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Stability Properties in Ring Theory

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by

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The dissertation of John Dominic Farina is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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To my parents.

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ABSTRACT OF THE DISSERTATION

Stability Properties in Ring Theory

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Professor Lance Small, Chair

This thesis is primarily concerned with the behavior of various ring-theoretic properties under base field extension, and in particular with algebras for which such properties are preserved upon extension of scalars. We begin with an investigation of chain conditions on one-sided ideals, and specifically with a new class of rings which we have christened *stably noetherian*. This is a property which, though fairly natural, has been largely neglected until now. It is a mild enough restriction to encompass many important classes of noncommutative algebras, and yet it is sufficiently restrictive to allow one to prove (or improve upon) theorems of some interest. The first two parts of this thesis make this last statement precise. In the third section we study *just infinite* rings, which are infinite dimensional algebras for which all homomorphic images are finite dimensional. We prove some new structure-theoretic results, and then investigate the behavior of this property under scalar extension.

1 Introduction

We consider algebras over a field k, and all our rings will have an identity and thus contain the field k. In fact, for the most part we use the terms "ring" and "algebra" interchangeably. Since in this thesis (almost) all rings are k-algebras, we hope that this will not cause undue confusion. On those rare occasions when we deal with rings which are not k-algebras we are careful to point this out to the reader.

Modules and ring homomorphisms, unless otherwise stated, will be unital, and subrings will share the same identity element. We declare once and for all our preference to work "on the right," and accordingly functions will be written on the left, so that the condition for a function to be a module homomorphism resembles the associative law. We use the term *affine* to describe algebras which are finitely generated *as algebras.* The term *finitely generated* is reserved for groups, (bi)modules, field extensions, and division algebras. In addition to its ordinary usage as a partial order on $\mathbb{R}$, the symbol $\leq$ denotes a (not necessarily proper) subobject in the relevant category, which should be clear from context. Moreover, the symbol $\otimes$, unadorned, will stand for $\otimes_k$.

We are concerned primarily with tensor products, and in particular, with the behavior of various ring-theoretic properties upon extension of scalars by field extensions. Much of what we prove can actually be generalized to the case of algebras, with or without 1, over more general commutative rings, but the theorems and proofs tend to gain a certain economy when the base ring is a field. For example, modules over a field are automatically free, which often makes statements about tensor products less cumbersome. While tensor products over more general commutative rings can be useful, the marginal increase in generality is often outweighed by a disproportionate increase in headaches. And despite its relative simplicity, the theory of tensor products of algebras over fields is still not fully understood.

This thesis is organized as follows: Chapter 2 deals with the noetherian property in relation to scalar extension. We introduce a new class of algebras, which we refer to as *stably noetherian,* and study some of their basic properties. The impetus for much of this work, and perhaps still the central idea in the subject, is a theorem of Vámos [60] which shows that a field extension L/k is stably noetherian iff L is a finitely generated field extension of k (see Theorem 2.24). One would like to generalize this theorem as far as possible to the case of noncommutative algebras, and while there are partial results in the literature, (see [59] and [48] in particular), some of the most fundamental questions remain unanswered.

The philosophy of Chapter 2 is that most noetherian rings that one encounters "in nature" are in fact stably noetherian. For example, we show that rings resembling commutative polynomial rings are almost always stably noetherian. To wit, skew polynomial rings, skew Laurent polynomial rings, and finite almost normalizing extensions of stably noetherian rings are all stably noetherian. Thus, many important examples of noetherian rings, such as Weyl algebras and their quotient division rings, coordinate rings of quantum planes and quantum tori, Sklyanin algebras, twisted homogeneous coordinate rings, and enveloping algebras of finite dimensional Lie algebras are all stably noetherian. A theorem of De Jong, see [5, Theorem 5.1], shows that most "reasonable" graded noetherian rings are stably noetherian, and Bell [8] has recently shown that a similar result holds in the ungraded case. In fact, aside from a few easy constructions, in particular power series rings like $k[[x]]$ and their ilk, one must usually expend some effort to produce noetherian rings which are not stably noetherian.

What's more, the class of stably noetherian rings is not so large that good results become impossible to obtain. In Chapter 3 we study embedding problems for algebras which satisfy a polynomial identity. Specifically, if A is a PI k-algebra, one would like to find necessary and sufficient conditions for A to embed in matrices over some field extension of k. Considerable work on this problem has been done over the past 40 years, leading to the following conjecture: "Is there always such an embedding provided that A is right noetherian?" While we remain unable to answer this question in full generality, we show that a positive solution can be obtained for stably right noetherian rings. To date, the best known result in this direction was a theorem of Ananin [4], which we improve upon in Theorem 3.23.

In Chapter 4 we switch gears and study a generalization of the class of simple algebras. We allow for the presence of nonzero proper two-sided ideals, but we insist that they have finite codimension. Such rings are called *just infinite* [1]. The literature on just infinite rings is still rather sparse, ([6], [17], [18], [43], [55], [46], [61]), but there is a recent increase in interest. With C. Pendergrass [17], we have recently proved some interesting structure theoretic results about these rings; that all such rings are prime (Proposition 4.5), and that the PI case is essentially subsumed by the commutative case (Proposition 4.7). We first review these results, and then turn our attention to the case of just infinite rings which satisfy no polynomial identity. Here the picture is less clear. We show that in the non-PI case the center is (or more precisely can be assumed to be) reduced to scalars, and then we begin the investigation of how the class of just infinite algebras behaves with respect to extension of scalars. Pleasing results are obtained by employing a construction of Martindale [39], though it should be noted that many of the most interesting problems in this area remain unsolved, and indeed one aim of this thesis is to function as something of an advertisement for the subject.

[1] The idea comes from group theory, where the notion of a just infinite pro-finite p-group (meaning an infinite pro-finite p-group for which all nontrivial normal subgroups have finite index) has proved useful. See, eg. [63]

2 Stably Noetherian Rings

In this chapter we are interested primarily in the noetherian property and how it behaves with respect to extension of scalars. One of the more well-developed ideas in this direction is the notion of a *strongly right noetherian* algebra. These are right noetherian k-algebras A for which $A \otimes_k C$ is again right noetherian for an arbitrary commutative noetherian algebra C. An extremely useful attribute of strongly noetherian rings is the fact that they satisfy *generic flatness*, and thus the Nullstellensatz (see [5]).

