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Estimability and Efficiency in Nearly Orthogonal $2^{m_1} \times 3^{m_2}$ Deletion Designs

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1 Abstract

This article considers single replicate factorial experiments in incomplete blocks. A single replicate 2^{m_1} x 3^{m_2} deletion design in 3 incomplete blocks is obtained from a single replicate 3^m, where $m = m_1 + m_2$, preliminary design by deleting all runs (or treatment combinations) with the first m_1 factors at the level two. A systematic method for determining the unbiasedly estimable (u.e.) and not unbiasedly estimable (n.u.e.) factorial effects is provided. It is shown that for $m_2 > 0$ all factorial effects of the type $F(\alpha_1 \cdots \alpha_{m_1}, \alpha_{m_1+1} \cdots \alpha_m)$, where $\alpha_i = 0, 1$ for $i = 1, \cdots, m_1$, $\alpha_i = 0, 1, 2$ for $i = m_1 + 1, \cdots, m$, with $(\alpha_1 \cdots \alpha_m) \neq (0 \cdots 0)$, and $(\alpha_{m_1+1} \cdots \alpha_m) \neq \alpha(1 \cdots 1)$ for $\alpha = 1, 2$, are u.e. and the remaining factorial effects are n.u.e. It is noted that $(2^{m_1} - 1)$ factorial effects of 2^{m_1} factorial experiments and $(3^{m_2} - 3)$ factorial effects of 3^{m_2} factorial experiments, which are embedded in 2^{m_1} x 3^{m_2} factorial experiments, are u.e. The 2 x 3^{m-1} deletion designs were considered in the work of Voss (1986). Defining factorial effects of a 2^{m_1} x 3^{m_2} factorial experiment in a form different than in Voss (1986), we develop a simple representation of u.e. and n.u.e. factorial effects. In this representation, there are $(2^{m_1+1} + 1)$ n.u.e. factorial effects of the type $F(\alpha_1 \cdots \alpha_{m_1}, \alpha \cdots \alpha)$. This number is smaller than the corresponding number of n.u.e. factorial effects in the representation of Voss (1986). The relative efficiency expressions, and their bounds, in the estimation of factorial effects of 2^{m_1} x 3^{m_2} deletion designs are also given.

KEY WORDS: Confounding, Factorial experiment, Single replicate, Unbiasedly estimable.

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2 Introduction

There is a vast literature on the construction of single replicate asymmetrical factorial designs in incomplete blocks. The reader is referred to Voss (1986) and Raktoe, Hedayat and Federer (1981) for the list of references. The concept of deletion designs was introduced in Kishen and Srivastava (1959). The deletion technique in deletion designs was then used by many authors, among them Addleman (1962, 1972) and Voss (1986). This article considers 2^{m_1} x 3^{m_2} deletion designs in three incomplete blocks and presents a systematic method for finding u.e. and n.u.e. factorial effects. While the smaller values of m_1 and m_2 are the most practically important cases, we do not consider the case when $m_2 = 0$ since there the blocks are of unequal sizes and main effects are confounded. This work is based on Mahoney(1988) and Ghosh and Mahoney (1988), where several generalizations are discussed.

The model assumed is the linear fixed effects model. A factorial effect is estimable if, and only if, it can be unbiasedly estimated with a linear combination of the observations. An unadjusted estimator of a factorial effect is simply the factorial effect with the treatment effects replaced by the observed response at the corresponding treatment combination. The unadjusted estimators can be unbiased or biased. When they are biased, then under the assumption that certain higher order factorial effects are negligible, it is possible to adjust them to be unbiased in minimum variance fashion. The unbiased estimators of factorial effects obtained in this fashion are called adjusted estimators.

The relative efficiency in the estimation of a factorial effect is the ratio of the variance of its unadjusted estimator divided by the variance of its adjusted estimator. Under the assumption that certain higher order factorial effects are negligible, the relative efficiency considered in this paper is identical to the standard efficiency factor. (See for example, John (1987), equation (2.1) on page 24.) If the unadjusted estimator is unbiased there is no need for adjustment and hence the relative efficiency is unity. Otherwise, the relative efficiency is less than unity, and the closer the value of the relative efficiency to unity, the lesser the effect of adjustment on the variance of the estimator.

