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Coloring triangle-free graphs and network games

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Coloring Triangle-Free Graphs and Network Games

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in

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by

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2011

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Chapter 3 of this dissertation is based on my paper ,“A brooks’ theorem for triangle-free graphs”, submitted to the *Journal of Combinatorial Theory.* Chapter 4 is based on my paper, “A coloring algorithm for triangle-free graphs”, submitted to journal *Random Structures and Algorithms.* Chapter 5 is essentially a reprint of the paper “A network coloring game”, which appeared in *WINE 2008, Lecture Notes in Computer Science, Volume 5385.* Professor Kamalika Chaudhuri, Professor Fan Chung Graham, and myself contributed roughly equally to this co-authored paper. My contribution consists primarily of Sections 5.2 and 5.3.

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ABSTRACT OF THE DISSERTATION

Coloring Triangle-Free Graphs and Network Games

by

Mohammad Shoaib Jamall
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A *proper vertex coloring* of a graph is an assignment of colors to all vertices such that adjacent vertices have distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required for a proper vertex coloring.

In this dissertation, we give some background on graph coloring and applications of the probabilistic method to graph coloring problems. We then give three results about graph coloring.

- Let G be a triangle-free graph with maximum degree $\Delta(G)$. We show that the chromatic number $\chi(G)$ is less than $67(1 + o(1))\Delta/\log \Delta$. This number is best possible up to a constant factor for triangle-free graphs.

- We give a randomized algorithm that properly colors the vertices of a triangle-free graph G on n vertices using $O(\Delta(G)/\log \Delta(G))$ colors. The algorithm takes $O(n\Delta^2(G) \log \Delta(G))$ time and succeeds with high probability, provided $\Delta(G)$ is greater than $log^{1+\epsilon}n$ for a positive constant ϵ.

- We analyze a network(graph) coloring game. In each round of the game, each player, as a node in a network G, randomly chooses one of the available colors that is different from all colors played by its neighbors in the previous round. We show that the coloring game converges to its Nash equilibrium if the number of colors is at least $\Delta(G) + 2$. Examples are given for which convergence does not happen with $\Delta(G) + 1$ colors. We also show that with probability at least $1 - \delta$, the number of rounds required is $O(\log(n/\delta))$.

Chapter 1

A brief introduction to graph coloring

Session Number	Interested Participants
100	A, B, C
101	B, D, E
102	A, F, G
103	I, H, G
104	I, J, K
105	C, K, L
106	D, J, M
107	H, J, O
108	F, M, N
109	L, N, O

Figure 1.1: Who wants to attend which conference session.

Figure 1.1 lists sessions of a conference with the participants interested in attending each session. The conference organizers must assign a time slot to each session so that for each participant, all sessions she is interested in are assigned to distinct time slots. This will guarantee that all participants can attend the sessions they are interested in.

We can represent the information in the table above as a graph G, with vertex set corresponding to sessions. We place an edge between 2 vertices if there is a participant interested in both the corresponding sessions. The original scheduling problem of assigning a time slot to each session is equivalent to finding an assignment of colors to all vertices such that adjacent vertices have distinct colors. Such a coloring of a graph is called a *proper vertex coloring*. The *chromatic number* $\chi(G)$ is the minimum number of colors required to color graph G properly. It corresponds to the minimum number of time slots to schedule our sessions so that all participants can attend all sessions they find interesting.

Our original scheduling problem corresponds to the peterson graph(Figure 1.2) which has chromatic number 3. Thus 3 time slots are enough for our conference, but less will not work.

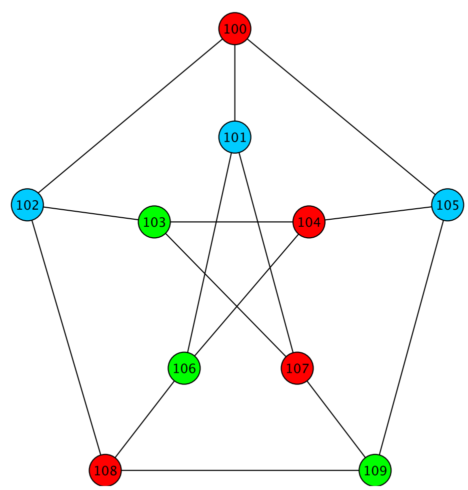

Figure 1.2: A graph where the vertices are sessions of the conference in Figure 1.1 and an edge between 2 vertices represents a scheduling conflict.

The above toy example gives a taste of how graph coloring appears in many constrained resource allocation problems. Listed below are some other applications of graph coloring.

- Wireless network protocols for selecting communication channels: Channels using a small spectrum of frequency must be assigned to transmitters so that they can communicate without interference from each other.

- Register allocation of variables in computer programs: Variables in a program must be assigned to a limited number of registers(by a compiler) so that any two variables that are used at overlapping time intervals have different registers.

- Parallel sparse matrix computations: Blocks of matrices must be assigned to parallel processors so that computation can proceed with little information exchanged between processors.

There are numerous generalizations of the graph coloring problem. For example we may color edges instead of vertices (*edge coloring*), or we may assign a list of colors to each vertex and require that the color chosen for each vertex be from its list (*list coloring*). Unless mentioned otherwise, we will mean "vertex coloring" when we mention "graph coloring" in this treatise. For a general discusson on graph coloring see [38, 33].

In this chapter we look at the complexity of graph coloring and some bounds on the chromatic number. In the next chapter we will see how tools from probability theory are used to prove theorems on graph coloring(and discrete structures in general). In Chapter 3 we will give an upper bound on the chromatic number of triangle-free graphs which is best possible up to a constant. Following that, in Chapter 4, we will give an algorithm for coloring triangle-free graphs. Finally, in Chapter 5, we will look at the coloring problem in a game theoretic setting.

1.1 Graph coloring is hard

Computational complexity theory classifies problems according to their usage of resources(time, storage, etc.). We discuss some issues in complexity theory that relate to this treatise and point the reader to [34] for a comprehensive treatment.

The *Turing machine* is a well known formal model of a general purpose computer. It is an imaginary machine on which algorithms are run. The complexity class P is the set of *decision problems* (with a yes-no answer) which can be solved on a turing machine in *polynomial time* in the size of input. The class NP is the set of decision problems in which"yes" answers can be verified on a turing machine in polynomial time. Clearly P is contained in NP. Whether P and NP are the same, is one of seven Millenium Prize Problems published by the Clay Mathematics Institute.

The decision problem 3-*Colorable* asks the following question. Can a given graph G be properly colored using 3 colors? If G is indeed properly colorable with 3 colors, one such proper coloring is a proof of this and it is verifiable in polynomial time. Thus 3-*Colorable* is in NP. In fact it is in NP-*Complete* [15], the subset of NP problems such that if there is a problem in it which is also in P, then P is NP. The NP-*Complete* problems are the most complex problems in NP.

A problem X is considered NP-*Hard* if there is a NP-*Complete* problem Y such that an instance of Y can be converted into an instance of X in polynomial time. NP-*Hard* problems are at least as complex as NP problems. Even approximating the chromatic number to within $n^{1-\epsilon}$, where n is the number of vertices in G and ϵ is any positive number, is NP-*Hard* (see [25]).

1.2 Brooks' theorem

Given a graph G with maximum degree $\Delta(G)$, we can color it properly using $\Delta(G) + 1$ colors by going through the list of vertices (order does not matter) and choosing a color for each vertex which is not already used for coloring its neighbor. Thus $\Delta(G) + 1$ is a trivial upper bound to the chromatic number of G.

Certainly this upper bound holds with equality for complete graphs and odd cycles. Brooks' theorem says that for a connected graph which is neither an odd cycle nor complete, the upper bound is lower(see [10]).

Theorem A (Brooks' Theorem)**.** *If graph G is connected, not complete and not an odd cyle, then $\chi(G) \leq \Delta(G)$.*

We outline the proof here. If G is not regular, then we choose a vertex of degree less that Δ as v_1, and grow a spanning tree of G rooted at v_1 assigning indices(in increasing order) as we reach each vertex. Then we start with v_n, the last vertex which was reached, go down to v_1 coloring each vertex so that it has a different color from all its higher indexed neighbors. We will never run out if we have Δ colors to choose from.

A *cut vertex* x is a vertex such that the graph $G - x$ is disconnected. If the graph is regular and has a cut vertex x then $G-x$ has more than one components. Let G' be a component of $G - x$ augmented with x and edges between x and the component. Then G' is not Δ regular as the degree of x in G' is less than Δ. We may thus color G' properly with Δ colors using the technique in the previous paragraph. In fact, each component of $G - x$ can be colored in this manner and the color name permuted so that the color of x in each case is 1. We then get a proper coloring of G using only Δ colors.

If G is regular and has no cut-vertex then the proof shows that there are vertices v_1, v_{n-1}, and v_n such that v_{n-1} and v_n are adjacent to v_1 but not to each other, and $G - \{v_{n-1}, v_n\}$ is connected. Then we can grow a spanning tree of $G - \{v_{n-1}, v_n\}$ assigning increasing indices as we reach each vertex. Once again we now go down the indices, coloring each vertex as before but making sure to assign the same color to both v_{n-1} and v_n. If we start with Δ colors we will never run out of colors.

1.3 Four colors suffice for a planar graph

A *planar graph* is one that can be drawn on a page so that edges intersect only at vertices. Francis Guthrie (see [8]) was probably the first to ask the following question. How many colors are required to color any planar graph? The complete graph on 4 vertices is planar and clearly requires 4 colors. In 1977, Apel and Haken [6] proved that 4 colors were enough to color any planar graph. Their proof proceeded by cases, and the cases were so many that a computer was required to go through each case. This seminal work is considered the first major theorem to be proved using a computer.

Chapter 2

Using probability to investigate graph coloring

The probabilistic method proves the existence of a structure with a desired property by defining a probability space of structures and then showing that the event corresponding to the desired property holds with positive probability. Erdös is widely accepted as being the first to realize the potential of the probabilistic method.

The *Ramsey number* $R(r, b)$ is the minimum number such that any edge coloring of a complete graph on $R(r, b)$ vertices with red and blue colors, contains either a complete red subgraph on r vertices or a complete blue subraph on b; Erdös [12] proved that

$$R(k, k) > \frac{1}{e\sqrt{2}}(1 + o(1))2^{k/2} \tag{2.1}$$

for $k \geq 3$ using the probabilistic method. He studied the probability space induced on all possible edge colorings of the complete graph on n vertices by the algorithm that independently and randomly choose color red or blue with equal probability for each edge. Whats more, the proof can be interpreted as a randomized algorithm for finding a red-blue edge coloring of a complete graph with less than $(1 + o(1))2^{k/2}/e\sqrt{2}$ vertices, which has no complete subgraph on k vertices with just one color. The algorithm runs in polynomial time with the probability of producing the desired coloring approaching 1 as k approaches ∞.

An *independent set* of vertices of a graph G is one for which there is no edge in G connecting two vertices in the set. The *independence number* of a G is the largest size of an independent set of vertices in G. Consider the uniform probability space on the independent sets of a triangle-free graph G with n vertices. The following theorem is obtained by applying the probabilistic method on this probability space [31].

Theorem B. *The independence number of a triangle-free graph G with maximum degree Δ is greater than $(1 + o(1))n \log \Delta/\Delta$.*

This is a significantly stronger result than the general lower bound on independence number of $n/(\Delta + 1)$. Much has been written on the probabilistic method. We merely give a taste of it in this chapter and suggest [31, 4] for further study.

2.1 The Lovasz Local Lemma

We use the following version of the Lovasz Local Lemma [31]

Theorem C (Lovasz Local Lemma)**.** *Consider a set $\mathcal{E}$ of events such that for each $A \in \mathcal{E}$*

- $Pr(A) \leq p < 1$,
- *A is mutually independent of a set of all but at most d of the other events.*

If $4pd \leq 1$, then with positive probability, none of the events in $\mathcal{E}$ occur.

If the events A in $\mathcal{E}$ are independent then clearly with positive probability, none of the events in $\mathcal{E}$ occur. The local lemma states that this happens even when there is limited dependence between the events.

When the local lemma is applied to the edge coloring procedure which colors each edge red or blue independently and randomly, it improves the lower bound on the diagonal Ramsey number by a factor of 2 when compared to the bound in inequality (2.1), the result of Erdös' initial analysis (see [4]). We get

$$R(k, k) > \frac{\sqrt{2}}{e} 2^{k/2}(1 + o(1))kw^{k/2}.$$

2.2 Talagrand's inequality and a bound on the chromatic number of triangle-free graphs

The following version of Talagrand's inequality is stated without proof [31]

Theorem D (Talagrand's inequality)**.** *Let X be a non-negative random variable, which is determined by n independent trials $T_1, \ldots, T_n$, and satisfying the following for some $c, r > 0$.*

- *Changing the outcome of any of the T_i affects X by at most c.*
- *If $X \geq s$, then there is a set of at most rs trials whose outcomes certify that $X \geq s$.*

Then for $0 \leq t \leq E[X]$,

$$Pr\{|X - E[X]| > t + 60c\sqrt{rE(X)}\} \leq 4e^{-\frac{t^2}{8c^2 rE[X]}}.$$

Consider the simple coloring algorithm where each vertex in a graph G with maximum degree Δ is assigned a color from $\{1, \ldots, \Delta/2\}$ uniformly and independently. Then all vertices with neighbors which are assigned the same colors are uncolored. Molloy and Reed [31] show that the partial coloring obtained can be greedily completed using less than Δ colors if Δ is large enough. We outline their analysis here.

Let random variable X_v be the number of colors which are assigned to at least two neighbors of vertex v. It is straightforward to show that

$$E[X_v] \geq \frac{\Delta}{e^6} - 1.$$

Observe that the choice of color at each vertex within a distance of 2 edges from v. has an effect of at most 2 on X_v. Further a color contributes to X_v if

- it is chosen by at least 2 neighbors of v, and
- it is not removed from one of the neighbors due to a conflicting color assignment.

Each of these events can be certified by the choice of color of at most 3 vertices within distance 2 of v.

Let $c = 2$, $r = 3$, and let the T_i be the choice of colors in vertices within distance 2 of v. They then use Talagrand's inequality (Theorem D) to conclude that

$$Pr\{|X_v - E[X_v]| > \log\Delta\sqrt{E[X_v]}\} < \frac{1}{4\Delta^5}.$$

Let A_v be the event that X_v is less than $1+(\Delta/2e^6)$. Then $A_v \subset \{|X_v - E[X_v]| > \log\Delta\sqrt{E[X_v]}\}$ and thus

$$Pr(A_v) < \frac{1}{f\Delta^5}.$$

Now for any vertex v, the event A_v depends on at most Δ^2 events of the form A_w where w is a vertex in G. Using the local lemma (Theorem C) we conclude that with positive probability none of the events A_v occur and therefore, there are at least $1 + (\Delta/2e^6)$ colors which appear more than once in the neighborhood of vertex v. We can now finish the coloring greedily using only $\Delta - \Delta/2e^6$ colors. We can now get the following theorem immediately.

Theorem E. *Let G be a triangle-free graph with maximum degree Δ, then*

$$\chi(G) \leq (1 - \frac{1}{2e^6})\Delta.$$

On the other hand the probabilistic method has also been used to show that there exist graphs of arbitrarily large girth with chromatic number at least $\frac{1}{2}(1 + o(1))\frac{\Delta}{\log\Delta}$ (see [31]).

