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Author
Sandholtz, Will

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Essays in Matching

by

Will Sandholtz

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Committee in charge:

Professor Haluk Ergin, Chair
Professor Chris Shannon
Professor Gabriel Zucman

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Abstract

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Within economic theory, the study of matching generally focuses on the problem of allocating heterogeneous, indivisible resources to agents, often without the use of prices. In this dissertation I focus on one-sided matching problems, in which a set of agents have preferences over a set of objects which are to be potentially allocated.

In Chapter 1, coauthored with Andrew Tai, we study the classic house swapping problem of Shapley and Scarf (1974) but relax the usual assumption that agents have strict preferences over the objects. Top trading cycles with fixed tie-breaking (TTC) has been suggested to deal with indifferences in these kinds of object allocation problems. Unfortunately, under general indifferences, TTC is neither Pareto efficient nor group strategyproof. Furthermore, it may not select an allocation in the core of the market, even when the core is nonempty. However, when indifferences are agreed upon by all agents ("objective indifferences"), TTC maintains Pareto efficiency, group strategyproofness, and core selection. Further, we characterize objective indifferences as the most general setting where TTC maintains these properties.

In Chapter 2, coauthored with Andrew Tai, we investigate object assignment problems in which objects may have minimum and maximum requirements. We describe a new random assignment mechanism, the minimums probabilistic serial (MPS) mechanism, which generalizes the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) to this more general setting. The random allocation produced by MPS is guaranteed to be Pareto efficient; that is, there is no other implementable allocation that all agents prefer via first order stochastic dominance. We also show that MPS is i) envy free, in that no agent will strictly prefer another agent's assignment, and ii) weak strategyproof, in that agents cannot achieve a better assignment by misreporting their preferences.

In Chapter 3, coauthored with Andrew Tai, we note that the proof of Bird (1984), the first to show group strategyproofness of top trading cycles (TTC), requires a correction. We provide a counterexample to a critical claim, and a corrected proof in the spirit of the original.

Contents

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Gabriel has been with me since the very beginning of this journey; it was his undergraduate course on inequality that first inspired me to pursue a PhD in economics. I also need to highlight his role in guiding me through difficult times during the pandemic. Gabriel truly went above and beyond to help me restart after a year away from the PhD program.

As my research interests shifted, Haluk and Chris were pivotal to my development as a theorist. Haluk's field course was my first introduction to the study of matching, which would ultimately become my primary area of research. As my main advisor, Haluk has been an exceptional mentor and friend. Over the last few years, I have enjoyed bonding over our shared connection to Türkiye. Likewise, Chris's caring guidance and support were instrumental to my progress on this dissertation and in my job search.

I also want to thank my friend and coauthor, Andrew. I have found that it is much more productive – and perhaps more importantly, much more enjoyable – to do research with a colleague. Beyond our research collaboration, however, I count Andrew as one of my closest friends from my time at Berkeley.

My partner, Serra, has been a constant source of encouragement, motivation, and inspiration. While I am excited for the next chapter of my career, I will miss being on campus together each day. I feel very lucky to have had such a positive and loving companion by my side.

Last, I want to thank my parents, Wayne and Judith, and my sister, Sarah. It is impossible to overstate how much their support helped to lift me up when I needed it. I am proud to be the final piece of our PhD family.

Chapter 1

Shapley-Scarf Markets with Objective Indifferences

Preface

In Chapter 1, coauthored with Andrew Tai, we study the classic house swapping problem of Shapley and Scarf (1974) but relax the usual assumption that agents have strict preferences over the objects. Top trading cycles with fixed tie-breaking (TTC) has been suggested to deal with indifferences in these kinds of object allocation problems. Unfortunately, under general indifferences, TTC is neither Pareto efficient nor group strategyproof. Furthermore, it may not select an allocation in the core of the market, even when the core is nonempty. However, when indifferences are agreed upon by all agents ("objective indifferences"), TTC maintains Pareto efficiency, group strategyproofness, and core selection. Further, we characterize objective indifferences as the most general setting where TTC maintains these properties.

1.1 Introduction

Important markets including living donor organ transplants, public housing assignments, and school choice can be modeled as Shapley-Scarf markets: each agent is endowed with an indivisible object and has preferences over the set of objects. Monetary transfers are not allowed, and participants have property rights to their own endowments. The goal is to reallocate these objects among the agents to achieve efficiency and stability. The usual stability notion is the core: an allocation is in the core if no subset of agents would prefer to trade their endowments among themselves. In the original setting of Shapley and Scarf (1974), agents have strict preferences over the houses, and Gale's top trading cycles (TTC) algorithm finds an allocation in the core. Roth and Postlewaite (1977) further show that the core is nonempty, unique, and Pareto efficient. Roth (1982) shows that TTC is strategyproof; Moulin (1995), Bird (1984), Sandholtz and Tai (2024), and Pápai (2000) show that it is group strategyproof. These properties make TTC an attractive algorithm for practical applications.

However, the assumption that preferences are strict is strong. In particular, if any objects are essentially identical, agents should naturally be indifferent between them. Consider the problem of assigning students to college dormitories. It seems reasonable to assume that two units with the same floor plan in the same building are equivalent in the eyes of a student. For example, the undergraduate dorm application process at UC Berkeley applies this same logic. On housing applications, students rank their five most preferred *housing complex* $\times$ *floor plan* pairs. It is implicitly assumed that students have strict preferences over *housing complex* $\times$ *floor plan* pairs, but are indifferent between dorm units of the same type. Beyond college dormitory assignment problems, there are a host of real-world object assignment problems (military occupational specialty assignment, school choice, class assignment, etc.) that can be modeled with a similar structure.

Motivated by these examples, we study a model of Shapley-Scarf markets where there may be indistinguishable copies (which we will call "houses") of the objects (which we will call "types" or "house types"). Our model restricts agents to be indifferent between houses of the same type, but never indifferent between houses of different types. We call these preferences "objective indifferences." Objective indifferences is a minimal model of indifferences, capturing the most basic and plausible form of indifferences.

In the fully general setting where agents' preferences may contain indifferences, TTC with fixed tie-breaking has been proposed as an intuitive extension. Ties in the agents' preference rankings are broken by some external rule, then TTC is executed on the resulting strict preference rankings. Ehlers (2014) is the first to formalize this procedure for Shapley-Scarf markets, though similar ideas date to earlier work. For example, Abdulkadiroğlu and Sönmez (2003) deal with tie-breaking between students in the school choice setting via an exogenous priority rule. However, TTC with fixed tie-breaking has several drawbacks. First, it is not Pareto efficient nor group strategyproof. In fact, there is an inherent tension between these two properties: Ehlers (2002) shows that under general indifferences, there does not exist a Pareto efficient and group strategyproof mechanism for Shapley-Scarf markets. Second, even when the core of a market is nonempty, TTC with fixed tie-breaking may not select a core allocation.

Objective indifferences adds structure to the case of general indifferences by constraining any indifferences to be universal among agents. We show that objective indifferences characterizes maximal domains on which TTC with fixed tie-breaking is Pareto efficient, group strategyproof, and core selecting. This statement contains three distinct results. First, TTC with fixed tie-breaking recovers these three properties on the restriction of objective indifferences. Second, any more general setting causes TTC to lose Pareto efficiency and core selection. Third, any setting in which TTC maintains either of these properties is a subset of an objective indifferences domain. The analogous results hold for group strategyproofness among preference domains that include both directions of any strict ranking. We will argue that this is not an onerous restriction in market design applications. The last two results are remarkable – objective indifferences and more restrictive settings are all of the settings on which TTC with fixed tie-breaking is Pareto efficient, core selecting, and group strategyproof. In other words, it is subjective indifferences that may differ between agents, and not indifferences in general, which cause TTC with fixed tie-breaking to lose these properties.

To our knowledge, the first use of the term "objective indifferences" was due to Bogomolnaia and Moulin (2001), who (almost tangentially) note that their probabilistic serial mechanism for random object assignment can accommodate this case. Fekete et al. (2003) and Cechlárová and Schlotter (2010) deal with competitive equilibrium in Shapley-Scarf markets with objective indifferences. Objective indifferences is similar to models where objects have capacities, as is sometimes applied in school choice; for example, see Morrill (2015) and Abdulkadiroğlu and Sönmez (2003). However, there are important differences which we illustrate in Appendix A.2. Object capacities give rise to a priority structure rather than an endowment structure, which changes the way TTC operates. More importantly, our emphasis here is on the objective indifferences setting as a domain of preferences, as will be clear in the results.

Others have have studied Shapley-Scarf markets with indifferences. Ehlers (2014) gives a characterization of TTC with fixed tie-breaking under general indifferences. Quint and Wako (2004) are the first to provide an algorithm finding the strict core in the presence of indifferences. Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) provide strategyproof, Pareto efficient, and core selecting families of trading cycle mechanisms. Aziz and De Keijzer (2012) and Plaxton (2013) propose further generalizations. Saban and Sethuraman (2013) develop conditions for identifying strategyproof mechanisms among a broader class of mechanisms that select Pareto efficient allocations in the weak core. Fundamentally, the challenge for any mechanism in a Shapley-Scarf market with indifferences is determining which trading cycles to execute from among the many potential trading cycles that indifferences may induce. Using a fixed tie-breaking rule is intuitive and easy to implement, but as Ehlers (2014) shows, it comes at the expense of certain desirable properties, including Pareto efficiency and core selection. Though we too study Shapley-Scarf markets with indifferences, we place additional structure on the agents' indifferences and demonstrate how this resolves many of the challenges that indifferences pose.

Our paper makes important contributions to the research in Shapley-Scarf markets along two lines. First, we contribute to the understanding of TTC. While there already exist TTC-like mechanisms that deal with indifferences, TTC is more intuitive and more widely known. Our paper captures exactly when its most important properties are maintained – under objective indifferences. We thus rationalize its use in many settings where the Shapley-Scarf model is actually applicable, such as in housing or school assignment. Second, we contribute to the understanding and awareness of the objective indifferences setting. While many have implicitly incorporated this model (e.g., objects with capacities), few have explicitly considered objective indifferences as a domain of preferences. Indeed, literature in mechanism design often considers only strict preferences or the full domain of indifferences. We show that objective indifferences adds an interesting and important intermediate case. It realistically models many object allocation problems and has the potential to preserve important properties relative to general indifferences.

Section 1.2 presents the formal notation. Section 1.3 explains TTC with fixed tie-breaking. Section 1.4 provides the main results. Section 1.5 concludes. Proofs are in Appendix A.1.

1.2 Model

In this section, we present the model primitives. First we recount the classical Shapley and Scarf (1974) setting. Afterwards we introduce the "objective indifferences" domain.

We first present the classical Shapley-Scarf model. Let $N = \{1, \dots, n\}$ be a finite set of agents, with generic member i. Let $H = \{h_1, \dots, h_n\}$ be a set of houses, with generic member h. Every agent is endowed with one object, given by a bijection $w : N \to H$. An allocation is an assignment of an object to each agent, also a bijection $x : N \to H$. The set of all allocations given (N, H) is $X(N, H)$, or simply X. For any $i \in N$, we use w_i as shorthand for $w(i)$ and x_i for $x(i)$. Similarly, for any $Q \subseteq N$ we use w_Q for $w(Q) = \{w_i : i \in Q\}$ and x_Q for $x(Q) = \{x_i : i \in Q\}$.

Each agent i has some preference ordering over H, denoted R_i. That is, R_i is a transitive, complete, reflexive binary relation. As usual, we let P_i denote the strict relation and I_i denote the indifference relation associated with R_i. A set of allowable preference orderings in a problem is denoted $\mathcal{R}$, which we call a **domain**. A preference profile is $R = (R_1, \dots, R_n) \in \mathcal{R}^n$. For any $Q \subseteq N$ and preference profile R, we denote $R_Q = (R_q)_{q \in Q}$. In this paper we restrict attention to settings where allowable preference profiles are drawn from some $\mathcal{R}^n$. That is, every agent has the same set of possible preference relations.

If $\mathcal{R}$ is the set of strict preference relations over H, then this is the **strict preferences domain**. To demonstrate the notation, the strict preference domain can be denoted

$$\mathcal{R}_{\text{strict}} = \{R_i : hI_ih' \iff h = h'\}.$$

If $\mathcal{R}$ is the set of weak preference relations over H, it is the **general indifferences domain**.

Our main domain is **objective indifferences**. Let $\mathcal{H} = \{H_1, H_2, \dots, H_K\}$ be a partition of H. An element H_k of a partition is a **block**. Given H and $\mathcal{H}$, let $\eta : H \to \mathcal{H}$ be the mapping from a house to the partition element containing it; that is, $\eta(h) = H_k$ if $h \in H_k$. Given $\mathcal{H}$, we denote the objective indifferences domain as

$$\mathcal{R}(\mathcal{H}) := \{R_i : hI_ih' \iff \eta(h) = \eta(h')\}.$$

We sometimes suppress $(\mathcal{H})$ from the notation when context makes it clear. Note that an agent is indifferent between houses in the same block of $\mathcal{H}$ and has strict preferences between houses in different blocks. Since all $i \in N$ draw from the same set of allowable preferences, these indifferences are "objective"; all agents must be indifferent between houses in the same block. Thus, we sometimes refer to the "indifference classes" of the domain with the understanding that every agent shares the same indifference classes. The next example illustrates.

Example 1.1. Let $H = \{h_1, h_2, h_3\}$ and $\mathcal{H} = \{\{h_1, h_2\}, \{h_3\}\}$. Then the two possible preference orderings in $\mathcal{R}(\mathcal{H})$ are given by:

R_α	R_β
h_1, h_2	h_3
h_3	h_1, h_2

That is, $h_1 I_\alpha h_2 P_\alpha h_3$ and $h_3 P_\beta h_1 I_\beta h_2$. Given a set of agents $N = \{1, 2, 3\}$, objective indifferences requires $R_i \in \{R_\alpha, R_\beta\}$ for each $i \in N$.

1.2.1 Mechanisms

This subsection recounts formalities on mechanisms and top trading cycles. Familiar readers may safely skip this subsection.

A **market** is a tuple (N, H, w, R). A **mechanism** is a function $f : \mathcal{R}^n \to X$; given a preference profile, it produces an allocation. When it is unimportant or clear from context, we suppress inputs from the notation. For any $i \in N$, let $f_i(R)$ denote i's allocated house under $f(R)$. Similarly, for any $Q \subseteq N$, let $f_Q(R) = \{f_i(R) : i \in Q\}$.

Fix a mechanism f, a tuple (N, H, w), and a preference domain $\mathcal{R}$. We work with the following axioms.

A mechanism is Pareto efficient if it always selects Pareto efficient allocations.

Pareto efficiency (PE). For all $R \in \mathcal{R}^n$, there is no other allocation $x \in X$ such that $x_i R_i f_i(R)$ for all $i \in N$ and $x_i P_i f_i(R)$ for at least one $i \in N$.

Group strategyproofness requires that no coalition of agents can collectively improve their outcomes by submitting false preferences. Note that in the following definition, we require both the true preferences and misreported preferences to come from the preference domain $\mathcal{R}$.[1]

Group strategyproofness (GSP). For all $R \in \mathcal{R}^n$, there do not exist $Q \subseteq N$ and $R' = (R'_Q, R_{-Q}) \in \mathcal{R}^n$ such that $f_i(R')R_if_i(R)$ for all $i \in Q$ with $f_i(R')P_if_i(R)$ for at least one $i \in Q$.

Individual rationality models the constraint of voluntary participation. It requires that agents receive a house they weakly prefer to their endowment.

Individual rationality (IR). For all $R \in \mathcal{R}^n$, $f_i(R)R_iw_i$ for all $i \in N$.

We also define the core of a market: an allocation is in the core if there is no subset of agents who could benefit from trading their endowments among themselves.

Definition 1.1. An allocation x is **blocked** if there exists a coalition $Q \subseteq N$ and allocation y such that $y_Q = w_Q$ and $y_iR_ix_i$ for all $i \in Q$, with $y_iP_ix_i$ for at least one $i \in Q$. An allocation x is in the **core** of the market if it is not blocked.

The weak core requires that all members of a potential coalition are strictly better off.

Definition 1.2. An allocation x is **weakly blocked** if there exists a coalition $Q \subseteq N$ and allocation y such that $y_Q = w_Q$ and $y_iP_ix_i$ for all $i \in Q$. An allocation x is in the **weak core** if it is not weakly blocked.

Of course, the weak core also contains the core. The (weak) core property models the restriction imposed by property rights. For example, in housing allocation, existing tenants might have the right to stay in their status quo allocations. Notice that individual rationality excludes blocking coalitions of size 1. The last axiom is core-selecting.

Core-selecting (CS). For all $R \in \mathcal{R}$, if the core of the market is nonempty then $f(R)$ is in the core.

1.2.2 Maximality

In Section 1.4, we characterize maximal domains on which TTC with fixed tie-breaking satisfies the axioms. By a "maximal" domain, we mean the following.

Definition 1.3. A domain $\mathcal{R}$ is **maximal** for Pareto efficiency and a class of mechanisms F if

1. every $f \in F$ is Pareto efficient on $\mathcal{R}$, and

2. for any $\tilde{\mathcal{R}} \supset \mathcal{R}$, there is some $f \in F$ that is *not* Pareto efficient on $\tilde{\mathcal{R}}$.

[1]This differs from other studies, e.g., Ehlers (2002).

The definition for any axiom besides Pareto efficiency is analogous. Notice that this definition of maximality depends on both the axiom and the class of mechanisms, which differs from elsewhere in the literature.[2] We define maximality for a class of mechanisms, since our claims deal with TTC for *all* tie-breaking rules. Again note that we only consider preference profiles drawn from $\mathcal{R}^n$, which is common.[3]

1.3 Top trading cycles with fixed tie-breaking

In this paper, we analyze top trading cycles (TTC) with fixed tie-breaking on the domains defined in the previous section. For an extensive history of TTC, we refer the reader to Morrill and Roth (2024). We briefly define TTC and TTC with fixed tie-breaking.

Top Trading Cycles. Consider a market (N, H, w, R) under strict preferences. Draw a graph with N as nodes.

1. Draw an arrow from each agent i to the owner (endowee) of his favorite remaining object.
2. There must exist at least one cycle; select one of them. For each agent in this cycle, give him the object owned by the agent he is pointing at. Remove these agents from the graph.
3. If there are remaining agents, repeat from step 1.

We denote the resulting allocation as $\text{TTC}(R)$.

TTC is well-defined only with strict preferences, as Step 1 requires a unique favorite object. In practice, a **fixed tie-breaking profile** $\succ$ is often proposed to resolve indifferences. Given N, let $\succ = (\succ_1, \ldots, \succ_n)$, where each $\succ_i$ is a strict linear order over N. This linear order will be used to break indifferences between objects (based on their owners). For any preference relation R_i and tie-breaking rule $\succ_i$, let $R_{i,\succ}$ be given by the following. For any $j \neq j'$, let $w_j R_{i,\succ} w_{j'}$ if

1. $w_j P_i w_{j'}$, or
2. $w_j I_i w_{j'}$ and $j \succ_i j'$.