For the general definition of estimable parametric functions, the reader is referred to Scheffé (1959), page 13 and Lehmann (1983), page 75. In this paper the parametric functions are factorial effects and contrasts of block effects. Definitions of factorial effects and deletion designs are given in section 3. The term orthogonal block design means the block design has the property that the least squares estimators of all factorial effects are not only orthogonal to each other but also orthogonal to the least squares estimators of a complete orthogonal set of block effect contrasts [see Raktoe, Hedayat and Federer (1981), Definition 8.1, page 102]. For a single replicate factorial design in incomplete blocks, the existence of such an orthogonal design is impossible. It is however observed in section 5, under the assumption that two of the highest order factorial effects are zero, the deletion designs are nearly orthogonal. Section 4 presents the systematic method of determining which factorial effects are unbiasedly estimable (u.e.) by their unadjusted estimators. Section 5 discusses the relative efficiency with an illustrative example. Section 6 presents an example from reliability and life testing of a 2×3^2 experiment which uses a 2×3^2 deletion design.

3 Definition and Notation

Consider a single replicate 2^{m_1} x 3^{m_2} factorial experiment in incomplete blocks. There are m factors ($m = m_1 + m_2$) in the experiment. Runs and their effects are denoted by the same notation, $(x_1 \cdots x_{m_1}, x_{m_1+1} \cdots x_m)$, where $x_i = 0, 1$ for $i = 1, \cdots, m_1$, and $x_i = 0, 1, 2$ for $i = m_1 + 1, \cdots, m$.

The observation on the run $(x_1 \cdots x_m)$ is denoted by $y(x_1 \cdots x_m)$. With this notation, the model can be written as

$$E(y(x_1 \cdots x_m)) = (x_1 \cdots x_m) + \beta_j$$

$$Var(y(x_1 \cdots x_m)) = \sigma^2 > 0$$

$$Cov(y(x_1 \cdots x_m), y(x_1{}' \cdots x_m{}')) = 0, \tag{1}$$

where β_j is the fixed effect of the jth block containing the run $(x_1 \cdots x_m)$, and $(x_1 \cdots x_m) \neq (x_1{}' \cdots x_m{}')$. The model assumed is equivalent to the linear fixed effects model:

$$Y_{(n\times 1)} = \tau_{(n\times 1)} + N_{n\times k} B_{(k\times 1)} + \epsilon_{(n\times 1)}.$$

Here, $\tau = (\tau_x)$ is the vector of run or treatment effects, where $x = (x_1 \cdots x_m)$ is a treatment combination and is ordered by x in lexicographical order, with x_i being the level of factor i. The matrix N is the incidence matrix. That is, $N = (\delta_{x,h})$ where

$$\delta_{x,h} = \begin{cases} 1, & \text{treatment combination } x \text{ appears in block } h \\ 0, & \text{otherwise.} \end{cases}$$

This model can also be expressed in more familiar form as

$$Y = X\Theta + \epsilon, \qquad \Theta^t = (\tau^t \mid B^t), \tag{2}$$

where $X = (I_{(n\times n)} \mid N_{(n\times k)})$, B^t is the vector of block effects, $E(\epsilon) = \underline{0}$, and $E(\epsilon\epsilon^t) = \sigma^2 I$.

