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**Permalink**

**ISBN**

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**Publication Date**

2015-12-01

**DOI**

Peer reviewed

# Constructing distance functions and piecewise quadratic Lyapunov functions for stability of hybrid trajectories

J. J. Benjamin Biemond, W. P. Maurice H. Heemels, Ricardo G. Sanfelice and Nathan van de Wouw

*Abstract*— Characterising the distance between hybrid trajectories is crucial for solving tracking, observer design and synchronisation problems for hybrid systems with state-triggered jumps. When the Euclidean distance function is used, the so-called "peaking phenomenon" for hybrid systems arises, which forms a major obstacle as trajectories cannot be stable in the sense of Lyapunov using such a distance. Therefore, in this paper, a novel and systematic way of designing appropriate distance functions is proposed that overcomes this hurdle and enables the derivation of sufficient Lyapunov-type conditions, using minimal or maximal average dwell-time arguments, for the stability of a hybrid trajectory. A constructive design method for piecewise quadratic Lyapunov functions is presented for hybrid systems with affine flow and jump maps and a jump set that is a hyperplane. Finally, we illustrate our results with an example.

## I. INTRODUCTION

Hybrid system models combine continuous-time dynamics with discrete events or jumps and are invaluable in numerous application domains [9], [10]. Many results on hybrid systems exist and, particularly, the stability of isolated points or closed sets of hybrid systems is well understood [9], [10]. However, the stability of time-varying trajectories received significantly less attention and many issues are presently unsolved. Given the importance of stability of trajectories in, e.g., tracking control, observer design and synchronisation problems, it is important to address these open issues.

One of the main complications to study the stability of hybrid trajectories is the "peaking phenomenon," as discussed, e.g., in [14], [17], [3], [12]. "Peaking" of the Euclidean error occurs when two solutions from close initial conditions do not jump at the same time instant. If before the first jump the Euclidean error is small, then the Euclidean error approximately equals the jump distance directly after the first jump. As the amplitude of the resulting peak in the Euclidean error cannot be reduced to zero by taking closer initial conditions, trajectories of hybrid systems with state-triggered jumps are generically not asymptotically stable

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with respect to the Euclidean measure, see also Fig. 1 below. This behaviour (including the "peaking" in the Euclidean measure) might still represent desirable stability behaviour. Consequently, the Euclidean error is not a good measure to analyse when jumps are state-triggered.

Focussing on mechanical systems with unilateral position constraints, two approaches have been presented in the literature to avoid the "peaking behaviour". Firstly, the Zhuravlev-Ivanov method, cf. [5] and the related approach in [7] are proposed, where tracking control and observer problems require asymptotic stability of a set that consists of the real system and the mirrored images. As a second approach, in [8], the standard Euclidean state error is employed away from the impacts times, while near impacts, only the position error, and no velocity error is considered.

The "peaking" of the Euclidean error is partially due to the comparison of the value of two trajectories at the same continuous-time instant. Alternatively, the graphs of complete trajectories can be compared. Based on this approach, continuity of trajectories with respect to initial conditions is investigated in [6], [15]. However, analysis tools and an appropriate definition for the stability of a trajectory are hard to formulate as complete trajectories are considered. To overcome the mentioned issues, in this paper, a distance function is formulated between states of the system (as opposed to complete trajectories).

The comparison of trajectories is facilitated by a distance function that takes the jumping nature of the hybrid system into account, therewith avoiding the "peaking phenomenon." In [3], conditions on this distance function are presented such that stability in this distance function corresponds to an intuitively correct stability notion: when this distance is small, time mismatches between jumps of trajectories with close initial conditions are small, and away from the jump times, states are close. Focussing on a class of constrained mechanical systems, a similar distance function was employed in [19] to study continuity of trajectories with respect to initial conditions. Both in [3] and in [19], ad-hoc techniques were used to design the distance function.

As a first main contribution, we present a constructive and general design method for the distance function. The proposed distance function provides a good comparison between two hybrid trajectories and we show that an intuitively correct definition of asymptotic stability, with respect to the new distance, is attained. As a second contribution, sufficient conditions for asymptotic stability are presented that rely on Lyapunov functions that may increase during either flow or jump, as long as the Lyapunov function eventually decreases along solutions. For this purpose, maximal

or minimal average dwell-time arguments are employed, as proposed in the context of impulsive systems in [11]. The third contribution consists of a constructive piecewise quadratic Lyapunov function design for a class of hybrid systems where the jump map is an affine function of the state, the jump set is a hyperplane, and the continuous-time dynamics can be influenced by a bounded control input. This class of systems contains certain models of mechanical systems with unilateral constraints. Finally, the results of this paper are illustrated with an example.

