1}(d_i)$ and a rank-ℓ vector bundle $\oplus_{j=1}^\ell \mathscr{O}_{\mathbb{P}^1}(e_j)$
- a section σ of $\left[\oplus_{i=1}^k \mathscr{O}_{\mathbb{P}^1}(d_i)^{\oplus n}\right] \oplus \left[\oplus_{j=1}^\ell \oplus_{i=1}^k \mathscr{O}_{\mathbb{P}^1}(e_j - d_i)^{\oplus m}\right]$, written as a $k \times n$ and $\ell \times mk$ matrix of polynomials, such that all but finitely many points $\mathbf{x} \in \mathbb{P}^1$ satisfy $\sigma(\mathbf{x}) \in Z^s$.

This data defines a quasimap to $Z/\!\!/T$ of class $\tilde\beta = (d_1, \dots, d_k, e_1, \dots, e_\ell) \in \operatorname{Hom}(\chi(T), \mathbb{Z})$; the class as a quasimap to $Z/\!\!/G$ is $\beta = (\sum d_i, \sum e_j) \in \operatorname{Hom}(\chi(G), \mathbb{Z})$. In order to have finitely many basepoints, a stable quasimap must have $d_i \geq 0$, hence also $e_j \geq 0$. Now we can check that (78) is satisfied: if $\beta = (d, e)$ is I-effective, then

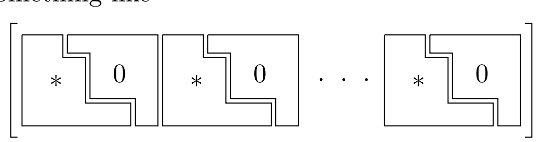
$$\beta(\det(T_Q)) - \beta(\det(\mathscr{D})) = (n - \ell m)d + (km - 1)e \geq 2d + 2e \geq 2$$

when $(d, e) \neq (0, 0)$ (the first inequality uses our assumptions in Theorem 6.5.1).

Finally we describe $F_{\tilde\beta}(Z/\!\!/G)$. For simplicity assume that the sequences $d_1, \dots, d_k$ and $e_1, \dots, e_\ell$ are ordered from smallest to largest. The subspace $Z_{\tilde\beta} \subset X$ is $M_{k\times n} \times Z'_{\tilde\beta}$, where $Z'_{\tilde\beta}$ is the subspace of $M_{\ell\times km}$ consisting of matrices (m_{ij}) where

$$m_{ij} = 0 \qquad \text{if} \qquad e_i - d_{(j \bmod m)+1} < 0.$$

Such a matrix looks something like



where the entries labeled "0" are required to be zero and the entries labeled "*" are not. The same $\ell \times k$ pattern of *'s and 0's is repeated m times. The group $P_{\tilde\alpha} \subset G$ is $P_1 \times P_2$ where P_1 is the parabolic subgroup of GL_k equal to block lower triangular matrices with blocks determined by the multiplicities of the d_i, and $P_2 \subset GL_\ell$ is similarly defined by the e_j. Hence in Proposition 4.0.1 we have a series of maps

$$F_{\tilde\beta}(Z/\!\!/G) = Z^s_{\tilde\beta}(G)/P_{\tilde\alpha} \hookrightarrow Z^s(G)/P_{\tilde\alpha} \to Z^s(G)/G = Z/\!\!/G$$

whose composition is $ev_\bullet$. The first arrow is a closed embedding and the second is a flag bundle.

6.5.3. *A good torus action.* The target Q has a torus action with isolated fixed points. Let $S = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^m$ act on Z as follows: if s_1 is an $n \times n$ diagonal matrix and s_2 is a $km \times km$ diagonal matrix with m constant $k \times k$ diagonal blocks, and both s_1 and s_2 are filled with numbers from $\mathbb{C}^*$ then

$$(s_1, s_2) \cdot (X, Y) = (Xs_1, Ys_2).$$

This action commutes with the action of G. We can extend it to a linearization of $\mathscr{D}$ as follows. The total space of $\mathscr{D}$ is $X \times_G \mathbb{C}$, where G acts on $\mathbb{C}$ via the character (80). Define $(s_1, s_2) \cdot (X, Y, z) = (Xs_1, Ys_2, z)$ for $(X, Y) \in Z$ and $z \in \mathbb{C}$.

