UC Berkeley
UC Berkeley Previously Published Works

Title
Asymptotic phase, shadowing and reaction-diffusion systems

Permalink
https://escholarship.org/uc/item/988178zs

Author
Hirsch, MW

Publication Date
1993

Peer reviewed

eScholarship.org Powered by the California Digital Library
University of California

7 Asymptotic Phase, Shadowing and Reaction-Diffusion Systems

Morris W. Hirsch University of California at Berkeley, Berkeley, California

0 Introduction

It is with pleasure and gratitude that we honor Professor Larry Markus for his contributions to mathematics. In 1956 he published *Asymptotically autonomous differential systems* [6] in a series called *Contributions to the Study of Nonlinear Oscillations*— a subject which today would be called "Dynamical Systems". The present article is a direct descendant of Markus' influential paper, through Conway, Hoff and Smoller [2].

Consider a smooth flow $\{\Phi_t\}$ having an attracting limit cycle γ. It is well known that if γ is a hyperbolic attractor— all Floquet exponents having negative real parts— then every trajectory $\Phi_t x$ attracted to γ is asymptotic with the trajectory of a unique point of γ. If γ is parameterized by the interval $[0, 2\pi]$ then y can be interpreted as an angle, called the *asymptotic phase* of x.

I abstract this notion as follows. Consider a trajectory $\Phi_t x$ attracted to some positively invariant set A. If $y \in A$ is such that $\lim_{t\to\infty} ||\Phi_t x - \Phi_t y|| = 0$ then I call y an *asymptotic phase* for x. (For clarity I use $||a - b||$ to denote the distance between points a, b in any metric space.) Notice that uniqueness of y is not required here.

If A is not negatively invariant it may happen that x does not have an asymptotic phase in A, but that $\Phi_s x$ does, for some $s > 0$. In this case I say x has an *eventual asymptotic phase* in A.

It is frequently incorrectly assumed that every orbit approaching an attractor has an eventual asymptotic phase in the attractor. A common situation is that of a cascade of two systems, that is, a system of the form:

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(y).$$

If $(x(t),y(t))$ is a particular solution such that $y(t) \to c$, it is often asserted without justification that $x(t)$ is asymptotic to a solution of $dz/dt = F(z,c)$. A simple counter-example in the plane is:

$$\frac{dx}{dt} = xy$$
$$\frac{dy}{dt} = -y^3.$$

The goal of this paper is to find conditions ensuring existence of an eventual asymptotic phase.

1 Main Results

Let $a = a(q)$ denote a nonnegative real-valued function of a variable q whose domain is understood to be a terminal segment of either the positive reals or the positive integers. Define

$$\mathcal{R}a = \mathcal{R}_{q\to\infty}a(q) = \limsup_{q\to\infty} a(q)^{\frac{1}{q}}.$$

Then

$$\mathcal{R}(a+b) = \max(\mathcal{R}a, \mathcal{R}b),$$

and for any constant $\kappa > 0$:

$$\mathcal{R}(\kappa a) = \mathcal{R}(a).$$

Let $F = \{F_t\}_{t\geq 0}$ be a flow (more precisely, a partial semiflow) on a metric space X (ususally, a Banach space). For clarity the distance between points $x, y \in X$ is denoted $||x-y||$. In applications X is usually a subset of a Banach space. I shall always assume the maps F_t have the following *local Lipschitz property:* For any $t_0 \geq 0, x_0 \in X$ there exist $L \geq 0$ and neighborhoods $N \in \mathbf{R}_+$of t_0 and $U \in X$ of x_0 such that

$$||F_t x - F_t y|| \leq L||x-y||$$

for all $t \in N, y \in U$.

Denote by $A \subset X$ a closed subspace having the following properties:

(a) A is positively invariant under F.

(b) A has the structure of a Riemannian manifold without boundary homeomorphically embedded in X. The norm of a tangent vector Y in the Riemannian metric is denoted by $||Y||$.

(c) There is a smooth ($=C^1$) tangent vector field G on A whose flow $\Phi = \{\Phi_t\}_{t\in\mathbf{R}}$ coincides with $F_t|A$ for $t \geq 0$.