2.3 The semi-random method

The so-called *semi-random method*, also known as the *pseudo-random method*, or the *Rödl nibble*, appeared first in Ajtai, Komlós and Szemerédi [2] and was applied to problems in hypergraph packings, Ramsey theory, colorings, and list colorings [14, 22, 23, 27, 32]. In general, given a set S_1, the goal is to show that there is an object in S_1 with a desired property $\mathcal{P}$. This is done by locating a sequence of non-empty subsets $S_1 \supseteq \cdots \supseteq S_\tau$ with S_τ having property $\mathcal{P}$. A randomized algorithm is applied to S_t, which guarantees that S_{t+1} will be obtained with some non-zero(often small) probability. For upper bounds on chromatic number, the semi-random method is used to prove the existence of a proper coloring with a limited number of colors. We will use the semi-random method in the next chapter to give an upper bound on the chromatic number of triangle-free graphs.

Chapter 3

A Brooks' theorem for triangle-free graphs

3.1 Introduction

For general graphs, $\Delta(G) + 1$ is a trivial upper bound for chromatic number $\chi(G)$. Brooks' Theorem [10] shows that $\chi(G)$ can be $\Delta(G) + 1$ only if G has a component which is either a complete subgraph or an odd cycle.

A natural question is: can this bound be improved for graphs without large complete subgraphs? In 1968, Vizing [36] had asked what the best possible upper bound for the chromatic number of a triangle-free graph was. Borodin and Kostochka [9], Catalin [11], and Lawrence [29] independently made progress in this direction; they showed that for a K_4-free graph, $\chi(G) \leq 3(\Delta(G) + 2)/4$. On the other hand, Kostochka and Masurova [28], and Bollobás [7] separately showed that there are graphs of arbitrarily large *girth*(length of a shortest cycle) with $\chi(G)$ of order $\Delta(G)/\log \Delta(G)$.

In 1995, Kim [26] proved that

$$\chi(G) \leq (1 + o(1)) \frac{\Delta(G)}{\log \Delta(G)}$$

when G has girth greater than 4. Later on, Johansson [21] showed that

$$\chi(G) \leq O(\frac{\Delta(G)}{\log \Delta(G)})$$

when G is a triangle-free graph(girth greater than 3). Both Kim and Johansson used the semi-random method to show that the chromatic number of graphs with large girth is $O(\Delta(G)/\log\Delta(G))$. Alon, Krivelevich and Sudakov [3], and Vu [37] extended the method of Johansson to prove bounds on the chromatic number for graphs in which no subgraph on the set of all neighbors of a vertex has too many edges.

In this paper we prove that the chromatic number of a triangle-free graph G is less than $67(1+o(1))\Delta/\log\Delta$ using the semi-random method. As we will indicate in Section 3.2, our proof is inspired by Kim's proof of an upper bound to the chromatic number of graphs with girth greater than 4. We believe our technique is simpler than Johanssons' which follows a different approach to that of Kims (see [31] for a comparison of both).

We give our proof by analyzing an iterative algorithm for graph coloring. To analyze this algorithm we identify a collection of random variables. The expected changes to these random variables after a round of the algorithm are written in terms of the values of the random variables before the round. We thus obtain a set of recurrence relations and prove that our random variables are concentrated around the solutions to the recurrence relations with some positive probability.

We describe our algorithm in Section 3.2. Section 3.2.1 contains motivation, which is followed by a formal description of the algorithm in Section 3.2.2. We give an outline of the analysis in Section 3.3. Section 3.4 contains some useful lemmas which we use in Section 3.5 to give details of the analysis.

3.2 An Iterative Algorithm for Coloring a Graph

Our algorithm takes as input a triangle-free graph G on n vertices, its maximum degree Δ, and the number of colors to use Δ/k where k is a positive number. It goes through rounds and assigns colors to more vertices each round. Initially all vertices are *uncolored*(no color assigned), at the end we have a proper vertex coloring of G with some probability.

Definition 1. *Let t be a natural number. We define the following:*

G_t	*The graph induced on G by the vertices that are uncolored at the beginning of round t.*
$N_t(u)$	*The set of vertices adjacent to vertex u in G_t. That is, the set of uncolored neighbors of u at the beginning of round t.*
$S_t(u)$	*The list of colors that may be assigned to vertex u in round t, also called the* palette *of u. For all u in $V(G)$,* $$S_0(u) = \{1, \ldots, \Delta/k\}.$$
$D_t(u,c)$	*The set of vertices adjacent to u that may be assigned color c in round t. That is,* $$D_t(u,c) := \{v \in N_t(u) \mid c \in S_t(v)\}.$$

It will be useful to define variables for the sizes of the sets $N_t(u)$, $S_t(u)$, and $D_t(u,c)$.

Definition 2.

$$\eta_t(u) = |N_t(u)|$$

$$s_t(u) = |S_t(u)|$$

$$d_t(u,c) = |D_t(u,c)|$$

Observe that for every round t, vertex u in $V(G)$, and color c in $\{1, \ldots, \Delta/k\}$,

$$d_t(u,c), \eta_t(u) \leq \Delta, \quad s_0(u) = \Delta/k, \quad s_t(u) \leq \Delta/k.$$

3.2.1 A Sketch of the Algorithm and the Ideas behind its Analysis

We say that a sequence $x(n)$ is $O(f(n))$ if there is a positive number M such that $|x(n)| \leq M|f(n)|$. All sequences in the big-oh are indexed by Δ, the maximum degree of graph G. Remember that each occurrence of the big-oh comes with a distinct constant M. We start by considering an algorithm that colors Δ-regular graphs with girth greater than 4. The coloring produced is proper with positive probability.

Let (d_t) and (s_t) be sequences defined recursively as

$$\begin{aligned} d_0 &:= \Delta & \qquad d_{t+1} &:= d_t(1 - c_1 \tfrac{s_t}{d_t})c_2 \\ s_0 &:= \Delta/k & \qquad s_{t+1} &:= s_t c_2 \end{aligned}$$

where c_1 and c_2 are constants between 0 and 1, which are determined by the analysis of the algorithm.

Repeat at every round t, until $d_t/s_t < 1/2$

Phase I - Coloring Attempt

For each vertex u in G_t:

Awake vertex u with probabilitity s_t/d_t.

If *awake*, assign to u a color chosen from $S_t(u)$ uniformly at random.

Phase II - Conflict Resolution

For each vertex u in G_t:

If u *is awake*,

uncolor u if an adjacent vertex is assigned the same color in the coloring attempt phase.

Remove from $S_t(u)$, all colors assigned to adjacent vertices.

$S_{t+1}(u) = S_t(u)$.

end repeat

Permanently color each vertex u in $V(G_t)$ with a color picked independently and uniformly at random from its palette $S_t(u)$.

Observe that $d_0/s_0 = k$ and

$$\frac{d_{t+1}}{s_{t+1}} = \frac{d_t}{s_t} - c_1.$$

In $O(k)$ rounds d_t/s_t will be less than 1/2, and this marks the end of the repeat-until block.

The algorithm above is derived from Kim [26]. After some modifications, his analysis tells us that if graph G has girth greater than 4, then there are constants c_1 and c_2 less than 1 such that at each round t, $\forall u \in V(G_t), \forall c \in S_t(u)$

$$s_t(u) = s_t(1 + o(1)), \quad d_t(u, c) = d_t(1 + o(1)) \tag{3.1}$$

with probability greater than 0. The equations above imply that after the repeat-until block, if Δ is large enough, then with positive probability $s_t(u) > 2d_t(u, c)$ for all uncolored vertices u and colors c in their palette. Now, applying the result of Haxell [17], we find that the random assignment of colors to all uncolored vertices in the final step of the algorithm gives a properly colored graph with positive probability.

The problem with cycles of length 4.

The analysis for the above algorithm is probabilistic and proves the property in equation (3.1) by induction, showing concentration of the variables around their expectations. It fails for graphs with 4-cycles. An example illustrates why: Consider a vertex u whose 2-neighborhood, the graph induced by vertices within distance 2 of u, is the complete bipartite graph $K_{\Delta,\Delta}$ with partitions X and Y. Suppose that u and another vertex v are in X. If v is colored with c in round 0 while u remains uncolored, then the set $D_1(u, c) = \emptyset$; this violates equation (3.1) since $d_1 \geq 1$ if for example $k \geq 2$ and $\Delta \geq 2/c_2$. So, when the graph has 4-cycles, $d_{t+1}(u, c)$ is not necessarily concentrated around its expectation with positive probability, given the state of the algorithm at the beginning of round t. We must modify the algorithm in two ways.

First Modification: Dealing with 4-cycles. While $d_t(u, c)$ is not concentrated enough when the graph has 4-cycles, our analysis will show that the average of $d_{t+1}(u, c)$ over all colors in the palette of a vertex u is concentrated enough. How does this benefit us? Markov's famous inequality may be interpreted as: a list of s positive number which average d has at most s/q numbers larger than qd for any positive number q. We modify the algorithm so that at the end of each round t, every vertex u removes from its palette every color c

with $d_{t+1}(u, c)$ larger than $2d_{t+1}$. Look at what happens in round $t = 1$. By a straightforward application of Markov's inequality, instead of equation (3.1) we will have the less stringent property: $\forall u \in V(G_t), \forall c \in S_t(u)$

$$s_t(u) \geq \frac{1}{2} s_t(1 - o(1)), \quad d_t(u, c) \leq 2d_t(1 + o(1)). \tag{3.2}$$

with positive probability. In fact, using a generalization of Markov's inequality, the analysis will show that with a few more modifications our algorithm maintains, with positive probability, a slightly stronger property(still weaker than equation (3.1)).

Equation (3.1) implies that the $s_t(u)$ and $d_t(u)$ at all uncolored vertices u are about the same. It is a strong statement and helps in the proofs, but is too much to maintain on graphs with 4-cycles. Equation (3.2) is weaker and is obtained by our algorithm with positive probability. Moreover, it is sufficient to ensure that after the repeat-until block, with positive probability, for each uncolored vertex u and color c in its palette, $s_t(u) \geq 2d_t(u, c)$. This is a key idea in our algorithm.

Second Modification: Independent random variables for easier analysis. Instead of waking up a vertex with some probability, and then choosing a color from its palette uniformly at random; for each uncolored vertex u and color c in its palette, we will assign c to u independently with some probability. In case multiple colors remain assigned to the vertex after the conflict resolution phase, we will arbitrarily choose one of them to permanently color the vertex. This modification, adapted from Johansson [21], will make concentration of our random variables simpler.

Next we provide a formal description of the algorithm we have just motivated.

3.2.2 A Formal Description of the Algorithm

Let (d_t) and (s_t) be sequences defined recursively as

$$\begin{aligned} d_0 &:= \Delta & d_{t+1} &:= d_t(1 - \frac{1}{16}e^{-1/2}\frac{s_t}{d_t})e^{-1/2} \\ s_0 &:= \Delta/k & s_{t+1} &:= s_t e^{-1/2}. \end{aligned} \tag{3.3}$$

For round t, vertex u, and color c,

$$\mathcal{F}_t(u, c) := \{c \text{ is not assigned to any vertex adjacent to } u \text{ in round } t\} \tag{3.4}$$

is an event in the probability space generated by the random choices of the algorithm in round t, given the state of all data structures at the beginning of the round.

Let

$$Desired\_\mathcal{F}_t := e^{-1/2}.$$

Repeat at every round t, until $d_t/s_t < 1/8$

Phase I - Coloring Attempt

For each vertex u in G_t, and color c in $S_t(u)$:

Assign c to u with probability $\frac{1}{4}\frac{1}{d_t}$.

Phase II - Conflict Resolution

For each vertex u in G_t:

Phase II.1

Remove from $S_t(u)$, all colors assigned to adjacent vertices.

Phase II.2

For each color c in $S_t(u)$, remove c from $S_t(u)$ with probability

$$1 - min(1, \frac{Desired\_\mathcal{F}_t(u,c)}{Pr(\mathcal{F}_t(u,c)}).$$

If $S_t(u)$ has at least one color which is assigned to u,

then arbirarily pick an assigned color from $S_t(u)$ to permanently color u.

Phase III - Cleanup(discard all colors c with $d_{t+1}(u,c) \gtrsim 2d_{t+1}$ from palette)

For each vertex u in G_t:

$S_{t+1}(u) = S_t(u)$.

Let $\alpha = 1 - |S_{t+1}(u)|/s_{t+1}$.

If $\alpha < 0$, then $\alpha = 0$, otherwise if $\alpha > 1/2$, then $\alpha = 1/2$.

Let γ be the smallest number in $[1, \infty)$ so that

$$Average_{c \in S_{t+1}(u)} d_{t+1}(u,c) \leq \frac{1-2\alpha}{1-\alpha} \gamma d_{t+1}.$$

Remove all colors c with $d_{t+1}(u,c) \geq 2\gamma d_{t+1}$ from $S_{t+1}(u)$.

end repeat

Permanently color each vertex u in $V(G_t)$ with a color picked independently and uniformly at random from its palette $S_t(u)$.

3.3 The Main Theorem

Theorem 1 (Main Theorem)**.** *Given $67\Delta/\log\Delta$ colors and a triangle-free graph G with maximum degree Δ large enough, our algorithm finds a proper coloring of the graph with positive probability.*

We need some lemmas to prove the Main Theorem and before that we need the following definition.

Definition 3. $d_t(v) = Average_{c\in S_t(v)} d_t(v,c)$

Lemma 3.3.1 (Main Lemma)**.** *Given $\psi > 1$ and a triangle-free graph G with maximum degree Δ, there is a positive constant β such that for the sequence (e_t) defined by*

$$e_0 = 0, \quad e_{t+1} = 3e_t + \beta(\sqrt{\frac{\psi}{s_t}}) \quad \text{for } t > 0, \tag{3.5}$$

if $s_t \gg \psi$ and $e_t \ll 1$ at round t, then

$$\forall u \in V(G_t), \exists \alpha \in [0,1/2], \forall c \in S_t(u),$$

$$\begin{aligned} s_t(u) &\geq (1-\alpha)s_t(1-e_t) \\ d_t(u) &\leq \frac{1-2\alpha}{1-\alpha} d_t(1+e_t) \\ d_t(u,c) &\leq 2d_t(1+e_t) \end{aligned}$$

with positive probability.

We will prove the Main Lemma in Section 3.5 and assume it in this section. Using it we can immediately conclude the following.

Corollary 1. *Given the setup of the Main Lemma(Lemma 3.3.1), if $s_t \gg \psi$ and $e_t \ll 1$ at round t, then $\forall u \in V(G_t)$,*

$$\begin{aligned} s_t(u) &\geq \frac{1}{2} s_t(1-e_t) \\ d_t(u) &\leq d_t(1+e_t) \end{aligned}$$

with positive probability.

Lemma 3.3.2. *The repeat-until block finishes in* $16e^{1/2}k$ *rounds.*

Proof. By the definition of sequences (d_t) and (s_t) in equation (3.3), we have

$$\begin{aligned}\frac{d_{t+1}}{s_{t+1}} &= \frac{d_t}{s_t}(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})\\ &= \frac{d_t}{s_t} - \frac{1}{16}e^{-1/2}.\end{aligned}$$

Since $\frac{d_0}{s_0} = k$ we get $\frac{d_{t_1}}{s_{t_1}} \leq \frac{1}{4}$ after $16e^{1/2}k$ rounds. □

Let

$$t_1 = 16e^{1/2}k$$

be the last round of the repeat-until block. Then the following lemma is a straightforward application of equation (3.3).

Lemma 3.3.3.