The S-action on Q has isolated fixed points as follows. For $I \subset \{1, \ldots, n\}$ let D_I denote the $k \times n$ matrix which has the identity matrix in the I-columns and zeros elsewhere. Similarly, for $J \subset \{1, \ldots, km\}$, let D_J denote the $\ell \times km$ matrix which has the identity matrix in the I-columns and zeros elsewhere. Then the fixed points of the S-action are (D_I, D_J) for all possible combinations of I and J.

Now we can read off the twisted I-function using Corollary 6.2.1. We apply the mirror theorem in [13, Cor 7.3.2] (stated here as Theorem 6.4.1) to conclude that (79) holds.

References

[1] J. Alper, J. Hall, and D. Rydh. A Luna étale slice theorem for algebraic stacks. *Ann. of Math. (2)*, 191(3):675–738, 2020.

[2] V. Balaji. Lectures on principal bundles. 3rd cycle. Guanajuato (Mexique), 2006.

[3] A. Bertram, I. t. Ciocan-Fontanine, and B. Kim. Two proofs of a conjecture of Hori and Vafa. *Duke Math. J.*, 126(1):101–136, 2005.

[4] A. Bertram, I. t. Ciocan-Fontanine, and B. Kim. Gromov-Witten invariants for abelian and nonabelian quotients. *J. Algebraic Geom.*, 17(2):275–294, 2008.

[5] M. Brion. The push-forward and Todd class of flag bundles. In *Parameter spaces (Warsaw, 1994)*, volume 36 of *Banach Center Publ.*, pages 45–50. Polish Acad. Sci. Inst. Math., Warsaw, 1996.

[6] M. Brion. Equivariant cohomology and equivariant intersection theory. In *Representation theories and algebraic geometry (Montreal, PQ, 1997)*, volume 514 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 1–37. Kluwer Acad. Publ., Dordrecht, 1998. Notes by Alvaro Rittatore.

[7] M. Brion. On automorphism groups of fiber bundles. *Publ. Mat. Urug.*, 12:39–66, 2011.

[8] H.-L. Chang, Y.-H. Kiem, and J. Li. Torus localization and wall crossing for cosection localized virtual cycles. *Adv. Math.*, 308:964–986, 2017.

[9] Q. Chen, F. Janda, and R. Webb. Virtual cycles of stable (quasi-)maps with fields. *Advances in Mathematics*, 385:107781, 2021.

[10] D. Cheong, I. Ciocan-Fontanine, and B. Kim. Orbifold quasimap theory. *Math. Ann.*, 363(3-4):777–816, 2015.

[11] I. Ciocan-Fontanine and B. Kim. Big I-functions. In *Development of moduli theory—Kyoto 2013*, volume 69 of *Adv. Stud. Pure Math.*, pages 323–347. Math. Soc. Japan, [Tokyo], 2016.

[12] I. t. Ciocan-Fontanine and B. Kim. Moduli stacks of stable toric quasimaps. *Adv. Math.*, 225(6):3022–3051, 2010.

[13] I. t. Ciocan-Fontanine and B. Kim. Wall-crossing in genus zero quasimap theory and mirror maps. *Algebr. Geom.*, 1(4):400–448, 2014.

[14] I. t. Ciocan-Fontanine, B. Kim, and D. Maulik. Stable quasimaps to GIT quotients. *J. Geom. Phys.*, 75:17–47, 2014.

[15] I. t. Ciocan-Fontanine, B. Kim, and C. Sabbah. The abelian/nonabelian correspondence and Frobenius manifolds. *Invent. Math.*, 171(2):301–343, 2008.

[16] I. t. Ciocan-Fontanine, M. z. Konvalinka, and I. Pak. Quantum cohomology of $\mathrm{Hilb}_n(\mathbb{C}^2)$ and the weighted hook walk on Young diagrams. *J. Algebra*, 349:268–283, 2012.