For all j, let $w_j R_{i,\succ} w_j$.

Then $R_{i,\succ}$ is a strict linear order over the individual houses. Note that $R_{i,\succ}$ is actually $R_{i,\succ_i}$ as it depends only on i's tie-breaking rule, but we suppress this at minimal risk of confusion. Example 1.2 illustrates how we combine an agent's preferences and the tie-breaking rule to construct tie-broken preferences.

[2]Commonly in the literature, a domain is maximal for Pareto efficiency if there exists *some* mechanism which is Pareto efficient on it, and none on any larger domain.

[3]This excludes settings where different agents may be able to report different sets of preferences.

Example 1.2. Let $N = \{1, 2, 3, 4\}$. Agent 1's preferences R_1 and tie-breaking rule $\succ_1$ are shown below. In our lists of preference relations, each line represents an indifference class, and the houses on any line are strictly preferred to houses on lines below them. For example, the representation of R_1 below indicates $w_3 I_1 w_4 P_1 w_1 I_1 w_2$.

R_1		$\succ_1$		$R_{1,\succ}$
w_3, w_4		1		w_3
w_1, w_2	+	2	$\to$	w_4
		3		w_1
		4		w_2

Since $w_3 I_1 w_4$ and $3 \succ_1 4$, we have $w_3 P_{1,\succ} w_4$. Likewise, since $w_1 I_1 w_2$ and $1 \succ_1 2$, we have $w_1 P_{1,\succ} w_2$. Therefore agent 1's complete tie-broken preferences $R_{1,\succ}$ are given by $w_3\ P_{1,\succ}\ w_4\ P_{1,\succ}\ w_1\ P_{1,\succ}\ w_2$.

Given a preference profile $R \in \mathcal{R}^n$ and a tie-breaking profile $\succ$, let $R_\succ = (R_{1,\succ}, \dots, R_{n,\succ})$. **TTC with fixed tie-breaking (TTC$_\succ$)** is $\text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$. That is, the tie-breaking profile is used to generate strict preferences, and TTC is applied to the resulting strict preference profile. Formally, each tie-breaking profile $\succ$ generates a different TTC$_\succ$ mechanism, so TTC with fixed tie-breaking is a class of mechanisms, $\{\text{TTC}_\succ\}_\succ$. For a given R and $\succ$, we use $\text{TTC}_\succ(R)$ to refer both to the step-by-step procedure of TTC$_\succ$ and to the final allocation it generates. The following example illustrates how TTC$_\succ$ works in the objective indifferences domain.

Example 1.3. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{H} = \{\{w_1\}, \{w_2, w_3\}, \{w_4\}\}$. The preference profile R and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3, \succ_4)$ are shown below. R and $\succ$ are combined in the manner shown in Example 1.2 to construct the tie-broken preference profile $R_\succ$. Recall that $\text{TTC}_\succ(R)$ is equivalent to $\text{TTC}(R_\succ)$.

R_1	R_2	R_3	R_4
w_2, w_3	w_1	w_1	w_2, w_3
w_1	w_2, w_3	w_4	w_4
w_4	w_4	w_2, w_3	w_1

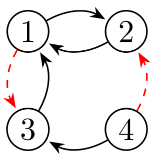
Preference profile R

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
2	1	3	3
1	2	2	1
3	3	1	2
4	4	4	3

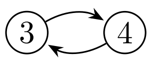
Tie-breaking profile $\succ$

$R_{1,\succ}$	$R_{2,\succ}$	$R_{3,\succ}$	$R_{4,\succ}$
w_2	w_1	w_1	w_3
w_3	w_2	w_4	w_2
w_1	w_3	w_3	w_4
w_4	w_4	w_2	w_1

Tie-broken preference profile $R_\succ$



Step 1 of $\text{TTC}_{\succ}(R)$:
Each agent points to the owner of their favorite house according to their *tie-broken* preferences, represented by black arrows. Red dashed arrows represent indifferences between w_2 and w_3. Agents 1 and 2 form a cycle and therefore swap houses.



Step 2 of $\text{TTC}_{\succ}(R)$:
After removing the agents assigned in Step 1, the remaining agents (3 and 4) point to the owner of their favorite remaining house. They form a cycle, and therefore swap houses. Since every agent has been assigned to a house, the $\text{TTC}_{\succ}$ procedure ends. The resulting allocation is $x = (w_2, w_1, w_4, w_3)$.

1.4 Results

With general indifferences, $\text{TTC}_{\succ}$ mechanisms are not Pareto efficient, core-selecting, nor group strategyproof. We provide simple examples below to illustrate these failures. However, we show that in the objective indifferences domain, $\text{TTC}_{\succ}$ mechanisms satisfy all three properties. We also show that expanding beyond this domain results in the loss of each property. Strikingly, any domain satisfying any of these properties must be a subset of objective indifferences. That is, objective indifferences characterizes the set of maximal domains on which $\text{TTC}_{\succ}$ mechanisms are PE and CS and characterizes the set of "symmetric-maximal" (defined below) domains on which $\text{TTC}_{\succ}$ mechanisms are GSP.

1.4.1 Pareto efficiency and core-selecting

When we relax the assumption of strict preferences and allow for general indifferences, $\text{TTC}_{\succ}$ loses two of its most appealing properties: Pareto efficiency and core-selecting. However, in the intermediate case of objective indifferences, $\text{TTC}_{\succ}$ retains these two properties. Moreover, on *any* larger domain, $\text{TTC}_{\succ}$ loses both Pareto efficiency and core-selecting. Thus, we show that it is not indifferences per se, but rather *subjective* evaluations of indifferences which cause $\text{TTC}_{\succ}$ to lose these properties.

We first demonstrate that $\text{TTC}_{\succ}$ mechanisms are not Pareto efficient under general indifferences. Example 1.4 gives the simplest case.

Example 1.4. Let $N = \{1, 2\}$. The preference profile $R = (R_1, R_2)$, tie-breaking profile $\succ = (\succ_1, \succ_2)$, and tie-broken preference profile $R_{\succ} = (R_{1,\succ}, R_{2,\succ})$ are shown below.

R_1	R_2
w_1, w_2	w_1
	w_2

Preference profile R

$\succ_1 = \succ_2$
1
2

Tie-breaking profile $\succ$

$R_{1,\succ}$	$R_{2,\succ}$
w_1	w_1
w_2	w_2

Tie-broken preference profile $R_\succ$

The TTC$_\succ$ allocation is $x = (w_1, w_2)$, which is Pareto dominated by $y = (w_2, w_1)$.

This example demonstrates the underlying reason that TTC$_\succ$ fails PE under general indifferences: tie-breaking rules may not take advantage of Pareto gains made possible by the agents' indifferences. However, under objective indifferences, if any agent is indifferent between two houses, then all agents are indifferent between those two houses. Consequently, objective indifferences rules out situations like the one shown in Example 1.4.

Under general indifferences, the set of core allocations may not be a singleton; there may be no core allocations or there may be multiple. As Example 1.4 demonstrates, even when the core of the market is nonempty, TTC$_\succ$ may still fail to select a core allocation.[4] However, under objective indifferences, if the core of a market is nonempty, then TTC$_\succ$ always selects a core allocation.

In fact, the objective indifferences setting characterizes the entire set of maximal domains on which TTC$_\succ$ mechanisms are Pareto efficient or core-selecting. That is, if all TTC$_\succ$ mechanisms are PE or CS on a domain $\mathcal{R}$, then $\mathcal{R}$ must be a subset of some objective indifferences domain. Conversely, for any superset of an objective indifferences domain, there is some TTC$_\succ$ mechanism (that is, some $\succ$) that is not PE or CS.

Theorem 1.1. *The following are equivalent:*

1. *$\mathcal{R}$ is an objective indifferences domain.*

2. *$\mathcal{R}$ is a maximal domain on which TTC$_\succ$ mechanisms are Pareto efficient.*

3. *$\mathcal{R}$ is a maximal domain on which TTC$_\succ$ mechanisms are core-selecting.*

Proof. Appendix A.1.1. □

The full proof is in the appendix, but the intuition is simple. The objective indifferences domain precludes possibilities such as Example 1.4, and any larger domain inevitably introduces the possibility of such a pair.

It follows from Sönmez (1999) that under objective indifferences, the core of a market is *essentially single-valued* when it exists. That is, for any two allocations x and y in the core of a market, we have $x_i I_i y_i$ for all agents i. In our proof of Theorem 1.1, we also prove this claim directly. Since the core is essentially single-valued, under objective indifferences the

[4]It is straightforward to see that $y = (w_2, w_1)$ is in the core of the market and $x = (w_1, w_2)$ is not.

core can be interpreted as a unique mapping from agents to house types. In other words, the core allocations are permutations of one another, where in each core allocation an agent always receives a house from the same indifference class.

Corollary 1.2. *If x and y are in the core of an objective indifferences market, then $x_i I_i y_i$ for all $i \in N$.*

Proof. Appendix A.1.1. □

Though all TTC$_\succ$ mechanisms are core-selecting under objective indifferences, the core of the market may still be empty, as the following simple example shows.

Example 1.5. Let $N = \{1, 2, 3\}$. It is easy to verify that for the preference profile $R = (R_1, R_2, R_3)$ shown below, there are no core allocations.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2, w_3	w_2, w_3

Any allocation x such that $x_1 \in \{w_1, w_2\}$ is blocked by $Q = \{1, 3\}$ and $y_Q = (w_3, w_1)$. Similarly, any allocation x such that $x_1 = w_3$ is blocked by $Q = \{1, 2\}$ and $y_Q = (w_2, w_1)$.

Even when the core of an objective indifferences market is empty, all TTC$_\succ$ mechanisms select an allocation in the weak core of the market. This is a known fact even under general indifferences; we reproduce it here and provide a proof for convenience.

Proposition 1.3. *For any market, the weak core is nonempty and TTC$_\succ$ mechanisms select an allocation in the weak core.*

Proof. Appendix A.1.2. □

1.4.2 Group strategyproofness

TTC$_\succ$ also loses group strategyproofness once we move from strict preferences to weak preferences. However, in the intermediate case of objective indifferences, TTC$_\succ$ recovers group strategyproofness. Further, TTC$_\succ$ mechanisms are not GSP in any larger "symmetric" domain. We say that a domain $\mathcal{R}$ is symmetric if, when $h_1 P_i h_2$ is possible, then so is $h_2 P_i h_1$. We will informally argue that this is not an onerous modeling restriction.

First we present a simple example demonstrating that under general indifferences, TTC$_\succ$ mechanisms are not group strategyproof. Notice that in Example 1.4, agent 1 can misreport $w_2 R_1' w_1$ to benefit agent 2 without harming himself. As a less trivial example, we also provide Example 1.6.

Example 1.6. Let $N = \{1, 2, 3\}$ and let $Q = \{1, 3\}$. For the preference profile $R = (R_1, R_2, R_3)$ and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3)$ shown below, the TTC$_\succ$ allocation is $x = (w_2, w_1, w_3)$. However, if agent 1 were to report R_1', then for $R' = (R_1', R_2, R_3)$ the TTC$_\succ$ allocation is $y = (w_3, w_2, w_1)$. Note that $y_3 P_3 x_3$ and $y_1 I_1 x_1$, so TTC$_\succ$ is not GSP.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2	w_2
	w_3	w_3

Preference profile R

R_1'	R_2	R_3
w_3	w_1	w_1
w_2	w_2	w_2
w_1	w_3	w_3

Preference profile R'

$\succ_1 = \succ_2 = \succ_3$
1
2
3

Tie-breaking profile $\succ$

Objective indifferences excludes situations like Example 1.6 in two ways. First, it eliminates the possibility that one agent is indifferent between two houses while another agent has a strict preference. Second, it constrains the possible set of misreports available to a manipulating coalition, since agents can *only* report indifference among all houses of the same type. Our next result characterizes the set of symmetric-maximal domains on which TTC$_\succ$ mechanisms are GSP.

Before presenting our result, we define "symmetric" and "symmetric-maximal" domains.

Definition 1.4. A domain $\mathcal{R}$ is **symmetric** if for any $h_1, h_2 \in H$, if there exists $R_i \in \mathcal{R}$ such that $h_1 P_i h_2$, then there also exists $R_i' \in \mathcal{R}$ such that $h_2 P_i' h_1$.

Definition 1.5. A domain $\mathcal{R}$ is **symmetric-maximal** for Pareto efficiency and a class of mechanisms F if

1. $\mathcal{R}$ is symmetric,
2. every $f \in F$ is Pareto efficient on $\mathcal{R}$, and
3. for any symmetric $\tilde{\mathcal{R}} \supset \mathcal{R}$, there is some $f \in F$ that is *not* Pareto efficient on $\tilde{\mathcal{R}}$.

The definition for any axiom besides Pareto efficiency is again analogous.

In most practical applications, symmetry is a natural restriction to place on the domain. If it is possible that agents might report strictly preferring some house h to another house h', we should not preclude the possibility they strictly prefer h' to h. Indeed, a central principle of mechanism design is that preferences are unknown and must be elicited. It is easy to see that objective indifferences domains are symmetric. Compared to maximality, symmetric-maximality restricts the possible expansions of objective indifferences domains that we must consider. The symmetry restriction is relatively weak; it does not enforce strict preferences nor general indifferences, as the next example illustrates.

Example 1.7. Let $H = \{h_1, h_2, h_3\}$ and $\mathcal{H} = \{\{h_1\}, \{h_2, h_3\}\}$. Consider the objective indifferences domain $\mathcal{R}(\mathcal{H}) = \{h_1 P h_2 I h_3, h_2 I h_3 P h_1\}$. Let $R_\alpha = h_1 P h_2 P h_3$ and $R_\beta = h_1 P h_3 P h_2$. Then $\mathcal{R}(\mathcal{H}) \cup \{R_\alpha\}$ is not a symmetric domain, since it allows $h_2 P h_3$ but not $h_3 P h_2$. However, $\mathcal{R}(\mathcal{H}) \cup \{R_\alpha, R_\beta\}$ *is* a symmetric domain. Further, notice that $\mathcal{R}(\mathcal{H}) \cup \{R_\alpha, R_\beta\}$ is not general indifferences nor strict preferences. Nor does it belong to some other objective indifferences domain, as it allows agents to report both $h_2 I h_3$ and $h_2 P h_3$.

Our second main result is that objective indifferences domains are the only symmetric-maximal domains on which every $\text{TTC}_\succ$ mechanism is group strategyproof.

Theorem 1.4. *$\mathcal{R}$ is a symmetric-maximal domain on which $TTC_\succ$ mechanisms are group strategyproof if and only if $\mathcal{R}$ is an objective indifferences domain.*

Proof. Appendix A.1.3. □

Our proof uses similar reasoning to the proof that TTC is group strategyproof under strict preferences contained in Sandholtz and Tai (2024). Any coalition requires a "first mover" to misreport, but this agent must receive an inferior house to the one he originally received. Under objective indifferences, this "inferior" house must actually be inferior according to his true preferences. Under more general indifferences, the situation in Example 1.4 again inevitably arises.

In the following example, we note that objective indifferences domains are *not* maximal domains on which $\text{TTC}_\succ$ mechanisms are GSP.

Example 1.8. Let $N = \{1, 2\}$, $H = \{h_1, h_2\}$, and $\mathcal{H} = \{\{h_1, h_2\}\}$. Suppose $\mathcal{R}' = \mathcal{R}(\mathcal{H}) \cup \{h_1 P h_2\} = \{h_1 I h_2, h_1 P h_2\}$. That is, expand the objective indifferences domain induced by $\mathcal{H}$ to include the ordering $h_1 P h_2$. Note that this expanded domain is not symmetric, since $\mathcal{R}'$ does not also contain the preference ordering $h_2 P h_1$.

Let $\succ = ((1 \succ_1 2), (1 \succ_2 2))$. We will show that for any market (N, H, w, R), $\text{TTC}_\succ$ is group strategyproof. It is straightforward to show that the same is true for the remaining 3 possible tie-breaking profiles.

Without loss of generality, assume $w_i = h_i$. If both agents have the same preferences, then there is clearly no profitable group manipulation. Consider the following two possible preference profiles:

R_1	R_2
h_1	h_1, h_2
h_2	

or

R_1	R_2
h_1, h_2	h_1
	h_2

In the first case, the $\text{TTC}_\succ$ allocation is $x = (h_1, h_2)$, so both agents receive one of their most preferred houses. Therefore, it is not possible for either agent to strictly improve. In the second case, the $\text{TTC}_\succ$ allocation is $x = (h_1, h_2)$. It would benefit agent 2 for agent 1 to rank h_2 above h_1, since agent 1 is indifferent between h_1 and h_2. However, this is not possible since $(h_2 P h_1) \notin \mathcal{R}'$.

Even though objective indifferences domains are not maximal for GSP, we have already argued that the symmetric restriction is not onerous. In most mechanism design settings, it is strange to allow participants to strictly prefer one object to another, but not vice versa.[5]

1.5 Conclusion

Our main set of results show that objective indifferences domains are the maximal domains on which $\text{TTC}_{\succ}$ mechanisms are Pareto efficient, core-selecting, and group strategyproof. While it may not be surprising that objective indifferences preserves these properties, it is quite remarkable that the maximal domains on which $\text{TTC}_{\succ}$ satisfies these three distinct properties essentially coincide.

In markets where one could reasonably assume that any indifferences are shared among all agents, $\text{TTC}_{\succ}$ is a sensible choice of mechanism. Regardless of the tie-breaking rule, $\text{TTC}_{\succ}$ will be Pareto efficient, core-selecting, and group strategyproof. While a number of other mechanisms generalize TTC and retain various properties in general indifferences, $\text{TTC}_{\succ}$ remains computationally efficient and easier to explain. However, in any more general domain of preferences, TTC inevitably loses some of its appeal. We do not interpret our results as necessarily prescriptive nor proscriptive. We simply characterize exactly when TTC retains its most desirable properties.

Our paper opens several new lines of inquiry. First, we believe that studying matching markets with constrained indifferences is an exciting avenue for future research. In many real-world matching markets, agents have indifferences, but often with a structure imposed by the particular market. Adding structure to general indifferences is not only theoretically interesting, but could improve policy choices. For instance, there may be tradeoffs in the selection of the partition $\mathcal{H}$ given the set of objects H. In some cases, there may be some ambiguity: are two dorms with the same floor plan but on different floors of the same building equivalent? Inappropriately combining indifference classes might lead to efficiency losses in the spirit of Example 1.4. On the other hand, splitting indifference classes might allow group manipulations like in Example 1.6. We leave formal results as future work. We also leave an axiomatic characterization of $\text{TTC}_{\succ}$ on objective indifferences domains as future work.