Factorial effects are denoted by $F(\alpha_1 \cdots \alpha_{m_1}, \alpha_{m_1+1} \cdots \alpha_m)$, $\alpha_i = 0, 1$, for $i = 1, \cdots, m_1$, and $\alpha_i = 0, 1, 2$ for $i = m_1 + 1, \cdots, m$. [This notation is equivalent to $F^{\alpha_1} \cdots F^{\alpha_{m_1}} F^{\alpha_{m_1+1}} \cdots F^{\alpha_m}$ which is only used in examples.] A factorial effect is a contrast in τ, $c^t\tau$, where $c^t\underline{1} = 0$ and $\underline{1}(\underline{0})$ represents a column vector of 1's (0's) whose dimension will be clear from context. The factorial effect $c^t\tau$ is estimable if, and only if, there is a vector d such that $d^t X\Theta = (c^t \mid \underline{0}^t)\Theta$ identically in Θ. This in turn is equivalent to $d^t(I \mid N) = (c^t \mid \underline{0}^t)$ which entails $d = c$ and $c^t N = \underline{0}$. This result, due to Dean (1978), simply states that if $c^t N = \underline{0}$ (the block effects cancel), then $c^t Y$ is an unbiased estimator of $c^t\tau$ which, by the Gauss Markov theorem, is the Best Linear Unbiased Estimator (BLUE). In general, the estimator $c^t Y$ of the factorial effect $c^t\tau$ will be called the unadjusted estimator of the effect, and will be denoted by $\widehat{c^t\tau}$. When $c^t N \neq \underline{0}$, then $E(\widehat{c^t\tau}) = c^t\tau + c^t NB$. For our problem, each row of N contains exactly one entry equal to 1 and the rest zero's. Hence, $c^t N\underline{1} = c^t\underline{1} = 0$, so $\psi^t = c^t N$ is a contrast. Now, if e_i are vectors and $e_1{}^t\tau = e_2{}^t\tau = \cdots e_r{}^t\tau = 0$ are negligible factorial effects, $e_i{}^t c = \underline{0}$, for $1 \le i \le r$, $e_i{}^t e_j = 0$, $i \neq j$, and $E(\widehat{e_i{}^t\tau}) = k_i\psi^t B$, $1 \le i \le r$, then it is possible to adjust $\widehat{c^t\tau}$ to be unbiased in minimum variance fashion, by subtracting a linear combination of the $\widehat{e_i{}^t\tau}$, $1 \le i \le r$. The resulting estimator, denoted by $(\widehat{c^t\tau})_{adj}$, is given by

$$(\widehat{c^t\tau})_{adj} = \widehat{c^t\tau} - \sum_{i=1}^{r} \frac{w_i(\widehat{e_i{}^t\tau})}{k_i}, \text{ where } w_i = \frac{(e_i{}^t e_i)^{-1}}{\sum_{i=1}^{r} (e_i{}^t e_i)^{-1}}, \text{ for } 1 \le i \le r. \tag{3}$$

The notation $\{\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1\}$ represents the sum of all 2^{m_1-1} points $(x_1 \cdots x_{m_1})$ which are solutions of $\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1$ over the Galois Field GF(2), $u_1 = 0, 1$, and the notation $\{\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2\}$ represents the sum of all 3^{m_2-1} points $(x_{m_1+1} \cdots x_m)$ which are solutions of $\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2$ over the Galois Field GF(3), $u_2 = 0, 1, 2$.

The product of $\{\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1\}$ and $\{\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2\}$ is denoted by:

$$\{\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1\} \otimes \{\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2\}.$$

It represents the sum of all $2^{m_1-1}3^{m_2-1}$ run effects $(x_1 \cdots x_{m_1}, x_{m_1+1} \cdots x_m)$, where $(x_1 \cdots x_{m_1})$ is a solution of $\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1$ over GF(2) and $(x_{m_1+1} \cdots x_m)$ is a solution of $\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2$ over GF(3).

Example 1. Consider a 2^2 x 3^2 factorial experiment. We have $m_1 = 2$, $m_2 = 2$, and $m = m_1 + m_2 = 4$.

Notation	Represents the Sum
$\{x_1 + x_2 = 0\}$	$(00) + (11)$
$\{x_3 + 2x_4 = 1\}$	$(10) + (02) + (21)$
$\{x_1 + x_2 = 0\} \otimes \{x_3 + 2x_4 = 1\}$	$(0010) + (0002) + (0021) + (1110) + (1102) + (1121)$

The factorial effects of a 2^{m_1} x 3^{m_2} factorial experiment are defined as contrasts of run effects. Let the coefficients c_0, c_1, d_0, d_1, and d_2 be as given in Table 1.

