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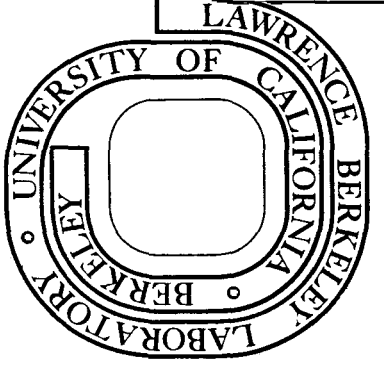

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THE USE OF ANTICOMMUTING INTEGRALS IN STATISTICAL MECHANICS I *

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ABSTRACT

Integrals over anticommuting variables are used to rewrite partition functions as fermionic field theories. In particular, the method is applied to the two-dimensional Ising and dimer models.

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I. INTRODUCTION

This paper introduces a new method of attacking certain problems in statistical mechanics. It uses integrals over anticommuting variables to express partition functions in terms of field theories.

The interplay of field theory and statistical mechanics is important. Many complicated field theories have simple underlying statistical mechanics analogues[1]). This supplies physical insight into these complicated field theoretic structures and allows one to extract the key concepts. On the other hand, when a statistical mechanics model is expressed as a field theory, various field theory techniques can be used such as perturbation theory, operator methods, variational methods, functional methods, etc. These are powerful avenues of attack, especially for extracting numbers. In short, the statistical mechanics point of view allows one physical insight whereas the field theory point of view supplies the powerful mathematical tools. It is therefore important to understand the connections between statistical mechanics and field theory. It is in this direction that this paper is written.

I shall use integrals over anticommuting variables. They were introduced to handle fermionic degrees of freedom in a path integral formulation[2]). Until recently[3]), they were usually used in formal ways, rarely being employed in actually calculations. In this paper and the following ones they will be used in a practical manner to obtain numbers. They are, without a doubt, powerful mathematical tools. They supply relations, relate unrelated models, organized unruly algebra, and evoke rapid

calculations often in a few steps.

I will try to follow a logical development with a pedagogical touch. First, this paper will introduce and review integrals over anticommuting variables (Sec. II). I have tried to summarize their key properties. Further details may be found in the references. Next (Sec. III) I will show how several problems may be expressed in terms of anticommuting variable integrals. This is a brute force method involving no elegance or ingenuity. Often a model has several different representations. It is important, therefore, to find the "best" and "efficient" ones. The fourth section will present a couple models in solvable form. Finally, I will discuss what these variables mean in the context of operator field theory.

This paper and the next deal only with solvable models. This is deliberate since it forms a testing ground on how these methods work. In the next paper, the actually solution of the two solvable models presented in Sec. IV will be carried out.

II. INTEGRALS OVER ANTICOMMUTING VARIABLES

This section will review[4]) some properties of integrals over Grassmann variables. More details may be found in reference four. A set of N Grassmann (or anticommuting) variables are objects, η_α ($\alpha = 1,2, \cdots, N$), satisfying

$$\eta_\alpha\eta_\beta + \eta_\beta\eta_\alpha = 0 \ . \qquad (2.1)$$

In particular, $\eta_\alpha^2 = 0$. Taking sums and products the most general

construct is

$$f = a_0 + \sum_\alpha a_\alpha\eta_\alpha + \sum_{\alpha<\beta} a_{\alpha\beta}\eta_\alpha\eta_\beta + \cdots + a_{123\cdots N}\eta_1\eta_2\cdots\eta_N \ , \qquad (2.2)$$

with the a's real or complex numbers. Functions of these variables are defined via Taylor series, which because of eq.(2.1) terminate at the Nth order. Equation (2.2) is the most general function, an Nth order polynomial.

The anticommuting variable integral of a function, f, of the form of eq. (2.2) is defined by

$$\int d\eta f \equiv \int d\eta_1 d\eta_2 \cdots d\eta_N f \equiv a_{123\cdots N} \ . \qquad (2.3)$$

The only term which contributes is the one where each η occurs precisely once, the sign being determined by the order (for example, $\int d\eta_1 d\eta_2 \eta_2 \eta_1 = -1$). Often η's are associated in pairs (or conjugates), one of which will have a dagger (i.e. η_α and $\eta_\alpha^\dagger$). This is convenient for determining the sign of an integral. For these the measure is defined as $\int d\eta d\eta^\dagger \equiv \int d\eta_1 d\eta_1^\dagger \cdots d\eta_N d\eta_N^\dagger$.

Statistical mechanics problems will involve spins, atoms, bonds, etc. at sites, $\vec{x}$, to which anticommuting variables will be assigned. The variable, $\vec{x}$, will range over the region of interest; for a cubic crystal this might be a three dimensional lattice so that $\vec{x} = (\alpha,\beta,\gamma)$ has integer coordinates. Often several variables are needed at a site, in which case, an additional label, r, is required, and the η's will appear as

$\eta^{r}_{\rightarrow x}$, $\eta^{r\dagger}_{\rightarrow x}\{r = 1,2,\cdots, T\}$ for T types. Graphically $\eta_{\rightarrow x}$ and $\eta^{\dagger}_{\rightarrow x}$ may be represented by an "o" and an "x" at $\vec{x}$. Different types may be distinguished by using different colors. The important point to remember is that a contribution to an integral occurs only if each site is covered by one "o" and one "x" of each color (type).

Key properties of these integrals which are consequences of eq.(2.3) are the following:

1. Shift of variable. Given J_α which anticommute with themselves and with all the η's,

$$\int d\eta\ f(\{\eta_\alpha\}) = \int d\eta\ f(\{\eta_\alpha + J_\alpha\})\ . \tag{2.4}$$

2. Change of Variables. Let $\psi_\alpha = \sum_\beta A_{\alpha\beta}\eta_\beta$ (with A invertible) be linear combinations of η's and hence an equivalent set of anticommuting variables. Then

$$\int d\eta\ f(\eta) = (\det A) \int d\psi\ f(A^{-1}\psi)\ . \tag{2.5}$$

Constrast this with normal (i.e. Riemann) integration where there is a factor $(\det A)^{-1}$ rather than $(\det A)$ in eq. (2.5).

3. Quadratic and Quadratic-like Actions.

$$\int d\eta d\eta^\dagger \exp\left(\sum_{\alpha\beta} \eta_\alpha A_{\alpha\beta}\eta^\dagger_\beta\right) = \det A\ . \tag{2.6}$$

$$\int d\eta\ \exp\left(\tfrac{1}{2}\sum_{\alpha\beta} \eta_\alpha A_{\alpha\beta}\eta_\beta\right) = \mathrm{Pf}\ A. \tag{2.7}$$

$$\int d\eta d\eta^\dagger \int d\psi d\psi^\dagger\ \exp\left(\sum_{\alpha\beta} \eta_\alpha\eta^\dagger_\alpha A_{\alpha\beta}\psi_\beta\psi^\dagger_\beta\right) = \mathrm{perm}\ A\ . \tag{2.8}$$

$$\int d\eta d\eta^\dagger\ \exp\left(\tfrac{1}{2}\sum_{\alpha\beta} \eta_\alpha\eta^\dagger_\alpha A_{\alpha\beta}\eta_\beta\eta^\dagger_\beta\right) = \mathrm{hf}\ A. \tag{2.9}$$

These are respectively the determinant, Pfaffian[5]), permanent, and hfaffian of A. Permanents and hfaffians are determinants and Pfaffians without the sign of permutation factor. In eqs. (2.7) and (2.9) A must be even dimensional. In eq. (2.7) A may be chosen to be antisymmetric. In eq. (2.9) it may be chosen to be symmetric, but must have zero's along the diagonal. These equations are easily proved by expanding the exponents: permutations of products of $A_{\alpha\beta}$ are obtained with the appropriate combinatorial and sign factors. Equation (2.6), however, is easier to prove by transforming $\eta^\dagger \to A^{-1}\eta^\dagger$ and using eq. (2.5).