$s_{t_1} = \dfrac{\Delta}{k}exp(-8e^{1/2}k)$ *and, if* $k \leq \dfrac{1}{9e^{1/2}}\log\Delta$ *and* Δ *is large enough, then* $s_{t_1} \gg 1$.

3.3.1 Bounding the Error Estimate in all Concentration Inequalities

Now we look at s_t, which is used to bound the error term e_t.

Lemma 3.3.4.

$$e_t \leq 3^t O(\sqrt{\frac{k\ exp(8e^{1/2}k)\psi}{\Delta}}).$$

Proof. By Lemma 3.3.3, we have

$$s_{t_1} = \frac{\Delta}{k}exp(-8e^{1/2}k).$$

Note that in equation (3.5), the recurrence for e_t, the largest term is $O(\sqrt{\psi/s_t})$. Since the sequence (s_t) is decreasing, we use Lemma 3.3.3 to conclude that

$$O(\sqrt{\frac{\psi}{s_{t_1}}}) = O(\sqrt{\frac{k\ exp(8e^{1/2}k)\psi}{\Delta}})$$

is the maximum this term can be. Thus we can simplify the recurrence for e_t to

$$e_{t+1} = 3e_t + \sqrt{\frac{k\ exp(8e^{1/2}k)\psi}{\Delta}}).$$

Since $e_0 = 0$, a simple upper bound for e_t is given by

$$e_t \leq 3^t O(\sqrt{\frac{k\ exp(8e^{1/2}k)\psi}{\Delta}})$$

where α is some positive constant. □

Lemma 3.3.5. *Given Δ/k colors where $k \leq \frac{1}{67}(\log \Delta)$ and a triangle-free graph G with maximum degree Δ, our algorithm reaches the end of the repeat-until block at round $t_1 = O(k)$ with $e_{t_1} \ll 1$, and $\forall u \in V(G_{t_1}), c \in S_{t_1}(u)$*

$$s_{t_1}(u) \geq \frac{1}{2}s_{t_1}(1 - e_{t_1})$$

$$d_{t_1}(u, c) \leq 2d_{t_1}(1 + e_{t_1})$$

with positive probability.

Proof. Let $\psi = 3\log \Delta$, and let t_1 be the number of rounds to reach the end of the repeat-until block. Using Lemma 3.3.4, we get

$$e_{t_1} \leq 3^{t_1} O(\sqrt{\frac{k\ exp(8e^{1/2}k)\psi}{\Delta}}).$$

Using Lemma 3.3.3 it is straightforward to show that

$$e_{t_1} \ll 1 \quad \text{and} \quad s_{t_1} \gg \psi$$

if $k \leq \frac{1}{67}\log \Delta$. Applying Corollary 1 completes the proof. □

We may now prove the Main Theorem.

of the Main Theorem. Using Lemma 3.3.5, we get

$$\forall u \in V(G_{t_1}), c \in S_{t_1}(u) \quad s_{t_1}(u) \geq 2d_{t_1}(u, c)$$

with positive probability. Now Haxell [17] shows that the final step of our algorithm finds(randomly coloring all uncolored vertices) finds a proper coloring with positive probability. □

3.4 Several Useful Inequalities

Now we look at some preliminaries which will be used in the proof details. The next lemma describes what happens to the average value of a finite subset of real numbers when large elements are removed. As shown in the statement of the lemma, it implies Markov's Inequality [4].

Lemma 3.4.1. *Consider a set of positive real numbers of size n and average value μ. If we remove αn elements with value atleast $q\mu$ for some $q > 1$, then the remaining points have average*

$$\mu' \leq \mu \frac{1 - q\alpha}{1 - \alpha}.$$

In particular, $\alpha \leq \frac{1}{q}$ since $\mu' \geq 0$.

Proof. The conclusion is obtained by a trivial manipulation of the following inequality which relates μ and μ'.

$$q\mu\alpha + \mu'(1 - \alpha) \leq \mu$$

□

The next lemma describes what happens when we add large elements to a finite subset of real numbers.

Lemma 3.4.2. *Given the setup of Lemma 3.4.1, if we add αn points with value $q\mu$ to the sample, then the resulting larger sample has average*

$$\mu' = \mu \frac{1 + q\alpha}{1 + \alpha}$$

Proof. The conclusion is easily obtained from the following equation relating μ and μ'.

$$\mu'(1 + \alpha) = \mu + q\mu\alpha$$

□

We use the following lemma for computations with error factors.

Lemma 3.4.3. *Let* (A_n) *be a sequence such that* $0 < A_n < c < 1$*(where* c *is a constant), and let* (e_n) *be another sequence. Then*

$$1 - A_n(1 + e_n) = (1 - A_n)(1 + O(e_n))$$

Proof.

$$\begin{aligned}
1 - A_n(1 + e_n) &= (1 - A_n)(1 + e_n) - e_n \\
&= (1 - A_n)(1 + e_n) - (1 - A_n)\frac{e_n}{(1 - A_n)} \\
&= (1 - A_n)(1 + e_n) - (1 - A_n)O(e_n) \\
&= (1 - A_n)(1 + O(e_n))
\end{aligned}$$

□

We use the following version of Azuma's inequality [31] to prove concentration of random variables.

Theorem F (Azuma's inequality)**.** *Let* X *be a random variable determined by* n *trials* $T_1, \ldots, T_n$*, such that for each* i*, and any two possible sequences of outcomes* $t_1, \ldots, t_i$ *and* $t_1, \ldots, t_{i-1}, t'_i$*:*

$$|E[X|T_1 = t_1, \ldots, T_i = t_i] - E[X|T_1 = t_1, \ldots, T_i = t'_i]| \leq \alpha_i$$

then

$$Pr(|X - E[X]| > t) \leq 2e^{-t^2/(\sum \alpha_i^2)}$$

3.5 Proof of the Main Lemma

The following assumptions are repeatedly used in the lemmas of this section.

Assumption 1. *Assume*

$$s_t \gg \psi, e_t \ll 1$$

and with positive probability

$$\forall u \in V(G_t), \forall c \in S_t(u), \exists \alpha \in [0, 1/2],$$

$$
\begin{aligned}
s_t(u) &\geq (1-\alpha)s_t(1-e_t) \\
d_t(u) &\leq \frac{1-2\alpha}{1-\alpha}d_t(1+e_t) \\
d_t(u,c) &\leq 2d_t(1+e_t).
\end{aligned}
$$

All events in this section are in the probability space generated by our randomized algorithm in round $t+1$, given the state of all data structures at the beginning of the round.

of the Main Lemma. The proof is by induction on the round number t using lemmas that follow. The base case, when $t = 0$, is trivially true. If we assume Assumption 1, the induction hypothesis, for round t then for each vertex u in $V(G_{t+1})$, by Lemma 3.5.5 we have

$$
\begin{aligned}
&Pr\{d_{t+1}(u) \leq d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})\} \\
&\qquad \geq 1 - e^{-\psi}O(1).
\end{aligned}
$$

For each c in $S_{t+1}(u)$, by Lemma 3.5.6, we have

$$
\begin{aligned}
&Pr\{\exists \alpha \in [0, \frac{1}{2}] \text{ such that} \\
&\quad s_{t+1}(u) \geq (1-\alpha)s_{t+1}(1 - 3e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \\
&\quad d_{t+1}(u) \leq \frac{1-2\alpha}{1-\alpha}d_{t+1}(1 + 3e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \\
&\quad d_{t+1}(u,c) \geq (1-\alpha)2d_{t+1}(1 - 3e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\} \\
&\qquad \geq 1 - e^{-\psi}O(1).
\end{aligned}
$$

Each of the events in the probabilities above is dependent on at most $O(\Delta^2)$ other such events. If $\psi \geq 3\log\Delta$ and Δ is large enough, then we use Theorem C to conclude that Assumption 1 holds for round $t+1$. □

The above proof of the Main Lemma required Lemmas 3.5.5 and 3.5.6. The rest of this section will prove these lemmas. Next we consider the state of the palettes just before the cleanup phase of round t.

Definition 4. *Let $\tilde{S}_t(u)$ be the list of colors in the palette of vertex u in round t just before the cleanup phase, and let $\tilde{s}_t(u)$ be the size of $\tilde{S}_t(u)$. That is, $\tilde{S}_t(u)$ is obtained from $S_t(u)$ by removing colors discarded in the conflict resolution phase.*

Lemma 3.5.1. *Given Assumption 1, for each vertex u in $V(G_{t+1})$ we have*

$$\begin{aligned} Pr\{s_t(u)e^{-1/2}(1-\frac{1}{2}e_t - O(\sqrt{\frac{\psi}{s_t}})) \leq \tilde{s}_t(u) \leq s_t(u)e^{-1/2}(1+O(\sqrt{\frac{\psi}{s_t}}))\} \\ \geq 1 - e^{-\psi}O(1). \end{aligned}$$

Proof. Suppose u is an uncolored vertex at the beginning of round t, and c a color in its palette.

$$\begin{aligned}
&Pr\{c \text{ is removed from } S_t(u) \text{ in phase II.1}\} \\
&\quad = 1 - Pr\{\text{no neighbor of } u \text{ is assigned } c\} \\
&\quad = 1 - \prod_{v \in D_t(u,c)} (1 - Pr\{v \text{ is assigned } c\}) \\
&\quad = 1 - \prod_{v \in D_{u,c}} (1 - \frac{1}{4}\frac{1}{d_t}) \\
&\quad \leq 1 - (1 - \frac{1}{4}\frac{1}{d_t})^{d_t(u,c)} \\
&\quad \leq 1 - (1 - \frac{1}{4}\frac{1}{d_t})^{2d_t(1+e_t)} \\
&\quad \leq 1 - e^{\log(1-\frac{1}{4}\frac{1}{d_t})2d_t(1+e_t)} && \langle \log(1+x) = x + O(x^2) \rangle \\
&\quad \leq 1 - e^{(-\frac{1}{4}\frac{1}{d_t}+O(\frac{1}{d_t})^2)2d_t(1+e_t)} && \langle \text{Assumption 1} \rangle \\
&\quad \leq 1 - e^{-1/2}(1 - \frac{1}{2}e_t + O(\frac{1}{d_t}))
\end{aligned}$$

In phase II.2 of round $t+1$ we remove colors from the palette using an appropriate bernoulli variable, to get

$$Pr\{c \notin \tilde{S}_t(u)\} = 1 - e^{-1/2}(1 - \frac{1}{2}e_t + O(\frac{1}{d_t})). \tag{3.6}$$

Using linearity of expectation

$$\forall u \in V(G_{t+1}), E[\tilde{s}_t(u)] = s_t(u)e^{-1/2}(1 - \frac{1}{2}e_t + O(\frac{1}{d_t})).$$

For concentration of $\tilde{s}_t(u)$, suppose $s_t(u) = m$. Let $c_1, \ldots, c_m$ be the colors in $S_t(u)$. Then $\tilde{S}_t(u)$ may be considered a random variable determined by m trials

$T_1, \ldots, T_m$ where T_i is the set of vertices in G_t that are assigned color c_i in round t. Observe that T_i affects $\tilde{S}_t(u)$ by at most 1 given $T_1, \ldots, T_{i-1}$. Now using Theorem F we get,

$$Pr\{|\tilde{s}_t(u) - E[\tilde{s}_t(u)]| \geq \sqrt{\psi s_t(u)}\} \leq e^{-\psi} O(1).$$

□

We now focus on the sets $D_t(u, c)$. The following two lemmas will help.

Lemma 3.5.2. *Let u be an uncolored vertex, and c be a color in its palette at the beginning of round t. Then given Assumption 1, we have*

$$Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} = \frac{1}{4}\frac{1}{d_t} e^{-1/2}(1 - \frac{1}{2}e_t + O(\frac{1}{d_t})).$$

Proof.

$$\begin{aligned} &Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} \\ &\quad = Pr\{u \text{ is assigned } c\} Pr\{c \in \tilde{S}_t(u)\} \\ &\quad = \frac{1}{4}\frac{1}{d_t} e^{-1/2}(1 - \frac{1}{2}e_t + O(\frac{1}{d_t})) \qquad \langle \text{Equation (3.6)} \rangle \end{aligned}$$

□

The following lemma is a consequence of the previous one.

Lemma 3.5.3. *Let u be an uncolored vertex at the beginning of round t. Then given Assumption 1, we have*

$$Pr\{u \text{ is colored}\} \geq \frac{1}{16}\frac{s_t}{d_t} e^{-1/2}(1 - 3e_t + O(\frac{1}{d_t})).$$

Proof. Consider the event

$$\{u \text{ is colored}\} = \bigcup_{c \in S_t(u)} \{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}.$$

Since the events in the union on the right hand side of the equation above are independent,

$$Pr\{u \text{ is colored}\} = 1 - \prod_{c \in S_t(u)} (1 - Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}).$$

Now using Lemma 3.5.2, we get

$$
\begin{aligned}
&Pr\{u \text{ is colored}\} \\
&\quad \geq 1-(1-\frac{1}{4}\frac{1}{d_t}e^{-1/2}(1-\frac{1}{2}e_t+O(\frac{1}{d_t})))^{s_t(u)} \\
&\quad \geq 1-(1-\frac{1}{4}\frac{1}{d_t}e^{-1/2}(1-\frac{1}{2}e_t+O(\frac{1}{d_t})))^{\frac{1}{2}s_t(1-e_t)} \qquad \langle\text{Assumption 1}\rangle \\
&\quad \geq 1-exp(-\frac{1}{8}\frac{s_t}{d_t}e^{-1/2}(1-\frac{3}{2}e_t+O(\frac{1}{d_t}))) \\
&\quad \geq 1-(1-\frac{1}{16}\frac{s_t}{d_t}e^{-1/2}(1-\frac{3}{2}e_t+O(\frac{1}{d_t}))) \\
&\quad = \frac{1}{16}\frac{s_t}{d_t}e^{-1/2}(1-\frac{3}{2}e_t+O(\frac{1}{d_t})).
\end{aligned}
$$

□

We will need the following definitions.

Definition 5.