A strictly weaker condition, which as yet has been largely overlooked in the literature, is that of being *stably right noetherian.* These are right noetherian rings which remain right noetherian upon extension of scalars by arbitrary (commutative) fields. Of course, every strongly right noetherian ring is stably right noetherian, but the converse is false. In [49], Rogalski constructs examples of affine, connected, $\mathbb{N}$-graded, stably right noetherian algebras over algebraically closed fields which fail to be strongly right noetherian. In short:

$$\{\text{strongly noetherian rings}\} \subsetneq \{\text{stably noetherian rings}\}.$$

We begin with a formal definition.

Definition 2.1. A right noetherian k-algebra A is called *stably right noetherian over k* if $A \otimes_k K$ is right noetherian for every field extension K of k.

Similarly, we say that a left noetherian k-algebra A is *stably left noetherian over k* if A remains left noetherian upon extension of scalars by an arbitrary field extension K of k. A is called *stably noetherian over k* if it is both stably right noetherian and stably left noetherian over k. We shall refer to such algebras simply as stably right noetherian (resp. stably left noetherian, resp. stably noetherian) if there is no risk of confusion about the ground field k.

Since we have agreed to work on the right, we will usually state and prove results about stably right noetherian rings. Of course, an algebra A is stably right noetherian iff A^{op} is stably left noetherian, and all of the theorems we present in this chapter have left-handed analogs. In fact, most examples of interest to us have enough left-right symmetry to guarantee the stably noetherian property holds on both sides, but to avoid unwieldy hypotheses, we will usually state only right-handed versions.

In addition to rings, one may also be interested in studying how modules behave with respect to scalar extension, and so we require a definition for modules as well.

Definition 2.2. Let A be a k-algebra and let M be a noetherian right A-module. M is called *stably noetherian* if the right $A\otimes_k K$-module $M\otimes_k K$ is right noetherian, for all field extensions K of k.

Note that A is stably right noetherian iff A_A is a stably noetherian right A-module, so this definition agrees with our previous one. More generally, every finitely generated right module over a stably right noetherian ring is stably noetherian:

Proposition 2.3. *Let A be a stably right noetherian k-algebra and let M_A be a finitely generated right A-module. Then M is stably noetherian.*

Proof. If M_A is generated by $\{m_1, \ldots, m_n\}$, then $M \otimes_k K$ is generated as a right $A \otimes_k K$-module by $\{m_1 \otimes 1, \ldots, m_n \otimes 1\}$. Thus $M \otimes_k K$ is a finitely generated module over a right noetherian ring, and is thus noetherian. □

Proposition 2.4. *If M is a stably noetherian right A-module, then every submodule and homomorphic image of M is stably noetherian.*

Proof. Suppose M is stably noetherian. If $N \leq M$, then submodules of $N\otimes_k K$ are, a fortiori, submodules of $M \otimes_k K$, so N is stably noetherian. Also, $M/N \otimes_k K \cong (M \otimes_k K)/(N \otimes_k K)$ as right $A \otimes_k K$-modules, from which it follows that M/N is also stably noetherian. □

Let $N \leq M$ be right A-modules. The noetherian property passes to submodules and homomorphic images, and so if M is noetherian, then N and M/N are also noetherian. The converse of this statement is a very useful result which is referred to as *noetherian induction*, the proof of which can be found in any introductory algebra text e.g [33] or [52].

Proposition 2.5. *Suppose $N \leq M$ are right A-modules. M is noetherian iff N and M/N are noetherian.*

The utility of this result lies in the following observation: If we wish to prove some proposition about a noetherian module M, we may, by the noetherian condition, choose a submodule $N \leq M$, which is maximal in the sense that M/N does not satisfy the proposition. By replacing M by M/N we may then assume, by way of a contradiction, that all homomorphic images of M have the property in question. This technique is used so frequently in ring theory that it seems fitting to extend it to the case of stably noetherian modules. This is the next

Proposition 2.6. *Let $N \leq M$ be right A-modules. Then M is stably noetherian iff N and M/N are stably noetherian.*

Proof. Note that $(M \otimes_k K)/(N \otimes_k K) \cong M/N \otimes_k K$ as right $A \otimes_k K$-modules, and so the theorem follows from ordinary noetherian induction. □

2.1 Some Homological Remarks

Given a pair of arbitrary algebras $A \subseteq B$, there is often no way of transferring salient properties of A to B or vice versa. However, in certain favorable situations one can pass features of one ring to the other. For example, if B is a matrix ring over A, then the (one and two-sided) ideal structure of B can be well understood from that of A. In the case of scalar extension, there is a rather tight connection between spec(A) and spec$(A \otimes_k K)$, the details of which we remind the reader of now.

Definition 2.7. Let $A \subseteq B$ be k-algebras. Given $P_1 \subseteq P_2$ in spec(A), define the following possible situations:

1. $\mathrm{LO}(P_1)$ = "lying over" means there is $P' \in \mathrm{spec}(B)$ with $P' \cap A = P_1$.

2. $\mathrm{GU}(P_1, P_2)$ = "going up" means given any $P_1' \in \mathrm{spec}(B)$ lying over P_1 there is a P_2' lying over P_2 such that $P_1' \subseteq P_2'$.

3. $\mathrm{INC}(P_1)$ = "incomparability" means that one cannot have $P_1' \subsetneq P_1''$ in $\mathrm{spec}(B)$ each lying over P_1.

4. We say that LO, GU, or INC holds from $A \subseteq B$ if $\mathrm{LO}(P)$, $\mathrm{GU}(-, P)$, or $\mathrm{INC}(P)$ holds for all $P \in \mathrm{spec}(A)$, respectively.

Proposition 2.8 ([52, 2.12.50 & 3.4.13])**.** *For every field extension K/k, $A \otimes_k K$ satisfies* LO *and* GU *over A. Moreover, if K/k is an algebraic extension, then $A \otimes_k K$ also satisfies* INC *over A.*

Remark 2.9. $A \otimes_k K$ is generated, as an A-module, by elements (of K) which centralize A in $A \otimes_k K$. Thus if I is a two-sided ideal of A, then $I \otimes_k K := I(A \otimes_k K)$ is a two-sided ideal of $A \otimes_k K$. It follows that $\mathrm{PrimeRad}(A) = A \cap \mathrm{PrimeRad}(A \otimes_k K)$. (Let $N' = \mathrm{PrimeRad}(A \otimes_k K)$. Then

$$A \cap N' = \bigcap \{ A \cap P' \mid P' \in \mathrm{spec}(A) \} \subseteq \bigcap \{ P \in \mathrm{spec}(A) \},$$

and in fact equality holds since LO is satisfied.)