Anticommuting variables are powerful objects. Let us demonstrate some of their power by proving the well known result[6]) that $(\mathrm{Pf}\ A)^2 = \det A$ for an antisymmetric even dimensional matrix. Usual proofs are quite cumbersome. Use eq. (2.6) and rewrite $\eta_\alpha = \sqrt{\tfrac{1}{2}}\ (\eta^{(1)}_\alpha + i\eta^{(2)}_\alpha)$, $\eta^+_\alpha = \sqrt{\tfrac{1}{2}}\ (\eta^{(1)}_\alpha - i\eta^{(2)}_\alpha)$,

$d\eta_\alpha d\eta_\alpha^\dagger = i d\eta_\alpha^{(1)} d\eta_\alpha^{(2)}$. Since A is antisymmetric $\eta_\alpha A_{\alpha\beta}\eta_\beta^\dagger = \frac{1}{2}\eta_\alpha^{(1)} A_{\alpha\beta}\eta_\beta^{(1)} + \frac{1}{2}\eta_\alpha^{(2)} A_{\alpha\beta}\eta_\beta^{(2)}$ (the cross terms cancel). The exponent factors into two exponents and the integral factorizes into two integrals, each of the form of eq. (2.7).

Finally, one may take derivatives of anticommuting variables. For example, $\frac{d}{d\eta_1}\eta_1 = 1$, $\frac{d}{d\eta_1}\eta_2 = 0$. All the usual rules of differentiation hold except for minus signs in the product rule due to anticommutation relations. Thus $\frac{d}{d\eta_1}(\eta_2\eta_1)$ $= (\frac{d}{d\eta_1}\eta_2)\eta_1 - \eta_2 \frac{d}{d\eta_1}\eta_1 = -\eta_2$. These derivatives act to the right. Derivatives acting to the left are defined analogously: $\eta_1 \frac{\overleftarrow{d}}{d\eta_1} = 1$.

4. Integration by parts. Given two functions, f and g,

$$\int d\eta\, f \frac{\vec{d}}{d\eta} g = \int d\eta\, f \frac{\overleftarrow{d}}{d\eta} g \,. \qquad (2.10)$$

In conclusion, anticommuting variables may be manipulated, integrated, and differentiated much like ordinary variables except that anticommutation must be taken into account.

III. SAMPLE REPRESENTATIONS

In a dimer problem[6,7,8] there are a set of sites and a set bonds connecting certain pairs of sites. The bonds may absorb dimers. If E_b is the energy of a particular dimer, $z_b = \exp(-\beta E_b)$ is the Boltzmann factor associated with an absorption. A site may be used only once, so that no two dimers may overlap or even touch. Effectively any two dimers are infinitely repulsive. There are two kinds of problems: the close-packed

problem in which every site must be covered exactly once, and the usual dimer problem where some sites may be left uncovered.

The statistical mechanics of this system is determined by the partition function. This partition function may be represented as an anticommuting integral. As an example, let us consider the two dimensional close-packed dimer problem. The sites are the integer lattice points (α,β) in a two dimensional plane. Bonds occur between nearest neighbors in the vertical and horizontal directions; z_v is associated with vertical bonds and z_h with horizontal bonds. The partition function is

$$Z(z_h,z_v) = \int d\eta d\eta^\dagger \exp\left[\sum_{\alpha,\beta}\left(z_h \eta_{\alpha\beta}\eta_{\alpha\beta}^\dagger \eta_{\alpha+1\beta}\eta_{\alpha+1\beta}^\dagger + z_v \eta_{\alpha\beta}\eta_{\alpha\beta}^\dagger \eta_{\alpha\beta+1}\eta_{\alpha\beta+1}^\dagger\right)\right] . \qquad (3.1)$$

There is an η and $\eta^\dagger$ for each site, and the total measure is a product over all sites of the measure at each site. The operator $\exp(z_h \eta_{\alpha\beta}\eta_{\alpha\beta}^\dagger \eta_{\alpha+1\beta}\eta_{\alpha+1\beta}^\dagger) = 1 + z_h \eta_{\alpha\beta}\eta_{\alpha\beta}^\dagger \eta_{\alpha+1\beta}\eta_{\alpha+1\beta}^\dagger$ has the option of placing a dimer on the bond between (α,β) and $(\alpha+1,\beta)$ (see fig.1). If the option is exercised, a weight z_h results and no more dimers may be placed on sites involving (α,β) and $(\alpha+1,\beta)$. Since the integral is zero unless every site is covered exactly once, eq. (3.1) is the partition function for the two dimensional close-packed dimer problem. This model (and, in general, any close-packed dimer model) is by eq. (2.9) a hfaffian.

Modifying the measure of eq. (3.1) by

$$\int d\eta d\eta^\dagger \to \int d\eta d\eta^\dagger \exp\left(\sum_{\alpha\beta} \eta_{\alpha\beta}\eta_{\alpha\beta}^\dagger\right) , \qquad (3.2)$$

would produce the (usual) dimer partition function, since the $\eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}$ piece of $\exp(\eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}) \equiv 1 + \eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}$ would cover any uncovered site (α,β). On the other hand, for sites already covered by a dimer the 1 term would be used. The new action would be

$$A_{\text{dimer}}(z_h, z_v) = \sum_{\alpha\beta} \eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}(1 + z_h\eta_{\alpha+1\beta}\eta^{\dagger}_{\alpha+1\beta} + z_v\eta_{\alpha\beta+1}\eta^{\dagger}_{\alpha\beta+1}). \quad (3.3)$$

Equation (3.3) may be interpreted as the partition function of monomers and dimers where the enery of a monomer is zero. If E_m is the energy of a monomer, then

$$A_{\text{dimer + monomer}}(z_m, z_h, z_v) = \sum_{\alpha\beta} \eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}(z_m + z_h\eta_{\alpha+1\beta}\eta^{\dagger}_{\alpha+1\beta} + z_v\eta_{\alpha\beta+1}\eta^{\dagger}_{\alpha\beta+1}), \quad (3.4)$$

with $z_m = \exp(-\beta E_m)$, is the partition function for dimers and monomers. By rescaling $\eta_{\alpha\beta} \to \sqrt{\frac{1}{z_m}}\,\eta_{\alpha\beta}$ $\eta^{\dagger}_{\alpha\beta} \to \sqrt{\frac{1}{z_m}}\,\eta^{\dagger}_{\alpha\beta}$ one obtains

$$Z_{\text{dimer + monomer}}(z_m, z_h, z_v) = z_m^N Z_{\text{dimer}}(z_h/z_m^2,\ z_v/z_m^2), \quad (3.5)$$

where N is the number of sites. This result (that the partition function for dimers and monomers is simply related to the partition function for dimers alone) is easily derived using physical considerations. In general, there will be transformations on the Grassmann integral which yield results in a few steps that, unlike

this example, are difficult to obtain using physical arguments. This is one reason why anticommuting variables are powerful.

To deal with a general dimer problem, let α be a labelling of sites. The set of bonds, B, is a set of pairs (α,β) having Boltzmann factors $z_{\alpha\beta}$. Then

$$Z = \int d\eta d\eta^{\dagger} \exp\left(\sum_{\alpha} \eta_{\alpha}\eta^{\dagger}_{\alpha} + \sum_{(\alpha,\beta)\in B} z_{\alpha\beta}\eta_{\alpha}\eta^{\dagger}_{\alpha}\eta_{\beta}\eta^{\dagger}_{\beta} \right). \quad (3.6)$$

Dimer models are equivalent to $\eta\eta^{\dagger}\eta\eta^{\dagger}$ field theories with a kinetic energy term consisting only of a mass piece, $\eta\eta^{\dagger}$. The field theory methods that deal with $\eta\eta^{\dagger}\eta\eta^{\dagger}$ theories may be applied to dimer problems.

Almost all partition functions which have a graphical representation are expressible as anticommuting integrals. The d-dimensional Ising model[9] has such a graphical representation[6,7,8], where one sum's over closed non-overlapping but (possibly) intersecting polygonal curves; in two dimensions this is obtained by starting with configurations where all spins are down and drawing curves around regions of up spin. There is a Boltzmann factor for each unit of "Bloch" wall. Alternatively, one may use bond variables[10] (which works in any dimension) for which there is a similar representation with different Bloch wall Boltzmann factors.

Let us consider $d = 2$. Then

$$Z_{\text{Ising}}(J_h, J_v) = f\, Z_{\text{closed polygons}}(z_h, z_v), \quad (3.7)$$

where $Z_{\text{Ising}}(J_h, J_v)$ is the Ising model partition function, with

horizontal and vertical spin couplings J_h and J_v, $Z_{\text{closed polygons}}(z_h, z_v)$ is the partition function for closed non-overlapping polygons with Boltzmann weights, z_h and z_v for horizontal and vertical Bloch walls, and f is a multiplicative factor. For the first representation

$$f = \exp\left[N(\beta J_v + \beta J_h)\right] \quad ,$$
$$z_h = \exp(-2\beta J_v) \quad , \tag{3.8}$$
$$z_v = \exp(-2\beta J_h) \quad ,$$

where N is the number of sites. For the bond variable representation

$$f = (2\cosh \beta J_v \cosh \beta J_h)^N,$$
$$z_h = \tanh \beta J_h \quad , \tag{3.9}$$
$$z_v = \tanh \beta J_v \quad .$$

Duality is the well known fact that the Ising model has these two representations relating low and high temperatures, one using bond variables on the lattice and one using disorder variables on the dual lattice.

