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Numerical invariants of Quot schemes of curves

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Numerical invariants of Quot schemes of curves

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by

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The Dissertation of Shubham Sinha is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2023

DEDICATION

In loving memory of my father Pankaj Kumar Sinha.

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• Dragos Oprea and Shubham Sinha, "*Euler characteristics of tautological bundles over Quot schemes of curves*", Advances in Mathematics, Volume 418 (2023), 108943.

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Chapter 4, in part is currently being prepared for submission for publication of the material. Ming Zhang and Shubham Sinha "*Quantum K-invariants of Grassmannian via Quot scheme*". I would like to thank the coauthor Ming Zhang for permitting to include the material in the paper to the thesis.

Chapter 5 and 6, in part, has been submitted for publication. The dissertation author was the primary investigator and author of the material below.

• Shubham Sinha "*The virtual intersection theory of isotropic Quot Schemes*".

VITA

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ABSTRACT OF THE DISSERTATION

Numerical invariants of Quot schemes of curves

by

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Doctor of Philosophy in Mathematics

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Professor Dragos Oprea, Chair

I present formulas for the Euler characteristics of tautological sheaves over the punctual Quot scheme, which parameterizes zero-dimensional quotients of a fixed vector bundle over curves. We observe a striking similarity with the formulas for the Hilbert scheme of points on surfaces. Furthermore, we study the Quot schemes of higher rank quotients for a genus-zero curve. We calculate the holomorphic Euler characteristics of Schur bundles and tautological bundles over Quot schemes. These formulas can be considered a generalization of the formulas for Grassmannians, which were obtained using the Borel-Weil-Bott theorem. Additionally, we show non-trivial vanishing results using these formulas.

The symplectic (or orthogonal) Grassmannian parameterizes isotropic subspaces of a vector space endowed with a symplectic (or symmetric) bilinear form. I study the intersection theory of the symplectic and orthogonal isotropic Quot schemes. In particular, I construct a virtual fundamental class for these Quot schemes and find explicit formulas for certain intersection numbers. I also calculate the Gromov-Ruan-Witten invariants of the corresponding Grassmannians and compare the answers with those for the isotropic Quot schemes.

Chapter 1

Preliminaries

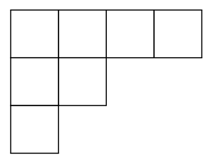
Mathematicians have been interested in counting geometric objects for centuries. The earliest questions to be studied include counting number of conics passing through five general points and Apollonius's problem of determining the number of circles tangent to three circles in general position. Even in the Euclidean geometry, the early mathematicians understood the importance of parameter space. Projective algebraic geometry over the complex numbers is often the most convenient place to study enumerative problems, and several classical problems can be solved using the machinery developed here.

The intersection theory of the Grassmannian, known as Schubert calculus, is an important development in enumerative geometry, representation theory and combinatorics from nineteenth century. It helps solve several counting problems in projective geometry and still an active area of research. In late twentieth century, physicists and mathematicians started enumerating curves on Grassmannians and other projective varieties. This led to the construction and the study of many moduli spaces.

The Quot scheme is a natural generalization of the Grassmannian. In particular, the Quot scheme provides a compactification of the space of morphisms from a smooth projective curve C to the Grassmannian. Quot schemes play an important role in constructing and understanding the moduli space of vector bundles (or sheaves) and other moduli spaces of interest. The intersection theory of the Quot scheme is related to many important topics in enumerative geometry and mathematical physics such as Gromov-Witten theory, Verlinde numbers, and topological quantum field theory. In this chapter, I give basic definitions and theorems on Grassmannian and Quot schemes.

1.1 Schur polynomials

An integer partition λ is a non-increasing finite sequence of positive integers $(\lambda_1,\ldots,\lambda_r)$. A partition λ is graphically represented using Young diagrams, in which we place λ_i boxes in the i'th row. For example, the Young diagram (in English notation) of the partition $(4,2,1)$ is



The number of parts of a partition $\lambda = (\lambda_1,\ldots,\lambda_r)$ is r and the size of a partition is the number of boxes in its Young diagram and denoted by $|\lambda| = \lambda_1 + \cdots + \lambda_r$. The conjugate partition of λ, denoted λ', is the obtained by taking the transpose of the Young diagram of λ. For example, the transpose of the partition $(4,2,1)$ is $(3,2,1,1)$.

Definition 1.1.1. *For any r variables $x_1, x_2, \ldots, x_r$ and an integer partition λ with at most r parts, the Schur polynomials associated to λ is defined using the Jacobi bialternant formula*

$$s_\lambda(x_1,\ldots,x_r) = \frac{1}{\det(x_i^j)} \begin{vmatrix} x_1^{\lambda_1+r-1} & & \cdots & & x_r^{\lambda_1+r-1} \\ x_1^{\lambda_2+r-2} & & \cdots & & x_r^{\lambda_2+r-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^{\lambda_r} & & \cdots & & x_r^{\lambda_r} \end{vmatrix}. \tag{1.1}$$

where the denominator is the $r \times r$ Vandermonde determinant. If the number of parts of λ is strictly less than r, concatenate required number of zeros 0's at the end to define the above matrix.

Schur polynomials are symmetric polynomials in the variables $x_1, x_2, \ldots, x_r$. Furthermore, any symmetric polynomial in $x_1, x_2, \ldots, x_r$ can be uniquely expressed as linear sum of Schur polynomials corresponding to partitions with at most r parts (i.e they form a basis for the space of symmetric polynomials).

The ring of symmetric polynomials is generated by elementary symmetric polynomials $e_i(x_1, \ldots, x_r)$ for $0 \leq i \leq r\}$ where

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k}.$$

The (second) Jacobi-Trudi formula expresses the Schur polynomials in terms of elementary symmetric polynomials and is given by the $\ell \times \ell$ determinant

$$s_\lambda = \det \begin{bmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & \cdots & e_{\lambda'_1+\ell-1} \\ e_{\lambda'_1-1} & e_{\lambda'_1} & \cdots & e_{\lambda'_1+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_\ell-\ell+1} & e_{\lambda'_\ell-\ell+2} & \cdots & e_{\lambda'_\ell} \end{bmatrix} \tag{1.2}$$

where λ' is the conjugate partition of λ and $\ell = \lambda_1$ is the number of parts of λ'. Here we define $e_k = 0$ when k is negative or $k > r$.

Note that $s_{(1^n)}(x_1, \ldots, x_r) = e_n(x_1, \ldots, x_r)$, where $(1^n) = (1, 1, \ldots, 1)$ (repeated n times). Similarly, $s_{(n)}(x_1, \ldots, x_r) = h_n(x_1, \ldots, x_r)$ is the complete homogeneous symmetric polynomial.

The product of Schur polynomials can be expressed in terms of Schur polynomials using the Littlewood Richardson rule, which states that

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

where $c^\nu_{\lambda,\mu}$ is the number of Littlewood Richardson tableaux (see 4.1.7) of skew shape ν/λ and weight μ.

1.2 Schubert Calculus

Let V be a rank$N$ vector space over $\mathbb{C}$. The Grassmannian $\mathrm{Gr}(r,V)$ parameterizes r dimensional subspaces of V. There exists a universal short exact sequence of vector bundles on $\mathrm{Gr}(r,V)$

$$0 \to S \to V \times \mathrm{Gr}(r,V) \to Q \to 0.$$

For any point $q \in \mathrm{Gr}(r,V)$, the corresponding subspace of q equals $S|_{\{q\}} \subset V \times \{q\}$.

The cohomology ring of Grassmannian $H^*(\mathrm{Gr}(r,V),\mathbb{C})$ can be described using the Chern classes of the tautological subbundle (or equivalently the tautological quotient bundle). The chern classes of S, denoted by

$$a_i := c_i(S^\vee) \in H^{2i}(\mathrm{Gr}(r,V)),$$

forms a multiplicative generator of the cohomology ring $H^*(\mathrm{Gr}(r,V))$. All the relations are derived from the identity

$$c(S) \cdot c(Q) = 1.$$

More concretely, the Chern classes of Q can be viewed as a polynomial in a_i's and is given by Segre ploynomials $b_i = c_i(Q)$, where the polynomials s_i recursively obtained by solving

$$(1 - a_1 + a_2 - \cdots + (-1)^k a_r)(1 + b_1 + b_2 + \cdots) = 1.$$

Note that Q is a rank$N - r$ vector bundles, thus $b_i = c_i(Q) = 0$ for $i \geq N - r + 1$. This gives a presentation for the cohomology ring of $\mathrm{Gr}(r,V)$:

$$H^*(\mathrm{Gr}(r,V),\mathbb{C}) = \mathbb{C}[a_1, a_2, \ldots, a_r]/\langle b_{N-r+1}, \ldots, b_N \rangle.$$

Schubert calculus describes a linear basis for the cohomology ring $H^*(\mathrm{Gr}(r,V),\mathbb{C})$ and multiplication of these basis elements. This is explicitly understood using the combinatorics of Schur polynomials. Let $x_1, x_2, \ldots, x_r$ denote the Chern roots of $S^\vee$, that is, $c_i(S^\vee) = e_i(x_1, \ldots, x_r)$ where e_i's are the elementary symmetric polynomials. The linear basis of the cohomology ring $H^*(\mathrm{Gr}(r,V), \mathbb{C})$ is given by the Schur functions. Let $\mathcal{P}^{r,\ell}$ denote set of integer partition λ contained in $r \times \ell$ rectangular box, i.e.

$$\mathcal{P}^{r,\ell} = \{\lambda = (\lambda_1, \ldots, \lambda_r) \mid 0 \leq \lambda_r \leq \cdots \leq \lambda_1 \leq \ell\}.$$

Then $H^*(\mathrm{Gr}(r,V), \mathbb{C})$ is generated (as a vector space) by $\{s_\lambda(x_1, \ldots, x_r) : \lambda \in \mathcal{P}^{r,N-r}\}$. Furthermore, these classes can be represented in terms of the multiplicative generators $\{a_1, a_2, \ldots, a_r\}$ using the Jacobi-Trudi formula (see (1.2)), and the multiplication rule for these classes are given by the Littlewood Richardson rule.

1.3 Symmetric powers of curves

Let C be a smooth projective curves over the field of complex numbers (or a compact Riemann surface). The cohomology ring of C, $H^*(C, \mathbb{Z})$, admits symplectic basis $\{1, \delta_1, \ldots, \delta_{2g}, \omega\}$ with the relations

$$\delta_i \delta_{i+g} = \omega = -\delta_{i+g} \delta_i$$

for all $1 \leq i \leq g$. Here ω is Poincaré dual of the single point class in C.

The symmetric power of the curve $C^{(d)}$ is isomorphic to the Hilbert scheme of d points on C, denoted $C^{[d]}$ (since the Hilbert-Chow morphism $C^{[d]} \to C^{(d)}$ is an isomorphism for curves). There is a universal sequence over $C^{[d]} \times C$

$$0 \to \mathcal{K} \to \mathcal{O}_{C^{[d]} \times C} \to \mathcal{T} \to 0.$$

Consider the Künneth decomposition of the cohomology classes $c_1(\mathcal{K}^\vee)$ in $C^{[d]} \times C$ with respect to a chosen symplectic basis of $H^*(C,\mathbb{Z})$,

$$c_1(\mathcal{K}^\vee) = x\otimes 1 + \sum_{k=1}^{2g} y^k \otimes \delta_k + d\otimes \omega. \tag{1.3}$$

The cohomology classes $x \in H^2(C,\mathbb{Z})$ and $y^k \in H^1(C,\mathbb{Z})$ for $1 \le k \le 2g$ generate the cohomology ring $H^*(C,\mathbb{Z})$. There is a natural map

$$\phi : C^{[d]} \to \mathrm{Pic}^d, \quad D \to \mathcal{O}_C(D)$$

where $Pic^d(C)$ denote the Picard group parameterizing degree d line bundles on C. By abuse of notation, we let $\theta \in H^2(C,\mathbb{Z})$, is the pullback of the usual theta class on Pic^d under the map ϕ. We have the following relation (explained in [ACGH])

$$\left(\sum_{k=1}^{2g} (y^k \otimes \delta_k)\right)^2 = -2\theta \otimes \omega.$$

The following are some known facts about the x, θ and y classes (see [ACGH] and [Tha]) over $C^{[d]}$:

- The intersections of x and θ are given by:

$$\int_{C^{[d]}} \theta^\ell x^{d-\ell} = \begin{cases} \frac{g!}{(g-\ell)!} & \ell \le g \\ 0 & \ell > g \end{cases}.$$

  In particular, for any polynomial P, and $\ell \le g$

$$\int_{C^{[d]}} \theta^\ell P(x) = \frac{g!}{(g-\ell)!} \int_{C^{[d]}} x^\ell P(x). \tag{1.4}$$

- The non-zero integrals in the y classes over $C^{[d]}$ satisfy

  (i) y^k appears with exponent at most 1 because these are odd classes.

(ii) y^k appears if and only if y^{k+g} appears.

(iii) For any choice of choice of distinct integers $k_1, \dots, k_s \in \{1, \dots g\}$ and a polynomial P in two variables,

$$\int_{C^{[d]}} y^{k_1} y^{k_1+g} \cdots y^{k_s} y^{k_s+g} P(x, \theta) = \frac{(g-s)!}{g!} \int_{C^{[d]}} \theta^s P(x, \theta). \tag{1.5}$$

1.4 Quot scheme of curves

Definition 1.4.1. *Let E be a vector bundle (or a locally free sheaf) of rank N over C. The **punctual Quot scheme** $\mathsf{Quot}_d(E)$ parameterizes degree d rank 0 quotient sheaves of E. Here, a sheaf has rank 0 if it is supported on a divisors of C. Equivalently, $\mathsf{Quot}_d(E)$ parameterizes short exact sequences of sheaves*

$$0 \to S \to E \to Q \to 0,$$

such that S is a locally free sheaf of rank N and $\deg Q = d$.

The punctual Quot scheme $\mathsf{Quot}_d(E)$ is a smooth projective scheme. There is a map from $\mathsf{Quot}_d(E) \to C^{[d]}$ sending the quotient Q to the support of Q. When E is a line bundle, this map is an isomorphism.

Several geometric properties of $\mathsf{Quot}_d(E)$, such as the Poincaré polynomial and motives [Bif, BFP, Che, Ric], stabilization of cohomology [Moc], the automorphism group [BDH], the Picard group and certain cones of divisors [GS] have been studied. Recently, the derived categories of $\mathsf{Quot}_d(E)$ was studied in [Tod]. In joint work with Dragos Oprea, we study K-theoretic invariants of $\mathsf{Quot}_d(E)$ (see Chapter 3).

Definition 1.4.2. *Fix a vector bundle E over a smooth projective curve C. The **Quot scheme** $\mathsf{Quot}_d(E, r)$ parameterizes rank r subsheaves of E of degree $-d$. We denote $\mathsf{Quot}_d(N, r, C)$ for*

$\mathsf{Quot}_d(\mathcal{O}^{\oplus N}, r)$.

In general, the Quot scheme $\mathsf{Quot}_d(N,r,C)$ provides a compactification of the space of degree d morphisms $\mathrm{Mor}_d(C, G(N,r))$ from C to the Grassmannian. This approach was pioneered by Bertram and collaborators [Ber, BDW]. A geometric comparison of the Quot compactification to the stable map compactification was studied by [PR].

Explicit expressions for the count of maps to the Grassmannian subject to incidence conditions with Schubert subvarieties are given by Vafa-Intriligator formulas. In the mathematics literature, these formulas were obtained in [Ber], [ST] and [MO 3] using the two compactifications mentioned above, see also [Int] for the physics reference.

One of the most spectacular applications of the Vafa-Intriligator formula for the Quot scheme appears in its connection to the Verlinde numbers [MO 2, Wit]. Furthermore, other invariants over the Quot scheme are also related to the invariants over the moduli space of vector bundles [MO 1, BDW, RZ].

1.5 Vafa-Intriligator formula

Below we describe the Vafa-Intriligator formula in the context Quot schemes. Let C be a smooth projective curve of genus g. The Quot scheme $\mathsf{Quot}_d(N,r,C)$ is not smooth in general. The intersection theory of the Quot scheme was studied in [MO 3] by constructing the virtual fundamental class and virtual $\mathbb{C}^*$ localization.

Theorem 1.5.1 ([MO 3])**.** *The scheme* $\mathsf{Quot}_d(N,r,C)$ *admits a virtual fundamental class*

$$[\mathsf{Quot}_d(N,r,C)]^{\mathrm{vir}} \in H_{2\,\mathrm{vd}}(\mathsf{Quot}_d(N,r,C), \mathbb{C}),$$

where vd *is the expected (or virtual) dimension given by* $\mathrm{vd} = Nd + (1-g)r(N-r)$.

In this subsection, we describe the virtual invariants and postpone the definition of virtual fundamental class to the next subsection.

Here, we consider the universal exact sequence over $C \times \mathsf{Quot}_d(N, r, C)$,

$$0 \to \mathcal{S} \to p^* \mathcal{O}_C^{\oplus N} \to \mathcal{Q} \to 0,$$

where p and π are the projection maps to C and $\mathsf{Quot}_d(N, r, C)$ respectively. For any point $x \in C$, let $\mathcal{S}_x$ be the restriction of $\mathcal{S}$ to $\{x\} \times \mathsf{Quot}_d(N, r, C)$. Then we define

$$a_i = c_i(\mathcal{S}_x^\vee) \in H^{2i}(\mathsf{Quot}_d(N, r)).$$

Theorem 1.5.2 ([MO 3]). *Let $P(z_1, \ldots, z_r)$ be a polynomial in r variables of weighted degree* vd, *where the variable z_i has degree i. Define*

$$J(x_1, \ldots, x_r) := N^r x_1^{-1} \cdots x_r^{-1} \left(\det(x_i^j)\right)^{-2}$$

where $\det(x_i^j)$ is the $r \times r$ Vandermonde determinant. Then

$$\int_{[\mathsf{Quot}_d]^{\mathrm{vir}}} P(a_1, \ldots, a_r) = u \cdot \sum_{\xi_1, \ldots, \xi_r} R(\xi_1, \ldots, \xi_r) J^{g-1}(\xi_1, \ldots, \xi_r) \tag{1.6}$$

where $\{\xi_1, \ldots, \xi_r\}$ runs over $\binom{N}{r}$ tuples of distinct N^{th} roots of unity. Here

$$u = (-1)^{(g-1)\binom{r}{2} + d(r-1)},$$

and R is the symmetric polynomial obtained by expression $P(a_1, \ldots, a_r)$ in terms of the Chern roots of $\mathcal{S}_x^\vee$.

Note that a_i's generate the ring of symmetric polynomials in the Chern roots of $\mathcal{S}_x^\vee$, thus we may replace P with product of Schur polynomials of Chern roots of $\mathcal{S}_x^\vee$ to obtain an analogous formulas. When $d = 0$ and $C = \mathbb{P}^1$, the Quot scheme $\mathsf{Quot}_d(N, r)$ is isomorphic to the Grassmannian $\mathrm{Gr}(r, \mathbb{C}^N)$. In this case, the formula (1.6) gives a new approach to Schubert

Calculus.

Let $\{1, \delta_1, \dots \delta_{2g}, \omega\}$ be a symplectic basis for the cohomology of C. Let the Künneth decomposition of $\mathcal{S}^\vee$ over $C \times \mathsf{Quot}_d(N,r,C)$ be

$$c_i(\mathcal{S}^\vee) = a_i \otimes 1 + \sum_{k=1}^{2g} b_i^k \otimes \delta_k + f_i \otimes \omega,$$

where $a_i \in H^{2i}(\mathsf{Quot}_d, \mathbb{C})$, $b_i^k \in H^{2i-1}(\mathsf{Quot}_d, \mathbb{C})$ and $f_i \in H^{2i-2}(\mathsf{Quot}_d, \mathbb{C})$. When the Quot scheme $\mathsf{Quot}_d(N,r,C)$ is smooth, for example punctual Quot scheme or when $C = \mathbb{P}^1$, the classes a_i, b_i^k and f_i forms a generator the cohomology ring. In [MO 3], formulas for finding intersection numbers involving the above classes were also obtained.

1.6 Perfect obstruction theory

We will briefly describe the results pertaining to the construction of virtual fundamental classes in [BF]. Let X be a scheme (or a stack) over a scheme (or a stack) S and $\mathbb{L}_{X/S}$ be the relative cotangent complex.

Definition 1.6.1. *A 2-term relative perfect obstruction theory is a morphism in the derived category*

$$\phi : E^\bullet \to \tau_{[-1,0]}\mathbb{L}_{X/S},$$

where $E^\bullet = [E^{-1} \to E^0]$ is a complex of vector bundles over X of amplitude contained in $[-1,0]$ and satisfies:

- *h^0 is an isomorphism and*
- *h^{-1} is a surjection.*

Let $[E_0 \to E_1]$ be the dual of $E^\bullet$. Given a 2-term perfect obstruction theory, [BF] and [LT] define a cone inside E_1. The virtual fundamental class is then defined to be an element in $H_{2e}(X)$ given by the refined intersection of the cone with the zero section of E_1. Here $e = \operatorname{rank} E_0 - \operatorname{rank} E_1$ is called the virtual dimension of X.

Let X be a projective scheme. The group $K_0(X)$ (resp. $K^0(X)$) denotes the Grothendieck group of coherent sheaves (resp. locally free sheaves) on X. For practical purposes, we only need the description of the virtual tangent (or cotangent) bundle, which is an element in the K-theory

$$T_X^{\mathrm{vir}} = [E_0] - [E_1] \in K^0(X).$$

The simplest case is when X is a closed subscheme of a smooth scheme Y cut out by a section s of a vector bundle V over Y. In this case, there is a natural 2-term perfect obstruction theory given by $[V^\vee|_X \to \Omega_Y|_X]$. Note that when s is a regular section, we get the usual fundamental class.

1.7 Schur bundles on Grassmannian

We move our attention to K-theoretic invariants. In Section 1.2, we observed that the Chern classes of the universal subbundle S (or its dual $S^\vee$) generated the cohomology ring of $\mathrm{Gr}(r,V)$. It turns out that the K-theory of Grassmannian, $K^0(\mathrm{Gr}(r,V))$, admits a basis consisting of the Schur bundles associated to S.

Definition 1.7.1. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an integer partition and $V = \mathbb{C}^n$ (standard representation of $GL_n(\mathbb{C})$). The Schur functor $\mathbb{S}^\lambda$ associates $\mathbb{S}^\lambda(V)$, the unique irreducible representation of $GL_n(\mathbb{C})$ of highest weight λ. For any $g \in GL_n(\mathbb{C})$, the trace of g on $\mathbb{S}^\lambda(V)$ is given by*

$$\chi_{\mathbb{S}^\lambda(V)}(g) = s_\lambda(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ are eigenvalues of g. In particular, $\dim \mathbb{S}^\lambda(V) = s_\lambda(\underbrace{1,1,\ldots,1}_{n \text{ times}})$.

The Schur functors also associates to a partition λ and a vector bundles $V \to X$, for a variety X, another vector bundles denoted $\mathbb{S}^\lambda(V) \to X$ (see [Wey] for detailed description).

Recall that $\mathcal{P}^{r,N-r}$ denote the set of partitions λ inside the rectangular partition $(N-r,\ldots,N-r)$ (repeated r times). Then the Grothendieck group $K^0(\mathrm{Gr}(r,V))$ (equivalently $K^0(\mathrm{Gr}(r,V))$) admits a $\mathbb{Z}$-basis $\{\mathbb{S}^\lambda(S) : \lambda \in \mathcal{P}^{r,N-r}\}$ consisting of Schur bundles of the universal subbundle S. The cohomology groups of the these Schur bundles are explicitly described using Borel-Weil-Bott theorem on flag manifolds. The precise statements are noted below:

Proposition 1.7.2. *For any partition non-empty partition $\lambda \in \mathcal{P}^{r,N-r}$,*

(a) For all $i \geq 0$, $H^i(Gr(r,V), \mathbb{S}^\lambda(S)) = 0$.

(b) For all $i > 0$, $H^i(Gr(r,V), \mathbb{S}^\lambda(S^\vee)) = 0$, and

$$H^0(Gr(r,V), \mathbb{S}^\lambda(S^\vee)) \cong \mathbb{S}^\lambda(V^\vee),$$

where $V^\vee$ is the dual representation of the standard representation.

Let E be a coherent sheaf over scheme X, then the holomorphic Euler characteristics of E is

$$\chi(X,E) = \sum_{i=0}^{n} \dim\, H^i(X,E).$$

When higher cohomology vanish $\chi(X,E)$ equals the dimension of the space of global sections of E. The Euler characteristics is well behaved in flat families, and it is often easier to compute than individual cohomology groups. In particular, the above proposition implies that for a non-empty partition $\lambda \in \mathcal{P}^{r,N-r}$, $\chi(Gr(N,V), \mathbb{S}^\lambda(S)) = 0$ and

$$\chi(Gr(N,V), \mathbb{S}^\lambda(S^\vee)) = s_\lambda(\underbrace{1,1,\ldots,1}_{N \text{ times}}).$$

In Chapter 4, I prove a generalization of the above formula for $\mathsf{Quot}_d(N,r,\mathbb{P}^1)$.

Chapter 2

Summary of results

2.1 Punctual Quot scheme

Let E be a vector bundle of rank N over C. The punctual Quot scheme $\mathsf{Quot}_d(E)$ parameterizes short exact sequences

$$0 \to S \to E \to Q \to 0$$

where Q has rank zero and degree d. It is a smooth projective scheme of dimension Nd.

Define the vector bundle $L^{[d]} := \pi_*(p^*L \otimes \mathcal{Q})$ for any line bundle $L \to C$. Here $\mathcal{Q}$ is the universal quotient over $C \times \mathsf{Quot}_d$, and p and π are the first and the second projection. For any vector bundle V, we can package all exterior powers of V into the polynomial

$$\wedge_y V := \sum_k y^k \wedge^k V.$$

Theorem 2.1.1. *Let $E \to C$ be a vector bundle and let $L \to C$ be a line bundle. Then*

$$\sum_{d=0}^{\infty} q^d \chi(\mathsf{Quot}_d(E), \wedge_y L^{[d]}) = (1-q)^{-\chi(\mathcal{O}_C)}(1+qy)^{\chi(E\otimes L)}. \tag{2.1}$$

There is an analogous result for the Hilbert scheme of points on surfaces. Let X be a smooth projective surface, and L a line bundle over X. The tautological bundle $L^{[d]}$ is defined in the same fashion as above. Then

$$\sum_{d=0}^{\infty} q^d \chi(X^{[d]}, \wedge_y L^{[d]}) = (1-q)^{-\chi(\mathcal{O}_S)}(1+qy)^{\chi(L)}.$$

The case of surfaces was proven by Luca Scala and Andreas Krug in [Sca 1, Kru] using the celebrated Bridgeland-King-Reid equivalence $D^b(X^{[d]}) \cong D^b_{S_d}(X^d)$, and was obtained in [Arb] using Donaldson-Thomas theory of toric Calabi-Yau 3-folds.

We have a generalization of Theorem 2.1.1 to with multiple insertions.

Theorem 2.1.2. *For any line bundles $M_1, M_2, \ldots, M_r$ and L over C, where $0 \le r \le rk\, E - 1$, we have*

$$\sum_{d=0}^{\infty} q^d \chi\left(\mathsf{Quot}_d(E), \wedge_y L^{[d]} \otimes_{i=1}^{r} \left(\wedge_{x_i} M_i^{[d]}\right)^{\vee}\right)$$
$$= (1-q)^{-\chi(\mathcal{O}_C)}(1+qy)^{\chi(E\otimes L)} \prod_{i=1}^{r} (1-qx_i y)^{-\chi(M_i^{\vee}\otimes L)}.$$

For Hilbert scheme of points on surface, the formula for the Euler characteristics lifts to an isomorphism between the cohomology groups. It is natural to ask if a similar result holds for the punctual Quot schemes $\mathsf{Quot}_d(E)$. In particular, we formulate a conjecture (see Subsection 3.1.4 for the notation):

Conjecture 2.1.3. *For any line bundle $L \to C$,*

$$H^{\bullet}\left(\mathsf{Quot}_d(E), \wedge^k L^{[d]}\right) = \wedge^k H^{\bullet}(E \otimes L) \otimes \mathsf{Sym}^{d-k} H^{\bullet}(\mathcal{O}_C), \tag{2.2}$$

where the above exterior and symmetric powers are understood in graded sense.

The symmetric powers over the Hilbert scheme of points on surfaces were studied by [Dan, Arb, Sca 2]. We obtain an analogous formula for the Quot scheme:

Theorem 2.1.4. *For $C = \mathbb{P}^1$ and $d \geq k$, we have*

$$\chi\left(\mathsf{Quot}_d(E), \mathsf{Sym}^k L^{[d]}\right) = \binom{\chi(E \otimes L) + k - 1}{k}.$$

Let $\mathsf{Sym}_y V = \sum_{k=0}^{\infty} y^k \mathsf{Sym}^k V$. In arbitrary genus, we use the cobordism argument of [EGL] to show that there exist universal series A and B in $\mathbb{Q}(y)[[q]]$ such that

$$\sum_d q^d \chi\left(\mathsf{Quot}_d, \mathsf{Sym}_y L^{[d]}\right) = \mathsf{A}^{\chi(\mathcal{O}_C)} \cdot \mathsf{B}^{\chi(E \otimes L)}.$$

that depend on N, but not on the triple (C, E, L). Our results give precise information about the series B. While we can determine A in principle, we do not have a closed-form expression.

Theorem 2.1.5. *We have*

$$\mathsf{B} = f\left(\frac{qy}{(1-y)^{N+1}}\right)$$

where $f(z)$ is the analytic solution to the equation $f(z)^N - f(z)^{N+1} + z = 0$ with $f(0) = 1$.

In the special case $N = 2$, we obtain

$$f(z) = 1 + \frac{4}{3}\sinh^2\left(\frac{1}{3}\operatorname{arcsinh}\left(\frac{3\sqrt{3z}}{2}\right)\right).$$

2.2 Quot scheme of $\mathbb{P}^1$

The Quot scheme $\mathsf{Quot}_d(N, r, \mathbb{P}^1)$ (denoted $\mathsf{Quot}_d(N, r)$) is a smooth scheme. Recall that the cohomological invariants of $\mathsf{Quot}_d(N, r)$ were calculated using the Vafa-Intriligator formula (see Section 1.5). We further the study to obtain K-theoretic formulas for the Quot scheme over $\mathbb{P}^1$.

Recall that there is universal exact sequence over $\mathbb{P}^1 \times \mathsf{Quot}_d(N, r)$,

$$0 \to \mathcal{S} \to p^* \mathcal{O}_{\mathbb{P}^1}^{\oplus N} \to \mathcal{Q} \to 0,$$

where p and π are the projection maps to $\mathbb{P}^1$ and $\mathsf{Quot}_d(N,r)$ respectively. For any point $x \in \mathbb{P}^1$, the restriction of $\mathcal{S}$ to $\{x\} \times \mathsf{Quot}_d(N,r)$, denoted $\mathcal{S}_x$, is a rank r vector bundle. For any partition λ, let $\mathbb{S}^{\lambda}(\mathcal{S}_x)$ denote the associated Schur bundle. We prove the following theorems, extending the known results for Grassmannian (see Proposition 1.7.2):

Theorem 2.2.1. *For any non-empty partition $\lambda \in \mathcal{P}^{r,N-r+d}$,*

$$\chi(\mathsf{Quot}_d(N,r), \mathbb{S}^{\lambda}(\mathcal{S}_x)) = 0.$$

Theorem 2.2.2. *For any partition λ with at most r parts, we have*

$$\chi(\mathsf{Quot}_d(N,r), \mathbb{S}^{\lambda}(\mathcal{S}_x^{\vee})) = [t^d] s_{\Lambda}(z_1, \ldots, z_N)$$

where $z_1, \ldots, z_N$ are roots of $(z-1)^N + (-1)^r z^{N-r} t = 0$, and the partition

$$\Lambda = (d+\lambda_1, d+\lambda_2, \ldots, d+\lambda_r).$$

Remark 2.2.3. *Recall that the Schur polynomial $s_{\Lambda}(z_1, \ldots, z_N)$ is a symmetric polynomial that can be expressed in terms of the elementary symmetric polynomials in $z_1, \ldots, z_N$ using Jacobi-Trudi formula. The elementary symmetric polynomials are given by*

$$e_m(z_1, \ldots, z_N) = \begin{cases} \binom{N}{m} & m \neq r \\ \binom{N}{r} + t & m = r \end{cases}.$$

This implies that that $s_{\Lambda}(z_1, \ldots, z_N)$ is a polynomial in t (that depends on d, N and λ).

Corollary 2.2.4. *We have*

$$\sum_{d=0}^{\infty} q^d \chi(\mathsf{Quot}_d(N,r), \wedge^m(\mathcal{S}_x^\vee)) = \begin{cases} \binom{N}{m}\frac{1}{1-q} & m \neq r \\ \binom{N}{r}\frac{1}{(1-q)^2} & m = r \end{cases}.$$

Remark 2.2.5. *The vanishing result in the Theorem 2.2.1 and the Littlewood-Richardson rule implies that for any partitions $\lambda^1, \dots, \lambda^m$, the power series*

$$F(q;\lambda^1,\dots,\lambda^m) := \sum_{i=0}^{\infty} q^d \chi(\mathsf{Quot}_d(N,r), \mathbb{S}^{\lambda^1}(\mathcal{S}_x) \otimes \cdots \otimes \mathbb{S}^{\lambda^m}(\mathcal{S}_x))$$

is a polynomial in q of degree at most $\lambda_1^1 + \cdots + \lambda_1^m - (N-r)$. The bound on the degree can be improved by imposing extra conditions.

Proposition 2.2.6. *Let $r < N$. For any non-trivial partition λ with exactly r parts (i.e $\lambda_r \neq 0$) and $\lambda_1 \leq d + 2(N-r)$, we have $\chi(\mathsf{Quot}_d(N,r), \mathbb{S}^\lambda(\mathcal{S}_x)) = 0$.*

Corollary 2.2.7. *For any partitions λ and μ contained in the rectangular partition $(N-r,\dots,N-r)$ where $N-r$ is repeated r times, and $d > 0$,*

$$\chi\big(\mathsf{Quot}_d(N,r), \det\mathcal{S}_x \otimes \mathbb{S}^\lambda(\mathcal{S}_x) \otimes \mathbb{S}^\mu(\mathcal{S}_x)\big) = 0.$$

Remark 2.2.8. *The genus 0, 3-pointed Quantum K-invariants of Grassmannian $X = \mathrm{Gr}(r,\mathbb{C}^N)$ are defined as follows. Let $ev_i : \overline{M}_{0,3}(X,d) \to X$ for the evaluation maps from the moduli space of* 3 *pointed* $\deg d$ *stables maps from $\overline{M}_{0,3}(X,d)$. The quantum K-invariants are defined by*

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,d} := \chi\left(\overline{M}_{0,3}(X,d), \mathcal{O}^{\mathrm{vir}}_{\overline{M}_{0,3}(X,d)} \cdot \prod_{i=1}^{3} ev_i^*(\alpha_i)\right).$$

In the upcoming work with Ming Zhang, we show that

$$\langle \mathbb{S}^\nu(S), \mathbb{S}^\lambda(S), \mathbb{S}^\mu(S) \rangle_{0,3,d} = \chi\big(\mathsf{Quot}_d(N,r), \mathbb{S}^\nu(\mathcal{S}_x) \otimes \mathbb{S}^\lambda(\mathcal{S}_x) \otimes \mathbb{S}^\mu(\mathcal{S}_x)\big),$$

where S is the universal subbundle on X and $\nu, \lambda, \mu \in \mathcal{P}^{r,N-r}$. The new formulas for Quot scheme in this section give a new way to study the quantum K-theory of Grassmannian. For example, Corollary 2.2.7 implies that for all $d > 0$ and any $F, G \in K^0(X)$, $\langle \det S, F, G \rangle_{0,3,d} = 0$.

In Proposition 4.1.1, we calculate Euler Characteristics of $\det(\pi_* \mathcal{S}^\vee)^\ell \otimes \mathbb{S}^\lambda(\mathcal{S}_x)$ where λ is a partition with at most r parts and $-(N-r) < \ell \leq r$. The case $\ell = -1$ gives us formula for the tautological line bundles over $\mathsf{Quot}_d(N, r)$.

For any line bundle M over $\mathbb{P}^1$, we define the tautological K-theory class is defined by

$$M^{[d]} = \pi_! [p * L \otimes \mathcal{Q}],$$

where $\mathcal{Q}$ is the universal quotient. The formula for all the exterior powers of $M^{[d]}$ is calculated in [OS]. I prove the following formula as a corollary of Proposition 4.1.1:

Theorem 2.2.9. *Let M be a line bundle over $\mathbb{P}^1$ of degree m and $-r \leq e < N - r$,*

$$\chi(\mathsf{Quot}_d(N, r), (\det M^{[d]})^e) = [t^d] s_\Lambda(z_1, \ldots, z_N)$$

for rectangular partition $\Lambda = ((m+1)e + d, \ldots, (m+1)e + d)$ (repeated r times), and $z_1, \ldots, z_N$ are roots of the equation

$$(z-1)^N + (-1)^r t z^{N-r-e} = 0.$$

2.3 Isotropic Quot schemes

Let E be a vector bundle over a smooth projective curve C endowed with L-valued non-degenerate symplectic form

$$\sigma : E \otimes E \to L,$$

where L is a line bundle. A subsheaf $S \subset E$ is said to be isotropic if $\sigma|_{S \otimes S} = 0$. The isotropic Quot scheme, $\mathsf{IQ}_d(E, \sigma, r)$ (IQ_d for short) parameterizes isotropic subsheaves of E of rank r and degree $-d$. Several geometric properties of isotropic Quot scheme was studied in [KT, CCH 2].

When E is a trivial vector bundle, IQ_d provides a compactification of the morphism space $\mathrm{Mor}_d(C, SG(N,r))$ to the symplectic Grassmannian $SG(N,r)$. I find a Vafa-Intriligator type formula for the isotropic Quot scheme of rank 2 subsheaves. This answers a question posed in [CCH 1]. Moreover, I study the stable map compactification and compare the invariants. The precise results are given below.

Virtual fundamental class

The isotropic Quot scheme is almost always singular. When $C = \mathbb{P}^1$ and E is a trivial rank N vector bundle, the usual Quot_d is a smooth space and IQ_d can be described as zero locus of a section of a vector bundle. In arbitrary genus, we have:

Theorem 2.3.1. $\mathsf{IQ}_d(E,\sigma,r)$ *admits a 2-term perfect obstruction theory induced by a morphism in the derived category from* $\left(\mathbf{R}\pi_*(J^\bullet)\right)^\vee$ *to the truncated cotangent complex* $\tau_{[-1,0]}\mathbb{L}_{\mathsf{IQ}_d}$ *where* $J^\bullet = [\mathcal{H}om(\mathcal{S},\mathcal{Q}) \to \mathcal{H}om(\wedge^2\mathcal{S}, p^*L)]$.

Here $\mathcal{S}$ and $\mathcal{Q}$ denote the universal subsheaf and the universal quotient sheaf respectively over $C \times \mathsf{IQ}_d$, and p and π are projections to C and IQ_d respectively.

Over a closed point $[0 \to S \to E \to Q \to 0]$ in IQ_d, the tangent space and the obstruction space are given by the hypercohomology of the complex of sheaves $[\mathcal{H}om(S,Q) \to \mathcal{H}om(\wedge^2 S, L)]$. The virtual dimension is

$$\mathrm{vd} = \begin{cases} \chi(C, S^\vee \otimes Q) - \chi(C, \wedge^2 S^\vee \otimes L) & \text{when } \sigma \text{ is symplectic} \\ \chi(C, S^\vee \otimes Q) - \chi(C, \mathrm{Sym}^2 S^\vee \otimes L) & \text{when } \sigma \text{ is symmetric} \end{cases},$$

These are easy to calculate as an application of the Riemann-Roch formula. The above theorem gives a virtual fundamental class $[\mathsf{IQ}_d]^{\mathrm{vir}} \in H_{2\,\mathrm{vd}}(\mathsf{IQ}_d)$ using the construction of Behrend-Fantechi [BF] and Li-Tian [LT].

Remark 2.3.2. *When $2r = N$ and σ is symplectic, the isotropic Quot scheme is irreducible and generically smooth [CCH 2] for $d >> 0$ and its dimension equals the virtual dimension obtained above. In this case, the virtual fundamental class agrees with the fundamental class.*

We note that the method in [MO 3] for constructing the virtual fundamental class for $\mathsf{Quot}_d(E,r)$ does not suffice for the isotropic case. When E is trivial, IQ_d can be realized as the moduli of quasi-maps from a fixed curve to the isotropic Grassmannian $SG(N,r)$. The 2-term perfect obstruction theory constructed here matches the one obtained using [CFKM].

Compatibility of virtual fundamental classes

The group $G = Sp(N)$ (or $G = SO(N)$) acts on the isotropic Quot scheme with σ symplectic (resp. symmetric). The perfect obstruction theory we construct is equivariant under any one-parameter subgroup $\mathbb{C}^* \subset G$. In this case, we use the virtual localization theorem [GP] to study the virtual intersection theory of IQ_d. This has been done extensively for Quot_d in [MO 3].

We first show a compatibility result for the virtual fundamental classes. Fix a point $q \in C$. There is a natural embedding

$$i_q : \mathsf{IQ}_d \to \mathsf{IQ}_{d+r}$$

which sends a subsheaf $S \subset \mathbb{C}^N \otimes \mathcal{O}$ to the composition

$$S(-q) \to S \to \mathbb{C}^N \otimes \mathcal{O},$$

which is also an isotropic subsheaf of degree $-(d+r)$.

Theorem 2.3.3. *We have the following identity in the homology $H_*(\mathsf{IQ}_{d+r})$:*

$$i_{q*}(c_{top}(\wedge^2 \mathcal{S}_q^\vee)^2 \cap [\mathsf{IQ}_d]^{\mathrm{vir}}) = c_{top}(\mathcal{S}_q^\vee)^N \cap [\mathsf{IQ}_{d+r}]^{\mathrm{vir}} \tag{2.3}$$

where we assume that σ is symplectic. The corresponding identity for symmetric form is obtained by replacing $\wedge^2$ with Sym^2.

This means that the virtual fundamental classes we construct, $[\mathsf{IQ}_d]^{\mathrm{vir}}$, are related as we vary the degree d by a multiple of r. An analogous result was proven in the case of the Quot scheme in [MO 3].