Let $K \subset A$ denote a nonempty compact set positively invariant under F, so that K is also Φ-invariant. The *inset* of K under F is the set

$$In(K) = In(K,F) = \{x \in X : \lim_{t\to\infty} \text{dist}(F_t x, K) = 0\}.$$

For $x \in In(K)$ define the *rate of approach to* K of x to K under F to be the number

$$\mathcal{P}(x,K,F) = \mathcal{R}_{t\to\infty}\text{dist}(F_t x, K).$$

Evidently $0 \leq \mathcal{P}(x,K,F) \leq 1$. If $\mathcal{P}(x,K,F) < 1$ I say x is *exponentially attracted to* K.

Fix a Riemannian metric on A. The closed ball in A with radius $\rho \geq 0$ centered at $x \in A$ is denoted by $B(\rho, x)$.

For a diffeomorphism h between open subsets of A, the *expansion constant of h at $x \in A$* is the positive number

$$EC(h,x) = ||T_x h^{-1}||^{-1} = \min_{||Y||=1} ||T_x h(Y)||.$$

Here Y denotes tangent vectors to A at x, and $||T_x h||$ denotes the operator norm of the differential of h at x (defined by the Riemannian metric). Thus $EC(h,x) \geq \mu$ iff $||T_x z|| \geq \mu ||z||$ for all $z \in T_x$.

Now for any compact subset $K \subset A$ define

$$EC(h,K) = \min_{x\in K} EC(h,x)$$

If $EC(h,K) > \nu > 0$ then it is not hard to see that there exists $\rho_* > 0$ such that if $x \in K$ and $0 < \rho \leq \rho_*$ then

$$hB(\rho,x) \supset B(\mu\rho, h(x));$$

see Hirsch and Pugh [4].

The *expansion rate of* Φ *at* K is the nonnegative number

$$\mathcal{E}(\Phi, K) = \sup_{t>0} EC(\Phi_t, K)^{\frac{1}{t}}.$$

Since $[T_x\Phi_t]^{-1} = T_{\Phi_t x}\Phi_{-t}$, we have

$$\mathcal{E}(\Phi, K) = \sup_{t>0} \min_{x\in K} ||T_{\Phi_t x}\Phi_{-t}||^{-\frac{1}{t}}$$

The expansion rate is is the largest $\mu > 0$ having the following property: If $0 < \nu < \mu$ then there exist $s > 0, \rho_* > 0$ such that

$$\Phi_s B(\rho, x) \supset B(\nu^s \rho, \Phi_s x)$$

provided $x \in K$ and $0 < \rho \leq \rho_*$.

The expansion rate depends on the dynamics and the Riemannian metric. In some cases it is possible to estimate it from a formula for the vector field, from the dynamics of its flow, or from estimates using other metrics. Here are several such estimates.

(i) Assume that $A = \mathbf{R}^n$ with the standard inner product $\langle \cdot, \cdot \rangle$, and denote $T_x\Phi_t$ by $D\Phi_t(x)$. The variational equation along orbits of the reversed time flow Φ_{-t}, generated by the vector field $-G$ on A, gives the following matrix differential equation:

$$\frac{d}{dt}D\Phi_{-t}(x) = -DG(\Phi_{-t}x)D\Phi_{-t}(x)$$

Therefore for every nonzero vector $Y \in \mathbf{R}^n$ and every $t \geq 0, y \in K$ we have, setting $y = \Phi_t x \in K$:

$$\frac{d}{dt}||D\Phi_{-t}(y)Y|| = ||D\Phi_{-t}(y)Y||^{-1}\langle -DG(\Phi_{-t}y)D\Phi_{-t}(y)Y, D\Phi_{-t}(y)Y\rangle$$