[5] One reasonable exception is contracts where the objects may include payments; e.g., a seat at a school with or without a scholarship.

Chapter 2

Random Matching with Minimums

Preface

Building on our study of one-sided matching markets in Chapter 1, in Chapter 2, coauthored with Andrew Tai, we investigate object assignment problems in which objects may have minimum and maximum requirements. We describe a new random assignment mechanism, the minimums probabilistic serial (MPS) mechanism, which generalizes the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) to this more general setting. The random allocation produced by MPS is guaranteed to be Pareto efficient; that is, there is no other implementable allocation that all agents prefer via first order stochastic dominance. We also show that MPS is i) envy-free, in that no agent will strictly prefer another agent's assignment, and ii) weak strategyproof, in that agents cannot achieve a better assignment by misreporting their preferences.

2.1 Introduction

In the Book of Numbers, Moses casts lots to divide territory among the tribes of Israel – thus object allocation via lottery goes at least as far back as the Hebrew Bible. In modern settings, public housing, dorm allocation, public school allocation, jury selection, and even immigration are examples of "object" allocation via randomization. When the objects vary significantly but monetary transfers are infeasible, randomization can impose some degree of fairness.

Going further, social choice mechanisms attempt to incorporate the preferences of the agents to achieve efficiency. As a simple example, random serial dictatorship (RSD) randomizes the order in which agents select their favored objects. Bogomolnaia and Moulin (2001) point out the shortcomings of RSD with respect to ordinal preferences (which we will recount later) and propose the probabilistic serial mechanism (PS). PS has agents "eat" probability of their favored object until it is exhausted, yielding a lottery over the objects. Once agents have lotteries over objects, the Birkhoff-von Neumann theorem guarantees that it can be translated to a lottery over deterministic allocations.

Often the objects assigned are bound by minimum constraints in addition to capacities. For example, consider assigning projects to workers; a project may require a minimum number of workers to be successful. Another example is the school club system in Japanese secondary schools; clubs are often mandatory and incorporated as class credit. A club may have a maximum capacity of students, and it may also have a minimum constraint. For example, sports teams are often organized as clubs – the basketball team requires at least 5 students.

This paper considers randomized object allocation with minimums. While random serial dictatorship[1] suffices to select an ex post efficient outcome, it inherits the same problems from Bogomolnaia and Moulin (2001)'s setting; the lotteries are not efficient. We therefore generalize the probabilistic serial mechanism to accommodate minimums and maximums while maintaining efficiency and fairness.

Towards a formal model, we have a set of agents, N, each of whom must be assigned to exactly one object in O. The allocation is denoted $[M_{ij}]_{i\in N, j\in O}$, which is a matrix of integers.[2] A given object o_j has some maximum capacity c_j and a minimum m_j. The final allocation requires $m_j \leq \sum_i M_{ij} \leq c_j$. We seek to find a matrix $[\mu_{ij}]$ where each row represents a lottery over objects. The advantage of working explicitly with lotteries is that they can be more efficient (in a precise sense to be discussed in the next section). Of course, it is both critical and nontrivial to ensure that $[\mu_{ij}]$ can indeed be implemented via a lottery over feasible deterministic allocations.

We apply results from Balbuzanov (2022) and Budish et al. (2013) to construct our generalization of the probabilistic serial mechanism, which we call *Minimums Probabilistic Serial (MPS)*. To do so, we convert the constraints for minimums, $\sum_i M_{ij} \geq m_j$, into

[1]With proper modifications to ensure respecting the constraints; see Monte and Tumennasan (2013).

[2]Note it is not bistochastic.

linear constraints of the form $A\mu \leq b$, where all entries of A and b are weakly positive. These are constraints that agents must obey as they "eat" objects. Since these are positive inequalities, agents consume favored objects until a relevant inequality becomes binding. The resulting random allocation is guaranteed to be a convex combination of feasible deterministic allocations, so it is implementable. The random allocation is also guaranteed to be Pareto efficient, in that there is no other implementable random allocation that all agents prefer via first order stochastic dominance. We further show that MPS is envy free, meaning no agent will strictly prefer another agent's allocation, and weak strategyproof, meaning a misreport cannot result in a stochastically dominant allocation. While efficiency is inherited as a special case from Balbuzanov (2022), the other properties are not.

Bogomolnaia and Moulin (2001) introduce the probabilistic serial (PS) mechanism, the first "eating" algorithm for random allocations. They show that it is efficient and strategyproof with respect to stochastic dominance for the special case of 1:1 matching problems. Budish et al. (2013) (henceforth BCKM13) deal with many-to-many matching problems and allow constraints they term bihierarchies. They also present a generalization of PS, termed *Generalized Probabilistic Serial* (GPS). Minimums fall under the bihierarchical constraints of BCKM13, enabling implementation of random allocations via their results. However, their GPS mechanism does not accommodate minimum quotas; it accepts only maximum quotas. In general, minimum requirements on objects cannot be represented with bihierarchical constraints using only maximum quotas. With three or more objects, at least two of which have nonzero minimums, the bihierarchical constraint structure cannot be maintained. Details and an example are in Appendix B.2. Balbuzanov (2022) (henceforth B22) generalizes the setting to allow arbitrary constraints. His primitive is a list of allowable deterministic allocations, which yields positive linear inequalities to input into the continuous-time eating process, called *Generalized Constrained Probabilistic Serial (GCPS)*. However, it is nontrivial to convert constraints phrased as minimums to an enumeration of allowable deterministic allocations; this is a factorial complexity problem, as Example 2.1 notes.

Example 2.1. Let there be n agents in N and k objects in O, with $n < k$. Let $m_j = 0$ and $c_j = 1$ for all $j \in O$. Then there are $\frac{k!}{(k-n)!} = n!\binom{k}{n}$ different allowable deterministic allocations.

While the allowed deterministic allocations are easy to describe, the list is factorial in size. Further, computing the positive inequalities from the list of allowed allocations induces the *facet enumeration problem (FEP)*. FEP procedures have worst-case exponential complexity in the inputs and outputs (which are proportional to the potentially factorial number of deterministic allocations).[3] We therefore take minimum and capacity constraints as our primitives and provide a polynomial-time algorithm. By providing the first feasible algorithm for random matching with minimums, we hope that our results help to bridge the gap between the literature and implementation in real applications.

[3]Balbuzanov notes that in practical terms for tried examples, the computations seem to be well-behaved.

In addition to BCKM13 and B22, many have worked on probabilistic allocation. Katta and Sethuraman (2006) extend Probabilistic Serial to indifferences; Yılmaz (2009) also considers endowments. Kojima (2009) deals with multi-unit demand. Heo (2014) and Hashimoto et al. (2014) offer characterizations in this setting. Papers including Che and Kojima (2010), Kojima and Manea (2010), and Liu and Pycia (2016) deal with probabilistic allocations in large markets. A number of papers deal with constraints such as minimums in matching. A non-exhaustive list includes Biró et al. (2010), Hamada et al. (2011), and Ehlers et al. (2014).

2.2 Model

Let $N = \{1, ..., N\}$ be the set of agents and $O = \{o_1, ..., o_O\}$ be the set of objects. We generally use $j \in O$ to refer a generic object. We say that $M \in \{0,1\}^{N \times O}$ is a **deterministic allocation**, where $M_{ij} = 1$ means that agent i is assigned to object j. We require:

$$\sum_j M_{ij} = 1 \qquad \forall i \in N \qquad \text{(Unit demand)}$$

That is, each agent is assigned to exactly one object.

Each object j has two characteristics: a maximum capacity of agents, $c_j \in \mathbb{Z}_+$, and a minimum required number of agents, $m_j \in \mathbb{Z}_+$. In other words, we require:

$$\sum_i M_{ij} \leq c_j \qquad \forall j \in O \qquad \text{(Capacity)}$$

$$\sum_i M_{ij} \geq m_j \qquad \forall j \in O \qquad \text{(Minimum-I)}$$

As shorthand, we refer to $O_m := \{j : m_j > 0\}$ as "minimum objects." We say a deterministic allocation $M = [M_{ij}]$ is **allowable** if it satisfies the unit demand, capacity, and minimum conditions shown above. Given a problem, the set of allowable deterministic allocations is $D(N, O)$, or simply D. When we refer to the set of deterministic allocations, we mean D unless otherwise noted.

We say that $\mu \in [0,1]^{N \times O}$ is a **random allocation**, where μ_{ij} represents agent i's probability of being assigned to object j. A random allocation is **implementable** if it is a convex combination of allowable deterministic allocations. A feature of our model is that many constraints on allowable deterministic allocations can be written in terms of marginal sums over the agents. As shorthand, we also denote $M_j = \sum_i M_{ij}$ and $\mu_j = \sum_i \mu_{ij}$. Note that we can rewrite our condition for capacity constraints as $M_j \leq c_j$ and our condition for minimum constraints as $M_j \geq m_j$. We will make use of this notation throughout the paper.

Each agent $i \in N$ is endowed with a strict preference ranking $\succ_i$ over the objects. A **mechanism** is $f : (N, O, \succ) \to \Delta D$. We typically suppress the first inputs and write $f(\succ)$. In other words, a mechanism takes in the model primitives and agent preferences over objects and reports an implementable random allocation.

A random allocation is **ex post efficient** if it is a convex combination of Pareto efficient deterministic allocations. This is not difficult to achieve – random serial dictatorship (sometimes called random priority) yields ex post efficient allocations. Using first order stochastic dominance (FOSD), we also endow agents with a partial preference ordering, $\geq_i^{FOSD}$, over random allocations. Without loss of generality, label the objects such that $o_1 \succ_i o_2 \succ_i \cdots \succ_i o_O$. Given random allocations μ and ν, we say $\mu_i \geq_i^{FOSD} \nu_i$ if

$$\sum_{j=1}^{k} \mu_{ij} \geq \sum_{j=1}^{k} \nu_{ij} \qquad \forall k = 1, ..., O.$$

Additionally, $\mu_i >_i^{FOSD} \nu_i$ if $\mu_i \geq_i^{FOSD} \nu_i$ and $\mu_i \neq \nu_i$. A random allocation μ is **SD efficient** if there does not exist ν such that $\nu_i \geq_i^{FOSD} \mu_i$ for all $i \in N$ and $\nu_i >_i^{FOSD} \mu_i$ for at least one i. We say that a mechanism is SD efficient if it always selects SD efficient random allocations.[4]

Random serial dictatorship is ex post efficient but not SD efficient. We reproduce an example from Bogomolnaia and Moulin (2001) below to illustrate.

Example 2.2. Let $N = \{1, 2, 3, 4\}$ and $O = \{o_1, o_2, o_3, o_4\}$. Let preferences be given by:

$$1, 2 : o_1 \succ o_2 \succ o_3 \succ o_4$$
$$3, 4 : o_2 \succ o_1 \succ o_4 \succ o_3$$

Random serial dictatorship (with uniform probability on order of selection) results in the random allocation ν, shown below:

ν	o_1	o_2	o_3	o_4
1	5/12	1/12	5/12	1/12
2	5/12	1/12	5/12	1/12
3	1/12	5/12	1/12	5/12
4	1/12	5/12	1/12	5/12

For example, agent 1 receives o_1 if he is first priority or second priority *after 3 or 4*, which occurs with probability 5/12. However, each agent's lottery under ν is stochastically dominated by his lottery under the implementable random allocation μ, shown below:

μ	o_1	o_2	o_3	o_4
1	1/2	0	1/2	0
2	1/2	0	1/2	0
3	0	1/2	0	1/2
4	0	1/2	0	1/2

Here, each agent's probability under ν on his second (fourth) favorite is moved to his first (third) favorite.

[4] SD efficiency is called *ordinal efficiency* in Bogomolnaia and Moulin (2001).

SD efficiency is an ordinal notion of efficiency for random allocations. As Bogomolnaia and Moulin (2001) note, its definition does not assume agents have von Neumann-Morgenstern (vNM) expected utilities. Therefore, from the perspective of the market designer, SD efficiency is attractive because it does not require eliciting vNM utilities. It is straightforward to see that if an allocation is SD efficient, it is Pareto efficient according to *some* profile of vNM utilities.[5]

We also consider two notions of fairness. A mechanism is **anonymous** if re-indexing the agents simply re-indexes their allocations in the same way. Formally, if π is a permutation, let $\succ^{\pi}$ be the permutation of $\succ$ and μ^{π} be the same permutation of a random allocation. The mechanism f is anonymous if $f(\succ^{\pi}) = f(\succ)^{\pi}$. A random allocation is **envy free** if for all $i, i' \in N$, we have $f_i(R) \geq_i^{FOSD} f_{i'}(R)$. That is, i does not prefer another agent's lottery. We say that a mechanism is envy free if it only selects envy free random allocations.

Finally, we consider incentives. A mechanism f is **weak strategyproof** if, for any (true) preferences $\succ$ and any $i \in N$ with misreport $\succ_i'$,

$$f_i(\succ_i', \succ_{-i}) \geq_i^{FOSD} f_i(\succ_i, \succ_{-i}) \Rightarrow f_i(\succ_i', \succ_{-i}) = f_i(\succ_i, \succ_{-i}).$$

In words, if i unilaterally misreports his preferences, it does not give him a first order stochastically dominant lottery. Note that the misreport may result in a FOSD noncomparable allocation.

2.3 Implementable random allocations

In this section, we recount relevant results chiefly from Balbuzanov (2022). From a pure mathematics standpoint, our setting is a special case of B22. However, from an applied mathematics standpoint, the difference is significant, as Example 2.1 demonstrates.

The first step is to characterize implementable random allocations ΔD from our model primitives. This is not as straightforward as it may initially seem, because our primitive is *not* D, the set of allowable deterministic allocations. While it is theoretically trivial to translate our primitive constraints to a set of deterministic allocations, in practice this set can be very large. We will therefore give an analytic description of ΔD that avoids enumerating the set of deterministic allocations.

We next consider the **lower contour set** of ΔD defined as

$$\mathrm{lcs}(\Delta D) := \left\{\mu' \in \mathbb{R}_+^{N\times O} : \exists \mu \in \Delta D \text{ where } \mu' \leq \mu\right\}.$$

Intuitively, $\mathrm{lcs}(\Delta D)$ are sub-random allocations that can be "completed" to implementable random allocations. This characterizes valid "paths" for an eating algorithm to assign probability to agents.

[5] This is Bogomolnaia and Moulin (2001) Lemma 2.

The key insight of B22 is that for any general constraints (including those beyond the present paper), $\text{lcs}(\Delta D)$ can be expressed as the intersection of the half-spaces defined by a set of **positive linear inequalities**:

$$\text{lcs}(\Delta D) = \{\mu \in \mathbb{R}_+^{N\times O} : A\mu \leq b, A \geq 0, b \geq 0\}.$$

This also implies $\text{lcs}(\Delta D)$ is a convex polytope. For convenience we reproduce B22's result below.

Proposition 2.1 (Balbuzanov, 2022)**.** *For any D, there exists an essentially unique minimal collection Ω^D of pairs (a, b) comprising a function $a : N \times O \to \mathbb{R}_+$ and a scalar $b \in \mathbb{R}_+$ such that*

$$lcs(\Delta D) = \bigcap_{(a,b)\in\Omega^D} \left\{\mu \in \mathbb{R}_+^{N\times O} : \sum_{i,j} a(i,j)\mu_{ij} \leq b\right\}.$$

Each pair (a, b) is a linear inequality that applies to all (sub-)random allocations. These inequalities lead naturally to a greedy algorithm like PS. An agent consumes his most preferred object until one of these inequalities binds, and then must move on to his next favored object. Since all inequalities are positive, an inequality that begins to bind will bind forever.

While the above result gives a theoretical guarantee for the existence of these positive linear inequalities, they must still be computed in practice. The points in D along with their projections onto the axes constitute the vertices of the convex polytope $\text{lcs}(\Delta D)$. Finding the inequalities defining the facets of the polytope from the vertices is the *facet enumeration problem (FEP)*. This problem is known to be NP-hard and has non-polynomial complexity in the number of vertices and dimensions, which both grow quickly by introducing new objects or agents. We refer readers to Seidel (2018) for a reference text.

We adapt this machinery to our setting. We start with a list of constraints $(m_j, c_j)_{j\in O}$ as our primitive, not the set of allowable deterministic allocations, D. Even in our setting, which contains much more structure than B22, the size of D (and complexity of enumerating it) can be factorial. This factorial size D would then be fed into the exponential complexity FEP. Thus, even in moderately sized problems, this approach may be impractical. Our approach avoids this issue – we directly find the analytic description of $\text{lcs}(\Delta D)$ without needing to enumerate D.

Once we have expressed $\text{lcs}(\Delta D)$ using a system of positive linear inequalities, we can immediately apply GCPS. For convenience, we give the definition of GCPS below.

Definition 2.1 (Balbuzanov, 2022)**. Generalized Constrained Probabilistic Serial (GCPS) mechanism**. Time runs continuously from $t \in [0, 1]$; each t is associated with a sub-random allocation μ^t. Given a preference profile $\succ$ and constraints in Ω^D that define $\text{lcs}(\Delta D)$, object j is available to agent i and time t if none of the inequalities in Ω^D involving μ_{ij} bind at that time. At time t, each agent claims with uniform speed remaining probability shares of his favorite reported object j among those available to him at t. The procedure ends at time $t = 1$ with output μ^1, which is an implementable random allocation.

However, even after finding lcs(ΔD), one more barrier exists with regards to practical implementation: GCPS defines a continuous-time eating process, not an algorithm in the classical sense. Therefore, in the following section we express our minimums probabilistic serial mechanism as an algorithm (i.e., a finite sequence of discrete steps).

2.4 Minimums probabilistic serial mechanism

We now describe our mechanism and its properties.

As before, we have a set of agents N and objects O. Each object j has a capacity $c_j > 0$ and a minimum $m_j \geq 0$. Our intermediate goal is to find the system of positive linear inequalities referenced by Proposition 2.1, $A\mu \leq b$, which define lcs(ΔD). First, we present positive linear inequalities that allowable deterministic allocations must respect.

Proposition 2.2. *Given a market* $(N, O, \succ)$*, the set* D *of allowable deterministic allocations* $M \in \{0,1\}^{N \times O}$ *is characterized by the following system:*

$$\sum_j M_{ij} = 1 \qquad \forall i \in N \qquad \text{(Unit demand)}$$

$$M_j \leq c_j \qquad \forall j \in O \qquad \text{(Capacity)}$$

$$\sum_{k \neq j} M_k \leq N - m_j \qquad \forall j \in O_m \qquad \text{(Minimum-II)}$$

Proof. Appendix B.1.1. □

The intuition is straightforward. The first two conditions are exactly primitives of the model. The third rephrases the minimum condition in terms of maximums on the other objects. If j has a minimum of m_j, we can assign at most $N - m_j$ agents to the rest of the objects.