Intersection numbers

Virtual invariants are obtained by integrating natural cohomology classes over the virtual fundamental class of IQ_d. Let $\{1, \delta_1, \ldots, \delta_{2g}, \omega\}$ be a symplectic basis for the cohomology of C. The standard tautological classes are obtained by considering the Künneth decomposition of the Chern classes of $\mathcal{S}^\vee$ over $C \times \mathsf{IQ}_d$:

$$c_i(\mathcal{S}^\vee) = 1 \otimes a_i + \sum_{k=1}^{2g} \delta_k \otimes b_i^k + \omega \otimes f_i.$$

We have the following algebro-geometric description: $f_i = \pi_* c_i(\mathcal{S}^\vee)$ and $a_i = c_i(\mathcal{S}_x^\vee)$ for any $x \in C$. We obtain a Vafa-Intriligator type formula for the virtual intersection numbers over isotropic Quot scheme when $r = 2$. The virtual dimension in this case is $\mathrm{vd} = (N-1)d - (2N-5)\bar{g}$.

Theorem 2.3.4. *When $E = \mathcal{O}^{\oplus N}$ and $m_1 + 2m_2 = \mathrm{vd} \geq 0$,*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} a_1^{m_1} a_2^{m_2} = T_{d,g}(N) \sum_{\zeta \neq \pm 1} (1+\zeta)^{m_1+d} \zeta^{m_2} J(\zeta)^{g-1}, \tag{2.4}$$

where the sum is taken over N^{th} roots of unity $\zeta \neq \pm 1$. Here

$$J(\zeta) = -N^2 \zeta^{-1}(1-\zeta)^{-2}(1+\zeta)^{-1} \qquad \textit{and} \qquad T_{d,g}(N) = (-1)^d \tfrac{N}{2} \textstyle\sum_{i=0}^{d} \binom{g}{i} (-N)^{-i}.$$

We use virtual equivariant localization with respect to a torus action [GP] to prove the above theorem. Localization is a standard technique, but this is attempted on IQ_d for the first time. Extending these methods to the higher rank is combinatorially cumbersome .

Example 2.3.5. *When $N = 4$, the virtual dimension* $\mathrm{vd} = 3d - 3\bar{g}$. *The above theorem specializes* *to*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} a_1^{m_1} a_2^{m_2} = \begin{cases} 2^{2d-m_2-\bar{g}}3^g & \mathrm{vd}>0 \\ 2^{\bar{g}}(3^g+(-1)^{\bar{g}}) & \mathrm{vd}=0. \end{cases}$$

When $\mathrm{vd}=0$*, the resulting invariant can be interpreted as a ‘virtual’ count of isotropic subsheaves of E. This virtual count matches the enumerative count [CCH 1] of the rank two maximal degree isotropic subbundle of a general rank 4 stable bundle endowed with an* $\mathcal{O}$*-valued symplectic form. It is natural to ask if the formula in Theorem 2.3.4 give an enumerative count of maximal degree rank 2 isotropic subbundles of a stable rank N symplectic bundle E.*

I also obtain an explicit formula for the intersection numbers of the form $f_2^{\ell}a_1^{m_1}a_2^{m_2}\cap [\mathsf{IQ}_d]^{\mathrm{vir}}$ in [Sin]. The analogous formula for Quot schemes was found for $\ell=1$ in [MO 3]. Compared to [MO 3], the combinatorics here is different and enables calculation for higher exponents ℓ. The following is a specialization of Theorem 5.7.1 to $\ell=1$:

Theorem 2.3.6. *Let* $m_1+2m_2+1=\mathrm{vd}$ *and* $d>g$*, then*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} f_2 a_1^{m_1} a_2^{m_2} = \left(1-\frac{1}{N}\right)^g \sum_{\zeta\neq\pm 1}\left(D\circ B(1,\zeta)-\frac{\zeta B(1,\zeta)}{(1+\zeta)}\right).$$

where

$$D\circ R(z_1,z_2)=\frac{z_1z_2}{2}\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)R(z_1,z_2)$$

is a differential operator and

$$B(z_1,z_2)=u(z_1+z_2)^{m_1}(z_1z_2)^{m_2}\frac{(z_1+z_2)^{d-\bar{g}}}{(z_1-z_2)^{2\bar{g}}}\prod_{i=1}^{2}(Nz_i^{N-1})^{\bar{g}}.$$

I also consider the case where the vector bundle E is endowed with a non-degenerate symmetric L-valued form and construct a virtual fundamental class here as well. Even when $r=1$, we obtain new formulas.

Proposition 2.3.7. *Let $r=1$, let N be even and let σ be a symmetric form. Then*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} a_1^{\mathrm{vd}} = (N-2)^g 2^{2d-\bar{g}},$$

where $\mathrm{vd}=(N-2)(d-\bar{g})$ *is the virtual dimension and $d \geq g$.*

The orthogonal Grassmannian $OG(N,1)$ is a quadric in $\mathbb{P}^{N-1}$. The invariants obtained above are related to the Tevelev degree for quadrics found in [LP].

When $r=2$, we obtain a Vafa-Intriligator type formula for the virtual intersection numbers over (symmetric) isotropic Quot scheme $\widetilde{\mathsf{IQ}}_d$. The virtual dimension of IQ_d in this case is $\mathrm{vd}=(N-3)d-\bar{g}(2N-7)$.

Theorem 2.3.8. *Let $m_1+2m_2=\mathrm{vd}$ and $N=2n+2$. When $m_2>0$, then*

(i) When $m_2>0$, then

$$\int_{[\widetilde{\mathsf{IQ}}_d]^{\mathrm{vir}}} a_1^{m_1} a_2^{m_2} = T_{d,g}(2n) \sum_{\zeta \neq \pm 1} (1+\zeta)^{m_1+d} \zeta^{m_2} J(1,\zeta)^{\bar{g}}$$

(ii) When $m_2=0$,

$$\int_{[\widetilde{\mathsf{IQ}}_d]^{\mathrm{vir}}} a_1^{m_1} = T_{d,g}(2n)\left(4n^{\bar{g}} + \sum_{\zeta \neq \pm 1} (1+\zeta)^{m_1+d} J(1,\zeta)^{\bar{g}}\right),$$

where the sum is taken over $2n^{th}$ roots of unity $\zeta \neq \pm 1$. Here

$$J(\zeta) = -n^2(1+\zeta)^{-1}(1-\zeta)^{-2} \qquad \textit{and} \qquad T_{d,g}(N) = (-4)^d \tfrac{N}{2} \textstyle\sum_{i=0}^{d} \binom{g}{i}(-N)^{-i}.$$

Virtual Euler Characteristic

Let $N=2n$.The topological Euler characteristics of schemes IQ_d is given by

$$\sum_{d=0}^{\infty} e(\mathsf{IQ}_d) q^d = 2^r \binom{n}{r} (1-q)^{r(2g-2)}.$$

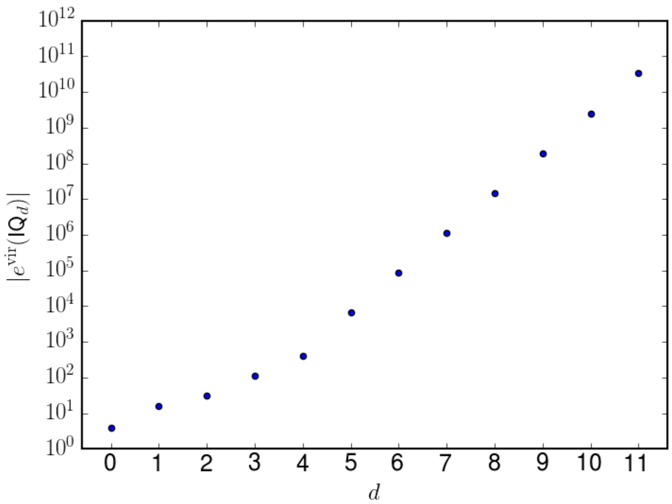


Figure 2.1. The absolute value of the virtual Euler characteristic of IQ_d in log scale, where $r = 2$ and σ is the standard symplectic form on $\mathbb{C}^4 \otimes \mathcal{O}$ over $\mathbb{P}^1$.

Let X be a scheme admitting a 2-term perfect obstruction theory. The virtual Euler characteristic is defined [FG], [CFK]

$$e^{\mathrm{vir}}(X) = \int_{[X]^{\mathrm{vir}}} c(T_X^{\mathrm{vir}}).$$

The virtual Euler characteristic of Quot scheme parameterizing zero dimensional quotients over surfaces were calculated in [OP].

When X is smooth and the obstruction bundle vanishes, the virtual Euler characteristic $e^{\mathrm{vir}}(X)$ matches the topological Euler characteristic of X. The isotropic Quot schemes, IQ_1, are smooth for $C = \mathbb{P}^1$ and all values of $N = 2n$ and r. By contrast, the isotropic Quot schemes IQ_d are not smooth for $d > 1$ even when $C = \mathbb{P}^1$. Thus the virtual Euler characteristics, $e^{\mathrm{vir}}(\mathsf{IQ}_d)$, are new invariants. While we do not a have a closed form expression for these power series, nonetheless we find a finite number of values using Sagemath [The]. We provide a small list of these invariants in Section 5.8.

When $r = 2$, $N = 4$ and σ is symplectic, we plot a log scale graph for the absolute value of $e^{\mathrm{vir}}(\mathsf{IQ}_d)$. The plot (see Figure 2.1) indicates an exponential growth in contrast with the polynomial expression for the topological Euler characteristics.

2.4 Gromov-Ruan-Witten invariants

In the previous subsections, we considered the Quot scheme compactification of the morphism space $\mathrm{Mor}_d(C, \mathrm{SG}(2,N))$ and $\mathrm{Mor}_d(C, \mathrm{OG}(2,N))$.

Let (M, ω) be a compact symplectic manifold with a generic almost complex structure J tamed by ω (i.e. $\omega(v, Jv) > 0$ for all non-zero $v \in TM$). We will further assume that $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$ and M is positive in the sense that $c_1(TM, J) \cdot f_*[\mathbb{P}^1] > 0$ for all non-constant J-holomorphic maps $f : \mathbb{P}^1 \to M$.

The morphism space of J-holomorphic maps from C to (M, ω) can be compactified by letting the curve C 'bubble' [RT]. The boundary of this compactification includes C with finitely many trees of rational curves. This leads to the definition of quantum cohomology and Gromov-Ruan-Witten (GRW) invariants. We briefly describe these terms, but a detailed description is available in [ST] and [MS].

Let $\alpha \in H^2(M, \mathbb{Z})$ be a positive generator. Define the index e of M by $c_1(M) = e\alpha$. Let $d \in H^2(M, \mathbb{Z})$ and $\alpha_1, \ldots, \alpha_s$ be cohomology classes in $H^*(M, \mathbb{Z})$ satisfying

$$\frac{1}{2} \sum_{i=1}^{s} \deg \alpha_i = ed + \dim(M)(1-g). \tag{2.5}$$

The right side of the above expression is the expected dimension of the moduli space of maps $f : C \to M$ with $f_*(C) = d \in H_2(M, \mathbb{Z})$.

Let $B_1, \ldots, B_s$ be a generic choice of the Poincaré dual homology classes of $\alpha_1, \ldots, \alpha_s$. Then for s generic points $p_1, \ldots, p_s \in C$, the GRW invariants

$$\Phi_{g,d}(\alpha_1, \ldots, \alpha_s)$$

is the algebraic count (considering sign and multiplicities) of J-holomorphic curves $f : C \to X$

such that $f(p_i) \in B_i$ and $f_*([C]) = d$. The GRW invariants depend on the genus but not the complex structure of the curve.

Quantum cohomology packages the information of 3-point genus zero GRW invariants giving a deformation of the usual cohomology ring (see [MS] for more details). A presentation of quantum cohomology of $\mathrm{SG}(r,N)$ and $\mathrm{OG}(r,N)$ was described in [Tam] and [BKT]. In [CMMPS], the authors gave a simpler presentation for $\mathrm{SG}(2,N)$. We extend their result obtaining a similar presentation for $\mathrm{OG}(2,N)$.

Let $N = 2n+2$. We have the universal exact sequence

$$0 \to \mathcal{S} \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0$$

over $\mathrm{OG}(2,N)$. Let $\mathcal{S}^\perp \subset \mathbb{C}^N \otimes \mathcal{O}$ be the rank $N-2$ orthogonal complement.

We have the following cohomology classes :

- The Chern classes $a_i = c_i(\mathcal{S}^\vee)$ for $i \in \{1,2\}$.
- Let $b_i = c_{2i}(\mathcal{S}^\perp/\mathcal{S})$ for $i \in \{1,\dots,n-1\}$. The bundle $\mathcal{S}^\perp/\mathcal{S}$ is self dual, hence all the odd Chern classes vanish.
- Let ξ be the Edidin-Graham square root class [EG] of the bundle $\mathcal{S}^\perp/\mathcal{S}$. In particular, it satisfies
$$(-1)^{n-1}\xi^2 = b_{n-1}.$$

Proposition 2.4.1. *The quantum cohomology ring $QH^*(\mathrm{OG}(2,2n+2),\mathbb{C})$ is isomorphic to the quotient of the ring $\mathbb{C}[a_1,a_2,b_1,\dots,b_{n-2},\xi,q]$ by the ideal generated by the relations*

$$\xi a_2 = 0$$

and

$$(1+(2a_2-a_1^2)x^2+a_2^2x^4)(1+b_1x^2+\cdots+b_{n-2}x^{2n-4}+(-1)^{n-1}\xi^2x^{2n-2})=1+4qa_1x^{2n},$$

where x is a formal variable.

Define the GRW invariant

$$\langle a_1^{m_1}a_2^{m_2}\rangle_g=\Phi_{g,d}(a_1,\ldots,a_1,a_2,\ldots,a_2),$$

where a_1 and a_2 appear m_1 and m_2 times respectively; and d is chosen (if possible) such that it satisfies (2.5).

In [ST], Siebert and Tian gave a remarkable technique to compute the higher genus GRW invariants using a given presentation for the quantum cohomology. We explicitly calculate the GRW invariants for $\mathrm{SG}(2,N)$ and $\mathrm{OG}(2,N)$ in Theorems 6.4.3 and 6.5.3 respectively. In particular, we prove the following theorem.

Theorem 2.4.2. *Let d, m_1 and m_2 be non-negative integers such that $\mathrm{vd}=m_1+2m_2$ is the expected dimension. The GRW invariants for $\mathrm{SG}(2,N)$ (and $\mathrm{OG}(2,N)$)*

$$\langle a_1^{m_1}a_2^{m_2}\rangle_g=\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}}a_1^{m_1}a_2^{m_2},$$

where IQ_d is the symplectic (respectively symmetric) isotropic Quot scheme.

Chapter 3

Punctual Quot schemes

In this chapter, we prove explicit formulas for the Euler characteristics of exterior powers and symmetric powers of tautological vector bundles over Punctual Quot schemes of curves.

Proposition 3.0.1. *For any rank N vector bundle E over C, $\mathsf{Quot}_d(E)$ is a smooth projective scheme of dimension Nd.*

Proof. Quot schemes are, by general construction, always projective. The deformation theory of $\mathsf{Quot}_d(E)$ is given by $\mathrm{Ext}^\bullet(S,Q)$ for any point $q = [0 \to S \to E \to Q \to 0]$. Since Q is supported on a zero dimensional scheme, $\mathrm{Ext}^i(S,Q) = 0$ for all $i > 0$. Moreover, we note that the tangent space at the point q equals $\dim(\mathrm{Hom}(S,Q)) = Nd$. □

The Quot scheme admits a universal exact sequence

$$0 \to \mathcal{S} \to p^*E \to \mathcal{Q} \to 0$$

over $C \times \mathsf{Quot}_d(E)$, and we let p and π denote the two projections over the factors of $C \times \mathsf{Quot}_d(E)$.

Definition 3.0.2. *For any line bundle $L \to C$, there is an induced tautological vector bundle over $\mathsf{Quot}_d(E)$ given by*

$$L^{[d]} = \pi_*(p^*L \otimes \mathcal{Q}).$$

Proposition 3.0.3. *For any line bundle $L \to C$, $L^{[d]}$ is a rank d vector bundle.*

Proof. Let $q = [0 \to S \to E \to Q \to 0]$ be a point in $\mathsf{Quot}_d(E)$. Since Q is supported on a degree d divisor, $H^0(L \otimes Q)$ is a rank d vector space and $H^i(L \otimes Q) = 0$ for all $i > 0$. The proposition follows using Grauert's theorem. $\square$

3.1 Exterior powers

We first study the holomorphic Euler characteristics of all exterior powers $\wedge^k L^{[d]}$. For any vector bundle V over a scheme Y, we set

$$\wedge_y V := \sum_k y^k \wedge^k V.$$

We show the following two theorems. The second theorem is a generalization of the first theorem. To ensure clarity, we will prove the theorems in the specified sequence.

Theorem 3.1.1. *Let $E \to C$ be a vector bundle over a smooth projective curve, and let $L \to C$ be a line bundle. Then*

$$\sum_{d=0}^{\infty} q^d \chi(\mathsf{Quot}_d(E), \wedge_y L^{[d]}) = (1-q)^{-\chi(\mathcal{O}_C)}(1+qy)^{\chi(E \otimes L)}.$$

Example 3.1.2. *Theorem 2.1.1 in higher genus immediately implies*

$$\chi\left(\mathsf{Quot}_d(E), \wedge^k L^{[d]}\right) = 0 \ \textit{ if } d \geq k+g,\ g \geq 1.$$

This follows by examining the coefficient of $q^d y^k$ in the expression $(1-q)^{-\chi(\mathcal{O}_C)}(1+qy)^{\chi(E \otimes L)}$.

The same methods will establish a slightly stronger result:

Theorem 3.1.3. *For any line bundles* $M_1, M_2, \ldots, M_r$ *and* L *over* C*, where* $0 \le r \le rk\, E - 1$*, we have*

$$\sum_{d=0}^{\infty} q^d \chi\left(\mathsf{Quot}_d(E), \wedge_y L^{[d]} \otimes_{i=1}^{r} \left(\wedge_{x_i} M_i^{[d]}\right)^\vee\right) = \frac{(1+qy)^{\chi(E\otimes L)}}{(1-q)^{\chi(\mathcal{O}_C)} \prod (1-x_i y q)^{\chi(L\otimes M_i^\vee)}}$$

The proof of the above theorems can broadly be divided into three steps.

- **Universality:** Using the arguments similar in spirit to [EGL], we show that there exists universal powers series A, B, and C in y and q (depends only on E by its rank N) such that

  $$\sum_d q^d \chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = A^{\chi(\mathcal{O}_C)} B^{\deg L} C^{\deg E}.$$

  Universality reduces the calculations to Quot scheme over $\mathbb{P}^1$, where the vector bundle E splits as a direct sum of line bundles.

- **Localization:** Over Quot schemes over $\mathbb{P}^1$, we use equivariant Atiyah-Bott localization (using a torus action) to reduce the calculations to integrals over the fixed loci. Since the fixed loci are comprised of products of projective spaces, we can simplify the problem to a tedious summation.

- **Combinatorics:** We use several combinatorial identities, such as Lagrange-Bürmann formula, to realize the expression as a Schur polynomial evaluated at roots of a polynomial with coefficients involving q and y. We then use Jacobi-Trudi identities to obtain explicit formulas.

3.1.1 Universality

Relying on the ideas of [EGL], we show how the calculations for $C = \mathbb{P}^1$ imply Theorems 2.1.1 and 3.1.3 for arbitrary genus. We explain this for Theorem 2.1.1, the case of Theorem 3.1.3 being entirely similar. The argument is also noted and used in [OP] over surfaces for punctual quotients of trivial bundles, and extended to quotients of arbitrary vector bundles in [Sta 1]. The case of curves is analogous, but we record the details for the benefit of the readers who seek a self-contained account.

Proposition 3.1.4. *For any line bundles $L \to C$ and vector bundle E*

$$\mathsf{Z}(C,L,E) := \sum_{d=0}^{\infty} q^d \chi\left(\mathsf{Quot}_d(E), \wedge_y L^{[d]}\right) = \mathsf{A}^{\chi(C,\mathcal{O}_C)} \cdot \mathsf{B}^{\deg L} \cdot \mathsf{C}^{\deg E}$$

where A, B and C are universal series in $\mathbb{Q}[y][[q]]$ (that may depend on N).

Proof. Consider a disconnected curve $C = C_1 \sqcup C_2$, $E = E_1 \sqcup E_2$ and $L = L_1 \sqcup L_2$. We compare the Quot schemes of C, C_1, C_2 and the tautological bundles over them:

$$\mathsf{Quot}_d(E) = \bigsqcup_{d_1+d_2=d} \mathsf{Quot}_{d_1}(E_1) \times \mathsf{Quot}_{d_2}(E_2), \quad L^{[d]} = \bigsqcup_{d_1+d_2=d} L_1^{[d_1]} \boxplus L_2^{[d_2]}.$$

Since $\wedge_y(L_1^{[d_1]} \boxplus L_2^{[d_2]}) = \wedge_y(L_1^{[d_1]}) \cdot \wedge_y(L_2^{[d_2]})$, this implies

$$\mathsf{Z}(C,L,E) = \mathsf{Z}(C_1,L_1,E_1) \cdot \mathsf{Z}(C_2,L_2,E_2). \tag{3.1}$$

Using Lemma 3.1.6 we know that the function Z is a composition of $h : \mathbb{Z}^3 \to \mathbb{Q}[y][[q]]$ (each exponent of q is a polynomial in the inputs and y) and $\gamma(C,L,E) = (\chi(C,\mathcal{O}_C), \deg L, \deg E)$. The image of γ is Zariski dense $\mathbb{Z}^3$, and over this image h satisfy $h(z_1+z_2) = h(z_1)h(z_2)$. Thus $\log h$ is a linear function, hence proving the theorem. □

The same proof as above also gives us the following proposition:

Proposition 3.1.5. *For any line bundles $M_1, M_2, \ldots, M_r$ and $L \to C$ and vector bundle $E \to C$*

$$\sum_{d=0}^{\infty} q^d \chi\left(\mathsf{Quot}_d(E), \wedge_y L^{[d]} \otimes_{i=1}^{r} \left(\wedge_{x_i} M_i^{[d]}\right)^{\vee}\right) = \mathsf{A}^{\chi(C,\mathcal{O}_C)} \cdot \mathsf{B}^{\deg L} \cdot \mathsf{C}^{\deg E} \prod_{i=1}^{r} D_i^{\deg M}.$$

where A, B, C and D_i's are universal series in $\mathbb{Q}[y][[q]]$ (that may depend on N).

Lemma 3.1.6. *Let P be a polynomial in the Chern classes of the tangent bundle of* $\mathsf{Quot}_d(E)$ *and tautological bundles* $L_1^{[d]}, \ldots, L_r^{[d]}$*. Then*

$$\int_{\mathsf{Quot}_d(E)} \mathsf{P} \tag{3.2}$$

is a polynomial in $\deg E$*,* $\deg L_1, \ldots, \deg L_r$ *and* $\chi(\mathcal{O}_C)$ *(that may depend on N and d).*

Proof. We first analyze the case of split vector bundles

$$E = \bigoplus_{i=1}^{N} F_i, \quad \operatorname{rk} F_i = 1.$$

For such a vector bundle, we can use the action of $\mathbb{C}^\star$ on the summands of E (with distinct weights) to evaluate (3.2). The fixed loci consists of product of symmetric powers of the curve

$$C^{[d_1]} \times \cdots \times C^{[d_N]}$$

where $d_1 + \cdots + d_N = d$. The points in $C^{[d_i]}$ corresponds to the short exact sequences $0 \to K_i \to F_i \to T_i \to 0$ such that $\deg T_i = d_i$. Let $\mathcal{K}_i$ denote the universal subbundle on $C^{[d_i]} \times C$ (and by abuse of notation its pullback to the product $C^{[d_1]} \times \cdots \times C^{[d_N]}$). Note that the restriction of the Chern classes of the tangent bundle of $\mathsf{Quot}_d(E)$ and tautological bundles to the fixed loci, and the normal bundle can be represented (in the K-theory of the fixed loci) using Chern classes of

$$\pi_\star\left(\mathcal{K}_i \otimes p^* M\right), \quad \pi_\star\left(\mathcal{K}_i^\vee \otimes p^* M\right), \quad \pi_\star\left(\mathcal{K}_i^\vee \otimes \mathcal{K}_j \otimes p^* M\right) \tag{3.3}$$

where M are the classes of the form $M = F_i^\vee \otimes F_j$ or $M = L_j \otimes F_i$. Here p and π denote the projections from $C \times C^{[d_1]} \times \cdots \times C^{[d_N]}$ to C and $C^{[d_1]} \times \cdots \times C^{[d_N]}$ respectively. Using Atiyah-Bott localization, we are led to considering integrals of the form

$$\int_{C^{[d_1]} \times \cdots \times C^{[d_N]}} \mathsf{Q} \tag{3.4}$$

where Q is a polynomial involving Chern class of elements in (3.3). With the aid of Grothendieck-Riemann-Roch, we can express the above as an integrals over $C^{[d_1]} \times \cdots \times C^{[d_N]} \times C$ of the first Chern class of $\mathcal{K}_i$'s and classes from C. The integrals (3.4) can be pulled back via the finite map

$$C^d \times C \to C^{[d_1]} \times \cdots \times C^{[d_N]} \times C.$$

The pullbacks of $\mathcal{K}_i^\vee$ over $C^d \times C$ correspond to sums of diagonals $\Delta_{\bullet,d+1}$, and thus (3.4) takes the form

$$\frac{d_1! \cdots d_N!}{d!} \int_{C^d \times C} \widetilde{\mathsf{Q}}$$

where $\widetilde{\mathsf{Q}}$ is a universal expression in the diagonals and classes from C. In general, monomials in diagonals and classes from C can be evaluated explicitly using that for $\Delta \hookrightarrow C \times C$ we have

$$\Delta^2 = 2\chi(\mathcal{O}_C), \quad \Delta \cdot M = \deg M,$$

for all smooth projective possibly disconnected curves $C, M \to C$. Therefore (3.2) is a polynomial in $\deg F_i$, $\deg L_j$ and $\chi(\mathcal{O}_C)$.

We next argue that the above polynomial only depends on $\deg E = \sum_i \deg F_i$, $\deg L_j$ and $\chi(\mathcal{O}_C)$. This requires additional considerations. We write

$$x_i = \deg F_i, \quad y_j = \deg L_j, \quad z = \chi(\mathcal{O}_C),$$

and $\mathsf{R}(x_1, \ldots, x_N, y_1, \ldots y_r, z)$ for the universal polynomial found above. The polynomial R is certainly symmetric in $x_1, \ldots, x_N$.

We claim that if x_i are sufficiently large, $\mathsf{R}(x_1, \ldots, x_N, y_1, \ldots y_r, z)$ is in fact a polynomial in $\sum_{i=1}^N x_i$. Indeed, for large degrees, the line bundles F_i are globally generated (over connected curves C). Thus we can write E as a quotient

$$0 \to K \to W \to E \to 0,$$

where W is a trivial bundle (whose rank depends on $\deg E$). By [Sta 2, Theorem 5], modified from the original setting of surfaces to the case of curves, there is an embedding

$$\mathsf{Quot}_d(E) \hookrightarrow \mathsf{Quot}_d(W) \tag{3.5}$$

cut out by a canonical section of the bundle $(K^\vee)^{[d]}$. With this observation, the integral (3.2) rewrites as

$$\int_{\mathsf{Quot}_d(E)} \mathsf{P} = \int_{\mathsf{Quot}_d(W)} \widetilde{\mathsf{P}} \tag{3.6}$$

where $\widetilde{\mathsf{P}}$ is a polynomial in the Chern classes of the tangent bundle of $\mathsf{Quot}_d(W)$ and the tautological bundles $(K^\vee)^{[d]}$ and $L_j^{[d]}$. Applying the localization argument as earlier once again, this time to $\mathsf{Quot}_d(W)$, we see that (3.6) only depends on

$$\deg K^\vee = \deg E, \quad \deg L_j, \quad \chi(\mathcal{O}_C).$$

Thus, the polynomial $\mathsf{R}(x_1, \ldots, x_N, y_1, \ldots y_r, z)$ is a function of $\sum_{i=1}^N x_i, y_1, \ldots y_r, z$, when x_i are large. Hence

$$\mathsf{R}(x_1, \ldots, x_N, y_1, \ldots y_r, z) = \mathsf{S}(x_1 + \ldots + x_N, y_1, \ldots y_r, z),$$

for a new universal polynomial S. This proves the statement we need about (3.2) when the bundle E splits.

The general case follows from the following observation. Assume E sits in an extension

$$0 \to E_1 \to E \to E_2 \to 0.$$

Considering the universal extension

$$0 \to p^\star E_1 \to \mathcal{E} \to p^\star E_2 \to 0$$

over $p : C \times \mathrm{Ext}^1(E_2, E_1) \to C$, and constructing the relative Quot scheme $\mathsf{Quot}_d(\mathcal{E})$ over the extension space, we see that

$$\int_{\mathsf{Quot}_d(E)} \mathsf{P} = \int_{\mathsf{Quot}_d(E_1 \oplus E_2)} \mathsf{P}.$$

To reduce to the case of split E, consider M a line bundle such that

$$0 \to M \to E \to F \to 0$$

is exact and F is a vector bundle of smaller rank. By the above observation we can replace E by $M \oplus F$, and then continue inductively. □

3.1.2 Localization

The proofs of the two Theorems 2.1.1 and 3.1.3 are similar. The calculations for Theorem 2.1.1 are however simpler and already illustrate the main points.

Torus action

We first establish Theorem 2.1.1 when $C = \mathbb{P}^1$ and $E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_N)$, and for L such that

$$\deg L + a_i + 1 \geq 0$$

for all i. The arbitrary genus case follows from here by universality arguments, see Section 3.1.1 below.

Under the above assumptions, we seek to show that

$$\sum_{d=0}^{\infty} q^d \chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = (1-q)^{-1}(1+qy)^{\chi(E\otimes L)}. \tag{3.7}$$

Here, for simplicity, we wrote Quot_d instead of $\mathsf{Quot}_d(E)$.

We evaluate expression (3.7) via Hirzebruch-Riemann-Roch

$$\chi\left(\mathsf{Quot}_d, \wedge_y L^{[d]}\right) = \int_{\mathsf{Quot}_d} \mathrm{ch}(\wedge_y L^{[d]})\,\mathrm{Td}\left(\mathsf{Quot}_d\right),$$

and we use $\mathbb{C}^*$-equivariant localization to compute the integral. [1]

To this end, we let $\mathbb{C}^*$ act on E with weight $-w_i$ on the summand $\mathcal{O}(a_i)$. This induces a $\mathbb{C}^\star$-action on Quot_d. The fixed subbundles correspond to split inclusions

$$S = \bigoplus_{i=1}^{N} K_i(a_i) \hookrightarrow E = \bigoplus_{i=1}^{N} \mathcal{O}(a_i).$$

Thus, the fixed loci are products of projective spaces

$$\mathrm{F}_{\vec{d}} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_N}$$

for vectors $\vec{d} = (d_1, \ldots, d_N)$ such that $d_1 + \cdots + d_N = d$. The factor $\mathbb{P}^{d_i}$ corresponds to the Hilbert scheme of d_i points of $\mathbb{P}^1$ parameterizing short exact sequences

$$0 \to K_i \to \mathcal{O} \to T_i \to 0$$

such that T_i is a torsion sheaf of length d_i.

There is a universal exact sequence

$$0 \to \mathcal{K}_i \to \mathcal{O} \to \mathcal{T}_i \to 0$$

[1] It is natural to attempt localization directly in K-theory, but we were unable to establish the result in this fashion.

over the product $\mathbb{P}^1 \times \mathbb{P}^{d_i}$, with the universal kernel given by

$$\mathcal{K}_i = \mathcal{O}_{\mathbb{P}^1}(-d_i) \boxtimes \mathcal{O}_{\mathbb{P}^{d_i}}(-1). \tag{3.8}$$

For future reference, we note that the universal exact sequence $0 \to \mathcal{S} \to p^*E \to \mathcal{Q} \to 0$ over $\mathbb{P}^1 \times \mathsf{Quot}_d$ restricts to $\mathbb{P}^1 \times \mathrm{F}_{\vec{d}}$ as

$$0 \to \bigoplus_{i \in [N]} \mathcal{K}_i(a_i) \to \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_N) \to \bigoplus_{i \in [N]} \mathcal{T}_i(a_i) \to 0, \tag{3.9}$$

where pullbacks from the factors are understood above. We also set $[N] = \{1, 2, \ldots, N\}$.

By Atiyah-Bott localization, we have

$$\chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = \sum_{|\vec{d}|=d} \int_{\mathrm{F}_{\vec{d}}} \mathrm{ch}(\wedge_y L^{[d]}) \frac{\mathsf{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(\mathrm{N}_{\vec{d}})} \bigg|_{\mathrm{F}_{\vec{d}}}. \tag{3.10}$$

Here $\mathrm{N}_{\vec{d}}$ denotes the normal bundle of the fixed locus $\mathrm{F}_{\vec{d}}$.

Explicit calculations

We proceed to calculate the expressions appearing in the localization sum (3.10). In the next subsections, we record the Todd genera, the normal bundle contributions and the Chern characters of the tautological bundles.

Todd Classes

By (3.9), the tangent bundle $T\mathsf{Quot}_d = \mathrm{Hom}_\pi(\mathcal{S}, \mathcal{Q})$ restricts to

$$\bigoplus_{i,j \in [N]} \pi_* \left(\mathcal{K}_i^\vee(-a_i) \otimes \mathcal{T}_j(a_j) \right)$$

over the fixed locus $\mathrm{F}_{\vec{d}} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_N}$. Here $\pi : \mathsf{Quot}_d \times \mathbb{P}^1 \to \mathsf{Quot}_d$ denotes the projection. In K-theory, the above expression equals

$$\bigoplus_{i,j\in[N]} \pi_* \left(\mathcal{K}_i^\vee(a_j - a_i)\right) - \bigoplus_{i,j\in[N]} \pi_* \left(\mathcal{K}_i^\vee \otimes \mathcal{K}_j(a_j - a_i)\right).$$

Therefore the Todd class of Quot_d restricted to each fixed locus is

$$\prod_{i,j\in[N]} \mathrm{Td}\left(\pi_* \left(\mathcal{K}_i^\vee(a_j - a_i)\right)\right) \left(\prod_{i,j\in[N],\, i\neq j} \mathrm{Td}\left(\pi_* \left(\mathcal{K}_i^\vee \otimes \mathcal{K}_j(a_j - a_i)\right)\right)\right)^{-1}.$$

The above (i, j)-terms carry the weight $w_i - w_j$. The assumption $i \neq j$ in the second product can be made since the term $i = j$ is trivial in genus 0.

Equivariant normal bundles

Over each fixed locus, the normal bundle is given by the moving part of the tangent bundle:

$$\begin{aligned} \mathrm{N}_{\vec{d}} = T^{\mathrm{mov}}\Big|_{\mathrm{F}_{\vec{d}}} &= \bigoplus_{i\neq j} \pi_* \left(\mathcal{K}_i^\vee(-a_i) \otimes \mathcal{T}_j(a_j)\right) \\ &= \bigoplus_{i\neq j} \pi_* \left(\mathcal{K}_i^\vee(a_j - a_i)\right) - \bigoplus_{i\neq j} \pi_* \left(\mathcal{K}_i^\vee \otimes \mathcal{K}_j(a_j - a_i)\right), \end{aligned} \tag{3.11}$$

where we continue to keep track of the weights $w_i - w_j$. Therefore, we find the Euler classes

$$\frac{1}{e_{\mathbb{C}^*}(\mathrm{N}_{\vec{d}})} = \prod_{i,j\in[N],\, i\neq j} \left(e_{\mathbb{C}^*}\left(\pi_\star \left(\mathcal{K}_i^\vee(a_j - a_i)\right)\right)\right)^{-1} \prod_{i,j\in[N], i\neq j} e_{\mathbb{C}^*}\left(\pi_\star \left(\mathcal{K}_i^\vee \otimes \mathcal{K}_j(a_j - a_i)\right)\right).$$

Collecting all expressions above, we obtain that over the fixed locus $\mathrm{F}_{\vec{d}}$, the factor

$\frac{\mathrm{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(\mathrm{N}_{\vec{d}})}$ in the localization expression (3.10) restricts to

$$\prod_{i\in[N]} \mathrm{Td}\left(\pi_*\left(\mathcal{K}_i^\vee\right)\right) \prod_{i,j\in[N],\, i\neq j} \frac{\mathrm{Td}}{e_{\mathbb{C}^*}}\left(\pi_*\left(\mathcal{K}_i^\vee(a_j-a_i)\right)\right) \prod_{i,j\in[N], i\neq j} \frac{e_{\mathbb{C}^*}}{\mathrm{Td}}\left(\pi_\star\left(\mathcal{K}_i^\vee\otimes\mathcal{K}_j(a_j-a_i)\right)\right). \tag{3.12}$$

Explicit contributions

The terms in (3.12) can be made explicit. For the first term, recalling (3.8), we immediately compute

$$\pi_\star(\mathcal{K}_i^\vee) = \mathbb{C}^{d_i+1}\otimes\mathcal{O}_{\mathbb{P}^{d_i}}(1) \implies \mathrm{Td}(\pi_*(\mathcal{K}_i^\vee)) = \left(\frac{h_i}{1-e^{-h_i}}\right)^{d_i+1},$$

where h_i is the hyperplane class on $\mathbb{P}^{d_i}$ (by abuse of notation also pulled back to $\mathbb{P}^{d_1}\times\cdots\times\mathbb{P}^{d_N}$). The equivariant weights vanish for this term. (This is the Todd genus of the projective space, as it should.)

Turning to the remaining terms, more generally, equation (3.8) straightforwardly yields

$$c(\pi_*(\mathcal{K}_i^\vee(a_j-a_i))) = (1+h_i)^{d_i+a_j-a_i+1}$$
$$c(\pi_*(\mathcal{K}_i^\vee\otimes\mathcal{K}_j(a_j-a_i))) = (1+(h_i-h_j))^{d_i-d_j+a_j-a_i+1}.$$

In the equivariant cohomology, recalling that the above sheaves carry the weight w_i-w_j, we obtain

$$c(\pi_*(\mathcal{K}_i^\vee(a_j-a_i))) = (1+(h_i+w_i\varepsilon-w_j\varepsilon))^{d_i+a_j-a_i+1}$$
$$c(\pi_*(\mathcal{K}_i^\vee\otimes\mathcal{K}_j(a_j-a_i))) = (1+(h_i+w_i\varepsilon-h_j-w_j\varepsilon))^{d_i-d_j+a_j-a_i+1}.$$

Here, ε denotes the equivariant parameter. This implies the following expressions for the equivariant Todd genera

$$\mathrm{Td}(\pi_*(\mathcal{K}_i^\vee(a_j-a_i))) = \left(\frac{h_i+w_i\varepsilon-w_j\varepsilon}{1-e^{-(h_i+w_i\varepsilon-w_j\varepsilon)}}\right)^{d_i+a_j-a_i+1}$$
$$\mathrm{Td}(\pi_*(\mathcal{K}_i^\vee\otimes\mathcal{K}_j(a_j-a_i))) = \left(\frac{h_i+w_i\varepsilon-h_j-w_j\varepsilon}{1-e^{-(h_i+w_i\varepsilon-h_j-w_j\varepsilon)}}\right)^{d_i-d_j+a_j-a_i+1}.$$

Similarly we obtain the Euler classes

$$e_{\mathbb{C}^*}(\pi_*(\mathcal{K}_i^\vee(a_j-a_i))) = (h_i+w_i\varepsilon-w_j\varepsilon)^{d_i+a_j-a_i+1}$$
$$e_{\mathbb{C}^*}(\pi_*(\mathcal{K}_i^\vee\otimes\mathcal{K}_j(a_j-a_i))) = (h_i+w_i\varepsilon-h_j-w_j\varepsilon)^{d_i-d_j+a_j-a_i+1}.$$

Simplification. All told, substituting the above expressions into (3.12) and cancelling terms, we obtain

$$\left.\frac{\mathrm{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(\mathrm{N}_{\vec{d}})}\right|_{\mathrm{F}_{\vec{d}}} = \prod_{i\in[N]} h_i^{d_i+1} \prod_{i,j\in[N]}\left(\frac{z_i}{z_i-\alpha_j}\right)^{d_i+a_j-a_i+1} \prod_{i,j\in[N],\, i\neq j}\left(\frac{z_i-z_j}{z_i}\right)^{d_i-d_j+a_j-a_i+1} \tag{3.13}$$

where we set for notational convenience

$$z_i = e^{h_i+w_i\varepsilon}, \quad \alpha_i = e^{w_i\varepsilon}.$$

We rewrite this in a slightly more convenient form in terms of the polynomial

$$R(z) = \prod_{j\in[N]}(z-\alpha_j).$$

Combining the (i,j) and (j,i)-factors in the last product appearing in (3.13), and judiciously accounting for the remaining terms, we eventually obtain

$$\left.\frac{\mathsf{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(\mathrm{N}_{\vec{d}})}\right|_{\mathrm{F}_{\vec{d}}} = \mathsf{u}\cdot\prod_{i\in[N]}\left(\frac{h_i}{R(z_i)}\right)^{d_i+1} z_i^{d+1}\left(\frac{R(z_i)}{\prod_{j\in[N]}(z_j-\alpha_i)}\right)^{a_i+\ell+1}\cdot\prod_{i,j\in[N],\, i<j}(z_i-z_j)^2 \tag{3.14}$$

for the sign

$$\mathsf{u} = (-1)^{(N-1)(d+\sum(a_i+\ell+1))+\binom{N}{2}}.$$

The integer ℓ included in the above expression will be useful later on. For now, the value of ℓ plays no role. Any ℓ will work since

$$\prod_i \frac{R(z_i)}{\prod_j(z_j-\alpha_i)} = 1.$$

Chern classes

For the remaining term in (3.10), we record the following

Lemma 3.1.7. *The equivariant restrictions of the Chern characters of the tautological bundles to the fixed loci are given by*

$$\left.\mathrm{ch}(\wedge_y L^{[d]})\right|_{\mathrm{F}_{\vec{d}}} = \prod_i\left(\frac{z_i(\alpha_i+y)}{\alpha_i(z_i+y)}\right)^{a_i+\ell+1}\left(\frac{z_i+y}{z_i}\right)^{d_i} \tag{3.15}$$

$$\left.\mathrm{ch}\left((\wedge_x M^{[d]})^\vee\right)\right|_{\mathrm{F}_{\vec{d}}} = \prod_i\left(\frac{1+\alpha_i x}{1+z_i x}\right)^{a_i+m+1}(1+z_i x)^{d_i} \tag{3.16}$$

where L and M are line bundles of degree ℓ and m respectively.