**Outline:** We present the class of hybrid systems considered in Section II and design the distance function in Section III. Subsequently, the stability of trajectories is defined in Section IV. A Lyapunov theorem to study the stability of a hybrid trajectory is presented in Section V and a constructive piecewise quadratic Lyapunov function is designed in Section VI for a class of hybrid systems. Finally, an example is presented in Section VII, followed by conclusions in Section VIII.

**Notation:** Let $\mathbb{N}$ and $\mathbb{N}_{>0}$ denote the set of nonnegative and positive integers, respectively. The set $\mathbb{B} \subset \mathbb{R}^n$ is the closed unit ball. Given a map $F$ with dom $F \subseteq \mathbb{R}^n$ and a set $S \subseteq$ dom $F$, $F(S) = \{y|y \in F(x), \text{ with } x \in S\}$ denotes its image; $F(y) = \emptyset$ for $y \notin$ dom $F$, $F^k(x)$, with $x \in \mathbb{R}^n$, $k \in \mathbb{N}_{>0}$, denotes $F(F^{k-1}(x))$ and for all $x \in \mathbb{R}^n$, $F^0(x) = \{x\}$. Let $F^{-1}(S)$ denote the pre-image, namely, $F^{-1}(S) = \{x : F(x) \cap S \neq \emptyset\}$. Using Definition 1.4.11 in [2], an outer semicontinuous mapping $F : S \rightrightarrows Y$ is *proper* if for every sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ where $y_n \in F(x_n)$ and $y_n$ converges in $Y$, the sequence $\{x_n\}_{n \in \mathbb{N}}$ has a cluster point $\bar{x}$. For $n, m \in \mathbb{N}_{>0}$, let $I_n$ and $O_{mn}$ denote the identity matrix and the matrix of zeros of dimension $n \times n$ and $m \times n$, respectively, and for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $(x, y)$ denotes $(x^T, y^T)^T$. Given matrices $A, B \in \mathbb{R}^{n \times n}$, $A \succ B$ and $A \succeq B$ denote that $A, B$ are symmetric and that $A - B$ is positive definite or positive semidefinite, respectively. Given $A \in \mathbb{R}^{n \times n}$, $A \succ 0$ and $x \in \mathbb{R}^n$, $\|x\|_A^2$ denotes $x^T A x$.

## II. HYBRID SYSTEM MODEL

Consider the hybrid system

$$\dot{x} \in F(t, x) \qquad x \in C, \qquad (1a)$$
$$x^+ \in G(x) \qquad x \in D \qquad (1b)$$

with $F : [0, \infty) \times C \rightrightarrows \mathbb{R}^n$ and $G : D \rightrightarrows \mathbb{R}^n$, where $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$. We emphasize that the jump map $G$ is independent of the time $t$, which will be exploited in the design of the distance function below. In contrast to embedding continuous time as a state variable in the hybrid dynamics, we prefer to use explicit time-dependency of the flow map $F$, as this allows to study the perturbation of initial conditions without perturbing the initial time. The class of hybrid systems in the form (1) is quite general and permits modelling systems arising in many relevant applications.

To illustrate the "peaking behaviour," in Fig. 1, a reference trajectory $x_d$ and a trajectory $x$ are shown of a hybrid system with data

$$C = [0, \infty) \times \mathbb{R}, \quad D = \{0\} \times (-\infty, 0],$$
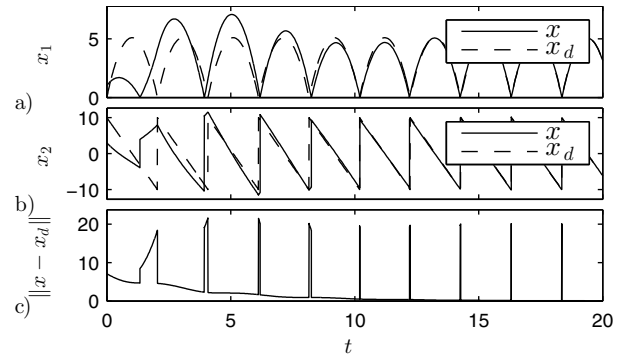$$F(t, x) = (x_2, -g + u(t)), \quad G(x) := -x, \qquad (2)$$



Fig. 1. a),b) Projection on the $t$-axis of trajectories $x$ and $x_d$ obtained for the hybrid system with data (2). c) Euclidean distance function.

with $g = 9.81$. The reference trajectory $x_d$ is generated by the hybrid system with input $u \equiv 0$ and $x_{d0}(0, 0) = (0, 10)$, while the input $u$, that generates the trajectory $x$ from initial condition $x(0, 0) = (0, 3)$, enforces convergence of $x$ to $x_d$ in the sense that when the time domain is projected onto the continuous-time axis, the graphs of both trajectories converge to each other. Indeed, the error between the jump times of both trajectories approaches zero over time, and, in addition, away from the jump times, the states of both systems approach each other. However, the Euclidean distance between the trajectories does not converge to zero, cf. Fig 1c. This "peaking phenomenon" renders the Euclidean distance not appropriate to compare these hybrid trajectories, thereby motivating this study towards systematic techniques to design proper distance functions that do converge to zero in situations as in Fig. 1.