One might naively expect that we could characterize lcs(ΔD) by relaxing the unit demand condition to $\sum_j \mu_{ij} \leq 1$ for all j. However, this is not true, as the following example illustrates.

Example 2.3. Let $N = \{1, 2, 3\}$ and $O = \{o_1, o_2, o_3, o_4\}$ with the following characteristics:

	c_j	m_j
o_1	3	1
o_2	3	1
o_3	3	0
o_4	3	0

If we relax the unit demand condition in the manner described above, the conditions of Proposition 2.2 become:

$$\sum_j \mu_{ij} \le 1 \qquad i \in \{1,2,3\} \qquad \text{(Unit demand - relaxed)}$$
$$\mu_j \le 3 \qquad j \in \{1,2,3,4\} \qquad \text{(Capacity)}$$
$$\sum_{j\in\{2,3,4\}} \mu_j \le 2, \sum_{j\in\{1,3,4\}} \mu_j \le 2 \qquad \text{(Minimum-II)}$$

A sub-random allocation μ' with $\sum_{j\in\{3,4\}} \mu'_j = 2$ might satisfy these conditions but cannot result in an implementable random allocation. In particular, for the sub-random allocation μ' shown below, there is no random allocation $\mu \in \Delta D$ such that $\mu' \le \mu$.

μ'	o_1	o_2	o_3	o_4
1	0	0	1	0
2	0	0	0	1
3	0	0	0	0

Note that the minimum requirements of objects o_1 and o_2 require $\sum_{j\in\{1,2\}} \mu_j \ge 2$ for any $\mu \in \Delta D$. But also if $\mu \ge \mu'$, then $\sum_{j\in\{3,4\}} \mu_j \ge 2$. However, this implies that $\sum_j \mu_{ij} > 1$ for some i. Thus, μ could not be in ΔD.

To characterize lcs(ΔD), we require a much richer set of inequalities.

Proposition 2.3. *Given a market $(N, O, \succ)$, the set lcs(ΔD) of allowable sub-random allocations $\mu \in [0,1]^{N\times O}$ is characterized by the following system:*

$$\sum_j \mu_{ij} \le 1 \qquad \forall i \in N \qquad \text{(Unit demand - relaxed)}$$
$$\mu_j \le c_j \qquad \forall j \in O \qquad \text{(Capacity)}$$
$$\sum_{j\in O\setminus S} \mu_j \le N - \sum_{j\in S} m_j \qquad \forall S \subseteq O_m \qquad \text{(Minimum-III)}$$

Proof. Appendix B.1.1. □

Remark. The third group of inequalities is for any set $S \subseteq O_m$. Since $m_j = 0$ for all $j \notin O_m$, it is equivalent to $\forall S \subseteq O$ instead.

Proposition 2.3 provides an analytic description of the inequalities which are guaranteed to exist by Proposition 2.1 for our setting. We can now apply GCPS to express our mechanism as a continuous-time eating process.

Definition 2.2. Minimums Probabilistic Serial (MPS) mechanism. Fix $(N, O, \succ)$ and the profile of constraints $(m_j, c_j)_{j\in O}$. Time runs continuously from $t \in [0,1]$; each t is associated with a sub-random allocation μ^t. Object j is available at time t if none of the inequalities in Proposition 2.3 involving μ_j bind at that time. At time t, each agent claims with uniform speed probability shares of his favorite reported object j among those available at t. The procedure ends at time $t = 1$ with output μ^1, which is an implementable random allocation.

We therefore describe MPS without the need to enumerate D from the primitive constraints nor to solve the facet enumeration problem, both of which are non-polynomial time procedures. An algorithmic definition is contained later in this section.

Theorem 2.4. *The MPS mechanism is SD efficient, anonymous, envy free, and weak strategyproof.*

Proof. SD efficiency and anonymity are inherited from B22. The remainder is in Appendix B.1.2 and Appendix B.1.3. □

As Theorem 1 shows, MPS maintains the most desirable properties of PS from Bogomolnaia and Moulin (2001). In particular, weak strategyproofness is lost in the more general setting of B22 but preserved in our setting. The intuition for SD efficiency matches that in Bogomolnaia and Moulin (2001) and B22. Agents consume their favored objects until they become unavailable. Since the linear inequalities governing availability are positive, an object becomes unavailable forever. Thus there is no possibility that an object closes then reopens later. Envy freeness has a similar intuition. Since objects are available to everyone or to no one, an agent can never envy another agent's allocation. Anonymity is also immediate for this reason.

Weak strategyproofness is much more challenging to prove. As becomes clear in the description of Algorithm 1, the total amount of an object that is available to consume may increase if agents' consumption of O_m is manipulated. This possibility is not present in Bogomolnaia and Moulin (2001) or in GPS, as the only constraints are maximum capacities. For example, it is possible that in a truthful run of MPS, the total consumption of object a is capped by m_a, but under a misreport run, total consumption of a might be able to exceed m_a if agents' consumption is shifted into O_m. The proof (contained in Appendix B.1.3) shows that while this is possible, it can never benefit the misreporter.

Remark. MPS is not an application of the Generalized Probabilistic Serial (GPS) mechanism contained in BCKM13. Given a bihierarchical set $\mathcal{H}$ of constraint sets $S \subseteq N \times O$, GPS allows for constraints of the form

$$\underline{q}_S \leq \sum_{(i,j)\in S} \mu_{ij} \leq \bar{q}_S$$

where $\underline{q}_S = 0$ for all $S \in \mathcal{H}$ that are not rows. In general, it is impossible to represent the constraints in our model this way. Details are in Appendix B.2.

The conditions in Proposition 2.3 can be re-written more compactly. The advantage of the following representation is that it leads easily to an algorithm that runs in a finite number of discrete steps.

Proposition 2.5. *Given a market $(N, O, \succ)$, lcs(ΔD) is the set of all $\mu \in [0,1]^{N\times O}$ satisfying the following conditions:*

$$\sum_j \mu_{ij} \leq 1 \qquad \forall i \in N \qquad \text{(Unit demand - relaxed)}$$

$$\mu_j \leq c_j \qquad \forall j \in O \qquad \text{(Capacity)}$$

$$\sum_j \max\{m_j, \mu_j\} \leq N \qquad \text{(Minimum-IV)}$$

Proof. Appendix B.1.1. □

While Bogomolnaia and Moulin (2001) contains an algorithm for the PS mechanism, B22 does not for the GCPS mechanism.[6] We take advantage of the additional structure in our model to express MPS as an algorithm.

The characterization of lcs(ΔD) in Proposition 2.5 suggests three things that can occur to end a step of the MPS algorithm. If the first inequality binds for any agent, it means that this agent has consumed his total share of probability and must stop eating. Since we restrict to uniform eating speed and unit demand, this occurs for all agents at time $t = 1$. When the second inequality binds for any object, it means that this object has reached its maximum capacity and can no longer be consumed. Lastly, when the third inequality binds, the only way for agents to continue eating is to consume objects for which $\mu_j < m_j$. That is, agents must consume from objects whose minimum requirements are not yet satisfied.

The algorithm will run in discrete steps, denoted s. For any $S \subseteq O$ and $j \in S$, let $N(j, S) = \{i \in N : j \succsim_i k \quad \forall k \in S\}$. That is, $N(j, S)$ is the set of agents who would consume j among the objects in S. Also, let $n(j, S) = |N(j, S)|$. The objective is to find the time t^s at which the current step s will end. Let O^{s-1} be objects available at step s, and $O_m^{s-1} \subseteq O^{s-1}$ be those that have not yet met their minimum requirements.

For any object $j \in O^{s-1}$, define

$$t_c^s(j) = \begin{cases} t^{s-1} + \frac{c_j - \mu_j^{s-1}}{n(j, O^{s-1})} & \text{if } n(j, O^{s-1}) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $t_c^s(j)$ expresses the time at which object j would reach its capacity if all agents who prefer j among O^{s-1} begin eating from j at time t^{s-1}. By construction, we know that $\mu_j^{s-1} < c_j$ for all $j \in O^{s-1}$.

For any object $j \in O_m^{s-1}$, define

$$t_m^s(j) = \begin{cases} t^{s-1} + \frac{m_j - \mu_j^{s-1}}{n(j, O^{s-1})} & \text{if } n(j, O^{s-1}) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

[6] Of course, his setting is significantly more complicated.

Note that $t^s_m(j)$ expresses the time at which object j would satisfy its minimum requirement if all agents who prefer j among O^{s-1} begin eating from j at time t^{s-1}. By construction, we know that $\mu^{s-1}_j < m_j$ for all $j \in O^{s-1}_m$.

Last, define

$$t^s_O = \begin{cases} t^{s-1} + \frac{N - \sum_{j \in O^{s-1}_m} m_j - \sum_{j \notin O^{s-1}_m} \mu_j}{\sum_{j \notin O^{s-1}_m} n(j, O^{s-1})} & \text{if } n(j, O^{s-1}) > 0 \text{ for some } j \notin O^{s-1}_m \\ +\infty & \text{otherwise.} \end{cases}$$

Note that t^s_O expresses the time at which the minimum condition from Proposition 2.5 binds, if this occurs at step s.

Algorithm 1 Minimums Probabilistic Serial (MPS)

Initialize $t^0 = 0$, $O^0_m = O_m$, $O^0 = O$, $\mu^0 = 0$, $s = 0$, and $d =$ False.
while $t^s < 1$ **do**
  Set $s = s + 1$.
  Set $t^s = \min\left\{\{t^s_c(j) : j \in O^{s-1} \setminus O^{s-1}_m\} \cup \{t^s_m(j) : j \in O^{s-1}_m\} \cup \{t^s_O, 1\}\right\}$.
  if $t^s = t^s_O$ **then**
    Set $d =$ True.
  end if
  Set $O^s_m = O^{s-1}_m \setminus \{j \in O^{s-1}_m : t^s_m(j) = t^s\}$.
  if d **then**
    Set $O^s = O^s_m$.
  else
    Set $O^s = O^{s-1} \setminus \{j \in O^{s-1} \setminus O^{s-1}_m : t^s_c(j) = t^s\}$.
  end if
  for $j \in O$ **do**
    for $i \in N$ **do**
      Set

$$\mu^s_{ij} = \begin{cases} \mu^{s-1}_{ij} + (t^s - t^{s-1}) & \text{if } j \in O^{s-1} \text{ and } i \in N(j, O^{s-1}), \\ \mu^{s-1}_{ij} & \text{otherwise.} \end{cases}$$

    end for
  end for
end while
return $[\mu^s_{ij}]$.

Note that $t^s = t^s_O$ at most once during the run of the algorithm, since $O^s = O^s_m$ and therefore $t^s_O = +\infty$ for every subsequent step s after the step at which $t^s = t^s_O$. We will denote this time where $t^s = t^s_O$ by τ; the role of τ in our mechanism is important in proving weak strategyproofness of MPS. By misreporting their preferences, agents may be able to alter τ. We show, however, that they can never benefit from doing so.

Intuitively, the algorithm works as follows. At time $t = 0$, agents begin eating from their favorite objects. If an object reaches its capacity, it is closed to all agents and agents begin eating from their next favorite remaining object. At some time – which corresponds to τ as we have defined it above – it may be necessary to close all objects once they have met their minimum requirements in order to ensure that all objects' minimum requirements are met. That is, at time τ, any object that has already met its minimum requirement is closed, even if it is not yet at its capacity. Thereafter, objects are closed upon reaching their minimums.

Thus we are able to use our results to describe MPS as an algorithm that runs in a finite number of discrete steps. We characterize this mechanism without enumerating the deterministic allocations nor solving the facet enumeration problem, which would be a non-polynomial complexity problem with factorial inputs. Each step of our algorithm requires calculating at most $|O| + 1$ values. At the end of each step, either 1) an object reaches its capacity; 2) an object's minimum is satisfied; 3) the minimums become binding across the market; or 4) $t = 1$. Therefore, there are at most $2|O| + 2$ steps.

There does not exist an algorithmic description of GCPS. Of course, B22's setting is much more complicated. However, in our setting, we overcome a significant hurdle to practical usage by providing a polynomial time algorithm.

2.5 Implementation

Once we arrive at an implementable random allocation via MPS, it remains to implement the allocation. Since implementable random allocations may not be bistochastic, the Birkhoff-von Neumann theorem is not applicable. Recall that MPS is not contained as a special case of BCKM13's Generalized Probabilistic Serial (GPS) mechanism, since GPS does not accept minimums on the constraint sets. However, we can still apply the implementation results in BCKM13. Appendix B.2 explains the relationship between our setting and their setting. Thus, after using MPS to arrive at an implementable random allocation, we are able to apply results from BCKM13 to randomize over allowable deterministic allocations. We refer the reader to their paper for details on the exact procedure.

2.6 Conclusion

This paper deals with the allocation of objects to agents in a unit demand setting. It is not difficult to motivate minimums in this setting. Minimums might reflect lower bounds on the sizes of viable classes, projects, or basketball teams. However, finding implementable allocations that respect the minimums is nontrivial. We construct a new mechanism, the Minimums Probabilistic Serial (MPS) mechanism, that yields an implementable random allocation when objects may have both minimum and maximum restrictions. We show that

MPS is SD efficient, anonymous, envy free, and weak strategyproof. The last property is preserved, unlike in the more general setting of Balbuzanov (2022). Importantly, MPS is also computationally feasible – we provide a polynomial time algorithmic description of MPS. We hope that our results help to bridge the gap between the literature and actual implementation in real applications.

Chapter 3

Group Incentive Compatibility in a Market with Indivisible Goods: A Comment

Preface

In Chapters 1 and 2, we examined whether the mechanisms are immune to manipulation by the agents in the market. Continuing in this vein, in Chapter 3, coauthored with Andrew Tai, we note that the proof of Bird (1984), the first to show group strategyproofness of top trading cycles (TTC), requires a correction. We provide a counterexample to a critical claim, and a corrected proof in the spirit of the original.

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3.1 Introduction

Strategyproof mechanisms are desirable because they are immune to an individual agent's misrepresentations. Agents' decisions are thus straightforward, because the optimal action for any individual is to report his or her true preferences. Group strategyproof mechanisms ensure this is also true for coalitions of agents, protecting less sophisticated and less well-connected agents.

Therefore, in problems such as school assignment, housing assignment, and organ exchange, group strategyproofness is valuable. These settings are applications of the canonical house swapping model, where each agent is endowed with a single indivisible good and has strict preferences over the set of goods. In this model, the top trading cycles (TTC) mechanism of Shapley and Scarf (1974) produces the strong core allocation. Roth and Postlewaite (1977) show that this is also the unique competitive equilibrium allocation.

Roth (1982) shows that TTC is strategyproof. Bird (1984) presents a proof that TTC is weakly group strategyproof. In this note, we show that Bird's proof requires correction. To our knowledge, we are the first to do so. While others have since provided alternative proofs that TTC is group strategyproof,[1] we present new proofs in the spirit of the originals in Bird (1984). We also prove a non-obvious claim about strong group strategyproofness, which Bird (1984) presents as a corollary.

In the next section, we present the model notation. In Section 3, we correct a critical claim, Lemma 1, which Bird (1984) uses to prove weak group strategyproofness. We then revise his proof of weak group strategyproofness. Finally, we present a new proof of strong group strategyproofness using the corrected Lemma 1.

3.2 Model and notation

We retain the notation in Bird (1984) and recount it briefly here. Let $N = \{1, ..., n\}$ be the set of agents and let $w = (w_1, ..., w_n)$ be the endowment, where agent i is endowed with w_i, which we call a house. Each agent i has strict preferences P_i over the houses. We denote the weak preferences R_i. Let $P = (P_1, ..., P_n)$ be the preference profile of all agents. An allocation is a vector $x = (x_1, ..., x_n)$ where each x_i corresponds to some w_j.

Let $T(N, P)$ denote the allocation resulting from TTC applied to (N, P). For convenience, we use $TTC(N, P)$ to denote the procedure of TTC applied to (N, P). Let $S_k(P) \subseteq N$ be the set of agents in the kth trading cycle of $TTC(N, P)$, and let $S_0 = \emptyset$.[2] Define $R_k(P) = \cup_{i=1}^{k} S_i(P)$.

Suppose a subset Q of agents report their preferences as $P'_Q \neq P_Q$. Let $P' = (P'_Q, P_{-Q})$. Denote $x = T(N, P)$ and $x' = T(N, P')$.

[1] See Moulin (1995) Lemma 3.3 and Pápai (2000).

[2] The order of cycles generated by TTC is not generally unique. It is possible that two or more cycles are formed at the same step. However, the results carry through under any ordering of these cycles.

We seek to show that TTC is weakly group strategyproof: for any Q and P'_Q, there is some agent $i \in Q$ such that $x_i P_i x'_i$ or $x_i = x'_i$. That is, at least one agent in Q is weakly worse off under the misrepresentation. Additionally, we show TTC is strongly group strategyproof: for any Q and P'_Q, there is some agent $i \in Q$ such that $x_i P_i x'_i$. That is, at least one agent in Q is strictly worse off under the misrepresentation.

3.3 Weak group strategyproofness

Bird (1984) makes the following claim, which is critical to the main result.

Claim 3.1 (Bird, 1984. Lemma 1)**.** *If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exist $j \in R_{k-1}(P)$ and $h \in N \setminus R_{k-1}(P)$ such that $w_h P'_j x_j$.*

He gives the following intuition:

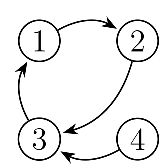
> [I]f any trader wants to get a more preferred good, he needs to get a trader in an earlier cycle to change his preference to a good that went in a later trading cycle. From this result, the group incentive compatibility follows easily.

The lemma as stated requires correction. We first give a counterexample.

Example 3.1 (Counterexample to Claim 3.1)**.** Let $N = \{1, 2, 3, 4\}$ with the following preferences.

P_1	P_2	P_3	P_4
w_2	w_3	w_1	w_3
w_1	w_2	w_4	w_4
		w_3	

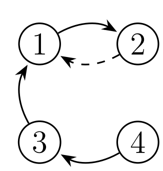
First step of $TTC(N, P)$



The TTC allocation is $x = (w_2, w_3, w_1, w_4)$. Now consider an alternative preference profile P':

P_1	P'_2	P_3	P_4
w_2	w_1	w_1	w_3
w_1	w_3	w_4	w_4
	w_2	w_3	

First step of $TTC(N, P')$



The new TTC allocation is $x' = (w_2, w_1, w_4, w_3)$.

In the notation of Claim 3.1, we have $i = 4$ and $k = 2$. That is, $4 \in S_2(P)$ and $x'_4 P_4 x_4$. Yet, there do not exist $j \in R_{k-1}(P) = S_1(P)$ and $h \in N \setminus R_{k-1}(P) = S_2(P)$ such that $w_h P'_j x_j$. The only candidate for j is $2 \in R_1(P) = S_1(P)$, since she is the only agent whose preferences change under P'. But she does not rank any houses from $N \setminus R_{k-1}(P) = S_2(P)$ above $x_2 = w_3$.