The inner product on the right hand side is bounded above by $-\beta||D\Phi_{-t}(y)Y||^2$ where $\beta = \beta(G, K)$ denotes the minimum over $x \in K$ and unit vectors $\xi \in \mathbf{R}^n$ of $\langle DG(x)\xi, \xi\rangle$. Equivalently, β equals the smallest eigenvalue of the symmetric matrix $\frac{1}{2}[DG(x)+DG(x)^T]$ where T denotes the transpose of a matrix. Therefore

$$\frac{d}{dt}||D\Phi_{-t}(x)|| \leq \beta||D\Phi_{-t}(x)||,$$

whence

$$||D\Phi_{-t}(x)|| \leq e^{-t\beta}.$$

This proves $EC(\Phi_t, x) \geq e^{t\beta}$ for all $t \geq 0, x \in K$. We get the convenient estimate:

$$\mathcal{E}(\Phi, K) \geq e^{\beta(G,K)}. \qquad (1)$$

(ii) Another estimate is obtained by noticing that

$$|\beta| \leq M = M(G,K) = \max_{x \in K} ||DG(x)||$$

(using the Schwarz inequality) so that $\beta \geq -M$. This yields the estimate:

$$\mathcal{E}(\Phi, K) \geq e^{-M(G,K)}. \tag{2}$$

which will be used in Section 2.

(iii) A different estimate can be obtained in case all forward and backward trajectoies in K are attracted to hyperbolic periodic orbit (possibly stationary). Suppose that the real parts of the Floquet exponents of these periodic orbits are all $\geq \gamma \in \mathbf{R}$. Then it can be proved that:

$$\mathcal{E}(\Phi, K) \geq e^{\gamma} \tag{3}$$

Suppose for example that the flow in A is the gradient flow of a function $g : A \longrightarrow \mathbf{R}$ having a finite set of critical points, and K is a compact attractor containing all the critical points. Then γ is the minimum of the eigenvalues of the Hessian of g at critical points in K.

(iv) More generally, it can be shown that if $L \subset K$ is a compact set containing all alpha and omega limit points in K, then $\mathcal{E}(\Phi, K) = \mathcal{E}(\Phi, L)$. The reason is that any semi-trajectory in K spends all but a finite amount of time in any given neighborhood of L.

(v) If K is a smooth submanifold and the flow in K is isometric for some Riemannian metric, then $\mathcal{E}(\Psi, K) = 1$. This is the case, for example, when K is a periodic orbit; when K is a smooth submanifold consisting of stationary points; or when the K is an n-dimensional torus and the flow is translation by a one parameter subgroup.

(vi) It seems reasonable to conjecture that if Ψ is generated by a vector field H on A of the form $H(x) = c(x)G(x)$ where c is a positive function on A, then $\mathcal{E}(\Phi, K) = \mathcal{E}(\Psi, K)$.

It would be very useful to know that $\mathcal{E}(\Phi, K)$ is preserved, or at least well controlled, by a smooth or continuous reparameterization of the trajectories, or by a topological conjugacy between flows. A key test case is a C^2 flow on a 2-torus without periodic orbits: Is the expansion rate equal to 1?

(vii) Clearly $\mathcal{E}(\Phi, K) \geq \mathcal{R}_{t\to\infty} EC(\Phi_t, K)$. The latter number is easier to estimate and in some ways is more natural. For example it is easy to prove that it is independent of the Riemannian metric on A.

The main result says roughly that x *is exponentially attracted to* K *at rate* λ*, while the expansion rate at* K *of the flow in* A *is* $\mu > \lambda$*, then* x *is eventually asymptotic at rate* λ *to a unique trajectory in* A*:*

Theorem 1.1 *Let* $\mathcal{E}(\Phi, K) = \mu$*. Suppose* $x \in In(K)$ *approaches* K *at rate*

$$\mathcal{P}(x, K, F) = \lambda < \min(1, \mu).$$

Then:

(a) *There exists* $r \geq 0, y \in A$ *such that*

$$\mathcal{R}_{t\to\infty}||\Phi_{t+r}x - \Phi_t y|| = \lambda.$$

(b) *Let* y *be as in (a). Suppose* $l > 0, z \in A$ *are such that*

$$\mathcal{R}_{t\to\infty}||\Phi_{t+l}x - \Phi_t z|| \leq \lambda.$$

Then z *and* y *are on the same orbit of* Φ.