To express the Ising model as a field theory, use four sets of anticommuting variables at a site (α,β), $\eta^r_{\alpha\beta}$ and $\eta^{r\dagger}_{\alpha\beta}$ with $r = R(\text{"right"}), L(\text{"left"}), U(\text{"up"}),$ or $D(\text{"down"})$ (see fig.2). To draw the sides of polygons use dimer operators

$\eta^R_{\alpha\beta}\eta^{R\dagger}_{\alpha\beta}\eta^L_{\alpha+1\beta}\eta^{L^\dagger}_{\alpha+1\beta}$ and $\eta^U_{\alpha\beta}\eta^{U\dagger}_{\alpha\beta}\eta^D_{\alpha\beta+1}\eta^{D\dagger}_{\alpha\beta+1}$ (see fig. 3). They give rise to a wall action

$$A_{\text{wall}} = \sum_{\alpha\beta}\left[z_h(\eta^R_{\alpha\beta}\eta^{R\dagger}_{\alpha\beta}\eta^L_{\alpha+1\beta}\eta^{L\dagger}_{\alpha+1\beta}) + z_v(\eta^U_{\alpha\beta}\eta^{U\dagger}_{\alpha\beta}\eta^D_{\alpha\beta+1}\eta^{D\dagger}_{\alpha\beta+1})\right]. \tag{3.10}$$

I next require "selection rules" at each (α,β) site. Suppose $\eta^D_{\alpha\beta}\eta^{D\dagger}_{\alpha\beta}\eta^R_{\alpha\beta}\eta^{R\dagger}_{\alpha\beta}$ is inserted in the integral, then only the configuration of (fig. 4a) may occur. Figure 4 illustrates the eight possibilities which can happen. To limit the graphs to these possibilities insert $1 + g$ with

$$g = N^R N^D + N^D N^L + N^L N^U + N^U N^R + N^U N^D + N^R N^L + N^R N^U N^L N^D \quad , \tag{3.11}$$

where $N^r = \eta^r\eta^{r\dagger}$. By using $1 + g = \exp\left[\ell n(1+g)\right] = \exp(g - \frac{1}{2}g^2)$ an action for these selection rules is obtained

$$A_{s.r.} = \sum_{\alpha\beta}(N^R_{\alpha\beta}N^D_{\alpha\beta} + N^D_{\alpha\beta}N^L_{\alpha\beta} + N^L_{\alpha\beta}N^U_{\alpha\beta} + N^U_{\alpha\beta}N^R_{\alpha\beta} + N^U_{\alpha\beta}N^D_{\alpha\beta} + N^R_{\alpha\beta}N^L_{\alpha\beta} - 2N^R_{\alpha\beta}N^U_{\alpha\beta}N^L_{\alpha\beta}N^D_{\alpha\beta}) \quad , \tag{3.12}$$

where again $N^r_{\alpha\beta} = \eta^r_{\alpha\beta}\eta^{r\dagger}_{\alpha\beta}$. The total action for the Ising model is $A = A_{\text{wall}} + A_{s.r.}$.

I call the above method of obtaining integral representations the "selection rule" method. By weighting the configurations of fig. 4, more general Ising-like models are obtained. Representations of more complicated models like the Baxter[11]) model can be derived in a similar manner.

The above representation of the Ising model is inefficient: It uses four sets of anticommuting variables per site; furthermore the action involves products of up to eight variables. Given a particular model, there will be many Grassmann integral representations. It is important to find efficient representations. Ingenuity in finding the "best" set of variables and the "best" actions will determine whether a model is exactly solvable and will determine how well approximation methods work. In the next section, efficient representations are found for these two 2-dimensional models.

IV QUADRATIC ACTIONS

Some models have quadratic action representations. I call these pseudo-free theories because they are exactly solvable by the techniques that solve free theories. In this section I will represent the two dimensional close-packed dimer and Ising models as pseudo-free theories. A later paper will calculate the partition functions and correlation functions.

The two dimensional dimer problem will be dealt with first. The method used to solve it closely follows the standard method[12]) of attack. In fact, I will be essentially reproducing the known method in integral form, circumventing a few algebraic steps along the way.

Take the lattice plane, group sites into units of four, and use the labelling indicated in fig. 5. Sites 1 and 3 are called odd sites; sites 2 and 4 are called even sites. For each unit (α,β), assign four sets of anticommuting variables, $\eta^{r}_{\alpha\beta}$, $\eta^{r\dagger}_{\alpha\beta}$ $\{r = 1,2,3,4\}$, one for each of the four original lattice points. It will be shown that

$$Z_{\text{dimer}}(z_h^A, z_v^A)\, Z_{\text{dimer}}(z_h^B, z_v^B) = \int d\eta d\eta^{\dagger} \exp(A), \qquad (4.1)$$

where

$$A = A^A + A^B ,$$

$$\begin{aligned} A^A = \sum_{\alpha\beta} \Big[& z_h^A(\eta^1_{\alpha\beta}\eta^{2\dagger}_{\alpha\beta} + \eta^{4\dagger}_{\alpha\beta}\eta^3_{\alpha\beta}) \\ & + z_v^A(\eta^{2\dagger}_{\alpha\beta}\eta^3_{\alpha\beta} + \eta^{4\dagger}_{\alpha\beta}\eta^1_{\alpha\beta}) \\ & + z_h^A(\eta^{2\dagger}_{\alpha\beta}\eta^1_{\alpha+1\beta} + \eta^3_{\alpha\beta}\eta^{4\dagger}_{\alpha+1\beta}) + z_v^A(\eta^1_{\alpha\beta+1}\eta^{4\dagger}_{\alpha\beta} + \eta^3_{\alpha\beta}\eta^{2\dagger}_{\alpha\beta+1}) \Big] , \end{aligned} \qquad (4.2)$$

and A^B is obtained by replacing z_h^A by z_h^B, z_v^A by z_v^B, $\eta^r_{\alpha\beta}$ by $\eta^{r\dagger}_{\alpha\beta}$, and $\eta^{r\dagger}_{\alpha\beta}$ by $\eta^r_{\alpha\beta}$. Equation (4.2) may look complicated, but it has a simple graphical representation in fig. 6. Each of the eight dimer-like operators of fig. 6a corresponds to a term of eq.(4.2). The dimer object, $\eta^1_{\alpha\beta}\eta^{2\dagger}_{\alpha\beta}$, produces an "o" at 1 and an "x" at 2 in the (α,β) unit. Arrows are used to indicate the order of the η's as illustrated in fig. 7. The dimers weighted by z^A factors are the ones with "o"'s on odd sites and "x"'s on even sites

and are called A-dimers. The B-dimers have "x"'s on odd sites and "o"'s on even sites.

We can now make contact with the usual method of solution. By the "golden rule" of Grassman integrals, each site must have an "x" and an "o". This means each site is covered by exactly one A-dimer and one B-dimer. Therefore, we have a simultaneous A and B dimer problem: Expand the B-action exponent and choose one configuration, b, which covers all sites with B-dimers. Let w_b be its weight (that is, the product of the z_h^B and z_v^B factors; for example, if $z_h^B = z_v^B \equiv z^B$ then $w_b = (z^B)^{N/2}$ where N is the number of sites). Expanding the A-action exponent, each A-dimer covering results in diagrams of closed non-overlapping polygons and overlapping isolated dimer pairs (see fig. 21, p. 233 of reference 7) with the proper weight (up to possibly a minus sign). A minus sign could result because of reorderings of anticommuting variables in evaluating integrals. It is proven in Appendix A, however, that all terms are positive. The reader is invited to check some examples by using the rules of fig.8. Each configuration, b, of B-dimers yields $w_b\, Z_{dimer}(z_h^A, z_w^A)$. Equation (4.1) results by summing over all B-coverings.

Every planar close-packed dimer problem, which is exactly solvable by the usual techniques, is expressible as an anticommuting integral over a quadratic action. At this stage, Grassmann integrals are used only as a bookkeeping device which organizes the algebra. No true progress has been made. The next example will obtain a quadratic action for the Ising model. Although similar to previous derivations, several simplifications are made.