The error in Bird's proof of Lemma 1 stems from the following erroneous claim.

Claim 3.2. *Let $x_m P'_m w_n$ for all $m \in R_{k-1}(P)$ and $n \in N \setminus R_{k-1}(P)$. That is, all members of the first $k-1$ cycles rank their original assignments above all houses from cycles k and later. Then $R_{k-1}(P') = R_{k-1}(P)$. That is, the set of agents assigned in the first $k-1$ cycles of $TTC(N, P')$ is the same as the set of agents assigned in the first $k-1$ cycles of $TTC(N, P)$.*

The above counterexample also serves as a counterexample to Claim 3.2, since $R_1(P) \neq R_1(P')$.

It is not necessary for an agent in an earlier cycle $\kappa < k$ to change her preference to a house in cycle k or later. She may change her preference to a house in cycle κ or later. This is the necessary addition; we present a corrected version.

Lemma 3.3 (Claim 3.1, Corrected)**.** *If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exist $j \in S_\kappa(P)$ where $\kappa < k$ and $h \in N \setminus R_{\kappa-1}(P)$ such that $w_h P'_j x_j$.*

That is, if an agent wants to get a more preferred good, he needs an agent in an earlier cycle to misrepresent her preferences to favor a good that went in her own cycle or a later cycle. Using the notation of Lemma 3.3, h may be in the same cycle κ as j. In the case of Example 3.1, $i = 4, k = 2, \kappa = 1, j = 2$, and $h = 1$.

Proof of Lemma 3.3. Suppose there exist k and $i \in S_k(P)$ such that $x'_i P_i x_i$. Toward a contradiction, suppose that for each $\kappa < k$, for all $j \in S_\kappa(P)$, we have that $x_j P'_j w_h$ for all $h \in N \setminus R_{\kappa-1}(P)$ and $h \neq j$. That is, all agents in cycles before k still rank their original allocation over any other house in their own cycle or later. We show by strong induction on the cycles t of $TTC(N, P)$ that there is some order of cycles of $TTC(N, P')$ such that $S_\kappa(P') = S_\kappa(P)$ for all $\kappa < k$.[3]

Step $t = 1$. For each $j \in S_1(P)$, x_j was top-ranked under P_j. By assumption, x_j is still top-ranked under P'_j. Then under $TTC(N, P')$, the same cycle exists in the graph at step 1, so there is an order of cycles of $TTC(N, P')$ such that $S_1(P') = S_1(P)$.

[3] The intuition is that the earlier cycles are all the same. But note that the order of cycles in TTC may not be unique; this is the case if multiple cycles are present in the graph at once. This does not substantially affect the intuition, but does require more careful notation.

Step $t < k$. Suppose there is some order of cycles of $TTC(N, P')$ such that $S_\tau(P') = S_\tau(P)$ for all $\tau < t$. Under this order, $N \setminus R_{t-1}(P') = N \setminus R_{t-1}(P)$. By assumption, for every $j \in S_t(P)$, $x_j P'_j w_h$ for all $h \in N \setminus R_{t-1}(P)$ where $w_h \neq x_j$. Thus x_j is top-ranked under P'_j among remaining houses for all $j \in S_t(P)$. Then under this order of $TTC(N, P')$, the cycle $S_t(P)$ also exists in the graph at this step, so there is an order such that $S_t(P') = S_t(P)$.

We have shown that there is some order of cycles of $TTC(N, P')$ such that $S_\kappa(P') = S_\kappa(P)$ for $\kappa < k$. Under this order, $R_{k-1}(P') = R_{k-1}(P)$. Since $i \in S_k(P)$, $x'_i P_i x_i$ implies that $x'_i = w_j$ for some $j \in R_{k-1}(P)$. Therefore, $j \in R_{k-1}(P')$ and $i \in R_{k-1}(P')$. But then $i \in R_{k-1}(P)$, contradicting the assumption that $i \in S_k(P)$. □

We now update the proof of Bird's main theorem using the corrected lemma. The argument proceeds in the same manner as the original.

Theorem 3.4 (Bird, 1984, Theorem)**.** *TTC is weakly group strategyproof.*

Proof of Theorem 3.4. Suppose there is a subset $Q \subseteq N$ reporting $P'_Q \neq P_Q$. Let agent $i \in S_k(P)$ be the first agent in Q to enter a trading cycle in $TTC(N, P)$. If there are multiple such agents, i.e. $|S_k(P) \cap Q| \geq 2$, let i be any such agent. We will show that i cannot strictly improve.

Toward a contradiction, suppose that $x'_i P_i x_i$. Note this requires $k \geq 2$, since agents in $S_1(P)$ top-rank x_i. By Lemma 3.3, there exist $j \in S_\kappa(P)$ and $h \in N \setminus R_{\kappa-1}(P)$, where $\kappa < k$, such that $w_h P'_j x_j$. Then $P'_j \neq P_j$ and $j \in Q$. But then i could not have been the first agent (or one of the first) in Q to enter a trading cycle in $TTC(N, P)$, a contradiction. □

3.4 Strong group strategyproofness

Bird (1984) also presents strong group strategyproofness as a corollary.

Theorem 3.5 (Bird, 1984, Corollary)**.** *TTC is strongly group strategyproof.*

He gives the following justification:

> [The corollary] follows directly. Trader j must misrepresent his preferences if trader i is to do better. Since the preferences are strict, trader j forms a cycle and receives a good that he does not prefer to the one he would receive under the original top trading cycle.

We feel this requires more elucidation. It is not immediate from strict preferences that j forms a cycle while pointing at the worse house. Moreover, the outcome is produced by a *group* misrepresentation, so there may be other deviating agents apart from j. Thus we provide a proof of strong group strategyproofness using the key insight from Lemma 3.3. While other proofs are available,[4] we provide a new one following the ideas laid out here.

We first state the following lemma.

Lemma 3.6. *Let $x = T(N, P)$ and $Q \subseteq N$. Let P''_Q be such that for all $q \in Q$, P''_q top-ranks x_q and ranks the remaining houses in any order. Denote $P'' = (P''_Q, P_{-Q})$. Then $T(N, P'') = x$.*

That is, if a subset of agents deviate and top-rank the houses they receive in TTC, the resulting TTC allocation is the same. Similar claims are proven in Miyagawa (2002) and Pápai (2000), but we provide a short proof for convenience.

Proof of Lemma 3.6. We show by strong induction on the steps of $TTC(N, P)$ that there is some order of cycles of $TTC(N, P'')$ such that $S_k(P'') = S_k(P)$ for all k.

Step $t = 1$. For each $i \in S_1(P)$, x_i is top-ranked under P_i. It is also top-ranked under P''_i. Therefore, the same cycle exists in the graph at step 1 of $TTC(N, P'')$, so there is an order of cycles of $TTC(N, P'')$ such that $S_1(P'') = S_1(P)$.

Step $t = k$. Suppose that there is an order of cycles of $TTC(N, P'')$ such that $S_t(P'') = S_t(P)$ for all $t < k$. Under this order, $N \setminus R_{k-1}(P'') = N \setminus R_{k-1}(P)$. That is, the remaining agents and houses at step k are the same under either preference profile. In particular, all $i \in S_k(P)$ remain at step k of $TTC(N, P'')$. For each $i \in S_k(P)$, x_i is i's top-ranked house under P_i among $N \setminus R_{k-1}(P)$. For any $i \in Q^C \cap S_k(P)$, since $P''_i = P_i$ and $N \setminus R_{k-1}(P'') = N \setminus R_{k-1}(P)$, x_i is i's top-ranked house under P''_i among remaining houses at step k of $TTC(N, P'')$. For any $i \in Q \cap S_k(P)$, x_i is i's top-ranked house under P''_i and remains at step k of $TTC(N, P'')$. Therefore, each $i \in S_k(P)$ remains at step k of $TTC(N, P'')$ and top-ranks x_i among the remaining houses. As a result, the same cycle exists in the graph at step k of $TTC(N, P'')$, so there is an order of cycles of $TTC(N, P'')$ such that $S_k(P'') = S_k(P)$.

Thus, there exists an order of cycles of $TTC(N, P'')$ such that $S_k(P'') = S_k(P)$ for all k. It follows immediately that $T(N, P'') = T(N, P)$. □

We now prove strong group strategyproofness by applying Lemmas 3.3 and 3.6.

Proof of Theorem 3.5. Let $Q \subseteq N$ and P'_Q be a misreport. Denote $P' = (P'_Q, P_{-Q})$, $x = T(N, P)$, and $x' = T(N, P')$. Suppose there exists $i \in Q$ such that $x'_i P_i x_i$. We seek to show that some $j \in Q$ is strictly worse off under x'.

Define P''_Q such that for each $q \in Q$, P''_q top-ranks x'_q and preserves the rest of the rankings in P_q. By Lemma 3.6, $T(N, P'') = x'$, where $P'' = (P''_Q, P_{-Q})$.

[4] Such as Moulin (1995).

Applying Lemma 3.3 to (N, P'') and x', there exists $j \in S_{\kappa(P)}$ such that $w_h P''_j x_j$ for some $h \in N \setminus R_{\kappa-1}(P)$. Since $j \in S_{\kappa(P)}$ and $h \in N \setminus R_{\kappa-1}(P)$, we have $x_j P_j w_h$. Therefore, $P_j \neq P''_j$ and $j \in Q$. The only change from P_j to P''_j is to top-rank x'_j under P''_j, so it must be that $w_h = x'_j$. Thus, $x_j P_j x'_j$ as desired. $\square$

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Appendix A

Appendix to Chapter 1

A.1 Proofs

We provide proofs for the results in the main text. Note that individual rationality (IR) of $\mathrm{TTC}_{\succ}$ follows immediately from IR of TTC and the fact that $\mathrm{TTC}_{\succ}(R) \equiv \mathrm{TTC}(R_{\succ})$. That is, TTC is IR according to $R_{\succ}$, which implies $\mathrm{TTC}_{\succ}$ is IR according to R.

Given a market (N, H, w, R) and mechanism $\mathrm{TTC}_{\succ}$, let $S_k(R)$ be the agents in the kth cycle executed in the process of $\mathrm{TTC}_{\succ}(R)$. Denote $\bar{S}_{k-1}(R) = \cup_{\ell=1}^{k-1} S_\ell(R)$. Where the risk of confusion is minimal (generally when not dealing with strategyproofness), we suppress the dependence on R and simply refer to S_k and $\bar{S}_{k-1}$. While $\mathrm{TTC}_{\succ}(R)$ may execute cycles in different orders, the cycles are always the same for a given market, so the order will be unimportant.

We will appeal to the following fact.

Fact A.1. *Fix a market (N, H, w, R) and a tie-breaking profile $\succ$. Let $x = TTC_{\succ}(R)$. If $i \in S_k$ and $x_j P_i x_i$, then $j \in \bar{S}_{k-1}$.*

In words, if i strictly prefers some object to x_i, that object must have been assigned in an earlier cycle. Fact A.1 follows from the definitions: $x_j P_i x_i$ implies $x_j P_{i,\succ} x_i$, and $\mathrm{TTC}_{\succ}(R) \equiv \mathrm{TTC}(R_{\succ})$. If a better object than x_i is available, i will point at it; thus i cannot be assigned to x_i until the better object is gone.

We also make use of the following technical lemma. We do not think it is of interest on its own, but it will be useful in the proceeding proofs.

Lemma A.2. *Fix (N, H, w) and a domain $\mathcal{R}$. For any preference relations $R_\alpha, R_\beta \in \mathcal{R}$ and (not necessarily distinct) houses $h_1, h_2 \in H$, suppose A and B satisfy*

$$\{i \in N : w_i P_\alpha h_1\} \subseteq A \subseteq \{i \in N : w_i R_\alpha h_1\}$$

and

$$\{i \in A^c : w_i P_\beta h_2\} \subseteq B \subseteq \{i \in A^c : w_i R_\beta h_2\}.$$

That is, A includes all agents whose endowments are strictly preferred to h_1 according to R_α, and perhaps some whose endowments are indifferent. Among the remainder, B includes all those whose endowments are strictly preferred to h_2 according to R_β, and perhaps some whose endowments are indifferent.

Let R be any preference profile in $\mathcal{R}^n$ such that $R_i = R_\alpha$ for all $i \in A$ and $R_i = R_\beta$ for all $i \in B$. Let $\succ$ be any tie-breaking profile such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. That *is, every agent's tie-breaking rule prioritizes himself first. Denote $x = TTC_{\succ}(R)$. Then $x_i = w_i$ for all $i \in A \cup B$.*

Proof of Lemma A.2. We first note that since $i \succ_i j$ for all $i \neq j$, an immediate consequence of $\text{TTC}_{\succ}$ is that if $x_i I_i w_i$, then $x_i = w_i$. To see this, observe that if w_i is available, i will never point to any other $h I_i w_i$.

First we show that $x_i = w_i$ for all $i \in A$. Toward a contradiction, suppose that $U := \{i \in A : x_i \neq w_i\}$ (for "upper") is nonempty. IR and the fact above imply $U = \{i \in A : x_i P_i w_i\}$. Fix some $i \in U$ such that $w_i R_\alpha w_{i'}$ for all $i' \in U$; informally, i's endowment is one of U's favorites among their endowments. Since $i \in A$, by construction $R_i = R_\alpha$; therefore $x_i P_i w_i$ implies $x_i P_\alpha w_i$. Consider $j \in N$ endowed with x_i; i.e., $x_i = w_j$. We have $(x_i =) w_j P_\alpha w_i$ and $w_i P_\alpha h_1$ by construction, so $j \in A$. We also have $x_j \neq w_j$, so $j \in U$. But since $w_j P_\alpha w_i$, $j \in U$ contradicts the assumption that $w_i R_\alpha w_{i'}$ for all $i' \in U$.

Next we show that $x_i = w_i$ for all $i \in B$. This follows nearly the same logic as above. Suppose that $U = \{i \in B : x_i \neq w_i\} = \{i \in B : x_i P_i w_i\}$ is nonempty. Fix some $i \in U$ such that $w_i R_\beta w_{i'}$ for all $i' \in U$. Since $i \in B$, by construction $R_i = R_\beta$; therefore, $x_i P_i w_i$ implies $x_i P_\beta w_i$. Consider $j \in N$ endowed with x_i; i.e., $x_i = w_j$. It must be that $j \in A^c$, because we showed that $x_i = w_i$ for all $i \in A$ and $x_j \neq w_j$. We have $w_j P_\beta w_i$ and $w_i P_\beta h_2$ by construction, so $j \in B$. We also have $x_j \neq w_j$, so $j \in U$. But since $w_j P_\beta w_i$, $j \in U$ contradicts the assumption that $w_i R_\beta w_{i'}$ for all $i' \in U$. $\square$

A.1.1 Proof of Theorem 1.1

Fix (N, H, w). The result is trivial for $|N| = 1$, so assume $|N| \geq 2$.

Proof of 1. $\iff$ 2. First we show that for any objective indifferences domain, all $\text{TTC}_{\succ}$ mechanisms are PE. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks.

Let $x = \text{TTC}_{\succ}(R)$, and suppose that some feasible allocation y Pareto dominates x. Let $U = \{i \in N : y_i P_i x_i\}$ be the set of agents who are strictly better off under y, which must be nonempty. Let k be the first step in the process of $\text{TTC}_{\succ}(R)$ that an agent in U is assigned. Formally, $\bar{S}_{k-1} \cap U = \emptyset$ and $S_k \cap U \neq \emptyset$. Consider some $j \in S_k \cap U$. We have $y_j P_j x_j$, so Fact A.1 implies that $\{i : x_i \in \eta(y_j)\} \subseteq \bar{S}_{k-1}$. That is, anyone assigned to a house in $\eta(y_j)$ was assigned before j. Because $\eta(y_j)$ is finite and $\eta(y_j) \neq \eta(x_j)$, there must be an agent $\ell \in \bar{S}_{k-1}$ for whom $x_\ell \in \eta(y_j)$ but $y_\ell \notin \eta(y_j)$. In words, in order to assign j to y_j, someone who originally received an object in $\eta(y_j)$ under x must receive something else under y. Therefore $\neg(y_\ell I_\ell x_\ell)$. Since y Pareto dominates x, it must be that $y_\ell P_\ell x_\ell$. But then $\ell \in U$, and ℓ was assigned before cycle k, a contradiction.

We now turn to the maximality claim. We show that for any domain $\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for any partition $\mathcal{H}$ of H, there is some $\text{TTC}_{\succ}$ mechanism which is not PE on $\tilde{\mathcal{R}}$. If $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two preference orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. The proof seeks to isolate a pair like in Example 1.4. Taking only the existence of $R_\alpha, R_\beta \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $\text{TTC}_\succ(R)$ is not PE. Without loss of generality, let $w_i = h_i$ for all $i \in N$. Let the preference profile R be given by

$$R_i = \begin{cases} R_\alpha & (\text{having } h_1 I_\alpha h_2) \quad \text{if } i \in A := \{i \in N : w_i R_\alpha w_1\} \setminus \{2\} \\ R_\beta & (\text{having } h_1 P_\beta h_2) \quad \text{if } i \in B := A^c. \end{cases}$$

Note that $R_1 = R_\alpha$ and $R_2 = R_\beta$.

Take any tie-breaking profile $\succ$ such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Denote $x = \text{TTC}_\succ(R)$. It follows from Lemma A.2 that $x = w$. However, note that $w_2 I_1 w_1$ and $w_1 P_2 w_2$, so x is Pareto dominated by $y = (w_2, w_1, w_3, \dots, w_n)$. $\square$

Proof of 1. $\iff$ 3. First we show that for any objective indifferences domain, $\text{TTC}_\succ$ mechanisms are CS. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks. Suppose that the core of (N, H, w, R) is nonempty and contains some allocation y. Let $x = \text{TTC}_\succ(R)$. It suffices to show that $x_i I_i y_i$ for all $i \in N$. We proceed by induction on the steps of $\text{TTC}_\succ(R)$.

Step 1. By definition of $\text{TTC}_\succ$, for all $i \in S_1$ we have $x_i R_i h$ for all $h \in H$; that is, agents in the first cycle receive (one of) their favorite houses. Therefore $x_i R_i y_i$ for all $i \in S_1$. Suppose there is some $j \in S_1$ such that $x_j P_j y_j$. Then S_1 and x block y, contradicting the assumption that y is in the core. Thus, $x_i I_i y_i$ for all $i \in S_1$.