We will propose such distance functions for systems (1) that satisfy the "hybrid basic conditions" as defined for autonomous systems in [9] adapted here to allow functions $F(t, x)$ in (1a) which depend on $t$. The conditions in Assumption 1 below are used, firstly, to employ Krasovskii-type solutions during flow, and, secondly, to enable a comparison between trajectories, as will become more clear in Theorem 1 below.

**Assumption 1** *The data of the hybrid system satisfies*
- *$C, D$ are closed subsets of $\mathbb{R}^n$ with $C \cup D \neq \emptyset$;*
- *the set-valued mapping $F(t, x)$ is non-empty for all $(t, x) \in [t_0, \infty) \times C$, measurable, and for each bounded closed set $S \subset [t_0, \infty) \times C$, there exists an almost everywhere finite function $m(t)$ such that $\|f\| \leq m(t)$ holds for all $f \in F(t, x)$ and for almost all $(t, x) \in S$;*
- *$G : D \rightrightarrows \mathbb{R}^n$ is nonempty, outer semicontinuous and locally bounded.*

We consider solutions $\varphi$ to (1) defined on hybrid time domains dom $\varphi \subset [t_0, \infty) \times \mathbb{N}$ as introduced in [9] (for $t_0 = 0$). The function $\varphi : $ dom $\varphi \to \mathbb{R}^n$ is a solution of (1) when dom $\varphi$ is a hybrid time domain, jumps satisfy (1b) and, for fixed $j \in \mathbb{N}$, the function $t \to \varphi(t, j)$ is locally absolutely continuous in $t$ and a solution to (1a). This means $\varphi(t, j) \in D$ and $\varphi(t, j+1) \in G(\varphi(t, j))$ for all $(t, j) \in$ dom $\varphi$ such that $(t, j+1) \in$ dom $\varphi$; $\varphi(t, j) \in C$, $\frac{d}{dt}\varphi(t, j) \in \bar{F}(t, \varphi(t, j))$ for almost all $t \in I_j := \{t| (t, j) \in$

dom $\varphi\}$ and all $j$ such that $I_j$ has nonempty interior. Herein, $\bar{F}(t,x) = \bigcap_{\delta \geq 0} \mathrm{co}\{F(t,(x+\delta\mathbb{B})\cap C)\}$ represents the Krasovskii-type convexification of the vector field which is restricted to $C$, cf. [18], where co denotes the closed convex hull operation. The solution $\varphi$ is said to be complete if dom $\varphi$ is unbounded. The hybrid time domain dom $\varphi$ is called unbounded in $t$-direction when for each $T \geq 0$ there exist $j$ such that $(T,j) \in$ dom $\varphi$. In this paper, we only consider maximal solutions, i.e., solutions $\varphi$ such that there are no solutions $\bar{\varphi}$ to (1) with $\varphi(t,j) = \bar{\varphi}(t,j)$ for all $(t,j) \in$ dom $\varphi$, and dom $\bar{\varphi}$ a hybrid time domain that strictly contains dom $\varphi$.

## III. Distance function design

A distance function will be presented that does not experience the "peaking behaviour" that can occur in the Euclidean distance between two trajectories of (1). We do so for hybrid systems that satisfy the following assumption.

**Assumption 2** *The data of the hybrid system* (1) *is such that $G$ is a proper function, there exists an integer $k > 0$ for which $G^k(D) \cap D = \emptyset$, and every maximal solution of* (1) *has a hybrid time domain that is unbounded in $t$-direction.*

We now formulate the distance function.

**Definition 1** *Consider the hybrid system* (1) *satisfying Assumptions 1 and 2 and let $\bar{k} > 0$ denote the minimum integer for which $G^{\bar{k}}(D) \cap D = \emptyset$. Let the distance function $d : (C \cup D)^2 \to \mathbb{R}_{\geq 0}$ be defined by*

$$d(x,y) = \inf_{z \in \mathcal{A}} \|(x,y) - z\|, \text{ with} \tag{3}$$
$$\mathcal{A} := \big\{(z_x, z_y) \in (C \cup D)^2 \big| \ \exists k_1, k_2 \in \{0,1,\ldots,\bar{k}\},$$
$$G^{k_1}(z_x) \cap G^{k_2}(z_y) \neq \emptyset \big\}. \tag{4}$$

Hence, $d$ vanishes on the set $\mathcal{A}$, which represents all pairs of states $x,y \in C \cup D$ that either are equal or that can jump onto each other by subsequent jumps characterised by (1b). The following theorem summarises particular properties of the distance function $d$ in Definition 1.