I shall use eq. (3.7) which relates the Ising model to a sum over closed non-overlapping but (possibly) intersecting polygons. I shall then use the anticommuting variables to "draw" these configurations. Two sets of variables will be used at each (α,β) site: $\eta^h_{\alpha\beta}$, $\eta^{h\dagger}_{\alpha\beta}$, and $\eta^v_{\alpha\beta}$, $\eta^{v\dagger}_{\alpha\beta}$. The superscripts "h" and "v" stand for horizontal and vertical. Consider

$$Z_{\text{closed polygons}}(z_h, z_v) = (-1)^N \int d\eta d\eta^+ \exp(A), \qquad (4.3)$$

where N is the number of sites and

$$A = A_{\text{Bloch wall}} + A_{\text{corner}} + A_{\text{monomer}},$$

$$A_{\text{Bloch wall}} = \sum_{\alpha\beta} \left(z_h \eta^{h\dagger}_{\alpha\beta} \eta^h_{\alpha+1\beta} + z_v \eta^{v\dagger}_{\alpha\beta} \eta^v_{\alpha\beta+1} \right),$$

$$A_{\text{corner}} = \sum_{\alpha\beta} \left(a_1 \eta^{h\dagger}_{\alpha\beta} \eta^v_{\alpha\beta} + a_3 \eta^{v\dagger}_{\alpha\beta} \eta^h_{\alpha\beta} + a_2 \eta^{v\dagger}_{\alpha\beta} \eta^{h\dagger}_{\alpha\beta} + a_4 \eta^v_{\alpha\beta} \eta^h_{\alpha\beta} \right), \qquad (4.4)$$

$$A_{\text{monomer}} = \sum_{\alpha\beta} \left(b_h \eta^h_{\alpha\beta} \eta^{h\dagger}_{\alpha\beta} + b_v \eta^v_{\alpha\beta} \eta^{v\dagger}_{\alpha\beta} \right).$$

The Bloch wall action produces a unit of Bloch wall in either the horizontal or vertical direction [see fig. (9)] weighted by the appropriate Boltzmann factor. The term A_{corner} produces the four corners of fig. (10) necessary to construct a ploygon. I have allowed for the most general quadratic form by weighting corners with the a_i. For the Ising model, set $a_i = 1$. Finally, $A_{monomer}$ fills all unoccupied "h" and "v" sites with monomer.

Again, for the Ising model, set b_h and $b_v = 1$. The eight possible configurations which can occur at a site are shown in fig. (11) with their weights. There is an extra (-1) for each site because of the $(-1)^N$ in eq. (4.3). The minus signs in configurations (b) through (g) always cancel in pairs and may be dropped. The extra minus sign in fig. (11h) is explained in Appendix B. This Appendix deals with minus signs due to reorderings of Grassmann variables. Finally, the double corners of fig. (12a,b) do not occur because a single corner uses up both horizontal and vertical variables. Equations (3.7), (4.3), and (4.4) form the quadratic action representation of the two-dimensional Ising model.

V THE OPERATOR ALGEBRA

This section dissusses the operator aspects of Grassmann variables and their probabilistic interpretation.

In the previous two sections, partition functions have been expressed as fermionic field theories. By taking expectation values of Grassmann variables (as well as functions of them) we may treat them as operators. They act like "local observables", measuring tools with probabilistic interpretations. Consider for example, the two dimensional dimer problem whose action is given by eq. (3.3).

$$Z\langle \eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}\rangle \equiv \int d\eta d\eta^{\dagger} \exp(A_{dimer})(\eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}) \ , \qquad (5.1)$$

is the sum over dimer configurations with the restriction that no dimer be placed on the (α,β) site. Therefore, $\langle \eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}\rangle$ is the probability that the (α,β) site is not covered by a dimer.

Likewise $\langle 1 - \eta_{\alpha\beta}\eta^{\dagger}_{\alpha\beta}\rangle$ is the probability that (α,β) is covered. In general, the expectation value of an operator will be the probability that a corresponding configuration will occur.

What do the equations of motion mean? The equation for $\eta_{\alpha\beta}$ is obtained by taking $\frac{d}{d\eta_{\alpha\beta}} A_{dimer}$. Let O be an operator (i.e. some function of the η's and $\eta^{\dagger}$'s) and use integration by parts [eq. (2.10)]:

$$\langle \frac{d}{d\eta_{\alpha\beta}} O\rangle = \langle O \frac{d}{d\eta_{\alpha\beta}} A\rangle \ . \qquad (5.2)$$

Equation (5.2), which involves the equation of motion of $\eta_{\alpha\beta}$, will generate many probability relations and is quite useful. For example, let O be $\eta_{\alpha\beta}$ and let $P^{o}_{(\alpha,\beta)}$, $P^{u}_{(\alpha,\beta)}$, $P^{u}_{(\alpha,\beta)and(\alpha,\beta+1)}$, etc. be respectively the probabilities that (α,β) is occupied, that (α,β) is unoccupied, that (α,β) and $(\alpha,\beta+1)$ are unoccupied, etc., then

$$P^{o}_{(\alpha,\beta)} = z_h\left[P^{u}_{(\alpha,\beta)and(\alpha+1,\beta)} + P^{u}_{(\alpha,\beta)and(\alpha-1,\beta)}\right]$$

$$+ z_v\left[P^{u}_{(\alpha,\beta)and(\alpha,\beta+1)} + P^{u}_{(\alpha,\beta)and(\alpha,\beta-1)}\right] \ . \qquad (5.3)$$

I invite the reader to derive this relation using physical considerations and compare it to the simple and powerful method of anticommuting variables.

The set of relations of eq. (5.2) along with the anti-commutation equations [eq. (2.1)] determine the model. They are

an equivalent expression of it, because A is obtainable from eqs. (5.2). The Ising-like or dimer-like problems are uniquely determined by a set of local probability relations. In field theory the equations of motion are foremost. The operator techniques used to attack such field theories may be used in statistical mechanics. I call this the operator method of local observables.

ACKNOWLEDGMENTS

I would like to thank Harry Morrison for delightful conversations, patient listening, and for reading the manuscript. I thank Korkut Bardakci for encouragement.

APPENDIX A

This Appendix treats the minus signs of the two dimensional close-packed dimer problem.

An isolated dimer pair between two neighboring sites r and s (see fig. 13) will be of the form $\eta_r\eta_s^\dagger\eta_r^\dagger\eta_s = +\eta_r\eta_r^\dagger\eta_s\eta_s^\dagger$; so it has has the correct sign.

To deal with a closed polygon, P, orient it counterclockwise and call the parity of P the number of minus signs which result from rule (b) of fig. 8. There are two types of polygons: type 1 (fig. 14a) and type 2 (fig. 14b). For type 1, there is a minus sign from rule (c) and no minus signs due to rule (a). Therefore, for type 1, the overall sign is opposite to the counterclockwise parity of P. For type 2, the identical conclusion is obtained using a similar approach. Kasteleyn's theorem[13]) (which is easily verified for test examples and easily proven by induction on the area of P) says that the counterclockwise parity is $(-1)^{I+1}$ (where I is the number of interior points) if all elementary polygons (ones with no interior points) have odd parity. In fig. 14, for example, I = 1 and the parity is even. With the arrow assignment of fig. 6, all elementary polygons are odd parity. We conclude all polygons having an even number of interior points have the correct sign. Fortunately only these kinds of polygons occur in a covering since dimers, covering two sites at a time, cannot cover regions of an odd number of sites. Therefore, all polygons have the correct sign.

APPENDIX B

In this Appendix, I will analyze the sign problem associated with eqs. (4.3) and (4.4). The conclusion will be that the sign of a configuration of polygons is equal to the number of intersections which occur. This explains the extra minus factor in the weight of fig. (11h). I will proceed in steps: first dealing with an isolated non-self-intersecting polygon, then with one that self-intersects, and finally dealing with a multipolygonal configuration.