Step k. Assume that $x_i I_i y_i$ for all $i \in \bar{S}_{k-1}$. Suppose that $y_j P_j x_j$ for some $j \in S_k$. By construction of objective indifferences, $\eta(y_j) \neq \eta(x_j)$. By Fact A.1, $\{i \in N : x_i \in \eta(y_j)\} \subseteq \bar{S}_{k-1}$. Because $\eta(y_j)$ is finite, if $\eta(y_j) \neq \eta(x_j)$, there must be an agent ℓ in $\bar{S}_{k-1}$ for whom $x_\ell \in \eta(y_j)$ but $y_\ell \notin \eta(y_j)$. Therefore $\neg(y_\ell I_\ell x_\ell)$, contradicting the assumption that $x_i I_i y_i$ for all $i \in \bar{S}_{k-1}$. Thus $x_i R_i y_i$ for all $i \in S_k$. Now suppose there is some $j \in S_k$ such that $x_j P_j y_j$. Then S_k and x block y, contradicting that y is in the core.

Thus $x_i I_i y_i$ for all $i \in N$, so x must also be in the core as desired. (Since y was an arbitrary allocation in the core, this also proves Corollary 1.2.)

We now turn to the maximality claim. We show that for any domain $\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for some partition $\mathcal{H}$, there is some $\text{TTC}_\succ$ which is not CS on $\tilde{\mathcal{R}}$. As previously, if $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. Without loss of generality, let h_1 be R_β's highest ranked house such that $h_1 I_\alpha h_2$ and $h_1 P_\beta h_2$ are true. Formally,

$$h_1 R_\beta h \text{ for all } h \in H \text{ such that } h I_\alpha h_2. \qquad (*)$$

Also without loss of generality, let $w_i = h_i$ for all $i \in N$.

The following construction is the same as in the previous part. Define $A = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$ and consider the preference profile where $R_i = R_\alpha$ for all $i \in A$ and $R_i = R_\beta$ for all $i \in B := A^c$. Take any tie-breaking profile $\succ$ such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Denote $x = \mathrm{TTC}_\succ(R)$. It follows from Lemma A.2 that $x = w$. As before, x is Pareto dominated by $y = (w_2, w_1, w_3, \dots, w_n)$, so x is not in the core (x is blocked by the grand coalition N and y).

It remains to show that y is in the core. Toward a contradiction, suppose there is a coalition Q and allocation z that blocks y. Let $U = \{i \in Q : z_i P_i y_i\}$ be the strict improvers under z, which must be nonempty. We proceed in steps.

Step 1. We show that $U \cap A = \emptyset$.

Intuitively, if any agent in A strictly prefers a house to his assignment, this house is also in w_A, preventing mutual improvements within A. Toward a contradiction, suppose $U_A := U \cap A$ is nonempty; note that all agents in U_A have the preference ranking R_α. Take $i \in U_A$ such that $z_i R_\alpha z_{i'}$ for all $i' \in U_A$. That is, i is the best off improver in U_A (or one of the best off improvers). Observe that $z_i P_\alpha w_i$ since $i \in U$ and $w_i R_\alpha w_1$ since $i \in A$. Thus by construction of R_α and A, we have $J := \left\{j \in Q : w_j I_\alpha z_i\right\} \subseteq A$. That is, objects equivalent to z_i are also owned by members of A. Also, J is nonempty since $z_i \in w_J$. Further, note $1 \notin J$ since $z_i P_\alpha w_1$. Agent 1 is the only agent in A who does not receive $y_i = w_i$, so $y_j = w_j$ for all $j \in J$.

Since w_J is finite and $i \notin J$, in order for i to receive $z_i \in w_J$, some $j \in J$ must have $z_j \notin w_J$. But then since $j \in Q \cap A$, it must be that $z_j P_\alpha w_j (= y_j)$. Then $j \in U_A$, contradicting the assumption that $z_i R_\alpha z_{i'}$ for all $i' \in U_A$.

Step 2. We show that $Q \cap A \neq \emptyset$.

We show that the coalition Q cannot generate improvements with w_Q if all its members have the same preference R_β. Toward a contradiction, suppose $Q \subseteq A^c$; then $R_i = R_\beta$ for all $i \in Q$. Let $i \in U$ be such that $z_i R_\beta z_{i'}$ for all $i' \in U$; that is, i is the best off improver.

Now consider $J := \{j \in Q : w_j I_\beta z_i\} \subseteq A^c$. Since w_J is finite and $i \notin J$, in order for i to receive $z_i \in w_J$, some $j \in J$ must have $z_j \notin w_J$. Then since $j \in Q$, it must be $z_j P_\beta w_j$. If $j \neq 2$, then $y_j = w_j$ so $z_j P_\beta y_j$. But then $j \in U$, contradicting $z_i R_\beta z_{i'}$ for all $i' \in U$.

Now suppose $j = 2$. Since $w_1 \in w_A$, we have $z_2 \neq w_1 (= y_2)$. Then $z_2 R_\beta w_1$ and $z_2 \neq w_1$. That is, agent 2 must receive a house besides w_1 endowed to $Q \subseteq A^c$. Consider $L := \{\ell \in Q : w_\ell R_\beta w_1\} \subseteq A^c$, and note that $2 \notin L$ since $w_1 P_\beta w_2$. Since w_L is finite, in order for $z_2 \in w_L$, some $\ell \in L$ must receive $z_\ell \notin w_L$. Then it must be that $w_\ell P_\beta z_\ell$. Since $\ell \in A^c$ and $\ell \neq 2$, we have $y_\ell = w_\ell$. This gives $y_\ell P_\beta z_\ell$, contradicting that every agent in Q weakly improves under z.

Step 3. We conclude by showing that the results in Steps 1 and 2 cannot both be true.

Without loss of generality, assume Q forms a single trading cycle under z.[1] Since Q contains agents in both A and A^c in the same cycle, there exists some agent $\bar{b} \in A^c$ such that $z_{\bar{b}} \in w_A$ and some agent $\bar{a} \in A$ such that $z_{\bar{a}} \in w_{A^c}$. It must be that $z_{\bar{a}} = w_2$, since $z_{\bar{a}} I_\alpha w_{\bar{a}}$ and by construction $w_{\bar{a}} P_\alpha w_i$ for any $i \in A^c \backslash \{2\}$. For the same reason, $\bar{a}$ must be the only agent in $A \cap Q$ who receives a house from w_{A^c}.

Since only one agent in $A \cap Q$ receives a house from w_{A^c}, only one agent from $A^c \cap Q$ receives a house from w_A. We can represent the trading cycle Q as

$$a_1 \to a_2 \to \ldots \to a_m \to \bar{a} \to 2 \to b_1 \to \ldots \to \bar{b} \to a_1$$

where $a_1, \ldots, a_m, \bar{a} \in A$ and $2, b_1, \ldots, \bar{b} \in A^c$, and $z_{a_1} = w_{a_2}$, etc.

Since $U \cap A = \emptyset$, no $a \in \{a_1, \ldots, a_m, \bar{a}\}$ strictly improves under z; that is, $z_a I_\alpha y_a$ for each a. Recall that $y_i I_\alpha w_i$ for all $i \in A$, since either 1) $y_i = w_i$ or 2) $i = 1$, $y_1 = w_2$ and $w_2 I_1 w_1$. Thus $w_{a_1} I_\alpha w_{a_2} I_\alpha \cdots I_\alpha w_2$.

So $z_b R_\beta y_b$ for all $b \in \{2, b_1, \ldots, \bar{b}\}$, with a strict relation for at least one b. Since $y_2 = w_1$, we have $w_{b_1} R_\beta w_1$. For all other b, we have $y_b = w_b$, giving us $w_{a_1} R_\beta w_{\bar{b}}$, ..., $w_{b_2} R_\beta w_{b_1}$. At least one of these relations is strict; combining these gives $w_{a_1} P_\beta w_1$.

Then $w_{a_1} I_\alpha w_2$ and $w_{a_1} P_\beta w_1$. But this contradicts the assumption $(*)$.

□

A.1.2 Proof of Proposition 1.3

Proof of Proposition 1.3. Fix a market (N, H, w, R) and a tie-breaking profile $\succ$. Denote $x = \text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$. Toward a contradiction, suppose there exists a coalition $Q \subseteq N$ and allocation y that weakly blocks x. That is, $y_Q = w_Q$ and $y_i P_i x_i$ for all $i \in Q$. Then we also have $y_i P_{i,\succ} x_i$ for all $i \in Q$, since the tie-breaking procedure preserves strict comparisons. Thus y and Q form a weak blocking coalition against x *according to the strict preferences* $R_\succ$. This violates the core property of TTC with strict preferences; thus $\text{TTC}_\succ$ always selects a mechanism in the weak core.

Since $\text{TTC}_\succ$ always produces an allocation, this also proves the existence of a weak core allocation. □

A.1.3 Proof of Theorem 1.4

Before proving Theorem 1.4, we review an important property of $\text{TTC}_\succ$ and state a useful lemma. Let $L(h, R_i) = \{h' \in H : h R_i h'\}$ be the lower contour set of a preference ranking R_i at house h.

Monotonicity (MON). A mechanism f is **monotone** if $f(R) = f(R')$ for any preference profiles R and R' such that $L(f_i(R), R_i) \subseteq L(f_i(R), R'_i)$ for all $i \in N$.

[1] If the agents in Q formed two or more trading cycles, then some cycle S must contain an agent i such that $z_i P_i y_i$, and this cycle S suffices to form a blocking coalition.

That is, a mechanism f is monotone if, whenever any set of agents move their allocations upwards in their preference rankings, the allocation remains the same. It is straightforward to show that TTC is monotone for strict preferences (e.g., Takamiya, 2001). Since $\text{TTC}_{\succ}(R) \equiv \text{TTC}(R_{\succ})$ for any R and $\succ$, it follows directly that $\text{TTC}_{\succ}$ mechanisms are monotone.

The following result is an immediate consequence of Lemma 1 from Sandholtz and Tai (2024) and the fact that $\text{TTC}_{\succ}(R) \equiv \text{TTC}(R_{\succ})$.

Lemma A.3 (Sandholtz and Tai, 2024)**.** *For any R, R', let $x = TTC_{\succ}(R)$ and $y = TTC_{\succ}(R')$. Suppose there is some i such that $y_i P_{i,\succ} x_i$. Then there exists some agent j and house h such that $x_j P_{j,\succ} h$ and $h P'_{j,\succ} x_j$.*

In other words, if i is to receive a better house, some other agent j must misreport his preferences and rank a worse house above his original allocation.

We now provide our proof of Theorem 1.4.

Proof of Theorem 1.4. Fix (N, H, w). The result is trivial for $|N| = 1$, so assume $|N| \geq 2$. First we show that for any objective indifferences domain, $\text{TTC}_{\succ}$ mechanisms are GSP. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so assume that $\mathcal{H}$ has at least two blocks. Suppose $Q \subseteq N$ reports R'_Q where $R' = (R'_Q, R_{-Q}) \in \mathcal{R}(\mathcal{H})^n$. Let $y = \text{TTC}_{\succ}(R')$. We will show that if $y_i P_i x_i$ for some $i \in Q$, then $x_j P_j y_j$ for some $j \in Q$.

Let $R'' = (R''_Q, R_{-Q})$ be the preference profile in $\mathcal{R}(\mathcal{H})^n$ such that for each $i \in Q$, R''_i top-ranks the houses in $\eta(y_i)$ and otherwise preserves the ordering of R_i. That is, for any $h \in \eta(y_i)$ and $h' \notin \eta(y_i)$, $h P''_i h'$; otherwise, $h R''_i h'$ if and only if $h R_i h'$. Let $z = \text{TTC}_{\succ}(R'')$. By monotonicity of $\text{TTC}_{\succ}$, $z = y$. Fix $i \in Q$ such that $y_i P_i x_i$. Since $z = y$, we have $z_i P_i x_i$ and consequently $z_i P_{i,\succ} x_i$. Applying Lemma A.3, there must be some $j \in Q$ and $h \in H$ such that $x_j P_{j,\succ} h$ but $h P''_{j,\succ} x_j$. Note that $h \notin \eta(x_j)$; if it were, then for any $R, R'' \in \mathcal{R}(\mathcal{H})^n$, $x_j P_{j,\succ} h$ if and only if $x_j P''_{j,\succ} h$. Therefore, $x_j P_j h$ and $h P''_j x_j$.[2] The only change from R_j to R''_j is to top-rank the houses in $\eta(y_j)$, so it must be that $h \in \eta(y_j)$. But then $x_j P_j y_j$, as desired.

Next we show that for any symmetric domain $\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for any partition $\mathcal{H}$ of H, $\text{TTC}_{\succ}$ mechanisms are not GSP on $\tilde{\mathcal{R}}$. As before, if $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. The symmetric requirement also necessitates that $\tilde{\mathcal{R}}$ contains some R_γ such that $h_2 P_\gamma h_1$. Taking only the existence of $R_\alpha, R_\beta, R_\gamma \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $\text{TTC}_{\succ}(R)$ is not GSP.

[2]This is where the restriction to objective indifferences is used. Under general indifferences, this is not necessarily true.

Without loss of generality, assume $w_i = h_i$ for all $i \in N$. Consider the preference profile $R \in \tilde{\mathcal{R}}^n$ where

$$R_i = \begin{cases} R_\alpha & \text{(having } w_1 I_\alpha w_2\text{)} \quad \text{if } i \in A := \{i \in N : w_i R_\alpha w_1\} \setminus \{2\} \\ R_\beta & \text{(having } w_1 P_\beta w_2\text{)} \quad \text{if } i \in B := \{i \in A^c : w_i R_\beta w_1\} \cup \{2\} \\ R_\gamma & \text{(having } w_2 P_\gamma w_1\text{)} \quad \text{if } i \in C := (A \cup B)^c. \end{cases}$$

Note that $R_1 = R_\alpha$ and $R_2 = R_\beta$. Let $\succ$ be any tie-breaking profile such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Additionally, let agent 2's tie-breaking order be $2 \succ_2 1 \succ_2 \cdots$. Let $x = \text{TTC}_\succ(R)$. Recall that by individual rationality of $\text{TTC}_\succ$ and our choice of $\succ$, for any agent i if $x_i \neq w_i$ then $x_i P_i w_i$. We proceed in two steps. In the first step, we show that under x, agent 1 keeps his endowment and agent 2 prefers agent 1's assignment to his own. In the second step, we show that agents 1 and 2 swap when agent 1 misreports his preferences.

Step 1. We show that $x_1 = w_1$ and $w_1 P_2 x_2$.

Let $\bar{A} = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$ and $\bar{B} = \{i \in \bar{A}^c : w_i R_\beta w_1\}$. Note that $R_i = R_\alpha$ for all $i \in \bar{A}$ and $R_i = R_\beta$ for all $i \in \bar{B}$. It follows from Lemma A.2 (with w_1 filling the roles of both h_1 and h_2) that $x_i = w_i$ for all $i \in \bar{A} \cup \bar{B}$. Therefore $x_1 = w_1$. Moreover since $R_2 = R_\beta$, if $x_2 R_2 w_1$ then $x_2 = w_i$ for some $i \in \bar{A} \cup \bar{B}$, a contradiction. So $w_1 P_2 x_2$.

Step 2. Suppose that agent 1 misreports $R'_1 = R_\gamma$. Let $R' = (R'_1, R_{-1})$ and $y = \text{TTC}_\succ(R')$. We show that $y_1 = w_2$ and $y_2 = w_1$.

Let $\bar{A} = \{i \in N : w_i R_\alpha w_1\} \setminus \{1, 2\}$ and $\bar{B} = \{i \in \bar{A}^c : w_i R_\beta w_1\} \setminus \{1\}$. Note that $R'_i = R_\alpha$ for all $i \in \bar{A}$ and $R'_i = R_\beta$ for all $i \in \bar{B}$. It follows from Lemma A.2 that $y_i = w_i$ for all $i \in \bar{A} \cup \bar{B}$.

Let $\bar{C} = \{i \in (\bar{A} \cup \bar{B})^c : w_i R_\gamma w_2\} \setminus \{2\}$. We will show that $y_i = w_i$ for all $i \in \bar{C}$. Note that $R'_i = R_\gamma$ for all $i \in \bar{C}$. Toward a contradiction, suppose $U := \{i \in \bar{C} : y_i \neq w_i\} = \{i \in \bar{C} : y_i P_\gamma w_i\}$ is nonempty. Fix $i \in U$ such that $w_i R_\gamma w_{i'}$ for all $i' \in U$. Consider j endowed with y_i; that is, $w_j = y_i$. Since $y_j \neq w_j$, it must be $j \notin \bar{A} \cup \bar{B}$, since $y_i = w_i$ for all $i \in \bar{A} \cup \bar{B}$. Then, since $y_j \neq w_j$ and $w_j P_\gamma w_i R_\gamma w_2$, it must be that $j \in U$. But this contradicts the assumption that $w_i R_\gamma w_{i'}$ for all $i' \in U$.

Thus we have shown that $y_i = w_i$ for all $i \in \bar{A} \cup \bar{B} \cup \bar{C}$. Therefore, $y_i = w_i$ for all $i \neq 1$ such that $w_i R_\beta w_1$ and for all $i \neq 2$ such that $w_i R_\gamma w_2$. Consequently, $w_2 R'_1 y_1$ (with indifference only if $y_1 = w_2$) and $w_1 R'_2 y_2$ (with indifference only if $y_2 = w_1$).

Toward a contradiction, suppose that $y_1 \neq w_2$. Then $w_2 P'_1 y_1$. By Fact A.1, this implies that agent 2 was assigned before agent 1 was assigned during the process of $\text{TTC}_\succ(R')$. By definition of $\text{TTC}_\succ$, $x_2 R_2 w_1$. Then $y_2 = w_i$ for some $i \neq 1$, and $w_i R_\beta w_1$. But this contradicts $x_i = w_i$ for all $i \in \bar{A} \cup \bar{B} \cup \bar{C}$. By an analogous argument, it is straightforward to show that $y_2 = w_1$.

Observe that $w_2 I_1 w_1$ and $w_1 P_2 w_2$. Thus, via the misrepresentation R', we have $y_1 I_1 x_1$ but $y_2 P_2 x_2$, showing $\text{TTC}_\succ(R)$ is not GSP. $\square$

A.2 Relation to capacities and priorities

We briefly note that TTC in the objective indifferences setting is not identical to TTC where the objects have capacities and priorities. Intuitively, in Shapley-Scarf markets with objective indifferences, the fixed tie-breaking rule determines which object an agent points at. Conversely, a priority ranking determines which agent an object points at. Consider the following example.

Example A.1. Let the set of schools be $H = \{A, B, C\}$, with C having 2 seats. Let the students be $N = \{a, b, c_1, c_2\}$, where a is "endowed" with A, and so on. Below are a possible priority ranking for the school choice setting and a possible tie-breaking profile for the Shapley-Scarf setting.