Proof. We only explain the first formula, the second assertion being entirely similar. We note that over $\mathrm{F}_{\vec{d}} = \mathbb{P}^{d_1}\times\cdots\times\mathbb{P}^{d_N}$, the bundle $L^{[d]}$ splits as contributions coming from each factor

$$L^{[d]} = \pi_\star(\mathcal{Q}\otimes p^\star L) = \bigoplus_{i\in[N]}\pi_\star(\mathcal{T}_i\otimes p^\star L(a_i)),$$

with each summand acted on with $\mathbb{C}^\star$-weight $-w_i$. In K-theory, we have by (3.8) that

$$\mathcal{T}_i = \mathcal{O} - \mathcal{K}_i = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^{d_i}} - \mathcal{O}_{\mathbb{P}^1}(-d_i) \boxtimes \mathcal{O}_{\mathbb{P}^{d_i}}(-1).$$

This yields

$$\pi_\star(\mathcal{T}_i \otimes p^\star L(a_i)) = \mathbb{C}^{a_i+\ell+1} \otimes \mathcal{O}_{\mathbb{P}^{d_i}} - \mathbb{C}^{a_i-d_i+\ell+1} \otimes \mathcal{O}_{\mathbb{P}^{d_i}}(-1).$$

The result follows immediately from here, using that $\wedge_y(V+W) = \wedge_y V \cdot \wedge_y W$ and accounting for all terms. $\square$

3.1.3 Proof of Theorem 2.1.1

With the above ingredients in place, the key steps of the argument are as follows:

(i) after judiciously accounting for all localization terms, the fixed point contributions are summed using the Lagrange-Bürmann formula;

(ii) next, the answer is recast as a quotient of suitable determinants. Schur polynomials evaluated at the roots of a certain algebraic equation arise at this step;

(iii) finally, an application of the Jacobi-Trudi formula to the Schur polynomials greatly simplifies the answer and gives the result.

To begin, we substitute equations (3.14) and (3.15) into the localization expression (3.10). We obtain that $\chi(\mathsf{Quot}_d, \wedge_y L^{[d]})$ equals

$$\mathsf{u} \sum_{|\vec{d}|=d} \left[h_1^{d_1} \ldots h_N^{d_N}\right] \left\{ \prod_i \left(\frac{z_i(\alpha_i+y)}{\alpha_i(z_i+y)}\right)^{b_i} \left(\frac{z_i+y}{z_i}\right)^{d_i} \left(\frac{h_i}{R(z_i)}\right)^{d_i+1} z_i^{d+1} \left(\frac{R(z_i)}{\prod_j (z_j-\alpha_i)}\right)^{b_i} \right. \tag{3.17}$$
$$\left. \cdot \prod_{i<j} (z_i-z_j)^2 \right\} \Bigg|_{\varepsilon=0}.$$

Here, we wrote for simplicity

$$b_i = a_i + \ell + 1.$$

The brackets indicate extracting the coefficient of $h_1^{d_1}\dots h_N^{d_N}$ in the relevant expression; this corresponds to integration over the product of projective spaces $\mathrm{F}_{\vec{d}} = \mathbb{P}^{d_1}\times\dots\times\mathbb{P}^{d_N}$. The equivariant parameter ε is set to 0 at the end.

The rest of this section is dedicated to the explicit combinatorial manipulations (i)-(iii) which bring the above expression into the form stated in Theorem 2.1.1. We first apply the multivariable Lagrange-Bürmann formula [WW]. The formulation we need in this case is as follows. Consider formal power series $\Phi_1(h_1),\dots,\Phi_N(h_N)$ with $\Phi_i(0)\neq 0$, and consider a power series $\Psi(h_1,\dots,h_N)$. We have

$$\sum_{(d_1,\dots,d_N)} q_1^{d_1}\cdots q_N^{d_N}\left[h_1^{d_1}\dots h_N^{d_N}\right]\Big(\Phi_1(h_1)^{d_1+1}\cdots\Phi_N^{d_N+1}(h_N)\cdot\Psi(h_1,\dots,h_N)\Big) = \frac{\Psi}{J} \tag{3.18}$$

for the change of variables

$$q_i = \frac{h_i}{\Phi_i(h_i)}$$

with Jacobian

$$J = \frac{dq_1}{dh_1}\cdots\frac{dq_N}{dh_N}.$$

This formula will be used to derive equation (3.23) below. The intermediate calculations are straightforward; nonetheless, we record the details for completeness.

Set

$$\Phi_i(h_i) = \frac{h_i}{R(z_i)}\frac{z_i+y}{z_i},$$

and let

$$\Psi = \mathsf{u}\prod_i\left(\frac{z_i(\alpha_i+y)}{\alpha_i(z_i+y)}\right)^{b_i}\left(\frac{z_i+y}{z_i}\right)^{-1} z_i^{d+1}\left(\frac{R(z_i)}{\prod_j(z_j-\alpha_i)}\right)^{b_i}\cdot\prod_{i<j}(z_i-z_j)^2$$

be determined by the remaining terms in (3.17). [2] Due to the factor $z_i-\alpha_i$ in $R(z_i)$ which has a

[2]In expression Ψ, we regard the exponent d in the term z_i^{d+1} as an independent parameter, foregoing for the moment the requirement that $d = d_1+\dots+d_N$. The careful reader may wish to replace the term z_i^{d+1} by a more general z_i^{e+1} for $e\geq d$ in the proof below. This leads to (3.26) written instead for the partition $\lambda_k = (e^N,k)$ or

simple zero at $h_i = 0$, we have $\Phi_i(0) \neq 0$. We apply (3.18) with

$$q_1 = \ldots = q_N = q.$$

Thus, letting z_i be the root of the equation

$$q(z_i + y) = z_i R(z_i), \quad z_i\Big|_{q=0} = \alpha_i,$$

and letting h_i be determined by $z_i = \alpha_i e^{h_i}$, we have $q = \frac{h_i}{\Phi_i(h_i)}$. It follows from (3.17) and (3.18) that

$$\chi\left(\mathsf{Quot}_d, \wedge_y L^{[d]}\right) = \left[q^d\right] \frac{\Psi}{J}\left(h_1(q), \ldots, h_N(q)\right).$$

Equivalently, $\chi\left(\mathsf{Quot}_d, \wedge_y L^{[d]}\right)$ equals

$$\left[q^d\right] \mathsf{u} \prod_i \left(\frac{z_i(\alpha_i + y)}{\alpha_i(z_i + y)}\right)^{b_i} \left(\frac{z_i + y}{z_i}\right)^{-1} \frac{dh_i}{dq} z_i^{d+1} \left(\frac{R(z_i)}{\prod_j (z_j - \alpha_i)}\right)^{b_i} \cdot \prod_{i<j} (z_i - z_j)^2 \Bigg|_{\varepsilon=0}. \tag{3.19}$$

Consider the polynomial

$$P(z) = zR(z) - q(z + y). \tag{3.20}$$

Note that P has degree $N+1$, so it admits $N+1$ roots, with $z_1, \ldots, z_N$ being N of them. Let z_{N+1} be the additional root of P which satisfies

$$z_{N+1}\Big|_{q=0} = 0.$$

We will greatly simplify (3.19) using the additional root z_{N+1}.

$(k-N, (e+1)^N)$. One then specializes back to the case of interest $e = d$ and continues by applying Lemma 3.1.8.

To this end, write $P(z) = (z - z_1)\cdots(z - z_{N+1})$. A simple calculation gives

$$\frac{dq}{dh_i} = \frac{dq}{dz_i}\cdot\frac{dz_i}{dh_i} = z_i\frac{dq}{dz_i} = \frac{z_i}{z_i+y}P'(z_i). \tag{3.21}$$

Here, we used that

$$q = \frac{z_i R(z_i)}{z_i+y} \implies \frac{dq}{dz_i} = \frac{R(z_i)}{z_i+y} + \frac{z_i R'(z_i)}{z_i+y} - \frac{z_i R(z_i)}{(z_i+y)^2} = \frac{P'(z_i)}{z_i+y},$$

where the definition of P was necessary in the last equality. The terms in (3.19) with exponent b_i further simplify since

$$\frac{z_i(\alpha_i+y)R(z_i)}{\alpha_i(z_i+y)\prod_{j=1}^{N}(z_j-\alpha_i)} = (-1)^N\frac{(z_{N+1}-\alpha_i)}{\alpha_i} \iff (\alpha_i+y)z_iR(z_i) = -(z_i+y)P(\alpha_i) \tag{3.22}$$

where the last identity holds true using (3.20) and the fact that $P(z_i) = 0$, $R(\alpha_i) = 0$. Substituting identities (3.21) and (3.22) into (3.19), we obtain the expression

$$\left[q^d\right](-1)^{(N-1)d}\prod_{i=1}^{N}\left(\frac{\alpha_i - z_{N+1}}{\alpha_i}\right)^{b_i}\frac{z_i^{d+1}}{P'(z_i)}\cdot\prod_{1\le i\ne j\le N}(z_i-z_j)\Bigg|_{\varepsilon=0}. \tag{3.23}$$

Having arrived at (3.23), a new idea is needed to go further. Note that

$$\prod_{i=1}^{N}P'(z_i) = \prod_{i=1}^{N}\prod_{\substack{j=1\\ j\ne i}}^{N+1}(z_i-z_j) \implies \prod_{i=1}^{N}\frac{1}{P'(z_i)}\cdot\prod_{1\le i\ne j\le N}(z_i-z_j) = \frac{\mathsf{V}_N}{\mathsf{V}_{N+1}}.$$

Here, we introduced the two Vandermonde determinants

$$\mathsf{V}_N = \begin{vmatrix} z_1^{N-1} & \cdots & z_1 & 1\\ z_2^{N-1} & \cdots & z_2 & 1\\ \vdots & \cdots & \vdots & \vdots\\ z_N^{N-1} & \cdots & z_N & 1\end{vmatrix}, \quad \mathsf{V}_{N+1} = \begin{vmatrix} z_1^{N} & \cdots & z_1 & 1\\ z_2^{N} & \cdots & z_2 & 1\\ \vdots & \cdots & \vdots & \vdots\\ z_{N+1}^{N} & \cdots & z_{N+1} & 1\end{vmatrix}.$$

We thus rewrite expression (3.23) in the form

$$\left[q^d\right] \frac{(-1)^{(N-1)d}}{\mathsf{V}_{N+1}} \begin{vmatrix} z_1^{d+N} & z_1^{d+N-1} & \cdots & z_1^{d+1} \\ z_2^{d+N} & z_2^{d+N-1} & \cdots & z_2^{d+1} \\ \vdots & \vdots & \cdots & \vdots \\ z_N^{d+N} & z_N^{d+N-1} & \cdots & z_N^{d+1} \end{vmatrix} \prod_{i=1}^{N} \left(\frac{\alpha_i - z_{N+1}}{\alpha_i} \right)^{b_i} \Bigg|_{\varepsilon=0}. \tag{3.24}$$

Next, recall that $[q^0]z_{N+1} = 0$, hence we may add terms which are multiples of z_{N+1}^{d+1} without changing the coefficient of q^d. Using this observation, we enlarge the determinant in the numerator by adding one more row and column. The answer is recast as the quotient of two determinants of size $(N+1) \times (N+1)$:

$$\left[\varepsilon^0 q^d\right] \frac{(-1)^{(N-1)d}}{\mathsf{V}_{N+1}} \begin{vmatrix} z_1^{d+N} & z_1^{d+N-1} & \cdots & z_1^{d+1} & \prod_{i=1}^N (\frac{\alpha_i - z_1}{\alpha_i})^{b_i} \\ z_2^{d+N} & z_2^{d+N-1} & \cdots & z_2^{d+1} & \prod_{i=1}^N (\frac{\alpha_i - z_2}{\alpha_i})^{b_i} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ z_{N+1}^{d+N} & z_{N+1}^{d+N-1} & \cdots & z_{N+1}^{d+1} & \prod_{i=1}^N (\frac{\alpha_i - z_{N+1}}{\alpha_i})^{b_i} \end{vmatrix}. \tag{3.25}$$

This does not change expression (3.24). Indeed, expanding along the last row, the first entries do not contribute by the above reasoning, while the rightmost corner contribution matches (3.24). The assumption $\deg L + a_i + 1 \geq 0$ made in the beginning of this section is also used here. This condition rewrites as $b_i \geq 0$, so the terms we added on the last column do not contribute poles at $q = 0$.

Expression (3.25) is symmetric in the roots $z_1, \ldots, z_{N+1}$ of $P(z)$. The answer can be rewritten in terms of the elementary symmetric functions in the z_i's which depend polynomially (hence continuously) on the α_i's. Thus (3.25) is a rational fraction in the α's, with denominator $\prod_{i=1}^N \alpha_i^{b_i}$ (coming from the last column). Since we are interested in the coefficient of ε^0, by continuity we may substitute $\alpha_i = 1$, noting that there are no poles in (3.25) at these values.

After the substitution, the z_i's solve $z(z-1)^N - q(z+y) = 0$. Furthermore, since

$$\sum_{i=1}^{N} b_i = \chi(E \otimes L) := \chi,$$

the entries in the last column of the determinant (3.25) become

$$(1-z_i)^{\chi} = \sum_{k \geq 0} (-1)^k \binom{\chi}{k} z_i^k.$$

Expanding the determinant along the last column yields sums over Schur polynomials. Specifically, we obtain

$$\left[q^d\right] \sum_{k \geq 0} (-1)^{(N-1)d+k} \binom{\chi}{k} s_{\lambda_k}(z_1, \ldots, z_{N+1}). \tag{3.26}$$

Here, we set

$$\lambda_k = (d^N, k) = (d, \ldots, d, k),$$

and $s_{\lambda_k}(z_1, \ldots z_{N+1})$ denotes the corresponding Schur polynomial, when $k \leq d$. The terms for $d < k \leq d+N$ have vanishing contribution due to repeating columns in the determinant. To account for the ordering of the exponents, the shape of the partition changes when $k > d+N$. In all cases, we find

$$\lambda_k = \begin{cases} (d^N, k) & \text{if } k \leq d \\ (k-N, (d+1)^N) & \text{if } k > d+N \end{cases}.$$

The lemma below identifies the coefficient of q^d in $s_{\lambda_k}(z_1, \ldots, z_{N+1})$. We obtain

$$\chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = \sum_{k=0}^{d} \binom{\chi}{k} y^k = \left[q^d\right] (1+qy)^{\chi} (1-q)^{-1}.$$

This completes the proof of Theorem 2.1.1 in genus 0 under the assumption $b_i \geq 0$ for all $1 \leq i \leq N$.

Lemma 3.1.8. *We have*

$$\left[q^d\right] s_{\lambda_k}(z_1,\ldots,z_{N+1}) = \begin{cases} (-1)^{d(N-1)}(-y)^k & \text{if } k \le d \\ 0 & \text{if } k > d+N \end{cases}.$$

Proof. Since the z_i's are the roots of the polynomial $P(z) = z(z-1)^N - q(z+y)$, the elementary symmetric functions in $z_1,\ldots,z_{N+1}$ are

$$e_j = \begin{cases} \binom{N}{j} & \text{if } j \neq N, N+1 \\ 1+(-1)^{N-1}q & \text{if } j = N \\ (-1)^N qy & \text{if } j = N+1. \end{cases}$$

Assume $k \le d$ so that $\lambda_k = (d^N, k)$. The Jacobi-Trudi formula expresses the Schur polynomial as a $d \times d$ determinant in the elementary symmetric functions. The entries are dictated by the conjugate partition $\lambda_k' = ((N+1)^k, N^{d-k})$, so that

$$s_{\lambda_k} = \left|\begin{array}{ccccc|cccc} e_{N+1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ e_N & e_{N+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ e_{N-1} & e_N & e_{N+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e_{N-k+2} & e_{N-k+3} & e_{N-k+4} & \cdots & e_{N+1} & 0 & \cdots & 0 & 0 \\ \hline e_{N-k} & e_{N-k+1} & e_{N-k+2} & \cdots & e_{N-1} & e_N & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e_{N-d+2} & e_{N-d+3} & e_{N-d+4} & \cdots & e_{N-d+k+1} & e_{N-d+k+2} & \cdots & e_N & e_{N+1} \\ e_{N-d+1} & e_{N-d+2} & e_{N-d+3} & \cdots & e_{N-d+k} & e_{N-d+k+1} & \cdots & e_{N-1} & e_N \end{array}\right|. \tag{3.27}$$

Each of the e_j's is at most linear in q. Since the determinant has size d, extracting the q^d coefficient is immediate. In fact, we can replace the e_j's by their linear terms in q; these are zero unless $j = N$ or $j = N+1$. We obtain that

$$\left[q^d\right] s_{\lambda_k} = \left[q^d\right] \left|\begin{array}{cccccc|cccccc} e_{N+1} & 0 & 0 & \cdots & 0 & 0 & & & & & & \\ e_N & e_{N+1} & 0 & \cdots & 0 & 0 & & & & & & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & \multicolumn{2}{c}{0} & & \\ 0 & 0 & 0 & \cdots & e_N & e_{N+1} & & & & & & \\ \hline & & & & & & e_N & e_{N+1} & 0 & \cdots & 0 & 0 \\ & & & & & & 0 & e_N & e_{N+1} & \cdots & 0 & 0 \\ & & \multicolumn{2}{c}{0} & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ & & & & & & 0 & 0 & 0 & \cdots & 0 & e_N \end{array}\right|.$$

Thus,

$$[q^d]s_{\lambda_k} = \left[q^d\right] e_{N+1}^k e_N^{d-k} = (-1)^{kN+(d-k)(N-1)} y^k.$$

The case $k > d+N$ changes the conjugate partition λ_k', but the reasoning is identical. $\square$

Proof of Theorem 2.1.1. We specialize to $(C,L) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\ell))$ with ℓ sufficiently large, and $E = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_N)$. Comparing (3.7) and (**??**), we obtain

$$\mathsf{A} = (1-q)^{-1} \cdot (1+qy)^N, \quad \mathsf{B} = (1+qy)^N, \quad \mathsf{C} = 1+qy.$$

Substituting these expressions back into (**??**), we obtain Theorem 2.1.1 for all genera:

$$\mathsf{Z}_{C,L,E} = \mathsf{A}^{\chi(C,\mathcal{O}_C)} \cdot \mathsf{B}^{\deg L} \cdot \mathsf{C}^{\deg E} = (1-q)^{-\chi(\mathcal{O}_C)} \cdot (1+qy)^{\chi(E\otimes L)}.$$

This completes the argument. $\square$

3.1.4 Conjecture 2.2

It is natural to inquire whether Theorem 2.1.1 can be refined to yield information about all cohomology groups of the tautological bundles $\wedge^k L^{[d]}$. We first explain some notations: If $V^\bullet = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space, we define the graded vector spaces

$$\wedge^k V^\bullet = \bigoplus_{i+j=k} \wedge^i V_0 \otimes \mathsf{Sym}^j V_1, \quad \mathsf{Sym}^k V^\bullet = \bigoplus_{i+j=k} \mathsf{Sym}^i V_0 \otimes \wedge^j V_1$$

where the summands have degree j. With the convention

$$\dim W^\bullet = \sum (-1)^j \dim W^j$$

for the superdimension of a graded vector space, the usual formulas hold true

$$\dim \wedge^k V^\bullet = \binom{\dim V^\bullet}{k}, \quad \dim \mathsf{Sym}^k V^\bullet = (-1)^k \binom{-\dim V^\bullet}{k}.$$

Conjecture 3.1.9. *Is it true that*

$$H^\bullet\left(\mathsf{Quot}_d(E), \wedge^k L^{[d]}\right) = \wedge^k H^\bullet(E \otimes L) \otimes \mathsf{Sym}^{d-k} H^\bullet(\mathcal{O}_C)? \tag{3.28}$$

Thus, taking dimensions in (2.2), we immediately match the expressions in Theorem 2.1.1, thus providing a geometric interpretation of our result. There is also a natural analogue for Theorem 3.1.3. The study of these questions may require understanding the derived category of $\mathsf{Quot}_d(E)$.

Evidence. Formula (2.2) is true in the following cases

(i) over the symmetric product of a curve, that is for rank $E = 1$. This was shown in [**?**, Section 3] using the derived category;

(ii) for $d = 1$ so that $\mathsf{Quot}_1(E) = \mathbb{P}(E)$, the projective bundle of length 1 quotients of E;

(iii) for $k=0$, the formula predicts the Hodge numbers $h^{p,0}(\mathsf{Quot}_d(E)) = \binom{g}{p}$ for $p \leq d$. This follows from [BFP, Ric] which give the Hodge polynomials

$$\sum h^{p,q}(\mathsf{Quot}_d(E))(-u)^p(-v)^q t^d = \prod_{i=0}^{\mathrm{rk}(E)-1} \frac{(1-u^i v^{i+1} t)^g (1-u^{i+1} v^i t)^g}{(1-u^i v^i t)(1-u^{i+1} v^{i+1} t)}.$$

3.1.5 Proof of Theorem 3.1.3

A similar but slightly more involved argument yields Theorem 3.1.3 in genus 0 when $b_i \geq 0$ for all $1 \leq i \leq N$. Specifically, we prove that

$$\chi\left(\mathsf{Quot}_d, \wedge_y L^{[d]} \otimes_{p=1}^r \left(\wedge_{x_p} M_p^{[d]}\right)^\vee\right) = \left[q^d\right](1-q)^{-1}(1+qy)^{\chi(E\otimes L)} \prod_{p=1}^r (1-x_p y q)^{-\chi(L\otimes M_p^\vee)}. \tag{3.29}$$

We indicate some of the steps.

Just as in Theorem 2.1.1, we begin by applying Hirzebruch-Riemann-Roch followed by Atiyah-Bott localization:

$$\chi\left(\mathsf{Quot}_d, \wedge_y L^{[d]} \otimes_{p=1}^r \left(\wedge_{x_p} M_p^{[d]}\right)^\vee\right) = \sum_{\vec{d}} \int_{\mathrm{F}_{\vec{d}}} \mathrm{ch}(\wedge_y L^{[d]}) \prod_{p=1}^r \mathrm{ch}\left(\left(\wedge_{x_p} M_p^{[d]}\right)^\vee\right) \frac{\mathrm{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(\mathrm{N}_{\vec{d}})}\Bigg|_{\mathrm{F}_{\vec{d}}}. \tag{3.30}$$

All terms that appear here have been computed in the previous subsections. Using (3.14), (3.15) and (3.16), we rewrite (3.30) as

$$\mathsf{u} \sum_{|\vec{d}|=d} \left[h_1^{d_1} \cdots h_N^{d_N}\right] \Bigg\{ \prod_{i=1}^N \left(\left(\frac{z_i(\alpha_i+y)}{\alpha_i(z_i+y)}\right)^{b_i} \left(\frac{z_i+y}{z_i}\right)^{d_i} \prod_{p=1}^r \left(\frac{1+\alpha_i x_p}{1+z_i x_p}\right)^{a_i+m_p+1} (1+z_i x_p)^{d_i} \left(\frac{h_i}{R(z_i)}\right)^{d_i+1} \right.$$
$$\left. z_i^{d+1} \left(\frac{R(z_i)}{\prod_{j=1}^N (z_j-\alpha_i)}\right)^{b_i}\right) \cdot \prod_{1\leq i<j\leq N} (z_i-z_j)^2 \Bigg\}\Bigg|_{\varepsilon=0}$$

where $b_i = a_i + \ell + 1$ and $\mathsf{u} = (-1)^{(N-1)(d+\sum b_i)+\binom{N}{2}}$. Here, we set $m_p = \deg M_p$.

Next, the Lagrange-Bürmann formula with the change of variable

$$q(z_i+y)\prod_{p=1}^{r}(1+z_ix_p)=z_iR(z_i) \tag{3.31}$$

turns (3.30) into the following unwieldy expression

$$\left[q^d\right]\,\mathsf{u}\prod_{i=1}^{N}\left[\left(\frac{z_i(\alpha_i+y)}{\alpha_i(z_i+y)}\right)^{b_i}\prod_{p=1}^{r}\left(\frac{1+\alpha_ix_p}{1+z_ix_p}\right)^{a_i+m_p+1}\left(\frac{z_i+y}{z_i}\prod_{p=1}^{r}(1+z_ix_p)\right)^{-1}\frac{dh_i}{dq}z_i^{d+1}\left(\frac{R(z_i)}{\prod_j(z_j-\alpha_i)}\right)^{b_i}\right]$$
$$\cdot\prod_{1\le i<j\le N}(z_i-z_j)^2\Big|_{\varepsilon=0}.$$

However, there are further simplifications. To this end, we define the polynomial

$$P(z)=zR(z)-q(z+y)\prod_{p=1}^{r}(1+zx_p).$$

Since $r\le N-1$, the degree of P is $N+1$, so there is an additional root z_{N+1} for P. Following the same steps that led to (3.23), we simplify the above expression to

$$\left[q^d\right](-1)^{(N-1)d}f(z_{N+1})\prod_{i=1}^{N}\frac{z_i^{d+1}}{P'(z_i)}\prod_{1\le i\ne j\le N}(z_i-z_j)\Big|_{\varepsilon=0} \tag{3.32}$$

where

$$f(z)=\prod_{p=1}^{r}(1+zx_p)^{m_p-\ell}\prod_{i=1}^{N}\left(\frac{\alpha_i-z}{\alpha_i}\right)^{b_i}.$$

We record the details of the simplification in the lemma below; the reader can also skip directly to (3.35).

Lemma 3.1.10. *We have*

$$\left(\frac{z_i+y}{z_i}\prod_{p=1}^{r}(1+z_ix_p)\right)\frac{dq}{dh_i}=P'(z_i) \tag{3.33}$$

and

$$\prod_{i=1}^{N}\left(\left(\frac{z_i(\alpha_i+y)}{\alpha_i(z_i+y)}\frac{R(z_i)}{\prod_{j=1}^{N}(z_j-\alpha_i)}\right)^{b_i}\prod_{p=1}^{r}\left(\frac{1+\alpha_i x_p}{1+z_i x_p}\right)^{a_i+m_p+1}\right)=(-1)^{(N-1)\sum b_i}f(z_{N+1}). \tag{3.34}$$

Proof. Equation (3.33) follows by differentiating the expression for q given in (3.31). For (3.34), recall $b_i=a_i+\ell+1$, and use the following identities

$$z_iR(z_i)=q(z_i+y)\prod_{p=1}^{r}(1+z_ix_p),$$

$$\prod_{j=1}^{N}(z_j-\alpha_i)=(-1)^{N+1}\frac{P(\alpha_i)}{z_{N+1}-\alpha_i}=(-1)^{N}q\frac{(\alpha_i+y)\prod_{p=1}^{r}(1+\alpha_ix_p)}{z_{N+1}-\alpha_i}.$$

In the last line we used the definition of P and the fact that $R(\alpha_i)=0$. Then (3.34) becomes

$$\prod_{i=1}^{N}\left(\left((-1)^{N}\frac{(z_{N+1}-\alpha_i)}{\alpha_i}\right)^{b_i}\prod_{p=1}^{r}\left(\frac{1+\alpha_ix_p}{1+z_ix_p}\right)^{m_p-\ell}\right).$$

Finally, recalling that α_i and z_i are roots of R and P, for each fixed p we have

$$\prod_{i=1}^{N}\frac{1+\alpha_ix_p}{1+z_ix_p}=\frac{R(-1/x_p)}{P(-1/x_p)}\left(-\frac{1}{x_p}-z_{N+1}\right)=(1+z_{N+1}x_p).$$

In the last equality, we used again the definition of P in terms of R. The lemma follows from here. □

Having arrived at (3.32), by the same reasoning as in (3.24) we rewrite the answer as the quotient of two determinants

$$\left[\varepsilon^0q^d\right]\frac{(-1)^{(N-1)d}}{\det(z_i^{N-j+1})}\begin{vmatrix} z_1^{d+N} & z_1^{d+N-1} & \cdots & z_1^{d+1}\\ z_2^{d+N} & z_2^{d+N-1} & \cdots & z_2^{d+1}\\ \vdots & \vdots & \cdots & \vdots\\ z_N^{d+N} & z_N^{d+N-1} & \cdots & z_N^{d+1}\end{vmatrix}f(z_{N+1}). \tag{3.35}$$

The denominator is the Vandermonde determinant of size $(N+1)\times(N+1)$, while the numerator has size $N\times N$. Using the previous arguments, in particular that $[q^0]z_{N+1}=0$, we enlarge the determinant appearing in the numerator of (3.35) by adding one row and one column:

$$[\varepsilon^0 q^d]\frac{(-1)^{(N-1)d}}{\det(z_i^{N-j+1})}\begin{vmatrix} z_1^{d+N} & z_1^{d+N-1} & \cdots & z_1^{d+1} & f(z_1) \\ z_2^{d+N} & z_2^{d+N-1} & \cdots & z_2^{d+1} & f(z_2) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ z_{N+1}^{d+N} & z_{N+1}^{d+N-1} & \cdots & z_{N+1}^{d+1} & f(z_{N+1}) \end{vmatrix}. \tag{3.36}$$

Since (3.36) is symmetric in $z_i's$, it can be written as a rational function in the α_i's whose denominator equals $\prod_{i=1}^N \alpha_i^{b_i}$ coming from the denominator of f. The substitution $\alpha_i=1$ therefore makes sense. After this substitution, the last column can be rewritten in terms of

$$f(t)|_{\alpha_i=1}=\prod_{p=1}^r (1+x_p t)^{m_p-\ell}\cdot(1-t)^{\chi}$$

for the values $t=z_1,z_2,\ldots,z_{N+1}$. Here $\chi=\chi(E\otimes L)$. We expand $f(t)$ into powers t^k, and then we expand the determinant (3.36) along the last column yielding

$$(-1)^{(N-1)d}\sum_{k\geq 0}\left[t^k\right] f(t)\cdot\left[q^d\right] s_{\lambda_k}(z_1,\ldots,z_{N+1}), \tag{3.37}$$

for the partition

$$\lambda_k=\begin{cases}(d^N,k) & \text{if } k\leq d\\ \\ (k-N,(d+1)^N) & \text{if } k>d+N.\end{cases}$$

By Lemma 3.1.11 below, for $k\leq d$ we have

$$\left[q^d\right] s_{\lambda_k}(z_1,\ldots,z_{N+1})=(-1)^{(N-1)d}(-y)^k\left[t^{d-k}\right]\frac{1}{(1-t)(1-x_1yt)\cdots(1-x_ryt)}.$$

Substituting the last formula into (3.37), we obtain that

$$
\begin{aligned}
&\chi\left(\mathsf{Quot}_d, \wedge_y L^{[d]} \otimes_{p=1}^r \left(\wedge_{x_p} M_p^{[d]}\right)^\vee\right) \\
&= \sum_{k=0}^d \left[t^k\right] \left((1-t)^{\chi(E\otimes L)} \prod_{p=1}^r (1+x_p t)^{m_p-\ell}\right) \cdot (-y)^k \left[t^{d-k}\right] \frac{1}{(1-t)\prod_{p=1}^r (1-x_p yt)} \\
&= \sum_{k=0}^d \left[t^k\right] \left((1+yt)^{\chi(E\otimes L)} \prod_{p=1}^r (1-x_p yt)^{m_p-\ell}\right) \cdot \left[t^{d-k}\right] \frac{1}{(1-t)\prod_{p=1}^r (1-x_p yt)} \\
&= \left[t^d\right] \frac{(1+yt)^{\chi(E\otimes L)}}{(1-t)\prod_{p=1}^r (1-x_p yt)^{\chi(L\otimes M_p^\vee)}}.
\end{aligned}
$$

This completes the proof of (3.29) and of Theorem 3.1.3 in genus 0 when $b_i \geq 0$ for all i.

Schur polynomials

Let $z_1, \ldots z_{N+1}$ be $N+1$ roots of

$$P(z) = z(z-1)^N - q(z+y)(1+zx_1)\cdots(1+zx_r),$$

where $0 \leq r \leq N-1$. We show

Lemma 3.1.11. *For the partition λ_k above, and $k \leq d$, we have*

$$\left[q^d\right] s_{\lambda_k}(z_1, \ldots, z_{N+1}) = (-1)^{(N-1)d}(-y)^k \left[t^{d-k}\right] \frac{1}{(1-t)(1-x_1 yt)\cdots(1-x_r yt)}.$$

If $k > d+N$, the coefficient vanishes.

Proof. The proof is similar to that of Lemma 3.1.8. Assume $k \leq d$, the other case being similar. Since the z_i's are the roots of the polynomial $P(z)$, the elementary symmetric functions are

$$e_j = \binom{N}{j} + (-1)^{j-1} q \left[z^{N+1-j}\right](y+z)(1+zx_1)\cdots(1+zx_r).$$

We examine again the Jacobi-Trudi determinant (3.27)

$$s_{\lambda_k} = \left| \begin{array}{cccc|cccc} e_{N+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ e_N & e_{N+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ e_{N-1} & e_N & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e_{N-k+2} & e_{N-k+3} & \cdots & e_{N+1} & 0 & \cdots & 0 & 0 \\ \hline e_{N-k} & e_{N-k+1} & \cdots & e_{N-1} & e_N & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e_{N-d+2} & e_{N-d+3} & \cdots & e_{N-d+k+1} & e_{N-d+k+2} & \cdots & e_N & e_{N+1} \\ e_{N-d+1} & e_{N-d+2} & \cdots & e_{N-d+k} & e_{N-d+k+1} & \cdots & e_{N-1} & e_N \end{array} \right| .$$

The e_j's are at most linear in q. To find the coefficient of q^d in the above $d \times d$ determinant, we may thus replace e_j with the coefficient of the linear term in q. Thus, we may take

$$e_j = (-1)^{j-1} \left[z^{N+1-j}\right] (y+z)(1+zx_1)\cdots(1+zx_r). \tag{3.38}$$

In particular $e_{N+1} = (-1)^N y$. Furthermore, note that the first $k \times k$ block of the determinant is lower triangular, hence

$$\left[q^d\right] s_{\lambda_k} = e_{N+1}^k \cdot T_{d-k} = (-1)^{Nk} y^k \cdot T_{d-k}$$

where T_m is the $m \times m$ determinant

$$T_m = \begin{vmatrix} e_N & e_{N+1} & 0 & \cdots & 0 & 0 \\ e_{N-1} & e_N & e_{N+1} & \cdots & 0 & 0 \\ e_{N-2} & e_{N-1} & e_N & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ e_{N-m+2} & e_{N-m+3} & e_{N-m+2} & \cdots & e_N & e_{N+1} \\ e_{N-m+1} & e_{N-m+2} & e_{N-m+1} & \cdots & e_{N-1} & e_N \end{vmatrix}.$$

The argument is completed using the Lemma below. $\square$

Lemma 3.1.12. *Assume $e_1, \ldots, e_{N+1}$ are given by* (3.38)*. For any $m \geq 0$, we have*

$$T_m = (-1)^{(N-1)m} [t^m] \frac{1}{(1-t)(1-x_1yt)\cdots(1-x_ryt)}. \tag{3.39}$$

Proof. We set $T_0 = 1$ and $T_\ell = 0$ for $\ell < 0$. By expanding the determinant T_m along the first column and then successively along the rows, we obtain the recursion

$$T_m = \sum_{j=0}^{r} (-1)^j e_{N+1}^j e_{N-j} T_{m-j-1} \quad \text{for all } m > 0.$$

Note that by (3.38), for degree reasons we have $e_{N-j} = 0$ if $j > r$. This explains the upper bound of the index j in the sum. Forming the generating series

$$T = \sum_{m=0}^{\infty} T_m t^m,$$

the above recursion immediately yields

$$T = \left(1 - \sum_{j=0}^{r} (-1)^j e_{N+1}^j e_{N-j} t^{j+1}\right)^{-1}.$$

Substituting the values of e_j from (3.38), we obtain for all $0 \le j \le r$ that

$$(-1)^{j+1}e_{N+1}^{j}e_{N-j} = (-1)^{N(j+1)}y^{j+1}\left[z^{j+1}\right]\left(\left(1+\frac{z}{y}\right)(1+zx_1)\cdots(1+zx_r)\right)$$
$$= \left[t^{j+1}\right]\left((1-(-1)^{N-1}t)(1-(-1)^{N-1}x_1yt)\cdots(1-(-1)^{N-1}x_ryt)\right),$$

where the substitution $z=(-1)^N yt$ was carried out in the last step. Therefore

$$T = \left(1-\sum_{j=0}^{r}(-1)^j e_{N+1}^{j}e_{N-j}t^{j+1}\right)^{-1} = \frac{1}{(1-(-1)^{N-1}t)\prod_{p=1}^{r}(1-(-1)^{N-1}x_pyt)}.$$

Taking the coefficient of t^m gives the required identity. $\square$

3.2 Symmetric Powers

3.2.1 Genus zero.

Theorem 2.1.4 concerns the symmetric powers of the tautological bundles $\mathsf{Sym}_y L^{[d]}$ in genus 0 and is proven in a similar fashion as Theorem 2.1.1. The calculations are however more involved. The higher genus case and Theorem 2.1.5 will be considered in Section 3.2.2.

By Section 3.1.1, for each d and k, the Euler characteristic of

$$\chi\left(\mathsf{Quot}_d, \mathsf{Sym}^k L^{[d]}\right)$$

depends polynomially on ℓ. To prove Theorem 2.1.4, it suffices to assume $b_i = \ell + a_i + 1 \ge d+1$ for all i.

By Hirzebruch-Riemann-Roch followed by Atiyah-Bott localization, we calculate

$$\chi\left(\mathsf{Quot}_d, \mathsf{Sym}_y L^{[d]}\right) = \int_{\mathsf{Quot}_d}\mathrm{ch}(\mathsf{Sym}_y L^{[d]})\,\mathrm{Td}\left(\mathsf{Quot}_d\right) = \sum_{\vec{d}}\int_{\mathrm{F}_{\vec{d}}}\mathrm{ch}(\mathsf{Sym}_y L^{[d]})\frac{\mathrm{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(\mathrm{N}_{\vec{d}})}\bigg|_{\mathrm{F}_{\vec{d}}}. \tag{3.40}$$

Instead of Lemma 3.1.7, for the current computation we use the expression

$$\mathsf{ch}(\mathsf{Sym}_y L^{[d]})\Big|_{\mathrm{F}_{\vec{d}}} = \prod_{i\in[N]} \left(\frac{\alpha_i(z_i-y)}{z_i(\alpha_i-y)}\right)^{a_i+\ell+1} \left(\frac{z_i}{z_i-y}\right)^{d_i}. \tag{3.41}$$

The Todd genera and the normal bundle contributions are found in (3.14). We substitute (3.14) and (3.41) into (3.40) and apply Lagrange-Bürmann. Carrying out these steps carefully, we arrive at the following. Consider the polynomial

$$P(z) = (z-y)R(z) - qz,$$

and let $z_1,\dots,z_{N+1}$ be its roots with $z_i(q=0)=\alpha_i$ for $1\le i\le N$. Then, just as in the derivation leading up to (3.23) for exterior powers, (3.40) turns into

$$(-1)^{(N-1)d}\left[q^d\right]\prod_{i\in[N]}\left(\frac{\alpha_i - z_{N+1}}{\alpha_i - y}\right)^{b_i}\left(\frac{z_i}{z_i-y}\right)^{-1}\frac{z_i^{d+1}}{P'(z_i)}\prod_{i,j\in[N],\ i\neq j}(z_i-z_j)\Big|_{\varepsilon=0}.$$

This simplification makes use of the fact that

$$\frac{dq}{dh_i} = P'(z_i).$$

As in (3.24), the above expression can be recast as the quotient of determinants

$$\left[\varepsilon^0 q^d\right]\frac{(-1)^{(N-1)d}}{\det(z_i^{N-j+1})}\begin{vmatrix}(z_1-y)z_1^{d+N-1} & \cdots & (z_1-y)z_1^d\\ (z_2-y)z_2^{d+N-1} & \cdots & (z_2-y)z_2^d\\ \vdots & \cdots & \vdots\\ (z_N-y)z_N^{d+N-1} & \cdots & (z_N-y)z_N^d\end{vmatrix}\prod_{i\in[N]}\left(\frac{\alpha_i-z_{N+1}}{\alpha_i-y}\right)^{b_i}.$$

The same derivation that led to (3.25) yields the enlarged $(N+1)\times(N+1)$ determinant

$$[\varepsilon^0 q^d]\frac{(-1)^{(N-1)d}}{\det(z_i^{N-j+1})}\begin{vmatrix} (z_1-y)z_1^{d+N-1} & (z_1-y)z_1^{d+N-1} & \cdots & (z_1-y)z_1^{d} & \prod_{i=1}^N(\frac{\alpha_i-z_1}{\alpha_i-y})^{b_i} \\ (z_2-y)z_2^{d+N-1} & (z_2-y)z_2^{d+N-1} & \cdots & (z_2-y)z_2^{d} & \prod_{i=1}^N(\frac{\alpha_i-z_2}{\alpha_i-y})^{b_i} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (z_{N+1}-y)z_{N+1}^{d+N-1} & (z_{N+1}-y)z_{N+1}^{d+N-1} & \cdots & (z_{N+1}-y)z_{N+1}^{d} & \prod_{i=1}^N(\frac{\alpha_i-z_{N+1}}{\alpha_i-y})^{b_i} \end{vmatrix}.$$

This uses $b_i \geq d+1$ for all i, and the fact that $\alpha_i - z_i$ has no free q-term, so in particular the first N entries of the last column do not contribute to the q^d-coefficient.

The expression above is symmetric in the roots of P, and as previously remarked the substitution $\alpha_i = 1$ is allowed to obtain the coefficient of ε^0. Thus $z_1,\ldots,z_{N+1}$ become roots of

$$P(z) = (z-1)^N(z-y) - qz.$$

This also turns the last column into the vector with entries

$$\frac{(1-z_i)^\chi}{(1-y)^\chi} = \frac{1}{(1-y)^\chi}\sum_{\ell=0}^{\chi}\binom{\chi}{\ell}(-1)^\ell z_i^\ell.$$

Here $\chi = \sum_i b_i = \chi(E\otimes L)$.

Using the additivity of the determinant with respect to the first N columns, we split the last determinant into a sum

$$\left[q^d\right]\sum_{\ell=0}^{\chi}\sum_{m=0}^{N}\frac{(-1)^{(N-1)d+\ell}}{(1-y)^\chi}\binom{\chi}{\ell}(-y)^m\frac{1}{\det(z_i^{N-j+1})}\begin{vmatrix} z_1^{d+N} & \cdots & z_1^{d+m+1} & z_1^{d+m-1} & \cdots & z_1^d & z_1^\ell \\ z_2^{d+N} & \cdots & z_2^{d+m+1} & z_2^{d+m-1} & \cdots & z_2^d & z_2^\ell \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ z_{N+1}^{d+N} & \cdots & z_{N+1}^{d+m+1} & z_{N+1}^{d+m-1} & \cdots & z_{N+1}^d & z_{N+1}^\ell \end{vmatrix}.$$

Indeed, from each of the first N columns we select N powers of z_i whose exponents range from d to $d+N$. Exactly one value $d+m$ must be skipped, giving a term in the sum. The contribution

$(-y)^m$ comes from terms with exponents between d and $d+m-1$.

Regarding the last sum, we make the following three remarks.