Consider a closed polygon, P, which does not intersect itself. I will show that its sign is positive. Choose a horizontal bond of P and proceed to the right (and eventually around the polygon). Start at the "x" and use the rules of fig.8. When moving upward or to the right no minus signs result from rules (a) or (b) because arrows are in the correct direction and "o"'s occur before "x"'s. When moving downward or to the left, each site has a minus sign from rule (a) and a minus sign from rule (b). They cancel in pairs. Next consider what happens, when one goes around a corner. There are eight different types (see fig. 15) [two orientations times the four basic corners of fig. (10)]. They are oriented because we are moving around the polygon in a particular direction. Figure 15 summarizes the results: only corners of types d and $\bar{d}$ lead to a minus sign. Now use the following theorem (which is easily proved by induction on the area of P): Let m_a, m_b, etc. be the number of type a, type b, etc. corners occurring in an oriented non-self-intersecting polygon, P. If P is counterclockwise oriented then

$$\begin{aligned} m_a - m_{\bar{a}} &= 1 , \\ m_b - m_{\bar{b}} &= 1 , \\ m_c - m_{\bar{c}} &= 1 , \\ m_d - m_{\bar{d}} &= 1 . \end{aligned} \tag{B.1}$$

This implies that the sign due to corners is $(-1)^{m_d}(-1)^{m_{\bar{d}}} = -1$. For clockwise oriented, P, the theorem holds with $a \longleftrightarrow \bar{a}$, $b \longleftrightarrow \bar{b}$, etc. Rules (a) and (b) therefore result in one minus sign which when combined with the minus sign of rule (c) gives an overall plus sign.

Now consider an oriented self-intersecting polygon, P. It may be constructed from non-intersecting ones by the pasting construction of fig. 16. The order of the operators in P is indicated in Figure 17a. When they are regrouped into the forms occurring in the non-self-intersecting polygons (Figures 17b and 17c) which "compose" P, a minus sign results for each intersection as Figure 17 illustrates.

Finally, the result holds for multipolygonal configurations because pairs of polygons can only intersect an even number of times. Summarizing, an extra minus occurs for each intersection (fig. 11h).

FOOTNOTES AND REFERENCES

1) S. Samuel, The Grand Partition Function in Field Theory with Applications to The Sine-Gordon Theory, to be published in Phys. Rev. D.

2) D. J. Chandlin, Nuovo Cimento 4, (1956) 231.

3) A practical application is S. Samuel, J. Math. Phys. 19, (1978) 1438.

4) F. A. Berezin, The Method of Second Quantization (Academic, New York, 1966). See also the appendices of reference 3.

5) The definition of a Pfaffian is

$$\text{Pf } A = \frac{1}{2^{N/2}} \frac{1}{(\frac{N}{2})!} \sum_{\sigma} (\text{sign } \sigma)\, A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(N-1)\sigma(N)}.$$

The sum is over all permutations, σ .

6) For example, B. M. McCoy and T. T. Wu, The Two-Dimensional Ising Model (Harvard University Press, Cambridge, 1973). The proof that $(\text{Pf } A)^2 = \det A$ can be found on pages 47-51.

7) E. W. Montroll, in Brandeis University Summer Institute in Theoretical Physics, 1966, edited by M. Chrétien, E. P. Gross, and S. Deser (Gordon and Breach, New York, 1968).

8) H. S. Green and C. A. Hurst, Order-Disorder Phenomena (Interscience, New York, 1964).

9) There is an enormous amount of literature on the Ising model. See the references of reference 6 for a partial list.

10) See for example, pages 218-221 of reference 7.

11) R. J. Baxter, Annals of Physics, 70, (1972) 193.

12) See pages 230-248 of reference 7.

13) See pages 236 and 237 of reference 7.

Figure 1. The dimer operators: (a) The horizontal dimer operator, $\eta_{\alpha\beta}^{\dagger}\eta_{\alpha\beta}\eta_{\alpha+1\beta}\eta_{\alpha+1\beta}^{\dagger}$, and (b) The vertical dimer operator $\eta_{\alpha\beta}^{\dagger}\eta_{\alpha\beta}\eta_{\alpha\beta+1}\eta_{\alpha\beta+1}^{\dagger}$.

Figure 2. The two-dimensional lattice used for the Ising model: (a) Each site has been replaced by four sites, and (b) The notation used to label sites. The pair, (α,β), labels the group and "Right", "Up", "Left", and "Down" are used to label types.

Figure 3. Bloch wall operators: (a) The horizontal dimer operator, $\eta_{\alpha\beta}^{R}\eta_{\alpha\beta}^{R\dagger}\eta_{\alpha+1\beta}^{L}\eta_{\alpha+1\beta}^{L\dagger}$, and (b) The vertical dimer operator, $\eta_{\alpha\beta}^{U}\eta_{\alpha\beta}^{U\dagger}\eta_{\alpha\beta+1}^{D}\eta_{\alpha\beta+1}^{D\dagger}$.

Figure 4. The eight possibilities which can happen at a vertex. In each case, the operator on the right will produce the dimer configuration on the left. Figures (a) - (g) represent the seven terms in eq. (3.11). To these one must add the last term which is the unity operator.

Figure 5. The lattice plane reorganized into groups of four sites each. Each unit is labelled by a pair of integers (α,β) and each of the four sites in a unit are labelled by 1,2,3, or 4. Sites of type 1 and 3 are called odd sites, whereas sites of type 2 and 4 are even sites.

Figure 6. (a) The eight bonds corresponding to the eight terms in eq. (4.2). Each of these operators create A-dimers and is weighted by a z^A factor. The arrows indicate the order of the η's (see fig. 7.). (b) The B-dimer operators which make up the B-dimer action, A^B .

Figure 7. The use of arrows to indicate operator ordering. On the left-hand side is the operator, $\eta^{\dagger}\eta$, which equals minus the right-hand side operator, $\eta\eta^{\dagger}$. The arrow originates from the first anticommuting variable and terminates on the second one.

Figure 8. Sign rules. The rules for evaluating the sign of a "dimer loop" are as follows: Pick an initial "o" or "x" (here, "o" is chosen at A) and proceed around the loop (here, counterclockwise). There is a) a minus sign for each "x" occurring before an "o" (the point, B), b) a minus sign for each arrow in the opposite direction (the bond, C), and finally c) a minus sign if one begins with an "x". In this figure the sign is positive.

Figure 9. Bloch wall operators: (a) is the graphical representation of $\eta^{h\dagger}_{\alpha\beta}\eta^{h}_{\alpha+1\beta}$ which occurs in eq. (4.4) and produces a horizontal Block wall; (b) is the vertical Bloch wall operator, $\eta^{v\dagger}_{\alpha\beta}\eta^{v}_{\alpha\beta+1}$.

Figure 10. The corner operators in eq. (4.4). In all cases they occur at the (α,β) site, that is corner operators only change the direction of a curve; they do not connect neighboring sites. Although one could use labels to distinguish horizontal and vertical variables, it's easier to use the following convention: if an "o" or an "x" has a horizontal line coming into or out of it, it is a horizontal variable; on the other hand vertical variables have vertical lines flowing into or

out of them. For example, (a) involves a horizontal "x" or $\eta^{h\dagger}_{\alpha\beta}$ and a vertical "o" or $\eta^{v}_{\alpha\beta}$. The arrow indicates the order, so that this term is $\eta^{h\dagger}_{\alpha\beta}\eta^{v}_{\alpha\beta}$, the first term in A_{corner} of eq. (4.4). (b), (c), and (d) are the other three terms.

Figure 11. The eight possible configurations that can occur at a site. When disorder variables are used [eq. (3.8)], the first two columns represent corresponding spin configurations. In obtaining the weights of column 4 a (-1) factor has been included from the $(-1)^{N}$ of eq. (4.3). The minus signs in (b) through (g) may be eliminated because i) there are always an even number of (b) and (c) configurationa and ii) corners (d) and (f) as well as (e) and (g) occur in pairs. Alternatively, one could redefine the b's and a's in eq. (4.4) to have minus signs. Configuration (h) has an extra minus sign due to reordering of anticommuting variables as described in Appendix B. The numbers in column 4 are easily obtained: For example, the b_h of (b) is obtained because a vertical bond enters and exits the vertical site and a horizontal monomer with b_h must fill the empty horizontal site.

Figure 12. Intersections. The double corners of figs. (a) and (b) are not allowed by eq. (4.3). When four lines meet at a site they must pass directly through as in fig. (c).

Figure 13. A typical dimer pair. The sign of this pair is plus since $\eta_r\eta_s^\dagger\eta_r^\dagger\eta_s = +\eta_r\eta_r^\dagger\eta_s\eta_s^\dagger$.