A	B	C
a	b	c_1
b	a	c_2
c_1	c_2	a
c_1	c_1	b

School priority for the school choice setting

$\succ_a$	$\succ_b$	$\succ_{c_1}$	$\succ_{c_2}$
c_1	c_2	c_1	c_1
c_2	c_1	c_2	c_2
a	a	a	a
b	b	b	b

Tie-breaking profile for Shapley-Scarf setting

Consider the preference profiles $R = (R_a, R_b, R_{c_1}, R_{c_2})$ and $R' = (R'_a, R'_b, R'_{c_1}, R'_{c_2})$, shown below.

R_a	R_b	R_{c_1}	R_{c_2}
C	C	A	A
A	A	B	B
B	B	C	C

Preference profile R

R'_a	R'_b	R'_{c_1}	R'_{c_2}
C	C	B	B
A	A	A	A
B	B	C	C

Preference profile R'

Note that under R and R', both c_1 and c_2 have the same preferences. TTC with school priorities results in $(A : c_1, B : c_2, C : ab)$ and $(A : c_2, B : c_1, C : ab)$ under R and R' respectively. Note that c_1 must receive his preferred school in either case, since his priority at school C is higher than c_2's priority at school C. In fact, in the school choice setting, since c_1 has higher priority at C than c_2 has, whenever c_1 and c_2 have the same preferences c_1 will weakly prefer his assignment to c_2's assignment. By contrast, in the Shapley-Scarf setting, $\text{TTC}_{\succ}$ results in $(A : c_1, B : c_2, C : ab)$ and $(A : c_1, B : c_2, C : ab)$ under R and R' respectively. Now, c_1 does not necessarily receive his top choice when c_1 and c_2 have the same preferences. Under R', c_2 gets his top choice at the expense of c_1, since $c_2 \succ_b c_1$.

It is not always possible to recreate a school priority order with a tie-breaking profile, and vice versa.

Appendix B

Appendix to Chapter 2

B.1 Proofs

B.1.1 Proofs of Propositions 2.2, 2.3, and 2.5

Proof of Proposition 2.2. The first two conditions are primitives of the model. Now, fix $j \in O_m$. Note that unit demand implies

$$\sum_{j' \neq j} M_{j'} = N - M_j.$$

Then can rewrite condition (Minimum-II) as

$$\begin{aligned} & \sum_{j' \neq j} M_{j'} \leq N - m_j \\ \iff & N - M_j \leq N - m_j \\ \iff & M_j \geq m_j \qquad \text{(Minimum-I)}. \end{aligned}$$

Since $j \in O_m$ is generic, this proves necessity and sufficiency of condition (Minimum-II). $\square$

Proof of Proposition 2.3. Let $X \subseteq [0,1]^{N \times O}$ denote the set of solutions to the system of inequalities in Proposition 2.3. We want to show that $X = \text{lcs}(\Delta D)$.

Let $X' \subseteq \mathbb{R}_+^{N \times O}$ denote the set of solutions to the following related system:

$$\begin{aligned} \sum_j \mu_{ij} &\leq 1 && \forall i \in N && \text{(Unit demand)} \\ -\sum_j \mu_{ij} &\leq -1 && \forall i \in N && \text{(Unit demand)} \\ \mu_j &\leq c_j && \forall j \in O && \text{(Capacity)} \\ -\mu_j &\leq -m_j && \forall j \in O && \text{(Minimum-I)} \end{aligned}$$

Step 1. We show that $X' = \Delta D$.

Observe that $D \subseteq X'$ and that X' is convex. Thus, $\Delta D \subseteq X'$. Since X' is also compact, by a simple version of the Krein-Milman theorem, in order to show that $X' \subseteq \Delta D$ it suffices to show that $\text{ext}(X') \subseteq D \subseteq \Delta D$. It is obvious that any integral solution $\mu \in X'$ is also in D. It remains to show that X' has no non-integral extreme points.

For any $\mu \in \mathbb{R}_+^{N\times O}$, vectorize it by stacking columns:

$$\text{vec}(\mu) = (\mu_{11}, \mu_{21}, \dots, \mu_{N1}, \mu_{12}, \mu_{22}, \dots, \mu_{N2}, \dots, \mu_{1O}, \dots, \mu_{NO}) \in \mathbb{R}_+^{NO}.$$

Let $m = (m_1, m_2, \dots, m_O) \in \mathbb{Z}_+^O$ and let $c = (c_1, c_2, \dots, c_O) \in \mathbb{Z}_+^O$. Note that we can write $X' = \{\text{vec}^{-1}(\mu) : \mu \in \mathbb{R}_+^{NO}, A\mu \leq b\}$, where

$$A = \begin{bmatrix} -(I_O \otimes \mathbf{1}_N^\top) \\ (I_O \otimes \mathbf{1}_N^\top) \\ (\mathbf{1}_O^\top \otimes I_N) \\ -(\mathbf{1}_O^\top \otimes I_N) \end{bmatrix} \quad b = \begin{bmatrix} -m \\ c \\ \mathbf{1}_N \\ -\mathbf{1}_N \end{bmatrix}$$

Consider the matrix

$$B = \begin{bmatrix} (I_O \otimes \mathbf{1}_N^\top) \\ (\mathbf{1}_O^\top \otimes I_N) \end{bmatrix}$$

Note that B is totally unimodular (TUM), since it is a $\{0,1\}$-matrix such that 1) each column has at most two non-zero entries, and 2) the rows can be partitioned into two sets such that each column has at most one 1 in each set. It follows that A is also TUM, since duplicating rows and multiplying rows by -1 preserves TUM. Since b is integral and A is TUM, by the Hoffman-Kruskal theorem the extreme points of X' are all integral.

Let $X'' \subseteq [0,1]^{N\times O}$ denote the set of solutions to the following system:

$$\sum_j \mu_{ij} \leq 1 \qquad \forall i \in N \qquad \text{(Unit demand)}$$

$$-\sum_j \mu_{ij} \leq -1 \qquad \forall i \in N \qquad \text{(Unit demand)}$$

$$\mu_j \leq c_j \qquad \forall j \in O \qquad \text{(Capacity)}$$

$$\sum_{j\in O\setminus S} \mu_j \leq N - \sum_{j\in S} m_j \qquad \forall S \subseteq O_m \qquad \text{(Minimum-III)}$$

Step 2. We show that $X'' = X'$.

It suffices to show that every μ satisfying unit demand and capacity constraints satisfies condition (Minimum-I) if and only if it satisfies condition (Minimum-III).

First, suppose μ satisfies condition (Minimum-I). Fix any $S \subseteq O_m$. Since μ satisfies condition (Minimum-I), we know that

$$-\sum_{j\in S} \mu_j \leq -\sum_{j\in S} m_j.$$

Therefore,

$$N - \sum_{j\in S} \mu_j \leq N - \sum_{j\in S} m_j.$$

But since μ satisfies unit demand, we can rewrite $N - \sum_{j \in S} \mu_j$ as $\sum_{j \in O \setminus S} \mu_j$. Thus,

$$\sum_{j \in O \setminus S} \mu_j \leq N - \sum_{j \in S} m_j.$$

Since $S \subseteq O_m$ was arbitrary, μ satisfies condition (Minimum-III).

Now, suppose μ satisfies condition (Minimum-III). Fix any $j \in O$. Since μ satisfies condition (Minimum-III), we know that

$$\sum_{j' \neq j} \mu_{j'} \leq N - m_j.$$

But since μ satisfies unit demand, we can rewrite $\sum_{j' \neq j} \mu_j$ as $N - \mu_j$. Thus, $N - \mu_j \leq N - m_j$, implying $-\mu_j \leq -m_j$. Since $j \in O$ was arbitrary, μ satisfies condition (Minimum-I).

Step 3. We conclude by showing that $X = \text{lcs}(X'') = \text{lcs}(\Delta D)$.

Note that $\text{lcs}(X'') \subseteq X$. It is obvious that $X'' \subseteq X$. Therefore, any $\mu \in \text{lcs}(X'')$ will also satisfy unit demand, capacity, and minimum constraints.

It remains to show that $X \subseteq \text{lcs}(X'')$. It is obvious that any $\mu \in X$ that also satisfies the relaxed unit demand condition is in $X'' \subseteq \text{lcs}(X'')$. Therefore, we will show that for every $\mu \in X$ such that $\sum_j \mu_{ij} < 1$ for some $i \in N$, there exists some $\mu' \in X''$ such that $\mu \leq \mu'$. In other words, any strictly sub-random $\mu \in X$ can be completed to an implementable random allocation $\mu' \in X''$.

Fix some strictly sub-random $\mu \in X$. Denote unassigned demand as $A^* = N - \sum_i \sum_j \mu_{ij}$. Since μ is strictly sub-random, $A^* > 0$. For any subset $S \subseteq O$, we can write

$$A^* = N - \sum_i \sum_{j \in S} \mu_{ij} - \sum_i \sum_{j \in O \setminus S} \mu_{ij}.$$

Let O^* be the set of objects that do not meet their minimum requirements. That is,

$$O^* := \left\{ j \in O : m_j - \sum_i \mu_{ij} > 0 \right\} \subseteq O_m.$$

Now, rewrite A^* as

$$A^* = N - \sum_i \sum_{j \in O^*} \mu_{ij} - \sum_i \sum_{j \in O \setminus O^*} \mu_{ij}.$$

Since μ satisfies condition (Minimum-III), we have

$$\sum_i \sum_{j \in O \setminus O^*} \mu_{ij} \leq N - \sum_{j \in O^*} m_j.$$

Combining the above two lines yields

$$\begin{aligned} A^* = N - \sum_i \sum_{j \in O^*} \mu_{ij} - \sum_i \sum_{j \in O \setminus O^*} \mu_{ij} &\geq N - \sum_i \sum_{j \in O^*} \mu_{ij} - \left(N - \sum_{j \in O^*} m_j \right) \\ \Rightarrow A^* &\geq \sum_{j \in O^*} m_j - \sum_i \sum_{j \in O^*} \mu_{ij}. \end{aligned}$$

That is, the unassigned demand A^* is sufficient to meet the remaining minimum requirements. If the relation is strict, any remaining unassigned demand after satisfying all minimum requirements can be allocated to objects with remaining capacity. (In the model primitives it is assumed that $\sum_j c_j \geq N$.) Thus any strictly sub-random $\mu \in X$ can be completed to an implementable random allocation in $X'' = \Delta D$. □

Proof of Proposition 2.5. Let $\mu \in [0,1]^{N\times O}$ satisfy unit demand (relaxed) and capacity constraints. It suffices to show that μ satisfies condition (Minimum-III) if and only if it satisfies condition (Minimum-IV). For all $S \subseteq O_m$, condition (Minimum-III) requires

$$\sum_{j\in O\setminus S} \mu_j \leq N - \sum_{j\in S} m_j \tag{Minimum-III}$$

$$\iff \sum_{j\in O} \mu_j - \sum_{j\in S} \mu_j + \sum_{j\in S} m_j \leq N$$

$$\iff \sum_{j\in O} \mu_j + \sum_{j\in S} (m_j - \mu_j) \leq N.$$

Consider the terms $(m_j - \mu_j)$ and the subset $S \subseteq O_m$. The above inequality is binding if and only if $\{j : m_j - \mu_j > 0\} \subseteq S \subseteq \{j : m_j - \mu_j \geq 0\}$. For all other S, this condition is redundant. Thus, we can equivalently write this condition as

$$\sum_{j\in O} \mu_j + \sum_{j\in O_m} \max\{m_j - \mu_j, 0\} \leq N$$

$$\iff \sum_{j\in O\setminus O_m} \mu_j + \sum_{j\in O_m} \max\{m_j, \mu_j\} \leq N$$

$$\iff \sum_{j\in O} \max\{m_j, \mu_j\} \leq N. \tag{Minimum-IV}$$

□

B.1.2 Proof of Theorem 2.4: Envy free

Proof that MPS is envy free. It is clear from the definition of our algorithm that it satisfies **common availability**; an object is always available to everyone or to no one. Formally, our capacity and minimum conditions can be written in terms of the marginals μ_j over agents. For $j \in O$, let $t_j \in [0,1]$ be the time at which object j becomes unavailable during the process of MPS. (By common availability, t_j is unique for each $j \in O$.) A formal definition of t_j is contained in the next section.

Without loss of generality, consider agent 1, and label objects in O such that $a \succ_1 b \succ_1 \cdots$. Since 1 consumes a over the time interval $[0, t_a)$, his allocation μ_a satisfies $\mu_{1a} \geq \mu_{ia}$ for any $i \in N$. Now consider $\{a, b\}$. Agent 1 consumes from $\{a, b\}$ over the interval $[0, \max\{t_a, t_b\})$, so $\mu_{1a} + \mu_{1b} \geq \mu_{ia} + \mu_{ib}$ for all $i \in N$. Iterating through all objects in O gives the desired result. □

B.1.3 Proof of Theorem 2.4: Weak strategyproof

Without loss of generality suppose that agent 1 misreports his preferences as $\succ_1'$, and let $\succ' = (\succ_1', \succ_{-1})$. Also without loss of generality, let $a \in O$ be 1's favorite object. Denote $\mu = MPS(\succ)$ and $\mu' = MPS(\succ')$.

Definition B.1. Let agent i's **eating function** be $e_i : [0,1] \to O$, denoting his consumption throughout $MPS(\succ)$. Note that e_i is right-continuous; that is, for all $t \in [0,1)$, $\exists \varepsilon > 0$ such that $e_i(s) = e_i(t)$ for all $s \in [t, t+\varepsilon)$. Let $e = (e_i)_{i \in N}$ denote an eating profile.

Let $e_i'(\cdot)$ be i's eating function under $MPS(\succ')$. Note we may have $e_i'(t) \neq e_i(t)$ for any $i \in N$, not just agent 1.

Finally, let $n_j(t,e) = |\{i \in N : e_i(t) = j\}|$.

Definition B.2. For any set of objects $S \subseteq O$, perhaps a singleton, let $n_S(t,e)$ be the number of agents consuming S at time t under eating profile e. Denote

$$\mu_S(t,e) = \int_0^t n_S(s,e)\, ds.$$

to be the amount consumed from S up to time t under e.

Similarly, for a set of agents $A \subseteq N$, let $n_{A,S}(t,e)$ be the number of agents among A consuming S at time t under eating profile e. Denote

$$\mu_{A,S}(t,e) = \int_0^t n_{A,S}(s,e)\, ds.$$

Recall from Proposition 2.5 that there are two possible constraints that can bind on an object:

$$\mu_j \leq c_j$$
$$\sum_{j \in O} \max\{m_j, \mu_j\} \leq N.$$

Definition B.3. We refer to two conditions on an object j, time t, and eating schedule e

$$\mu_j(t,e) < c_j \tag{B.1}$$
$$\exists \varepsilon > 0 \text{ s.t. } \max\{m_j, \mu_j(t,e) + \varepsilon\} + \sum_{k \neq j} \max\{m_k, \mu_k(t,e)\} \leq N \tag{B.2}$$

If condition (B.1) fails for j at (t,e), we say that j is bound by (B.1) at (t,e). That is, it is at its capacity. Likewise, if condition (B.2) fails for j at (t,e), we say that j is bound by (B.2) at (t,e). We say j is available at (t,e) if it satisfies both (B.1) and (B.2) at (t,e).

Let $\tau = \inf\{t : \exists j \in O \text{ s.t. } j \text{ fails } (B.2) \text{ at } (t,e)\}$. That is, τ is the time at which $\sum_{j \in O} \max\{m_j, \mu_j\} = N$. After τ, note that only objects such that $\mu_j(t,e) < m_j$ can be consumed further. Let τ' be the analogous time under the eating profile e'.

Finally, let $t_j = \inf\{t : j \text{ fails } (B.1) \text{ or } (B.2) \text{ at } (t, e)\}$. That is, t_j is the closing time of object j under e; no more can be consumed of j after t_j. Let t'_j be the analogous time under the eating profile e'.

Suppose that for an eating profile e, at some time $t \in [0, 1]$ there exists an object j that fails condition (B.2). That is, $\mu_j(t, e) + \sum_{k \neq j} \max\{m_k, \mu_k(t, e)\} = N$. Note that $t \geq \tau$. Define $S := \{j \in O : j \text{ fails (B.1) or (B.2) at } (t, e)\}$. Note then that $O \backslash S = \{j : \mu_j(t, e) < m_j\}$. Thus $\mu_S(t, e)$ is an upper bound on μ_S at any time and under any eating function, since we have $\mu_S(t, e) = N - \sum_{j \notin S} m_j$.

Definition B.4. Let $\delta(t)$ be the amount of time from $[0, t)$ which agent 1 has spent eating differently between the reports:

$$\delta(t) = \int_0^t 1\{e_1(s) \neq e'_1(s)\}\ ds.$$

Lemma B.1. *Suppose $t_a \geq \tau$ and $\tau' \geq \tau$. Then for any object $b \neq a$ we have*

$$\mu_b(\tau, e') \geq \mu_b(\tau, e).$$

Proof of Lemma B.1. If $\mu_b(\tau, e') = c_b$, then the claim is immediate. Now let $\mu_b(\tau, e') < c_b$. Toward a contradiction, suppose $\mu_b(\tau, e') < \mu_b(\tau, e)$ for some $b \neq a$. Then there must exist some agent i and time $t < \tau$ such that $e_i(t) = b$ but $e'_i(t) = c \neq b$. If not, then $n_b(s, e) \leq n_b(s, e')$ for all $s < \tau$. Then

$$\mu_b(\tau, e) = \int_0^\tau n_b(s, e)\ ds \leq \int_0^\tau n_b(s, e')\ ds = \mu_b(\tau, e'),$$

a contradiction. Note $i \neq 1$ since $e_1(s) = a$ for all $s < t_a$.

Since $t < \tau \leq \tau'$ and $\mu_b(t, e') \leq \mu_b(\tau, e') < c_b$, object b is available at (t, e'). Then since i eats truthfully and is consuming c at (t, e'), cP_ib and $t_c \leq t < t'_c$. Let $S := \{j \in O : t_j < t'_j\}$, which is nonempty since $c \in S$. Let $d \in S$ be the element of S such that t_d is minimal. Note that $t_d \leq t_c < \tau$ so $\mu_d(t_d, e) = c_d$. Note also $d \neq a$ since $t_d < \tau \leq t_a$.

Since $t_d < t'_d$, there must exist some agent ℓ and time $r < t_d$ such that $e_\ell(r) = d$ but $e'_\ell(r) = x \neq d$. If not, then $n_d(s, e) \leq n_d(s, e')$ for all $s < t_d$. But this would imply that

$$\begin{aligned} c_d &= \mu_d(t_d, e) \\ &= \int_0^{t_d} n_d(s, e)\ ds \\ &\leq \int_0^{t_d} n_d(s, e')\ ds \\ &< \int_0^{t_d} n_d(s, e')\ ds + \int_{t_d}^{t'_d} n_d(s, e')\ ds \\ &= \mu_d(t'_d, e'), \end{aligned}$$

which is not possible. The strict inequality in the fourth line follows from the fact that $n_d(\cdot, e')$ is non-decreasing.