This is proved in Section 3 below. The same argument yields the analogous result for mappings.

The proof of the following corollary is left to the reader:

Corollary 1.2 *If* $\mathcal{P}(x, K, F) = \lambda < \min(1, \mathcal{E}(\Phi, K))$*, then* x *has an eventual asymptotic phase* $y \in K$*. If* $\mathcal{E}(\Phi, K) \geq 1$ *then the* Φ*-trajectory of such a* y *is unique.*

As a simple example illustrating Theorem 1.1, consider a smooth flow in some manifold A having an invariant n-torus $K = T^n = (\mathbf{R}/2\pi\mathbf{Z})^n$ in which the flow is quasiperiodic, the generating vector field G in T^n being covered by a constant vector field in $\mathbf{R}^n$. It is clear that $\mathcal{E}(\Phi, T^n) = 1$, using the Riemannian metric covered by the Euclidean metric on $\mathbf{R}^n$. Therefore by Theorem 1.1, any orbit attracted to T^n at a rate of approach less that 1 has an asymptotic phase in T^n. It is not hard to show that the same conclusion holds if the flow in T^n is generated by gG where g is any smooth real-valued function on T^n. The proof is based on the fact that orbits of the lifted flow in $\mathbf{R}^n$ stay in parallel lines.

Remark 1.3 Suppose K is a normally hyperbolic submanifold, or a hyperbolic subset, for the flow in A (see [3, 4, 5, 7]). Then any point $x \in A$ attracted to K belongs to the strong stable manifold of some $y \in K$. Therefore x is exponentially asymptotic with y.

Remark 1.4 The main results apply equally to discrete-time systems, i. e. to a mapping f from an open subset $X_0 \subset X$ to X. Everything makes sense if t is restricted to the natural numbers, F_t is the t'th iterate of f, and Φ is replaced by the iterates of the map $h = f|A \cap X_0$, assumed to be a diffeomorhism from $A_0 = A \cap X_0$ onto a neighborhood of K in A. In fact the main part of the proof of the main theorem in Section 3 consists of a proof of the discrete-time case; this is applied to the mapping $f = F_s$ for suitable $s > 0$.

2 Reaction Diffusion Systems

Theorem 1.1 is applied to reaction diffusion systems of the following kind. Let $\overline{\Omega} \subset \mathbf{R}^m$ be a smooth (i. e. C^1) compact submanifold with interior Ω. We look for a continuous function $u(x,t)$, $x \in \overline{\Omega}$, $t \geq 0$ with values in $\mathbf{R}^n$satisfying for $t > 0$

$$\frac{\partial u}{\partial t} = B\Delta u + \sum_{j=1}^{m} C_j(x,u)\frac{\partial u}{\partial x_j} + f(u), \tag{4}$$

$$\frac{\partial u}{\partial \nu} = 0. \tag{5}$$

Here Δ is the Laplacean in the spatial variable $x \in \overline{\Omega}$, operating on each component u_j of u; B is a positive definite $n \times n$ matrix; each $n \times n$ matrix-valued function C_j is continuous in (x,u); f is a smooth vector field on $\mathbf{R}^n$; ν is the inward pointing unit vector field normal to the boundary of Ω.

It is known that solutions to this system form a *solution semiflow* $S = \{S\}_{t \geq 0}$ in the Sobelev space $H^1(\overline{\Omega}, \mathbf{R}^n)$: The solution taking initial values $u(x,0) = v(x)$ is $u(x,t) = (S_t v)(x)$.

Let $A \subset H^1(\overline{\Omega}, \mathbf{R}^n)$ denote the linear subspace of constant maps $\overline{\Omega} \to \mathbf{R}^n$, and identify A with $\mathbf{R}^n$ in the natural way. The form of Equation (4) shows A is positively invariant under S.

A trajectory of S in A defines a spatially homogeneous solution to Equations (4), (5). Such a solution has the form $u(x,t) = y(t)$ where y is a solution to the autonomous system $dy/dt = f(y)$.