This remark proves the first part of the next

Proposition 2.10 ([52, 2.12.52])**.** *Let $N = \mathrm{PrimeRad}(A)$ denote the prime radical of A and set $N' = \mathrm{PrimeRad}(A \otimes_k K)$. Then $N = A \cap N'$, and if $\mathrm{char}(k) = 0$ we also have $N' = NK$.*

Corollary 2.11 ([52, 2.12.53])**.** *Suppose that $\mathrm{char}(k) = 0$. If A is semiprime then $A \otimes_k K$ semiprime.*

One might hope that a similar result might hold for prime rings, but in general this is not the case. Note that in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, we have $(i \otimes i)^2 = 1$, and so this ring isn't a domain. This can be avoided if we agree to work over algebraically closed fields: If k is algebraically closed and A is a domain, then $A \otimes_k K$ is also a domain (Proposition 4.13). Moreover if A is prime, then $A \otimes_k K$ is again prime (Proposition 4.15).

The next two lemmas are standard, and can be found in [51, p. 106-107].

Lemma 2.12. *Let A be a k-algebra, let M be a finitely presented right A-module, and let N be any right $A \otimes_k K$-module. Then there is an isomorphism*

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{A \otimes_k K}(M \otimes_k K, N).$$

Lemma 2.13. *Let A be a k-algebra, let M be a finitely presented right A-module, and let N be any right A-module. Then there is an isomorphism*

$$\mathrm{Hom}_A(M, N) \otimes_k K \cong \mathrm{Hom}_A(M, N \otimes K).$$

An immediate application of the two preceding lemmas is the following proposition. This is also standard, but we reproduce the proof for the convenience of the reader.

Proposition 2.14. *Let A be any k-algebra and let M be a finitely presented right A-module. Then there is a ring isomorphism*

$$\mathrm{End}_A(M) \otimes_k K \cong \mathrm{End}_{A \otimes_k K}(M \otimes_k K).$$

Proof. We apply the previous two lemmas:

$$\begin{aligned} \mathrm{End}_A(M) \otimes_k K &\cong \mathrm{Hom}_A(M, M \otimes K) \\ &\cong \mathrm{Hom}_{A \otimes K}(M \otimes K, M \otimes K) \\ &= \mathrm{End}_{A \otimes K}(M \otimes K). \end{aligned}$$

□

Proposition 2.15. *Let A be a k-algebra, and let K be a field extension of k. If $A \otimes_k K$ is algebraic over K, then A is algebraic over k.*

Proof. Suppose that $t \in A$ is transcendental over k. So A contains $k[t]$ as a subring, and since ${}_kK$ is free and hence flat, we have $k[t] \otimes_k K \subseteq A \otimes_k K$. However, $k[t] \otimes_k K \cong K[t]$, and so the set $\{ t^n \otimes 1 \mid n \geq 0 \}$ is K-linearly independent. □

2.2 First Properties

Since the literature on stably noetherian rings is virtually nonexistent, this is a convenient time to set down some of their basic properties. Most of the proofs in this section are quite easy.

Proposition 2.16. *Let A be a stably right noetherian k-algebra, and let I be an ideal of A. Then A/I is stably right noetherian.*

Proof. Note that $A/I \otimes_k K \cong (A \otimes_k K)/(I \otimes_k K)$ as k-algebras, and since $A \otimes_k K$ is right noetherian, so is any homomorphic image. □

Stably right noetherian rings also allow for a certain amount of flexibility in what one chooses to call the ground field. This is the content of the next two propositions.

Proposition 2.17. *Let k and F be fields with $k \subseteq F \subseteq Z(A)$. If A is stably right noetherian over k, then A is stably right noetherian over F.*

Proof. We have a ring surjection $A \otimes_k K \twoheadrightarrow A \otimes_F K$, so if the former is right noetherian, so too is the latter. □

Of course, the converse is not true. The easiest example being to take $A = F$ to be any infinitely generated field extension of k. However, a proof of this requires an application of Vámos' theorem (Theorem 2.24). On the other hand, if the field extension F/k is finitely generated, then the converse does hold (see Proposition 2.29).

One might like to know what happens with finite direct sums of algebras. As in the noetherian case, the answer is as expected.

Proposition 2.18. *$A := A_1 \oplus \cdots \oplus A_n$ is stably right noetherian iff each A_i is stably right noetherian.*

Proof. Since A_i is a homomorphic image of the direct sum, one direction is obvious. Suppose now that each A_i is stably right noetherian. Tensor products commute with direct sums, and so

$$A \otimes_k K \cong \bigoplus_{i=1}^{n} (A_i \otimes_k K)$$

is right noetherian. □

In fact, the previous result generalizes to finite subdirect products. We recall the following

Definition 2.19. We call A a *subdirect product* of a family of algebras $\{A_i\}$ if there is a ring injection $A \hookrightarrow \prod_i A_i$ such that the image of A surjects onto each A_i under the natural projection maps. An alternate definition is to say that A is a subdirect product of a family of homomorphic images $\{A/I_i\}$ iff $\bigcap_i I_i = 0$, which amounts to the same thing.

The relevant fact about subdirect products is that a finite subdirect product of right noetherian rings is right noetherian. To see this, suppose that A is a finite subdirect product of $\{A/I_j\}$, with each A/I_j right noetherian. Then each A/I_j is noetherian as a right A-module, and so the product $\prod_j A/I_j$, and hence A, is a noetherian right A-module as well.

Remark 2.20. Since K is flat over k, the functor $- \otimes_k K$ preserves kernels. That is, if

$$\varphi\colon M \to N$$

is a map of k-modules, then we get a map on the tensor product

$$\varphi \otimes \mathrm{id}\colon M \otimes_k K \to N \otimes_k K$$

with the property that $\ker(\varphi \otimes 1) = \ker(\varphi) \otimes K$.

As an application of this remark, we can show that a (finite) subdirect product of stably right noetherian rings is stably right noetherian.