- *Let $\tilde{D}_t(u,c)$ be the set of uncolored vertices that have color c in their palettes and are uncolored in round t, just before the cleanup phase. That is,*

$$\tilde{D}_t(u,c) = D_t(u,c) \setminus (\{v|c \notin \tilde{S}_t(v)\} \cup \{v|v \text{ is colored in round } t\}).$$

- *Let $\tilde{d}_t(u,c)$ be the size of $\tilde{D}_t(u,c)$.*
- $\bar{d}_t(u) := \sum_{c\in\tilde{S}_t(u)} \tilde{d}_t(u,c) = \sum_{c\in S_t(u)} 1_{\{c\in\tilde{S}_t(u)\}}\tilde{d}_t(u,c)$
- $\tilde{d}_t(u) := \frac{\bar{d}_t(u)}{\tilde{s}_t(u)}$

Lemma 3.5.4. *Given Assumption 1, for each vertex u in $V(G_{t+1})$ we have*

$$
\begin{aligned}
Pr\{\tilde{d}_t(u) \leq d_t(u)(1-\frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1+2e_t+O(\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}+\sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\} \\
\geq 1-e^{-\psi}O(1).
\end{aligned}
$$

Proof. Let u be an uncolored vertex at the beginning of round t, and let c be a color in its palette. For a vertex v in $D_t(u,c)$, Lemma 3.5.2 implies that $Pr\{v$ is colored with $d\} = O(1/d_t)$ for any color d in $S_t(v)$. Thus,

$$\begin{aligned}
&Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is colored}\}) \\
&\quad = \sum_{d \in S_t(v)} Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is colored with } d\}) \\
&\quad = \sum_{d \in S_t(v)} Pr\{c \notin \tilde{S}_t(v) | v \text{ is colored with } d\} Pr\{v \text{ is colored with } d\} \\
&\quad = Pr\{c \notin \tilde{S}_t(v)\}(1 + O(\frac{1}{d_t})) \sum_{d \in S_t(v)} Pr\{v \text{ is colored with } d\} \\
&\quad = Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is colored}\}(1 + O(\frac{1}{d_t})).
\end{aligned}$$

A straightforward computation now shows that

$$\begin{aligned}
&Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}) \\
&\quad = Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is not colored}\}(1 + O(\frac{1}{d_t})).
\end{aligned} \tag{3.7}$$

Now, v is removed from the set $D_t(u,c)$ if either it is colored or color c is removed from its palette. This means that event

$$\{v \notin \tilde{D}_t(u,c)\} = \{v \text{ is colored}\} \cup (\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}).$$

Since G is triangle-free, u and v do not have any common neighbors. This implies that

$$
\begin{aligned}
&Pr\{v \notin \tilde{D}_t(u,c) | c \in \tilde{S}_t(u)\} \\
&\quad = Pr\{v \notin \tilde{D}_t(u,c)\}(1 + O(\frac{1}{d_t})) \\
&\quad = (Pr\{v \text{ is colored}\} \\
&\qquad + Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}))(1 + O(\frac{1}{d_t})) \\
&\quad = (Pr\{v \text{ is colored}\} \\
&\qquad + Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is not colored}\})(1 + O(\frac{1}{d_t})) \qquad \langle\text{equation (3.7)}\rangle \\
&\quad = (Pr\{v \text{ is colored}\} \\
&\qquad + (1 - e^{-1/2})(1 - Pr\{v \text{ is colored}\}))(1 + O(\frac{1}{d_t})) \qquad \langle\text{equation (3.6)}\rangle \\
&\quad = (1 - (1 - Pr\{v \text{ is colored}\})e^{-1/2})(1 + O(\frac{1}{d_t})) \\
&\quad \geq (1 - (1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2})(1 + 2e_t + O(\frac{1}{d_t})) \qquad \langle\text{Lemma 3.5.3}\rangle.
\end{aligned}
$$

Using linearity of expectation

$$
\begin{aligned}
E[\tilde{d}_t(u,c) | c \in \tilde{S}_t(u)] &= E[\tilde{d}_t(u,c)](1 + O(\frac{1}{d_t})) \\
&\leq d_t(u,c)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\frac{1}{d_t})).
\end{aligned}
$$

Now using the above bound

$$
\begin{aligned}
E[\bar{d}_t(u)] &= \sum_{c \in S_t(u)} Pr\{c \in \tilde{S}_t(u)\} E[\tilde{d}_t(u,c) | c \in \tilde{S}_t(u)] \\
&\leq e^{-1/2} \sum_{c \in S_t(u)} d_t(u,c)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\frac{1}{d_t})) \\
&\leq e^{-1/2} s_t(u) d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\frac{1}{d_t}))
\end{aligned}
$$

For concentration of $\bar{d}_t(u)$, suppose $s_t(u) = m$. Let $c_1, \ldots, c_m$ be the colors in $S_t(u)$. Then $\bar{d}_t(u)$ may be considered a random variable determined by the random trials $T_1, \ldots, T_m$, where T_i is the set of vertices in G_t that are assigned color c_i in round t. Observe that T_i affects $\bar{d}_t(u)$ by at most $d_t(u,c)$.

Thus $\sum \alpha_i^2$ in the statement of Theorem F is less that $\sum_{c \in S_t(u)} d_t^2(u,c)$. This upperbound is maximized when the $d_t(u,c)$ take the extreme values of $2d_t$ and 0 subject to $d_t(u) = \frac{1}{s_t(u)} \sum_{c \in S_t(u)} d_t(u,c)$. Thus

$$\sum \alpha_i^2 \leq O((d_t)^2 d_t(u) s_t(u)/d_t) \leq O(s_t(u) d_t d_t(u))$$

Using Theorem F, we get

$$Pr\{\bar{d}_t(u) - e^{-1/2} s_t(u) d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t} e^{-1/2}) e^{-1/2}(1 + 2e_t + O(\frac{1}{d_t})) \\ \geq O(\sqrt{\psi s_t(u) d_t d_t(u)})\} \leq e^{-\psi} O(1).$$

Lemma 3.5.1 says that

$$Pr\{s_t(u) e^{-1/2}(1 - \frac{1}{2} + O(\sqrt{\frac{\psi}{s_t}})) \leq \tilde{s}_t(u) \\ \leq s_t(u) e^{-1/2}(1 + O(\sqrt{\frac{\psi}{s_t}}))\} \geq 1 - e^{-\psi} O(1).$$

Combining the above two inequalities we have

$$Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} - d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t} e^{-1/2}) e^{-1/2}(1 + 2e_t + O(\frac{1}{d_t} + \sqrt{\frac{\psi}{s_t}})) \geq O(\sqrt{\frac{\psi d_t d_t(u)}{s_t}})\} \\ \leq e^{-\psi} O(1).$$

Therefore

$$Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t} e^{-1/2}) e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\} \\ \leq e^{-\psi} O(1).$$

$\square$

Note that $\frac{\bar{d}_t(u)}{\tilde{s}_t(u)}$ is the average $|\tilde{D}_t(u,c)|$ at a vertex u at the end phase II. Phase III only brings this average down by removing colors with large $d_{u,c}$. Thus we get the next lemma almost immediately.

Lemma 3.5.5. *Given Assumption 1, for each u in $V(G_{t+1})$ we have*

$$Pr\{d_{t+1}(u) \leq d_{t+1}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})\} \\ \geq 1 - e^{-\psi} O(1).$$

Proof. Let u be a vertex in $V(G_{t+1})$. By Lemma 3.5.4

$$Pr\{\tilde{d}_t(u) \leq d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}$$
$$\geq 1 - e^{-\psi}O(1).$$

Now

$$\begin{aligned}
&d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\\
&\quad = d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) + O(\sqrt{\frac{\psi d_t(u) d_t}{s_t}})\\
&\quad = d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) + d_t O(\sqrt{\frac{\psi d_t(u)}{s_t d_t}})\\
&\quad \leq d_t(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) + d_t O(\sqrt{\frac{\psi}{s_t}})\\
&\quad = d_t(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})).
\end{aligned}$$

Thus the event

$$\begin{aligned}
&\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\}\\
&\quad \subseteq \{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&e^{-\psi}O(1)\\
&\quad \leq Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\}\\
&\quad \leq Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(u)(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}\\
&\qquad\qquad (1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}.
\end{aligned}$$

□

Next we show that in the cleanup phase of round t, a vertex discards so many colors that its palette size in round $t+1$ becomes less than $\frac{1}{2}s_{t+1}(1-e_{t+1})$ with a very small probability.

Lemma 3.5.6. *Given Assumption 1, for each vertex u in $V(G_{t+1})$ we have*

$$\begin{aligned} &Pr\{\exists\alpha \in [0, \frac{1}{2}] \text{ such that } \forall c \in S_{t+1}(u) \\ &\quad s_{t+1}(u) \geq (1-\alpha)s_{t+1}(1-\frac{3}{2}e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \\ &\quad d_{t+1}(u) \leq \frac{1-2\alpha}{1-\alpha}d_{t+1}(1+\frac{3}{2}e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \\ &\quad d_{t+1}(u,c) \leq 2d_{t+1}(1+\frac{3}{2}e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\} \\ &\qquad \geq 1 - e^{-\psi}O(1). \end{aligned}$$

Proof. Consider vertex $u \in V(G_{t+1})$. Using Assumption 1, at round t, $\exists\alpha \in [0, \frac{1}{2}]$ such that $s_t(u) \geq (1-\alpha)s_t(1-e_t)$ and $d_t(u) \leq \frac{1-2\alpha}{1-\alpha}d_t(1+e_t)$. By Lemma 3.5.1 we get

$$\begin{aligned} &Pr\{s_t(u)e^{-1/2}(1-\frac{1}{2}e_t + O(\sqrt{\frac{\psi}{s_t}})) \leq \tilde{s}_t(u) \leq s_t(u)e^{-1/2}(1+\frac{1}{2}e_t + O(\sqrt{\frac{\psi}{s_t}}))\} \\ &\quad \geq 1 - e^{-\psi}O(1). \end{aligned}$$

Now

$$\begin{aligned} \tilde{s}_t(u) &= s_t(u)e^{-1/2}(1+\frac{1}{2}e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\ &\geq (1-\alpha)s_te^{-1/2}(1+\frac{1}{2}e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\ &\geq (1-\alpha)s_{t+1}(1+\frac{1}{2}e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})). \end{aligned}$$

By Lemma 3.5.4 we get

$$\begin{aligned} &Pr\{\tilde{d}_t(u) \leq d_t(u)(1-\frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1+\frac{3}{2}e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\} \\ &\quad \geq 1 - e^{-\psi}O(1). \end{aligned}$$

Now

$$\begin{aligned}\tilde{d}_t(u) &\le d_t(u)(1-\frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1+\frac{3}{2}e_t+O(\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))\\ &\le \frac{1-2\alpha}{1-\alpha}d_t(1-\frac{1}{16}\frac{s_t}{d_t}e^{-1/2})e^{-1/2}(1+\frac{3}{2}e_t+O(\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))\\ &\le \frac{1-2\alpha}{1-\alpha}\gamma d_{t+1}.\end{aligned}$$

where γ is the smallest number in $[1,\infty)$ for which the above inequality is true. Combining the preceding inequalities, we get

$$\gamma = 1+3e_t+O(\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}).$$

In the cleanup phase of our algorithm(given in Section 3.2.2), the change in palette is equivalent to the following process.

1. Add $\frac{\alpha}{1-\alpha}\tilde{s}_t(u)$ arbitrary colors to u's palette, with $\tilde{d}_t(u,c) = 2\gamma d_{t+1}$. This adjusts the palette size to $\tilde{s}_t(u) \ge s_{t+1}(1+3e_t+O(\sqrt{\psi/s_t}+1/d_t))$. Lemma 3.4.2 ensures that the adjusted new average is $\tilde{d}_t(u) \le \gamma d_{t+1}$

2. Remove all the colors with $d_t(u,c) \ge 2\gamma d_{t+1}$.

Now we use Lemma 3.4.1, setting μ to γd_{t+1} and $q\mu$ to $2\gamma d_{t+1}$, to get

$$s_{t+1}(u) \ge (1-\alpha)s_{t+1}(1+3e_t+O(\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))$$

and

$$d_{t+1}(u) \le \frac{1-2\alpha}{1-\alpha}d_{t+1}(1+3e_t+O(\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t})).$$

The result is obtained using Lemmas 3.5.1 and 3.5.4. $\square$

Chapter 4

An approximation algorithm for triangle-free graphs

4.1 Introduction

In Chapter 3 we proved that a triangle free graph G has a proper coloring using $O(\Delta(G)/\log\Delta(G))$ colors. In this chapter we will turn to the problem of finding such a coloring. In general the problem of finding the chromatic number of a graph is NP-Hard [15]. Approximating it to within a polynomial ratio is also hard [25].

Grable and Panconesi [16] gave an algorithm for properly coloring a Δ-regular graph with girth greater than 4 and $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ. The procedure uses $O(\Delta/\log\Delta)$ colors and has polynomial running time. This is a constructive version of the existential proof of Kim with extra conditions on the degree of the input graph. Section 5 of [16] asserts that the algorithm can be extended to triangle-free graphs. In Section 3.2.1 we will see a counter-example that shows that their analysis does not work on all triangle-free graphs. In particular, as will be seen, the analysis fails on a complete bipartite graph.

In this paper we give a method for coloring the larger class of triangle-free graphs. Our randomized algorithm properly colors a triangle-free graph G on n vertices with $\Delta(G) \geq \log^{1+\epsilon} n$ for a positive constant ϵ, using $O(\Delta(G)/\log\Delta(G))$ colors in $O(n\Delta^2(G)\log\Delta(G))$ time. The probability of failure goes to 0 as n

becomes large.

Our algorithm is obtained by extending the procedure from Section 3.2.2 to a three stage algorithm. Achlioptas and Molloy's study of the performance of a coloring algorithm on random graphs [1] is related to our analysis technique. They setup recurrence relations as we do, and then use the so-called differential equation method for random graph processes [39].

We describe our algorithm in Section 4.2. We give an outline of the analysis in Section 4.3 and details in Section 4.4.

4.2 A Three Stage Algorithm

In each round of the algorithm, some vertices are colored. The details of the coloring procedure vary depending on which of three stages the algorithm is in. We will continue to use Definitions 1 and 2 from Section 3.2.

First Stage Let q be a constant greater than 1, and let (η_t), (d_t), and (s_t) be sequences defined recursively as

$$\begin{aligned} \eta_0 &:= \Delta & \eta_{t+1} &:= \eta_t\left(1 - \frac{q-1}{2q^3} e^{-1/q} \frac{s_t}{d_t}\right) \\ d_0 &:= \Delta & d_{t+1} &:= d_t\left(1 - \frac{q-1}{2q^3} e^{-1/q} \frac{s_t}{d_t}\right) e^{-1/q} \\ s_0 &:= \Delta/k & s_{t+1} &:= s_t e^{-1/q}. \end{aligned} \tag{4.1}$$

For round t, vertex u, and color c,

$$\mathcal{F}_t(u, c) := \{c \text{ is not assigned to any vertex adjacent to } u \text{ in round } t\} \tag{4.2}$$

is an event in the probability space generated by the random choices of the algorithm in round t, given the state of all data structures at the beginning of the round.

Let

$$Desired\_\mathcal{F}_t := e^{-1/q}.$$

Repeat at every round t, until $d_t/s_t < 1/q^2$

Phase I - Coloring Attempt

For each vertex u in G_t, and color c in $S_t(u)$:

Assign c to u with probability $\frac{1}{q^2}\frac{1}{d_t}$.

Phase II - Conflict Resolution

For each vertex u in G_t:

Phase II.1

Remove from $S_t(u)$, all colors assigned to adjacent vertices.

Phase II.2

For each color c in $S_t(u)$, remove c from $S_t(u)$ with probability

$$1 - min(1, \frac{Desired\_\mathcal{F}_t(u,c)}{Pr(\mathcal{F}_t(u,c)}).$$

If $S_t(u)$ has at least one color which is assigned to u,

then arbirarily pick an assigned color from $S_t(u)$ to permanently color u.

Phase III - Cleanup(discard all colors c with $d_{t+1}(u,c) \gtrsim qd_{t+1}$ from palette)

For each vertex u in G_t:

$S_{t+1}(u) = S_t(u)$.

Let $\alpha = 1 - |S_{t+1}(u)|/s_{t+1}$.

If $\alpha < 0$, then $\alpha = 0$, otherwise if $\alpha > 1/q$, then $\alpha = 1/q$.