The restriction to A of the solution flow S of (4), (5) coincides for $t \geq 0$ with the flow Φ obtained by integrating the vector field f.

Suppose from now on that $\Gamma \subset \mathbf{R}^n$ is a compact invariant rectangle[1] (the product of n nondegenerate compact intervals.) We identify Γ with a compact subset of A, namely the constant functions with values in Γ. *Invariance* means that if the initial

[1]More generally, Γ can be an invariant region as defined in Conway, Hoff and Smoller [2].

map $v : \overline{\Omega} \to \mathbf{R}^n$ takes values in Γ then the same holds for every map $S_t v$. When B is a diagonal matrix, invariance holds provided that for every y on the boundary of Γ, the vector $f(y)$ does not point out of Γ.

In [2] a condition is given ensuring that Γ attracts every initial $v \in H^1(\overline{\Omega}, \mathbf{R}^n)$ taking values in Γ, or in other words, that the set $X = H^1(\overline{\Omega}, \Gamma)$ lies in the inset of Γ. This condition is given in terms of the real parameter

$$\sigma = b\Lambda - M - c\sqrt{m\lambda} \tag{6}$$

defined in terms of the following constants: The positive number b is the smallest eigenvalue of the positive definite matrix B; Λ (also positive) is the smallest eigenvalue of $-\Delta$ on Ω with homogeneous Neumann boundary conditions (5); c is the maximum matrix operator norm $||C_j(x,y)||$, $(1 \le j \le m,\ x \in \overline{\Omega}, y \in \Gamma)$; and as before, $M = \max_{y \in \Gamma} ||Df(y)||$.

It will also be convenient to consider the slightly different parameter:

$$\sigma_2 = \sigma - M = b\Lambda - 2M - c\sqrt{m\lambda} \tag{7}$$

For each $v \in X$ set $v_t = S_t v$, and denote by $\overline{v}_t \in \mathbf{R}$ the average of v_t over $\overline{\Omega}$. Notice that $\overline{v}_t$ is a curve in X, but it need not be a trajectory of the flow S, that is, $\overline{v}_t(x)$ need not be a solution to Equations (4, 5).

Let $||\cdot||_\infty$ denote the $L_\infty(\overline{\Omega}, \mathbf{R}^n)$ norm.

The following result is a corollary of Theorem 3.1 of [2][2]

Theorem 2.1 (Conway, Hoff, Smoller [2])
Assume $\sigma > 0$ and let $v \in X = H^1(\overline{\Omega}, \Gamma)$. Then:

(a) *There is a constant $c_1 > 0$ such that $||v_t - \overline{v}_t||_1 \le c_1 e^{-\sigma t}$ for all $t \ge 0$.*

(b) *If the matrices $C_1, \dots, C_n$ are zero, or if $C_1, \dots, C_n$ and B are diagonal, then there is a constant $c_2 > 0$ such that $||v_t - \overline{v}_t||_\infty \le c_2 e^{-\frac{2\sigma}{m}t}$ for all $t \ge 0$.*

This says that when σ is positive, in the appropriate norm trajectories of the reaction-diffusion system approach spatially homogeneous functions. In fact in [2] it is proved that the spatial averages $\overline{v}_t$ satisfy a nonautonomous system $d\overline{v}_t/dt = f(\overline{v}_t) + g(t)$ with $||g(t)||_1 \le c_3 e^{-\sigma t}$ for some constant $c_3 \ge 0$. Conway, Hoff and Smoller say that "because of a result of Markus [6] it follows that the *asymptotic* behavior of $\overline{v}_t$ is determined only by f".

In the terminology of Section 1 we have:

[2]The statements of Theorem 2.1 are proved but not stated in this form. The exponent in (b) is given as $-\dfrac{\sigma}{m}t$, I think incorrectly.

Corollary 2.2 *Under the same hypothesis as Theorem 2.1:*

(a) $\mathcal{R}_{t\to\infty}(||v_t - \overline{v_t}||_1) \leq e^{-\sigma}$.