Figure 14. The two types of polygons. Type 1 [fig. (a)] is characterized by the fact that $\eta\eta^\dagger$ occurs at each site when going counterclockwise arount the polygon. Alternatively, as one goes around the curve the "x" occurs before the "o" in a given dimer. For type 2 [fig. (b)], $\eta^\dagger\eta$ products occur at each site. The parity of this polygon will be even if all elementary polygons are counterclockwise odd. The assignment of arrows in fig. (6) does make all elementary squares of odd parity.

Figure 15. The eight oriented corners and the minus sign factors associated with them.

Figure 16. The pasting construction. Polygon, P, may be obtained from two (possibly self-intersecting) polygons, P_1 and P_2, by cutting open the corners and rejoining. There are four (two different types of pairs of corners times two orientations) possible pasting constructions.

Figure 17. How the minus sign arrises. This is just a "fermion" statistics effect. The order of operators in an intersection of P is indicated in Figure (a) and is $(\eta_1^\dagger\eta_3)(\eta_2^\dagger\eta_4)$. When P is decomposed into non-intersecting polygons as in Figure 16, the order of the operators is that of (b) or (c). For case (b), $(\eta_1^\dagger\eta_4)(\eta_2^\dagger\eta_3) = -(\eta_1^\dagger\eta_3)(\eta_2^\dagger\eta_4)$, that is, there is a minus sign relative to (a). For case (c), $(\eta_2^\dagger\eta_1^\dagger)(\eta_4\eta_3)$ is also $-(\eta_1^\dagger\eta_3)(\eta_2^\dagger\eta_4)$.

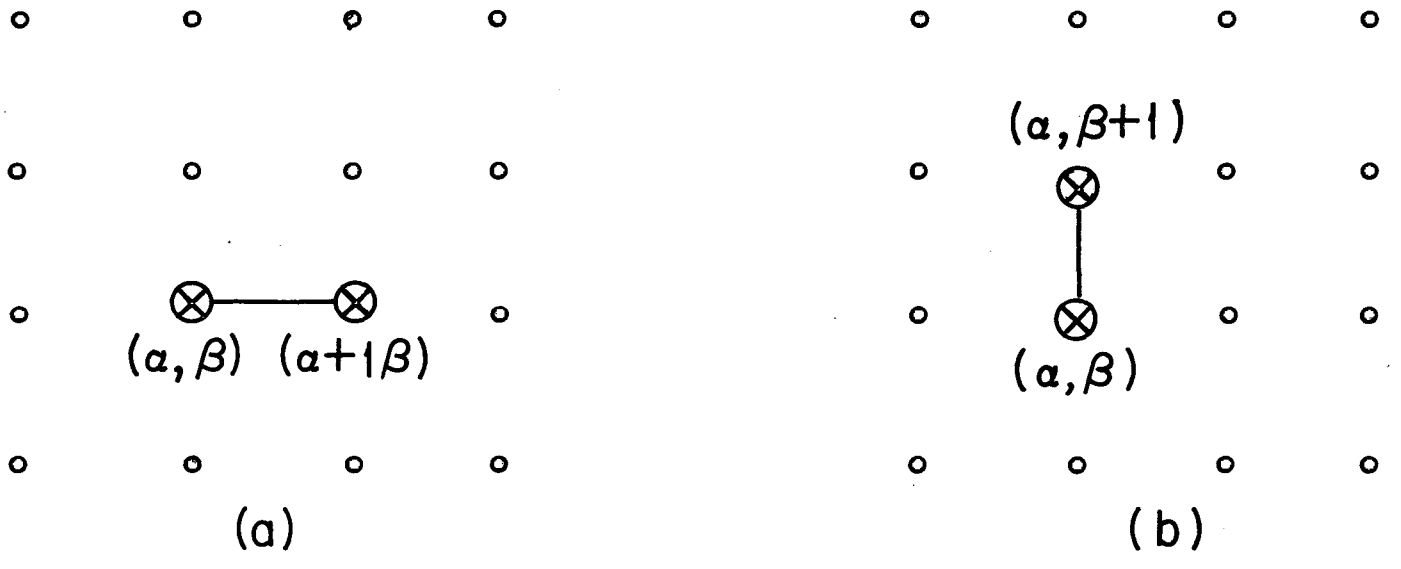

Figure 1

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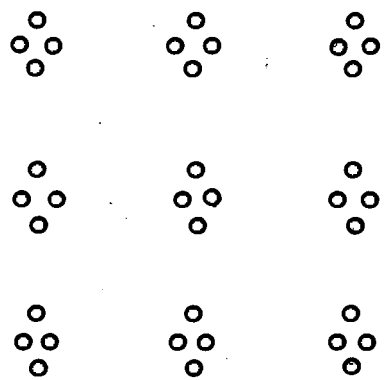
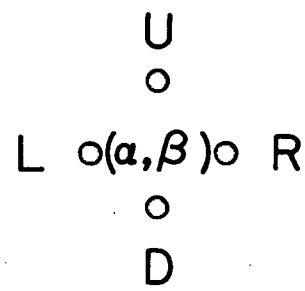


(a)



(b)

Figure 2

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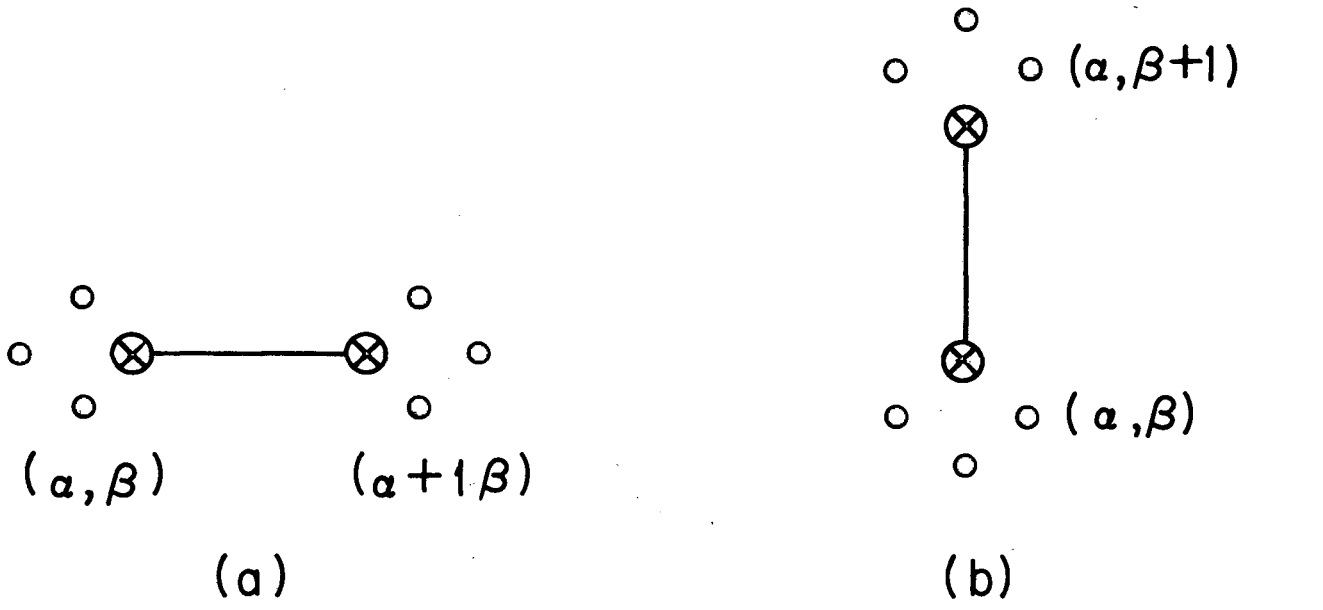