Proposition 2.21. *Let A be a subdirect product of $A_1, \ldots, A_n$, then A is stably right noetherian iff each A_i is stably right noetherian.*

Proof. We have $A_j = A/I_j$ where $I_j \lhd A$ and $\bigcap_j I_j = 0$. It is clear that each A_j is stably right noetherian provided that A is. For the other direction, suppose that A_j is stably right noetherian for all j and let φ denote the map $A \to \oplus A/I_j$. Since tensoring with K preserves kernels, we have

$$\bigcap_j (I_j \otimes_k K) = \ker(\varphi \otimes \mathrm{id}) = \ker(\varphi) \otimes_k K = (\bigcap_j I_j) \otimes_k K$$

and the latter is 0. Thus $A \otimes_k K$ is a subdirect product of the family $\{A_j \otimes_k K\}$, and since each $A_j \otimes_k K$ is right noetherian, so too is $A \otimes_k K$. □

This can't be extended to infinite subdirect products, as even infinite direct products of noetherian rings needn't be noetherian.

Proposition 2.22. *Let $A \subseteq B$ be k-algebras such that B is a finitely generated right A-module. If A is stably right noetherian, then B is stably right noetherian.*

Proof. Since B is a finitely generated right A-module, $B \otimes_k K$ is a finitely generated right $A \otimes_k K$-module for any field extension K/k. Thus $B \otimes_k K$ is noetherian as a right $A \otimes_k K$-module, and so also as a ring. □

Recall that two rings are called *Morita equivalent* if their right module categories are equivalent. Any ring-theoretic property which is preserved under Morita equivalence is called a *Morita invariant*. (See [40] for more background). Perhaps not surprisingly, the stably right noetherian property is in fact a Morita invariant, so being stably right noetherian is really a property of the right module category. By [40, Proposition 3.5.6], in order to show that it is a Morita invariant it suffices to prove the following

Proposition 2.23. *Let A be a stably right noetherian k-algebra, and let $e^2 = e$ be an idempotent element in A. Then*

1. *eAe is stably right noetherian.*

2. *For any natural number n, $M_n(A)$ is stably right noetherian.*

Proof.

1. Note that the image of e in $A \otimes_k K$, $e \otimes 1$, is idempotent, and we have a ring isomorphism $eAe \otimes_k K \cong (e \otimes 1)(A \otimes_k K)(e \otimes 1)$, and the latter is a corner in a right noetherian ring, hence right noetherian.

2. $M_n(A)$ is a finitely generated right module over A, and so the result follows from Proposition 2.22.

□

2.2.1 Some Examples

Perhaps the most obvious examples of stably noetherian algebras are finite dimensional algebras. Since $\dim_K(A \otimes_k K) = \dim_k(A)$, if A is finite dimensional over k, then $A \otimes_k K$ is finite dimensional over K, and hence noetherian. Of course, if all stably noetherian algebras were finite dimensional, it wouldn't be a terribly interesting class of rings to study (from a structure-theoretic point of view at least). Fortunately, stably noetherian algebras are plentiful, and as this is a relatively new definition, we aim to give as many examples of stably noetherian algebras as possible.

We begin our list of examples with commutative stably noetherian rings. If $k[x_1, \ldots, x_n]$ is a polynomial ring in finitely many commuting indeterminates, then it is clear that upon extending scalars we get

$$k[x_1, \ldots, x_n] \otimes_k K \cong K[x_1, \ldots, x_n]$$

which is again a polynomial ring. It then follows from Proposition 2.16 that all affine commutative rings are stably noetherian. Small [57] has generalized this to show that affine right noetherian PI algebras are stably right noetherian. See Theorem 2.51 for the proof.

Apart from affine commutative rings, perhaps the first thing that one might like to know is when a field extension of k is stably noetherian. This question was solved by Vámos in [60], and as it marks the historical beginning of the subject, we include his proof next.

Theorem 2.24 (Vámos). *If L is a field containing k, then L is stably noetherian over k iff L is a finitely generated field extension of k.*

Proof. Suppose first that $L = k(\alpha_1, \ldots, \alpha_n)$. In this case L is a localization of $k[\alpha_1, \ldots, \alpha_n]$, which is an affine commutative k-algebra and hence stably noetherian over k. On the other hand, suppose that L is infinitely generated, and choose a countably generated subfield $L := k(\alpha_1, \alpha_2, \ldots)$ of L. We claim that $L \otimes_k L$ is not noetherian. For each $n \in \mathbb{N}$, let k_n denote the field $k(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We then have a strictly increasing infinite chain of subfields of L

$$k \subsetneq k_1 \subsetneq k_2 \subsetneq \cdots .$$

Consider the map

$$\varphi_n \colon L \otimes_{k_n} L \to L \otimes_{k_{n+1}} L.$$

If x is any element of k_{n+1} which is not in k_n, then $1 \otimes x - x \otimes 1$ is a nonzero element of $L \otimes_{k_n} L$ which is in the kernel of φ_n, thus if I_n denotes the kernel of the map $L \otimes_k L \to L \otimes_{k_n} L$, then

$$I_1 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$$

is a strictly increasing chain of ideals in $L \otimes_k L$. □

Examples of non-affine commutative stably noetherian rings can be obtained by localization, vis. Proposition 2.26. Another, very different way of constructing non-affine commutative stably noetherian rings is via infinite blowups of affine space. We turn to these now.

2.2.2 Infinite Blowups

Infinite blowing-up is a way of building commutative stably noetherian domains (in fact, unique factorization domains) which are rather far from being affine, in the sense that they cannot be realized as localizations of affine rings. We will carry out the construction in dimension 2, but a similar construction works for commutative algebras of arbitrary (finite) Krull dimension.

Let $A_0 = k[x, y]$ be a commutative polynomial ring, and choose a sequence of points $\{ d_n \mid n \in \mathbb{N} \} \subseteq k^2$. Write $d_n = (a_n, b_n)$. The idea is to construct a sequence of rings by iteratively adjoining certain elements.