(b) *If the matrices* $C_1, \dots, C_n$ *are zero, or if* $C_1, \dots, C_n$ *and* B *are diagonal, then also* $\mathcal{R}_{t\to\infty}(||v_t - \overline{v_t}||_\infty) \leq e^{-\frac{2\sigma}{m}}$.

While the Conway-Hoff-Smoller theorem provides much information about such systems, it leaves open the question of whether trajectories have an asymptotic phase in A. The following result gives a sufficient condition for this.

Let $\mu = \mathcal{E}(\Phi, \Gamma)$, the expansion rate in Γ of the flow in $A = \mathbf{R}^n$ defined by $dy/dt = f(y)$.

Theorem 2.3 *Assume* $\sigma > 0$ *and* $e^{-\sigma} < \mu$. *Let* $v \in H^1(\overline{\Omega}, \mathbf{R}^n)$ *take values in the invariant rectangle* $\Gamma \subset \mathbf{R}^n$. *Then the trajectory* $S_t v$ *in* $In(\Gamma)$ *of the solution flow in* $H^1(\overline{\Omega}, \mathbf{R}^n)$ *of the reaction-diffusion system (4),(5) has an an eventual asymptotic phase in the space A of constant maps. More precisely, if* $S_t v(x) = u(x,t)$ *then for every sufficiently large* $s \geq 0$ *there is a unique solution to* $dy/dt = f(y)$ *such that:*

(a) $\mathcal{R}_{t\to\infty}(||u(\cdot, t+s) - y(t)||_1) \leq e^{-\sigma}$.

Moreover, if the matrices $C_1, \dots, C_n$ *are zero, or if* $C_1, \dots, C_n$ *and* B *are diagonal, then:*

(b) $\mathcal{R}_{t\to\infty}(||u(\cdot, t) - y(t)||_\infty) \leq e^{-\frac{2\sigma}{m}}$.

Corollary 2.4 *If* $\sigma_2 > 0$ *then the conclusions of Theorem 2.3 hold.*

Proof Corollary 2.2(a) implies v has rate of approach $\leq e^{-\sigma}$ to Γ. Therefore Theorem 2.3 follows from Theorem 1.1 (with $K = \Gamma$) and the assumption $e^{-\sigma} < \mu$.

To prove Corollary 2.4, assume $\sigma_2 > 0$. Then $\sigma > 0$ and $e^{-\sigma} < e^{-M}$ (see (7)). Since estimate (2) therefore implies $e^{-M} \leq \mathcal{E}(\Phi, \Gamma)$, the corollary is a consequence of Theorem 2.3.

QED

3 Shadowing

The main theorem will be derived from the results of this section. The same notations and assumptions as in Section 1 are in force, although at first the setting is quite general.

Let $X_0 \subset X$ be any subset and let $g : X_0 \to X$ be a map (g = some F_t in the application). Let $0 \leq \lambda < 1$. I call a sequence $\{y_k\}$ in K a *λ-pseudoorbit* for g if

$$\mathcal{R}_{k\to\infty}||g(y_{k-1}) - y_k|| \leq \lambda.$$

Lemma 3.1 *Suppose g is α-Hölder, $0 < \alpha \leq 1$. Let $\{y_k\}$ be a sequence in X which is λ-shadowed by a point $u \in X_0$. Then $\{y_k\}$ is a λ^α-pseudoorbit for h. In particular if g is Lipschitz then $\{y_k\}$ is a λ-pseudoorbit.*

Proof Fix $C > 0$ such that $||g(a) - g(b)|| \leq C||a - b||^\alpha$. Observe that

$$\begin{aligned} ||g(y_{m+k-1}) - y_{m+k}|| &\leq ||g(y_{m+k-1}) - g^k u|| + ||g^k u - y_{m+k}|| \\ &\leq C||y_{m+k-1} - g^{k-1}u||^\alpha + ||g^k u - y_{m+k}||. \end{aligned}$$

Therefore (see Section 1)

$$\begin{aligned} \mathcal{R}_{k\to\infty} C||g(y_{k-1}) - y_k|| &\leq \max(\mathcal{R}_{k\to\infty}||y_{m+k-1} - g^{k-1}u||^\alpha, \mathcal{R}_{k\to\infty}||g^k u - y_{m+k}||) \\ &\leq \max(\lambda^\alpha, \lambda) = \lambda^\alpha. \end{aligned}$$

QED

Now set $A_0 = A \cap X_0$, assume $g(A_0) \subset A$ and $g(K) \subset K$. Set $g|A_0 = h$ and assume from now on that h is a C^1 diffeomorphism of A_0 onto some neighborhood of K in A.