(i) When $\ell < d$, the above quotient of determinants is the Schur polynomial for the partition $\lambda = (d^{N-m}, (d-1)^m, \ell)$. Using Jacobi-Trudi as in Lemma 3.1.8, we obtain that

$$\left[q^d\right] s_\lambda(z_1, \ldots, z_{N+1}) = \begin{cases} (-1)^{(N-1)d} & \text{if } \ell = m = 0 \\ 0 & \text{otherwise} \end{cases}.$$

(ii) When $\ell > d+N$, the shape of the partition changes to $\lambda = (\ell - N, (d+1)^{N-m}, d^m)$, and we also acquire an additional $(-1)^N$ coming from permuting the columns to bring the last one to the front. Note that λ contains the rectangular partition (d^{N+1}) and a hook partition $\mu := (\ell - N - d, 1^{N-m})$. Examining the determinant, we can factor z_i^d from each column. Thus

$$s_\lambda = e_{N+1}^d \cdot s_\mu = y^d \cdot s_\mu.$$

Here we used that $e_{N+1} = y$ which can be seen from the expression $P(z) = (z-1)^N(z-y) - qz$.

(iii) Finally, for $d \leq \ell \leq d+N$, the only value that can contribute is $\ell = d+m$, in which case we can directly evaluate the corresponding quotient of determinants to be $(-1)^m y^d$. The coefficient of q^d vanishes in this case (for $d \neq 0$).

Putting everything together we conclude

$$\chi\left(\mathsf{Quot}_d, \mathsf{Sym}_y L^{[d]}\right) = \frac{1}{(1-y)^\chi} + O(y^d).$$

Consequently, for $d > k$, we have

$$\chi\left(\mathsf{Quot}_d, \mathsf{Sym}^k L^{[d]}\right) = \left[y^k\right] \frac{1}{(1-y)^\chi} = \binom{\chi + k - 1}{k}.$$

The result is also correct for $d = k$; this can be seen for instance from the result below. $\square$

With a bit more effort, the same ideas (combined with a residue calculation) yield a general expression in genus 0. We need this result in order to prove Theorem 2.1.5 in all genera in Section 3.2.2.

Theorem 3.2.1. *When $C = \mathbb{P}^1$ and $\chi = \chi(E \otimes L)$, we have*

$$\chi(\mathsf{Quot}_d(E), \mathsf{Sym}_y L^{[d]}) = \sum_{k=0}^{d} \binom{-\chi + d(N+1)}{k} \frac{(-y)^k}{(1-y)^{d(N+1)}}.$$

Proof. Since both sides depend polynomially on ℓ, see for instance the arguments in Section 3.1.1 for the left hand side, we may assume ℓ is sufficiently large. In this case, we have seen above that

$$\chi(\mathsf{Quot}_d(E), \mathsf{Sym}_y L^{[d]}) = \frac{1}{(1-y)^\chi} + \sum_{\ell > d+N}^{\chi} \frac{1}{(1-y)^\chi} (-1)^{(N-1)d+N+\ell} \binom{\chi}{\ell} y^d \left[q^d\right] \sum_{m=0}^{N} (-y)^m s_{\mu(\ell,m)},$$

for the partition $\mu(\ell,m) = (\ell - N - d, 1^{N-m})$.

Lemma 3.2.2 below evaluates the sum over m. We obtain

$$\begin{aligned}
\chi(\mathsf{Sym}_y L^{[d]}) &= \frac{1}{(1-y)^\chi} \left[1 + \sum_{\ell > d+N}^{\chi} (-1)^{(N-1)d+\ell} \binom{\chi}{\ell} y^{d+1} \left[t^{\ell-N-d}\right] \frac{t^{N(d-1)+1}}{(1-t)^{Nd}(1-yt)^{d+1}} \right] \\
&= \frac{1}{(1-y)^\chi} \left[1 + \sum_{\ell > d+N}^{\chi} (-1)^{(N-1)d+\ell} \binom{\chi}{\ell} y^{d+1} \mathrm{Res}_{t=0} \, \frac{t^{(N+1)d-\ell}}{(1-t)^{Nd}(1-yt)^{d+1}} \, dt \right].
\end{aligned}$$

We can allow all values $\ell \geq 0$ in the sum above since the residue vanishes in the range $\ell \leq N + d$. The binomial theorem evaluates the sum over ℓ. Letting

$$\omega = \frac{t^{(N+1)d-\chi}(1-t)^{\chi - Nd}}{(1-yt)^{d+1}} \, dt,$$

we conclude that

$$\chi(\mathsf{Sym}_y L^{[d]}) = \frac{1}{(1-y)\chi}\left[1+(-1)^{(N-1)d+\chi}y^{d+1}\mathrm{Res}_{t=0}\ \omega\right].$$

Lemma 3.2.3 finishes the proof. □

Lemma 3.2.2. *Let $z_1,\dots,z_{N+1}$ be the roots of $P(z)=(z-1)^N(z-y)-qz$. For $\ell>0$, we have*

$$\left[q^d\right]\sum_{m=0}^{N}(-y)^m s_{(\ell,1^{N-m})}(z_1,\dots,z_{N+1}) = (-1)^N\left[t^\ell\right]\frac{yt^{N(d-1)+1}}{(1-t)^{Nd}(1-yt)^{d+1}}.$$

Proof. Using Jacobi-Trudi, the left hand side of the expression in the lemma equals the $\ell\times\ell$ determinant

$$\sum_{m=0}^{N}(-y)^m\begin{vmatrix} e_{N+1-m} & e_{N+2-m} & e_{N+3-m} & \cdots & e_{N+\ell-m}\\ e_0 & e_1 & e_2 & \cdots & e_{\ell-1}\\ 0 & e_0 & e_1 & \cdots & e_\ell\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & e_1\end{vmatrix}.$$

Summing with respect to m, we obtain that the ith term in first row becomes

$$\begin{aligned} A_i = \sum_{m=0}^{N}(-y)^m e_{N+i-m} &= \left[t^{N+i}\right](1-yt+\cdots+(-1)^N y^N t^N)(1+e_1t+\cdots+e_{N+1}t^{N+1})\\ &= (-1)^{N+i}\left[t^{N+i}\right]\frac{(1-(yt)^{N+1})}{1-yt}\cdot t^{N+1}P(1/t). \end{aligned}$$

Expanding with respect to the first row, we obtain the required determinant equals

$$A_1h_{\ell-1}-A_2h_{\ell-2}+\cdots+(-1)^{\ell-1}A_\ell$$

where $h_j=s_{(j)}$ is the homogeneous symmetric polynomial. We know that the homogeneous symmetric polynomials are given by

$$h_i = \left[t^i\right] \frac{1}{(1-z_1 t)\cdots(1-z_{N+1}t)} = \left[t^i\right] \frac{1}{t^{N+1}P(1/t)}.$$

Thus the required sum equals

$$\begin{aligned}
\sum_{i=1}^{\ell}(-1)^{i-1}A_i h_{\ell-i} &= \sum_{i=1}^{\ell}(-1)^{N+1}\left[\left[t^{N+i}\right]\frac{(1-(yt)^{N+1})}{1-yt}t^{N+1}P(1/t)\right]\left[\left[t^{\ell-i}\right]\frac{1}{t^{N+1}P(1/t)}\right] \\
&\approx \sum_{j=0}^{N}(-1)^{N}\left[\left[t^{N-j}\right]\frac{(1-(yt)^{N+1})}{1-yt}t^{N+1}P(1/t)\right]\left[\left[t^{\ell+j}\right]\frac{1}{t^{N+1}P(1/t)}\right] \\
&\approx \sum_{j=0}^{N}(-1)^{N}\left[\left[t^{N-j}\right]\frac{t^{N+1}P(1/t)}{1-yt}\right]\left[\left[t^{\ell+j}\right]\frac{1}{t^{N+1}P(1/t)}\right],
\end{aligned} \tag{3.42}$$

where $\approx$ means equality of the q^d coefficients. To justify the second line, we note that the difference with the previous term equals

$$\left[q^d\right](-1)^N\left[t^{N+\ell}\right]\left(\frac{1-(yt)^{N+1}}{1-yt}t^{N+1}P(1/t)\cdot\frac{1}{t^{N+1}P(1/t)}\right) = 0$$

for $d>0$. Moreover, since j runs from 0 to N, we may also ignore the term $(yt)^{N+1}$ in the second line, thus yielding the third equality.

Note that

$$t^{N+1}P(1/t) = (1-yt)(1-t)^N - qt^N.$$

Thus

$$\left[t^{N-j}\right]\frac{t^{N+1}P(1/t)}{1-yt} = \left[t^{N-j}\right]\left((1-t)^N - \frac{qt^N}{1-yt}\right) = \begin{cases}(-1)^{N-j}\binom{N}{N-j} & \text{if } j>0 \\ (-1)^N - q & \text{if } j=0\end{cases}.$$

Hence the q^d-coefficient in the sum (3.42) equals

$$\left[q^d\right] \sum_{j=0}^{N} (-1)^j \binom{N}{j} \left[t^{\ell+j}\right] \frac{1}{(1-yt)(1-t)^N - qt^N} + (-1)^{N+1} \left[q^{d-1}\right] \left[t^{\ell}\right] \frac{1}{(1-yt)(1-t)^N - qt^N}$$
$$= (-1)^N \left[q^d\right] \left[t^{\ell+N}\right] \frac{(1-t)^N}{(1-yt)(1-t)^N - qt^N} + (-1)^{N+1} \left[q^{d-1}\right] \left[t^{\ell}\right] \frac{1}{(1-yt)(1-t)^N - qt^N}.$$

We note that the order in which we take the q^d and $t^{\ell+N}$-coefficients can be switched. This is allowed in our case since we are considering expressions of the form $(1-\mathsf{A}(q,t))^{-1}$ expanded near $q=t=0$, where A is a polynomial in q,t (and y). Thus, taking the respective coefficient of powers of q in the above expression we obtain

$$(-1)^N \left[t^{\ell+N}\right] \frac{t^{Nd}}{(1-yt)^{d+1}(1-t)^{Nd}} + (-1)^{N+1} \left[t^{\ell}\right] \frac{t^{N(d-1)}}{(1-yt)^{d}(1-t)^{Nd}}.$$

This immediately implies the lemma. □

Lemma 3.2.3. *For $\chi \geq Nd$, set*

$$\omega = \frac{t^{(N+1)d-\chi}(1-t)^{\chi-Nd}}{(1-yt)^{d+1}}\, dt.$$

We have

$$1 + (-1)^{(N-1)d+\chi} y^{d+1} Res_{t=0}\, \omega = \sum_{k=0}^{d} \binom{-\chi+(N+1)d}{k} \frac{(-y)^k}{(1-y)^{(N+1)d-\chi}}.$$

Proof. Since $\chi \geq Nd$, the form ω has poles at worst at $t=0$, $t=\infty$ and $t=\frac{1}{y}$. By the residue theorem, we have

$$\mathrm{Res}_{t=0}\, \omega = -\mathrm{Res}_{t=\infty}\, \omega - \mathrm{Res}_{t=1/y}\, \omega.$$

Changing variables $t=\frac{1}{s}$, we compute

$$\mathrm{Res}_{t=\infty}\, \omega = -\mathrm{Res}_{s=0}\, (s-1)^{\chi-Nd}(s-y)^{-d-1}\frac{ds}{s} = (-1)^{\chi-(N+1)d} y^{-d-1}.$$

Similarly, changing variables $t = \frac{1-s}{y}$, we find

$$\begin{aligned}
\operatorname{Res}_{t=\frac{1}{y}} \omega &= -\operatorname{Res}_{s=0} (1-s)^{-\chi+(N+1)d} (s+y-1)^{\chi - Nd} y^{-d-1} \frac{ds}{s^{d+1}} \\
&= -y^{-d-1} \left[s^d\right] (1-s)^{-\chi+(N+1)d} (s+y-1)^{\chi-Nd} \\
&= -y^{-d-1} \sum_{k=0}^{d} (-1)^k \binom{-\chi+(N+1)d}{k} \binom{\chi - Nd}{d-k} (y-1)^{\chi - Nd - d + k}.
\end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned}
1 + (-1)^{(N-1)d+\chi} y^{d+1} \operatorname{Res}_{t=0} \omega &= \sum_{k=0}^{d} \binom{-\chi+(N+1)d}{k} \binom{\chi - Nd}{d-k} (1-y)^{\chi - Nd - d + k} \\
&= \sum_{k=0}^{d} \binom{-\chi+(N+1)d}{k} \frac{(-y)^k}{(1-y)^{(N+1)d-\chi}}.
\end{aligned}$$

To justify the last equality, we write $u = -\chi + (N+1)d$ and show more generally

$$\sum_{k=0}^{d} \binom{u}{k} \binom{-u+d}{d-k} (1-y)^k = \sum_{k=0}^{d} \binom{u}{k} (-y)^k.$$

This follows by induction on d. Indeed, write L_d for the left hand side. Using Pascal's identity and then rewriting the binomials, we obtain

$$\begin{aligned}
\mathsf{L}_{d+1} - \mathsf{L}_d &= \sum_{k=0}^{d+1} \binom{u}{k} \left(\binom{-u+d+1}{d+1-k} - \binom{-u+d}{d-k} \right) (1-y)^k = \sum_{k=0}^{d+1} \binom{u}{k} \binom{-u+d}{d+1-k} (1-y)^k \\
&= \sum_{k=0}^{d+1} \binom{u}{d+1} \binom{d+1}{k} (-1)^{d-k+1} (1-y)^k = \binom{u}{d+1} (-y)^{d+1}.
\end{aligned}$$

The proof follows immediately from here. $\square$

3.2.2 Universal functions

Over a smooth projective curve C of arbitrary genus, let

$$\mathsf{W} = \sum_{d=0}^{\infty} q^d \chi \left(\mathsf{Quot}_d(E), \mathsf{Sym}_y L^{[d]} \right).$$

The arguments in Section 3.1.1 exhibit W as a product of universal series [3]

$$\mathsf{W} = \mathsf{A}^{\chi(\mathcal{O}_C)} \cdot \mathsf{B}^{\chi(E \otimes L)}. \tag{3.43}$$

In principle Theorem 3.2.1 determines both series A, B from the genus 0 answer. Theorem 2.1.5 asserts that more precisely we have

$$\mathsf{B} = f\left(\frac{qy}{(1-y)^{N+1}} \right)$$

where $f(z)$ is the solution to the equation

$$f(z)^N - f(z)^{N+1} + z = 0, \quad f(0) = 1.$$

Proof of Theorem 2.1.5. The function f is most conveniently expressed in terms of a change of variables. We have

$$f(z) = \frac{1}{1+t} \quad \text{for } z = -\frac{t}{(1+t)^{N+1}}.$$

We record the one-variable version of the general Lagrange-Bürmann formula (3.18).

[3]Strictly speaking, we only explained the factorization $\mathsf{W} = \mathsf{A}_1^{\chi(\mathcal{O}_C)} \cdot \mathsf{B}_1^{\deg E} \cdot \mathsf{B}_2^{\deg L}$ in terms of 3 universal series. An argument of [Sta 1] shows that only 2 series are needed. Indeed, tensorization by a line bundle $M \to C$ gives an isomorphism $\mathsf{Quot}_d(E) \simeq \mathsf{Quot}_d(E \otimes M)$ in such a fashion that $L^{[d]}$ gets identified with $\left(L \otimes M^{-1}\right)^{[d]}$. On the level of generating series this implies $\mathsf{B}_1^N = \mathsf{B}_2$, which then yields the result with $\mathsf{A} = \mathsf{A}_1 \cdot \mathsf{B}_1^{-N}$, $\mathsf{B} = \mathsf{B}_1$.

Assuming $\Phi(0) \neq 0$, for the change of variables $z = \frac{t}{\Phi(t)}$, the following general identity holds

$$\sum_{d=0}^{\infty} z^d \cdot \left(\left[t^d \right] \Phi(t)^d \cdot \Psi(t) \right) = \frac{\Psi(t)}{\Phi(t)} \cdot \frac{dt}{dz}. \tag{3.44}$$

We introduce two functions which will be useful in the argument. Write

$$\mathsf{F}_\chi(z) = \sum_{d=0}^{\infty} z^d \binom{-\chi + (N+1)d}{d} \implies F_\chi(z) = \sum_{d=0}^{\infty} z^d \left(\left[t^d \right] (1+t)^{-\chi+(N+1)d} \right). \tag{3.45}$$

An immediate application of (3.44) yields

$$\mathsf{F}_\chi(z) = \frac{(1+t)^{-\chi+1}}{1-Nt} \quad \text{for } z = \frac{t}{(1+t)^{N+1}}. \tag{3.46}$$

Setting $\chi = 0$ and integrating, we also obtain the expression

$$\mathsf{G}(z) = \sum_{d=1}^{\infty} z^d \cdot \frac{N}{d} \binom{(N+1)(d-1)}{d-1} = 1 - \frac{1}{(1+t)^N}, \tag{3.47}$$

for the same change of variables. With this understood, we note that for the function f in the theorem, we have

$$f(-z)^N = 1 - \mathsf{G}(z).$$

The statement to be proven thus becomes

$$\mathsf{B}^N = 1 - \mathsf{G}\left(-\frac{qy}{(1-y)^{N+1}} \right)$$

or equivalently

$$\mathsf{B}^N = 1 + \sum_{d=1}^{\infty} (-1)^{d+1} \frac{N}{d} \cdot \binom{(N+1)(d-1)}{d-1} \cdot \left(\frac{qy}{(1-y)^{N+1}} \right)^d. \tag{3.48}$$

Turning to the generating series (3.43), we specialize to genus 0 and we keep track on the dependence on $\deg L = \ell$ in the notation, so that

$$\mathsf{W}_\ell = \sum_{d=0}^{\infty} q^d \chi\left(\mathsf{Quot}_d(E), \mathsf{Sym}_y L^{[d]}\right) = \mathsf{A}^{-1} \cdot \mathsf{B}^{\chi}.$$

As usual, $\chi = \chi(E \otimes L)$. This yields

$$\mathsf{W}_{\ell+1} = \mathsf{W}_\ell \cdot \mathsf{B}^N. \tag{3.49}$$

By Theorem 3.2.1, we have

$$\mathsf{W}_\ell = \sum_{d=0}^{\infty} \mathsf{c}_d(\chi) \cdot q^d, \quad \mathsf{W}_{\ell+1} = \sum_{d=0}^{\infty} \mathsf{c}_d(\chi + N) \cdot q^d,$$

where for simplicity, we wrote

$$\mathsf{c}_d(\chi) = \sum_{k=0}^{d} \binom{-\chi + d(N+1)}{k} \frac{(-y)^k}{(1-y)^{d(N+1)}}. \tag{3.50}$$

Examining the coefficient of q^d in the identity (3.49), it follows that in order to confirm (3.48) it suffices to prove

$$\mathsf{c}_d(\chi + N) = \mathsf{c}_d(\chi) + \sum_{\ell=1}^{d} \mathsf{c}_{d-\ell}(\chi) \cdot (-1)^{\ell+1} \frac{N}{\ell} \binom{(N+1)(\ell-1)}{\ell-1} \left(\frac{y}{(1-y)^{N+1}}\right)^{\ell}.$$

We use the defining expressions (3.50) to verify this equality. After multiplying by $(1-y)^{d(N+1)}$ and extracting the coefficient of y^k on both sides, we need to show that for $0 \le k \le d$, we have

$$\binom{-\chi - N + d(N+1)}{k} = \binom{-\chi + d(N+1)}{k} - \sum_{\ell=1}^{k} \frac{N}{\ell} \binom{(N+1)(\ell-1)}{\ell-1} \binom{-\chi + (d-\ell)(N+1)}{k-\ell}. \tag{3.51}$$

Using Pascal's identity, it is easy to see that if (3.51) holds for k and all χ, then it also holds for

$k-1$ and all χ. Thus, by downward induction it suffices to assume $k = d$. In this case, we seek to show

$$\sum_{\ell=1}^{d} \frac{N}{\ell} \binom{(N+1)(\ell-1)}{\ell-1} \binom{-\chi+(d-\ell)(N+1)}{d-\ell} = \binom{-\chi+d(N+1)}{d} - \binom{-\chi-N+d(N+1)}{d}. \blacksquare$$

This is indeed correct. Recalling (3.45) and (3.47), we see that the two sides equal the z^d coefficient in the identity

$$\mathsf{G}(z) \cdot \mathsf{F}_{\chi}(z) = \mathsf{F}_{\chi}(z) - \mathsf{F}_{\chi+N}(z).$$

The latter equality is immediately justified using the explicit formulas (3.46) and (3.47) after changing variables from z to t as above.

Chapter 4

Quot schemes over $\mathbb{P}^1$

Let $\mathsf{Quot}_d(N,r)$ denote the Quot scheme $\mathsf{Quot}_d(\mathbb{C}^N, r, \mathbb{P}^1)$. Let $0 \to \mathcal{S} \to p^*\mathcal{O}_C \to \mathcal{Q} \to 0$ denote the universal exact sequence.

We first note that $\mathsf{Quot}_d(N,r)$ is a smooth projective scheme.

Proposition 4.0.1. *For any choice of N, r and d,* $\mathsf{Quot}_d(\mathbb{C}^N, r, \mathbb{P}^1)$ *is smooth.*

Proof. The deformation theory for Quot schemes is given by $\mathrm{Ext}^\bullet(S,Q)$. Since we work over curves it is enough to show that $\mathrm{Ext}^1(S,Q) = 0$. Using Serre duality, $\mathrm{Ext}^1(S,Q) = \mathrm{Ext}^0(Q, S(-2))^\vee$. Since $\mathbb{C}^N \otimes \mathcal{O} \to Q$ is a surjection and $S \to \mathbb{C}^N \times \mathcal{O}$ is an injection, it is enough to show that $\mathrm{Hom}(\mathbb{C}^N \otimes \mathcal{O}, \mathbb{C}^N \otimes \mathcal{O}(-2)) = 0$, which is clear. $\square$

The dimension of $\mathsf{Quot}(N,r)$ equals $\chi(\mathsf{Quot}_d(N,r), S^\vee \otimes Q) = Nd + (N-r)r$.

4.1 Torus action

We will use the Atiyah-Bott localization formula to obtain the Euler characteristics of schur bundles associated to $\mathcal{S}_x$ and $\mathcal{S}_x^\vee$ over $\mathsf{Quot}_d(N,r)$. The localization calculation is slight different from that for punctual Quot scheme.

Let $\mathbb{C}^*$ act on $\mathbb{C}^N \otimes \mathcal{O}_C$ with distinct weights $-w_1, \dots, -w_N$. This induces a $\mathbb{C}^*$-action on the Quot scheme $\mathsf{Quot}_d(N,r)$. The fixed loci of this action is parameterized by pairs $(\vec{d}, I)$ where $\vec{d} = (d_1, \dots, d_r)$ with $|\vec{d}| = d_1 + \cdots + d_r = d$, and $I \subset [N]$ is a subset of size r. Moreover, the fixed loci are isomorphic to products of symmetric products of $\mathbb{P}^1$:

$$\mathrm{F}_{\vec{d},I} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r}.$$

The factor $\mathbb{P}^{d_i}$ corresponds to the Hilbert scheme of d_i points parameterizing short exact sequences

$$0 \to K_i \to \mathcal{O}_{\mathbb{P}^1} \to T_i \to 0$$

such that T_i is a torsion sheaf of length d_i. The corresponding point in the fixed locus $\mathrm{F}_{\vec{d},I}$ is

$$0 \to S \to \bigoplus_{i \in I} \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i \in [N]} \mathcal{O}_{\mathbb{P}^1} \to Q \to 0,$$

where

$$S = K_1 \oplus \cdots \oplus K_r \quad \text{and} \quad \mathcal{Q} \cong T_1 \oplus \cdots \oplus T_r \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus N-r}.$$

Define $\mathcal{K}_i$ and $\mathcal{T}_i$ denote the tautological subbundle and the quotient bundle on $\mathbb{P}^1 \times \mathbb{P}^{d_i}$. We shall use the same notation for their pullback to $\mathbb{P}^1 \times \mathrm{F}_{\vec{d},I}$. Note that

$$\mathcal{K}_i = \mathcal{O}_{\mathbb{P}^1}(-d_i) \boxtimes \mathcal{O}_{\mathbb{P}^{d_i}}(-1).$$

Todd Calculations

We observe that $T_{\mathsf{Quot}_d} = \mathrm{Hom}_\pi(\mathcal{S}, \mathcal{Q})$ restricts to

$$\bigoplus_{i,j \in I} \pi_*[\mathcal{K}_i^\vee \otimes \mathcal{T}_j] \bigoplus_{i \in I, j \in [N] \setminus I} \pi_*[\mathcal{K}_i^\vee \otimes \mathcal{O}_{\mathbb{P}^1}]$$

over the fixed loci $\mathrm{F}_{\vec{d},I}$.

In K-theory, it equals

$$\bigoplus_{i\in I, j\in[N]} \pi_*[\mathcal{K}_i^\vee] - \bigoplus_{i,j\in I} \pi_*[\mathcal{K}_i^\vee \otimes \mathcal{K}_j].$$

Therefore the Todd class of T_{Quot_d} restricted to the fixed loci is

$$\prod_{i\in I, j\in[N]} \mathrm{Td}(\pi_*\mathcal{K}_i^\vee)\left(\prod_{i,j\in I} \mathrm{Td}(\pi_*\mathcal{K}_i^\vee \otimes \mathcal{K}_j)\right)^{-1}.$$

The above classes comes equiped with weight $w_i - w_j$ and $w_i - w_j$ respectively.

Equivariant normal bundle

We know that the normal bundle is given by the moving part of the Tangent bundle over the fixed loci. We observe that the moving part is

$$\begin{aligned}\mathcal{N}^{\mathrm{vir}} = T^{\mathrm{mov}}\Big|_{\mathrm{F}_{\vec{d}}} &= \bigoplus_{i,j\in I; i\neq j} \pi_*[\mathcal{K}_i^\vee \otimes \mathcal{T}_j] \bigoplus_{i\in I, j\in[N]\setminus I} \pi_*[\mathcal{K}_i^\vee] \\ &= \bigoplus_{i\in I, j\in[N], i\neq j} \pi_*[\mathcal{K}^\vee] - \bigoplus_{i,j\in I; i\neq j} \pi_*[\mathcal{K}_i^\vee \otimes \mathcal{K}_j].\end{aligned}$$

The above classes comes equiped with weight $w_i - w_j$ and $w_i - w_j$ respectively.

We know that taking Euler class is multiplicative in K-theory. Thus

$$\frac{1}{e_{\mathbb{C}^*}(N)} = \prod_{i\in I, j\in[N];\ k_i\neq j} \left(e_{\mathbb{C}^*}(\mathcal{K}^\vee)\right)^{-1} \prod_{i,j\in I; i\neq j} e_{\mathbb{C}^*}(\mathcal{K}_i^\vee \otimes \mathcal{K}_j)$$

Riemann Roch

Let $h_i \in H^2(\mathrm{F}_{\vec{d},I})$ denote the pull back of the hyperplane class in $\mathbb{P}^{d_i}$. In the equivariant cohomology,

$$c(\pi_* \mathcal{K}_i^\vee) = (1 + (h_i + w_i\varepsilon - w_j\varepsilon))^{d_i+1}$$
$$c(\pi_* \mathcal{K}_i^\vee \otimes \mathcal{K}_j) = (1 + (h_i + w_i\varepsilon - h_j - w_j\varepsilon))^{d_i-d_j+1},$$

where ε is the equivariant parameter.

Using Riemann-Roch, we can calculated corresponding equivariant Todd classes:

$$\mathrm{Td}(\pi_* \mathcal{K}_i^\vee) = \left(\frac{h_i + w_i\varepsilon - w_j\varepsilon}{1 - e^{-(h_j + w_j\varepsilon - w_j\varepsilon)}}\right)^{d_i+1}$$
$$\mathrm{Td}(\pi_* \mathcal{K}_i^\vee \otimes \mathcal{K}_j) = \left(\frac{h_i + w_i\varepsilon - h_j - w_j\varepsilon}{1 - e^{-(h_i + w_i\varepsilon - h_j - w_j\varepsilon)}}\right)^{d_i-d_j+1},$$

the later equals 1 when $i = j$. Similarly we obtain the Euler classes :

$$e_{\mathbb{C}^*}(\pi_* \mathcal{K}_i^\vee) = (h_i + w_i\varepsilon - w_j\varepsilon)^{d_i+1}$$
$$e_{\mathbb{C}^*}(\pi_* \mathcal{K}_i^\vee \otimes \mathcal{K}_j) = (h_i + w_i\varepsilon - h_j - w_j\varepsilon)^{d_i-d_j+1}.$$

Simplification

Observe that over the fixed locus $\mathrm{F}_{\vec{d},I}$, the factor $\frac{\mathrm{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(N)}$ restricts to

$$\prod_{i \in I} e_{\mathbb{C}^*}(\pi_* \mathcal{K}_i^\vee) \prod_{i \in I, j \in [N]} \frac{\mathrm{Td}}{e_{\mathbb{C}}^*}\left(\pi_* \mathcal{K}_i^\vee\right) \prod_{i,j \in I; i \neq j} \frac{e_{\mathbb{C}^*}}{\mathrm{Td}}\left(\mathcal{K}_i^\vee \otimes \mathcal{K}_j\right).$$

For notational convenience, we set

$$z_i = e^{h_i + w_i\varepsilon}, \quad \alpha_i = e^{w_i\varepsilon} \quad \text{and} \quad R(Z) = \prod_{i=1}^{N} (Z - \alpha_i).$$

Thus

$$\begin{aligned}
\left.\frac{\mathsf{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}(N)}\right|_{\mathrm{F}_{\vec{d},I}} =& \prod_{i\in I} h_i^{d_i+1}\left(\frac{z_i^N}{R(z_i)}\right)^{d_i+1} \cdot \prod_{i,j\in I; i\neq j}\left(\frac{z_i - z_j}{z_i}\right)^{d_i-d_j+1} \\
=& (-1)^{(r-1)d}\prod_{i\in I}\left(\frac{h_i z_i^{N-r}}{R(z_i)}\right)^{d_i+1} z_i^{d+1}\prod_{i,j\in I; i\neq j}(z_i - z_j).
\end{aligned}$$

4.1.1 Schur bundles

In this subsection, we use $\mathbb{C}^*$ localization to reduce the calculations of K-theoretic invariants of Quot schemes to the theory of symmetric functions. We are primarily concerned with the K-theory classes of the Schur functors associated to $\mathcal{S}_x$ (and its dual) over $\mathsf{Quot}_d(N,r)$. Let $a_1, a_2, \ldots, a_r$ be the Chern roots of $\mathcal{S}_x^\vee$. Note that the Chern character of the Schur bundle associated to $\mathcal{S}_x$ (and its dual) over $\mathsf{Quot}_d(N,r)$ are given by corresponding Schur polynomial in $e^{a_1}, e^{a_2}, \ldots, e^{a_r}$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be tuple of n (not necessarily ordered) integers. We define the corresponding Schur function using the bialternant formula

$$s_\lambda(z_1,\ldots,z_n) = \frac{1}{\det(z_i^j)}\begin{vmatrix} z_1^{\lambda_1+n-1} & & \cdots & & z_n^{\lambda_1+n-1} \\ z_1^{\lambda_2+n-2} & & \cdots & & z_r^{\lambda_2+n-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ z_1^{\lambda_n} & & \cdots & & z_n^{\lambda_n} \end{vmatrix}. \tag{4.1}$$

Note that any symmetric Laurent polynomial in $z_1, \ldots, z_n$ can be uniquely expressed as a linear combinations of s_λ's.

In the proposition below, we give a formula for the Euler characteristics of a K-theory class $F(S_x)$ and its twist with line bundles $\det(\pi_* \mathcal{S}^\vee)^\ell$.

Proposition 4.1.1. *Let $-(N-r) < \ell \leq r$ denote the level and let $F(S_x)$ be a K-theory such that*

$$\mathrm{ch}(F(S_x)) = s_\lambda(e^{a_1}, \dots, e^{a_r})$$

where $\lambda = (\lambda_1, \dots, \lambda_r)$ is an r-tuple of integers, then

$$\chi(\mathsf{Quot}_d(N,r), \det(\pi_* \mathcal{S}^\vee)^\ell \otimes F(\mathcal{S}_x)) = [t^d] s_\Lambda(z_1, \dots, z_N)$$

where $z_1, z_2, \dots, z_N$ are roots of the equation $(z-1)^N + (-1)^r z^{N-r+\ell} t = 0$; and $\Lambda = (\lambda_1 + d + \ell, \dots \lambda_r + d + \ell, 0, \dots, 0)$.

Proof. Using $\mathbb{C}^*$ Atiyah-Bott localization, the holomorphic Euler characteristic of $\det(\pi_* \mathcal{S}^\vee)^\ell \otimes F(\mathcal{S}_x)$ equals

$$\sum_{\vec{d},I} \int_{\mathrm{F}_{\vec{d},I}} \mathrm{ch}\left(\det(\pi_* \mathcal{S}^\vee)^\ell|_{\mathrm{F}_{\vec{d},I}}\right) \mathrm{ch}\left(F(\mathcal{S}_x)|_{\mathrm{F}_{\vec{d},I}}\right) \frac{\mathrm{Td}(\mathsf{Quot}_d)}{e_{\mathbb{C}^*}}\bigg|_{\mathrm{F}_{\vec{d},I}},$$

where $I = \{i_1, \dots, i_r\}$ runs over r element subsets of $\{1, 2, \dots, N\}$, and $\vec{d}$ runs over the tuples of non-negative integers $(d_1, \dots, d_r)$ that sum to d.

Recall that $\mathcal{K}_i = \mathcal{O}_{\mathbb{P}^1}(-d_i) \boxtimes \mathcal{O}_{\mathbb{P}^{d_i}}(-1)$. Over the fixed loci $\mathrm{F}_{\vec{d},I}$,

$$\mathcal{S}_x^\vee\bigg|_{\mathrm{F}_{\vec{d},I}} = \mathcal{O}_{\mathbb{P}^{d_1}}(1) \boxplus \cdots \boxplus \mathcal{O}_{\mathbb{P}^{d_r}}(1) \quad \text{and} \quad \det(\pi_\star S^\vee)\bigg|_{\mathrm{F}_{\vec{d},I}} = \mathcal{O}_{\mathbb{P}^{d_1}}(d_1+1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{d_r}}(d_r+1)$$

In particular, the Chern roots of the above (in the equivariant cohomology) equals $\{(h_i + w_i \varepsilon) : i \in I\}$. Thus

$$\mathrm{ch}\left(F(\mathcal{S}_x)\bigg|_{\mathrm{F}_{\vec{d},I}}\right) = s_\lambda(z_{i_1}, \dots, z_{i_r}) \quad \text{and} \quad \mathrm{ch}\left(\det(\pi_* \mathcal{S}^\vee)|_{\mathrm{F}_{\vec{d},I}}\right) = \prod_{i \in I} z_i^{d_i+1}$$

where $z_i = e^{h_i + w_i \varepsilon}$ and s_λ is the corresponding Schur polynomial.

We use the explicit calculations from the previous subsections to obtain

$$[\varepsilon^0]\sum_{\vec{d},I}[h^{\vec{d}}]\prod_{i\in I}z_i^{\ell(d_i+1)}s_\lambda(z_I)\prod_{i=I}\left(\frac{h_i z_i^{N-r}}{R(z_i)}\right)^{d_i+1}z_i^{d+1}(-1)^{(r-1)d}\prod_{i,j\in I,i\neq j}(z_i-z_j).$$

Here $z_I^{-1}=(z_{i_1}^{-1},\dots,z_{i_r}^{-1})$ and $[h^{\vec{d}}]$ denotes taking the coefficient of $\prod_{i\in I}h_i^{d_i}$; this corresponds to integrating over $\mathrm{F}_{\vec{d},I}=\mathbb{P}^{d_1}\times\cdots\times\mathbb{P}^{d_r}$.

We invoke the multivariate Lagrange-Bürmann formula to sum over $\vec{d}$ (see (3.18)). Recall that for formal power series $\Psi(h_1,\dots,h_r)$, and $\Phi_1(h_1),\dots,\Phi_N(h_r)$ with $\Phi_i(0)\neq 0$, we have

$$[h^{\vec{d}}]\Psi(h_1,\dots,h_r)\prod_{i=1}^{r}\Phi_i(h_i)^{d_i+1}=[t^{\vec{d}}]\psi(h_1,\dots,h_r)\prod_{i=1}^{r}\frac{dh_i}{dt_i},\tag{4.2}$$

where we use the change of variable $t_i=\frac{h_i}{\Phi_i(h_i)}$, and express h_i in terms of t_i in the right hand side of (4.2). In our problem, we use the change of variable

$$t_i=\frac{R(z_i)}{z_i^{N-r+\ell}}\tag{4.3}$$

where t_i is considered as a power series in h_i and furthermore

$$\frac{dt_i}{dh_i}=\frac{R'(z_i)-(N-r+\ell)z_i^{-1}R(z_i)}{z_i^{N-r+\ell-1}}.\tag{4.4}$$

The Lagrange-Bürmann formula implies that the previous expression equals

$$[\varepsilon^0]\sum_{\vec{d},I}[t^{\vec{d}}]s_\lambda(z_I)\prod_{i\in I}\frac{dh_i}{dt_i}z_i^{d+1}(-1)^{(r-1)d}\prod_{i,j\in I,i\neq j}(z_i-z_j).$$

In the above expression, we regard d appearing in the exponent z_i^d as an independent parameter. We thus observe that the summand is independent of d_i's. In particular, to evaluate the above sum, we let $t_1=\cdots=t_N=t$ and find the coefficient of t^d in the result sum. Furthermore, we note that $z_1, \dots, z_N$ are distinct solutions (see (4.3)) to

$$P(z) = R(z) - z^{N-r+\ell} t,$$

which is a degree N polynomial in z.

Using (4.3) and (4.4), we observe that

$$\frac{dh_i}{dt} = \frac{z_i^{N-r+\ell-1}}{P'(z_i)},$$

where the derivative $P_i'(z)$ is taken with respect to the variable z. We thus rewrite the required expression as

$$(-1)^{(r-1)d} [\varepsilon^0] \sum_I [t^d] s_\lambda(z_I) \prod_{i \in I} \frac{z_i^{d+N-r+\ell}}{P'(z_i)} \prod_{i,j \in I, i \neq j} (z_i - z_j).$$

The crucial observation is to view the above expression as ratio of $N \times N$ determinants

$$[\varepsilon^0][t^d] \frac{(-1)^{(r-1)d}}{\det(z_i^j)} \det \begin{vmatrix} z_1^{\lambda_1+N-1+d+\ell} & z_2^{\lambda_1+N-1+d+\ell} & \cdots & z_N^{\lambda_1+N-1+d+\ell} \\ z_1^{\lambda_2+N-2+d+\ell} & z_2^{\lambda_2+N-2+d+\ell} & \cdots & z_N^{\lambda_2+N-2+d+\ell} \\ \vdots & \cdots & \cdots \ \vdots & \vdots \\ z_1^{\lambda_r+N-r+d+\ell} & z_2^{\lambda_r+N-r+d+\ell} & \cdots & z_N^{\lambda_r+N-r+d+\ell} \\ z_1^{N-r-1} & z_2^{N-r} & & z_N^{N-r-1} \\ \vdots & \cdots & \cdots \ \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

Here $\det(z_i^j)$ in the denominator is the $N \times N$ Vandermonde determinant. We used generalized Laplace expansion of the determinant along first r rows to compare the previous expression.

Another crucial observation is that the above expression is a symmetric Laurent polynomial in z_i's given by

$$[\varepsilon^0][t^d](-1)^{(r-1)d}s_\Lambda(z_1,\dots,z_N),$$

which are in turn Laurent polynomial in α_j's and t. Here $\Lambda=(\lambda_1+d+\ell,\dots\lambda_r+d+\ell)$. This means we can set $\varepsilon=0$ and obtain $\alpha_j=1$ for all j. In particular, z_i's are roots of

$$P(z)=(z-1)^N-z^{N-r+\ell}t=0.$$

We finish the proof of Proposition 4.1.1 by substituting $t\to(-1)^{r-1}t$. $\square$

Note that for any integer partition λ, the Schur bundle

$$\mathrm{ch}\big(\mathbb{S}^\lambda(\mathcal{S}_x^\vee)\big)=s_\lambda(e^{a_1},\dots,e^{a_r}).$$

This gives us the following corollary.

Theorem 4.1.2. *For any partition λ with at most r parts, we have*

$$\chi(\mathsf{Quot}_d(N,r),\mathbb{S}^\lambda(\mathcal{S}_x^\vee))=[t^d]s_\Lambda(z_1,\dots,z_N)$$

where $z_1,\dots,z_N$ are roots of $(z-1)^N+(-1)^r z^{N-r}t=0$, and the partition

$$\Lambda=(d+\lambda_1,d+\lambda_2,\dots,d+\lambda_r).$$

Corollary 4.1.3. *We have*

$$\sum_{d=0}^{\infty}q^d\chi(\mathsf{Quot}_d(N,r),\wedge^m(\mathcal{S}_x^\vee))=\begin{cases}\binom{N}{m}\frac{1}{1-q} & m\neq r\\ \binom{N}{r}\frac{1}{(1-q)^2} & m=r\end{cases}.$$

Proof. Note that the elementary symmetric polynomials in z_i's are

$$e_m(z_1,\ldots,z_N) = \begin{cases} \binom{N}{m} & m \neq r \\ \binom{N}{r} + t & m = r \end{cases}.$$

Note that $\lambda = (1,1,\ldots,1)$, where 1 appears m times. We may express the Schur polynomial $s_\Lambda(z_1,\ldots,z_N)$ in terms of elementary symmetric polynomial using Jacobi-Trudi formula as a $(d+1)\times(d+1)$ determinant

$$\begin{vmatrix} e_r & e_{r+1} & e_{r+2} & \cdots & e_{r+d-1} & e_{r+d} \\ e_{r-1} & e_r & e_{r+1} & \cdots & e_{r+d-2} & e_{r+d-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e_{r-d+1} & e_{r-d+2} & e_{r-d+3} & \cdots & e_r & e_{r+1} \\ e_{m-d} & e_{m-d+1} & e_{m-d+2} & \cdots & e_{m-1} & e_m \end{vmatrix}.$$

When $m < r$, the above determinant is a polynomial in t of degree d with leading coefficient $e_m = \binom{N}{m}$. When $m = r$, it is a polynomial in t of degree $d+1$, with t^d coefficient $(d+1)\binom{N}{r}$. □

Corollary 4.1.4. *When $m < r$,*

$$\sum_{d=0}^{\infty} q^d \chi(\mathsf{Quot}_d(N,r), \mathrm{Sym}^m(\mathcal{S}_x^\vee)) = \binom{N+m-1}{m}\frac{1}{1-q}.$$

Proposition 4.1.5. *For any partition λ with at most r parts, we have*

$$\chi(\mathsf{Quot}_d(N,r), \mathbb{S}^\lambda(\mathcal{S}_x)) = [t^d] s_\Lambda(z_1,\ldots,z_N),$$

where $\Lambda = (d-\lambda_r,\ldots,d-\lambda_1,0,\ldots,0)$, and $z_1,z_2,\ldots,z_N$ are roots of the equation $(z-1)^N + (-1)^r z^{N-r} t = 0$.