**Theorem 1** *Consider the hybrid system* (1) *satisfying Assumption 1 and 2 and let $\bar{k} > 0$ denote the minimum integer for which $G^{\bar{k}}(D) \cap D = \emptyset$. The function $d$ in Definition 1 is continuous and satisfies*

1) $d(x,y) = 0$ *if and only if there exist $k_1, k_2 \in \{0,1,\ldots,\bar{k}\}$ such that $G^{k_1}(x) \cap G^{k_2}(y) \neq \emptyset$,*
2) $\{y \in C \cup D | \ d(x,y) < \beta\}$ *is bounded for all $x \in C \cup D$, and all $\beta > 0$, and*
3) $d(x,y) = d(y,x)$ *for all $x, y \in C \cup D$.*

*In addition, the set $\mathcal{A}$ in Definition 1 is closed.*

*Proof:* The proof can be found in [4]. ∎

To illustrate that this distance function $d(x,y)$ is non-peaking, in Fig. 2, the function $d(x,y)$ is evaluated along the trajectories of Fig. 1. While this function is discontinuous in continuous-time $t$ when jumps occur, the function does converge to zero for $t \to \infty$. Hence, the depicted behaviour
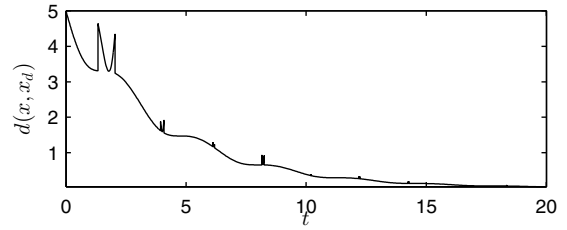


Fig. 2. Distance function $d$ in (3) evaluated along the trajectories shown in Fig. 1 of the hybrid system with data in (2).

corresponds to the intuitive observation that when the time domain is projected onto the continuous-time axis, the graphs of both trajectories converge towards each other.

## IV. Stability of hybrid trajectories

We now evaluate the distance function $d$ along two trajectories $\varphi_x$, $\varphi_y$ of (1). In order to enable the comparison of the states of two trajectories in terms of the distance $d$, following [3], we introduce an extended hybrid system, such that a combined hybrid time domain is created. Let $q = (x,y) \in (C \cup D)^2$, and

$$\dot{q} \in F_e(t,q) := (F(t,x), F(t,y)), \quad q \in C_e := C^2,$$
$$q^+ = G_e(q) := \begin{cases} (G(x),y) \text{ if } x \in D, y \in C \setminus D \\ (x, G(y)) \text{ if } x \in C \setminus D, y \in D \\ \{(G(x),y),(x,G(y))\} \text{ if } x,y \in D, \end{cases}$$
$$q \in D_e := \big\{(x,y) \in (C \cup D)^2 \big| \ x \in D \vee y \in D \big\}. \tag{5}$$

Given the initial conditions $\varphi_x(t_0,0)$ and $\varphi_y(t_0,0)$ at initial time $(t_0,0)$ for the individual trajectories $\varphi_x, \varphi_y$, respectively, we select the initial condition $\varphi_q(t_0,0) = (\varphi_x(t_0,0), \varphi_y(t_0,0))$. Solutions of this extended system generate a combined hybrid time domain. This allows to compare two trajectories of the hybrid system at every hybrid time instant $(t,j) \in$ dom $\varphi_q$. Hereto, let

$$\begin{aligned} \bar{\varphi}_x(t,j) &:= \begin{pmatrix} I_n & O_{nn} \end{pmatrix} \varphi_q(t,j), \\ \bar{\varphi}_y(t,j) &:= \begin{pmatrix} O_{nn} & I_n \end{pmatrix} \varphi_q(t,j). \end{aligned} \tag{6}$$

Given a trajectory $\varphi_x$ of (1), we say that a trajectory $(\bar{\varphi}_x, \bar{\varphi}_y)$ of (5) represents $\varphi_x$ in the first $n$ states when $\bar{\varphi}_x$ is a reparameterisation of $\varphi_x$, i.e., there exists a function $j_x : \mathbb{N} \to \mathbb{N}$ such that $\bar{\varphi}_x(t,j) = \varphi_x(t, j_x(j))$ for all $(t,j) \in$ dom $q$. Clearly, any trajectory to (5) represents $\varphi_x$ in the first $n$ states when both $\bar{\varphi}_x(t_0,0) = \varphi_x(t_0,0)$ holds and this initial condition corresponds to a unique solution to (1), as considered in [3].

The distance function defined in (3) allows to compare different points $x, y \in C \cup D$ while taking the jumping nature of the hybrid system (1) into account. Hence, following [3], system (5) allows to define stability of trajectories as follows.