Let γ be the smallest number in $[1, \infty)$ so that

$$Average_{c \in S_{t+1}(u)} d_{t+1}(u,c) \leq \frac{1 - q\alpha}{1 - \alpha} \gamma d_{t+1}.$$

Remove all colors c with $d_{t+1}(u,c) \geq q\gamma d_{t+1}$ from $S_{t+1}(u)$.

end repeat

Second Stage We change the recursive equations defining the constants η_t, d_t, and s_t.

$$\begin{aligned} \eta_{t+1} &:= \eta_t\left(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}\right) \\ d_{t+1} &:= d_t\left(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}\right) e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ s_{t+1} &:= s_t e^{-\frac{1}{q}\frac{d_t}{s_t}} \end{aligned} \tag{4.3}$$

Also

$$Desired\_\mathcal{F}_t := e^{-\frac{1}{q}\frac{d_t}{s_t}}.$$

All other details of the repeat-until loop of the first stage are the same for the second stage, except that an uncolored vertex u is assigned a color c from its palette with probability

$$\frac{1}{q^2}\frac{1}{s_t},$$

and we repeat until

$$\eta_t < \frac{q-1}{q} s_t.$$

Third Stage(Greedy Coloring) Color each uncolored vertex u, with an arbitrary color from its palette which has not been used to color an adjacent vertex.

4.3 The Main Theorem

The analysis for the first stage proceeds in parallel to the analysis of the procedure in Chapter 3. In addition we need to analyze two more stages of the algorithm.

Theorem 2 (Main Theorem)**.** *Given a triangle-free graph G on n vertices with maximum degree $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ, and $q = 7$, there is a positive constant c_ϵ such that starting with $c_\epsilon \Delta / \log \Delta$ colors, our algorithm finds a proper coloring of the graph in $O(n\Delta^2 \log \Delta)$ time with probability* $1 - O(1/n)$.

We need some lemmas to prove the Main Theorem and before that we need the following definition.

Definition 6. $d_t(v) = Average_{c \in S_t(v)} d_t(v,c)$

Lemma 4.3.1 (Main Lemma)**.** *Given a triangle-free graph G on n vertices with maximum degree $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ, $q \geq 2$, and $\psi > 1$, there are positive constants α_1 and α_2 such that for the sequences (e_t) defined by*

$$e_0 = 0, \quad e_{t+1} = \alpha_1 (e_t + \sqrt{\frac{\psi}{d_t}} + \sqrt{\frac{\psi}{s_t}}) \quad \text{for } t > 0, \tag{4.4}$$

and (f_t) defined by

$$f_t = 1 - \alpha_2 t n^2 e^{-\psi}, \tag{4.5}$$

if $s_t \gg \psi$ and $e_t \ll 1$ at round t, then

$$\forall u \in V(G_t), \exists \alpha \in [0, 1/q], \forall c \in S_t(u),$$

$$\begin{aligned} s_t(u) &\geq (1-\alpha) s_t (1 - e_t) \\ d_t(u) &\leq \frac{1 - q\alpha}{1-\alpha} d_t (1 + e_t) \\ d_t(u,c) &\leq q d_t (1 + e_t) \\ \eta_t(u) &\leq \eta_t (1 + e_t) \end{aligned}$$

with probability greater than f_t.

We will prove the Main Lemma in Section 4.4 and assume it in this section. Using it we can immediately conclude the following.

Corollary 2. *Given the setup of the Main Lemma(Lemma 4.3.1), if $s_t \gg \psi$ and $e_t \ll 1$ at round t, then $\forall u \in V(G_t)$,*

$$\begin{aligned} s_t(u) &\geq \frac{q-1}{q} s_t (1 - e_t) \\ d_t(u) &\leq d_t (1 + e_t) \end{aligned}$$

with probability greater than f_t.

Lemma 4.3.2. *The first stage finishes in $O(k)$ rounds.*

Proof. By the definition of sequences (d_t) and (s_t) in equation (4.1), we have

$$\begin{aligned}\frac{d_{t+1}}{s_{t+1}} &= \frac{d_t}{s_t}(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q}) \\ &= \frac{d_t}{s_t} - \frac{q-1}{2q^3}e^{-1/q}.\end{aligned}$$

Since $\frac{d_0}{s_0} = k$ we get $\frac{d_{t_1}}{s_{t_1}} \leq \frac{1}{q^2}$ for some round $t_1 = O(k)$. $\square$

Let

$$t_1 = O(k)$$

be the last round of the first stage. Then the following lemma is a straightforward application of equation (4.1).

Lemma 4.3.3.

$$s_{t_1} = \frac{\Delta}{k}e^{-O(k)} \text{ and } \exists c > 0 \text{ such that if } k \leq c\log\Delta, \text{ then } s_{t_1} \gg 1.$$

4.3.1 The Second Stage: Controlling the ratio of available colors to uncolored neighbors.

We must show that η_t decreases significantly faster than s_t in the second stage. Let

$$\rho = 1 - \frac{q-2}{2q^3}.$$

Lemma 4.3.4.

$$\eta_{t_1+t} < \eta_{t_1}\rho^t$$

Proof. By the definition of sequence (η_t) in equation (4.3), we have

$$\begin{aligned}
\eta_{t+1} &= \eta_t(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) \\
&< \eta_t(1 - \frac{q-1}{2q^3}(1 - \frac{1}{q}\frac{d_t}{s_t})) && \langle e^x \geq 1 + x \rangle \\
&< \eta_t(1 - \frac{q-1}{2q^3}(1 - \frac{1}{q})) && \langle \frac{d_t}{s_t} < 1 \rangle \\
&< \eta_t(1 - \frac{q-1}{2q^3} + \frac{1}{2q^3}) \\
&= \eta_t(1 - \frac{q-2}{2q^3}) \\
&= \eta_t \rho.
\end{aligned}$$

Thus $\eta_{t_1+t} < \eta_{t_1}\rho^t$. □

We now study the ratio d_t/s_t, which appears in the recursive equation (4.3) for the sequences (η_t), (d_t), and (s_t).

Lemma 4.3.5.

$$\frac{d_{t_1+t}}{s_{t_1+t}} < \frac{1}{q^2}\rho^t$$

Proof. By the definition of sequences (d_t) and (s_t) in equation (4.3), we have

$$\frac{d_{t+1}}{s_{t+1}} = \frac{d_t}{s_t}(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}).$$

Since the ratio of d_{t+1}/s_{t+1} to d_t/s_t is the same as the ratio of η_{t+1} to η_t, we use Lemma 4.3.4 to conclude that

$$\frac{d_{t_1+t}}{s_{t_1+t}} < \frac{d_{t_1}}{s_{t_1}}\rho^t < \frac{1}{q^2}\rho^t.$$

□

Lemma 4.3.6. *If q is greater than 2, then*

$$s_{t_1+t} \geq s_{t_1}(1 - \frac{4}{q-2}).$$

Proof. By the definition of sequence (s_t) in equation (4.3), we have

$$\begin{aligned} s_{t+1} &= s_t e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &\geq s_t e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &\geq s_t(1 - \frac{1}{q}\frac{d_t}{s_t}). \end{aligned}$$

Thus

$$\begin{aligned} s_{t_1+t} &\geq s_{t_1}\prod_{i=0}^{t}(1 - \frac{1}{q}\frac{d_t}{s_t}) \\ &\geq s_{t_1}\prod_{i=0}^{t}(1 - \frac{1}{q^3}\rho^i) && \langle\text{Lemma 4.3.5}\rangle \\ &\geq s_{t_1} exp(-\frac{2}{q^3}\sum_{i=0}^{t}\rho^i) && \langle\text{since } q > 2\rangle \\ &\geq s_{t_1} exp(-\frac{2}{q^3}\frac{1}{1-\rho}) && \langle\text{sum of geometric series}\rangle \\ &= s_{t_1} exp(-\frac{4q^3}{q^3(q-2)}) \\ &= s_{t_1} exp(-\frac{4}{q-2}) \\ &\geq s_{t_1}(1 - \frac{4}{q-2}). \end{aligned}$$

□

Lemma 4.3.7. *For any $\delta > 0$, there is a $c_\delta > 0$ such that if $q > 6$, $k \leq c_\delta \log\Delta$, and $\Delta \gg 1$, then the second stage takes at most $\delta\log\Delta$ rounds.*

Proof. By Lemma 4.3.3, we have

$$s_{t_1} = \frac{\Delta}{k}e^{-O(k)}.$$

Since $q > 6$, the above equation and Lemma 4.3.6 imply that

$$s_{t_1+t} = \Omega(\frac{\Delta}{k}e^{-O(k)})$$

for all t. Also, by Lemma 4.3.4, we have

$$\eta_{t_1+t} < \Delta\rho^t.$$

Since $\Delta \gg 1$, given a δ it is straightforward to find a $c_\delta > 0$ so that if $k \leq c_\delta \log\Delta$, then after $\delta\log\Delta$ rounds $\eta_t < \frac{q-1}{q}s_t$. □

4.3.2 Bounding the Error Estimate in all Concentration Inequalities

Now we look at d_t, which is used to bound the error term e_t. Let

$$\mu = 1 - \frac{1}{2q^2}.$$

Lemma 4.3.8.

$$d_{t_1+t} \geq d_{t_1}(1 - \frac{4}{q-2})\mu^t$$

Proof. By the definition of sequence (d_t) in equation (4.3), we have

$$\begin{aligned} d_{t+1} &= d_t(1 - \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}})e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &\geq d_t(1 - \frac{1}{2q^2})e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &= d_t\mu\, e^{-\frac{1}{q}\frac{d_t}{s_t}}. \end{aligned}$$

Combining the inequality above with the definition of sequence (s_t) in equation (4.3), we get

$$\frac{d_{t+1}}{d_t} \geq \mu\frac{s_{t+1}}{s_t}.$$

We now use Lemma 4.3.6 to conclude that

$$\frac{d_{t_1+t}}{d_{t_1}} \geq \mu^t\frac{s_{t_1+t}}{s_{t_1}} \geq \mu^t(1 - \frac{4}{q-2}).$$

□

Let t_2 be the number of rounds spent in the second stage.

Lemma 4.3.9. *There is a positive constant α such that in the first two stages of our algorithm,*

$$e_t \leq \alpha^t O(\sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}}).$$

Proof. By Lemma 4.3.3, we have

$$s_{t_1} = \frac{\Delta}{k}e^{-O(k)}.$$

Note that in equation (4.4), the recurrence for e_t, the largest term is $O(\sqrt{\psi/s_t})$ in the first stage, while $O(\sqrt{\psi/d_t})$ is larger in the second. At round t_1, the algorithm moves to the second stage and $d_{t_1} = \Theta(s_{t_1})$. The greedy stage starts at round $t_1 + t_2$. Since both sequences (d_t) and (s_t) are decreasing, we use Lemma 4.3.8 to conclude that

$$O(\sqrt{\frac{\psi}{d_{t_1+t_2}}}) = O(\sqrt{\frac{\psi}{s_{t_1}\mu^{t_2}}}) = O(\sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}})$$

is the maximum this error term can be. Thus we can simplify the recurrence for e_t to

$$e_{t+1} = O(e_t + \sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}}).$$

Since $e_0 = 0$, a simple upper bound for e_t is given by

$$e_t \leq \alpha^t O(\sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}})$$

where α is some positive constant. □

Lemma 4.3.10. *Given a triangle-free graph G on n vertices with maximum degree $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ, and $q = 7$, there is a positive constant c_ϵ such that, with Δ/k colors where $k \leq c_\epsilon \log \Delta$, our algorithm reaches the greedy stage at round $\tau = O(\log \Delta)$ with $e_\tau \ll 1$, and $\forall u \in V(G_\tau)$*

$$s_\tau(u) \geq \frac{q-1}{q} s_\tau(1 - e_\tau)$$

$$\eta_\tau(u) \leq \eta_\tau(1 + e_\tau)$$

with probability $1 - O(1/n)$.

Proof. Let $\psi = 3 \log n$, and let $\tau = t_1 + t_2$ be the number of rounds to reach the greedy stage. Using Lemma 4.3.9, we get

$$\exists \alpha, \beta > 0, \quad e_\tau \leq \alpha^{t_1}\beta^{t_2} O(\sqrt{\frac{ke^{O(k)}\psi}{\Delta}}).$$

Since $\Delta \geq \log^{1+\epsilon} n$, it is straightforward to show that

$$\exists c_1 > 0, \quad e_\tau \ll 1$$

if $t_1, t_2, k \leq c_1 \log \Delta$. Since $t_1 = O(k)$, by Lemma 4.3.7

$$\exists c_2 > 0, \quad t_1, t_2 \leq c_1 \log \Delta, \quad \text{and} \quad s_\tau \gg \psi$$

if $k \leq c_2 \log \Delta$.

Let $c_\epsilon = min(c_1, c_2)$. The above computations show that if $k \leq c_\epsilon \log \Delta$, then $\tau = t_1 + t_2 = O(k + \log \Delta) = O(\log \Delta)$ and $e_\tau \ll 1$. Applying Corollary 2 completes the proof. □

We may now prove the Main Theorem.

of the Main Theorem. Using Lemma 4.3.10, we only need to compute the time complexity of our algorithm. The number of steps taken in the greedy coloring stage is dominated by that of the first and second stages. In each round of these stages, we compute the probability of the event $\mathcal{F}_t(u, c)$ defined in equation (4.2). This requires iterating through each uncolored vertex u and color c in its palette, and examining all adjacent vertices.

Thus each round takes $O(n\Delta^2)$ steps and by Lemma 4.3.10, the algorithm requires $\tau = O(\log \Delta)$ rounds to reach the greedy coloring stage. With probability $1 - O(1/n)$, we get

$$\forall u \in V(G_\tau), \quad \eta_t(u) < s_\tau(u),$$

which implies that the greedy coloring stage will successfully complete the coloring. □

4.4 Proof of the Main Lemma

of the Main Lemma. The proof is by induction on the round number t using the lemmas that follow. The base case, when $t = 0$, is trivially true. Lemmas 4.4.5, 4.4.6, and 4.4.7 give us the induction step for the first stage; Lemmas 4.4.12, 4.4.13, and 4.4.14 give the induction step for the second. □

All events below are in the probability space generated by our randomized algorithm in round $t + 1$, given the state of all data structures at the beginning of the round. The following assumptions are the induction hypothesis and are repeatedly used in all the lemmas of this section.

Assumption 2. *Assume*

$$q \geq 2, s_t \gg \psi, e_t \ll 1$$

and

$$\forall u \in V(G_t), \forall c \in S_t(u), \exists \alpha \in [0, 1/q],$$

$$\begin{aligned} s_t(u) &\geq (1-\alpha) s_t (1-e_t) \\ d_t(u) &\leq \frac{1-q\alpha}{1-\alpha} d_t (1+e_t) \\ d_t(u,c) &\leq q d_t (1+e_t) \\ \eta_t(u) &\leq \eta_t (1+e_t). \end{aligned}$$

4.4.1 Expected Values and Concentration Inequalities for the First Stage

We next consider the state of the palettes just before the cleanup phase of round t.