Table 1. The Coefficients c_0, c_1, d_0, d_1, and d_2 in the Definition of Factorial Effects

		c_0	c_1	d_0	d_1	d_2
$(\alpha_1 \cdots \alpha_{m_1})^t = \underline{0}$	$(\alpha_{m_1+1} \cdots \alpha_m)^t = \underline{0}$	1	1	1	1	1
$(\alpha_1 \cdots \alpha_{m_1})^t \neq \underline{0}$	$(\alpha_{m_1+1} \cdots \alpha_m)^t = \underline{0}$	-1	1	1	1	1
$(\alpha_1 \cdots \alpha_{m_1})^t = \underline{0}$	$(\alpha_{m_1+1} \cdots \alpha_m)^t \neq \underline{0}$					
(i) the first nonzero element in $(\alpha_{m_1+1} \cdots \alpha_m)^t$ is 1.		1	1	-1	0	1
(ii) the first nonzero element in $(\alpha_{m_1+1} \cdots \alpha_m)^t$ is 2.		1	1	1	-2	1
$(\alpha_1 \cdots \alpha_{m_1})^t \neq \underline{0}$	$(\alpha_{m_1+1} \cdots \alpha_m)^t \neq \underline{0}$					
(i) the first nonzero element in $(\alpha_{m_1+1} \cdots \alpha_m)^t$ is 1.		-1	1	-1	0	1
(ii) the first nonzero element in $(\alpha_{m_1+1} \cdots \alpha_m)^t$ is 2.		-1	1	1	-2	1

Let

$$\begin{array}{rclllll}
\alpha^*_i & = & \alpha_i, & \text{for} & i = 1, \cdots, m_1 & \text{if} & (\alpha_1 \cdots \alpha_{m_1}) \neq (0 \cdots 0), \\
\alpha^*_i & = & 1, & \text{for} & i = 1, \cdots, m_1 & \text{if} & (\alpha_1 \cdots \alpha_{m_1}) = (0 \cdots 0), \\
\alpha^*_i & = & \alpha_i, & \text{for} & i = m_1 + 1, \cdots, m & \text{if} & (\alpha_{m_1+1} \cdots \alpha_m) \neq (0 \cdots 0), \\
\alpha^*_i & = & 1, & \text{for} & i = m_1 + 1, \cdots, m & \text{if} & (\alpha_{m_1+1} \cdots \alpha_m) = (0 \cdots 0).
\end{array}$$

4.2 Theorem 2.

The factorial effects $F(\alpha_1 \cdots \alpha_{m_1}, 1 \cdots 1)$ and $F(\alpha_1 \cdots \alpha_{m_1}, 2 \cdots 2)$ are n.u.e. under D (i.e., they are confounded with blocks in D).

Proof. When $(\alpha_{m_1+1} + \cdots + \alpha_m) = \alpha(1 \cdots 1)$, $\alpha = 1, 2$, it can be seen that there is a block in which the number of runs satisfying $\alpha_1^* x_1 + \cdots + \alpha_{m_1}^* x_{m_1} = 0$ and $x_{m_1+1} + \cdots + x_m = u_2$ is different from the number of runs satisfying $\alpha_1^* x_1 + \cdots + \alpha_{m_1}^* x_{m_1} = 1$ and $x_{m_1+1} + \cdots + x_m = u_2$. Moreover, for every two distinct values of u_2, the difference of the numbers of runs in the above two cases varies within a block and these blocks are different. It is now clear from definitions (2) and (4) that the block effects do not cancel in $E(\hat{F}(\alpha_1 \cdots \alpha_{m_1}, \alpha \cdots \alpha))$ for $(\alpha_1 \cdots \alpha_{m_1}) \neq (0 \cdots 0)$ and $\alpha = 1, 2$. It can also be seen that the number of runs satisfying $x_{m_1+1} + \cdots + x_m = u_2$ varies for values of u_2, $u_2 = 0, 1, 2$. Furthermore, for every two distinct values of u_2, the above numbers are distinct within a block and these blocks are different. Clearly,the block effects do not cancel in $E(\hat{F}(\alpha_1 \cdots \alpha_{m_1}, \alpha \cdots \alpha))$ for $(\alpha_1 \cdots \alpha_{m_1}) = (0 \cdots 0)$ and $\alpha = 1, 2$. This completes the proof.