Set $y_1 = (y - b_1)(x - a_1)^{-1}$, and assuming that y_{n-1} has already been defined, set $y_n = (y_{n-1} - b_n)(x - a_n)^{-1}$. We will now define a sequence of algebras inductively as follows: Let $A_1 = A_0[(y - b_1)(x - a_1)^{-1}]$ and, supposing that we have already constructed A_{n-1}, set

$$A_n = A_{n-1}[(y_{n-1} - b_n)(x - a_n)^{-1}].$$

Note that A_n is in fact isomorphic to a polynomial ring in 2 variables over k, namely $k[x, y_n]$. We illustrate the whole situation diagrammatically:

$$\begin{aligned}
A_0 &= k[x, y] \\
A_1 &= A_0[\underbrace{(y - b_1)(x - a_1)^{-1}}_{y_1}] = k[x, y_1] \\
A_2 &= A_1[\underbrace{(y_1 - b_2)(x - a_2)^{-1}}_{y_2}] = k[x, y_2] \\
A_3 &= A_2[\underbrace{(y_2 - b_3)(x - a_3)^{-1}}_{y_3}] = k[x, y_3] \\
&\vdots \\
A_n &= A_{n-1}[\underbrace{(y_{n-1} - b_n)(x - a_n)^{-1}}_{y_n}] = k[x, y_n]
\end{aligned}$$

This process yields an infinite strictly increasing chain of commutative integral domains

$$A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots ,$$

and we may form the union $A = \bigcup_n A_n$. We first claim that A is not a localization of any affine subalgebra, and so in a sense the algebra A is rather far removed from being affine.

Lemma 2.25. *A is not a localization of any affine subalgebra.*

Proof. Suppose that R is an affine subalgebra of A with $A = RS^{-1}$, $S \subseteq R - \{0\}$. Then $R \subseteq A_n$ for some n, and so $A = A_n S^{-1}$. We claim that this is impossible.

Note that in the above notation, $A_n = k[x, y_n]$, is a polynomial ring, so its group of units is just $k^\times$. Since A is constructed as the union of the A_n, the group of units of A is also just $k^\times$. However, were it the case that $A = A_n S^{-1}$, then the group of units of A would be strictly larger than that of A_n. □

To determine when the infinite blowup constructed above is stably noetherian, we need a few preliminary remarks. We call the set of points $\{d_n\} \subseteq k^2$ *critically dense* if there is no polynomial $f \in k[x, y]$ such that $f(d_n) = 0$ for infinitely many n. By [5, Theorem 1.5], the ring A is noetherian iff the sequence of points $\{d_n\}$ is a critically dense subset of k^2.

Now we are prepared to choose a sequence of points $\{d_n\}$ which will guarantee that A is stably noetherian. Choose p and q in k which are algebraically independent over the prime subfield of k, and set $d_n = (p^n, q^n)$. Rogalski has shown, [49, Theorem 12.3], that this set of points $\{d_n\}$ is critically dense in k^2. If K/k is a field extension, then since the prime subfields of K and k are the same, the set $\{d_n\}$ is also critically dense in K^2 by the same argument. Finally, since tensor products commute with direct limits, we have

$$A \otimes_k K = (\bigcup_n A_n) \otimes_k K \cong \bigcup_n (A_n \otimes_k K) = \bigcup_n K[x, y_n]$$

is an infinite affine blowup over K, so is again noetherian by [5, Theorem 1.5].

In order to give further examples of stably right noetherian algebras, we first need to prove some actual theorems.

2.3 Lifting Results

In this section we investigate sufficient conditions on a pair of k-algebras $A \subseteq B$ under which the stably right noetherian condition can be passed from A to B. When $A \subseteq B$ are k-algebras, we will refer to B as a *ring extension* of A. We show that most of the usual ring-theoretic constructions respect the stably right noetherian property. In particular, finite module extensions, iterated Ore extensions, and localizations of stably right noetherian rings are again stably right noetherian. As a consequence, we will greatly expand our stockpile of examples of stably right noetherian rings to include Weyl algebras, Zhang twists of polynomial rings [64], Sklyanin algebras, stratiform simple artinian rings of finite stratiform length [54], enveloping algebras of finite dimensional lie algebras, and all constructible algebras.

The stably right noetherian property is well-behaved with respect to localization in the sense that any localization of a stably right noetherian ring is again stably right noetherian.

Proposition 2.26. *Let A be a stably right noetherian k-algebra and let $S \subseteq A$ be a right Ore set. Then AS^{-1} is stably right noetherian.*

Proof. Denote the image of S inside $A \otimes_k K$ by $S \otimes_k 1$. Then $S \otimes_k 1$ is a right Ore set and $AS^{-1} \otimes_k K \cong (A \otimes_k K)(S \otimes_k 1)^{-1}$. Since a localization of a right noetherian ring by a right Ore set is again right noetherian, the result follows. □

Proposition 2.27. *Let $B = A[x_1, \ldots, x_n]$ be a polynomial extension of A. Then B is stably right noetherian iff A is.*

Proof. If K is a field extension of k, then

$$A[x_1, \ldots, x_n] \otimes_k K \cong (A \otimes_k K)[x_1 \otimes 1, \ldots, x_n \otimes 1].$$

Thus $B \otimes_k K$ is a polynomial ring over $A \otimes_k K$, and the result follows from the Hilbert basis theorem. □

In fact essentially the same proof works for iterated Ore extensions as well, but first we should remind the reader of the pertinent definitions.

Let $\sigma \colon A \to A$ be a k-algebra automorphism, and let $\delta \colon A \to A$ be a (left) σ-derivation. This means that δ is an additive map satisfying the additional property that for all $r, s \in A$, we have

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s.$$

Fix an indeterminate x and consider the free left A-module with basis $\{1, x, x^2, \ldots\}$. We wish to turn this module into a ring extension of A, which we will denote by $A[x; \sigma, \delta]$. We adopt the convention that polynomials are to be written with coefficients on the left. So $A[x; \sigma, \delta]$ consists of all finite sums of the form $\sum a_i x^i$, and two such sums are equal iff all of their coefficients agree (this is the same as saying that the above module is free with basis $\{1, x, x^2, \ldots\}$). Addition in $A[x; \sigma, \delta]$ is defined in the usual way,

$$(\sum_i a_i x^i) + (\sum_i a_i' x^i) = \sum_i (a_i + a_i')x^i,$$

and multiplication is defined via enforcing the rule

$$xa = \sigma(a)x + \delta(a), \quad a \in A$$

and extending linearly. The resulting k-algbera $A[x; \sigma, \delta]$ is called a *skew polynomial ring* (in the variable x) with coefficients in A. In case the derivation δ is absent, we can also form the *skew Laurent extension* $A[x, x^{-1}; \sigma]$, which is just a (right) localization of the skew polynomial ring $A[x; \sigma]$ by the multiplicatively closed subset generated by x. We can iterate these constructions to form *iterated skew polynomial rings* (and *iterated skew Laurent extensions*) of the form

$$A[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n],$$

and

$$A[x_1, x_1^{-1}; \sigma_1][x_2, x_2^{-1}; \sigma_2] \cdots [x_n, x_n^{-1}; \sigma_n]$$

where each σ_i is a k-algebra automorphism of the ring

$$A[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}],$$

or in the skew Laurent extension case, of the ring

$$A[x_1, x_1^{-1}; \sigma_1] \cdots [x_{i-1}, x_{i-1}^{-1}; \sigma_{i-1}]$$

and δ_i is a (left) σ_i-derivation.