[17] B. Conrad and T. Feng. Reductive groups over fields, 2020. virtualmath1.stanford.edu/ conrad/249BW16Page/handouts/249B_2016.pdf.

[18] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive Groups*. New Mathematical Monographs. Cambridge University Press, 2010.

[19] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[20] D. Edidin and W. Graham. Characteristic classes in the Chow ring. *J. Algebraic Geom.*, 6(3):431–443, 1997.

[21] D. Edidin and W. Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.

[22] G. Ellingsrud and S. A. Strømme. On the Chow ring of a geometric quotient. *Ann. of Math. (2)*, 130(1):159–187, 1989.
[23] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11:No. 4, 107–131, 2003.
[24] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
[25] W. Fulton and R. MacPherson. Categorical framework for the study of singular spaces. *Mem. Amer. Math. Soc.*, 31(243):vi+165, 1981.
[26] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.
[27] A. Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, volume 160 of *Progr. Math.*, pages 141–175. Birkhäuser Boston, Boston, MA, 1998.
[28] E. González and C. Woodward. Quantum Witten localization and abelianization for qde solutions, 2016. 0811.3358.
[29] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.
[30] J. Hall and D. Rydh. Perfect complexes on algebraic stacks. *Compos. Math.*, 153(11):2318–2367, 2017.
[31] J. Hall and D. Rydh. Coherent Tannaka duality and algebraicity of Hom-stacks. *Algebra Number Theory*, 13(7):1633–1675, 2019.
[32] A. (https://mathoverflow.net/users/97316/alexander). Fixed points of the weyl group action on a maximal torus and the center of a reductive group. MathOverflow. URL:https://mathoverflow.net/q/307868 (version: 2018-08-15).
[33] D. Inoue, A. Ito, and M. Miura. I-functions of Calabi-Yau 3-folds in Grassmannians. *Commun. Number Theory Phys.*, 11(2):273–309, 2017.
[34] F. Ischebeck. Über die Abbildung $\mathrm{Pic}A \to \mathrm{Pic}A_f$. *Math. Ann.*, 243(3):237–245, 1979.
[35] E. Kalashnikov. Four-dimensional Fano quiver flag zero loci. *Proc. Math. Phys. Eng. Sci.*, 475:20180791, 2019.
[36] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.
[37] F. C. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*, volume 31 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1984.
[38] C. Manolache. Virtual pull-backs. *J. Algebraic Geom.*, 21(2):201–245, 2012.
[39] S. Martin. Symplectic quotients by a nonabelian group and by its maximal torus, 2000. arXiv:0001002v1.
[40] V. B. Mehta and S. Subramanian. Principal bundles on the projective line. *Proc. Indian Acad. Sci. Math. Sci.*, 112(3):393–397, 2002.
[41] M. Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.
[42] M. C. Olsson. Deformation theory of representable morphisms of algebraic stacks. *Math. Z.*, 253(1):25–62, 2006.
[43] M. Romagny. Group actions on stacks and applications. *Michigan Math. J.*, 53(1):209–236, 2005.
[44] T. Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2019.
[45] B. Totaro. The resolution property for schemes and stacks. *J. Reine Angew. Math.*, 577:1–22, 2004.
[46] B. Totaro. *Group cohomology and algebraic cycles*, volume 204 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2014.
[47] R. Vakil. The rising sea: foundations of algebraic geometry, 2017. math.stanford.edu/ vakil/216blog/FOAGnov1817public.pdf.
[48] R. Webb. *Functoriality and the Moduli of Sections, with applications to quasimaps*. PhD thesis, University of Michigan, 5 2020.
[49] R. Webb. Quasimaps and some examples of stacks for everybody. In *Singularities, mirror symmetry, and the gauged linear sigma model*, volume 763 of *Contemp. Math.*, pages 1–17. Amer. Math. Soc., [Providence], RI, 2021.
[50] Y. Wen. K-theoretic I-function of $V /\!\!/_\theta G$ and application, 2019. arXiv:1906.00775.
[51] Y. Zhou. Quasimap wall-crossing for GIT quotients, 2020. arXiv:1911.02745.

(R. Webb) Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840, U.S.A.

Email address: `rwebb@berkeley.edu`