Proof. Note that the Chern character

$$\mathrm{ch}\big(\mathbb{S}^{\lambda}(\mathcal{S}_x)\big) = s_{\lambda}(e^{-a_1}, \dots, e^{-a_r}) = s_{\tilde{\lambda}}(e^{a_1}, \dots, e^{a_r}).$$

where $\tilde{\lambda} = (-\lambda_r, \dots, -\lambda_1)$. The result follows from Proposition 4.1.1. □

Theorem 4.1.6. *For any non-trivial partition λ with at most r parts and $\lambda_1 \leq d + N - r$,*

$$\chi(\mathsf{Quot}_d(N,r), \mathbb{S}^{\lambda}(\mathcal{S}_x)) = 0.$$

Proof. When $-(N-r) \leq d - \lambda_1 < 0$, the r^{th} row in the bialternant formula (see (4.1)) for s_{Λ}, where $\Lambda = (d - \lambda_r, \dots, d - \lambda_1, 0, \dots, 0)$, has exponents $0 \leq d - \lambda_1 + N - r \leq N - r - 1$. Thus the r^{th} row equals one of the last $(N-r)$ rows, hence the determinant is equal to zero .

When $d - \lambda_1 \geq 0$, $\tilde{\Lambda}$ is an integer partition strictly contained (since λ is not trivial) in the rectangular partition $(d, d \dots, d)$, where d appears r times. The highest exponent of e_r appearing in the Jacobi-Trudi expansion of $s_{\tilde{\Lambda}}$ in terms of elementary symmetric polynomial is strictly less than d for degree reasons. Thus $s_{\tilde{\Lambda}}(z_1, \dots, z_N)$ is a polynomial in t of degree at most $d-1$.

□

Lemma 4.1.7. *Let $\lambda^1, \dots, \lambda^m$ be m partitions, and let*

$$s_{\lambda^1} s_{\lambda^2} \cdots s_{\lambda^m} = \sum_{\nu} C^{\nu}_{\lambda^1, \dots, \lambda^m} s_{\nu}$$

be the expansion of the product of Schur polynomials in the Schur basis. Then $C^{\nu}_{\lambda^1, \dots, \lambda^m} = 0$ unless $\lambda^1_1 + \cdots + \lambda^m_1 \geq \nu$.

Proof. Using Littlewood-Richardson rule, for any two partitions λ and μ

$$s_{\lambda} \cdot s_{\mu} = \sum_{\nu} c^{\nu}_{\lambda,\mu} s_{\nu}$$

where $c^{\nu}_{\lambda,\mu}$ equal the number of Littlewood-Richardson tableaux of skew shape ν/λ of weight μ. The Littlewood-Richardson tableaux are skew semi-standard tableaux such that the word obtained by the concatenating the entries in each of ν/λ in the reverse order is a lattice word (i.e for each positive integer i, every initial part of the word contains more number of i's than $i+1$'s). This implies that the first row of ν/λ (if it is not empty) must only contain 1's. Since there are μ_1 number of 1's, for $c^{\nu}_{\lambda,\mu} = 0$ unless $\mu_1 + \lambda_1 \geq \nu_1$. This argument can be applied repeatedly to obtain finish the proof.

$\square$

Remark 4.1.8. *The vanishing result in the previous corollary and the Littlewood-Richardson rule implies that for any partitions $\lambda^1, \ldots, \lambda^m$, the power series*

$$F(q; \lambda^1, \ldots, \lambda^m) := \sum_{i=0}^{\infty} q^d \chi(\mathsf{Quot}_d(N,r), \mathbb{S}^{\lambda^1}(\mathcal{S}_x) \otimes \cdots \otimes \mathbb{S}^{\lambda^m}(\mathcal{S}_x))$$

is a polynomial in q of degree at most $\lambda^1_1 + \cdots + \lambda^m_1 - (N-r)$. The bound on the degree can be improved by imposing extra conditions.

Proposition 4.1.9. *Let $r < N$. For any non-trivial partition λ with exactly r parts (i.e $\lambda_r \neq 0$) and $\lambda_1 \leq d + 2(N-r)$,*

$$\chi(\mathsf{Quot}_d(N,r), \mathbb{S}^{\lambda}(\mathcal{S}_x)) = 0.$$

Proof. Let $z_1, \ldots, z_N$ be roots of the equation $(z-1)^N + (-1)^r z^{N-r} t = 0$. Note that $\prod_{i=1}^N z_i = 1$. Using Proposition 4.1.5 and the defining equation (4.1),

$$\begin{aligned}\chi(\mathsf{Quot}_d(N,r), \mathbb{S}^{\lambda}(\mathcal{S}_x)) &= [t^d] \left(\prod_{i=1}^{N} z_i \right)^{N-r} s_{\Lambda}(z_1, \ldots, z_N) \\ &= [t^d] s_{\Lambda + (N-r, \ldots N-r)}(z_1, \ldots, z_N).\end{aligned}$$

where $\Lambda + (N-r, \ldots N-r)$ is the N-tuple obtained by adding $(N-r)$ to each coordinate.

Note that $\Lambda + (N-r, \ldots N-r)$ is not in decreasing order. We reorder the rows of the determinant in its bialternant formula (in (4.1)) such that the corresponding exponents are in decreasing order (we may assume that the exponents of each rows are distinct, otherwise the determinant vanishes as desired). Recall that $\Lambda = (d-\lambda_r, \ldots, d-\lambda_1, 0, \ldots, 0)$. Let $0 \leq \ell \leq r$ be the largest index such that the exponent in the $(r-\ell+1)^{\text{th}}$ row is less than the exponents appearing in the last row of the bialternant formula for the $s_{\Lambda+(N-r,\ldots N-r)}(z_1, \ldots, z_N)$. By reordering the rows by placing the last $N-r$ rows above the $(r-\ell+1)^{\text{th}}$, we obtain that

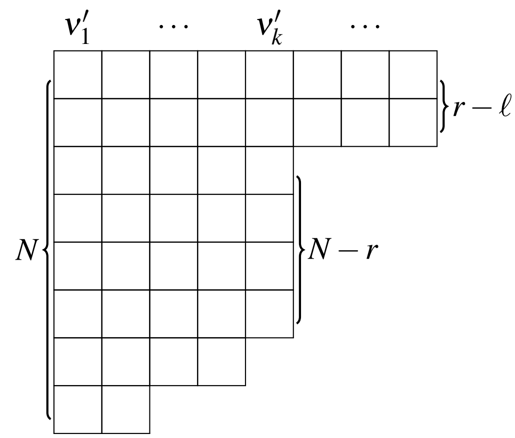
$$s_{\Lambda+(N-r,\ldots N-r)}(z_1, \ldots, z_N) = (-1)^{(N-r)\ell} s_\nu(z_1, z_2, \ldots, z_N),$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is the integer partition is given by

$$\nu_i = \begin{cases} (N-r)+d-\lambda_{r+1-i} & \text{when } 1 \leq i \leq r-\ell \\ N-r-\ell & \text{when } r-\ell < i \leq N-\ell \cdot \\ 2(N-r)+d-\lambda_{N+1-i} & \text{when } N-\ell < i \leq N \end{cases}$$

Here are a few important properties of the partition ν that we will use. Let $k = N-r-\ell$ and ν' denote the partition conjugate to ν. Then

$$\nu'_k \geq N-\ell$$
$$\nu'_{k+1} \leq r-\ell.$$



Moreover, the first part of ν is given by

$$\nu_1 = \begin{cases} (N-r+d)-\lambda_r & \ell < r \\ N-2r & \ell = r \end{cases}.$$

In either case, since $\lambda_r \geq 1$, the first part $\nu_1 \leq N-r+d-1$.

We will now use Jacobi-Trudi formula to express the above Schur polynomial in terms of the elementary symmetric polynomial. Note that the elementary symmetric polynomial are given by

$$e_m(z_1,\ldots,z_N) = \begin{cases} \binom{N}{m} & m \neq r \\ \binom{N}{r}+t & m = r \end{cases}.$$

The Schur polynomial $s_\nu(z_1,\ldots,z_N)$ equals the determinant of the following $\nu_1 \times \nu_1$. For notational convenience, let $M = \nu_1$.

$$\begin{vmatrix} e_{\nu'_1} & e_{\nu'_1+1} & \cdots & \cdots & e_{\nu'_1+k} & \cdots & e_{\nu'_1+M-1} \\ e_{\nu'_2-1} & e_{\nu'_2} & \cdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ e_{\nu'_k-k+1} & e_{\nu'_k-k+2} & \cdots & e_{\nu'_k} & \vdots & & \\ \vdots & & & & e_{\nu'_{k+1}} & \cdots & \vdots \\ \vdots & & & & \cdots & \ddots & \vdots \\ e_{\nu'_M-M+1} & \cdots & \cdots & \cdots & \cdots & \cdots & e_{\nu'_M} \end{vmatrix}.$$

We claim that e_r is not present as a entry in the first $N-r$ columns of the above matrix, thus the highest exponent of e_r in the expansion of the above determinant is at most $\nu_1 - (N-r) \leq d-1$. Thus the above determinant is a polynomial in t of degree at most $d-1$, hence proving $[t^d]s_\nu(z_1, \ldots, z_N) = 0$.

To see the claim, first note that first k columns does not contain e_r since

$$\nu'_k - k + 1 \geq r + 1$$

and $\nu'_{k+1} \leq (r-\ell) \leq r$. Furthermore, since $\nu'_{k+1} \leq r - \ell$, the next ℓ columns does not contain e_r as entry. Therefore, the first $k + \ell = N - r$ columns does not contain e_r as an entry.

□

Remark 4.1.10. *In the above Proposition, we may replace the assumption $\lambda_r \neq 0$ with the condition on the degree $d > r$. The last step of the proof has to be slightly modified.*

Theorem 4.1.11. *For any partitions λ and μ contained in the rectangular partition $(N-r, \ldots, N-r)$ where $N-r$ is repeated r times, and $d > 0$,*

$$\chi\big(\mathsf{Quot}_d(N,r), \det \mathcal{S}_x \otimes \mathbb{S}^\lambda(\mathcal{S}_x) \otimes \mathbb{S}^\mu(\mathcal{S}_x)\big) = 0.$$

Proof. Note that $\det S_x = \mathbb{S}^{(1^r)}(\mathcal{S}_x)$ where $(1^r) = (1,1,\dots,1)$. Let

$$\operatorname{ch}(\mathbb{S}^{(1^r)}(\mathcal{S}_x))\operatorname{ch}(\mathbb{S}^{\lambda}(\mathcal{S}_x))\operatorname{ch}(\mathbb{S}^{\mu}(\mathcal{S}_x)) = \sum_{\nu} C^{\nu}_{(1^r),\lambda,\mu}\operatorname{ch}(\mathbb{S}^{\nu}(\mathcal{S}_x))$$

By Lemma 4.1.7, $C^{\nu}_{(1^r),\lambda,\mu} = 0$ unless $\nu_1 \leq 1+\lambda_1+\mu_1 \leq 2(N-r)+1$. Moreover, the Littlewood Richardson rule also implies $C^{\nu}_{(1^r),\lambda,\mu} = 0$ unless that the partitions (1^r), λ and μ are contained in ν. We may thus apply Proposition 4.1.9 since $\nu_r > 0$, and $\nu_1 \leq 2(N-r)+d$ (as $d > 0$).

$\square$

4.2 Tautological classes

Let $M \to \mathbb{P}^1$ be a line bundle. We define the tautological class

$$M^{[d]} = R^0\pi_\star(p^\star M \otimes \mathcal{Q}) - R^1\pi_\star(p^\star M \otimes \mathcal{Q}),$$

where p and π continue to denote the projections over $C \times \mathsf{Quot}_d(E,r)$, and $\mathcal{Q}$ stands for the universal quotient. We calculate the Euler characteristic of the determinant of tautological classes using Proposition 4.1.1. The proof of Theorem 2.2.9 follows from the following lemma.

Lemma 4.2.1. *Let $M = \mathcal{O}_{\mathbb{P}^1}(m)$, then in the cohomology group of $\mathsf{Quot}_d(N,r)$,*

$$\det M^{[d]} = \det(\pi_\star \mathcal{S}^\vee)^{-1} \cdot \det(\mathcal{S}_x^\vee)^{m+2}.$$

Proof. We first note that in the K-theory,

$$M^{[d]} = \pi_!(p^*M \otimes \mathcal{O}^{\oplus N}) - \pi_!(p^*M \otimes \mathcal{S}).$$

The Chern classes of the first term $\pi_!(p^*M \otimes \mathcal{O}^{\oplus N})$ vanish since $\pi : \mathbb{P}^1 \times \mathsf{Quot}_d(N,r) \to \mathsf{Quot}_d(N,r)$ has connected fibers. To compute the first Chern class of the second term, we use Grothendieck-Riemann-Roch for π. Note that in the Künneth decomposition of $\mathbb{P}^1 \times \mathsf{Quot}_d(N,r)$,

$$\mathrm{ch}(\mathcal{S}) = 1 \otimes \mathrm{ch}(\mathcal{S}_x) + h \otimes [\mathrm{ch}(\pi_! \mathcal{S}) - \mathrm{ch}(\mathcal{S}_x)]$$

where h is the Poincaré dual of the points class in $\mathbb{P}^1$. Using Grothendieck-Riemann-Roch,

$$\begin{aligned}
\mathrm{ch}(\pi_!(p^*M \otimes \mathcal{S})) &= \pi_*(\mathrm{ch}(p^*M) \cdot \mathrm{ch}(\mathcal{S}) \cdot \mathrm{Td}(\mathbb{P}^1)) \\
&= \pi_*((1+mh) \cdot (\mathrm{ch}(\mathcal{S}_x) + h[\mathrm{ch}(\pi_! \mathcal{S}) - \mathrm{ch}(\mathcal{S}_x)]) \cdot (1+h)) \\
&= \mathrm{ch}(\pi_! \mathcal{S}) + m \cdot \mathrm{ch}(\mathcal{S}_x).
\end{aligned}$$

In the first two line, we have suppressed the $\otimes$ sign appearing in the Künneth decomposition. We finish the proof by noting $c_1(\mathcal{S}_x^\vee) = -c_1(\mathcal{S}_x)$ and $c_1(\pi_! \mathcal{S}) = c_1(\pi_! \mathcal{S}^\vee) - 2c_1(\mathcal{S}_x)$, and thus

$$c_1(M^{[d]}) = (m+2)c_1(\mathcal{S}_x^\vee) - c_1(\pi_! \mathcal{S}^\vee).$$

$\square$

Example 4.2.2. *An interesting specialization of Theorem 2.2.9 arises for $\ell = 0$. We show*

$$\chi\left(\mathsf{Quot}_d(N,r), \det \mathcal{O}^{[d]}\right) = \binom{N}{N-r+d}.$$

We have

$$\chi(\mathsf{Quot}_d, \det \mathcal{O}^{[d]}) = \left[t^d\right] s_\lambda(z_1, \ldots, z_N)$$

where $\lambda = ((d+1)^r)$. The elementary symmetric functions in $z_1, \ldots z_N$ are

$$e_j = \begin{cases} \binom{N}{j} & j \neq r+1 \\ \binom{N}{j} - t & j = r+1 \end{cases}.$$

Using Jacobi-Trudi, we have

$$s_\lambda(z_1,\dots,z_N) = \begin{vmatrix} e_r & e_{r+1} & e_{r+2} & \cdots & e_{r+d-1} & e_{r+d} \\ e_{r-1} & e_r & e_{r+1} & \cdots & e_{r+d-2} & e_{r+d-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e_{r-d+1} & e_{r-d+2} & e_{r-d+3} & \cdots & e_r & e_{r+1} \\ e_{r-d} & e_{r-d+1} & e_{r-d+2} & \cdots & e_{r-1} & e_r \end{vmatrix}.$$

In the $(d+1)\times(d+1)$ determinant, the only term yielding the power t^d is $(-1)^d e_{r+1}^d e_{r-d}$, coming from the lower left corner e_{r-d} and the terms e_{r+1} above the diagonal. To conclude, it remains to note that

$$\left[t^d\right] e_{r+1}^d e_{r-d} = (-1)^d \binom{N}{r-d}.$$

Example 4.2.3. *Assume $d > r(m+1)$. The Schur polynomial s_λ has weighted degree $|\lambda| = r(d+m+1) < (r+1)d$ in the elementary symmetric functions e_i, where we set $\deg e_i = i$. We noted in Example 4.2.2 that only e_{r+1} contains a linear t-term. By degree reasons, e_{r+1} appears in s_λ with exponent $< d$. Thus, in this case the t^d-coefficient vanishes, and*

$$\chi\left(\mathsf{Quot}_d(N,r), \det M^{[d]}\right) = 0.$$

Example 4.2.4. *Assume $d = r(m+1)$, so that $d+m+1 = (r+1)(m+1)$ and $|\lambda| = d(r+1)$ for $\lambda = ((d+m+1)^r)$. With these numerics, we claim that*

$$s_\lambda = (-1)^d e_{r+1}^d + \text{ lower order terms in } e_{r+1}. \tag{4.5}$$

Using that the only nonzero t-contribution in $e_j(z_1,\dots,z_N)$ is given by

$$[q]\, e_{r+1}(z_1,\dots,z_N) = -1,$$

we obtain $\left[t^d\right] s_\lambda(z_1,\ldots,z_N)=1,$ *and thus* $\chi\left(\mathsf{Quot}_d(N,r),\det M^{[d]}\right)=1.$

To justify (4.5)*, we let*

$$(x_1,\ldots,x_N)=(1,\zeta,\zeta^2,\ldots,\zeta^s,0,\ldots,0),$$

where ζ *is a primitive* $(r+1)$*-root of* 1*. In this case, we have*

$$e_{s+1}(x_1,\ldots,x_N)=(-1)^r,\quad e_j(x_1,\ldots,x_N)=0\,\textit{for}\ j\neq 0,\ j\neq r+1.$$

Thus, to confirm (4.5) *it remains to show that*

$$s_\lambda(x_1,\ldots,x_N)=1. \tag{4.6}$$

This follows from the (first) Jacobi-Trudi identity

$$s_\lambda=\begin{vmatrix} h_{(r+1)(m+1)} & h_{(r+1)(m+1)+1} & \cdots & h_{(r+1)(m+1)+(r-1)} \\ h_{(r+1)(m+1)-1} & h_{(r+1)(m+1)} & \cdots & h_{(r+1)(m+1)+(r-2)} \\ \vdots & \vdots & \cdots & \vdots \\ h_{(r+1)(m+1)-(r-2)} & h_{(r+1)(m+1)-(r-1)} & \cdots & h_{(r+1)(m+1)+1} \\ h_{(r+1)(m+1)-(r-1)} & h_{(r+1)(m+1)-(r-2)} & \cdots & h_{(r+1)(m+1)} \end{vmatrix},$$

where h_j *are the homogeneous symmetric functions. In our case, we have*

$$h_j(x_1,\ldots,x_N)=1\ \textit{if}\ j\equiv 0 \mod r+1,\quad h_j(x_1,\ldots,x_N)=0\ \textit{otherwise}.$$

Hence the above matrix evaluated at $(x_1,\ldots,x_N)$ *is the identity, yielding* (4.6)*.*

Chapter 5

Isotropic Quot scheme

The isotropic Grassmannian $\mathrm{SG}(r,\mathbb{C}^N)$ (or $\mathrm{OG}(r,\mathbb{C}^N)$) is the variety parameterizing r dimensional isotropic subspaces of a vector space $\mathbb{C}^N$ endowed with symplectic (or symmetric) non-degenerate bilinear form. The classical intersection theory of the Grassmannian $\mathrm{G}(r,\mathbb{C}^N)$ and isotropic Grassmannians has been an important subject connecting many areas of mathematics.

The Quot scheme $\mathsf{Quot}_d(E,r,C)$ (for short Quot_d) parameterizes degree $-d$, rank r sub-sheaves of a fixed vector bundle E over C. Let L be a line bundle over C and let σ be a symplectic or symmetric non-degenerate L-valued form on E:

$$\sigma : E \otimes E \to L.$$

A subsheaf $S \subset E$ is isotropic if the restriction $\sigma|_{S\otimes S} = 0$. The isotropic Quot scheme $\mathsf{IQ}_d(E,\sigma,r,C)$ (for short IQ_d) is the closed subscheme of Quot_d consisting of isotropic subsheaves.

5.1 Perfect Obstruction Theory

5.1.1 Genus 0

Over $\mathbb{P}^1$, the Quot scheme $\mathsf{Quot}_d(\mathbb{C}^N,r,\mathbb{P}^1)$ is smooth for any choice of N,r and d. The isotropic Quot scheme IQ_d is smooth for $d=0,1$ for all r and N, but it is singular for higher values of d.

The isotropic Quot schemes IQ_d can be described as the zero locus of a section of a vector bundle over Quot_d. Therefore, the virtual fundamental class exists and is given by the Euler class of the vector bundle.

Proposition 5.1.1. *Let $\pi : \mathsf{Quot}_d \times \mathbb{P}^1 \to \mathsf{Quot}_d$ be the projection. Then $\pi_*(\wedge^2 \mathcal{S}^\vee)$ is a locally free sheaf.*

Proof. Note that for any point $q = [0 \to S \to \mathcal{O}^N \to Q \to 0]$ in the Quot scheme, $\mathbb{C}^N \otimes \mathcal{O} \to S^\vee$ is generically surjective and so is

$$\phi : \wedge^2(\mathbb{C}^N \otimes \mathcal{O}) \to \wedge^2 S^\vee.$$

Observe that $\wedge^2(\mathbb{C}^N \otimes \mathcal{O}) = \mathbb{C}^{\binom{N}{2}} \otimes \mathcal{O}$. We have the following exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \to & \ker\phi & \to & \mathbb{C}^{\binom{N}{2}} \otimes \mathcal{O} & \to & \operatorname{im}\phi & \to & 0 \\ 0 & \to & \operatorname{im}\phi & \to & \wedge^2 S^\vee & \to & \operatorname{coker}\phi & \to & 0 \end{array}.$$

Since $\operatorname{coker}(\phi)$ is zero dimensional and $\mathbb{C}^{\binom{N}{2}} \otimes \mathcal{O}$ is a trivial vector bundle over $\mathbb{P}^1$, their first sheaf cohomology groups vanish. The first exact sequence implies $H^1(\mathbb{P}^1, \operatorname{im}\phi) = 0$. The second exact sequence gives us $H^1(\mathbb{P}^1, \wedge^2(S^\vee)) = 0$, hence $h^0(\wedge^2 S^\vee) = \chi(\wedge^2 S^\vee)$ is constant. Using Grauert's theorem we conclude that $\pi_*(\wedge^2(\mathcal{S}^\vee))$ is locally free. $\square$

The symplectic form $\sigma : \wedge^2(\mathbb{C}^N \otimes \mathcal{O}) \to \mathcal{O}$ induces an element of $H^0(\mathbb{P}^1, \wedge^2 S^\vee)$ given as the composition

$$\wedge^2 S \to \wedge^2 \mathbb{C}^N \otimes \mathcal{O} \xrightarrow{\sigma} \mathcal{O}$$

for any subsheaf S of $\mathbb{C}^N \otimes \mathcal{O}$. This induces a section, denoted as $\tilde{\sigma}$, of $\pi_*(\wedge^2 \mathcal{S}^\vee)$ over Quot_d.

Recall that IQ_d is the subscheme of Quot_d consisting of subsheaves S of $\mathbb{C}^N \otimes \mathcal{O}$ such that the above composition is zero, hence $\mathsf{IQ}_d = \operatorname{Zero}(\tilde{\sigma})$. Therefore, we have a natural perfect obstruction theory and a virtual fundamental class proving Theorem 2.3.1 in this case.

5.1.2 The Perfect Obstruction theory in general

In the general case, the two main aspects of the above proof break down, namely Quot_d is not always smooth and the sheaf $\pi_*(\wedge^2 \mathcal{S}^\vee)$ may not be locally free. To construct a perfect obstruction theory, we will have to make a few auxiliary constructions.

Fix E, L, r and d. Let **Bun** be the moduli stack of rank r and degree d vector bundles over C. There is a natural forgetful map $\mu : \mathsf{Quot}_d \to \mathbf{Bun}$ sending the exact sequence $0 \to S \to E \to Q \to 0$ to $[S^\vee] \in \mathbf{Bun}$.

We define another stack **WS** which parameterizes pairs (S, ϕ), where S is a vector bundle with $S^\vee \in \mathbf{Bun}$ and $\phi : \wedge^2 S \to L$ is a morphism of sheaves. This also comes equipped with a natural map $\eta : \mathbf{WS} \to \mathbf{Bun}$ sending the pair (S, ϕ) to $[S^\vee]$.

We have tabulated the situation in the following commutative diagram

$$\begin{tikzcd}
\mathsf{IQ}_d \arrow[r, hook, "i"] \arrow[d, "\mu\circ i"] & \mathsf{Quot}_d \arrow[dl, "\mu"] \arrow[d, "\tilde{\sigma}"] \\
\mathbf{Bun} \arrow[r, hook, "z"] & \mathbf{WS} \arrow[l, bend left, "\eta"]
\end{tikzcd}.$$

Here $\tilde{\sigma}$ is the map sending the short exact sequence $0 \to S \to E \to Q \to 0 \in \mathsf{Quot}_d$ to the pair (S, ϕ), where ϕ is the composition $\wedge^2 S \to \wedge^2 E \xrightarrow{\sigma} L$.

Recall IQ_d is precisely the closed locus in Quot_d which is sent to $(S, 0)$ under the map $\tilde{\sigma}$. There is a zero section $z : \mathbf{Bun} \to \mathbf{WS}$ sending $[S^\vee]$ to $(S, 0)$, and we see that IQ_d is the fiber product of the maps $\tilde{\sigma}$ and z.

The advantage of the above description is that we understand the cotangent complex of Quot_d and **Bun**, and the new stack **WS** is an abelian cone over **Bun**. We will first describe relative perfect obstruction theory for the maps μ and η, and use it to obtain a relative perfect obstruction theory for IQ_d relative to **Bun**. Since **Bun** is a smooth Artin stack, this standardly yields a global perfect obstruction theory for IQ_d, by [GP, Appendix B].

5.1.3 A perfect obstruction theory for WS

We will first carefully define the stack **WS** and show that it is an abelian cone over **Bun**. We will use the results in [Sca] and [Sca 3] to obtain perfect obstruction theory of **WS** over **Bun**.

Definition 5.1.2. *A Wedge system is a pair (S,ϕ) where S is a locally free sheaf on C and ϕ is a morphism of sheaves $\phi : \wedge^2 S \to L$ over C. A family of Wedge systems over a scheme T is $(\pi : C \times T \to T, \mathcal{S}, \phi : \wedge^2 \mathcal{S} \to p^*L)$ where $p : C \times T \to C$ is the first projection and $\mathcal{S}$ is a locally free sheaf over $C \times T$.*

An isomorphism of two families of Wedge system $(\pi : C \times T \to T, \mathcal{S}, \phi : \wedge^2 \mathcal{S} \to p^*L)$ and $(\pi : C \times T \to T, \mathcal{S}', \phi' : \wedge^2 \mathcal{S}' \to p^*L)$ over T is an isomorphism $\alpha : \mathcal{S} \to \mathcal{S}'$ over $C \times T$ such that $\phi = \phi' \circ \wedge^2 \alpha$.

Definition 5.1.3. *Let* **WS** *be the category fibered in groupoids defined by* $\mathbf{WS}(T)$ *being the families of Wedge systems over T. Let* $\eta : \mathbf{WS} \to \mathbf{Bun}$ *be the forgetful morphism.*

Proposition 5.1.4. *There is a natural isomorphism of* **Bun***-stacks*

$$\mathbf{WS} \to \operatorname{Spec}\operatorname{Sym}(\mathbf{R}^1\pi_*(\wedge^2 \mathcal{S} \otimes p^*L^\vee \otimes \omega_\pi)) \tag{5.1}$$

where ω_π is the relative dualising sheaf of $\pi : \mathbf{WS} \times C \to \mathbf{WS}$. In particular **WS** *is an abelian cone over* **Bun***. Thus* **WS** *is an algebraic stack.*

Proof. The proof is almost same as the proof of Prop 1.8 in [Sca]. Let T be a scheme, then $\mathbf{WS}(T) = \{t : T \to \mathbf{Bun}, \phi : \bar{t}^* \wedge^2 \mathcal{S} \to p^*L\}$, where $\bar{t}$ is the induced map from $C \times T \to C \times \mathbf{Bun}$. Using Grothendieck duality and base change there is a canonical bijection between $\operatorname{Hom}(\bar{t}^* \wedge^2 \mathcal{S}, p^*L)$ and $\operatorname{Hom}(t^*\mathbf{R}^1\pi_*(\wedge^2 \mathcal{S} \otimes p^*L^\vee \otimes \omega_\pi), \mathcal{O}_T)$ which is compatible with pull backs. $\square$

Corollary 5.1.5. *There is a relative perfect obstruction theory for η induced by*

$$\mathbf{R}\pi_*(\mathcal{H}om(\wedge^2 \mathcal{S}, p^*L))^\vee \to \tau_{[-1,0]}\mathbb{L}_\eta.$$

Proof. The corollary follows using Lemma 5.1.6 by observing that

$$RHom(\mathbf{R}\pi_*(\wedge^2 \mathcal{S} \otimes p^* L^\vee \otimes \omega_\pi[1]), \mathcal{O}_{\mathbf{WS}})$$

is isomorphic to $\mathbf{R}\pi_*(Hom(\wedge^2 \mathcal{S}, p^* L))$ in the derived category. $\square$

Lemma 5.1.6. *Let $\pi : Y' \to Y$ be a relative dimension one, flat, projective morphism of algebraic stacks and let $F \in Coh(Y')$ be flat over Y, then the abelian cone $\mathbf{WS} := \operatorname{Spec}\operatorname{Sym}(\mathbf{R}^1\pi_* F) \xrightarrow{\eta} Y$ has a relative perfect obstruction theory induced by the canonical morphism*

$$\mathbf{R}\bar{\pi}_*(\bar{F}[1]) \to \tau_{[-1,0]}\mathbb{L}_\eta \tag{5.2}$$

where $\bar{\pi} : Y' \times_Y \mathbf{WS} \to \mathbf{WS}$ and $\bar{F}$ is the induced sheaf on $Y' \times_Y \mathbf{WS}$.

Proof. We will briefly explain the argument assuming Y is a scheme. The complete proof is exactly the same as the proof of Proposition 2.4 in [Sca].

Under the given conditions, F can be shown to admit a resolution

$$0 \to K \to M \to F \to 0$$

where M is locally free, $\pi_* K = \pi_* M = 0$ and the first derived pushforwards $\mathbf{R}^1\pi_* M$ and $\mathbf{R}^1\pi_* K$ are locally free. Then η admits a factorization

$$\mathbf{WS} \xrightarrow{i} \operatorname{Spec}\operatorname{Sym}(\mathbf{R}^1\pi_* M) \xrightarrow{q} Y$$

where $\eta = q \circ i$, q is a smooth morphism and i is a closed embedding. Then $\tau_{[-1,0]}\mathbb{L}_\eta \cong [I|_{\mathbf{WS}} \to \Omega_q|_{\mathbf{WS}}]$, where I is the ideal sheaf of i. There is a natural isomorphism $\eta^*\mathbf{R}^1\pi_* M \to \Omega_q|_{\mathbf{WS}}$ and surjection $\eta^*\mathbf{R}^1\pi_* K \to I|_{\mathbf{WS}}$.

Therefore, it remains to show that $[\eta^*\mathbf{R}^1\pi_* K \to \eta^*\mathbf{R}^1\pi_* M]$ is quasi-isomorphic to

$\mathbf{R}\bar{\pi}_*(\bar{F}[1])$. By cohomology and base-change, $[\eta^*\mathbf{R}^1\pi_*K \to \eta^*\mathbf{R}^1\pi_*M]$ is isomorphic to $[\mathbf{R}^1\bar{\pi}_*\bar{\eta}^*\bar{K} \to \mathbf{R}^1\bar{\pi}_*\bar{\eta}^*\bar{M}]$, where

$$0 \to \bar{K} \to \bar{M} \to \bar{F} \to 0$$

is the induced resolution on $Y' \times_Y \mathbf{WS}$. The required statement is obtained by the distinguished triangle of the above short exact sequence. □

5.1.4 Perfect Obstruction theory

Recall that we have a map $\tilde{\sigma} : \mathsf{Quot}_d \to \mathbf{WS}$ which takes a subsheaf $[0 \to S \to E \to Q \to 0]$ to the point (S, ϕ) in $\mathbf{WS}$ where ϕ is the composition of $\wedge^2 S \to \wedge^2 E \to L$. This can be defined as a morphism of **Bun**-stacks.

Consider the morphisms

$$\mathsf{Quot} \xrightarrow{\tilde{\sigma}} \mathbf{WS} \xrightarrow{\eta} \mathbf{Bun}\,.$$

Let $\mu = \eta \circ \tilde{\sigma}$. There exists a distinguished triangle

$$\tilde{\sigma}^*\mathbb{L}_\eta \to \mathbb{L}_\mu \to \mathbb{L}_{\tilde{\sigma}} \to \tilde{\sigma}^*\mathbb{L}_\eta[1]. \tag{5.3}$$

Note that the Quot schemes over smooth curves have perfect obstruction theories as described in [MO 3]. In order to obtain the relative perfect obstruction theory over **Bun**, we consider Quot_d as an open substack of the abelian cone

$$\operatorname{Spec}\operatorname{Sym}(\mathbf{R}^1\pi_*(\mathcal{S} \otimes p^*E^\vee \otimes \omega_\pi)).$$

Therefore Lemma 5.1.6 and relative duality implies that the morphism

$$\mathbf{R}\pi_*(Hom(\mathcal{S}, p^*E))^\vee \to \tau_{[-1,0]}\mathbb{L}_\mu$$

induces a perfect obstruction theory for $\mu : \mathsf{Quot}_d \to \mathbf{Bun}$. We also recall Corollary 5.1.5. Thus we get a map of distinguished triangles completing (5.3) by the axioms of derived category:

$$\begin{array}{ccccc} \mathbf{R}\pi_*(\mathcal{H}om(\wedge^2\mathcal{S},p^*L))^\vee & \xrightarrow{d\tilde{\sigma}} & \mathbf{R}\pi_*(\mathcal{H}om(\mathcal{S},p^*E))^\vee & \longrightarrow & \mathbf{R}\pi_*(D^\bullet)^\vee \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\sigma}^*\mathbb{L}_\eta & \longrightarrow & \mathbb{L}_\mu & \longrightarrow & \mathbb{L}_{\tilde{\sigma}} \end{array} \tag{5.4}$$

where $D^\bullet = [\mathcal{H}om(\mathcal{S},p^*E) \xrightarrow{d\sigma} \mathcal{H}om(\wedge^2\mathcal{S},p^*L)]$. The description of $d\sigma$, given below, is important for proving Lemma 5.1.8.

Fix a vector bundle S in **Bun**, then the map $\tilde{\sigma}$ restricts to a quadratic map $\mathrm{Hom}(S,E) \to \mathrm{Hom}(\wedge^2 S,L)$ sending f to $\sigma\circ\wedge^2 f$. Vanishing of this map is precisely the locus of the fiber of IQ_d over S. Hence the tangent space at a point $f=[0\to S\xrightarrow{f}E\to Q\to 0]$ in IQ_d relative to **Bun** is given as kernel of the linear map $d\tilde{\sigma}:\mathrm{Hom}(S,E)\to\mathrm{Hom}(\wedge^2 S,L)$ sending g to the map $[u\wedge v\to\sigma(f(u)\wedge g(v)+g(u)\wedge f(v))]$. The corresponding map of sheaves $d\sigma:\mathcal{H}om(S,E)\to\mathcal{H}om(\wedge^2 S,L)$ over the fiber $C\times\{f\}$ is given by the same expression over each open sets of C.

Over $C\times\mathsf{IQ}_d$ we have the universal section f of the vector bundle $\mathcal{H}om(\mathcal{S},p^*E)$. The above description induces a morphism of locally free sheaves

$$d\sigma:\mathcal{H}om(\mathcal{S},p^*E)\to\mathcal{H}om(\wedge^2\mathcal{S},p^*L).$$

We have seen in Proposition 5.1.4 that **WS** is an abelian cone, therefore it comes equipped with the zero section $z:\mathbf{Bun}\to\mathbf{WS}$ which is a closed immersion. Recall that IQ_d sits inside the commutative diagram

$$\begin{tikzcd} \mathsf{IQ}_d \arrow[r, hook, "i"] \arrow[d, "\mu\circ i"] & \mathsf{Quot}_d \arrow[d, "\tilde{\sigma}"] \arrow[dl, "\mu"'] \\ \mathbf{Bun} \arrow[r, hook, "z"] & \mathbf{WS} \arrow[l, bend left, "\eta"] \end{tikzcd}.$$

Observe that IQ_d is the inverse image $\tilde{\sigma}^{-1}(z(\mathbf{Bun}))$. The perfect obstruction theory $\mathbf{R}\pi_*(D^\bullet)^\vee$ of σ induces a perfect obstruction theory of IQ_d relative to **Bun** using the map of cotangent complex

$$i^*\mathbb{L}_{\tilde{\sigma}} \to \mathbb{L}_{\mathsf{IQ}_d\,/\,\mathbf{Bun}}. \tag{5.5}$$

Lemma 5.1.7. *There is a perfect obstruction theory of* IQ_d *relative to* **Bun** *induced by*

$$\mathbf{R}\pi_*(D^\bullet)^\vee \to \tau_{[-1,0]}\mathbb{L}_{\mathsf{IQ}_d\,/\,\mathbf{Bun}}. \tag{5.6}$$

where $D^\bullet = [Hom(\mathcal{S}, p^*E) \xrightarrow{d\sigma} Hom(\wedge^2\mathcal{S}, p^*L)]$ *is the two term complex over vector bundles with amplitude in [0,1] over* $C \times \mathsf{IQ}_d$*.*

Proof. We obtain the perfect obstruction theory in (5.6) by restricting the perfect obstruction theory of $\tilde{\sigma}$ in (5.4) to IQ_d using (5.5).

Let $D^\bullet|_C = [Hom(S,E) \xrightarrow{d\sigma} Hom(\wedge^2 S, L)]$ be the restriction to a fibers, denoted as C, of $\pi : C \times \mathsf{IQ}_d \to \mathsf{IQ}_d$. Consider the hypercohomology long exact sequence

$$\cdots \to \mathrm{H}^1(Hom(S,E)) \to \mathrm{H}^1(Hom(\wedge^2 S, L)) \to \mathbb{H}^2(D^\bullet|_C) \to \mathrm{H}^2(Hom(S,E)) = 0.$$

Since $d\sigma$ is generically surjective (see Lemma 5.1.8) and C is one dimensional, $\mathrm{H}^1(Hom(S,E)) \to \mathrm{H}^1(Hom(\wedge^2 S, L))$ is surjective. Thus we conclude that $\mathbb{H}^2(D^\bullet|_C)$ vanishes. □

Lemma 5.1.8. *The restriction of* $d\sigma$ *to each fiber* $C = C \times \{f\}$*, where* $[0 \to S \xrightarrow{f} E \to Q \to 0]$ *is an element in* IQ_d*, is generically surjective.*

Proof. Note that f is morphism of vector bundle over $C \backslash A$ where A is finite set of points in C. We will show that the linear map of vector spaces

$$\begin{aligned} \phi : Hom(S_x \to E_x) &\to Hom(\wedge^2 S_x, L_x) \\ g &\to [u \wedge v \to \sigma(f(u) \wedge g(v) + g(u) \wedge f(v))] \end{aligned}$$

is surjective for all $x \in C \backslash A$. This is now an exercise in linear algebra.

Let $N = 2n$. We can choose symplectic coordinates $\{e_1, \ldots e_N\}$ of E_x such that $\sigma(e_i, e_{n+i}) = 1$ and f identifies the isotropic subspace S_x with $span\{e_1, \ldots, e_r\}$. An element $g \in Hom(S_x \to E_x)$ can be identified with an $N \times r$ matrix $(B_{i,j})$. A simple calculation shows that $g \in \ker \phi$ if and only if $B_{i,n+k} = B_{k,n+i}$ for all $1 \leq i, k \leq r$. Thus the rank of $\ker \phi$ is $Nr - \binom{r}{2}$, hence ϕ is surjective. □

Proof of Theorem 2.3.1. In Lemma 5.1.7, we constructed a relative perfect obstruction theory. We follow the arguments in [GP, Appendix B] verbatim to obtain an absolute perfect obstruction theory. Here we use the fact that **Bun** is a smooth Artin stack with obstruction theory given by $\mathbf{R}\pi_*(Hom(\mathcal{S}, \mathcal{S}))^\vee[-1] \to \mathbb{L}_{\mathbf{Bun}}$. □

Remark 5.1.9. *We note that when E and L are trivial and σ is induced from a standard symplectic or symmetric form on $\mathbb{C}^N$, there is another way to construct the virtual fundamental class for IQ_d using the theory of quasi-maps to GIT quotients as discussed in [CFKM].*

Indeed, IQ_d can be considered as the moduli space of quasi maps from C to $\mathrm{SG}(r,N)$ (or $\mathrm{OG}(r,N)$). The isotropic Grassmannian can be realized as a GIT quotient of $W /\!\!/_\theta G$, where $\theta = \det^{-1}$ is the multiplicative character of $G = GL_r$ and $W = \{f \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N) : \sigma(f(u), f(v)) = 0 \,\forall u, v \in \mathbb{C}^r\}$ is a closed subscheme of the affine space $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)$.

5.2 Symplectic isotropic Quot schemes

Throughout this section we will assume that σ is the standard symplectic form on $\mathbb{C}^N \otimes \mathcal{O}$; i.e., it is induced by the block matrix

$$\sigma = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where $N = 2n$.

There is a natural action of $Sp(2n)$ on IQ_d induced by the respective action on $\mathbb{C}^{2n}$. We consider the subtorus $G = \mathbb{C}^* \subseteq Sp(2n)$ given by $(t^{-w_1}, \ldots, t^{-w_N})$ where $w_i = -w_{i+n}$ for

$1 \leq i \leq n$. The weights w_i are assumed to be distinct, unless stated otherwise.

5.2.1 Fixed Loci

Each summand $\mathcal{O}$ of $\mathbb{C}^N \otimes \mathcal{O}$ is acted upon with different weights. A point $[0 \to S \to \mathbb{C}^N \otimes \mathcal{O} \to Q \to 0]$ in IQ_d is fixed under the action of G if and only if :

(i) S splits as a direct sum of line bundles

$$S = \oplus_{j=1}^{r} L_j,$$

where L_j is subsheaf of one of the N copies of $\mathcal{O}$ of $\mathbb{C}^N \otimes \mathcal{O}$. Denote k_j by the position of this copy of $\mathcal{O}$.