**Definition 2** *Consider a hybrid system* (1) *satisfying Assumption 2 and let $d$ be given in* (3). *The trajectory $\varphi_x$ of* (1) *is said to be* stable with respect to $d$ *if for all $\epsilon > 0$ there exists $\delta > 0$ such that for every initial condition $\varphi_y(t_0,0)$ satisfying $d(\varphi_x(t_0,0), \varphi_y(t_0,0)) \leq \delta$, it holds that*

$$d(\bar{\varphi}_x(t,j), \bar{\varphi}_y(t,j)) < \epsilon \text{ for all } (t,j) \in \text{dom } \varphi_q,$$

with $\varphi_q(t,j) = (\bar{\varphi}_x(t,j), \bar{\varphi}_y(t,j))$ *being any traject-ory of the combined system* (5) *with initial condition* $(\varphi_x(t_0,0), \varphi_y(t_0,0))$ *that represents* $\varphi_x$ *in the first* $n$ *states. It is called* asymptotically stable with respect to $d$ *if it is stable with respect to* $d$ *and, for sufficiently small* $\bar{\delta} > 0$, $d(\varphi_x(t_0,0), \varphi_y(t_0,0)) \leq \bar{\delta}$ *implies* $\lim_{t+j \to \infty} d(\bar{\varphi}_x(t,j), \bar{\varphi}_y(t,j)) = 0$.

## V. Lyapunov conditions for stability of trajectories with respect to $d$

We now present sufficient conditions for stability of a trajectory of the system (1) in the sense of Definition 2, that are based on the existence of an appropriate Lyapunov function. In order to allow the Lyapunov function to increase during flow, and decrease during jumps, or vice versa, the following definitions of minimal and maximal average inter-jump time are adapted from [17].

**Definition 3** *A hybrid time domain $E$ is said to have minimal average inter-jump time $\tau > 0$ if there exists $N_0 > 0$ such that for all $(t,j) \in E$ and all $(T,J) \in E$ where $T+J \geq t+j$, it holds that $J - j \leq N_0 + \frac{T-t}{\tau}$. It has maximal average inter-jump time $\tau > 0$ if there exists $N_0 > 0$ such that for all $(t,j) \in E$ and all $(T,J) \in E$, where $T + J \geq t + j$, it holds that $J - j \geq \frac{T-t}{\tau} - N_0$. We say that a hybrid trajectory $\varphi_q$ has a minimal (or maximal) average inter-jump time if $\mathrm{dom}\ \varphi_q$ has a minimal (or maximal, respectively) average inter-jump time.*

The following theorem presents Lyapunov-based sufficient conditions for the stability of a trajectory of (1). We are interested in the stability of a trajectory, so these conditions are imposed locally near this trajectory. For this purpose, we recall that given a function $V : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ and scalar $v_L > 0$, $V^{-1}([0,v_L])$ denotes the set $\{q \in \mathbb{R}^{2n} |\ V(q) \in [0,v_L]\}$.

**Theorem 2** *Consider a hybrid system* (1) *satisfying Assumptions 1 and 2. Let $d$ be given in* (3). *The trajectory $\varphi_x$ of system* (1) *is asymptotically stable with respect to $d$ if there exist a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2$, a scalar $v_L > 0$ and scalars $\lambda_c, \lambda_d$ such that $V$ is differentiable on an open domain containing $V_L := V^{-1}([0,v_L])$ and, for all $(t,j) \in \mathrm{dom}\ \varphi_x$, it holds that*

$$\alpha_1(d(\varphi_x(t,j),y)) \leq V(\varphi_x(t,j),y) \leq \alpha_2(d(\varphi_x(t,j),y)),$$
$$\textit{for all } y \textit{ such that } (\varphi_x(t,j),y) \in C_e \cup D_e, \quad (7)$$
$$V(g) \leq e^{\lambda_d} V(q), \textit{ for all } g \in G_e(q),$$
$$\textit{and } y \textit{ such that } q = (\varphi_x(t,j),y) \in D_e \cap V_L, \quad (8)$$
$$\left\langle \left.\frac{\partial V}{\partial q}\right|_q, f \right\rangle \leq \lambda_c V(\varphi_x(t,j),y) \textit{ for all } f \in \bar{F}_e(t,q)$$
$$\textit{and } y \textit{ such that } q = (\varphi_x(t,j),y) \in C_e \cap V_L, \quad (9)$$

*and at least one of the following conditions are satisfied:*

1) $\lambda_c < 0, \lambda_d \leq 0$;
2) *all trajectories of* (1) *have minimal average inter-jump time $2\tau > 0$, $\lambda_c \leq 0$ and $\lambda_d + \lambda_c \tau < 0$;*