Example 7. In Example 3, the following factorial effects are n.u.e. in addition to the general mean $\mu = \underline{1}^t \tau$.

F_3F_4	${F_3}^2{F_4}^2$	$F_1F_3F_4$	$F_1{F_3}^2{F_4}^2$
		$F_2F_3F_4$	$F_2{F_3}^2{F_4}^2$
		$F_1F_2F_3F_4$	$F_1F_2{F_3}^2{F_4}^2$

4.3 Theorem 3.

Under D, $F(\alpha_1 \cdots \alpha_{m_1})X$ and $F(\alpha_1 \cdots \alpha_{m_1})Y$ with $(\alpha_1 ... \alpha_{m_1}) \neq (0...0)$, defined in (5) are u.e.

Proof. In the uth $(u = 0, 1, 2)$ block of D, $2^{m_1}3^{m_2-1}$ runs can be divided into 2 sets of $2^{m_1-1}3^{m_2-1}$ runs each satisfying $\alpha_1{}^* x_1 + \cdots + \alpha_m{}^* x_m = i$, $i = 0, 1$. It now follows from (2) and (5) that in $E(\hat{F}(\alpha_1 \cdots \alpha_{m_1})X)$ and $E(\hat{F}(\alpha_1 \cdots \alpha_{m_1})Y)$ the block effects cancel. The rest is clear. This completes the proof.

Observe that μ, X, Y are confounded with the blocks in D. The $(2^{m_1}(3^{m_1} - 2) - 1)$ factorial effects $F(\alpha_1 \cdots \alpha_{m_1}, \alpha_{m_1+1} \cdots \alpha_m)$ with $(\alpha_{m_1+1} ... \alpha_m) \neq \alpha(1...1)$, $\alpha = 1, 2$ and $(\alpha_1 ... \alpha_m) \neq (0...0)$, are u.e. under D. The $(2^{m_1} - 1)2$ linear functions of factorial effects $F(\alpha_1 \cdots \alpha_{m_1})X$ and $F(\alpha_1 \cdots \alpha_{m_1})Y$ with $(\alpha_1 ... \alpha_{m_1}) \neq (0...0)$, are u.e. under D. Thus, we have $[3 + (2^{m_1}(3^{m_2} - 2) - 1) + (2^{m_1} - 1)2] = 2^{m_1}3^{m_2}$ linear functions of factorial effects which are also orthogonal linear functions of run effects.

5 Relative Efficiency

When certain higher order factorial effects are assumed negligible, then n.u.e. factorial effects become estimable through adjustment as discussed earlier. In this section the relative effeciencies of adjusted estimators of n.u.e. factorial effects are calculated. It can be easily verified that for $\alpha = 1$ or 2,

$$Var(\hat{F}(\alpha_1 \cdots \alpha_{m_1}, \alpha \cdots \alpha)) = \sigma^2 2^{(m_1+1)} 3^{(m_2+\alpha-2)}. \tag{6}$$

Let $S = wt(\alpha_1 \cdots \alpha_{m_1})$ be the number of non zero elements in $(\alpha_1 \cdots \alpha_{m_1})$, $B^t = (\beta_0, \beta_1, \beta_2)$ the block effects, and $[\cdot]$ the usual greatest integer function. It follows after some straightforward, but tedious, computations that for $\alpha = 1$ or 2,

$$E(\hat{F}(\alpha_1 \cdots \alpha_{m_1}, \alpha \cdots \alpha)) = F(\alpha_1 \cdots \alpha_{m_1}, \alpha \cdots \alpha)) + 3^{m_2-1+[\frac{(S+\alpha-1)}{2}]} \beta_{S,\alpha}(B). \tag{7}$$