(ii) $k_j - k_i \not\equiv 0 \mod n$ for any $1 \leq i < j \leq r$: This ensures that S is isotropic.

Let $\underline{k} = \{k_1, \dots, k_r\}$ and $\vec{d} = (d_1, \dots, d_r)$ where $d_j = \deg L_j$ and

$$d_1 + \cdots + d_r = d.$$

We require $\{i, i+n\} \not\subset \underline{k}$ for any $1 \leq i \leq n$. Let $\mathrm{F}_{\vec{d},\underline{k}}$ be the set of fixed points with the numerical data $\vec{d}$ and $\underline{k}$. Note that there are $2^r \binom{n}{r}$ possible values of $\underline{k}$ and $\binom{d+r-1}{r-1}$ choices of $\vec{d}$.

Denote $\mathcal{O}_{k_i}$ be the k_i'th copy of $\mathcal{O}$ in $\mathbb{C}^N \otimes \mathcal{O}$. The short exact sequence

$$0 \to L_i \to \mathcal{O}_{k_i} \to T_i \to 0$$

defines an element of $C^{[d_i]}$, the Hilbert scheme of d_i points on C. Therefore we have

$$\mathrm{F}_{\vec{d},\underline{k}} = C^{[d_1]} \times C^{[d_2]} \times \cdots \times C^{[d_r]}.$$

5.2.2 The Equivariant Normal bundle

Let $0 \to \mathcal{K}_i \to \mathcal{O}_{k_i} \to \mathcal{T}_i \to 0$ be the universal exact sequence over $C \times C^{[d_i]}$. We use the same notation for the pull-back exact sequence over $C \times \mathrm{F}_{\vec{d},\underline{k}}$.

Let $0 \to \mathcal{S} \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0$ be the universal exact sequence over $C \times \mathsf{IQ}_d$. This restricts to

$$0 \to \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_r \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_r \oplus \mathbb{C}^{N-r} \otimes \mathcal{O} \to 0$$

on $C \times \mathrm{F}_{\vec{d},\underline{k}}$.

Let $\pi_!$ be the derived pushforward $\mathbf{R}^0\pi_* - \mathbf{R}^1\pi_*$ in the K-theory. Recall that in Theorem 2.3.1, we provided a perfect obstruction theory for the isotropic Quot scheme. In the K-theory of IQ_d, the corresponding virtual tangent bundle is given by

$$T^{\mathrm{vir}} = \pi_![(RHom(\mathcal{S},\mathcal{Q}))] - \pi_![(Hom(\wedge^2\mathcal{S},\mathcal{O}))].$$

The restriction of the virtual tangent bundle in the $\mathbb{C}^*$-equivariant K-theory of $\mathrm{F}_{\vec{d},\underline{k}}$ is given by the following formula

$$\pi_!\Bigg(\sum_{i,j\in[r]}[\mathcal{K}_i^\vee \otimes \mathcal{T}_j] + \sum_{i\in[r],k\in\underline{k}^c}[\mathcal{K}_i^\vee] - \sum_{1\le i<j\le r}[\mathcal{K}_i^\vee \otimes \mathcal{K}_j^\vee]\Bigg),$$

where the above three groups of elements have $\mathbb{C}^*$ weights $(w_{k_i} - w_{k_j})$, $(w_{k_i} - w_k)$ and $(w_{k_i} + w_{k_j})$ respectively.

Note that the fixed part of the restriction of T^{vir} to $\mathrm{F}_{\vec{d},\underline{k}}$ is

$$\sum_{i\in[r]} \pi_![\mathcal{K}_i^\vee \otimes \mathcal{T}_i],$$

which matches the tangent bundle of $\mathrm{F}_{\vec{d},\underline{k}}$. The induced virtual class $[\mathrm{F}_{\vec{d},\underline{k}}]^{\mathrm{vir}} = [\mathrm{F}_{\vec{d},\underline{k}}]$ agrees with the usual fundamental class.

The virtual equivariant normal bundle $\mathcal{N}^{\text{vir}}$ is given by the moving part of the restriction of T^{vir}. Using the identity in K-theory,

$$[\mathcal{K}_i^\vee \otimes \mathcal{T}_j] = [\mathcal{K}_i^\vee \otimes \mathcal{O}_{k_j}] - [\mathcal{K}_i^\vee \otimes \mathcal{K}_j],$$

we obtain the following equality

$$\mathcal{N}^{\text{vir}} = \pi_! \Bigg(\sum_{\substack{i\in[r],k\in[N] \\ k\neq k_i}} [\mathcal{K}_i^\vee] - \sum_{\substack{i,j\in[r] \\ i\neq j}} [\mathcal{K}_i^\vee \otimes \mathcal{K}_j] - \sum_{1\leq i<j\leq r} [\mathcal{K}_i^\vee \otimes \mathcal{K}_j^\vee] \Bigg), \tag{5.7}$$

where the terms are acted on with wights $(w_{k_i} - w_k)$, $(w_{k_i} - w_{k_j})$ and $(w_{k_i} + w_{k_j})$ respectively.

5.2.3 Chern polynomials

In the subsection we briefly describe certain Grothendieck-Riemann-Roch calculations for the map $\pi : C \times X \to X$, where

$$X = C^{[d_1]} \times C^{[d_2]} \times \cdots \times C^{[d_r]}.$$

Let $\{1, \delta_1, \ldots, \delta_{2g}, \omega\}$ be the symplectic basis for the cohomology ring of C with the relations $\delta_i \delta_{i+g} = \omega = -\delta_{i+g}\delta_i$ for all $1 \leq i \leq g$. Consider the Künneth decomposition of the cohomology classes $c_1(\mathcal{K}^\vee)$ in $C \times C^{[d_i]}$ with respect to a chosen symplectic basis of $\mathrm{H}^*(C)$,

$$c_1(\mathcal{K}_i^\vee) = x_i \otimes 1 + \sum_{k=1}^{2g} y_i^k \otimes \delta_k + d_i \otimes \omega. \tag{5.8}$$

The theta class, $\theta_i \in \mathrm{H}^*(C^{[d_i]})$, is the pullback of the usual theta class under the map

$$C^{[d_i]} \to \mathrm{Pic}^{d_i}.$$

We have the following relation (explained in [ACGH])

$$\left(\sum_{k=1}^{2g}(y_i^k\otimes\delta_k)\right)^2=-2\theta_i\otimes\omega.$$

We will use the same notation for the pullback of x_i, y_i^k and θ_i under the map

$$pr_i : X\to C^{d_i}.$$

Let E be a vector bundle of rank m and let $c_t(E)=1+c_1(E)t+\cdots+c_m(E)t^m$ be its Chern polynomial. We extend the definition of c_t to the K-theory in the usual way. We can use Grothendieck-Riemann-Roch to obtain expression for the Chern polynomials $c_t(\pi_![\mathcal{K}_i^\vee])$, $c_t(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_j])$ and $c_t(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_j^\vee])$:

$$\begin{aligned} c_t(\pi_![\mathcal{K}_i^\vee]) &= (1+tx_i)^{d_i-\bar{g}}e^{-\frac{t\theta_i}{(1+tx_i)}} \\ c_t(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_j]) &= (1+t(x_i-x_j))^{d_i-d_j-\bar{g}}e^{-\frac{t(\theta_i+\theta_j+\phi_{ij})}{1+t(x_i-x_j)}} \\ c_t(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_j^\vee]) &= (1+t(x_i+x_j))^{d_i+d_j-\bar{g}}e^{-\frac{t(\theta_i+\theta_j-\phi_{ij})}{1+t(x_i+x_j)}} \\ c_t(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_i^\vee]) &= (1+2tx_i)^{2d_i-\bar{g}}e^{-\frac{4t\theta_i}{1+2tx_i}} \end{aligned} \tag{5.9}$$

where $\phi_{ij}=-\sum_{k=1}^{g}(y_i^ky_j^{k+g}+y_j^ky_i^{k+g})$. The detailed calculation for the first two expression can be found in [ACGH] and [MO 3]. The other two expressions are obtained in a similar way. We will briefly explain the last one for completeness: The first Chern class is $c_1(\mathcal{K}^\vee\otimes\mathcal{K}^\vee)=2c_1(\mathcal{K}^\vee)$, therefore the Chern character

$$ch(\mathcal{K}^\vee\otimes\mathcal{K}^\vee)=e^{2x}\otimes 1+e^{2x}(2d-4\theta)\otimes\omega+2\sum_k y^k\otimes\delta_k.$$

We may further apply Grothendieck Riemann Roch to obtain the Chern characters of $\pi_![\mathcal{K}^\vee\otimes\mathcal{K}^\vee]$

and then covert it into Chern polynomials to obtain the required result. The Chern character is

$$\begin{aligned} ch(\pi_![\mathcal{K}^\vee \otimes \mathcal{K}^\vee]) &= \pi_*(ch(\mathcal{K}^\vee \otimes \mathcal{K}^\vee)(1+(1-g)\omega)) \\ &= e^{2x}(2d+(1-g)-4\theta). \end{aligned}$$

5.2.4 The Euler class of virtual normal bundle

Next we would like to find the equivariant Euler class of $\mathcal{N}^{\mathrm{vir}}$ in the equivariant cohomology ring $H^*(\mathrm{F}_{\vec{d},\underline{k}})[[t,t^{-1}]]$. This will be useful in the virtual localization formula.

Let E be one of the line bundles appearing in the formula for $\mathcal{N}^{\mathrm{vir}}$ in (5.7). We evaluated the formula for the total Chern classes $c_q(\pi_!E)$ in (5.9). Let $\pi_!E$ be acted on with weight w, then the equivariant Euler class is a homogeneous element in $H^*(\mathrm{F}_{\vec{d},\underline{k}})[t,t^{-1}]$ and is given by

$$e_{\mathbb{C}^*}(\pi_!E) = (wt)^m c_{\frac{1}{wt}}(\pi_!E)$$

where $m = \chi(\pi_!E)$ is the virtual rank.

Consider the polynomial $P(X) = \prod_{i=1}^N (X - w_it)$. Let $Y_i = x_i + w_{k_i}t$ be a change of variable over $\mathbb{C}[[t]]$. Then

$$\begin{aligned} \prod_{\substack{i\in[r],k\in[N]\\ k\neq k_i}} \frac{1}{e_{\mathbb{C}^*}(\pi_![\mathcal{K}_i^\vee])} &= \prod_{\substack{i\in[r],k\in[N]\\ k\neq k_i}} (Y_i - w_kt)^{-d_i+\bar{g}} e^{\frac{\theta_i}{(Y_i-w_kt)}} \\ &= \prod_{i\in[r]} \left(\frac{P(Y_i)}{x_i}\right)^{-d_i+\bar{g}} e^{\theta_i\left(\frac{P'(Y_i)}{P(Y_i)} - \frac{1}{x_i}\right)} \end{aligned} \tag{5.10}$$

Here we are using the elementary identity

$$\frac{P'(X)}{P(X)} = \sum_{k=1}^N \frac{1}{X - w_kt}.$$

For the remaining classes, we obtain

$$\prod_{\substack{i,j\in[r]\\ i\neq j}} e_{\mathbb{C}^*}(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_j]) = \prod_{\substack{i,j\in[r]\\ i\neq j}} (Y_i - Y_j)^{d_i - d_j - \bar{g}} e^{-\frac{(\theta_i+\theta_j+\phi_{ij})}{Y_i - Y_j}} \tag{5.11}$$

$$= (-1)^{\bar{g}\binom{r}{2}+d(r-1)} \prod_{i<j} (Y_i - Y_j)^{-2\bar{g}}$$

$$\prod_{\substack{i,j\in[r]\\ i<j}} e_{\mathbb{C}^*}(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_j^\vee]) = \prod_{i<j} (Y_i + Y_j)^{d_i + d_j - \bar{g}} e^{-\frac{(\theta_i+\theta_j-\phi_{ij})}{Y_i + Y_j}} \tag{5.12}$$

Using the multiplicative property for the Euler classes, we have the the following expression for the equivariant Euler class of the virtual normal bundle :

$$\frac{1}{e_{\mathbb{C}^*}(\mathcal{N}^{\text{vir}})} = u \prod_i h_i^{d_i - \bar{g}} e^{\theta_i z_i} \cdot \prod_{i<j} \frac{(Y_i + Y_j)^{d_i + d_j - \bar{g}}}{(Y_i - Y_j)^{2\bar{g}}} e^{-\frac{\theta_i+\theta_j-\phi_{ij}}{Y_i + Y_j}} \tag{5.13}$$

where $u = (-1)^{\bar{g}\binom{r}{2}+d(r-1)}$, $h_i = \frac{x_i}{P(Y_i)}$ and

$$z_i = \left(\frac{P'(Y_i)}{P(Y_i)} - \frac{1}{x_i}\right). \tag{5.14}$$

5.3 Symmetric isotropic Quot scheme

Throughout this section we will assume $N = 2n$, $E = \mathbb{C}^N \otimes \mathcal{O}$ is the trivial vector bundle over C and σ is induced by a non-degenerate symmetric form on $\mathbb{C}^N$. We may assume that the symmetric form σ is given by the block matrix

$$\sigma = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

There is a natural action of $SO(N)$ on the IQ_d induced by the respective action on $\mathbb{C}^N$. The subtorus $G = \mathbb{C}^* \subset SO(N)$ given by $(t^{-w_1}, \ldots, t^{-w_N})$ also acts on IQ_d where the weights $w_i = -w_{i+n}$ for $1 \leq i \leq n$.

5.3.1 Fixed Loci

When the weights are distinct, we get the same description of fixed loci as in the case of σ symplectic. Thus the fixed loci of the $\mathbb{C}^*$ action are isomorphic to a disjoint union of

$$\mathrm{F}_{\vec{d},\underline{k}} = C^{[d_1]} \times C^{[d_2]} \times \cdots \times C^{[d_r]}$$

for each possible tuple of positive integers $\vec{d} = (d_1, d_2, \ldots, d_r)$ such that $d_1 + d_2 + \cdots + d_r = d$ and $\underline{k} = \{k_1, \ldots, k_r\} \subset \{1, \ldots, N\}$ such that $\{i, i+n\} \not\subset \underline{k}$ for any $1 \leq i \leq n$.

We will use the localization formula with distinct weights to show compatibility of the virtual fundamental classes in Theorem 2.3.3. We will use non-distinct weights to obtain the Vafa-Intriligator type formula in Theorem 2.3.8. In the latter case, we will obtain different fixed loci; we will describe it in Section 5.6. The description of the equivariant normal bundle will be crucial in proving both the theorems.

5.3.2 Equivariant Normal bundle

Let $0 \to \mathcal{S} \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0$ be the universal exact sequence over $C \times \mathsf{IQ}_d$. This restricts to

$$0 \to \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_r \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_r \oplus \mathbb{C}^{N-r} \otimes \mathcal{O} \to 0$$

on $C \times \mathrm{F}_{\vec{d},\underline{k}}$, where $0 \to \mathcal{K}_i \to \mathcal{O} \to \mathcal{T}_i \to 0$ is the universal exact sequence over $C \times C^{[d_i]}$ at the position k_i.

Recall that in Theorem 2.3.1, we provided a perfect obstruction theory for the isotropic

Quot scheme. In the K-theory of IQ_d, the corresponding virtual tangent bundle is given by

$$T^{\mathrm{vir}} = \pi_![(RHom(\mathcal{S},\mathcal{Q}))] - \pi_![(Hom(\mathrm{Sym}^2\mathcal{S},\mathcal{O}))].$$

The restriction of the virtual tangent bundle in the $\mathbb{C}^*$ equivariant K-theory of $\mathrm{F}_{\vec{d},\underline{k}}$ is given by

$$\pi_!\Bigg(\sum_{i,j\in[r]}[\mathcal{K}_i^\vee\otimes\mathcal{T}_j] + \sum_{i\in[r],k\in\underline{k}^c}[\mathcal{K}_i^\vee] - \sum_{1\le i\le j\le r}[\mathcal{K}_i^\vee\otimes\mathcal{K}_j^\vee]\Bigg).$$

where the above three summands have $\mathbb{C}^*$ weights $(w_{k_i}-w_{k_j})$, $(w_{k_i}-w_k)$ and $(w_{k_i}+w_{k_j})$ respectively.

The fixed part of the restriction of T^{vir} to $\mathrm{F}_{\vec{d},\underline{k}}$ is

$$\sum_{i\in\underline{k}}\pi_![\mathcal{K}_i^\vee\otimes\mathcal{T}_i]$$

which matches with the K-theory class of the tangent bundle of $\mathrm{F}_{\vec{d},\underline{k}}$.

The virtual normal bundle $\mathcal{N}^{\mathrm{vir}}$ is given by the moving part of the restriction of T^{vir}. In the K-theory of $\mathrm{F}_{\vec{d}}$,

$$\mathcal{N}^{\mathrm{vir}} = \pi_!\Bigg(\sum_{\substack{i\in[r]\\ k\neq k_i}}[\mathcal{K}_i^\vee] - \sum_{\substack{i,j\in[r]\\ i\neq j}}[\mathcal{K}_i^\vee\otimes\mathcal{K}_j] - \sum_{1\le i\le j\le r}[\mathcal{K}_i^\vee\otimes\mathcal{K}_j^\vee]\Bigg). \tag{5.15}$$

Next we would like to determine the equivariant Euler class of $\mathcal{N}^{\mathrm{vir}}$ in the equivariant cohomology ring $H^*(\mathrm{F}_{\vec{d},\underline{k}})[t,t^{-1}]$.

Let $P(X) = \prod_{k=1}^N (X - w_k t)$ and $Y_i = x_i + w_{k_i}t$. Using (5.10), (5.11) and (5.12)and the identity

$$\prod_{i\in[r]} e_{\mathbb{C}^*}(\pi_![\mathcal{K}_i^\vee\otimes\mathcal{K}_i^\vee]) = \prod_{i\in[r]}(2Y_i)^{2d_i-\bar{g}}e^{-\frac{2\theta_i}{Y_i}}, \tag{5.16}$$

we obtain the expression for the equivariant Euler class of $\mathcal{N}^{\mathrm{vir}}$:

$$\frac{1}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})} = u2^{2d-r\bar{g}} \prod_{i=1}^{r} h_i^{d_i-\bar{g}} Y_i^{2d_i-\bar{g}} e^{\theta_i z_i} \prod_{i<j} \frac{(Y_i+Y_j)^{d_i+d_j-\bar{g}}}{(Y_i-Y_j)^{2\bar{g}}} e^{-\frac{\theta_i+\theta_j-\phi_{ij}}{Y_i+Y_j}} \tag{5.17}$$

where $u = (-1)^{d(r-1)+\binom{r}{2}\bar{g}}$ and

$$\begin{aligned} h_i &= \frac{x_i}{P(Y_i)} \\ z_i &= \frac{P'(Y_i)}{P(Y_i)} - \frac{2}{Y_i} - \frac{1}{x_i}. \end{aligned} \tag{5.18}$$

5.4 Compatibility of virtual fundamental classes

In this section we only consider IQ_d with E, L trivial and N even. Fix a point $q \in C$. Then there is a natural embedding

$$i_q : \mathsf{IQ}_d \to \mathsf{IQ}_{d+r} \tag{5.19}$$

which sends a subsheaf $S \subset \mathbb{C}^N \otimes \mathcal{O}$ to the composition $S(-q) \to S \to \mathbb{C}^N \otimes \mathcal{O}$. Observe that $S(-q)$ is an isotropic subsheaf because the composition

$$S(-q) \to S \to \mathbb{C}^N \otimes \mathcal{O} \xrightarrow{\sigma} \mathbb{C}^N \otimes \mathcal{O} \to S^\vee \to S(-q)^\vee$$

is zero.

Proof of Theorem 2.3.3. We work with the symmetric isotropic Quot scheme. The argument in the symplectic case is similar.

Let j be the inclusion of the fixed loci into IQ_d. The virtual localization formula [GP] asserts that

$$[\mathsf{IQ}_d]^{\mathrm{vir}} = j_* \sum_{\vec{d},\underline{k}} \frac{[\mathrm{F}_{\vec{d},\underline{k}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})}$$

in $A_*^{\mathbb{C}^*}(\mathsf{IQ}_d) \otimes \mathbb{Q}[t,t^{-1}]$ where t is the generator of the equivariant ring of $\mathbb{C}^*$. Note that $[\mathrm{F}_{\vec{d},\underline{k}}]^{\mathrm{vir}} = [\mathrm{F}_{\vec{d},\underline{k}}]$ in our case. We will show the compatibility of the virtual fundamental classes by equating the fixed loci contributions.

We denote $\bar{\mathrm{F}} = \mathrm{F}_{\vec{d}+(1,\ldots,1),\underline{k}}$ and $\mathrm{F} = \mathrm{F}_{\vec{d},\underline{k}}$ for notational convenience. These are fixed loci on IQ_d and IQ_{d+r} respectively.

The map i_q restricts to the natural map over the fixed locus $\tilde{i}_q : \mathrm{F} \to \bar{\mathrm{F}}$. This sends the fixed point $L_1 \oplus \cdots \oplus L_r \subset \mathbb{C}^N \otimes \mathcal{O}$ to $L_1(-q) \oplus \cdots \oplus L_r(-q) \subset \mathbb{C}^N \otimes \mathcal{O}$. We have the identity (see [MO 3] for more details)

$$\tilde{i}_{q*}[\mathrm{F}] = \prod_{\ell=1}^{r} \bar{x}_i \cap [\bar{\mathrm{F}}],$$

where $\bar{x}_i$ are the cohomology classes on $\bar{\mathrm{F}}$ defined in (5.8).

In the equivariant cohomology of the fixed loci F,

$$c_{\mathrm{top}}(\mathrm{Sym}^2 \mathcal{S}_q^\vee)|_{\mathrm{F}} = \prod_{1 \le i \le j \le r} (Y_i + Y_j)$$

where $Y_i = x_i + w_{k_i} t$, and over $\bar{\mathrm{F}}$ we have

$$c_{\mathrm{top}}(Hom(\mathcal{S}_q, \mathbb{C}^N \otimes \mathcal{O}))|_{\bar{\mathrm{F}}} = \prod_{i=1}^{r} \bar{x}_i \cdot \prod_{i=1}^{r} \bar{h}_i^{-1}. \tag{5.20}$$

Using the description of the Euler class of the equivariant normal bundle in (5.17), we have

$$\prod_{1\leq i\leq j\leq r}(Y_i+Y_j)^2\cdot\frac{1}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}}{}_{\mathrm{F}/\mathsf{IQ}_d})}=\tilde{i}_q^*\prod_{i=1}^{r}h_i^{-1}\cdot\tilde{i}_q^*\frac{1}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}}{}_{\bar{\mathrm{F}}/\mathsf{IQ}_{d+r}})}.$$

Hence the fixed loci contribution matches in the application of equivariant virtual localization in [GP] to IQ_{d+r} for the fixed loci of the kind $\bar{\mathrm{F}}=\mathrm{F}_{\vec{d},\underline{k}}$ with $d_i>0$ for any $1\leq i\leq r$ with the corresponding contribution over IQ_d. When $d_i=0$ for some i, the fixed point contribution vanishes since $\bar{x}_i$ appears in (5.20). $\square$

5.5 Symmetric powers of curves

In this section we will describe the intersection theory of the products of symmetric powers of curves

$$X_{\vec{d}}=C^{[d_1]}\times\cdots\times C^{[d_r]},$$

where $\vec{d}=(d_1,\ldots,d_r)$. This will be needed to obtain the Vafa-Intriligator type formula for the intersection of a and f classes over isotropic Quot schemes.

There are two difficulties in the calculation of the virtual intersection numbers involving the above classes : knowing how to intersect θ, ϕ_{ij} and x (defined in section 5.2.4), and summing over all the fixed loci. Note that the number of fixed loci increases as d increases. Moreover, the expressions for the Euler class of the virtual normal bundles (5.13) and (5.17) over the fixed loci involve many complicated terms.

We describe techniques to evaluate intersection numbers involving the above terms. For the summation, we will use a beautiful combinatorial technique called multivariate Lagrange-Bürman formula.

For $1\leq i\leq r$, define the cohomology classes x_i, y_i^k and θ_i on $X_{\vec{d}}$ obtained by pulling back the corresponding classes from $C^{[d_i]}$ (see Section 1.3 for known intersection numbers).

Proposition 5.5.1. *Let P be a polynomial in $2r$ variables, then*

$$\int_{X_{\vec{d}}} \phi_{12}^{2\ell} P(\underline{x},\underline{\theta}) = (-1)^{\ell} \binom{2\ell}{\ell} \binom{g}{\ell}^{-1} \int_{X_{\vec{d}}} (\theta_1\theta_2)^{\ell} P(\underline{x},\underline{\theta}) \tag{5.21}$$

where $\underline{x} = (x_1,\ldots,x_r)$ and $\underline{\theta} = (\theta_1,\ldots,\theta_r)$.

Proof. Recall that

$$\phi_{12} = -\sum_{k=1}^{g} (y_1^k y_2^{k+g} + y_2^k y_1^{k+g}).$$

For parity reasons, ϕ_{12} must appear with even exponent.

Using (1.5), $\phi_{12}^{2\ell}$ can be replaced by a constant multiple of $\theta_1^{\ell}\theta_2^{\ell}$, where the constant is $\frac{(g-\ell)!^2}{g!^2}$ times the sum of coefficients of

$$y_1^{k_1} y_1^{k_1+g} \ldots y_1^{k_\ell} y_1^{k_\ell+g} \cdot y_2^{k_1} y_2^{k_1+g} \ldots y_2^{k_\ell} y_2^{k_\ell+g}$$

in the multinomial expansion of $\phi_{12}^{2\ell}$. We observe that

$$\begin{aligned}(y_1^k y_2^{k+g} + y_2^k y_1^{k+g})^2 &= y_1^k y_2^{k+g} y_2^k y_1^{k+g} + y_2^k y_1^{k+g} y_1^k y_2^{k+g} \\ &= -2 y_1^k y_1^{k+g} y_2^k y_2^{k+g}.\end{aligned}$$

Thus the required sum of coefficients is

$$(-2)^{\ell} \binom{g}{\ell} \binom{2\ell}{2,\ldots,2},$$

where $\binom{g}{\ell}$ is the number of choices for $\{k_{i_1},\ldots,k_{i_\ell}\}$ and $\binom{2\ell}{2,\ldots,2}$ is the number of ways of picking ℓ pairs of factors in $\phi_{12}^{2\ell}$ each of which contributes (-2). The binomial identity

$$(-2)^{\ell} \binom{g}{\ell} \binom{2\ell}{2,\ldots,2} \frac{(g-\ell)!^2}{(g!)^2} = (-1)^{\ell} \binom{2\ell}{\ell} \binom{g}{\ell}^{-1} \tag{5.22}$$

completes the proof. $\square$

In Section 5.6 and 5.7, we will use the localization formula to calculate the tautological intersection numbers. We use the independence of the weights in the localization formula. We will describe how to sum over the fixed point contributions for a special choice of weights. The following two Propositions are crucial for our argument.

Let $w_1, \ldots, w_r$ be r distinct N^{th} roots of unity and let $P(Y) = Y^N - 1$.

Proposition 5.5.2. *Let $p_1, \ldots, p_r$ and d be non-negative integers and $R(Y_1, \ldots, Y_r)$ be a homogeneous rational function of degree $s = Nd - r\bar{g}(N-1) - p$ where $p_1 + \cdots + p_r = p$. Let $B(Y) = \frac{aY^N + b}{Y}$, $Y_i = x_i + w_i$, $h_i = \frac{x_i}{P(Y_i)}$ and*

$$z_i = \frac{B(Y_i)}{P(Y_i)} - \frac{1}{x_i}.$$

Then we have the following identity

$$\begin{aligned} &\sum_{|\vec{d}|=d} \int_{X_{\vec{d}}} R(Y_1, \ldots, Y_r) \prod_{i=1}^{r} \frac{\theta_i^{p_i}}{p_i!} e^{\theta_i z_i} h_i^{d_i - \bar{g}} \\ &= N^{-r} \frac{R(w_1, \ldots, w_r)}{(w_1 \cdots w_r)^{\bar{g}}} \prod_{i=1}^{r} \binom{g}{p_i} w_i^{p_i} [q^d](a + b + aq)^{rg-p}(1+q)^{d-rg} q^p. \end{aligned} \tag{5.23}$$

Proof. The expression inside the integral is considered in the power series ring $\mathbb{Q}[[x_1, \ldots, x_r, \theta_1, \ldots, \theta_r]]$ We will first single out the terms containing θ_i. We know that $\theta^k = 0$ for $k > g$ thus

$$\frac{\theta_i^{p_i}}{p_i!} e^{\theta_i z_i} = \sum_{\ell=0}^{g-p_i} \frac{\theta_i^{p_i+\ell}}{p_i! \ell!} \left(\frac{B(Y_i)}{P(Y_i)} - \frac{1}{x_i} \right)^{\ell}$$

We replace $\theta_i^{p_i+\ell}$ by $\frac{g!}{(g-p_i-\ell)!} x_i^{p_i+l}$ using (1.4). We further simplify

$$\sum_{\ell=0}^{g-p_i} \frac{g! x_i^{p_i+\ell}}{p_i!(g-p_i-\ell)!} \frac{1}{\ell!} \left(\frac{B(Y_i)}{P(Y_i)} - \frac{1}{x_i} \right)^{\ell} = \binom{g}{p_i} \cdot x_i^{p_i} \cdot \left(\frac{x_i B(Y_i)}{P(Y_i)} \right)^{g-p_i}.$$

Plugging this back in (5.23), we obtain the following integral of a power series in the variables $x_1, \ldots, x_r$

$$\sum_{|\vec{d}|=d} \int_{X_{\vec{d}}} R(Y_1, \ldots, Y_r) \prod_{i=1}^{r} \binom{g}{p_i} \cdot x_i^{p_i} \cdot \left(\frac{x_i B(Y_i)}{P(Y_i)} \right)^{g-p_i} h_i^{d_i - \bar{g}}.$$

We now have to find the coefficient of $x_1^{d_1} \ldots x_r^{d_r}$ in the above expression and sum it over $|\vec{d}| = d_1 + \cdots + d_r = d$. For such problems, we have a very useful result from combinatorics, the Lagrange-Bürmann formula [WW], which states

$$\sum_{|\vec{d}|} q_1^{d_1} \cdots q_2^{d_2} ([x_1^{d_1} \cdots x_r^{d_r}] f(x_1, \ldots, x_r) \prod_{i=1}^{r} h_i^{d_i}) = f(x_1, \ldots, x_r) \cdot \prod_{i=1}^{r} \frac{1}{h_i} \frac{dx_i}{dq_i} \tag{5.24}$$

where $q_i = \frac{x_i}{h_i}$ and $h_i := h_i(x_i)$ are power series with $h_i(0) \neq 0$.

We can apply this formula to

$$\begin{aligned} h_i &= \frac{x_i}{P(Y_i)} \\ f(x_1, \ldots x_r) &= R(Y_1, \ldots, Y_r) \prod_{i=1}^{r} \binom{g}{p_i} \cdot x_i^g \cdot \left(\frac{B(Y_i)}{P(Y_i)} \right)^{g-p_i} \left(\frac{x_i}{P(Y_i)} \right)^{-\bar{g}} \\ &= R(Y_1, \ldots, Y_r) \prod_{i=1}^{r} \binom{g}{p_i} B(Y_i)^{g-p_i} P(Y_i)^{p_i} h_i. \end{aligned}$$

We have the change of variable

$$q_i = \frac{x_i}{h_i} = P(Y_i) = Y_i^N - 1 = (x_i + w_i)^N - 1,$$

and the inverse is given by

$$x_i = Y_i - w_i = w_i (1 + q_i)^{1/N} - w_i.$$

Observe that the derivative

$$\frac{dx_i}{dq_i} = \frac{1}{P'(Y_i)}.$$

By direct computation

$$f(x_1,\ldots,x_r)\cdot\prod_{i=1}^{r}\frac{1}{h_i}\frac{dx_i}{dq_i} = R(Y_1,\ldots,Y_r)\prod_{i=1}^{r}\binom{g}{p_i}\frac{B(Y_i)^{g-p_i}P(Y_i)^{p_i}}{P'(Y_i)}. \tag{5.25}$$

In (5.23), we are interested in finding the sum over the coefficients of $q_1^{d_1}\cdots q_r^{d_r}$ where $d_1+\cdots+d_r = d$. To find this sum, we will substitute

$$q_1 = \cdots = q_r = q$$

to obtain a power series in one variable q and find the coefficient of q^d.

In this situation,

$$\begin{aligned} &Y_i = w_i(1+q)^{1/N}, \quad B(Y_i) = \frac{(aq+(a+b))}{w_i(1+q)^{1/N}}, \\ &P'(Y_i) = Nw_i^{-1}(1+q)^{\frac{N-1}{N}}. \end{aligned}$$

Note that R is a homogeneous rational function of degree s, thus $R(Y_1,\ldots Y_r) = R(w_1,\ldots,w_r)(1+q)^{s/N}$. Substituting, the power series (5.25) becomes

$$\begin{aligned} &R(w_1,\ldots,w_r)(1+q)^{\frac{s}{N}}\prod_{i=1}^{r}\binom{g}{p_i}\frac{w_i^{p_i-\bar{g}}}{N}\frac{(a+b+aq)^{g-p_i}}{(1+q)^{\frac{g-p_i}{N}+\frac{N-1}{N}}}q^{p_i} \\ &= (a+b+aq)^{rg-p}(1+q)^{d-rg}q^pN^{-r}\frac{R(w_1,\ldots,w_r)}{(w_1\cdots w_r)^{\bar{g}}}\prod_{i=1}^{r}\binom{g}{p_i}w_i^{p_i}, \end{aligned}$$

where $p = p_1+\cdots+p_r$. □

Remark 5.5.3. *When $p \geq rg$ then $p_i > g$ for some i, thus the integral is 0 since $\theta_i^p = 0$. Therefore* *we may assume that the first term is a polynomial. Moreover, when $d \geq rg$ or $b = 0$ and $d \geq p$ then the answer in* (5.23) *is given by*

$$\frac{a^{rg}}{N^r}\frac{R(w_1,\ldots,w_r)}{(w_1\cdots w_r)^{\bar{g}}}\prod_{i=1}^{r}\binom{g}{p_i}\frac{w_i^{p_i}}{a^{p_i}}.$$

Remark 5.5.4. *The above proposition, specialized to $B(Y) = P'(Y)$ and $p = 0$, greatly simplifies the combinatorics used in finding the Vafa-Intriligator formula for Quot schemes in Section 4 of [MO 3].*

The previous result does not suffice for the calculation of virtual intersection numbers over isotropic Quot schemes. When rank $r = 2$, the following proposition can be used to find Vafa-Intriligator type formulas for IQ_d.

Proposition 5.5.5. *Let $R(Y_1,Y_2)$ be a homogeneous rational function of degree $s = Nd - 2\bar{g}(N-1)$. We borrow the notation $X_{\vec{d}}$, Y_i, $P(Y)$, $B(Y)$, h_i and z_i from Proposition 5.5.2. Let $T(q) = (a+b+aq)/q$. Then we have the following identity*

$$\sum_{|\vec{d}|=d}\int_{X_{\vec{d}}} R(Y_1,Y_2)e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}}\prod_{i=1}^{2}e^{\theta_i z_i}h_i^{d_i-\bar{g}}$$
$$= \frac{1}{N^2}\frac{R(w_1,w_2)}{(w_1w_2)^{\bar{g}}}[q^d](1+q)^d\left(\frac{qT(q)}{1+q}\right)^{2g}\left(1-\frac{1}{T(q)}\right)^{g}.$$

In particular, when $d \geq 2g$ the above value is

$$\frac{a^g(a-1)^g}{N^2}\frac{R(w_1,w_2)}{(w_1w_2)^{\bar{g}}}.$$

Proof. We will first replace exponents of ϕ_{12} with the exponents of $\theta_1\theta_2$ using Proposition 5.5.1. For parity reasons ϕ_{12} must appear with an even power to obtain a non-zero number. Thus we can make following replacements:

$$
\begin{aligned}
e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}} &\to \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(Y_1+Y_2)^p} \left(\sum_{2\ell+r+s=p} \binom{p}{2\ell, r, s} \theta_1^r \theta_2^s \phi_{12}^{2\ell} \right) \\
&\to \sum_{p=0}^{\infty} \sum_{2\ell+r+s=p} \frac{(-1)^{p-\ell}}{p!} \binom{p}{2\ell, r, s} \frac{\binom{2\ell}{\ell}}{\binom{g}{\ell}} \frac{\theta_1^{r+\ell} \theta_2^{s+\ell}}{(Y_1+Y_2)^p} \\
&= \sum_{p=0}^{\infty} \sum_{2\ell+r+s=p} \frac{(-1)^{p-\ell}}{(Y_1+Y_2)^p} \frac{\binom{p}{2\ell, r, s}}{\binom{p}{r+\ell}} \frac{\binom{2\ell}{\ell}}{\binom{g}{\ell}} \frac{\theta_1^{r+\ell} \theta_2^{s+\ell}}{(r+\ell)!(s+\ell)!}.
\end{aligned}
$$

Now we use Proposition 5.5.2 to reduce the problem to finding

$$
\begin{aligned}
\sum_{2\ell+r+s=p} (-1)^{p-\ell} \frac{\binom{p}{2\ell, r, s}}{\binom{p}{r+\ell}} \frac{\binom{2\ell}{\ell}}{\binom{g}{\ell}} \cdot \frac{1}{N^2} \frac{R(w_1,w_2) w_1^{r+\ell} w_2^{s+\ell}}{(w_1+w_2)^p (w_1 w_2)^{\bar{g}}} \binom{g}{r+\ell} \binom{g}{s+\ell} \\
\cdot [q^d](1+q)^d \left(\frac{a+b+aq}{1+q} \right)^{2g} \left(\frac{q}{a+b+aq} \right)^p
\end{aligned}
$$

where the sum is taken over r, s, ℓ such that $r+\ell, s+\ell \leq g$. Rearranging the binomial coefficients, the above expression is same as

$$
\begin{aligned}
[q^d](1+q)^d \left(\frac{a+b+aq}{1+q} \right)^{2g} \frac{1}{N^2} \frac{R(w_1,w_2)}{(w_1 w_2)^{\bar{g}}} \\
\cdot \sum_{2\ell+r+s=p} (-1)^{\ell} \binom{g}{\ell} \binom{g-\ell}{r} \binom{g-\ell}{s} \frac{(-w_1)^{r+\ell} (-w_2)^{s+\ell}}{T(q)^p (w_1+w_2)^p}.
\end{aligned}
$$

The summation in the above expression greatly simplifies via the following lemma. $\square$

Lemma 5.5.6. *Let g and d be integers, then*

$$
\sum_{2\ell+r+s=p} (-1)^{\ell} \binom{g}{\ell} \binom{g-\ell}{r} \binom{g-\ell}{s} \frac{(-w_1)^{r+\ell} (-w_2)^{s+\ell}}{T(q)^p (w_1+w_2)^p} = \left(1 - \frac{1}{T(q)} \right)^g.
$$

Proof. The lemma follows by observing that the given expression simplifies as

$$\sum_{\ell} \binom{g}{\ell} \frac{(-1)^{\ell}}{T(q)^{2\ell}} \frac{(-w_1)^{\ell}(-w_2)^{\ell}}{(w_1+w_2)^{2\ell}} \left(1 - \frac{w_1}{T(q)(w_1+w_1)}\right)^{g-\ell} \left(1 - \frac{w_2}{T(q)(w_1+w_1)}\right)^{g-\ell}$$
$$= \left(\left(1 - \frac{w_1}{T(q)(w_1+w_1)}\right)\left(1 - \frac{w_2}{T(q)(w_1+w_1)}\right) - \frac{w_1 w_2}{T(q)^2(w_1+w_2)^2}\right)^g$$
$$= \left(1 - \frac{1}{T(q)}\right)^g.$$

□

5.6 Intersection of a-classes

In this section we will prove Theorem 2.3.4 and 2.3.8, which are explicit expressions for the intersections of a-classes in the symplectic and symmetric case respectively.

5.6.1 a-class intersections for σ symplectic

Let $r = 2$. In this case the virtual dimension of IQ_d is given by

$$\mathrm{vd} = (N-1)d - (2N-5)\bar{g}.$$

Let us define

$$T_{d,g}(N) = [q^d](1+q)^{d-g}\left(1 + \frac{N-1}{N}q\right)^g. \tag{5.26}$$

In particular, when $d \geq g$, we get $T_{d,g}(N) = (1-1/N)^g$. A simple usage of Lagrange inversion theorem implies

$$T_{d,g}(N) = [q^d](1-q/N)^g(1-q)^{-1}$$

and hence $T_{d,g}(N)$ is the sum of the first d terms in the binomial expansion of $(1-1/N)^g$.

Theorem 5.6.1. *Let $Q(X_1, X_2)$ be a polynomial of weighted degree* vd*, where the variables X_i*

have degree i. Then,

$$\int_{[\mathrm{IQ}_d]^{\mathrm{vir}}} Q(a_1,a_2) = uT_{d,g}(N)\sum_{w_1,w_2} S(w_1,w_2)J(w_1,w_2)^{\bar{g}}(w_1+w_2)^d \tag{5.27}$$

where the sum is taken over all the pairs of N^{th} roots of unity $\{w_1,w_2\}$ with $w_1 \neq \pm w_2$. Here $u = (-1)^{\bar{g}+d}$ and

$$J(w_1,w_2) = N^2 w_1^{-1} w_2^{-1}(w_1-w_2)^{-2}(w_1+w_2)^{-1},$$

and $S(w_1,w_2) = Q(w_1+w_2, w_1w_2)$.

Proof. The equivariant pull back of a_i to the fixed loci is the ith elementary symmetric function $\sigma_i((w_1t+x_1),(w_2t+x_2))$, hence $Q(a_1,a_2)$ pulls back to $S(w_1t+x_1,w_2t+x_2)$. We are in a position to apply the equivariant virtual localization formula [GP] which yields

$$\int_{[\mathrm{IQ}_d]^{\mathrm{vir}}} Q(a_1,a_2) = \sum_{d_1+d_2=d}\sum_{w_1,w_2}\int_{\mathrm{F}_{\vec{d},\underline{k}}} \frac{S(Y_1,Y_2)}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})}, \tag{5.28}$$

where the sum is taken over all the prescribed choices for $\{w_1,w_2\}$ and $Y_i = x_i + w_i t$.

After appropriately replacing θ and ϕ_{12} classes with x classes as described in Section 5.5, the above expression can be written as a rational function in x_1,x_2 and t of with total degree d . The integral can thus be evaluated by finding coefficient of $x_1^{d_1}x_2^{d_2}$. The homogeneity and the identity $d_1+d_2=d$ ensures that resulting element in $\mathbb{C}[t,t^{-1}]$ has t degree 0. Hence we can safely assume $t=1$ for the purpose of our calculation without changing the value of integral.