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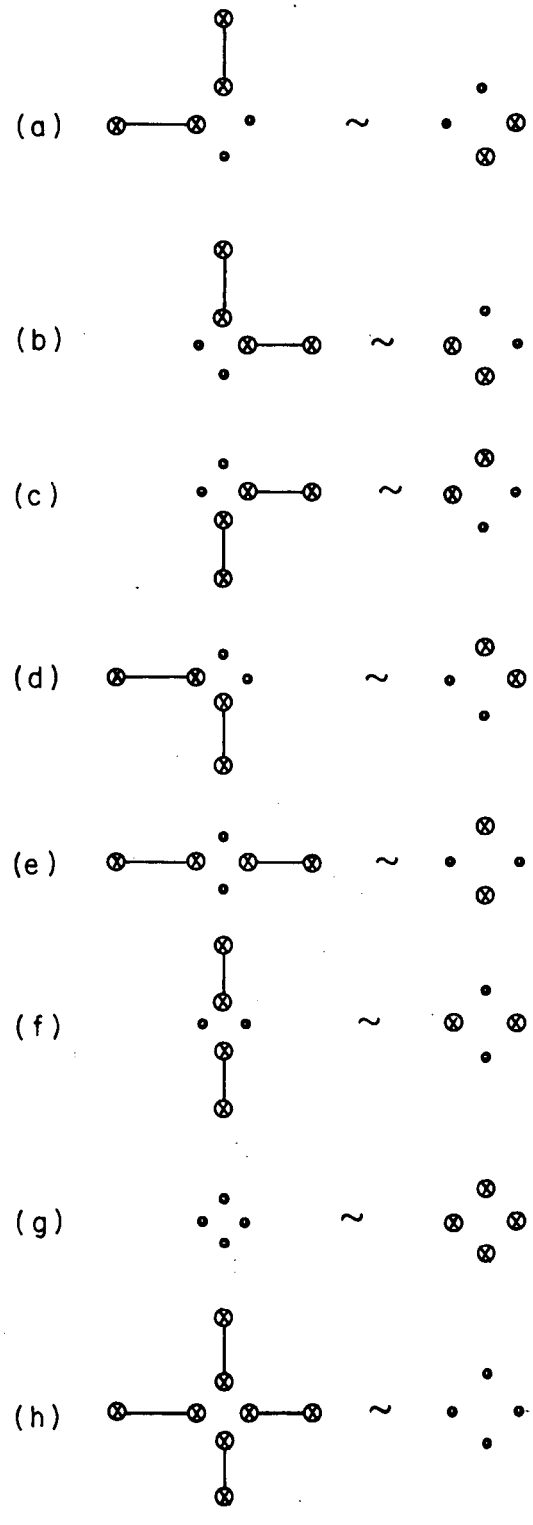

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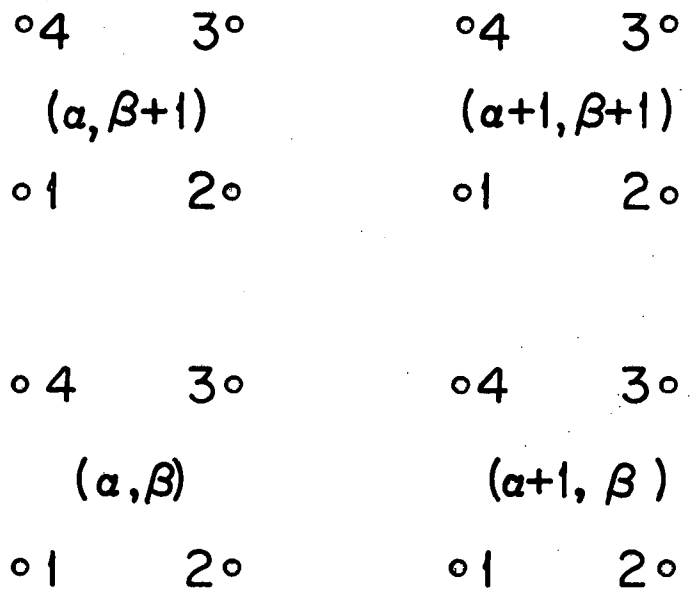

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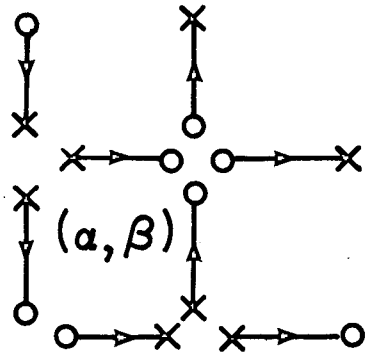
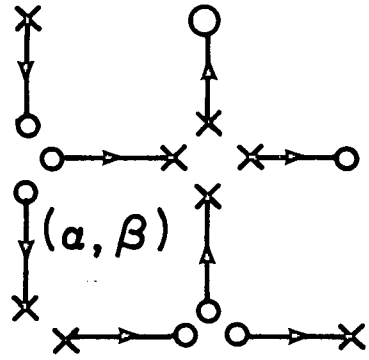


(a)



(b)

Figure 6

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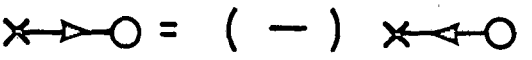

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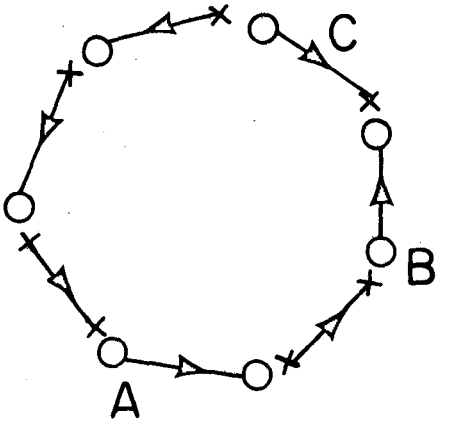

Figure 8 XBL7810-11570

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Figure 9

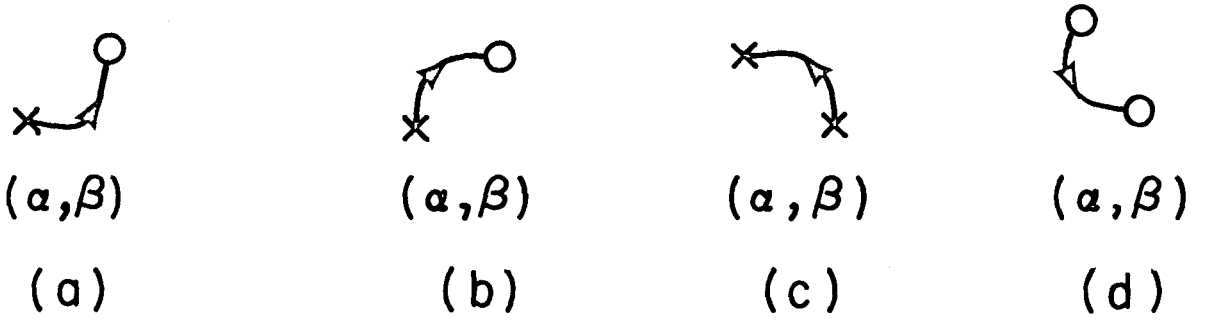

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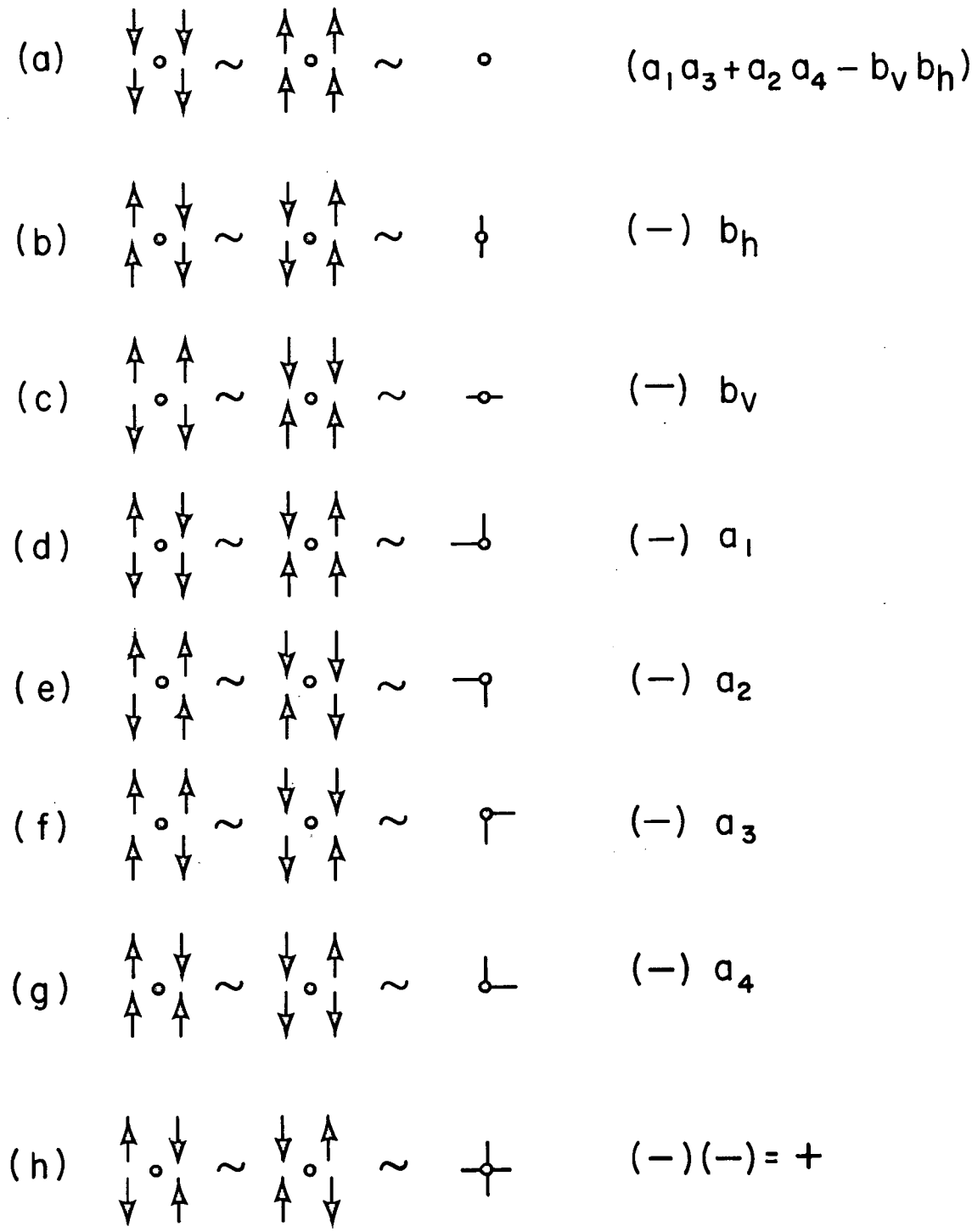
Figure 10