In (7), the absolute value of the term $\beta_{S,\alpha}(B)$ is

$$|\beta_{S,\alpha}(B)| = \left|\{Q_{S,\alpha}(1,-2,1) + (1-Q_{S,\alpha})(-1,0,1)\}P_{m_1 mod(3)}B\right| \tag{8}$$

where $Q_{S,\alpha} = (S+\alpha-1)$ mod(2), and P_0, P_1, and P_2 are the permutation matrices

$$P_0 = I_{(3\times 3)}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It follows from (7) and (8) that if $F(1\cdots 1, 1\cdots 1)$ and $F(1\cdots 1, 2\cdots 2)$ are zero, the bias in the unadjusted estimator of $F(\alpha_1\cdots\alpha_{m_1}, \alpha\cdots\alpha)$, $(\alpha_1\cdots\alpha_{m_1}) \neq (1\cdots 1)$, can be corrected by subtracting from it a constant multiple of the unadjusted estimator of either (but not both) $F(1\cdots 1, 1\cdots 1)$ or $F(1\cdots 1, 2\cdots 2)$. Whether $\hat{F}(1\cdots 1, 1\cdots 1)$ or $\hat{F}(1\cdots 1, 2\cdots 2)$ is used depends on whether or not $(S-m_1) = 0$ mod(2). The adjusted estimator will thus be of the form

$$\hat{F}(\alpha_1\cdots\alpha_{m_1}, \alpha\cdots\alpha)_{adj} = \hat{F}(\alpha_1\cdots\alpha_{m_1}, \alpha\cdots\alpha) + w\hat{F}(1\cdots 1, \gamma\cdots\gamma) \tag{9}$$

where $\gamma = 1$ or 2. Hence, using (6), and for $(\alpha_1\cdots\alpha_{m_1}) \neq (1\cdots 1), \alpha = 1,2$, the relative efficiency in the estimation of $F(\alpha_1\cdots\alpha_{m_1}, \alpha\cdots\alpha)$ is

$$RE = \frac{Var(\hat{F}(\alpha_1\cdots\alpha_{m_1}, \alpha\cdots\alpha))}{Var(\hat{F}(\alpha_1\cdots\alpha_{m_1}, \alpha\cdots\alpha)_{adj})} = \frac{1}{1+w^2 3^{(\gamma-\alpha)}}. \tag{10}$$

It is easily seen from (7) and (8) by considering the two cases $(m_1 - S) = 0, 1$ mod(2) that

$$w = 3^{\frac{(S-m_1+\alpha-\gamma)}{2}} \tag{11}$$

It follows from (10) that in every case,

$$\frac{3}{4} \leq RE = \frac{3^{m_1}}{(3^{m_1}+3^S)} \leq \frac{3^{m_1}}{(3^{m_1}+1)}. \tag{12}$$

It is noted that under the assumption that $F(1\cdots 1, \alpha\cdots\alpha)$, $\alpha = 1,2$, are negligible, the RE in (12) is the standard relative efficiency or the efficiency factor for the factorial effect $F(\alpha_1\cdots\alpha_{m_1}, \alpha\cdots\alpha), (\alpha_1\cdots\alpha_{m_1}) \neq (1\cdots 1), \alpha = 1,2$. [See in John (1987) the equation (2.1) on page 24].

We thus observe that under the assumption $F(1\cdots 1, \alpha\cdots\alpha)$ are negligible, all the factorial effects (except the general mean) are estimable in these deletion designs. Furthermore, the unbiased estimators which are unadjusted are mutually orthogonal and also orthogonal to the unbiased estimators which are adjusted. Pairs of unbiased estimators which have been adjusted are orthogonal when they are adjusted with different bias corrections. Hence the deletion design is a nearly orthogonal design under the assumption that $F(1\cdots 1, \alpha\cdots\alpha), \alpha = 1,2$ are negligible.