We use the term *Ore extension of A* to refer to algebras of the form $A[x; \sigma, \delta]$ or $A[x, x^{-1}; \sigma]$. One can of course iterate this construction finitely many times as well, in which case the resulting algebra is called a *(finite) iterated Ore extension* of A. The useful fact to note about these rings is that Hilbert's original proof of his famous "basis theorem" goes through essentially unchanged. That is, if A is right noetherian and B is a (finite) iterated Ore extension of A, then B is right noetherian. As a consequence, the analogous result is also true for stably right noetherian algebras.

Proposition 2.28. *Let B be an Ore extension of A of the form $A[x;\sigma,\delta]$ or $A[x,x^{-1};\sigma]$. Then A is stably right noetherian iff B is.*

Proof. We will do the proof for Ore extensions of the first kind. The other case follows from this by Proposition 2.26 because $A[x,x^{-1};\sigma]$ is a localization of $A[x;\sigma]$.

If K is any field extension of k, then

$$A[x;\sigma,\delta]\otimes_k K \cong (A\otimes_k K)[x\otimes 1;\sigma\otimes \mathrm{id},\delta\otimes \mathrm{id}],$$

where id denotes the identity transformation on K. Thus $B\otimes_k K$ is a skew polynomial ring over $A\otimes_k K$, and the rest is the Hilbert basis theorem. □

We alluded to the next proposition in Section 2.2 and we are now in a position to give the proof.

Proposition 2.29. *Let F/k be a finitely generated field extension, then A is stably right noetherian over k iff A is stably right noetherian over F.*

Proof. The easy direction is Proposition 2.17. For the other direction, write $F = k(\alpha_1,\dots,\alpha_n)$. Note that $A\otimes_k K \cong (A\otimes_k F)\otimes_F K$. Moreover, $A\otimes_k F$ is a localization of

$$A\otimes_k k[\alpha_1,\dots,\alpha_n]\cong A[\alpha_1,\dots,\alpha_n]$$

which is a finite centralizing extension of A. Since A is stably right noetherian over F, so too is $A\otimes_k F$ by Propositions 2.28 and 2.27, and hence $A\otimes_k K$ is right noetherian. □

Example 2.30 (Lorenz)**.** There are affine (primitive) stably right noetherian rings of infinite Gelfand-Kirillov dimension, and now that we have defined Ore extensions, we are in a position to give such an example, due to Lorenz [36]. Set $R = k[y,y^{-1},z,z^{-1}]$. Let σ be the automorphism of R defined by

$$\sigma(z) = y \text{ and } \sigma(y) = zy^2,$$

and set $A = R[x,x^{-1};\sigma]$. A is a stably right noetherian k-algebra (in fact, stably noetherian) by Proposition 2.28, and one easily sees that $\mathrm{GKdim}(R) = 2$. That $\mathrm{GKdim}(A) = \infty$ is proved, eg. in [41].

2.3.1 Filtered and Graded Techniques

We wish to show that rings which "resemble skew polynomial rings" are stably right noetherian. First we need a generalized version of Hilbert's basis theorem which works for ring extensions which are more general than Ore extensions.

Definition 2.31. An algebra B is called a *finite almost normalizing extension* of A provided that B is generated *as a ring* by A together with finitely many elements $x_1, \ldots, x_n$ which satisfy the conditions

1. $Ax_i + A = x_iA + A$
2. $[x_i, x_j] \in \sum_t Ax_t + A$.

For example, Ore extensions are finite almost normalizing extensions of A, though this construction is more general. A better example is the following enveloping algebra. Let B denote the k-algebra generated by elements $\{x, y, z\}$ and relations

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y.$$

Then B is an almost normalizing extension of k, but B is not an Ore extension.

The proof of the more general version of Hilbert's basis theorem that we want to present uses techniques from filtered and graded rings, so we make a small digression here in order to introduce the relevant terminology. As a bonus, we will also obtain two pleasant results about stably right noetherian rings. The first is that a filtered algebra whose associated graded ring is stably right noetherian is itself stably right noetherian. The second shows that to check whether a right noetherian algebra is stably right noetherian, it is often possible to reduce to the prime case.

Since there is no complete agreement on what precisely the term *filtered algebra* should mean, we will adopt the convention that filtrations are always increasing. Moreover, we restrict our attention to filtrations indexed by $\mathbb{Z}_{\geq 0}$, rather than considering rings filtered by arbitrary semigroups. As is customary, we refer to such algebras as $\mathbb{N}$-filtered, though they could more correctly be called $\mathbb{Z}_{\geq 0}$-filtered.

Definition 2.32. An algebra A is called $\mathbb{N}$-*filtered*, if there is a sequence of k-vector subspaces of A

$$k \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

such that

$$A = \bigcup_{n \geq 0} A_n, \quad \text{and} \quad A_n A_m \subseteq A_{n+m}.$$

If B is a k-algebra which is an almost normalizing extension of A, generated by $\{x_1, \ldots, x_n\}$, then an element of B of the form

$$a_0 x_{i_1} a_1 x_{i_2} \cdots x_{i_n} a_n$$

with $a_i \in A$ is called an *word* of length n. We let F_n denote the k-subspace of B generated by all words of length n, and we identify F_0 with A. In this way we see that B is naturally $\mathbb{N}$-filtered. One sees immediately that the associated graded ring of B is a ring extension of A and the images of the x_i in $\mathrm{gr}(B)$, denoted $\overline{x}_i$, are homogeneous elements of degree 1. Moreover, $\mathrm{gr}(B) = A[\overline{x}_1, \ldots, \overline{x}_n]$ with

$$A\overline{x}_i = \overline{x}_i A, \quad \text{and} \quad \overline{x}_i \overline{x}_j = \overline{x}_j \overline{x}_i.$$

Proposition 2.33. *Let A be an $\mathbb{N}$-filtered k-algebra with associated graded ring $\mathrm{gr}(A)$. If $\mathrm{gr}(A)$ is stably right noetherian, then A is stably right noetherian.*

Proof. Let $A_0 \subseteq A_1 \subseteq \cdots$ denote the filtration of A. This induces a filtration $A_0 \otimes_k K \subseteq A_2 \otimes_k K \subseteq \cdots$ of $A \otimes_k K$, and it suffices to show that $\mathrm{gr}(A \otimes_k K)$ is noetherian. In fact, $\mathrm{gr}(A \otimes_k K)$ is isomorphic to $\mathrm{gr}(A) \otimes_k K$, and the latter is right noetherian by hypothesis.