Definition 7. *Let $\tilde{S}_t(u)$ be the list of colors in the palette of vertex u in round t just before the cleanup phase, and let $\tilde{s}_t(u)$ be the size of $\tilde{S}_t(u)$. That is, $\tilde{S}_t(u)$ is obtained from $S_t(u)$ by removing colors discarded in the conflict resolution phase.*

Consider the event

$$\begin{aligned} \tilde{\mathcal{S}} = \{\forall u \in V(G_{t+1}), & s_t(u) e^{-1/q} (1 - O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \leq \tilde{s}_t(u) \\ & \leq s_t(u) e^{-1/q} (1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\}. \end{aligned} \tag{4.6}$$

Lemma 4.4.1. *Given Assumption 2, we have*

$$Pr(\tilde{\mathcal{S}}) \geq 1 - e^{-\psi} O(n).$$

Proof. Suppose u is an uncolored vertex at the beginning of round t, and c be a color in its palette.

$$\begin{aligned}
&Pr\{c \text{ is removed from } S_t(u) \text{ in phase II.1}\}\\
&\quad = 1 - Pr\{\text{no neighbor of } u \text{ is assigned } c\}\\
&\quad = 1 - \prod_{v \in D_t(u,c)} (1 - Pr\{v \text{ is assigned } c\})\\
&\quad = 1 - \prod_{v \in D_{u,c}} (1 - \frac{1}{q^2}\frac{1}{d_t})\\
&\quad \leq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t})^{d_t(u,c)}\\
&\quad \leq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t})^{qd_t(1+e_t)}\\
&\quad \leq 1 - e^{\log(1-\frac{1}{q^2}\frac{1}{d_t})qd_t(1+e_t)} && \langle \log(1+x) = x + O(x^2) \rangle\\
&\quad \leq 1 - e^{(-\frac{1}{q^2}\frac{1}{d_t}+O(\frac{1}{d_t})^2)qd_t(1+e_t)} && \langle \text{Assumption 2} \rangle\\
&\quad \leq 1 - e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))
\end{aligned}$$

In phase II.2 of round $t+1$ we remove colors from the palette using an appropriate bernoulli variable, to get

$$Pr\{c \notin \tilde{S}_t(u)\} = 1 - e^{-1/q}(1 + O(e_t + \frac{1}{d_t})). \tag{4.7}$$

Using linearity of expectation

$$\forall u \in V(G_{t+1}), E[\tilde{s}_t(u)] = s_t(u)e^{-1/q}(1 + O(e_t + \frac{1}{d_t})).$$

For concentration of $\tilde{s}_t(u)$, suppose $s_t(u) = m$. Let $c_1, \ldots, c_m$ be the colors in $S_t(u)$. Then $\tilde{S}_t(u)$ may be considered a random variable determined by m trials $T_1, \ldots, T_m$ where T_i is the set of vertices in G_t that are assigned color c_i in round t. Observe that T_i affects $\tilde{S}_t(u)$ by at most 1 given $T_1, \ldots, T_{i-1}$. Now using Theorem F we get,

$$Pr\{|\tilde{s}_t(u) - E[\tilde{s}_t(u)]| \geq \sqrt{\psi s_t(u)}\} \leq e^{-\psi}O(1).$$

We end the proof using the union bound for probabilities. □

We now focus on the sets $D_t(u, c)$. The following two lemmas will help.

Lemma 4.4.2. *Let u be an uncolored vertex, and c a color in its palette at the beginning of round t. Then given Assumption 2, we have*

$$Pr\{u \textit{ is assigned } c \textit{ and } c \in \tilde{S}_t(u)\} = \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})).$$

Proof.

$$\begin{aligned} &Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} \\ &\quad = Pr\{u \text{ is assigned } c\}Pr\{c \in \tilde{S}_t(u)\} \\ &\quad = \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})) && \langle\text{Equation (4.7)}\rangle \end{aligned}$$

□

The following lemma is a consequence of the previous one.

Lemma 4.4.3. *Let u be an uncolored vertex at the beginning of round t. Then given Assumption 2, we have*

$$Pr\{u \textit{ is colored}\} \geq \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})).$$

Proof. Consider the event

$$\{u \text{ is colored}\} = \bigcup_{c \in S_t(u)} \{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}.$$

Since the events in the union on the right hand side of the equation above are independent,

$$Pr\{u \text{ is colored}\} = 1 - \prod_{c \in S_t(u)} (1 - Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}).$$

Now using Lemma 4.4.2, we get

$$\begin{aligned} &Pr\{u \text{ is colored}\} \\ &\quad \geq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})))^{s_t(u)} \\ &\quad \geq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})))^{\frac{q-1}{q}s_t(1-e_t)} && \langle\text{Assumption 2}\rangle \\ &\quad \geq 1 - exp(-\frac{q-1}{q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))) \\ &\quad \geq 1 - (1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))) && \langle\text{since } q \geq 2\rangle \\ &\quad = \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})). \end{aligned}$$

□

Using Definition 5 consider the event

$$\begin{aligned}\tilde{\mathcal{A}} := \{&\forall u \in V(G_{t+1}),\\ &\tilde{d}_t(u) \le d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}\\ &(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}. \end{aligned} \tag{4.8}$$

Lemma 4.4.4. *Given Assumption 2, we have*

$$Pr(\tilde{\mathcal{A}}) \ge 1 - e^{-\psi}O(n^2).$$

Proof. Let u be an uncolored vertex at the beginning of round t, and let c be a color in its palette. For a vertex v in $D_t(u,c)$, Lemma 4.4.2 implies that $Pr\{v$ is colored with $d\} = O(1/d_t)$ for any color d in $S_t(v)$. Thus,

$$\begin{aligned}&Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is colored}\})\\ &\quad= \sum_{d \in S_t(v)} Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is colored with } d\})\\ &\quad= \sum_{d \in S_t(v)} Pr\{c \notin \tilde{S}_t(v) | v \text{ is colored with } d\} Pr\{v \text{ is colored with } d\}\\ &\quad= Pr\{c \notin \tilde{S}_t(v)\}(1 + O(\frac{1}{d_t})) \sum_{d \in S_t(v)} Pr\{v \text{ is colored with } d\}\\ &\quad= Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is colored}\}(1 + O(\frac{1}{d_t})).\end{aligned}$$

A straightforward computation now shows that

$$\begin{aligned}&Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\})\\ &\quad= Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is not colored}\}(1 + O(\frac{1}{d_t})).\end{aligned} \tag{4.9}$$

Now, v is removed from the set $D_t(u,c)$ if either it is colored or color c is removed from its palette. This means that event

$$\{v \notin \tilde{D}_t(u,c)\} = \{v \text{ is colored}\} \cup (\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}).$$

Since G is triangle-free, u and v do not have any common neighbors. This implies that

$$\begin{aligned}
&Pr\{v \notin \tilde{D}_t(u,c) | c \in \tilde{S}_t(u)\} \\
&\quad = Pr\{v \notin \tilde{D}_t(u,c)\}(1 + O(\frac{1}{d_t})) \\
&\quad = (Pr\{v \text{ is colored}\} \\
&\qquad + Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}))(1 + O(\frac{1}{d_t})) \\
&\quad = (Pr\{v \text{ is colored}\} \\
&\qquad + Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is not colored}\})(1 + O(\frac{1}{d_t})) \qquad \langle\text{equation (4.9)}\rangle \\
&\quad = (Pr\{v \text{ is colored}\} \\
&\qquad + (1 - e^{-1/q})(1 - Pr\{v \text{ is colored}\}))(1 + O(e_t + \frac{1}{d_t})) \quad \langle\text{equation (4.7)}\rangle \\
&\quad = (1 - (1 - Pr\{v \text{ is colored}\})e^{-1/q})(1 + O(e_t + \frac{1}{d_t})) \\
&\quad \geq (1 - (1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q})(1 + O(e_t + \frac{1}{d_t})) \qquad \langle\text{Lemma 4.4.3}\rangle.
\end{aligned}$$

Using linearity of expectation

$$E[\tilde{d}_t(u,c) | c \in \tilde{S}_t(u)] = E[\tilde{d}_t(u,c)](1 + O(\frac{1}{d_t})) \tag{4.10}$$

$$\leq d_t(u,c)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t})). \tag{4.11}$$

Now using the above bound

$$\begin{aligned}
E[\bar{d}_t(u)] &= \sum_{c \in S_t(u)} Pr\{c \in \tilde{S}_t(u)\} E[\tilde{d}_t(u,c) | c \in \tilde{S}_t(u)] \\
&\leq e^{-1/q} \sum_{c \in S_t(u)} d_t(u,c)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t})) \\
&\leq e^{-1/q} s_t(u) d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))
\end{aligned}$$

For concentration of $\bar{d}_t(u)$, suppose $s_t(u) = m$. Let $c_1, \ldots, c_m$ be the colors in $S_t(u)$. Then $\bar{d}_t(u)$ may be considered a random variable determined by the random trials $T_1, \ldots, T_m$, where T_i is the set of vertices in G_t that are assigned color c_i in round t. Observe that T_i affects $\bar{d}_t(u)$ by at most $d_t(u,c)$.

Thus $\sum \alpha_i^2$ in the statement of Theorem F is less that $\sum_{c\in S_t(u)} d_t^2(u,c)$. This upperbound is maximized when the $d_t(u,c)$ take the extreme values of qd_t and 0 subject to $d_t(u) = \frac{1}{s_t(u)} \sum_{c\in S_t(u)} d_t(u,c)$. Thus

$$\sum \alpha_i^2 \leq O((d_t)^2 d_t(u) s_t(u)/d_t) \leq O(s_t(u) d_t d_t(u))$$

Using Theorem F, we get

$$Pr\{\bar{d}_t(u) - e^{-1/q} s_t(u) d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))$$
$$\geq O(\sqrt{\psi s_t(u) d_t d_t(u)})\} \leq e^{-\psi}O(1).$$

Lemma 4.4.1 says that the event $\tilde{\mathcal{S}}$ occurs with probability $1 - e^{-\psi}O(n)$. Thus

$$Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} - d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t} + \sqrt{\frac{\psi}{s_t}}))$$
$$\geq O(\sqrt{\frac{\psi d_t d_t(u)}{s_t}})\} \leq e^{-\psi}O(n).$$

Therefore

$$Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)}$$
$$\geq d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}$$
$$\leq e^{-\psi}O(n).$$

We end the proof using the union bound for probabilities. □

Note that $\frac{\bar{d}_t(u)}{\tilde{s}_t(u)}$ is the average $|\tilde{D}_t(u,c)|$ at a vertex u at the end phase II. Phase III only brings this average down by removing colors with large $d_{u,c}$. Thus we get the next lemma almost immediately. Consider the event

$$\mathcal{A} := \{\forall u \in V(G_{t+1}), d_{t+1}(u) \leq d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})\}.$$

Lemma 4.4.5. *Given Assumption 2, we have*

$$Pr(\mathcal{A}) \geq 1 - e^{-\psi}O(n^2).$$

Proof. Assume the occurrence event $\tilde{\mathcal{A}}$, as defined in equation (4.8), and let u be a vertex in $V(G_{t+1})$. Then

$$\begin{aligned}
& d_t(u)(1-\frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}+\sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\\
&\quad = d_t(u)(1-\frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))+O(\sqrt{\frac{\psi d_t(u)d_t}{s_t}})\\
&\quad = d_t(u)(1-\frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))+d_tO(\sqrt{\frac{\psi d_t(u)}{s_t d_t}})\\
&\quad \leq d_t(1-\frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))+d_tO(\sqrt{\frac{\psi}{s_t}})\\
&\quad = d_t(1-\frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t})).
\end{aligned}$$

Thus the event

$$\begin{aligned}
&\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)}\geq d_t(1-\frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))\}\\
&\quad\subseteq\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)}\geq d_t(u)(1-\frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}+\sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}.
\end{aligned}$$

Therefore

$$Pr(\mathcal{A})\geq Pr(\tilde{\mathcal{A}})\geq e^{-\psi}O(n^2).$$

□

Next we show that in the cleanup phase of round t, no vertex discards so many colors that its palette size in round $t+1$ becomes less than $\frac{q-1}{q}s_{t+1}(1-e_{t+1})$. Consider the event

$$\begin{aligned}
\mathcal{S}:=\{\forall u\in V(G_{t+1}),&\exists\alpha\in[0,\frac{1}{q}]\text{ such that }\forall c\in S_{t+1}(u)\\
&s_{t+1}(u)\geq(1-\alpha)s_{t+1}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t})),\\
&d_{t+1}(u)\leq\frac{1-q\alpha}{1-\alpha}d_{t+1}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t})),\\
&d_{t+1}(u,c)\leq qd_{t+1}(1+O(e_t+\sqrt{\frac{\psi}{s_t}}+\frac{1}{d_t}))\}.
\end{aligned}\tag{4.12}$$

Lemma 4.4.6. *Given Assumption 2, we have*

$$Pr(\mathcal{S}) \geq 1 - e^{-\psi} O(n^2).$$

Proof. Consider vertex $u \in V(G_{t+1})$. Using Assumption 2, at round t, $\exists \alpha \in [0, \frac{1}{q}]$ such that $s_t(u) \geq (1-\alpha)s_t(1-e_t)$ and $d_t(u) \leq \frac{1-q\alpha}{1-\alpha} d_t(1+e_t)$. Assuming the occurrence of event $\tilde{\mathcal{S}}$, as defined in equation (4.6), we get

$$\begin{aligned} \tilde{s}_t(u) &= s_t(u) e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\ &\geq (1-\alpha) s_t e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\ &\geq (1-\alpha) s_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})). \end{aligned}$$

Assuming event $\tilde{\mathcal{A}}$ occurs,

$$\begin{aligned} \tilde{d}_t(u) &\leq d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t} e^{-1/q}) e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\ &\leq \frac{1-q\alpha}{1-\alpha} d_t(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t} e^{-1/q}) e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\ &\leq \frac{1-q\alpha}{1-\alpha} \gamma d_{t+1}. \end{aligned}$$

where γ is the smallest number in $[1, \infty)$ for which the above inequality is true. Combining the preceding inequalities, we get

$$\gamma = 1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}).$$

In the cleanup phase of our algorithm(given in Section 4.2), the change in palette is equivalent to the following process.