Example 8. In Example 3, m_1 equals 2. For the factorial effects F_3F_4 and ${F_3}^2{F_4}^2$, we have $S = 0$. For the factorial effects $F_iF_3F_4$ and $F_i{F_3}^2{F_4}^2$, $i = 1,2$, we have $S = 1$. The REs for estimating F_3F_4 and ${F_3}^2{F_4}^2$, attain the maximum value .90. The REs for estimating $F_iF_3F_4$ and $F_i{F_3}^2{F_4}^2$ attain the minimum value .75. Under the assumption that $F_1F_2F_3F_4$ and $F_1F_2{F_3}^2{F_4}^2$ are negligible, there are $(2^2 \times 3^2 - 1 - 2) = 33$ other factorial effects (notice that we have excluded the general mean). Out of these 33 factorial effects, all but 4 factorial effects attain the maximum values of RE, 1 (for u.e. factorial effects) or .90 (for n.u.e. factorial effects). These 4 factorial effects are all three factor interactions.

6 Example

An experiment is to be designed to study the effects of three factors on the reliability of a radar transmitter. The three factors to be studied are:

F_1:	Burn-in period.
F_2:	Operating temperature.
F_3:	Vibration in usage environment.

The burn-in period is the length of continuous failure-free operating time in a fixed environment that a unit experiences prior to delivery of the unit to a consumer. Units that do not survive the burn-in period are not delivered. The average temperature of the air surrounding the components of the unit in its usage environment is the operating temperature. Vibration in the usage environment is measured in units of root-mean-square G's (g-rms). This is the rms value of the power spectral density over the vibration frequency spectrum.

The transmitters will be manufactured according to three different quality grades as follows:

Grade 0:	Commercial (for civilian use).
Grade 1:	Military standard (for use in armed forces).
Grade 2:	High Reliability (for space-flight applications).

The burn-in times of interest are 24, 48, and 72 hours. The temperature range of interest is 0 degrees Centigrade to 38 degrees Centigrade. The period of testing is limited to one year, and the vibration experienced over this period is to range between .005 and .01 g-rms. Of primary interest are the main effects of the three factors, and interactions to a lesser degree.

Initially a single replicate 3^m design in three blocks can be recommended with the factor levels set as follows:

Factor	Low	Medium	High
F_1	24 hr.	48 hr.	72 hr.
F_2	0° C	22° C	38° C
F_3	.005 g-rms	.0075 g-rms	.01 g-rms

and the quality grades representing blocks. The runs in each block are determined by the equations:

$$x_1 + x_2 + x_3 = 0 \pmod 3 \quad \text{for block 0} \quad \text{(low quality)}$$
$$x_1 + x_2 + x_3 = 1 \pmod 3 \quad \text{for block 1} \quad \text{(medium quality)}$$
$$x_1 + x_2 + x_3 = 2 \pmod 3 \quad \text{for block 2} \quad \text{(high quality)}$$

where $x_i = 0, 1, 2$ according to factor i having low, medium, or high level, $i = 1, 2, 3$. The runs planned for the three blocks are depicted as follows:

Block 0	Block 1	Block 2
0 0 0	0 0 1	0 0 2
0 1 2	0 1 0	0 1 1
0 2 1	0 2 2	0 2 2
1 0 2	1 0 0	1 0 0
1 1 1	1 1 2	1 1 2
1 2 0	1 2 1	1 2 1
2 0 1	2 0 2	2 0 2
2 1 0	2 1 1	2 1 1
2 2 2	2 2 0	2 2 0

Because the units are expensive, and due to lack of facilities to place 27 transmitters on test, a deletion design may be considered. It is believed that the burn-in period has no significant effect on reliability beyond 48 hours, so deletion of all runs with F_1 high is reasonable. Since such a deletion design leaves all effects estimable except $F(011), F(022), F(111), F(122)$, and the block effects, this design is adopted. In this design, observations are taken on the runs above the lines in the previous description of the blocks.