For each $n \geq 0$, we have an exact sequence of right k-modules

$$0 \to A_{n-1} \to A_n \to A_n/A_{n-1} \to 0.$$

Since k is a field, K is a free left k-module, and so the functor $- \otimes_k K$ is exact, and we get an exact sequence of K-modules

$$0 \to A_{n-1} \otimes_k K \to A_n \otimes_k K \to A_n/A_{n-1} \otimes_k K \to 0.$$

Let φ_n denote the natural K-module isomorphism

$$A_n/A_{n-1} \otimes_k K \to (A_n \otimes_k K)/(A_{n-1} \otimes_k K),$$

and denote by φ the induced K-module isomorphism from $\mathrm{gr}(A) \otimes_k K$ to $\mathrm{gr}(A \otimes_k K)$. So $\varphi = \oplus_n \varphi_n$. We claim that φ is actually a k-algebra isomorphism. Since φ is already k-linear, we need only check that it respects multiplication, and for this it suffices to consider simple tensors. So choose $a = x + A_{n-1} \otimes \alpha \in A_n/A_{n-1} \otimes_k K$ and $b = y + A_m \otimes \beta \in A_m/A_{m-1} \otimes_k K$. Then

$$\varphi(xy) = \varphi(xy + A_{n+m-1} \otimes \alpha\beta) = (xy \otimes \alpha\beta) + A_{n+m-1} \otimes_k K$$

and

$$\begin{aligned} \varphi(x)\varphi(y) &= (x \otimes \alpha + A_{n-1} \otimes_k K)(y \otimes \beta + A_{m-1} \otimes_k K) \\ &= (xy \otimes \alpha\beta) + A_{n+m-1} \otimes_k K. \end{aligned}$$

□

As an immediate consequence, we see that Weyl algebras and enveloping algebras of finite dimensional lie algebras are stably right noetherian. This follows because these rings have associated graded rings which are polynomial rings in finitely many indeterminates.

The converse to Proposition 2.33 is not true. For example, define a filtration on $A = k[x]$ by setting $A_0 = k$, and $A_n = k[x]$ for all $n \geq 1$. In this case we have that $\mathrm{gr}(A) = k \oplus k[x]/k$ (as vector spaces). Note that if the symbol $^-$ denotes passage to $\mathrm{gr}(A)$, then $\overline{x}^2 = \overline{0}$, whereas $\overline{x^2} \neq \overline{0}$. Ideals of $\mathrm{gr}(A)$ are the same as k-subspaces of $\mathrm{gr}(A)$, and so $\mathrm{gr}(A)$ is not noetherian, hence not stably noetherian.

Proposition 2.33 may also fail in case the filtration is decreasing, as the example $k[[x]]$ demonstrates. If we endow $k[[x]]$ with the usual decreasing filtration, then $\mathrm{gr}(k[[x]]) \cong k[x]$, which we know is stably noetherian. However, were $k[[x]]$ itself stably noetherian, then it would follow from Proposition 2.26 that its quotient field $k((x))$ is stably noetherian, which would contradict Vámos' theorem. The issue here is that the right noetherian property needn't pass from $\mathrm{gr}(A)$ to A if the filtration is decreasing. However, for finite filtrations there is no problem, and indeed we have the following interesting result.

Proposition 2.34. *Let A be a right noetherian k-algebra with nilradical* $\operatorname{Nil}(A)$*. Then A is stably right noetherian iff $A/\operatorname{Nil}(A)$ is stably right noetherian.*

Proof. We saw in Proposition 2.16 that homomorphic images of stably right noetherian rings are stably right noetherian, so the forward implication is clear. For the other direction, set $N = \operatorname{Nil}(A)$ and suppose that A/N is stably right noetherian. Since A is right noetherian, N is nilpotent with index of nilpotence n say, so we get a finite filtration of A by powers of N:

$$A \supsetneq N \supsetneq N^2 \supsetneq \cdots \supsetneq N^n = 0.$$

Fix $i \geq 0$. Since A is right noetherian, N^i/N^{i+1} is a finitely generated right A-module, and since it is annihilated by N, it is also a finitely generated right A/N-module.

Associated to this filtration of A we have a graded ring

$$\operatorname{gr}(A) = A/N \oplus N/N^2 \oplus \ldots \oplus N^{n-1}$$

and we see that $\operatorname{gr}(A)$ is a finitely generated right A/N-module. Since A/N is stably right noetherian, $\operatorname{gr}(A)$ is a stably right noetherian right A/N-module by Propoistion 2.3, and since $A/N \subseteq \operatorname{gr}(A)$, we see that $\operatorname{gr}(A)$ is stably right noetherian by Proposition 2.22. That A is then stably right noetherian follows by the preceding remarks. □

Remark 2.35. Suppose that A is right noetherian, with nilradical $\operatorname{Nil}(A)$. Then $\operatorname{Nil}(A) = \operatorname{PrimeRad}(A)$, the prime radical of A, which is by definition the intersection of all prime ideals. In a right noetherian ring, there are only finitely many minimal primes, and so $\operatorname{Nil}(A)$ is a finite intersection of minimal primes. It then follows from Proposition 2.21 that A is stably right noetherian iff A/P is stably right noetherian for all minimal primes P. Thus, when trying to show that certain types of rings are stably right noetherian, it suffices to treat the case where A is prime. For example, this technique is used in the proof of Theorem 2.51 to show that affine right noetherian PI algebras are stably right noetherian.

We are now in a position to generalize Hilbert's basis theorem. The essence of the argument is the following lemma. It appears in [40] as a corollary to another result, but our definitions differ slightly, and so for the convenience of the reader we include the proof.