Moreover, the localization formula is independent of the choice of the weights $(w_1,\dots w_N)$ as long as these are distinct and satisfy $w_i = -w_{i+n}$ for $1 \leq i \leq n$. Hence we may assume these to be distinct roots of the polynomial $P(X) = X^N - 1$.

We substitute the expression (5.13) of the Euler class of $\mathcal{N}^{\mathrm{vir}}$ into (5.28) to get

$$\sum_{w_1,w_2}\sum_{d_1+d_2=d}\int_{\mathrm{F}_{\vec{d},\underline{k}}} R(Y_1,Y_2)e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}}\prod_{i=1}^{2} e^{\theta_i z_i} h_i^{d_i-\bar{g}},$$

where by (5.14) $z_i = \frac{P'(Y_i)}{P(Y_i)} - \frac{1}{x_i}$, $h_i = \frac{x_i}{P(Y_i)}$ and

$$R(Y_1,Y_2) = uS(Y_1,Y_2)\frac{(Y_1+Y_2)^{d-\bar{g}}}{(Y_1-Y_2)^{2\bar{g}}}.$$

The homogeneous degree of R is $\mathrm{vd}+(d-3\bar{g}) = Nd-2\bar{g}(N-1)$, therefore Proposition 5.5.5 gives the required intersection number

$$\sum_{w_1,w_2}\frac{1}{N^2}\frac{R(w_1,w_2)}{(w_1w_2)^{\bar{g}}}[q^d]N^{2g}(1+q)^{d-g}\left(1+\frac{N-1}{N}q\right)^{g}, \tag{5.29}$$

completing the proof. □

Proof of Theorem 2.3.4. In the statement of Theorem 5.6.1, the expression

$$S(w_1,w_2)J(w_1,w_2)^{\bar{g}}(w_1+w_2)^d$$

is homogeneous of degree $N(d-2\bar{g})$, hence this equals $S(1,\zeta)J(1,\zeta)^{\bar{g}}(1+\zeta)^d$, where $\zeta = w_2/w_1$. □

Example 5.6.2. *When $g=1$, the virtual dimension* $\mathrm{vd} = (N-1)d$. *Then*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} a_1^{\mathrm{vd}} = \begin{cases} (-1)^d\frac{N-1}{2}[q^{Nd}]\left(\frac{N(1-q)^{N-1}}{(1-q)^N-q^N}-\frac{1}{1+2q}\right) & d>0 \\ \frac{N(N-2)}{2} & d=0 \end{cases}.$$

5.6.2 a-class intersections for σ symmetric

Define

$$\tilde{T}_{d,g}(N) = [q^d]\left(1+\frac{N-2}{N}q\right)^g (1+q)^{d-g}.$$

Proposition 5.6.3. *Over IQ_d, where N is even, $r=1$ and σ is symmetric, the top intersection of the tautological class is given by*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} a_1^{\mathrm{vd}} = N^g \tilde{T}_{d,g}(N) 2^{2d-\bar{g}} \tag{5.30}$$

where $\mathrm{vd} = (N-2)(d-\bar{g})$ *is the virtual dimension.*

Proof. The restriction of a_1 to the fixed locus $\mathrm{F}_{d,i} = C^{[d]}$ is $Y_i = x_i + w_i t$. The Euler class of the equivariant normal bundle of the fixed locus is given by (5.17)

$$\frac{1}{e_{\mathbb{C}^*}^{\mathrm{vir}}(\mathcal{N}^{\mathrm{vir}})} = 2^{2d-\bar{g}} Y_i^{2d-\bar{g}} h_i^{d-\bar{g}} e^{\theta_i z_i}$$

where $z_i = (B(Y_i)/P(Y_i) - 1/x_i)$ and

$$\frac{B(Y)}{P(Y)} = \frac{P'(Y)}{P(Y)} - \frac{2}{Y}.$$

The equivariant virtual localization formula gives

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} a_1^{\mathrm{vd}} = \sum_{i=1}^{N} \int_{\mathrm{F}_{d,i}} \frac{Y_i^{\mathrm{vd}}}{e_{\mathbb{C}^*}^{\mathrm{vir}}(\mathcal{N}^{\mathrm{vir}})}.$$

We choose the weight of the action to be N^{th} roots of unity, thus $P(X) = X^N - 1$, hence $B(Y) = \frac{(N-2)Y^N+2}{Y}$, and we obtain the integral as a special case of Proposition 5.5.2 by putting $r=1$ and $p=0$. $\square$

Remark 5.6.4. *Similar results can be obtained when N is odd, $r=1$ and σ symmetric. In* *particular, when the virtual dimension is non-zero,*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} a_1^{\mathrm{vd}} = (N-1)^g 2^{2d-\bar{g}} T_{d,g}(N-1). \tag{5.31}$$

When $r = 2$, localizing with distinct weights makes combinatorics very difficult. However using two equal weights enable us to find a simple formula for these intersections. Using exactly two equal weights results in getting $C^{[d_1]} \times \mathsf{IQ}_{d_2}(\mathbb{C}^2 \otimes \mathcal{O}, r = 1, \sigma)$ as part of the fixed loci. We will first show that

$$\mathsf{IQ}_d(\mathbb{C}^2 \otimes \mathcal{O}, r = 1, \sigma) = C^{[d]} \sqcup C^{[d]},$$

and the two components $C^{[d]}$ come equipped with a non-standard virtual structure. We will use Proposition 5.6.3 to understand how to intersect over these non-standard loci.

Recall that the virtual dimension of IQ_d is

$$\mathrm{vd} = (N-3)d - \bar{g}(2N-7).$$

Let $N = 2n$. Let $G = \mathbb{C}^*$ act on IQ_d with weights

$$(w_1, \ldots, w_N) = (\zeta, \zeta^2, \ldots \zeta^{n-1}, 0, \zeta^n, \ldots, \zeta^{2n-2}, 0),$$

where ζ is a primitive $(N-2)$'th root of unity. A point $[0 \to S \to \mathbb{C}^N \otimes \mathcal{O} \to Q \to 0]$ in IQ_d is fixed under the action of G if and only if one of the following is satisfied:

(i) The sheaf S splits as $L_1 \oplus L_2$ where L_i is a subsheaf of one of the $N-2$ copies of $\mathcal{O}$, at position $k_i \notin \{n, 2n\}$, in $\mathbb{C}^N \otimes \mathcal{O}$ such that $k_1 - k_2 \not\equiv 0 \mod n$. The corresponding fixed locus is

$$\mathrm{F}_{\vec{d},\underline{k}} \cong C^{[d_1]} \times C^{[d_2]},$$

where $\deg L_i = d_i$ and $\underline{k} = (k_1, k_2)$.

(ii) The sheaf S splits as $L_1 \oplus E$ where L_1 is a subsheaf of one the copies of $\mathcal{O}$, at position $k \notin \{n, 2n\}$, in $\mathbb{C}^N \otimes \mathcal{O}$ and E is an isotropic rank one subsheaf of $\mathcal{O}_n \oplus \mathcal{O}_{2n}$, the sum of copies of $\mathcal{O}$ at positions n and $2n$. Let $\mathrm{F}_{\vec{d},k}$ be the component of the fixed loci consisting of (L_1, E), where $d_1 = \deg L_1$, $d_2 = \deg E$ and k is the position mentioned above. Note that

$$\mathrm{F}_{\vec{d},k} \cong C^{[d_1]} \times \mathsf{IQ}_{d_2}(\mathcal{O} \otimes \mathbb{C}^2, r = 1, \sigma).$$

Theorem 5.6.5. *Let $Q(X_1, X_2)$ be a polynomial of weighted degree* vd*, where the variables X_i have degree i. Then,*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} Q(a_1, a_2) = I_1 + I_2$$

where $S(X_1, X_2) = Q(X_1 + X_2, X_1 X_2)$,

$$I_1 = u4^d T_{d,g}(N-2) \sum_{w_1 \neq \pm w_2} S(w_1, w_2) J(w_1, w_2)^{\bar{g}} (w_1 + w_2)^d,$$
$$I_2 = (-1)^d 2^{2d+2-g} T_{d,g}(N-2)(N-2)^g \cdot Q(1, 0),$$

and $J(w_1, w_2) = \frac{(N-2)^2}{4}(w_1 + w_2)^{-1}(w_1 - w_2)^{-2}$.

Proof. Using equivariant virtual localization formula, we can write

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} Q(a_1, a_2) = I_1 + I_2,$$

where

$$I_1 = \sum_{\substack{k_1, k_2 \notin \{n, 2n\} \\ |k_1 - k_2| \neq n}} \sum_{d_1 + d_2 = d} \int_{\mathrm{F}_{\vec{d},\underline{k}}} \frac{i^*(Q(a_1, a_2))}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}} \mathrm{F}_{\vec{d},\underline{k}})}$$
$$I_2 = \sum_{\substack{k \in [N] \\ k \notin \{n, 2n\}}} \sum_{d_1 + d_2 = d} \int_{\mathrm{F}_{\vec{d},k}} \frac{i^*(Q(a_1, a_2))}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}} \mathrm{F}_{\vec{d},k})}.$$

Here we denote i^* the restriction to the fixed loci. The next two subsections will be devoted to the calculation of I_1 and I_2 respectively. $\square$

Fixed loci of the first kind

$\mathrm{F}_{\vec{d},\underline{k}} = C^{[d_1]} \times C^{[d_2]}$. In Section 5.3.2 we noted that the $\mathbb{C}^*$ equivariant virtual tangent bundle is given by

$$T^{\mathrm{vir}} = \pi_![(RHom(\mathcal{S},\mathcal{Q}))] - \pi_![(Hom(\mathrm{Sym}^2\mathcal{S},\mathcal{O}))].$$

The non-moving part of the restriction of T^{vir} to $\mathrm{F}_{\vec{d},\underline{k}}$ matches the K-theory class the tangent bundle of $\mathrm{F}_{\vec{d},\underline{k}}$. The virtual normal bundle

$$\mathcal{N}^{\mathrm{vir}} = \pi_*\Bigg(\sum_{\substack{i=1,2\\1\le k\le N\\k_i\neq k}}[\mathcal{K}_i^\vee] - \sum_{\substack{i,j\in[2]\\i\neq j}}[\mathcal{K}_i^\vee\otimes\mathcal{K}_j] - \sum_{1\le i\le j\le 2}[\mathcal{K}_i^\vee\otimes\mathcal{K}_j^\vee]\Bigg).$$

Therefore using (5.17), we have

$$\frac{1}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})} = u2^{2d-2\bar{g}}\frac{(Y_1+Y_2)^{d-\bar{g}}}{(Y_1-Y_2)^{2\bar{g}}}(Y_1Y_2)^{\bar{g}}e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}}\prod_{i=1}^{2}h_i^{d_i-\bar{g}}e^{\theta_i z_i} \tag{5.32}$$

where $P_0(X) = X^{N-2}-1$ and

$$\begin{aligned} h_i &= \frac{x_iY_i^2}{P(Y_i)} = \frac{x_i}{P_0(Y_i)}, \qquad B(Y_i) = P_0'(Y_i),\\ z_i &= \frac{P'(Y_i)}{P(Y_i)} - \frac{2}{Y_1} - \frac{1}{x_i} = \frac{B(Y_i)}{P_0(Y_i)} - \frac{1}{x_i}. \end{aligned}$$

Proposition 5.6.6. *We have*

$$I_1 = u4^dT_{d,g}(N-2)\sum_{w_1,w_2}S(w_1,w_2)J(w_1,w_2)^{\bar{g}}(w_1+w_2)^d \tag{5.33}$$

where the sum is taken over pairs of $(N-2)^{th}$ roots of unity $\{w_1, w_2\}$ with $w_1 \neq \pm w_2$, and

$$J(w_1, w_2) = \frac{(N-2)^2}{4}(w_1+w_2)^{-1}(w_1-w_2)^{-2}.$$

In particular when $d \geq g$, $T_{d,g}(N-2) = (N-3)^g(N-2)^{-g}$.

Proof. For notational convenience, we assume $\underline{k} = (1,2)$. The classes a_1 and a_2 restrict to Y_1+Y_2 and Y_1Y_2 respectively, where $Y_i = x_i + w_i t$ in the equivariant cohomology ring $H^*(\mathrm{F}_{\vec{d},\underline{k}} = C^{[d_1]} \times C^{[d_2]})[[t]]$.

We are interested in evaluating the following sum

$$\sum_{d_1+d_2=d} \sum_{w_1,w_2} \int_{\mathrm{F}_{\vec{d},\underline{k}}} \frac{S(Y_1,Y_2)}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}}{\mathrm{F}_{\vec{d},\underline{k}}})},$$

where $S(Y_i,Y_i) = Q(Y_1+Y_2, Y_1Y_2)$. After replacing the classes θ_i and ϕ_{12} as in the proof of Theorem 2.3.4, the above expression becomes a homogeneous degree rational function of degree $d = d_1+d_2$ in the variables x_i and t and a power series in x_1 and x_2 with coefficients in $\mathbb{C}[[t,t^{-1}]]$. Integrating over $C^{[d_1]} \times C^{[d_2]}$ amounts to finding the coefficient of $x_1^{d_1}x_2^{d_2}$.

Using the calculation of $e(\mathcal{N}^{\mathrm{vir}})$ in (5.32), we reduce our problem to finding

$$\sum_{d_1+d_2=d} \sum_{w_1,w_2} \int_{\mathrm{F}_{\vec{d},\underline{k}}} R(Y_1,Y_2) e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}} \prod_{i=1}^{2} h_i^{d_i-\bar{g}} e^{\theta_i z_i}$$

where (w_1,w_2) are the prescribed pair of $(N-2)$'th roots of unity and

$$R(Y_1,Y_2) = u 2^{2d-2\bar{g}} S(Y_1,Y_2)(Y_1Y_2)^{\bar{g}} \frac{(Y_1+Y_2)^{d-\bar{g}}}{(Y_1-Y_2)^{2\bar{g}}}.$$

We apply Proposition 5.5.5 to find

$$I_1 = \sum_{w_1,w_2} \frac{1}{(N-2)^2} \frac{R(w_1,w_2)}{(w_1,w_2)^{\bar{g}}} [q^d](N-2)^{2g}(1+q)^{d-g}\left(1+\frac{N-3}{N-2}q\right)^g.$$

$\square$

Fixed Loci of second kind

We will first understand the virtual geometry of the isotropic Quot scheme $\mathsf{IQ}_d^\circ = \mathsf{IQ}_d(\mathcal{O}\otimes\mathbb{C}^2, r=1,\sigma)$.

Lemma 5.6.7. *The isotropic Quot scheme IQ_d° is isomorphic to the disjoint union $C^{[d]} \sqcup C^{[d]}$. The virtual tangent bundle of IQ_d° restricted to either copy of $C^{[d]}$ is given by*

$$T^{\mathrm{vir}} = \pi_!([\mathcal{K}^\vee \otimes (\mathcal{T}\oplus\mathcal{O})] - [\mathcal{K}^\vee \otimes \mathcal{K}^\vee]),$$

where π is the projection $\pi : C \times C^{[d]} \to C^{[d]}$ and $0 \to \mathcal{K} \to \mathcal{O} \to \mathcal{T} \to 0$ is the universal exact sequence on $C \times C^{[d]}$.

Proof. A subsheaf $E \subset \mathbb{C}^2 \otimes \mathcal{O}$ is isotropic if and only if E factors through a copy of $\mathcal{O}$ in $\mathbb{C}^2 \otimes \mathcal{O}$, hence $\mathsf{IQ}_d^\circ \cong C^{[d]} \sqcup C^{[d]}$. The universal short exact sequence over $C \times \mathsf{IQ}_d^\circ$ restricts to

$$0 \to \mathcal{K} \to \mathbb{C}^2 \otimes \mathcal{O} \to \mathcal{T} \oplus \mathcal{O} \to 0$$

over each copy of $C \times C^{[d]}$. The lemma follows using the description of T^{vir} of IQ_d° in Theorem 2.3.1. $\square$

Therefore we see that the virtual fundamental class $[C^{[d]}]^{\mathrm{vir}}$ induced over each component $C^{[d]}$ of IQ_d° is different from the usual fundamental class $[C^{[d]}]$. We also observe that the virtual dimension for $C^{[d]}$ is zero.

Lemma 5.6.8. *Let $C^{[d]}$ be equipped with the non-standard virtual structure as described above, then*

$$\int_{[C^{[d]}]^{\mathrm{vir}}} 1 = 2^{2d}(-1)^d \binom{\bar{g}}{d}.$$

Proof. We have a natural automorphism obtained by swapping the copies of the $\mathcal{O}$ in $\mathbb{C}^2 \otimes \mathcal{O}$. Therefore the above intersection number is independent of the copy of $C^{[d]}$ we have chosen. The Proposition 5.6.3 tells us

$$\int_{[C^{[d]}]^{\mathrm{vir}}} 1 = \frac{1}{2} \int_{[\mathsf{IQ}_d^\circ]^{\mathrm{vir}}} 1 = 2^{2d}[q^d](1+q)^{d-g}.$$

$\square$

Now we are ready to prove

Proposition 5.6.9. *We have*

$$I_2 = (-1)^d 2^{2d+2-g}(N-2)^g T_{d,g}(N-2) \cdot Q(1,0)$$

Proof. We are working over the fixed loci $\mathrm{F}_{\vec{d},k,\varepsilon} = C^{[d_1]} \times \mathrm{C}_\varepsilon{}^{[d_2]}$ where $k \notin \{n, 2n\}$ and the first factor corresponds to the copy of $\mathcal{O}$ at position k and the index ε differentiates between the two components of $\mathsf{IQ}^0_{d_2} = C^{[d_2]} \sqcup C^{[d_2]}$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the pullbacks of the universal subsheaves over $C^{[d_1]}$ and $\mathrm{C}_\varepsilon{}^{[d_2]}$ to the product $\mathrm{F}_{\vec{d},k,\varepsilon}$. The virtual normal bundle is the moving part of the restriction of the T^{vir} and is given by

$$\mathcal{N}^{\mathrm{vir}} = \pi_!\Bigg(\sum_{j\in[N]-\{k\}} [\mathcal{K}_1^\vee] + \sum_{\substack{j\in[N]\\ j\notin\{n,2n\}}} [\mathcal{K}_2^\vee] - [\mathcal{K}_1^\vee \otimes \mathcal{K}_2] - [\mathcal{K}_1 \otimes \mathcal{K}_2^\vee] - [\mathcal{K}_1^\vee \otimes \mathcal{K}_2^\vee] - [\mathcal{K}_1^\vee \otimes \mathcal{K}_1^\vee] \Bigg),$$

where the above terms have $\mathbb{C}^*$ weights $(w_k - w_j)$, $-w_j$, w_k, $-w_k$, w_k and $2w_k$ respectively.

We may assume $t=1$ (see the proof of Theorem 5.6.1. Let $Y_1 = x_1 + w_k$, $u = (-1)^{d+\bar{g}}$ and $P(X) = X^{N-2} - 1$. A careful calculation using (5.10), (5.11) and (5.12) gives

$$\begin{aligned}\frac{1}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})} = &\left(\frac{Y_1^2 P(Y_1)}{x_1}\right)^{-d_1+\bar{g}} e^{\theta_1\left(\frac{P'(Y_1)}{P(Y_1)} + \frac{2}{Y_1} - \frac{1}{x_1}\right)} \cdot P(x_\varepsilon)^{-d_2+\bar{g}} e^{\theta_\varepsilon \frac{P'(x_\varepsilon)}{P(x_\varepsilon)}} \\ &\cdot u(Y_1 - x_\varepsilon)^{-2\bar{g}} \cdot (Y_1 + x_\varepsilon)^{d-\bar{g}} e^{\left(-\frac{\theta_1+\theta_\varepsilon-\phi_{12}}{(Y_1+x_\varepsilon)}\right)} \cdot (2Y_1)^{2d_1-\bar{g}} e^{-\frac{2\theta_1}{Y_1}}\end{aligned}$$

Since $\mathrm{C}_\varepsilon{}^{[d_2]}$ has virtual dimension zero, x_ε and θ_ε yield zero when intersected with the virtual fundamental class $[\mathrm{C}_\varepsilon{}^{[d_2]}]^{\mathrm{vir}}$. Thus for the purpose of our calculation, we may substitute $x_\varepsilon = \theta_\varepsilon = \phi_{12} = 0$ in the above expression to get

$$u2^{2d_1-\bar{g}}Y_1^{d-2\bar{g}}h_1^{d_1-\bar{g}}e^{\theta_1 z_1}\cdot(-1)^{(\bar{g}-d_2)},$$

where $h_1 = x_1/P(Y_1)$ and $z_1 = P'(Y_1)/P(Y_1) - 1/Y_1 - 1/x_1$.

Note that a_1 and a_2 restrict to $Y_1 + x_\varepsilon$ and $Y_1 x_\varepsilon$ respectively over the fixed loci. We want to calculate

$$I_2 = \sum_{k=1}^{N-2}\sum_{d_1+d_2=d}\sum_{\varepsilon=1}^{2}\int_{[\mathrm{F}_{\vec{d},k,\varepsilon}]^{\mathrm{vir}}}\frac{i^*(Q(a_1,a_2))}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}}{}_{\mathrm{F}_{\vec{d},k}})}$$

Substituting $x_\varepsilon = 0$, we get

$$I_2 = Q(1,0)\sum_{k=1}^{N-2}\sum_{d_1+d_2=d}\sum_{\varepsilon=1}^{2}\int_{[\mathrm{F}_{\vec{d},k,\varepsilon}]^{\mathrm{vir}}}\frac{Y_1^{\mathrm{vd}}}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})}. \tag{5.34}$$

Simplifying further using Lemma 5.6.8, we get

$$\begin{aligned}
I_2 &= Q(1,0)\sum_{k=1}^{N-2}\sum_{\varepsilon=1}^{2}\sum_{d_1+d_2=d} u2^{2d_1-\bar{g}}(-1)^{\bar{g}-d_2}\int_{C^{[d_1]}} Y_1^{\mathrm{vd}+d-2\bar{g}}h_1^{d_1-\bar{g}}e^{\theta_1 z_1}\int_{[C^{[d_2]}]^{\mathrm{vir}}}1\\
&= Q(1,0)\sum_{k=1}^{N-2}\sum_{\varepsilon=1}^{2}\sum_{d_1+d_2=d} u2^{2d-\bar{g}}(-1)^{\bar{g}}\binom{\bar{g}}{d_2}\int_{C^{[d_1]}} Y_1^{\mathrm{vd}+d-2\bar{g}}h_1^{d_1-\bar{g}}e^{\theta_1 z_1}\\
&= Q(1,0)\sum_{k=1}^{N-2}\sum_{\varepsilon=1}^{2} u2^{2d-\bar{g}}(-1)^{\bar{g}}(N-2)^{\bar{g}}[q^d](1+q)^{d-g}\left(1+\frac{N-3}{N-2}q\right)^{g}.
\end{aligned}$$

The last equality follows from noting that $\binom{\bar{g}}{d_2} = [q^{d_2}](1+q)^{\bar{g}}$ and the following Lemma. $\square$

Lemma 5.6.10.

$$\int_{C^{[d_1]}} Y_1^{\mathrm{vd}+d-2\bar{g}} h_1^{d_1-\bar{g}} e^{\theta_1 z_1} = (N-2)^{\bar{g}}[q^{d_1}](1+q)^{d-\bar{g}-g}\left(1+\frac{N-3}{N-2}q\right)^g$$

Proof. Proposition 5.5.2 does not directly apply here due to shape of d_1. However, we closely follow the proof of Proposition 5.5.2. Correctly replacing $e^{\theta_1 z_1}$ yield

$$\int_{C^{[d_1]}} Y_1^{\mathrm{vd}+d-2\bar{g}} h_1^{d_1-\bar{g}} \left(\frac{x_1 B(Y_1)}{P'(Y_1)}\right)^g.$$

Applying the Lagrange-Bürmann formula, we obtain

$$[q^{d_1}] Y_1^{\mathrm{vd}+d-2\bar{g}} \frac{B(Y_1)^g}{P'(Y_1)}$$

where $Y_1 = w_1(1+q)^{\frac{1}{N-2}}$ and $Y_1 B(Y_1) = (N-3)Y_1^{N-2}+1$. Therefore, it equals

$$(N-2)^{\bar{g}}[q^{d_1}](1+q)^{d-\bar{g}-g}\left(1+\frac{N-3}{N-2}q\right)^g.$$

□

5.7 Intersection of f classes

We will find an explicit expression for the intersection numbers of polynomials in a and f classes in terms of multivariate generating functions. We obtain Theorem 2.3.6 as a corollary. While the computations are more involved, the basic ideas are similar to those in Section 5.6.

We will only work with symplectic isotropic Quot scheme IQ_d with $r=2$. A similar analysis can be carried out when σ is symmetric.

Over the fixed loci $\mathrm{F}_{\vec{d},\underline{k}}$, the equivariant restriction of the f classes are given by $f_1 = d$ and $f_2 = \phi_{12} + d_1(x_2+w_2t) + d_2(x_1+w_1t)$. The formula for the intersection of f classes with a polynomial in a classes involves differential operators.

Let $P(X) = X^N - 1$ and

$$T_g(t, Y_1, Y_2) = \left(\prod_{i=1}^{2} (1 - \eta_i) - \prod_{i=1}^{2} t^2 \eta_i \right)^g,$$

where $\eta_i = \frac{P(Y_i)}{P'(Y_i)(Y_1+Y_2)}$. When $Y_i = w_i(1+q_i)^{\frac{1}{N}}$, $T_g(t, Y_1, Y_2)$ is a power series in q_1 and q_2 over $\mathbb{C}[t]$. This should be considered as an analogue of $T_{d,g}(N)$ in (5.26). In particular,

$$T_g(1, w_1(1+q)^{\frac{1}{N}}, w_2(1+q)^{\frac{1}{N}}) = \left(1 - \frac{q}{N(1+q)}\right)^g.$$

Let ∂_i and ∂_t be the partial derivatives with respect to Y_i and t respectively. Define the differential operators $\mathfrak{d}_t = -(Y_1 + Y_2)\partial_t$,

$$\begin{aligned} \Delta^u &:= \sum_{i=0}^{u} \binom{u}{i} (q_1\partial_1)^i (q_2\partial_2)^{u-i} Y_2^i Y_1^{u-i}, \\ (\Delta + \mathfrak{d}_t)^m &:= \sum_{u=0}^{m} \binom{m}{u} \Delta^u \mathfrak{d}_t^{m-u}. \end{aligned}$$

Note that Δ^u defined above is not u^{th} power of the operator Δ.

Theorem 5.7.1. *Let $Q(X_1, X_2)$ be a weighted homogeneous polynomial and m be a positive integer satisfying* $\mathrm{vd} = m + \deg Q$, *where* $\deg Q$ *is the weighted degree. Then*

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}} f_2^m Q(a_1, a_2) = \sum_{w_1, w_2} [q^d](\Delta + \mathfrak{d}_t)^m B(Y_1, Y_2) T_g(t, Y_1, Y_2)\Big|_{t=1, q=q_1=q_2}$$

where the sum is taken over N^{th} roots of unity $\{w_1, w_2\}$ such that $w_1 \neq \pm w_2$, $u = (-1)^{\bar{g}+d}$, $Y_i = w_i(1+q_i)^{1/N}$ and

$$B(Y_1, Y_2) = uQ(Y_1 + Y_2, Y_1 Y_2) \frac{(Y_1 + Y_2)^{d-\bar{g}}}{(Y_1 - Y_2)^{2\bar{g}}} \prod_{i=1}^{2} P'(Y_i)^{\bar{g}}.$$

Proof. Using the same arguments as in the proof of Theorem 5.6.1, we see that the required intersection number equals

$$\sum_{w_1,w_2}\sum_{|\vec{d}|=d}\sum_{k=0}^{m}\binom{m}{k}\int_{\mathrm{F}_{\vec{d},\underline{k}}}\phi_{12}^{k}(d_1Y_2+d_2Y_1)^{m-k}R(Y_1,Y_2)e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}}\prod_{i=1}^{2}e^{\theta_i z_i}h_i^{d_i-\bar{g}},$$

where $z_i=\frac{P'(Y_i)}{P(Y_i)}-\frac{1}{x_i}$ and $h_i=\frac{x_i}{P(Y_i)}$ and

$$R(Y_1,Y_2)=uQ(Y_1+Y_2,Y_1Y_2)\frac{(Y_1+Y_2)^{d-\bar{g}}}{(Y_1-Y_2)^{2\bar{g}}}.$$

We pursue this calculation in Proposition 5.7.3 and Proposition 5.7.4 below.

□

When $m=0$, we recover Theorem 2.3.4. We specialize to the case $m=1$ to obtain a simple expression.

Corollary 5.7.2. *Recall the definition of $T_{d,g}(N)$ from Theorem 2.3.4. Let Q be a homogeneous polynomial such that* $\mathrm{vd}=m+\deg Q$*, where* $\deg Q$ *is the weighted degree. Then*

$$\begin{aligned}\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}}f_2Q(a_1,a_2)=&\frac{2}{N}\sum_{w_1,w_2}\Bigg(T_{d-1,g}(N)D\circ B(w_1,w_2)+\\&\frac{1}{N}\frac{w_1w_2B(w_1,w_2)}{(w_1+w_2)}(T_{d-2,\bar{g}}(N)-NT_{d-1,\bar{g}}(N))\Bigg)\end{aligned}$$

where $D\circ B(z_1,z_2)=\frac{z_1z_2}{2}(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2})B(z_1,z_2)$ and the sum is taken over all the pairs of N^{th} roots of unity $\{w_1,w_2\}$ with $w_1\neq\pm w_2$.

In particular, when $d>g$ we get

$$\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}}f_2Q(a_1,a_2)=\frac{2}{N}\left(1-\frac{1}{N}\right)^{g}\sum_{w_1,w_2}\left(D\circ B(w_1,w_2)-\frac{w_1w_2B(w_1,w_2)}{(w_1+w_2)}\right).$$

Proof. Since B is a homogeneous rational function in variables Y_1 and Y_2 of degree $Nd-1$, substituting $Y_1/w_1 = Y_2/w_2 = (1+q)^{\frac{1}{N}}$ gives a constant multiple of $(1+q)^{d-1/N}$. We use product rule to split the calculation.

First we see that

$$[q^d]T_g(t,Y_1,Y_2)\Delta B(Y_1,Y_2)\bigg|_{q_1=q_2=q} = \frac{2}{N}T_{d-1,g}(N)D\circ B(w_1,w_2), \tag{5.35}$$

since substituting $Y_1/w_1 = Y_2/w_2 = (1+q)^{\frac{1}{N}}$ in $\Delta B(Y_1,Y_2)$ gives us a constant times $q(1+q)^{d-1}$. The rest follows from the definition of D and $T_{d,g}(N)$.

Now we will find $[q^d]B(Y_1,Y_2)(\Delta + \mathfrak{d}_t)T_g(t,Y_1,Y_2)$. Let us define

$$T_g(q) = \left(1 - \frac{q}{N(1+q)}\right)^g$$

for notational convenience. Note that

$$\mathfrak{d}_t T_g(t,Y_1,Y_2) = -(Y_1+Y_2)gT_{g-1}(t,Y_1,Y_2)(-2t\eta_1\eta_2)$$

therefore

$$\mathfrak{d}_t T_g(t,Y_1,Y_2)|_{t=1,q_1=q=q_2} = 2g\frac{w_1w_2}{w_1+w_2}\frac{q^2T_{g-1}(q)}{N^2(1+q)^2}(1+q)^{\frac{1}{N}},$$

hence the the corresponding contribution is

$$[q^d]B(Y_1,Y_2)\mathfrak{d}_t T_g(t,Y_1,Y_2)|_{t=1,q_1=q=q_2} = \frac{2}{N^2}\frac{w_1w_2B(w_1,w_2)}{w_1+w_2}T_{d-2,g-1}(N). \tag{5.36}$$

The other term simplifies as

$$\Delta T_g(1,Y_1,Y_2) = -gT_{g-1}(1,Y_1,Y_2)\big(q_1Y_2(\partial_1\eta_1+\partial_1\eta_2)+q_2Y_1(\partial_2\eta_1+\partial_2\eta_2)\big), \tag{5.37}$$

where we evaluate the partial derivatives

$$\partial_1\eta_1 = \left(\frac{1}{Y_1+Y_2} - \frac{P(Y_1)P''(Y_1)}{P'(Y_i)^2(Y_1+Y_2)} - \frac{P(Y_i)}{P'(Y_1)(Y_1+Y_2)^2}\right)\partial_1 Y_1$$
$$\partial_1\eta_2 = -\frac{P(Y_2)}{P'(Y_2)(Y_1+Y_2)^2}\partial_1 Y_1.$$

Similar expressions hold for $\partial_2\eta_1$ and $\partial_2\eta_2$. Note that we also know that $\partial_i Y_i = \frac{1}{NY_i^{N-1}} = \frac{1}{P'(Y_i)}$. Using this we find the following identities:

$$\left.\frac{q_1Y_2}{(Y_1+Y_2)P'(Y_1)} + \frac{q_2Y_1}{(Y_1+Y_2)P'(Y_2)}\right|_q = \frac{2}{N}\frac{w_1w_2}{(w_1+w_2)}\frac{q(1+q)^{\frac{1}{N}}}{(1+q)}$$
$$\left.\frac{q_1Y_2P(Y_1)P''(Y_1)}{(Y_1+Y_2)P'(Y_1)^3} + \frac{q_2Y_1P(Y_2)P''(Y_2)}{(Y_1+Y_2)P'(Y_1)^3}\right|_q = \frac{2(N-1)}{N^2}\frac{w_1w_2}{(w_1+w_2)}\frac{q^2(1+q)^{\frac{1}{N}}}{(1+q)^2}$$
$$\left.\frac{q_1Y_2P(Y_1)}{(Y_1+Y_2)^2P'(Y_1)^2} + \frac{q_2Y_1P(Y_2)}{(Y_1+Y_2)^2P'(Y_2)^2}\right|_q = \frac{1}{N^2}\frac{w_1w_2}{(w_1+w_2)}\frac{q^2(1+q)^{\frac{1}{N}}}{(1+q)^2}$$
$$\left.\frac{1}{Y_1+Y_2}\left(\frac{q_1Y_2P(Y_2)}{P'(Y_2)P'(Y_1)} + \frac{q_2Y_1P(Y_1)}{P'(Y_1)P'(Y_2)}\right)\right|_q = \frac{1}{N^2}\frac{w_1w_2}{(w_1+w_2)}\frac{q^2(1+q)^{\frac{1}{N}}}{(1+q)^2}.$$

Substituting the above expressions back in (5.37), we obtain

$$\left.\Delta T_g(1,Y_1,Y_2)\right|_{q_1=q_2=q} = gT_{g-1}(q)\frac{w_1w_2}{w_1+w_2}\frac{2}{N}\frac{-q}{(1+q)^2}(1+q)^{\frac{1}{N}}.$$

Therefore

$$[q^d]B(Y_1,Y_2)\Delta T_g(1,Y_1,Y_2)|_{,q_1=q=q_2} = \frac{-2}{N}\frac{w_1w_2B(w_1,w_2)}{w_1+w_2}T_{d-1,g-1}(N). \tag{5.38}$$

We get the required expression by summing (5.35), (5.36) and (5.38). □

The following results are crucially used to obtain Theorem 5.7.1. They are analogue of Proposition 5.5.2 and 5.5.5.

Proposition 5.7.3. *Let R be a homogeneous polynomial with weighted degree $Nd-2\bar{g}(N-$*

$1)-p-u$. Let $R(Y_1,Y_2)$ be a homogeneous rational function of degree $s=Nd-2\bar{g}(N-1)$. We borrow the notation $X_{\vec{d}}$, Y_i, $P(Y)$, $B(Y)$, h_i and z_i from Proposition 5.5.2. Then

$$\begin{aligned}&\int_{X_{\vec{d}}}(d_1Y_2+d_2Y_1)^u R(Y_1,Y_2)\prod_{i=1}^{2}\frac{\theta_i^{p_i}}{p_i!}e^{\theta_i z_i}h_i^{d_i-\bar{g}}\\&=[q_1^{d_1}q_2^{d_2}]\Delta^u\left(R(Y_1,Y_2)\prod_{i=1}^{2}\binom{g}{p_i}\frac{B(Y_i)^{g-p_i}P(Y_i)^{p_i}}{P'(Y_i)}\right)\end{aligned}$$

where $Y_i=w_i(1+q_i)^{\frac{1}{N}}$ as a power series in q_i on the right hand side.

Proof. Let $g(x)=\sum a_d x^d$. The generating functions of the form $f(x)=\sum d^k a_d x^d$ can be evaluated as

$$f(x)=\left(x\frac{\partial}{\partial x}\right)^k g(x).$$

This holds true for multivariate generating functions (by using partial derivatives). Using the proof of Proposition 5.5.2, specifically equation 5.25, we get the required expression. □

Proposition 5.7.4. *The following identity holds*

$$\begin{aligned}&\int_{X_{\vec{d}}}\phi_{12}^k(d_1Y_2+d_2Y_1)^{m-k}R(Y_1,Y_2)e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}}\prod_{i=1}^{2}e^{\theta_i z_i}h_i^{d_i-\bar{g}}\\&=[q_1^{d_1}q_2^{d_2}]\Delta^{m-k}\mathfrak{d}_t^k F_t(Y_1,Y_2)\Big|_{t=1}\end{aligned}$$

where $\eta_i=\frac{P(Y_i)}{B(Y_i)(Y_1+Y_2)}$, $\mathfrak{d}_t=-(Y_1+Y_2)\partial_t$ and

$$F_t(Y_1,Y_2)=R(Y_1,Y_2)\prod_{i=1}^{2}\frac{B(Y_i)^g}{P'(Y_i)}\left(\prod_{i=1}^{2}(1-\eta_i)-\prod_{i=1}^{2}t^2\eta_i\right)^g.$$

Proof. Using Proposition 5.5.1 we may replace even powers of ϕ_{12} with suitable expression in

θ_i's. Therefore we can make the following replacement

$$\phi_{12}^k e^{-\frac{\theta_1+\theta_2-\phi_{12}}{Y_1+Y_2}} \to \sum_{p=0}^{\infty} \frac{(-1)^{p+\ell}}{p!(Y_1+Y_2)^p} \Bigg(\sum_{\substack{\ell+r+s=p \\ \ell\equiv k \mod 2}} \binom{p}{\ell,r,s} \theta_1^r \theta_2^s \phi_{12}^{\ell+k} \Bigg)$$
$$\to \sum_{p=0}^{\infty} \sum_{\substack{\ell+r+s=p \\ \ell\equiv k \mod 2}} \frac{(-1)^{p+k-\frac{\ell+k}{2}}}{p!} \binom{p}{\ell,r,s} \binom{\ell+k}{\frac{\ell+k}{2}} \binom{g}{\frac{\ell+k}{2}}^{-1} \frac{\theta_1^{r+\frac{\ell+k}{2}} \theta_2^{s+\frac{\ell+k}{2}}}{(Y_1+Y_2)^p}$$

We use Proposition 5.7.3 and binomial identities to obtain that the required expression is

$$\sum_{p=0}^{\infty} \sum_{\substack{\ell+r+s=p \\ \ell\equiv k \mod 2}} (-1)^{p+k-\frac{\ell+k}{2}} \binom{p}{\ell,r,s} \frac{(p+k)!}{p!} \binom{p+k}{r+\frac{\ell+k}{2}}^{-1} \binom{\ell+k}{\frac{\ell+k}{2}} \binom{g}{\frac{\ell+k}{2}}^{-1}$$
$$\cdot \binom{g}{r+\frac{\ell+k}{2}} \binom{g}{s+\frac{\ell+k}{2}} [q_1^{d_1} q_2^{d_2}] \Delta^{m-k} \frac{J(Y_1,Y_2)}{(Y_1+Y_2)^p} \left(\frac{Y_1}{h(Y_1)}\right)^{r+\frac{\ell+k}{2}} \left(\frac{Y_2}{h(Y_2)}\right)^{s+\frac{\ell+k}{2}},$$

where $h(Y_i) = Y_i B(Y_i)/P(Y_i)$ and

$$J(Y_1,Y_2) = R(Y_1,Y_2) \prod_{i=1}^{2} \frac{B(Y_i)^g}{P'(Y_i)}.$$

The binomial factor simplifies to give us

$$[q_1^{d_1} q_2^{d_2}] \Delta^u \sum_{p=0}^{\infty} \sum_{\substack{\ell+r+s=p \\ 2|\ell-k}} (-1)^{\frac{\ell+k}{2}} \frac{(k+\ell)!}{\ell!} \binom{g}{\frac{\ell+k}{2}} \binom{g-\frac{\ell+k}{2}}{r} \binom{g-\frac{\ell+k}{2}}{s}$$
$$\cdot \frac{J(Y_1,Y_2)}{(Y_1+Y_2)^p} \left(\frac{-Y_1}{h(Y_1)}\right)^{r+\frac{\ell+k}{2}} \left(\frac{-Y_2}{h(Y_2)}\right)^{s+\frac{\ell+k}{2}}$$

We sum over r and s keeping ℓ fixed after pulling out the terms independent of r,s and ℓ to obtain

$$[q_1^{d_1} q_2^{d_2}] \Delta^{m-k} (-1)^k (Y_1+Y_2)^k J(Y_1,Y_2) \sum_{2|(\ell-k)} \frac{(k+\ell)!}{\ell!} \binom{g}{\frac{\ell+k}{2}} (-1)^{\frac{\ell+k}{2}}$$
$$\cdot \prod_{i=1}^{2} (-\eta_i)^{\frac{\ell+k}{2}} (1-\eta_i)^{g-\frac{\ell+k}{2}}$$

The result follows by noting that

$$\sum_{2|(\ell-k)} \frac{(k+\ell)!}{\ell!} \binom{g}{\frac{\ell+k}{2}} (-1)^{\frac{\ell+k}{2}} \prod_{i=1}^{2} (-\eta_i)^{\frac{\ell+k}{2}} (1-\eta_i)^{g-\frac{\ell+k}{2}} = \partial_t^k \left(\prod_{i=1}^{2} (1-\eta_i) - \prod_{i=1}^{2} t^2 \eta_i \right)^g \Bigg|_{t=1}.$$

□

5.8 Virtual Euler characteristics

The Euler characteristic of the symmetric product of curves is given by the well known formula

$$e(C^{[d]}) = [q^d](1-q)^{2g-2}.$$

Let $\vec{d} = (d_1, \ldots, d_r)$ and $X_{\vec{d}} = C^{[d_1]} \times \cdots \times C^{[d_r]}$. Then the multiplicative property of Euler characteristic implies

$$\sum_{|\vec{d}|=d} e(X_{\vec{d}}) = [q^d](1-q)^{r(2g-2)}.$$

Let IQ_d be the symplectic isotropic Quot scheme with $N = 2n$. The fixed loci under the $\mathbb{C}^*$ action described in Section 5.2.1. The localization formula give us explicit expression for the Euler characteristics:

$$\sum_{d=0}^{\infty} e(\mathsf{IQ}_d) q^d = 2^r \binom{n}{r} (1-q)^{r(2g-2)}.$$

Since the isotropic Quot scheme are not necessarily smooth, the virtual Euler characteristic $e^{\mathrm{vir}}(\mathsf{IQ}_d)$ may not coincide with the topological Euler characteristic. Define the formal power series

$$A_{N,r,g}(q) = \sum_{d=0}^{\infty} e^{\mathrm{vir}}(\mathsf{IQ}_d) q^d.$$

The virtual localization formula gives

$$e^{\mathrm{vir}}(\mathsf{IQ}_d) = \sum_{d_1+d_2=d} \sum_{w_1,w_2} \int_{\mathrm{F}_{\vec{d},\underline{k}}} c(\mathrm{F}_{\vec{d},\underline{k}}) \frac{c_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})}{e_{\mathbb{C}^*}(\mathcal{N}^{\mathrm{vir}})}.$$

We know how to evaluate the above integral (see Section 5.2.4), but the details are computationally challenging. We do not a have a closed form expression or a conjecture for $A_{N,r,g}(q)$.