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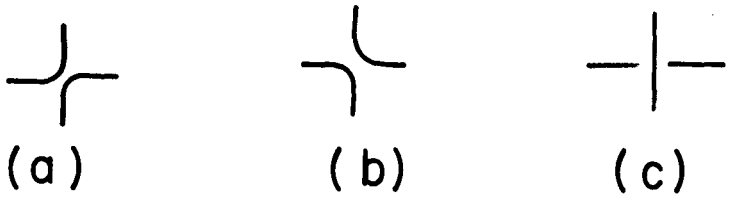

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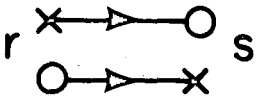

Figure 13 XBL7810-11575

Figure 14 XBL 7810-11559

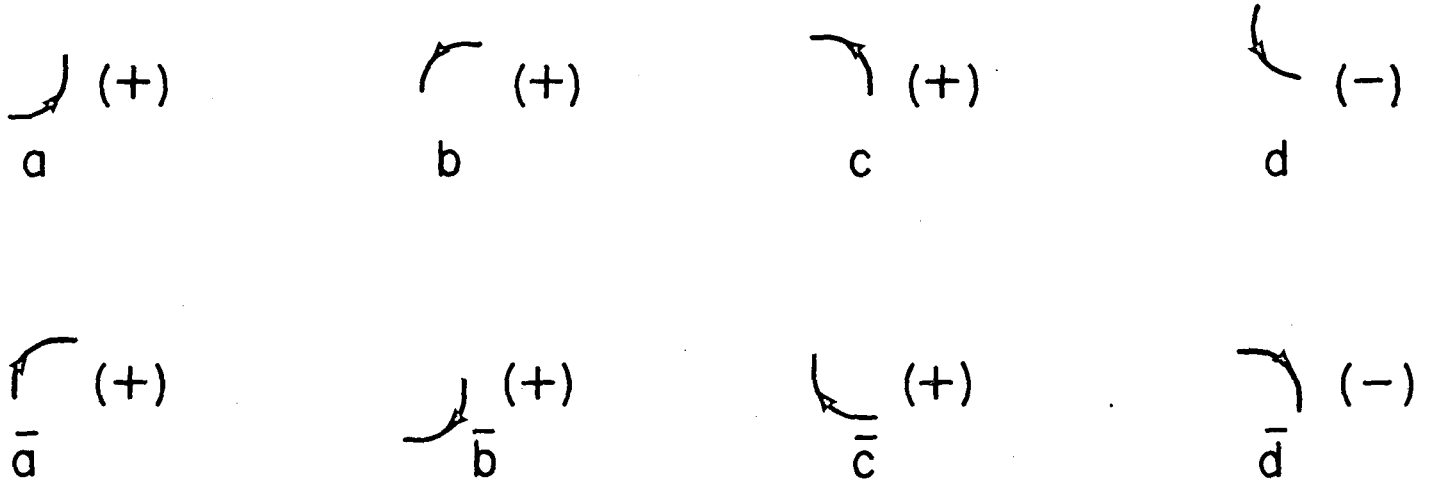

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Figure 15

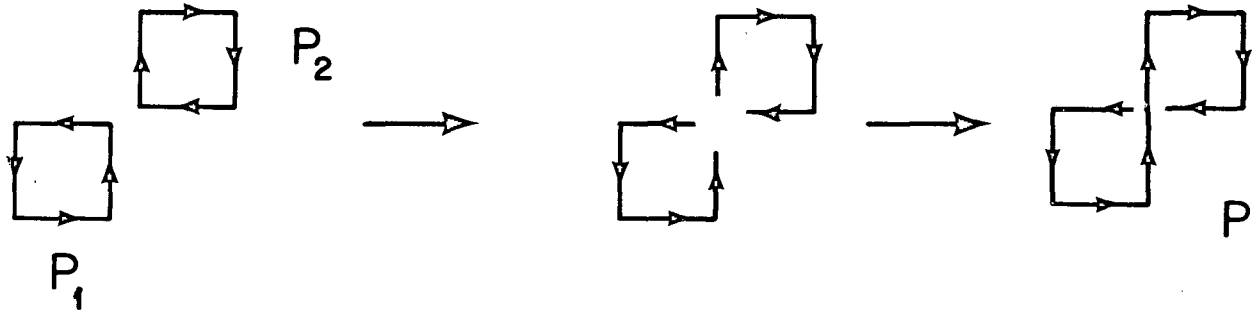

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Figure 16

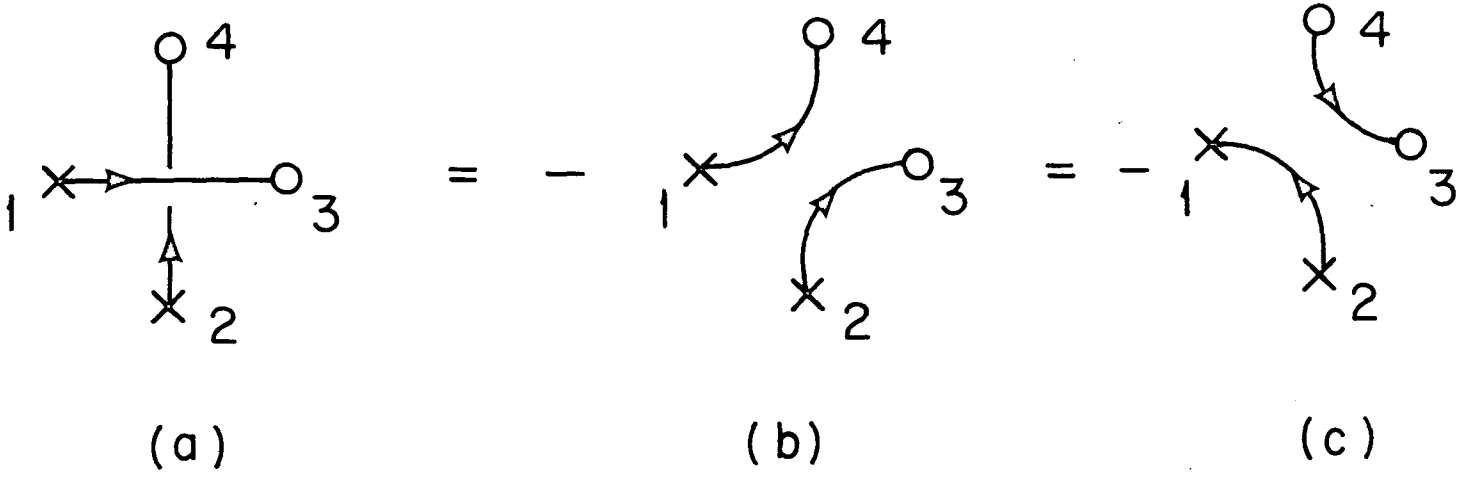

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Figure 17

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