A point $u \in A_0$ (or its orbit) is said to *λ-shadow* the sequence $\{y_k\}$ in case $h^k(u)$ is defined for all $k \in \mathbf{N}$, and:

$$\mathcal{R}_{k\to\infty}||h^k(u) - y_{k+m}|| \leq \lambda$$

for some $m \geq 0$.

Theorem 3.2 *Assume the expansion rate of h in K is $EC(h, K) = \mu > 0$. Let $\{y_k\}$ be a λ-pseudoorbit in K such that*

$$0 < \lambda < \min(1, \mu).$$

Then:

(a) *There exists $z \in A_0$ which λ-shadows $\{y_k\}$.*

(b) *If $z, w \in A_0$ both λ-shadow $\{y_k\}$ then there exist natural numbers l, r such that $h^l z = h^r w$.*

Remark 3.3 The proof shows that z in the theorem can be chosen in K if K is a smooth compact submanifold without boundary, or if K is an attractor for h, or if the pseudoorbit $\{y_k\}$ is eventually bounded away from the boundary of K in A. In any case the forward orbit of z is attracted to K and its omega limit set is in K.

Remark 3.4 The theorem is valid under the more general hypothesis where μ denotes $\sup_{k>0} EC(h^k, K)^{\frac{1}{k}}$.

Proof Fix $\rho_* > 0$ so small that if $0 \leq \rho \leq \rho_*$ then

$$hB(\rho, x) \supset B(\mu\rho, h(x)) \tag{8}$$

for all $x \in K$, where B refers to closed balls in A. Then this also holds for all x in some neighborhood $N \subset A_0$ of K, since K is compact.

Choose ν such that

$$0 < \lambda < \nu < \min(1, \mu).$$

Pick δ such that

$$\nu < \delta < \min(1, \mu).$$

I claim that for all sufficiently large positive integers k we have:

$$hB(\delta^{k-1}, y_{k-1}) \supset B(\delta^k, y_k). \tag{9}$$

To see this observe that $\delta^j < \rho$ and $B(\delta^{k-1}, y_{k-1}) \subset N$ for large j. Therefore by (8) it suffices to prove for sufficiently large k that

$$B(\mu\delta^{k-1}, h(y_{k-1})) \supset B(\delta^k, y_k). \tag{10}$$

And this last will hold by the triangle inequality provided we show

$$\mu\delta^{k-1} \geq \delta^k + ||h(y_{k-1}) - y_k||. \tag{11}$$

Because $\{y_k\}$ is a λ-pseudoorbit, for large k we have

$$||h(y_{k-1}) - y_k|| < \nu^k. \tag{12}$$

Therefore it suffices to show

$$\mu\delta^{k-1} \geq \delta^k + \nu^k \tag{13}$$

or equivalently

$$\mu \geq \delta + (\frac{\nu}{\delta})^{k-1}\nu \tag{14}$$

for sufficiently large k. This is true, say for $k \geq m$, because $\mu > \delta > \nu$.

Therefore estimate (9) holds for $k \geq m$. This implies that for $n \geq m$ the set

$$Q_n = \cap_{i\geq 0}(h|B(\delta^n, y_n))^{-i}B(\delta^{i+n}, y_{i+n})$$

is not empty, and the orbit of any point in Q_m λ-shadows $\{y_k\}$. This proves statement (a) of the theorem.