3) *all trajectories of* (1) *have maximal average inter-jump time $2\tau > 0$, $\lambda_d \leq 0$ and $\lambda_d + \lambda_c \tau < 0$.*

*Proof:* The proof can be found in [4]. ∎

**Remark 1** *The Lyapunov conditions in this theorem are closely related to the Lyapunov conditions used for* incre-mental stability, *see e.g.* [1], [16], [13], [20]. *In fact, if the conditions of Theorem 2 hold for any solution $\varphi_x(t,j)$ of* (1), *then they imply an incremental stability property with respect to the distance $d$. Sufficient conditions for this more restrictive/powerful system property are attained by replacing $\varphi_x(t,j)$ in* (7)-(9) *with $x$ and requiring the conditions to hold for all $(x,y) \in C_e \cup D_e$.*

## VI. Constructive Lyapunov function design for hybrid systems with affine jump map

In this section, for a specific class of hybrid systems, a piecewise quadratic Lyapunov function is presented which satisfies the requirements (7) and (8) by design. Hereby, we provide a constructive Lyapunov-based approach for the (local) stability analysis of trajectories.

For this purpose, we focus on the class of hybrid systems that have single-valued, affine, and invertible jump maps and have jump sets characterised by a hyperplane. In addition, the boundary of the flow set $C$ contains the jump set $D$ and its image $G(D)$, and the jump set $D$ is contained in a hyperplane, or a halfspace of a hyperplane. These assumptions are satisfied for a relevant class of hybrid systems, such as models of mechanical systems with impacts, see, for instance, the example in Section VII. To be precise, we focus on the class of hybrid systems given by

$$\dot{x} = f(t,x), \qquad x \in C, \qquad (10a)$$
$$x^+ = Lx + H, \qquad x \in D \qquad (10b)$$

with the function $f$ measurable in its first argument and Lipschitz in its second argument, the matrix $L \in \mathbb{R}^{n \times n}$ being invertible, and $H \in \mathbb{R}^n$. Furthermore, the sets

$$C \subseteq \{x \in \mathbb{R}^n |\ Jx + K \leq 0 \wedge$$
$$(JL^{-1}x + K - JL^{-1}H)s \leq 0\}, \quad (10c)$$
$$D := \{x \in C |\ Jx + K = 0 \wedge z_1 x + z_2 \leq 0\} \quad (10d)$$

are non-empty and closed, where the parameters $J^T, z_1^T \in \mathbb{R}^n \setminus \{0\}$, $K \in \mathbb{R}, z_2 \in \mathbb{R}$ characterise the half hyperplane containing $D$, and $s \in \{-1,1\}$ is selected such that $n_{gd} := s(L^{-1})^T J^T$ is a normal vector to $G(D)$ pointing out of $C$, cf. Fig. 3. We note that $G(D) \subset \{x \in \mathbb{R}^n |\ JL^{-1}x + K - JL^{-1}H = 0\}$ follows from the definitions of $D$ and $G$. Let $G(D) \subset C$ and let the following assumption hold.

**Assumption 3** *The data of* (10) *is such that there exist scalars $z_3, z_4, z_5 > 0$ such that*

- $z_1 x + z_2 \geq z_3$ *for all $x \in G(D)$,*
- $Jx + K < -z_4$ *for all $x \in C$ that satisfy $|z_1 x + z_2| \leq z_3$,*
- *for all $x \in C$ with $z_1 x + z_2 \leq 0$, there exists $y \in D$ such that $Jx + K \leq -z_5 \|x - y\|$,*
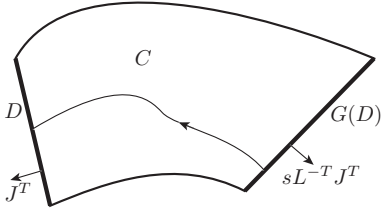- *all maximal solutions of* (10) *are complete.*

Fig. 3. Pictorial illustration of the phase space of (10) for the case $n = 2$.



Fig. 4. Dissipative mechanical system with impacts.

Note that this assumption directly implies $D \cap G(D) = \emptyset$, as $D$ and $G(D)$ are positioned at opposite sides of the hyperplane $\{x \in \mathbb{R}^n | z_1 x + z_2 = 0\}$. We observe that, with Assumption 3, all solutions to (10) have a time domain that is unbounded in $t$-direction, as, firstly, $G(D) \cap D = \emptyset$ excludes Zeno-behaviour since $D$ is closed, secondly, $G$ is linear and, thirdly, $f$ is Lipschitz in its second argument. Hence, the hybrid system (10) satisfies Assumptions 1 and 2.

In order to present a constructive Lyapunov function design, we first introduce the function $\bar{G} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\bar{G}(x) := Lx + H + M(Jx + K) + sLJ^T \max(0, z_1 x + z_2), \quad (11)$$

where the parameter $M \in \mathbb{R}^n$ is to be designed. Note that if $x \in D$, then $\bar{G}(x) = G(x) = Lx + H$.