Over $\mathbb{P}^1$, we find a finite number of values using computers. We used Sagemath [The] for these calculations:

$$\begin{aligned}
A_{4,2,0}(q) =& 4+16q+32q^2+112q^3+(-396)q^4+6800q^5+(-85856)q^6+1122544q^7+ \\
& (-14660608)q^8+192011264q^9+(-2520726176)q^{10}+33164547968q^{11}+\cdots \\
A_{6,2,0}(q) =& 12+48q+96q^2+228q^3-3246q^4+\cdots \\
A_{8,2,0}(q) =& 24+96q+192q^2+464q^3+\cdots
\end{aligned}$$

We observe that $e^{\mathrm{vir}}(\mathsf{IQ}_d)$ differs from the topological Euler characteristic when $d \geq 2$, which indicates that IQ_d is not smooth. When $d = 0,1$, the space IQ_d is always smooth.

Chapter 6

Gromov-Ruan-Witten Invariants

In this section we will compare the sheaf theoretic invariants obtained using isotropic Quot schemes and Gromov-Ruan-Witten invariants for Isotropic Grassmannians. We will denote by $\mathrm{SG}(2,N)$ and $\mathrm{OG}(2,N)$ the symplectic Grassmannian and orthogonal Grassmannian respectively.

6.1 Quantum Cohomology

The small quantum cohomology of the Isotropic Grassmannian and its presentation are known (see [BKT], [Tam]). However, the explicit expressions for the high genus and large degree Gromov-Ruan-Witten invariants require further arguments.

When the rank $r = 2$, a simpler presentation for the quantum cohomology of $\mathrm{SG}(2,2n)$ was obtained in [CMMPS]. We will briefly describe their result and find a similar presentation for the quantum cohomology of $\mathrm{OG}(2,2n+2)$.

Let $N = 2n$. We have the universal exact sequence $0 \to \mathcal{S} \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0$ over $\mathrm{SG}(2,N)$. Let $\mathcal{S}^\perp \subset \mathbb{C}^N \otimes \mathcal{O}$ be the rank $N-2$ vector bundle consisting of vectors perpendicular to $\mathcal{S}$.

Moreover, $\mathcal{S}^\perp$ is the kernel of the composition $\mathbb{C}^N \otimes \mathcal{O} \xrightarrow{\sigma} (\mathbb{C}^N)^\vee \otimes \mathcal{O} \to \mathcal{S}^\vee$ which gives us an identity for the Chern polynomial $c_t(\mathcal{S}^\vee)c_t(\mathcal{S}^\perp) = 1$. This implies

$$c_t(\mathcal{S})c_t(\mathcal{S}^\vee)c_t(\mathcal{S}^\perp/\mathcal{S}) = 1. \tag{6.1}$$

The above identity suggests us to define the following cohomology classes :

- The Chern classes $a_i = c_i(\mathcal{S}^\vee)$ for $i \in \{1,2\}$.
- Let $b_i = c_{2i}(\mathcal{S}^\perp/\mathcal{S})$ for $i \in \{1,\dots,n-2\}$. The bundle $\mathcal{S}^\perp/\mathcal{S}$ is self dual, hence all the odd Chern classes vanish.

The cohomology ring $H^*(\mathrm{SG}(2,2n))$ is isomorphic to the quotient of the ring $\mathbb{C}[a_1,a_2,b_1,\dots,b_{n-2}]$ by the ideal generated by

$$(1+(2a_2-a_1^2)x^2+a_2x^4)(1+b_1x^2+\cdots+b_{n-2}x^{2n-4}) = 1. \tag{6.2}$$

The above identity is simply a restatement of (6.1). The quantum cohomology ring is $H^*(\mathrm{SG}(2,2n))\otimes \mathbb{C}[[q]]$, where the quantum products is described in the following theorem. Note that $\deg(q) = 2n-1$ is the index of $\mathrm{SG}(2,2n)$.

Theorem 6.1.1 ([CMMPS])**.** *The quantum cohomology ring $QH^*(\mathrm{SG}(2,2n))$ is isomorphic to the quotient of the ring $\mathbb{C}[a_1,a_2,b_1,\dots,b_{n-2},q]$ by the ideal generated by*

$$(1+(2a_2-a_1^2)x^2+a_2x^4)(1+b_1x^2+\cdots+b_{n-2}x^{2n-4}) = 1+qa_1x^{2n} \tag{6.3}$$

The detailed proof of the above result can be found in [CMMPS]. Now we will describe a similar presentation for the orthogonal Grassmannian $\mathrm{OG}(2,N)$, where $N = 2n+2$. We will assume $n \geq 3$, otherwise $H^2(\mathrm{OG}(2,N),\mathbb{C})$ may have rank greater than one.

We have the universal exact sequence $0 \to \mathcal{S} \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0$ over $\mathrm{OG}(2,N)$. Let $\mathcal{S}^\perp \subset \mathbb{C}^N \otimes \mathcal{O}$ be the rank $N-2$ vector bundle consisting of vectors perpendicular to $\mathcal{S}$.

Unlike the symplectic case, there is a cohomology class which is not obtained using the universal exact sequence. Let $\mathrm{Q} \subset \mathbb{P}(\mathbb{C}^N)$ be the quadric of isotropic lines in $\mathbb{C}^N$ equipped with a non-degenerate symmetric bilinear form σ. Let $\pi : \mathbb{P}(\mathcal{S}) \to \mathrm{OG}(2,N)$ be the projective bundle. We have the natural the map $\theta : \mathbb{P}(\mathcal{S}) \to \mathrm{Q}$.

Note that $O(2n+2)$ acts on $\mathbb{C}^{2n+2}$. There are precisely two $SO(2n+2)$ orbits of maximal isotropic subspaces. Two maximal isotropic subspaces E and F lie in different orbits if and only if $\dim E \cap F$ is even. Let e and f be the cohomology classes corresponding to $\mathbb{P}(E)$ and $\mathbb{P}(F)$ inside the quadric $\mathrm{Q} \subset \mathbb{P}(\mathbb{C}^N)$. The classes e and f corresponds to two rulings of Q.

The cohomology ring of Q is generated by the hyper plane class h and ruling classes e and f (see [EG]).

Over $\mathrm{OG}(2,N)$, we have the following cohomology classes :

- The Chern classes $a_i = c_i(\mathcal{S}^\vee)$ for $i \in \{1,2\}$.
- Let $b_i = c_{2i}(\mathcal{S}^\perp/\mathcal{S})$ for $i \in \{1,\dots,n-1\}$. The bundle $\mathcal{S}^\perp/\mathcal{S}$ is self dual, hence all the odd Chern classes vanish.
- Let $\pi : \mathbb{P}(\mathcal{S}) \to \mathrm{OG}$ be the projection, then we define

$$\xi = \pi_* \theta^*(e-f).$$

The above classes still satisfy the identity (6.1), but two new identities involving ξ are required. We will briefly describe these for readers convenience.

Lemma 6.1.2. *The cohomology class ξ satisfy $\xi a_2 = 0$ and $\xi^2 = (-1)^{n-1} b_{n-1}$.*

Proof. Let $h = c_1(\mathcal{O}(1))$ on $\mathbb{P}(S)$, then $h\theta^*(e-f) = 0$. Multiplying $\theta^*(e-f)$ to the identity

$$h^2 - hc_1(\pi^* \mathcal{S}^\vee) + c_2(\pi^* \mathcal{S}^\vee) = 0,$$

we obtain $\theta^*(e-f)\pi^* a_2 = 0$. The projection formula implies $\xi a_2 = 0$.

Using the identities $c_t(\mathcal{S})c_t(\mathcal{S}^\vee)c_t(\mathcal{S}^\perp/\mathcal{S}) = 1$ and $c_t(\mathcal{S})c_t(\mathcal{Q}) = 1$, we obtain $c_t(\mathcal{S}^\perp/\mathcal{S}) = c_t(\mathcal{Q})c_{-t}(\mathcal{Q})$. In particular, for all $1 \leq k \leq n-1$

$$(-1)^k b_k = c_k(\mathcal{Q})^2 + 2\sum_{i=1}^{k}(-1)^i c_{k+i}(\mathcal{Q})c_{k-i}(\mathcal{Q}).$$

When $k = n-1$, the right side of the above equality is ξ^2 by [BKT].

$\square$

Remark 6.1.3. *The class ξ is the Edidin-Graham characteristic square root class for the quadratic bundle $\mathcal{S}^\perp/\mathcal{S}$.*

Proposition 6.1.4. *The cohomology ring $H^*(\mathrm{OG}(2,2n+2))$ is isomorphic to the quotient of the ring $\mathbb{C}[a_1,a_2,b_1,\ldots,b_{n-2},\xi]$ by the ideal generated by the relations $\xi a_2 = 0$ and*

$$(1+(2a_2-a_1^2)x^2+a_2^2x^4)(1+b_1x^2+\cdots+b_{n-2}x^{2n-4}+(-1)^{n-1}\xi^2x^{2n-2}) = 1.$$

Proof. Note that the topological Euler characteristic of OG is the vector space dimension of $H^*(\mathrm{OG})$ and is given by $2^2\binom{n+1}{2}$. This is obtained by counting the number of fixed points under $\mathbb{C}^*$ action on OG.

We can unpack the relations to obtain the generators of the ideal:

$$\begin{aligned}
f_0 &= \xi a_2 \\
f_1 &= b_1 + (2a_2 - a_1^2) \\
&\vdots \\
f_{n-1} &= (-1)^{n-1}\xi^2 + b_{n-2}(2a_2 - a_1^2) + b_{n-3}a_2^2 \\
f_n &= (-1)^{n-1}\xi^2(2a_2 - a_1^2) + b_{n-2}a_2^2
\end{aligned} \tag{6.4}$$

Define $R' = \mathbb{C}[a_1,a_2,b_1,\ldots,b_{n-2},\xi]/\langle f_0,\ldots,f_n\rangle$.

Using Lemma 6.1.2 and $c_t(\mathcal{S})c_t(\mathcal{S}^\vee)c_t(\mathcal{S}^\perp/\mathcal{S}) = 1$, we know that $f_i = 0$ for all $0 \leq i \leq n$

in $H^*(\mathrm{OG})$. Moreover, the classes a_1, a_2 and ξ generates $H^*(\mathrm{OG})$ (see [BKT]). Therefore we get the surjective ring homomorphism

$$R' \to H^*(\mathrm{OG}).$$

It is enough to show that R' is a vector space of dimension at most $2^2\binom{n+1}{2}$. We bound the dimension of R' using the exact sequence

$$0 \to \langle \xi \rangle \to R' \to R'/\langle \xi \rangle \to 0.$$

Using (6.2), we observe that $R'/\langle \xi \rangle = H^*(\mathrm{SG}(2,2n))$. Thus $R'/\langle \xi \rangle$ has dimension $2n^2 - 2n$, which is the Euler characteristic of $\mathrm{SG}(2,2n)$.

Note that $b_i \in a_1^{2i} + \langle a_2 \rangle$, $\xi^2 \in a_1^{2n-2} + \langle a_2 \rangle$ and $\xi^2 a_1^2 \in \langle a_2 \rangle$. Hence $\dim R'/\langle a_2 \rangle \leq |\{1, a_1 \ldots, a_1^{2n-1}, \xi, \ldots \xi a_1^{2n-1}\}| = 4n$. Consider the exact sequence

$$0 \to \ker \to R' \xrightarrow{\cdot a_2} R' \to R'/\langle a_2 \rangle \to 0.$$

Note that $\langle \xi \rangle \subset \ker$, thus

$$\dim \langle \xi \rangle \leq \dim \ker = \dim R'/\langle a_2 \rangle \leq 4n.$$

$\square$

Now we will turn our attention to the small quantum cohomology.

Proposition 6.1.5. *Let $n > 2$. The small quantum cohomology ring $QH^*(\mathrm{OG}(2,2n+2))$ is isomorphic to the quotient of the ring $\mathbb{C}[a_1, a_2, b_1, \ldots, b_{n-2}, \xi, q]$ by the ideal generated by the relations $\xi a_2 = 0$ and*

$$(1 + (2a_2 - a_1^2)x^2 + a_2^2 x^4)(1 + \cdots + b_{n-2}x^{2n-4} + (-1)^{n-1}\xi^2 x^{2n-2}) = 1 + 4qa_1 x^{2n}. \qquad (6.5)$$

Proof. The degrees of the relations in the given presentation of $H^*(\mathrm{OG})$ are

$$\deg f_i = \begin{cases} n+1 & i=0 \\ 2i & 1 \leq i \leq n. \end{cases}$$

Since q has degree $2n-1$, the quantum term can appear only in degree $2n$ in the above presentation of the cohomology. Therefore,

$$(-1)^{n-1}\xi^2(2a_2 - a_1^2) + b_{n-2}a_2^2 = cqa_1$$

for some constant c. Recall that $(-1)^{n-1}\xi^2 = b_{n-1} = c_{2n-2}(\mathcal{S}^\perp/\mathcal{S})$. The first term $\xi^2 a_2 = 0$ since $\xi a_2 = 0$. Note that we have the following Schubert classes

$$b_{n-1}a_1 = c_{2n-1}(\mathcal{Q})$$
$$b_{n-2}a_2 + b_{n-1} = c_{2n-2}(\mathcal{Q}).$$

It is enough to show that the three point GRW invariants

$$\Phi_{0,1}(a_1, c_{2n-1}(\mathcal{Q}), a_1^*) = 2, \qquad \Phi_{0,1}(a_2, c_{2n-2}(\mathcal{Q}), a_1^*) = 2,$$

where a_1^* corresponds to the class of a line. It follows by carefully applying the quantum Pieri rule stated in [BKT], which describes the three term genus zero GWR invariants (equivalently the quantum product) of the Schubert classes. $\square$

6.2 Jacobian Calculation

We can unpack (6.3) to write that the ideal of relations is generated by

$$\begin{aligned}
\tilde{f}_1 =& b_1 + (2a_2 - a_1^2) \\
\tilde{f}_2 =& b_2 + b_1(2a_2 - a_1^2) + a_2^2 \\
&\vdots \\
\tilde{f}_{n-2} =& b_{n-2} + b_{n-3}(2a_2 - a_1^2) + b_{n-4}a_2^2 \\
\tilde{f}_{n-1} =& b_{n-2}(2a_2 - a_1^2) + b_{n-3}a_2^2 \\
\tilde{f}_n =& b_{n-2}a_2^2 - qa_1.
\end{aligned} \tag{6.6}$$

Let $R = \mathbb{C}[a_1, a_2, b_1, \ldots, b_{n-2}, q]/\langle \tilde{f}_1, \ldots, \tilde{f}_n \rangle$ be the quantum cohomology ring of $\mathrm{SG}(2,2n)$ over $\mathbb{C}[q]$.

In order to calculate the Gromov-Ruan-Witten invariants, we are required to compute the Jacobian

$$J = \det \begin{bmatrix} \frac{\partial \tilde{f}_1}{\partial a_1} & \cdots & \frac{\partial \tilde{f}_n}{\partial a_1} \\ \vdots & & \vdots \\ \frac{\partial \tilde{f}_1}{\partial b_{n-2}} & \cdots & \frac{\partial \tilde{f}_n}{\partial b_{n-2}} \end{bmatrix}$$

at the vanishing locus of $(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$. Substituting $b_1 = (a_1^2 - 2a_2)$, this determinant equals

$$-4a_1 \det \begin{bmatrix}
1 & b_1 & b_2 & b_3 & \ldots & b_{n-2} & \frac{q}{2a_1} \\
1 & (a_2+b_1) & (a_2b_1+b_2) & (a_2b_2+b_3) & \ldots & (a_2b_{n-3}+b_{n-2}) & a_2b_{n-2} \\
1 & -b_1 & a_2^2 & 0 & \ldots & 0 & 0 \\
0 & 1 & -b_1 & a_2^2 & \ldots & 0 & 0 \\
0 & 0 & 1 & -b_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & & & \\
0 & 0 & 0 & \ldots & 1 & -b_1 & a_2^2
\end{bmatrix}.$$

After subtracting first two rows, we observe that the above equals

$$-4a_1a_2 \det \begin{bmatrix} 1 & b_1 & b_2 & b_3 & \dots & b_{n-2} & \frac{q}{2a_1} \\ 0 & 1 & b_1 & b_2 & \dots & b_{n-3} & b_{n-2} - \frac{q}{2a_1a_2} \\ 1 & -b_1 & a_2^2 & 0 & \dots & 0 & 0 \\ 0 & 1 & -b_1 & a_2^2 & \dots & 0 & 0 \\ 0 & 0 & 1 & -b_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 & -b_1 & a_2^2 \end{bmatrix}.$$

Let $v_0, v_1, \dots, v_{n-1}$ be the column vectors in the above matrix. Then the

$$\det[v_0, \dots, v_{n-1}] = \det[V_0, \dots V_{n-1}]$$

where $V_i = v_ib_0 + v_{i-1}b_1 + \cdots + v_0b_i$. Using the identity, $a_2^2b_{i-2} - b_1b_{i-1} + b_i = 0$, we observe that

$$[V_0, \dots, V_{n-1}] = \begin{bmatrix} 1 & B_1 & B_2 & B_3 & \dots & B_{n-2} & B_{n-1} + \frac{q}{2a_1} \\ 0 & 1 & B_1 & B_2 & \dots & B_{n-3} & B_{n-2} - \frac{q}{2a_1a_2} \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \end{bmatrix}$$

where $b_{n-1} := 0$ and $B_i := b_ib_0 + b_1b_{i-1} + b_2b_{i-2} + \cdots + b_0b_i$. Therefore the required Jacobian is given by

$$J = -4a_1a_2 \det \begin{bmatrix} B_{n-2} & B_{n-1} + \frac{q}{2a_1} \\ B_{n-3} & B_{n-2} - \frac{q}{2a_1a_2}. \end{bmatrix}. \tag{6.7}$$

6.3 Residues

We will use the presentation of the quantum cohomology in (6.3) and (6.5) to obtain the higher genus GRW invariants for $\mathrm{SG}(2,2n)$ and $\mathrm{OG}(2,2n+2)$ using the techniques in [ST]. We will briefly describe the result we require from [ST].

Let $F \in \mathbb{C}[x_1,\dots,x_n]$ be a polynomial, and $f = (f_1,\dots,f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a tuple of polynomials such that $f^{-1}(0)$ is finite. For any $p \in f^{-1}(0)$, we define

$$Res_f(p;F) := \frac{1}{(2\pi i)^n} \int_{\Gamma_p^\varepsilon} \frac{F}{f_1 \cdots f_n} dx_1 \dots dx_n$$

with $\Gamma_p^\varepsilon = \{q \in U(p) : |f(q)| = \varepsilon\}$, $U(p)$ small neighborhood of a with $f^{-1}(0) \cap U(p) = \{p\}$ and Γ_p^ε relatively compact in $U(p)$. We may further define

$$Res_f(F) = \sum_{p \in f^{-1}(0)} Res_f(p;F).$$

Note that when p is a regular point, i.e. the Jacobian $J = \det\left(\partial f_i/\partial x_j\right) \neq 0$ at p, then

$$Res_f(p;F) = \left(\frac{F}{J}\right)(p).$$

Let M be a Fano manifold with $h^2(M,\mathbb{C}) = 1$ and the cohomology ring $H^*(M,\mathbb{C}) = \mathbb{C}[x_1,\dots,x_n]/\langle f_1,\dots,f_n\rangle$, where each x_i corresponds to a pure dimensional cohomology class. Let

$$QH^*(M,\mathbb{C}) = \mathbb{C}[x_1,\dots,x_n,q]/\langle \tilde{f}_1,\dots,\tilde{f}_n\rangle$$

be the quantum cohomology as an algebra over $\mathbb{C}[q]$.

Substitute q for a complex number, and let $\tilde{f}^q = (\tilde{f}_1^q, \ldots, \tilde{f}_n^q)$ be the corresponding tuple of polynomials in $x_1, \ldots, x_n$. Let $R_q = QH_q^*(M, \mathbb{C})$ be the corresponding quantum cohomology ring. Note that R_q and $H^*(M, \mathbb{C})$ are isomorphic as vector spaces. The ring R_q is equipped with a quantum multiplication that matches the usual multiplication of cohomology classes when $q = 0$.

Theorem 6.3.1. *[ST] Let M and $\tilde{f}^q$ be defined as above. Let $F \in \mathbb{C}[x_1, \ldots, x_n]$ be a weighted homogeneous polynomial satisfying the dimension condition* (2.5) *for a natural number d. Then*

$$\langle F \rangle_g q^d = c^{\bar{g}} Res_{\tilde{f}^q}(J_q^g F) = \lim_{y \to 0} \sum_{x \in (\tilde{f}^q)^{-1}(y)} ((cJ_q)^{\bar{g}} F)(x)$$

where the limit is taken over regular points y, c is a constant and $J_q = \det\left(\partial \tilde{f}_i^q / \partial x_j\right)$ is the Jacobian.

6.4 GRW invariants for $\mathrm{SG}(2, 2n)$

We will the apply Theorem 6.3.1 to the presentation of the quantum cohomology $R = QH^*(\mathrm{SG}(2, 2n))$ in (6.3). To be precise, let $(x_1, x_2, x_3, \ldots, x_n) = (a_1, a_2, b_1, \ldots, b_{n-2})$ and let $\tilde{f}$ defined by (6.6).

Fix $q = -1$ (or any non-zero number). Equation (6.3) can be rephrased as

$$(z^2 - z_1^2)(z^2 - z_2^2)Q(z) = z^{2n} + q(z_1 + z_2)$$

where $a_1 = z_1 + z_2$, $a_2 = z_1 z_2$ and $Q(z) = z^{2n-4} + b_1 z^{2n-6} + \cdots + b_{n-2}$. Observe that b_i can be represented in terms of a_1 and a_2 for all $1 \leq i \leq n-2$.

Evaluating at z_1 and z_2, we obtain

$$z_1^{2n} = -q(z_1 + z_2)$$
$$z_2^{2n} = -q(z_1 + z_2).$$

The structure of R_q is described in [CMMPS]. The set $(\tilde{f}^q)^{-1}(0)$ has two types of points:

- Reduced points: The points described by the unordered pair $\{z_1, z_2\}$ satisfying

$$z_2 = \zeta z_1 \tag{6.8}$$
$$z_1 = \omega(1+\zeta)^{\frac{1}{2n-1}},$$

  where $\omega^{2n-1} = -q$, $\zeta^{2n} = 1$ and $\zeta \neq \pm 1$. Since $\{z_1, z_2\}$ is an unordered, (ω, ζ) and (ω, ζ^{-1}) yields the same point. Thus there are $(n-1)(2n-1)$ such points. The non-vanishing of the Jacobian computed below implies that these points are reduced.

- Fat point : The origin is the only other point in $(\tilde{f}^q)^{-1}(0)$. Since the vector space dimension $\dim(R_q) = 2n(n-1)$, the origin is a non-reduced point of order $(n-1)$ in $\mathrm{Spec}(R_q)$.

Thus $R_q = A_1 \times A_2$ where $A_1 \cong \mathbb{C}[\varepsilon]/\langle \varepsilon^{n-1} \rangle$ corresponds to the fat point at origin in $Spec(R_q)$ and $\mathrm{Spec}(A_2)$ consists of $(n-1)(2n-1)$ distinct reduced points.

Proposition 6.4.1. *Let $p \in A_2$ be a reduced point described using* (6.8)*. The Jacobian at p is*

$$J_q(p) = 2n(2n-1)\zeta^{-1}(1+\zeta)^{-1}(1-\zeta)^{-2} z_1^{4n-5}. \tag{6.9}$$

Proof. We recursively calculate a concise expression for $b_1, \ldots, b_{n-2}$:

$$b_i = z_1^{2i}(1 + \zeta^2 + \cdots + \zeta^{2i}).$$

We define b_i for all $i \in \mathbb{N}$ using the above identity. Note that $b_{n-1} = 0$ and $b_0 = 1$.

We are now going to give a simple formula for the convolution products B_i, and use it to find the Jacobian.

Let $t = z_1^2$. Let $P(x) = 1 + b_1 x + b_2 x^2 + \cdots$ be the power series in x. Then

$$\begin{aligned}(1-\zeta^2)P(x) &= \sum_{i=0}^{\infty}(1-\zeta^{2i+2})(tx)^i \\ &= \frac{1}{1-tx} - \frac{\zeta^2}{1-\zeta^2 tx}.\end{aligned}$$

Observe that $P(x)^2 = 1 + B_1 x + B_2 x^2 + \cdots$, which can be expressed as

$$P(x)^2 = \frac{1}{(1-\zeta^2)^2}\left(\frac{1}{(1-tx)^2} + \frac{\zeta^4}{(1-\zeta^2 tx)^2} - \frac{2\zeta^2}{1-\zeta^2}\left(\frac{1}{1-tx} - \frac{\zeta^2}{1-\zeta^2 tx}\right)\right).$$

Extracting the coefficient of x^i in the above expression gives

$$\begin{aligned}B_i &= \frac{1}{(1-\zeta^2)^2}\left((i+1)t^i + (i+1)\zeta^{2i+4}t^i - \frac{2\zeta^2}{1-\zeta^2}(t^i - \zeta^{2i+2}t^i)\right) \\ &= \left(\frac{(i+1)(1+\zeta^{2i+4})}{(1-\zeta^2)^2} - \frac{2\zeta^2(1-\zeta^{2i+2})}{(1-\zeta^2)^3}\right)t^i.\end{aligned}$$

In particular, we have

$$\begin{aligned}B_{n-1} &= n\frac{1+\zeta^2}{(1-\zeta^2)^2}t^{n-1}, \quad B_{n-2} = \frac{2n}{(1-\zeta^2)^2}t^{n-2}, \\ B_{n-3} &= \frac{n(1+\zeta^2)}{\zeta^2(1-\zeta^2)^2}t^{n-3}.\end{aligned}$$

Substituting $q = b_{n-2}a_2^2/a_1$ and using $a_1^2 = t(1+\zeta)^2$, $b_{n-2} = -t^{n-2}/\zeta^2$ and $a_2 = t\zeta$ we get the expression for Jacobian for $\tilde{f}^q = (\tilde{f}_1^q, \tilde{f}_2^q, \dots, \tilde{f}_n^q)$ at p:

$$
\begin{aligned}
J_q(p) &= -4a_1a_2\left(\det\begin{bmatrix} B_{n-2} & B_{n-1} \\ B_{n-3} & B_{n-2}\end{bmatrix} + \det\begin{bmatrix} B_{n-2} & \frac{b_{n-2}a_2^2}{2a_1^2} \\ B_{n-3} & -\frac{b_{n-2}a_2}{2a_1^2}\end{bmatrix}\right) \\
&= -4a_1a_2\left(-\frac{n^2}{\zeta^2(1-\zeta^2)^2} + \frac{n}{2\zeta^2(1-\zeta^2)^2}\right)t^{2n-4} \\
&= 2n(2n-1)\zeta^{-1}(1+\zeta)^{-1}(1-\zeta)^{-2}z_1^{4n-5}.
\end{aligned}
$$

□

Proposition 6.4.2. *Let* $\mathrm{vd} = (2n-1)d - \bar{g}(4n-5)$ *and* $F = a_1^{m_1}a_2^{m_2}$ *such that* $m_1 + 2m_2 = \mathrm{vd}$*, then*

$$
\sum_{p\in A_2} Res_{\tilde{f}^q}(p; J_q^g F) = \frac{2n-1}{2}\sum_{\zeta\neq\pm1}(1+\zeta)^{m_1}\zeta^{m_2}J(\zeta)^{\bar{g}}(1+\zeta)^d(-q)^d \tag{6.10}
$$

where $\zeta \neq \pm1$ *is an* $2n^{th}$ *root of unity and* $J(\zeta) := 2n(2n-1)\zeta^{-1}(1+\zeta)^{-1}(1-\zeta)^{-2}$.

Proof. Let p be given by (ω,ζ). Using Proposition 6.4.1

$$
\begin{aligned}
Res_{\tilde{f}^q}(p; J_q^g F) &= (J_q^{g-1}F)(p) \\
&= J(\zeta)^{\bar{g}}(1+\zeta)^{m_1}\zeta^{m_2}z_1^{\mathrm{vd}+\bar{g}(4n-5)}.
\end{aligned}
$$

Observe that $z_1^{\mathrm{vd}+\bar{g}(4n-5)} = (1+\zeta)^d(-q)^d$, thus

$$
\sum_{p\in A_2}\mathrm{Res}_{\tilde{f}^q}(p; J_q^g F) = \sum_{(\omega,\zeta)}(1+\zeta)^{m_1}\zeta^{m_2}J(\zeta)^{\bar{g}}(1+\zeta)^d(-q)^d
$$

where the latter is summed over pairs (ω,ζ) such that $\omega^{2n-1} = (-q)$ and ζ is a $2n^{\mathrm{th}}$ root of unity with strictly positive imaginary part. The above expression does not depend on the choice of ω and it is invariant under $\zeta \to \zeta^{-1}$. When summed over these choices the required formula is obtained. $\square$

Theorem 6.4.3. *Let $m_1+2m_2=\mathrm{vd}=(2n-1)d-(4n-5)\bar{g}$. The GRW invariants for $\mathrm{SG}(2,2n)$ equal the top virtual intersections of the a-classes on the corresponding isotropic Quot scheme:*

$$\langle a_1^{m_1}a_2^{m_2}\rangle_g=\int_{[\mathsf{IQ}_d]^{\mathrm{vir}}}a_1^{m_1}a_2^{m_2} \tag{6.11}$$

Proof. The origin $y=0:=(0,\dots,0)$ is not necessarily a regular point for the function $\tilde{f}^q=(\tilde{f}_1^q,\dots,\tilde{f}_n^q)$. We will evaluate the limit

$$\lim_{y\to 0}\sum_{p\in(\tilde{f}^q)^{-1}(y)}(J^{\bar{g}}F)(p), \tag{6.12}$$

where the limit $y\to 0$ is taken over regular values of y. Let ε be a non-zero complex number with small absolute value, and let $y_\varepsilon=(0,\dots,0,\varepsilon^{n-1},0)$. We will see that y_ε is regular for ε small enough.

Reduced points : Since the Jacobian for each point $p\in A_2$ is non-zero, the inverse function theorem implies that for small enough ε, there is exactly one reduced point p_ε near p satisfying $f(p_\varepsilon)=y_\varepsilon$. Thus y_ε is a regular value for all ε in a neighborhood of 0.

Let A_2^ε be the set of unique points p_ε near $p\in A_2$. Observe that the residue contribution is

$$\lim_{\varepsilon\to 0}\sum_{p_\varepsilon\in A_2^\varepsilon}(J^{\bar{g}}F)(p_\varepsilon)=\sum_{p\in A_2}\mathrm{Res}_f(p;J^gF). \tag{6.13}$$

This has been calculated in Proposition 6.4.2.

Fat point : The vanishing of $\tilde{f}_1^q,\dots,\tilde{f}_{n-2}^q$ implies that $b_1,\dots,b_{n-2}$ is a polynomial in a_1 and a_2. Observe that

$$b_i=(-1)^i(i+1)a_2^i+\langle a_1^2\rangle.$$

Since $q \neq 0$, the vanishing of $\tilde{f}_n^q$ implies

$$a_1 = q^{-1} a_2^n + \langle a_1^2 \rangle.$$

Therefore $a_1 = a_2^n h_1(a_2)$ for some power series h that defines a holomorphic function for an open set containing 0. A similar argument shows that $f_{n-1} = a_2^{n-1} h_2(a_2)$ where h_2 is holomorphic with non-zero constant term. Observe that $a_2^{n-1} h_2(a_2) = \varepsilon \neq 0$ has exactly $(n-1)$ simple zeros for all ε lying in a neighborhood of 0.

Note that $a_2 = O(\varepsilon)$, $a_1 = O(\varepsilon^n)$ and $b_i = O(\varepsilon^i)$ as ε approaches 0. Substituting the above orders in (6.7), we get $J = O(\varepsilon^{n-2})$. Thus the residue contributions of these $n-1$ points has order $O(\varepsilon^{nm_1+m_2+\bar{g}(n-2)})$, which vanishes in the limit $\varepsilon \to 0$ when the the exponent $nm_1 + m_2 + \bar{g}(n-2)$ is non-zero.

There are exactly two cases when the above exponent is zero: (i) $\mathrm{vd} = 0$, $d = g-1$, $N = 2n = 4$; and (ii) $\mathrm{vd} = d = 0$, $g = 1$. An easy calculation shows that the residue contribution are $(2q)^d$ and 1 respectively. These are the only instances where $\mathrm{vd} \geq 0$ and $d < g$.

We apply Theorem 6.3.1 to obtain the GRW invariant up to a constant c. When $g = d = 0$, the GRW invariants are the top intersections in the cohomology ring of $\mathrm{SG}(2,2n)$. Note that $\mathsf{IQ}_0 \cong \mathrm{SG}(2,2n)$ when $g = 0$, thus the virtual invariants in (2.4) must match the GRW invariants. Comparing the two we obtain $c = -1$.

Putting together all the terms, we get

$$\langle a_1^{m_1} a_2^{m_2} \rangle_g = \begin{cases} (-1)^{d+\bar{g}} \frac{2n-1}{2} \sum_\zeta (1+\zeta)^{m_1+d} \zeta^{m_2} J(\zeta)^{\bar{g}} & d \geq g \\ 2^{\bar{g}} 3^g + (-1)^{\bar{g}} 2^d & n = 2,\ d = \bar{g} \\ 2n(n-1) & g = 1, d = 0 \end{cases} \cdot$$

This match the expression in Theorem 2.3.4 (also see Examples 2.3.5 and 5.6.2) for all d, g and N. □

6.5 GRW invariants for $\mathrm{OG}(2,2n+2)$

Let $n \geq 3$. Recall the definition of $f_0, f_1, \ldots, f_n$ from (6.4). Let $\tilde{f}_i = f_i$ for $0 \leq i \leq n-1$ and let $\tilde{f}_n = f_n - 4qa_1$ as prescribed by (6.4). In particular,

$$
\begin{aligned}
\tilde{f}_0^q &= \xi a_2 \\
\tilde{f}_1^q &= b_1 + (2a_2 - a_1^2) \\
&\vdots \\
\tilde{f}_{n-1}^q &= (-1)^{n-1}\xi^2 + b_{n-2}(2a_2 - a_1^2) + b_{n-3}a_2^2 \\
\tilde{f}_n^q &= (-1)^{n-1}\xi^2(2a_2 - a_1^2) + b_{n-2}a_2^2 - 4qa_1
\end{aligned}
$$

Let $R' = \mathbb{C}[\xi, a_1, a_2, b_1, \ldots, b_{n-2}, q]/\langle \tilde{f}_0, \ldots, \tilde{f}_n \rangle$ be the presentation for the quantum cohomology of $\mathrm{OG}(2,2n+2)$ (see (6.5)). The Jacobian J' for $\tilde{f} = (\tilde{f}_0, \ldots, \tilde{f}_n)$ is calculated in similar fashion as it was done in the symplectic case. Observe that

$$
J' \in -4a_1a_2^2 \det \begin{bmatrix} B_{n-2} & B_{n-1} + \frac{4q}{2a_1} \\ B_{n-3} & B_{n-2} - \frac{4q}{2a_1a_2} \end{bmatrix} + \langle \xi \rangle, \tag{6.14}
$$

where $b_0 = 1$, $b_{n-1} := (-1)^{n-1}\xi^2$ and $B_i = b_ib_0 + \cdots + b_0b_i$.

Note that modulo $\langle a_2 \rangle$, we have

$$
\begin{aligned}
\tilde{f}_0 &= 0 \\
\tilde{f}_1 &= b_1 - a_1^2 \\
&\vdots \\
\tilde{f}_{n-1} &= (-1)^{n-1}\xi^2 - b_{n-2}a_1^2 \\
\tilde{f}_n &= (-1)^{n-1}\xi^2(-a_1^2) - 4qa_1
\end{aligned}
$$

An easy calculation shows that

$$J' \in -2b_{n-1}(2a_1B_{n-1}+4q)+\langle a_2\rangle.$$

Note that $b_i \in a_1^{2i}+\langle a_2\rangle$, thus we may further write

$$J' \in -2a_1^{2n-2}(2na_1^{2n-1}+4q)+\langle a_2\rangle. \tag{6.15}$$

Fix a non-zero number q. Note that $f_0 = 0$ implies that either $\xi = 0$ or $a_2 = 0$. The set $(\tilde{f}^q)^{-1}(0)$ has three types of points:

- Reduced points ($a_2 \neq 0$): The reduced points with $\xi = 0$ have almost the same description as that of $Spec(A_2)$ in the symplectic case. It is obtained by replacing $q \to 4q$ and letting a_1 and a_2 be described (similar to (6.8)) using Chern roots $\{z_1, z_2\}$ in this case.
- Reduced points ($\xi \neq 0$): Thus $a_2 = 0$ and hence $b_i = a_1^{2i}$. Moreover, $\tilde{f}^q_{n-1} = \tilde{f}^q_n = 0$ implies

$$\begin{aligned}(-1)^{n-1}\xi^2 &= a_1^{2n-2}\\ a_1^{2n} &= -4qa_1.\end{aligned}$$

  Thus there are $(4n-2)$ points given by $(\xi, a_1) = (\sqrt{-4q}\mu^{-1}, \mu^2)$ where μ is a $(4n-2)^{\text{th}}$ root of $(-4q)$. We observe that the Jacobian (see (6.15)) is non-zero.
- Fat point A_1: The origin is the non-reduced point of order $(n+1)$.

The Artinian ring R'_q is isomorphic to $A_1 \times A_2 \times A_3$ where $A_1 \cong \mathbb{C}[\varepsilon]/\langle \varepsilon^{n+1}\rangle$. The Spec of A_2 and A_3 corresponds to the distinct reduced points with $a_2 \neq 0$ and $\xi \neq 0$ respectively.

Over the points $p \in Spec(A_2)$ given by a choice of $\{z_1, z_2\}$ as defined in (6.8) by replacing

$q \to 4q$, the Jacobian

$$J_q'(p) = 2n(2n-1)(1+\zeta)^{-1}(1-\zeta)^{-2} z_1^{4n-3}.$$

We obtain an analogue of Proposition 6.4.2:

Proposition 6.5.1. *Let* $\mathrm{vd} = (2n-1)d - \bar{g}(4n-3)$ *and* $F = a_1^{m_1} a_2^{m_2}$ *such that* $m_1 + 2m_2 = \mathrm{vd}$*, then*

$$\sum_{p \in A_2} Res_{\tilde{f}^q}(p; J'^g F) = \frac{2n-1}{2} \sum_{\zeta \neq \pm 1} (1+\zeta)^{m_1+d} \zeta^{m_2} J'(\zeta)^{\bar{g}} (-4q)^d \tag{6.16}$$

where $\zeta \neq \pm 1$ *is* $2n^{th}$ *root of unity and* $J'(\zeta) := 2n(2n-1)(1+\zeta)^{-1}(1-\zeta)^{-2}$.

Proposition 6.5.2. *Let* $F = a_1^{m_1} a_2^{m_2}$*, where* $m_1 + 2m_2 = \mathrm{vd}$*. Then*

$$\sum_{p \in A_3} Res_{\tilde{f}^q}(p; J'^g F) = \begin{cases} (-1)^{\bar{g}} (4n-2)^g (-4q)^d & m_2 = 0 \\ 0 & m_2 > 0 \end{cases}. \tag{6.17}$$

Proof. Let $p \in A_3$ be determined by $(\xi, a_1) = (\sqrt{-4q}\mu^{-1}, \mu^2)$ where μ is a $(4n-2)^{\text{th}}$ root of unity. Note that $a_2 = 0$, thus the residues vanish when $m_2 > 0$.

We may assume $m_2 = 0$. Using (6.15) and the equality $a_1^{2n-1} + 4q = 0$, the Jacobian is $-2a_1^{4n-3}(2n-1)$. Thus

$$\begin{aligned} \mathrm{Res}_{\tilde{f}^q}(p; J'^g a_1^{\mathrm{vd}}) &= (-1)^{\bar{g}} (2(2n-1))^{\bar{g}} a_1^{(2n-1)d} \\ &= (-1)^{\bar{g}} (4n-2)^{\bar{g}} (-4q)^d. \end{aligned}$$

□

Theorem 6.5.3. *Let* $m_1 + 2m_2 = (2n-1)d - (4n-3)\bar{g}$ *and* $n \geq 3$*. The GRW invariants for* $\mathrm{OG}(2, 2n+2)$ *involving* a_1 *and* a_2 *equal the top virtual intersections of the a-classes on the* *corresponding isotropic Quot schemes.*

In particular, when $d \geq g$ and

(i) When $m_2 > 0$, then

$$\langle a_1^{m_1} a_2^{m_2} \rangle_g = u 4^d \frac{2n-1}{2} \sum_{\zeta \neq \pm 1} (1+\zeta)^{m_1+d} \zeta^{m_2} \left(\frac{J'(\zeta)}{4} \right)^{\bar{g}},$$

where $u = (-1)^{\bar{g}+d}$ and $J'(\zeta) = 2n(2n-1)(1+\zeta)^{-1}(1-\zeta)^{-2}$.

(ii) When $m_2 = 0$, then

$$\langle a_1^{m_1} \rangle_g = u 4^d \left(\frac{(-1)^{\bar{g}}(4n-2)^{\bar{g}}}{4^{\bar{g}}} + \frac{2n-1}{2} \sum_{\zeta \neq \pm 1} \frac{(1+\zeta)^{m_1+d} J'(\zeta)^{\bar{g}}}{4^{\bar{g}}} \right).$$

The proof of the above theorem is similar to that of Theorem 6.4.3.

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