Note that $\ell \neq 1$, since $e_1(s) = a$ for all $s < t_a$. Since $r < t_d < t'_d$, object d is available at (r, e'). Since ℓ eats truthfully and is consuming x at (r, e'), $xP_\ell d$ and $t_x \leq r < t'_x$. Then $x \in S$ as well. However, $t_x \leq r < t_d$, contradicting that t_d was minimal. □

Lemma B.2. *If $t_a > \tau$, then $\tau \geq \tau'$.*

Proof of Lemma B.2. This is a corollary of Lemma B.1. Suppose $t_a > \tau$. Let $S := \{j \in O : t_j \leq \tau\}$. Note $a \notin S$. By Lemma B.1, $\mu_j(\tau, e') \geq \mu_j(\tau, e)$ for all $j \in S$. Thus

$$\begin{aligned} \mu_S(\tau, e') &\geq \mu_S(\tau, e) \\ &= N - \sum_{j \notin S} m_j. \end{aligned}$$

Thus condition (B.2) must bind under e' weakly before time τ. So $\tau' \leq \tau$ as desired. □

Lemma B.3. *Denote $\mu_{-1,a}(t, e) = \sum_{i \neq 1} \mu_{i,a}(t, e)$. Let $t_a < t'_a$. Then*

$$\mu_{-1,a}(t_a, e) \leq \mu_{-1,a}(t_a, e').$$

Proof of Lemma B.3. Denote $n_{-1,j}(t, e) = |\{i \in N \backslash \{1\} : e_i(t) = j\}|$. Toward a contradiction, suppose $\mu_{-1,a}(t_a, e) > \mu_{-1,a}(t_a, e')$. Then there must be some agent $i \neq 1$ and time $t < t_a$ such that $e_i(t) = a$ but $e'_i(t) = b \neq a$. If not, then $n_{-1,a}(s, e) \leq n_{-1,a}(s, e')$ for all $s < t_a$, and

$$\mu_{-1,a}(t_a, e) = \int_0^{t_a} n_{-1,a}(s, e)\ ds \leq \int_0^{t_a} n_{-1,a}(s, e')\ ds = \mu_{-1,a}(t_a, e'),$$

a contradiction.

Since $t < t_a < t'_a$, a is available at (t, e'). Then since i eats truthfully, $bP_i a$ and $t_b \leq t < t'_b$. Let $S := \{j \in O : t_j < t'_j\}$, which is nonempty since $b \in S$. Let $c \in S$ be the element of S such that t_c is minimal. Note that $t_c \leq t_b \leq t < t_a$ and $t_a < 1$. Since $t_c < t_a$ we know $c \neq a$.

Since $t_c < t'_c$, there is some time $r < t_c$ and agent ℓ such that $e_\ell(r) = c$ but $e'_\ell(r) = d \neq c$. If not, then $n_c(s, e) \leq n_c(s, e')$ for all $s < t_c$, and

$$\begin{aligned} \mu_c(t_c, e) &= \int_0^{t_c} n_c(s, e)\ ds \\ &\leq \int_0^{t_c} n_c(s, e')\ ds \\ &\leq \int_0^{t_c} n_c(s, e)\ ds + \int_{t_c}^{t'_c} n_c(s, e)\ ds \\ &= \mu_c(t'_c, e'). \end{aligned}$$

In fact, we show that the third line is strict. It suffices to show that $n_c(s, e) \geq 1$ for some $s < t_c$. Suppose not; then $\mu_c(t_c, e) = 0$. Since $t_c < 1$, it must be that $m_c = 0$ and thus both $t_c = \tau$ and $t'_c = \tau'$. Since $\tau = t_c < t_a$, we know that $\tau < t_a$. Additionally, $t_c < t'_c$ gives $\tau < \tau'$. But this contradicts Lemma B.2. Thus, $\mu_c(t_c, e) < \mu_c(t'_c, e')$.

If $t_c < \tau$, then $\mu_c(t_c, e) = c_c$. But then $c_c < \mu_c(t'_c, e')$, a contradiction. Now suppose $t_c \geq \tau$. Since $\mu_c(t_c, e) \geq m_c$, we know that $\mu_c(t'_c, e') > m_c$. Then $t'_c \leq \tau'$. But then we have $t_a > t_c \geq \tau$ and $\tau' \geq t'_c > t_c \geq \tau$. But this contradicts Lemma B.2. Thus we have shown that there is some time $r < t_c$ and agent ℓ such that $e_\ell(r) = c$ but $e'_\ell(r) = d \neq c$.

Note that $\ell \neq 1$ since $t_c < t_a$ and $e_1(s) = a$ for all $s < t_a$. Since $r < t_c < t'_c$, c is available at (r, e'). Then since ℓ eats truthfully, $dP_\ell c$ and $t_d \leq r < t'_d$. Then $d \in S$ as well. But $t_d \leq r < t_c$, contradicting that t_c was minimal. □

Proof that MPS is weak strategyproof. The objective will be to show that if $\delta(t_a) > 0$, then $\mu'_{1a} < \mu_{1a}$. The same argument can then be repeated for other objects besides a.

Suppose that $\mu'_{1a} \geq \mu_{1a}$. Suppose also $\delta(t_a) > 0$, otherwise there is no change. Then $t'_a > t_a$, otherwise it could not be that $\mu'_{1a} \geq \mu_{1a}$. Additionally, $t_a < 1$, or else $\mu_{1a} = 1$ and no improvement is possible.

It suffices to show that $\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e') < \delta(t_a)$ for the following reason. By definition, $\mu_{1,a}(t_a, e') = t_a - \delta(t_a)$. Therefore, if $\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e') < \delta(t_a)$, then

$$\begin{aligned}
\mu_{1,a}(t'_a, e') &= \mu_{1,a}(t_a, e') + (\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e')) \\
&< \mu_{1,a}(t_a, e') + \delta(t_a) \\
&= t_a - \delta(t_a) + \delta(t_a) \\
&= t_a = \mu_{1,a}(t_a, e),
\end{aligned}$$

a contradiction.

We will repeatedly use the following fact. Note that

$$\begin{aligned}
\mu_a(t_a, e) - \mu_a(t_a, e') = \mu_{-1,a}(t_a, e) - \mu_{-1,a}(t_a, e') \\
+ \mu_{1,a}(t_a, e) - \mu_{1,a}(t_a, e').
\end{aligned}$$

By Lemma B.3, $\mu_{-1,a}(t_a, e) \leq \mu_{-1,a}(t_a, e')$. By definition, $\mu_{1,a}(t_a, e) - \mu_{1,a}(t_a, e') = \delta(t)$. Thus

$$\mu_a(t_a, e) - \mu_a(t_a, e') \leq \delta(t_a).$$

We proceed by cases relating t_a and τ.

Case 1. $t_a < \tau$.

Since $t_a < \tau$, we have $\mu_a(t_a, e) = c_a$. Then

$$\begin{aligned}
\mu_a(t'_a, e') - \mu_a(t_a, e') &\leq c_a - \mu_a(t_a, e') \\
&= \mu_a(t_a, e) - \mu_a(t_a, e') \\
&\leq \delta(t_a).
\end{aligned}$$

Since $t_a < 1$ and $\mu_a(t_a, e) = c_a \geq 1$, we know $\mu_{-1,a}(t_a, e) > 0$. Therefore, Lemma B.3 implies $\mu_{-1,a}(t_a, e') > 0$. That is, at least one agent $i \neq 1$ has consumed a before time t_a under eating profile e'. Since agent i will continue to consume a until it is closed, and $t'_a > t_a$, we know that $\mu_{-1,a}(t'_a, e') - \mu_{-1,a}(t_a, e') > 0$. Since $\mu_a(t'_a, e') - \mu_a(t_a, e') \leq \delta(t_a)$, this implies $\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e') < \delta(t_a)$ as desired.

Case 2. $t_a > \tau$.

By Lemma B.2, $\tau' \leq \tau$. In summary, $\tau' \leq \tau < t_a < t'_a$. Since object a closed after time τ under eating profile e and after time τ' under eating profile e', we know $\mu_a(t_a, e) = \mu_a(t'_a, e') = m_a \geq 1$. Then

$$\begin{aligned}\mu_a(t'_a, e') - \mu_a(t_a, e') &= m_a - \mu_a(t_a, e') \\ &\leq \mu_a(t_a, e) - \mu_a(t_a, e') \\ &\leq \delta(t_a).\end{aligned}$$

Since $t_a < 1$ and $\mu_a(t_a, e) = m_a \geq 1$, we know $\mu_{-1,a}(t_a, e) > 0$. The rest of the argument follows Case 1.

Case 3. $t_a = \tau$.

If $\tau' \leq \tau$, then $\tau' \leq \tau = t_a < t'_a$ and so $\mu_a(t'_a, e') = m_a \geq 1$. The remainder of the argument is identical to Case 2.

Instead, suppose $\tau' > \tau$. Let $S := \{j \in O : t_j \leq \tau\}$; note that $a \in S$. Then $S \equiv \{j \in O : j$ fails (B.1) or (B.2) at $(\tau, e)\}$, so $\mu_S(\tau, e) = N - \sum_{j \notin S} m_j$ is an upper bound for $\mu_S(t'_a, e')$. Then

$$\mu_S(t'_a, e') - \mu_S(\tau, e') \leq \mu_S(\tau, e) - \mu_S(\tau, e').$$

Recall $t_a = \tau$, so $\mu_a(t_a, e) - \mu_a(t_a, e') \leq \delta(t_a) \Rightarrow \mu_a(\tau, e) - \delta(\tau) \leq \mu_a(\tau, e')$. Combined with the previous inequality, we have

$$\begin{aligned}\mu_S(t'_a, e') - \mu_S(\tau, e') &\leq \mu_S(\tau, e) - \mu_S(\tau, e') \\ &= \mu_S(\tau, e) - \mu_{S\setminus\{a\}}(\tau, e') - \mu_a(\tau, e') \\ &\leq \mu_S(\tau, e) - \mu_{S\setminus\{a\}}(\tau, e') - \mu_a(\tau, e) + \delta(\tau).\end{aligned}$$

By Lemma B.1, $\mu_{S\setminus\{a\}}(\tau, e') \geq \mu_{S\setminus\{a\}}(\tau, e)$, so

$$\begin{aligned}\mu_S(t'_a, e') - \mu_S(\tau, e') &\leq \mu_S(\tau, e) - \mu_{S\setminus\{a\}}(\tau, e') - \mu_a(\tau, e) + \delta(\tau) \\ &\leq \mu_S(\tau, e) - \mu_{S\setminus\{a\}}(\tau, e) - \mu_a(\tau, e) + \delta(\tau) \\ &= \delta(\tau).\end{aligned} \tag{B.3}$$

Since $t_a = \tau$, we have $\mu_S(t'_a, e') - \mu_S(t_a, e') \leq \delta(t_a)$. In particular, since $a \in S$, $\mu_a(t'_a, e') - \mu_a(t_a, e') \leq \delta(t_a)$.

If $\mu_{-1,a}(t_a, e) > 0$, by Lemma B.3, we have $\mu_{-1,a}(t_a, e') > 0$. Agents consume an object until it closes, so our assumption that $t'_a > t_a$ implies $\mu_{-1,a}(t'_a, e') - \mu_{-1,a}(t_a, e') > 0$. Then since

$$\begin{aligned}\delta(t_a) &\geq \mu_a(t'_a, e') - \mu_a(t_a, e') \\ &= (\mu_{-1,a}(t'_a, e') - \mu_{-1,a}(t_a, e')) + (\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e')),\end{aligned}$$

we have $\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e') < \delta(t_a)$ as desired.

Now suppose $\mu_{-1,a}(t_a, e) = 0$. Since $t_a = \tau < 1$ and only agent 1 consumes a, we have $\mu_a(t_a, e) < 1$. Then a closed at $t_a = \tau$ having met its minimum, so it must be that $m_a = 0$. Therefore, $t'_a = \tau'$ also. Recall that $\mu_S(\tau, e) = N - \sum_{j \notin S} m_j$. Since $0 < \mu_a(\tau, e) < 1$ is not an integer, there is another object $b \in S$ such that $\mu_b(\tau, e)$ is not an integer. Since $t_b \leq \tau$, this implies $m_b < \mu_b(\tau, e) < c_b$.

Suppose $t'_b \leq \tau$. Since $\tau' > \tau$, this implies b must have closed because it reached its capacity; i.e., $\mu_b(t'_b, e') = c_b$. Then $\mu_b(t'_b, e') = \mu_b(\tau, e') = c_b$, so $\mu_b(\tau, e') > \mu_b(\tau, e)$. Together with Lemma B.1, this implies

$$\mu_{S \setminus \{a\}}(\tau, e') > \mu_{S \setminus \{a\}}(\tau, e).$$

Substituting into Equation B.3, we have $\mu_S(t'_a, e') - \mu_S(\tau, e') < \delta(\tau)$. Again, since $a \in S$ and $t_a = \tau$, it follows that $\mu_a(t'_a, e') - \mu_a(t_a, e') < \delta(t_a)$ and therefore

$$\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e') < \delta(t_a)$$

as desired.

Now suppose $t'_b > \tau$. By Lemma B.1, $\mu_b(\tau, e') \geq \mu_b(\tau, e) > 0$. Thus there is an agent consuming b at (τ, e'). Since this agent will consume b until $t'_b > \tau$ and since $t'_a > t_a = \tau$, we have

$$\mu_b(t'_a, e') - \mu_b(\tau, e') > 0.$$

By Equation B.3, we already have $\mu_S(t'_a, e') - \mu_S(\tau, e') \leq \delta(\tau)$. Since $b \in S$ and $t_a = \tau$, the fact that $\mu_b(t'_a, e') - \mu_b(\tau, e') > 0$ implies $\mu_a(t'_a, e') - \mu_a(t_a, e') < \delta(\tau)$. It follows that

$$\mu_{1,a}(t'_a, e') - \mu_{1,a}(t_a, e') < \delta(t_a)$$

as desired.

This completes the proof. □

B.2 Relation to generalized probabilistic serial

We first recount relevant definitions from BCKM13, though we refer readers to the original paper for full details.

Definition B.5. Given $N \times O$ and a **constraint set** $S \subseteq N \times O$, a **constraint** is a restriction on deterministic allocations of the form

$$\underline{q}_S \leq \sum_{(i,j) \in S} M_{ij} \leq \bar{q}_S$$

where $\underline{q}_S, \bar{q}_S \in \mathbb{N}$. That is, a constraint imposes a floor and ceiling on a set of allocations between objects and agents. A **constraint structure** $\mathcal{H}$ is a set of constraint sets. A constraint structure $\mathcal{H}$ is a **hierarchy** if, for any $S, S' \in \mathcal{H}$, we have $S \subseteq S'$, $S' \subseteq S$, or $S \cap S' = \emptyset$. A constraint structure $\mathcal{H}$ is a **bihierarchy** if there exist hierarchies $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$.

Notice that the constraints in our model can be written as a bihierarchical constraints:

- Capturing unit demand:
$$1 \leq \sum_{j \in O} M_{ij} \leq 1$$
The corresponding constraint sets are $\{i\} \times O$ for all $i \in N$, forming $\mathcal{H}_1$.

- Capturing minimums and capacities:
$$m_j \leq \sum_{i \in N} M_{ij} \leq c_j$$
The constraint sets are analogously $N \times \{j\}$ for all $j \in O$, forming $\mathcal{H}_2$.

Therefore, we can apply the implementation results contained in BCKM13.

BCKM13 also generalizes Bogomolnaia and Moulin (2001)'s Probabilistic Serial mechanism to a more general environment within bihierarchies, called Generalized Probabilistic Serial (GPS). GPS accepts constraints of the form

$$\begin{aligned} 1 &\leq \sum_{(i,j) \in S} M_{ij} \leq 1 \quad \forall S = \{i\} \times O \\ 0 &\leq \sum_{(i,j) \in S} M_{ij} \leq \bar{q}_S \quad \forall S \neq \{i\} \times O \end{aligned} \tag{B.4}$$

where each $S \in \mathcal{H}$ and $\mathcal{H}$ is a bihierarchy. The first line imposes unit demand, and the second allows only ceiling constraints on remaining sets.

Even though our primitive constraints are bihierarchical, minimums are not accepted by GPS. Further, even though we are able to re-express our constraints as maximums over sets of objects in Proposition 2.2, these constraint sets do not form a bihierarchy. These constraints include ceilings on sets of the form $O \backslash \{j\}$, which in general will intersect but not have a subset relation.

Of course, this leaves open the possibility that there may be *some* expression of the primitive constraints as ceilings on constraint sets that form a bihierarchy. The next proposition shows this is not the case, and therefore MPS is not a special case of GPS.

Proposition B.4. *Let $|N| \geq 3, |O| \geq 3$. The constraints in the model (Section 2.2) cannot generally be expressed as bihierarchical constraints of the form in Equation B.4.*

Proof of Proposition B.4. It suffices to show the claim for a particular market. Let $N = \{1, 2, 3\}$ and $O = \{o_1, o_2, o_3\}$, with $c_j = 3$ for all $j \in O$, and $m_1 = 1, m_2 = 1, m_3 = 0$. For a deterministic allocation M and constraint set $S \subseteq N \times O$, denote $M(S) = \sum_{(i,j)\in S} M_{ij}$. Without loss, assume that each $i \times O \in \mathcal{H}_1$.

First, note that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are not allowable deterministic allocations. In A, o_2's minimum requirement is not satisfied, and in B, o_1's minimum requirement is not satisfied. Conversely, note that

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is indeed an allowable deterministic allocation.

Therefore, we know that there exists some constraint set $S \in \mathcal{H}$ such that $A(S) > \bar{q}_S$ and some constraint set $S' \in \mathcal{H}$ such that $B(S') > \bar{q}_{S'}$. However, since C is allowable, we also know that $C(S) \leq \bar{q}_S$ and $C(S') \leq \bar{q}_{S'}$.

Note that $(1, o_1) \in S$. If not, then for the allowable deterministic allocation

$$A' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we would have $A'(S) \geq A(S) > \bar{q}_S$, a contradiction. By similar arguments, it is easy to show that $(2, o_1), (3, o_3) \in S$ and that $(1, o_2), (2, o_2), (3, o_3) \in S'$. Note that $(3, o_3) \in S \cap S'$, so $S \cap S' \neq \emptyset$.

But note that $(1, o_1) \notin S'$. If it were, then $C(S') = B(S') > q_{S'}$, contradicting that C is allowable. Therefore $S \not\subseteq S'$. Also, note that $(2, o_2) \notin S$. If it were, then $C(S) = A(S) > q_S$, again contradicting that C is allowable. Therefore, $S' \not\subseteq S$.

Therefore, S and S' cannot be part of the same hierarchy. But also, neither S nor S' can be in $\mathcal{H}_1$. Thus, $\mathcal{H}$ cannot be a bihierarchy. $\square$