Since $G(D) \cap D = \emptyset$, Definition 1 implies that $d(x, y) = 0$ if and only if $x = y$, or $x = G(y)$ or $y = G(x)$. To design a Lyapunov function $V$, we note that (7) requires that $V(x, y) = 0$ if and only if $d(x, y) = 0$. Hence, we propose the following piecewise quadratic Lyapunov function

$$V(x, y) = \min(\|x - y\|_{P_0}^2, \|x - \bar{G}(y)\|_{P_s}^2, \|\bar{G}(x) - y\|_{P_s}^2), \quad (12)$$

where the positive definite matrices $P_0, P_s \in \mathbb{R}^{n \times n}$ are to be designed. While this function is not smooth, we will restrict our attention to a sufficiently small sub-level set where $V$ is locally differentiable.

*Design of Lyapunov function parameters*

The following theorem allows to design the parameters $P_0, P_s$ and $M$ of the Lyapunov function $V$ in (12).

**Theorem 3** *Consider the hybrid system* (10), *let* $M \in \mathbb{R}^n$ *satisfy* $(JL^{-1}M + 1)s < 0$, *let* $P_0, P_s \succ 0$ *and let Assumption 3 hold. Consider the function* $V$ *in* (11),(12). *If for some* $\lambda_d \in \mathbb{R}$ *it holds that*

$$(L + MJ)^T P_s (L + MJ) \preceq e^{\lambda_d} P_0, \quad (13)$$

$$P_0 \preceq e^{\lambda_d} P_s, \quad (14)$$

*then there exist* $\mathcal{K}_\infty$-*functions* $\alpha_1, \alpha_2$ *and* $v_L > 0$ *such that the conditions* (7) *and* (8) *in Theorem 2 are satisfied with* $V_L = V^{-1}([0, v_L])$ *and the function* $V$ *in* (12) *is smooth on an open domain containing* $V_L$.

*Proof:* The proof can be found in [4]. ∎

This theorem provides conditions on the data of the hybrid system (10) and on the Lyapunov function such that (7) and (8) are satisfied. Additionally, (9) in Theorem 2 imposes conditions on the evolution of the Lyapunov function along flows of (1). In the following section, for an example, we present a controller des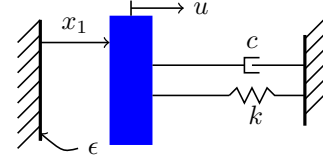ign such that these conditions are also satisfied. As shown in [4], for the class of systems considered here, such control laws can be designed in a constructive manner.

## VII. EXAMPLE

In this example, a hybrid system is considered and a control law is proposed for which a maximal dwell-time argument proves asymptotic stability of the reference trajectory, illustrating case 3) of Theorem 2. Consider a single degree-of-freedom mechanical system with a damper with damping constant $c > 0$ and a spring with stiffness $k > 0$ and unloaded position $x = \bar{x}_1 > 0$, as shown in Fig. 4. Impacts can only occur at the constraint at $x_1 = 0$ and are modelled with a restitution coefficient $\varepsilon = 0.9$. Hence, the impacts are dissipative, which allows to study the stability of the trajectory using a maximal average inter-jump time result. Assuming that no persistent contact occurs on the trajectory of which stability is studied, the hybrid system is locally described by (10) with $f(t, x) = Ax + E + Bu(t, x)$,

$A = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $E = \begin{pmatrix} 0 \\ k\bar{x}_1 \end{pmatrix}$, $L = -\varepsilon I_2$, $J = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $K = 0$, $H = 0$, $s = -1$, $z_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $z_2 = 0$ and the set $C$ is selected to exclude the origin. Herein, $u(t, x)$ represents a control law and the parameters $\bar{x}_1 = 1$, $k = 1$ and $c = 0.02$ are used.

Let the reference trajectory $x_d$ be a solution to (10) for a feedforward function $u = u_{\text{ff}}(t) = 100 \cos(\omega t)$, with $\omega = 0.4$. This forcing is selected such that the reference trajectory $x_d$ with initial condition $x_d(0, 0) = (50, 0)$ has a maximal average inter-jump time $\tau_d > 0$. In addition, $x_d(t, j)$ stays away from the origin, and simulation of nearby trajectories suggests that without control, the trajectory is unstable.