1. Add $\frac{\alpha}{1-\alpha}\tilde{s}_t(u)$ arbitrary colors to u's palette, with $\tilde{d}_t(u,c) = q\gamma d_{t+1}$. This adjusts the palette size to $\tilde{s}_t(u) \geq s_{t+1}(1 + O(e_t + \sqrt{\psi/s_t} + 1/d_t))$. Lemma 3.4.2 ensures that the adjusted new average is $\tilde{d}_t(u) \leq \gamma d_{t+1}$

2. Remove all the colors with $d_t(u,c) \geq q\gamma d_{t+1}$.

Now we use Lemma 3.4.1, setting μ to γd_{t+1} and $q\mu$ to $q\gamma d_{t+1}$, to get

$$s_{t+1}(u) \geq (1-\alpha) s_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))$$

and

$$d_{t+1}(u) \leq \frac{1-q\alpha}{1-\alpha} d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})).$$

The result is obtained using Lemmas 4.4.1 and 4.4.4 to get

$$Pr(\tilde{\mathcal{S}} \cap \tilde{\mathcal{A}}) \geq 1 - e^{-\psi} O(n^2),$$

and noting that the preceding inequalities for $s_{t+1}(u)$ and $d_{t+1}(u)$ are true for every vertex u in $V(G_{t+1})$, given events $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{A}}$ occur. □

Now consider the event

$$\mathcal{D} := \{\forall v \in V(G_{t+1}), \eta_{t+1}(v) \leq \eta_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{d_t}} + \frac{1}{d_t}))\}.$$

Lemma 4.4.7. *Given Assumption 2, we have*

$$Pr(\mathcal{D}) \geq 1 - e^{-\psi} O(n).$$

Proof. Suppose u is an uncolored vertex at the beginning of round t. By Lemma 4.4.3 we have

$$Pr\{u \text{ is colored}\} \quad \geq \tfrac{q-1}{2q^3} \tfrac{s_t}{d_t} e^{-1/q}(1 + O(e_t + \tfrac{1}{d_t})).$$

Using linearity of expectation, $\forall u$ in $V(G_{t+1})$

$$\begin{aligned} E[\eta_{t+1}(u)] &\leq \eta_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t} e^{-1/q})(1 + O(e_t + \frac{1}{d_t})) \qquad \langle \text{Lemma 3.4.3} \rangle \\ &\leq \eta_{t+1}(1 + O(e_t + \frac{1}{d_t})). \end{aligned}$$

We again resort to Theorem F to study concentration of $\eta_{t+1}(u)$. Suppose $\eta_t(u) = m$. Let $v_1, \ldots, v_m$ be the vertices in $N_t(u)$. Then $\eta_{t+1}(u)$ may be considered a random variable determined by $T_1, \ldots, T_m$, where T_i is a random variable which indicates that v_i is colored in round t or not. The affect of each T_i given $T_1, \ldots, T_{i-1}$ is at most $O(\frac{s_t}{d_t})$. Thus $\sum \alpha_i^2$ in the statement of Theorem F is

$$O(\frac{\eta_t s_t^2}{d_t^2}) = O(\frac{\eta_t s_t}{d_t}).$$

Using Theorem F, we get

$$Pr\{\eta_{t+1}(u) \geq \eta_{t+1}(1 + O(e_t + \frac{1}{d_t})) + O(\sqrt{\psi \frac{s_t \eta_t}{d_t}})\} \leq e^{-\psi}.$$

Thus

$$Pr\{\eta_{t+1}(u) \geq \eta_{t+1}(1 + O(e_t + \sqrt{\psi \frac{s_t}{d_t \eta_t}} + \frac{1}{d_t}))\} \leq e^{-\psi}.$$

Since the above is true for every vertex u in $V(G_{t+1})$, the theorem is proved by applying the union bound on probabilties. □

4.4.2 Expected Values and Concentration Inequalities for the Second Stage

We now focus on the second stage of the algorithm, where $s_t \geq q^2 d_t$. The pattern of analysis for the second stage is similar to that of the first stage. But simply reproducing the previous section, with changes here and there, would make it difficult to understand the differences. With this in mind, we proceed much faster now and focus on the expectations of variables, leaving out all concentration calculations. These can be filled in by using the corresponding proofs in Section 4.4.1 as templates.

Consider now the event

$$\begin{aligned}\tilde{\mathcal{S}} := \{\forall u \in V(G_{t+1}), s_t(u) e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 - O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t}))\\ \leq \tilde{s}_t(u) \leq s_t(u) e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t}))\}.\end{aligned}$$

Lemma 4.4.8. *Given Assumption 2, we have*

$$Pr(\tilde{\mathcal{S}}) \geq 1 - e^{-\psi} O(n).$$

Proof. Suppose u is an uncolored vertex at the beginning of round t, and c a color in its palette.

$$\begin{aligned}&Pr\{c \text{ is removed from } S_t(u) \text{ in phase II.1}\}\\ &= 1 - \prod_{v \in D_t(u,c)} (1 - Pr\{v \text{ is assigned } c\})\\ &= 1 - \prod_{v \in D_{u,c}} (1 - \frac{1}{q^2}\frac{1}{s_t}(1 + O(e_t + \frac{1}{s_t})))\\ &\leq 1 - e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t + \frac{1}{s_t}))\end{aligned}$$

In phase II.2 we remove colors from the palette using an appropriate bernoulli variable, to get

$$Pr\{c \notin \tilde{S}_t(u)\} = 1 - e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

Using linearity of expectation

$$\forall u \in V(G_{t+1}), E[\tilde{s}_t(u)] = s_t(u)e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

The rest of the proof follows that of Lemma 4.4.1. □

We now focus on the sets $D_t(u, c)$. The following two lemmas will help.

Lemma 4.4.9. *Let u be an uncolored vertex at the beginning of round t, and c a color in its palette. Then given Assumption 2, we have*

$$Pr\{u \textit{ is assigned } c \textit{ and } c \in \tilde{S}_t(u)\} \geq \frac{1}{q^2}\frac{1}{s_t}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

Proof.

$$\begin{aligned} Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} &= Pr\{u \text{ is assigned } c\}Pr\{c \in \tilde{S}_t(u)\} \\ &= \frac{1}{q^2}\frac{1}{s_t}\prod_{v \in D_t(u,c)}(1 - \frac{1}{q^2}\frac{1}{s_t}) \\ &\geq \frac{1}{q^2}\frac{1}{s_t}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})) \end{aligned}$$

□

The following lemma is a consequence of the previous one.

Lemma 4.4.10. *Let u be an uncolored vertex at the beginning of round t. Then given Assumption 2, we have*

$$Pr\{u \textit{ is colored}\} \geq \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

Proof. Following the proof of Lemma 4.4.3, but using Lemma 4.4.9 instead of Lemma 4.4.2, we get

$$\begin{aligned} Pr\{u \text{ is colored}\} &\geq 1 - (1 - \frac{1}{q^2}\frac{1}{s_t}e^{\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})))^{s_t(u)} \\ &\geq \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})). \end{aligned}$$

□

Consider the event

$$\tilde{\mathcal{A}} := \{\forall u \in V(G_{t+1}),$$
$$\tilde{d}_t(u) \leq d_t(u)(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}.$$

Lemma 4.4.11. *Given Assumption 2, we have*

$$Pr(\tilde{\mathcal{A}}) \geq 1 - e^{-\psi} O(n^2).$$

Proof. Let u be an uncolored vertex at the beginning of round t, and let c be a color in its palette. For a vertex v in $D_t(u, c)$, the event

$$\{v \notin \tilde{D}_t(u, c)\} = \{v \text{ is colored}\} \cup (\{v \text{ is not colored}\} \cap \{c \notin \tilde{S}_t(v)\}).$$

Then as in the proof of Lemma 4.4.4, we get

$$\begin{aligned}
&Pr\{v \notin \tilde{D}_t(u, c) | c \in \tilde{S}_t(u)\} \\
&\quad = Pr\{v \notin \tilde{D}_t(u, c)\}(1 + O(\frac{1}{s_t})) \\
&\quad = (Pr\{v \text{ is colored}\} + (1 - e^{-\frac{1}{q}\frac{d_t}{s_t}})(1 - Pr\{v \text{ is colored}\}))(1 + O(\frac{1}{s_t})) \\
&\quad = (1 - (1 - Pr\{v \text{ is colored}\}) e^{-\frac{1}{q}\frac{d_t}{s_t}})(1 + O(e_t + \frac{1}{s_t})) \\
&\quad \geq (1 - (1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) e^{-\frac{1}{q}\frac{d_t}{s_t}})(1 + O(e_t + \frac{1}{s_t})).
\end{aligned}$$

By linearity of expectation, $\forall u \in V(G_{t+1}), \forall c \in \tilde{s}_t(u)$

$$\begin{aligned}
E[\tilde{d}_t(u, c) | c \in \tilde{s}_t(u)] &\leq E[\tilde{d}_t(u, c)](1 + O(\frac{1}{s_t})) \\
&\leq d_t(u, c)(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t + \frac{1}{s_t})).
\end{aligned}$$

Now using the above bound

$$\begin{aligned}
E[\bar{d}_t(u)] &= \sum_{c \in s_t(u)} Pr\{c \in \tilde{s}_t(u)\} E[\tilde{d}_t(u, c) | c \in \tilde{s}_t(u)] \\
&\leq e^{-\frac{1}{q}\frac{d_t}{s_t}} \sum_{c \in s_t(u)} d_t(u, c)(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t + \frac{1}{s_t})) \\
&\leq e^{-\frac{1}{q}\frac{d_t}{s_t}} s_t(u) d_t(u)(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t + \frac{1}{s_t})).
\end{aligned}$$

The rest of the proof follows that of Lemma 4.4.4. □

Consider the event

$$\mathcal{A} := \{\forall u \in V(G_{t+1}) d_{t+1}(u) \leq d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t})\}.$$

Lemma 4.4.12. *Given Assumption 2, we have*

$$Pr(\mathcal{A}) \geq 1 - e^{-\psi} O(n^2).$$

Proof. The proof follows that of Lemma 4.4.5. □

Consider the event

$$\begin{aligned} \mathcal{S} := \{\forall v \in V(G_{t+1}), \exists \alpha \in [0, \frac{1}{q}] \text{ such that } \forall c \in S_{t+1}(u) \\ s_{t+1}(u) &\geq (1-\alpha) s_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t})), \\ d_{t+1}(u) &\leq \frac{1 - q\alpha}{1-\alpha} d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t})), \\ d_{t+1}(u, c) &\leq q d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t}))\}. \end{aligned}$$

Lemma 4.4.13. *Given Assumption 2, we have*

$$Pr(\mathcal{S}) \geq 1 - e^{-\psi} O(n^2).$$

Proof. The proof follows that of Lemma 4.4.6. □

Now consider the event

$$\mathcal{D} := \{\forall v \in V(G_{t+1}), \eta_{t+1}(v) \leq \eta_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{d_t}}))\}.$$

Lemma 4.4.14. *Given Assumption 2, we have*

$$Pr(\mathcal{D}) \geq 1 - e^{-\psi} O(n)$$

Proof. Using Lemma 4.4.10

$$Pr\{u \text{ is colored}\} \ \geq \tfrac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t)).$$

Using linearity of expectation, $\forall u$ in G_{t+1}

$$\begin{aligned} E[\eta_{t+1}(u)] &\leq \eta_t(u)(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t)) \\ &\leq \eta_{t+1}(1 + O(e_t)). \end{aligned}$$

The rest of the proof follows that of Lemma 4.4.7. □

Chapter 5

A network coloring game

5.1 Introduction

We perform a theoretical study of the following coloring game on networks. In the coloring game, there is an associated network G, and each player is associated with a vertex in this network. Each player has a set of available strategies or colors; the payoff of a player is 1 if he plays a color different from the color played by any of his neighbors in G, and 0 otherwise. The goal of each player is to maximize his payoff, and we are interested in the behaviour and strategies of the players as a function of the network structure.

Our study is motivated by the fact that graph-coloring problems arise as natural formalizations of many conflict-resolution problems in practice. An example due to [24], is a scenario where faculty members wish to schedule classes in a limited number of classrooms, and must avoid conflicts with other faculty members. This can be modelled as a coloring problem, where the faculty members represent vertices, classrooms represent colors, and any two faculty members who have classes with overlapping times are connected by an edge. However, in this scenario, typically there is no centralized agency which assigns classrooms to faculty members, players coordinate among themselves to decide on a non-conflicting assignment, and it is also unreasonable to lay down a distributed protocol and expect the players to abide by the rules of this protocol. As a result, a game-theoretic formulation is an appropriate model for this scenario. A second example of a scenario where a game-theoretic formulation is appropriate is when players are employees in an organization, and colors are skills, and employees attempt to perfect skills that are different from the skills possessed by other employees in their department.

An experimental study of various coloring games was initiated by [24], which reports on behavioral experiments on human subjects who are incentivized to play the coloring game on specified networks. Comparisons were made among several different types of networks of moderate sizees. In addition, examples were given to illustrate the difficulties in analyzing the dynamics of large networks in which each node takes simple and selfish steps. The paper [24] has attracted much attention and pointed to the need for theoretical analysis, although there has not been any other prior work in this specific direction to the extent of our knowledge.

In this paper, we model the coloring game as a game played on a network over multiple rounds. We study the dynamics of the game when the players play a very simple, greedy strategy. At each round, each player picks a color uniformly at random from the set of colors unused by any of his neighbors in the previous round, and plays this color. We note that this is the *best response myopic strategy* [5] for the coloring game. We say that the coloring game converges, when the color played by each player is different from the color played by any of its neighbors. This is a *Nash equilibrium* of the coloring game, because no player has an incentive to change his strategy under this configuration. We are interested in the time taken by the players to converge, when each player adopts the greedy strategy.

Our main result in this paper is that for a coloring game played on a network on n vertices with maximum degree Δ, if the number of colors available to each vertex is $\Delta + 2$ or more, and if each player plays the greedy strategy, then the coloring game converges in $O(\log n)$ steps with high probability. Our result is also accompanied by a lower bound, in which we show a graph and a starting assignment of colors, such that, if the number of colors available to each vertex is $\Delta + 1$, and if each player plays the greedy strategy, then, the coloring game does not converge. Our upper bound holds even in the presence of non-participating vertices, which maintain the same color throughout the game. This indicates that if sufficient number of colors are available, then, even in the game-theoretic scenario, even when the players play a simple greedy strategy, convergence to the nash equilibrium is very rapid. In fact, this convergence bound is even comparable to the convergence bound for distributed protocols for graph coloring, which require the nodes to follow a distributed protocol and cooperate with each other (e.g., see, for example, the work of Luby[30]).

Related Work

The problem of graph coloring has a long history, and there is a lot of literature on centralized as well as distributed algorithms for this problem. The network coloring game has been studied experimentally by Kearns *et. al.*; however, to the best of our knowledge, this game has not been analyzed theoretically. On the experimental side, a number of behavior experiments with human subjects were conducted on the coloring game by Kearns, Suri, and Montford [24]. In their experiment, a group of human subjects were assigned to vertices of a graph, and were asked to play the coloring game for some amount of time. Each subject has access to the colors of their neighbors, and did not have any knowledge about the structure of the rest of the graph. Three graphs were studied – a cycle, a cycle with chords, as well as a random preferential attachment graph. In the experiment, the subjects found it more difficult to color the preferential attachment graph in their allotted amount of time.

In general, finding the minimum number of colors required to color a graph, even in a centralized manner, is NP-Hard [15], as well as hard to approximate [25]. Coloring a graph when the number of available colors is more than its maximum degree can be easily done in linear time by a centralized algorithm; however, the same problem becomes more challenging when the algorithm is required to be distributed.

There has also been a line of work on distributed graph-coloring. Luby [30] provides a distributed algorithm which finds a coloring of a graph in $O(\log n)$ rounds, when the number of available colors is $\Delta + 1$ or more. Notice that here Δ is the maximum degree of any vertex in the graph. [13] also provide theoretical as well as experimental results on a simple algorithm for coloring a graph in a distributed manner in $O(\log n)$ rounds when the number of available colors is

$\Delta + 1$. Both Luby's algorithm and the algorithm of [13] require each node to communicate to its neighbors its status, which indicates whether this node has any conflicts with its neighbors or not. In contrast, our scenario is purely game-theoretic : our algorithm does not require any cooperation among the vertices, and will succeed even if some nodes do not participate in the game.

Another line of work which is relevant to ours is the literature on Markov Chains for randomly sampling colorings of a graph. In this case, the goal is to bound the mixing time of a Markov Chain on colorings of a graph G. Pioneering work in this field was done by Mark Jerrum [20], who showed a way to randomly sample colorings of a graph G with maximum degree Δ in $O(n \log n)$ time when the number of colors available is at least $2\Delta + 1$. This was later improved by Vigoda [35], who could randomly sample colorings from graphs of degree Δ, when $\frac{11\Delta}{6}$ colors were available. Hayes and Vigoda [19] showed a better bound for triangle-free graphs when the number of colors needed was $\min(\Delta + O(1), O(\log n))$. Finally, it was shown in [18] that colorings from planar graphs can be sampled in $O(n \log n)$ time when the number of colors is at least $\Delta / \log \log \Delta$.

A Summary of Our Results

We consider the coloring game played on a graph G. Before we state our results, we need to define the following concepts.