The observation on each run will be the number of failures in a one year period. For a given transmitter, the number Y of failures in a one year period (repair time is not counted) is Poisson distributed with mean ν. Since the variance of Y is also ν, a variance stabilizing transformation is needed. Since $2(\sqrt{Y} - \sqrt{\nu}) \approx (\frac{1}{\sqrt{\nu}})(Y - \nu)$ by Taylor's theorem, and since the latter term is approximately normally distributed with mean 0 and variance 1, the variance stabilizing transformation is $T(Y) = 2\sqrt{Y}$. The postulated linear model is thus

$$2\sqrt{Y(x_1x_2x_3)} = (x_1x_2x_3) + \beta_j + \epsilon_{(x_1x_2x_3)}$$

where β_j is the effect of the *jth* block containing the run $(x_1x_2x_3)$, and $\epsilon_{(x_1x_2x_3)}$ are (approximately) iid N(0,1) random errors.

The observations are as follows:

$(x_1x_2x_3)$	$Y(x_1x_2x_3)$	$2\sqrt{Y(x_1x_2x_3)}$	Block No.	Effect	Adjusted Estimates
000	103	20.298	0	F_3	13.649
001	65	16.125	1	F_3^2	3.939
002	46	13.565	2	F_2	27.109
010	50	15.492	1	F_2F_3	-11.577^*
011	58	15.232	2	$F_2F_3^2$	1.896
012	173	26.306	0	F_2^2	-0.328
021	80	17.889	2	$F_2^2F_3$	10.693
021	160	25.298	0	$F_2^2F_3^2$	51.817^*
022	125	22.361	1	F_1	-21.613
100	50	14.142	1	F_1F_3	-3.457
101	25	10.000	2	$F_2F_3^2$	-1.263
102	101	20.100	0	F_1F_2	-4.012
110	33	11.489	2	$F_1F_2F_3$	38.857^*
111	116	21.541	0	$F_1F_2F_3^2$	-2.962
112	80	17.889	1	$F_1F_2^2$	-3.280
120	125	22.361	0	$F_1F_2^2F_3$	-8.306
121	84	18.330	1	$F1F_2^2F_3^2$	30.552^*
122	57	15.100	2		

(*: Denotes an adjusted estimate that is biased.)

Denote by Z the vector of transformed observations listed by the lexicographical order of the corresponding runs. Then, the total sum of squares corrected for the mean is $Z^tZ - \frac{(1^tZ)^2}{18}$. If $c_1, \cdots, c_{17}$ are the contrast vectors corresponding to the effects listed above, then the unadjusted estimates above are given by $c_j{}^tZ$, for $j = 1, \cdots, 17$. The total sum of squares breaks down into single degree of freedom chi-square variates according to

$$Z_t Z - \frac{(\underline{1}^t Z)^2}{18} = \sum_{j=1}^{17} \frac{(c_j{}^t Z^2)}{c_j{}^t c_j}$$

The analysis of variance table follows. Here, (*) denotes entries that are non-central chi-square variates with one degree of freedom. All other entries are central under the hypothesis that the factorial effect is zero, and can thus be compared to quantiles of the chi-square distribution with one degree of freedom. Entries marked by (**) are not significant at the .05 level of significance.

Source	d.f.	Sum of Squares
F_3	1	15.525
F_3^2	1	0.431**
F_2	1	61.243
F_2F_3	1	11.169*
$F_2F_3^2$	1	0.299**
F_2^2	1	0.003**
$F_2^2F_3$	1	3.176**
$F_2^2F_3^2$	1	74.583*
F_1	1	25.951
F_1F_3	1	0.996**
$F_2F_3^2$	1	0.044**
F_1F_2	1	1.341**
$F_1F_2F_3$	1	125.824*
$F_1F_2F_3^2$	1	0.731**
$F_1F_2^2$	1	0.299**
$F_1F_2^2F_3$	1	1.916**
$F_1F_2^2F_3^2$	1	25.929*
	17	349.460

Of those effects that can be tested, only the main effects F_1, F_2, and F_3 are (highly) significant.

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