We now design a control law such that, for the closed-loop system, the conditions of Theorem 2 hold with $\lambda_d = \log(\varepsilon) < 0$ and $\lambda_c = 0$, such that accurate tracking of the trajectory $x_d$ is attained. Let $\bar{x}_d(t) := x_d(t, \min_{(t,j) \in \text{dom } x_d} j)$ and consider the control law $u(t, y) = u_{\text{ff}}(t) + u_{fb}(t, y)$ with

$$u_{\text{fb}}(t, y) = \begin{cases} 0, & (\bar{x}_d(t), y) \in S_0 \\ -\frac{1+\varepsilon}{\varepsilon}(k\bar{x}_1 + u_{\text{ff}}(t)), & (\bar{x}_d(t), y) \in S_1 \\ -(1+\varepsilon)(k\bar{x}_1 + u_{\text{ff}}(t)), & (\bar{x}_d(t), y) \in S_2 \end{cases} \quad (15)$$

with $P_0 = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $P_s = \frac{1}{\varepsilon} P_0$, $M = 0$ and the sets $S_0, S_1, S_2$ given by

$$\begin{aligned} S_0 &:= \{(x, y) \in (C \cup D)^2 | V(x, y) = \|x - y\|_{P_0}^2\}, \\ S_1 &:= \{(x, y) \in (C \cup D)^2 | V(x, y) = \|x - \bar{G}(y)\|_{P_s}^2\}, \\ S_2 &:= \{(x, y) \in (C \cup D)^2 | V(x, y) = \|\bar{G}(x) - y\|_{P_s}^2\}. \end{aligned}$$
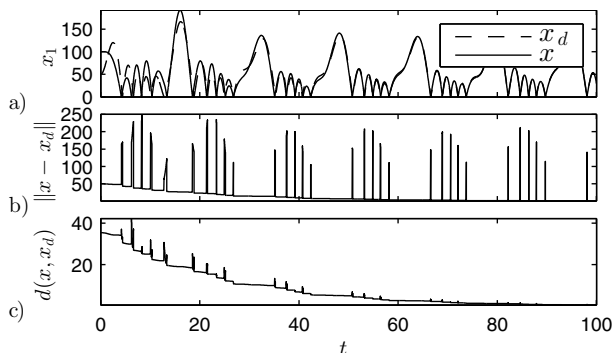
Fig. 5. a) First state of reference trajectory $x_d$ and plant trajectory $x$ for the dissipative mechanical system and periodic forcing. b) Euclidean tracking error. c) Distance function (3).

We observe that the conditions of Theorem 3 are satisfied with $\lambda_d = \log(\varepsilon) < 0$. Since $P_0 A + A^T P_0 \preceq 0$ and $P_s A + A^T P_s = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2c}{\varepsilon^2} \end{pmatrix} \preceq 0$, (9) follows with $\lambda_c = 0$.

As the trajectory $x_d$ has a maximal average inter-jump time, denoted $\tau_d$, it always crosses the set $D$ transversally. Consequently, any nearby trajectory $y$ will cross $D$ at nearby times, such that $y$ has a maximal average inter-jump time $\tau_y$ that is close to $\tau_d$ (as $(x_d, y)$ stays within a sufficiently small sublevelset of the Lyapunov function, $y$ remains close to $x_d$). Therewith, it can be shown that the trajectory of the embedded system (5) has a maximal average inter-jump time $\frac{\max(\tau_d, \tau_y)}{2} > 0$. Consequently, case 3) of Theorem 2 proves that the trajectory is asymptotically stabilised with respect to $d$ by the control law (15). In Fig. 5, the performance of this controller is illustrated with a trajectory with initial condition $x(0, 0) = (100, 0)$. Despite the fact that the Euclidian distance does not converge to zero (Fig. 5.b), the distance function d does (Fig. 5.c) and the graphs of the trajectories indeed converge to each other (Fig. 5.a).

From (15), we observe that no control is active when $V(\bar{\varphi}_y(t, j), x_d(t, j)) = \|\varphi_y(t, j) - x_d(t, j)\|_{P_0}^2$. The control input $u$ only needs to compensate the potentially destabilising effect of the forcing term $(E + Bu_{\text{ff}}(t))$ during the "peaks" of the Euclidean error.

## VIII. CONCLUSION

In this paper, we considered the asymptotic stability of time-varying and jumping trajectories of hybrid systems with state-triggered jumps. A general distance function design was proposed that allows to compare two trajectories of a hybrid system, thereby enabling the stability analysis for hybrid trajectories. Sufficient conditions for stability have been formulated using Lyapunov functions with sub-level sets that consist of disconnected pieces. Moreover, the stability conditions are formulated in terms of maximum or minimum average inter-jump time conditions to allow for increase of the Lyapunov function over flow or jumps, respectively. In case the jump map is an affine function and the jump set a hyperplane, a piecewise quadratic Lyapunov function was proposed that can be constructed systematically. Finally, we applied our results in an example that illustrated that the presented asymptotic stability notion indeed corresponds to desired tracking behaviour. As such, the proposed distance function and Lyapunov function design enable a good comparison between hybrid trajectories and have the potential to play an important role in tracking control, observer design and synchronisation problems for hybrid systems.

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