The Coloring Game In the coloring game, each vertex in G represents a player. Each player has a set of k available strategies or *colors*; the payoff of a player is 1 if he picks a color different from the colors picked by his neighbors in G, and 0 otherwise. The game is played in *rounds*; each round, the players choose their strategies simultaneously. It is assumed that the players only have a *local view* of G, which means that they only know who their neighbors are, and the colors picked by their neighbors, and they do not have any knowledge of the rest of the graph G.

Greedy Strategy In this paper, we study the dynamics of the coloring game when each player has a local view and plays the *greedy strategy*. A player is said to play the *greedy strategy* if, in each round, he picks a color uniformly at random from the set of colors unused by any of his neighbors in the previous round. We note that this is the *best response myopic strategy* for the coloring game.

Convergence of the Coloring Game The coloring game is said to *converge* if, for every vertex v in G, the color chosen by v is different from the color chosen by all its neighbors. We note that when the coloring game has converged, none of the players have any incentive to change their strategy.

Participants in the Coloring Game We say that a vertex v in graph G is a participant in the coloring game on G if v plays according to the greedy strategy, otherwise we say that v is a *non-participant*. An instance of a coloring game on a graph G which has non-participant nodes, is said to converge, if, for every participant vertex v in G, the color chosen by v is different from the color chosen by all its neighbors.

The main result of this paper can be summarized by the following theorem.

Theorem 5.1.1. *Let G be any graph on n vertices, and let Δ be the maximum degree of any vertex in G. If the number of colors available to each vertex is at least $\Delta + 2$, and if each player plays the greedy strategy, then, for any starting assignment of colors, the coloring game on G converges after at most $O(\log(\frac{n}{\delta}))$ rounds with probability at least $1 - \delta$.*

In addition, we show that if there exists a set S of non-participant vertices, such that any vertex $v \in S$ has a fixed color throughout the game, then, the convergence time of the game is still at most $O(\log(\frac{n-|S|}{\delta}))$ round.

Corollary 5.1.2. *Let G be any graph on n vertices, Δ be the maximum degree of any vertex in G, and S be a set of non-participant vertices. If the vertices in S do not change their color throughout the game, and the number of colors available to each vertex is at least $\Delta + 2$, and if each player plays the greedy strategy, then, for any starting assignment of colors, the coloring game on G converges after at most $O(\log(\frac{n-|S|}{\delta}))$ rounds with probability at least $1 - \delta$.*

We show that when the number of colors is $\Delta + 1$, there is a graph G and a starting assignment of colors such that the greedy strategy does not converge.

Theorem 5.1.3. *There exists a graph G and a starting assignment of colors $\mathcal{C}$*

such that if the number of colors is $\Delta + 1$, and if each player plays the greedy strategy, the coloring game on G never converges.

5.2 Several Lemmas

We use $c_t(u)$ to denote the color played by player u at round t, and $\mathcal{N}(u)$ to denote the set of neighbors of u in G. We say that player u at round t has a *conflict* if

$$\exists v \in \mathcal{N}(u), \;\; c_t(u) = c_t(v)$$

At any time t, we use *number of conflicts* to mean the number of vertices v which have a conflict. We observe that if a vertex u has no conflict at time t, then $c_{t+1}(u) = c_t(u)$. We use k to denote the number of colors available and Δ to denote the maximum degree of the graph G over which the game is played.

We use the following lemma which is a minor variant of Markov's inequality.

Lemma 5.2.1. *Let X be a random variable such that $0 \leq X \leq M$. Then, for any a,*

$$\Pr[X < a] \leq \frac{M - \boldsymbol{E}[X]}{M - a}.$$

We observe that if a vertex v has no conflict at time t, then at round $t+1$, v does not change its color; moreover, no neighbor of v picks color $c_t(v)$, and after round $t+1$, v still has no conflict. Thus we have the following lemma.

Lemma 5.2.2. *If a vertex v has no conflict at time t, then it has no conflict at any subsequent time.*

The main idea behind the proof of Theorem 5.1.1 is to show that if we consider any two successive rounds of the coloring game, then, the conflict at each vertex is resolved with constant probability. We note that considering the game over two successive rounds is essential; in a single round, it is possible that a player has as little as $(1 - \frac{1}{k-\Delta})^{\Delta}$ chance of getting its conflict resolved.

The main step in the proof of Theorem 5.1.1 is the following lemma.

Lemma 5.2.3. *Consider an instance of the coloring game played on a graph G, with maximum degree Δ, in which each node has k color choices, where $k \geq \Delta + 2$. If, after round t, some vertex v in G has a conflict, then, there exists some constant c such that*

$$\Pr[v \textit{ has no conflict after round } t+2] \geq c$$

Proof. For vertex v, Let M denote the neighbors of v which do not have a conflict after round t. We define

$$F = \cup_{u \in M}\{c_t(u)\}$$

and let $f = |F|$. Then, the number of choices available to v in rounds $t+1$ as well as $t+2$ is at most $k - f$. However, since t is any arbitrary round, the number of colors available to v after round t can possibly be much less.

The proof of this lemma proceeds in two steps. First, we show in Lemma 5.2.4 that after round $t+1$, with constant probability, the vertex v has at least $\frac{k-f}{6}$ color choices. Next, we show in Lemma 5.2.5 that, given that v has $\frac{k-f}{6}$ color choices after round $t+1$, with constant probability, v has no conflict after round $t+2$. Combining these two facts, we get a proof of the lemma.

After round t, for any color $i \in [k]$, we define the random variable Y_i as follows:

$$Y_i = \begin{cases} 1 & \text{if } c_{t+1}(u) \neq i, \text{ for all } u \in \mathcal{N}(v) \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

Let $Y = \sum_{i \in [k]} Y_i$; thus Y is the number of colors available to vertex v after round $t+1$.

For any $u \in \mathcal{N}(v) \setminus M$, we let $\chi(u)$ be the set of colors available to vertex u after round t. Further, let $p_u = \frac{1}{|\chi(u)|}$. For $u \in M$, we define $\chi(u) = \{c_t(u)\}$, and $p_u = 1$. Note that if $u \in \mathcal{N}(v) \setminus M$, p_u is the probability with which vertex u picks a fixed color available to it during round $t+1$. Also, as $k \geq \Delta + 2$, each vertex $u \in \mathcal{N}(v) \setminus M$ has at least two color choices available, and therefore $|\chi(u)| \geq 2$. We are interested in the probability that $Y \geq \frac{k-f}{6}$. We show the following lemma.

Lemma 5.2.4.

$$\Pr(Y \geq \frac{k-f}{6}) \geq \frac{1}{25}$$

Proof. From the definition of Y_i, we can write that:

$$\Pr(Y_i = 1) = \prod_{\{u \in \mathcal{N}(v) | i \in \chi(u)\}} (1 - p_u)$$

The expectation of Y can therefore be estimated as:

$$\begin{aligned}\mathbf{E}(Y) &= \sum_{i \in [k]} \prod_{\{u \in \mathcal{N}(v) | i \in \chi(u)\}} (1 - p_u) \\ &\geq \sum_{i \in [k] \setminus F} \prod_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} (1 - p_u)\end{aligned}$$

The second step follows by the definition of F and M. As, for each $u \in \mathcal{N}(v) \setminus M$, $|\chi(u)| \geq 2$, we can use the inequality $1 - x \geq e^{-\frac{3}{2}x}$ for $0 \leq x \leq \frac{1}{2}$ to write:

$$\mathbf{E}(Y) \geq \sum_{i \in [k] \setminus F} e^{-\frac{3}{2} \sum_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} p_u}$$

Using the convexity of the exponential function,

$$\mathbf{E}[Y] \geq (k - f) e^{-\frac{3}{2} \frac{1}{k-f} \sum_{i \in [k] \setminus F} \sum_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} p_u}$$

Observe that

$$\sum_{i \in [k] \setminus F} \sum_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} p_u \leq \sum_{u \in \mathcal{N}(v) \setminus M} p_u \chi(u)$$

and therefore, we can write that:

$$\begin{aligned}\mathbf{E}[Y] &\geq (k - f) e^{-\frac{3}{2} \frac{1}{k-f} \sum_{u \in \mathcal{N}(v) \setminus M} p_u \chi(u)} \\ &\geq (k - f) e^{-\frac{3}{2}} \geq \frac{k - f}{5}\end{aligned}$$

The final step follows from the fact that $|\{u \in \mathcal{N}(v) \setminus M\}| \leq k - f - 2$. To complete the lemma, we now use Lemma 5.2.1, which is a variant of Markov's inequality. $\square$

Now we analyze the dynamics in round $t+2$ given that $Y > \frac{k-f}{6}$.

For any color $i \in [k]$, we define variables $\tilde{Y}_i$ as follows:

$$\tilde{Y}_i = \begin{cases} 1 & \text{if } c_{t+2}(u) \neq i \text{ for all } u \in \mathcal{N}(v) \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

Let $\tilde{Y} = \sum_{i\in[k]} \tilde{Y}_i$. We are interested in the probability of the event that $\tilde{Y} \geq \frac{k-f}{7e^9}$.

Lemma 5.2.5.

$$\Pr(\tilde{Y} \geq \frac{k-f}{7e^9} | Y \geq \frac{k-f}{6}) \geq \frac{1}{42e^9}$$

Proof. We define $\tilde{M}$ to be the set of vertices in $\mathcal{N}(v)$ which have no conflicts after round $t+1$, and $\tilde{F}$ as the set $\cup_{u\in\tilde{M}}\{c_{t+1}(u)\}$. Note, by Lemma 5.2.2, that $\tilde{M} \supseteq M$ and $|\tilde{F}| \geq |F|$. Further, we let $\tilde{f} = |\tilde{F}|$. Also we define $H = [k] \setminus \cup_{u\in\mathcal{N}(v)}\{c_{t+1}(u)\}$. Thus H is the number of colors available to v in round $t+2$. Obviously, $|H| \leq k - \tilde{f}$; given that $Y \geq \frac{k-f}{6}$, $|H| \geq \frac{k-f}{6} \geq \frac{k-\tilde{f}}{6}$.

For any $u \in \mathcal{N}(v) \setminus \tilde{M}$, we define $\tilde{\chi}(u)$ as the set of colors available to vertex u after round $t+1$, and $\tilde{p}_u$ as $\frac{1}{|\tilde{\chi}(u)|}$ respectively. For $u \in \tilde{M}$, we define $\tilde{\chi}(u) = \{c_{t+1}(u)\}$ and $\tilde{p}_u = 1$. We can write:

$$\Pr(\tilde{Y}_i = 1) = \prod_{\{u\in\mathcal{N}(v)|i\in\tilde{\chi}(u)\}} (1 - \tilde{p}_u)$$

Similar to the proof of Lemma 5.2.4, we can write:

$$\begin{aligned} \mathbf{E}[\tilde{Y}] &= \sum_{i\in[k]} \prod_{\{u\in\mathcal{N}(v)|i\in\tilde{\chi}(u)\}} (1 - \tilde{p}_u) \\ &\geq \sum_{i\in H} \prod_{\{u\in\mathcal{N}(v)\setminus\tilde{M}|i\in\tilde{\chi}(u)\}} (1 - \tilde{p}_u) \end{aligned}$$

As, for each $u \in \mathcal{N}(v) \setminus \tilde{M}$, $|\tilde{\chi}(u)| \geq 2$, we can write:

$$\mathbf{E}(Y) \geq \sum_{i\in H} e^{-\frac{3}{2}\sum_{\{u\in\mathcal{N}(v)\setminus\tilde{M}|i\in\tilde{\chi}(u)\}} \tilde{p}_u}$$

Using the convexity of the exponential function,

$$\mathbf{E}[\tilde{Y}] \;\geq\; \frac{1}{6}(k-\tilde{f})e^{-\frac{3}{2}\frac{6}{k-\tilde{f}}\sum_{i\in H}\sum_{\{u\in\mathcal{N}(v)\setminus\tilde{M}|i\in\tilde{\chi}(u)\}}\tilde{p}_u}$$

Observe that

$$\sum_{i\in H}\sum_{\{u\in\mathcal{N}(v)\setminus\tilde{M}|i\in\tilde{\chi}(u)\}}\tilde{p}_u \leq \sum_{u\in\mathcal{N}(v)\setminus\tilde{M}}\tilde{p}_u\tilde{\chi}(u)$$

and therefore, we can write that:

$$\begin{aligned}\mathbf{E}[\tilde{Y}] \;&\geq\; \frac{1}{6}(k-\tilde{f})e^{-\frac{3}{2}\frac{1}{k-\tilde{f}}\sum_{u\in\mathcal{N}(v)\setminus\tilde{M}}\tilde{p}_u\tilde{\chi}(u)}\\ &\geq\; \frac{1}{6}(k-\tilde{f})e^{-9} \geq \frac{k-f}{6e^9}\end{aligned}$$

The lemma now follows by an application of Lemma 5.2.1. □

Now, given that the event $\tilde{Y} \geq \frac{k-\tilde{f}}{7e^9}$ occurs, there is a set $\mathcal{C}$ of $\frac{k-\tilde{f}}{7e^9}$ colors, such that the conflict of v is resolved if it picks a color from $\mathcal{C}$ in round $t+2$. Since v picks one out of at most $k-\tilde{f}$ colors, given the event $\tilde{Y} \geq \frac{k-\tilde{f}}{7e^9}$, v has no conflict after round $t+2$ with probability at least $\frac{1}{42e^9}$. Combining this with the probability of occurrence of the event $Y > \frac{k-f}{6}$, Lemma 5.2.3 follows for $c = \frac{1}{1050e^9}$. □

5.3 Proofs of the main theorems

First we prove Theorem 5.1.1.

Proof. (Of Theorem 5.1.1) For a vertex v, and a time t, we define random variables $X_v(t)$ as follows:

$$X_v(t) = \begin{cases} 1 & \text{if vertex } v \text{ has a conflict after round } t \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

From Lemma 5.2.3, for any vertex v, for some constant c,

$$\Pr(X_v(t+2) = 1 | X_v(t) = 1) \leq 1 - c \tag{5.4}$$

Using Lemma 5.2.2, for any τ,

$$\begin{aligned}
\Pr(X_v(2\tau) = 1 | X_v(0) = 1) &= \Pr(\bigcap_{i=1}^{\tau} X_v(2i) = 1 | X_v(0) = 1) \\
&= \prod_{i=1}^{\tau} \Pr(X_v(2i) = 1 | \bigcap_{j=1}^{i-1} X_v(2j) = 1) \\
&\leq (1-c)^{\tau} \leq e^{-c\tau}
\end{aligned}$$

Plugging in $\tau = \frac{1}{c}\log(\frac{n}{\delta})$, we get, for any vertex $v \in G$,

$$\Pr(X_v(2\tau) = 0 | X_v(0) = 1) \geq 1 - \frac{\delta}{n} \tag{5.5}$$

The theorem follows by applying an union bound over all vertices in G. $\square$

Proof. (Of Corollary 5.1.2) All steps and lemmas in the proof of Theorem 5.1.1 remain valid except that we use $n - |S|$ in place of n. $\square$

It remains to prove Theorem 2.

Proof. (Of Theorem 5.1.3) Let G be a cycle of length 5, and let $V = \{v_1, \ldots, v_5\}$. Let $k = 3$, and suppose the initial configuration is $(c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)) = (1, 0, 2, 2, 0)$. Then, only v_3 and v_4 have conflicts. If v_3 and v_4 follow the dynamics, then, in the next round, $c(v_3) = c(v_4) = 1$. This again causes them both to have conflicts, so in the following round, $c(v_3) = c(v_4) = 2$, and we have a cycle in the dynamics. $\square$

5.4 Acknowledgement

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