From the assumption $EC(h, K) > \lambda$ it follows easily that Q_n is a singleton for every $n \geq m$. This implies (b). **QED**

Proof of Theorem 1.1 With the notation and assumptions of Theorem 1.1, fix $r > 0$ so that

$$EC(\Phi_r, K) = \mu_0 > \lambda.$$

Set $h = \Phi_r : A_0 \to A$ where A_0 denotes the domain of Φ_r— a neighborhood of K in A. For $k \in \mathbf{N}$ let $y_k \in K$ be a point nearest to $h^k(x)$. It then follows from Lemma 3.1(a) with $u = x$ and $g = F_r$, and the standing assumption that each F_t is Lipschitz, that $\{y_k\}$ is a λ-pseudoorbit for h. By Theorem 3.2 $\{y_k\}$ is λ-shadowed by the orbit of some $z \in A_0$. It follows that for some $m \geq 0$ we have:

$$\mathcal{R}_{k\to\infty}||\Phi_{k+m}x - \Phi_k z|| = \lambda \ (k \in \mathbf{N}).$$

Continuity of the flow now implies:

$$\mathcal{R}_{t\to\infty}||\Phi_{t+m}x - \Phi_t z|| = \lambda \ (t \in \mathbf{R}).$$

This proves part (a) of Theorem 1.1.

Part (b) follows similarly from part (b) of Theorem 3.2. **QED**

Remark 3.5 The connection between asymptotic phase and shadowing is more extensive. For simplicity consider a diffeomorphism h. Suppose the orbit of some point x is attracted to a compact invariant set K, not necessarily at an exponential rate. By choosing $y_k \in K$ to be a point nearest to $h^k(x)$ we obtain a sequence $\{y_k\}$ in K with the property that $||h(y_{k-1}) - y_k|| \to 0$. If $h|K$ has the property of *unique shadowing*, described below, then it is easy to see that $\{y_k\}$ is asymptotic to the orbit of a unique point $z \in K$. Such a z would therefore be an asymptotic phase for x.

To say the map $h|K$ has unique shadowing means the following. For $\delta > 0$, $\{y_k\}$ is an *δ-pseudoorbit* in case $||h(y_{k-1}) - y_k|| < \delta$. "Unique shadowing" means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudoorbit $\{y_k\}$ there is a unique $z \in K$ such that $||y_k - h^k(z)|| < \epsilon$, or in other words $\{y_k\}$ is *ϵ-shadowed* by z.

R. Bowen [1] showed that if K is a hyperbolic invariant set, then $h|K$ has unique shadowing. Suppose for example that V is a compact smooth invariant submanifold of A and that $h|V$ is an Axiom A diffeomorphism in the sense of Smale [7]. If $x \in A$ is attracted to V then it is easy to see that in fact x is attracted to what Smale calls a basic set K for $h|V$, which is by definition a hyperbolic invariant set. Therefore Bowen's theorem implies that x has an asymptotic phase in K, hence also in V.

References

[1] R. Bowen, *On axiom A diffeomorphisms*, CBMS Regional Conference Series in Mathematics, vol. 35, American Mathematical Society Publications, Providence, Rhode Island, 1978.

[2] E. Conway, D. Hoff, and J. Smoller, *Large time behavior of solutions of systems of nonlinear reaction-diffusion equations*, SIAM Journal of Applied Mathematics **35** (1978), 1–16.

[3] M.W. Hirsch, J. Palis, C.C. Pugh, and M. Shub, *On invariant subsets of hyperbolic sets*, Inventiones Mathematicae **9** (1970), 121–134.

[4] M.W. Hirsch and C.C. Pugh, *Stable manifolds for hyperbolic sets*, Global Analysis, Proceedings of Symposia in Pure Mathematics, No. 14, American Mathematical Society, Providence, Rhode Island, 1970, pp. 133–163.

[5] M.W. Hirsch, C.C. Pugh, and M. Shub, *Invariant manifolds*, Springer Lecture Notes in Mathematics, vol. 583, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[6] L. Markus, *Asymptotically autonomous differential systems*, Contributions to the Theory of Nonlinear Oscillations, Vol. III (S. Lefschetz, ed.), Princeton University Press, 1956, Annals of Mathematics Study No. 36, pp. 17–30.

[7] Smale. S., *Differentiable dynamical systems*, Bulletin American Mathematical Society **73** (1967), 747–817.