Lemma 2.36. *Let A be a right noetherian k-algebra and let B be generated as a ring by A together with an element x such that $Ax = xA$. Then B is right noetherian.*

Proof. We view elements of B as *left* polynomials of the form $\sum a_i x^i$, where only finitely many of the $a_i \in A$ are nonzero. The relation $Ax = xA$ allows us to move elements of A to the right of x, bearing in mind that x needn't commute with elements of A. Given a right ideal $I \lhd_r B$, for each $n \in \mathbb{Z}_{\geq 0}$, let $I(n)$ be the set of leading coefficients of elements in I of degree $\leq n$, when represented with coefficients from A on the left. We claim that $I(n)$ is a right ideal of A. To see this, choose $a_j \in I(n)$ and $a \in A$; we need to show that $a_j a \in I(n)$. We may assume that $a_j a \neq 0$ for otherwise there is nothing to prove. Since a_j lives in $I(n)$, there is some element $p \in I$ with

$$p = a_j x^j + (\text{lower order terms}), \text{ and } j \leq n.$$

Though it's true that $pa \in I$, this doesn't help us since pa doesn't have the "correct" leading term. Instead, since $Ax = xA$, the same is true for x^j, so there is some $y \in A$ with $ax^j = x^j y$. Now, since I is a right ideal of B, we see that $py \in I$, and

$$\begin{aligned} py &= a_j x^j y + \text{ (lower order terms)} \\ &= a_j a x^j + \text{ (lower order terms)}, \end{aligned}$$

and so $a_j a \in I(n)$. Moreover, we clearly have $I(n) \subseteq I(n+1)$. Next, if $J \lhd_r B$ is some other right ideal of B with $I \subseteq J$ and $I(n) = J(n)$ for all n, then $I = J$.

Now, suppose

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

is a chain of right ideals in B and consider the collection of right ideals of A

$$\{\, I_i(n) \mid i, n \geq 0 \,\}.$$

It is easy to see that $I_i(j) \subseteq I_k(m)$ whenever $i \leq k$ and $j \leq m$. Moreover, since A is right noetherian, the ascending chain of right ideals $\{I_i(i)\}$ must stabilize, say with

$$I_j(j) = I_{j+1}(j+1) = \dots .$$

For each n with $0 \leq n \leq j-1$, the chain $\{\, I_i(n) \mid i \geq 0 \,\}$ also stabilizes, say at $i = k_n$. If we set

$$m = \max\{j, k_0, k_1, \dots, k_{j-1}\},$$

then for all $i \geq m$ and all $n \geq 0$ we have $I_i(n) = I_m(n)$. Thus $I_i = I_m$ and so B is right noetherian. □

Theorem 2.37 (Generalized Hilbert Basis Theorem)**.** *Let B be a finite almost normalizing extension of a right noetherian k-algebra A. So B is generated as a ring by A together with elements $x_1, \dots, x_n$ which normalize A in the sense that, for all i, j,*

1. $Ax_i + A = x_iA + A$
2. $[x_i, x_j] \in \sum_t Ax_t + A.$

Then B is right noetherian.

Proof. We have a chain of subalgebras of $\mathrm{gr}(B)$:

$$A \subseteq A[\overline{x}_1] \subseteq A[\overline{x}_1, A\overline{x}_2] \subseteq \dots \subseteq A[\overline{x}_1, \dots, \overline{x}_n] = \mathrm{gr}(B).$$

An application of Lemma 2.36 at each stage shows that $\mathrm{gr}(B)$ is right noetherian. It then follows that B is right noetherian as well. □

Combining these techniques with Proposition 2.33, one can easily prove

Proposition 2.38. *Let B be a finite almost normalizing extension of A. Then if A is stably right noetherian, so too is B.*

Remark 2.39. We say that a k-algebra is *constructible* if it can be obtained from k by a finite series of ring extensions, where each extension is either an almost normalizing extension or a finite module extension. The point is that many of the important examples of right noetherian rings are constructible algebras. In particular, Zhang twists, group algebras of polycyclic by-finite groups, and coordinate rings of quantum planes and quantum tori are all constructible. Combining Propositions 2.22 and 2.38 we see that all constructible algebras are stably right noetherian.

2.4 Descent Results

Dually to the results in the last section, we would like some conditions on a pair of algebras $A \subseteq B$ such that the stably right noetherian condition descends from B to A.

Remark 2.40. Let $I \lhd_r A$, so we have a short exact sequence of right A-modules

$$0 \to I \to A \to A/I \to 0.$$

If B is a flat left A-module then upon applying the functor $- \otimes_A B$ we arrive at the exact sequence

$$0 \to I \otimes_A B \to A \otimes_A B.$$

Moreover, if we naturally identify $A \otimes_A B \cong B$ via multiplication, then $I \otimes_A \cong IB$.

Definition 2.41. A left A-module ${}_AM$ is called *faithfully flat* if the tensor product functor $- \otimes_A M$ is both faithful and exact. This means that

$$0 \to B \to C \to D \to 0$$

is a short exact sequence of right A-modules if and only if

$$0 \to B \otimes_A M \to C \otimes_A M \to D \otimes_A M \to 0$$

is exact. The forward implication is the "flat" part, and the reverse implication is "faithfulness".

Lemma 2.42. *Let $A \subseteq B$ be k-algebras with B right noetherian and B a faithfully flat left A-module. Then A is right noetherian.*

Proof. Suppose that A is not right noetherian, so we can find an infinite chain of right ideals of A

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots .$$

We claim that this gives rise to a strictly increasing chain of right ideals in B, namely

$$I_1 B \subsetneq I_2 B \subsetneq I_3 B \subsetneq \cdots .$$

All that needs to be shown is that $I_{n+1}B/I_nB \neq 0$, and since ${}_AB$ is flat, we have that $I_nB \cong I_n \otimes_A B$ as right B-modules. Next,

$$I_{n+1}B/I_nB \cong (I_{n+1} \otimes_A B)/(I_n \otimes_A B) \cong I_{n+1}/I_n \otimes_A B,$$

and the latter is nonzero. Otherwise, by the faithful part of faithful flatness, $I_{n+1}/I_n = 0$, which is a contradiction. □

Proposition 2.43. *Let $A \subseteq B$ be k-algebras such that ${}_AB$ is a faithfully flat left A-module. Then if B is stably right noetherian, so too is A.*

Proof. This will follow from the previous lemma once we show that $B \otimes_k K$ is a faithfully flat left $A \otimes_k K$-module. This is probably well known, but we include the proof anyway.

Let C be a right $A \otimes_k K$-module. To avoid a surfeit of subscripts, we write $\otimes$ for $\otimes_k$. Then

$$\begin{aligned} C \otimes_A B &\cong [C \otimes_{A \otimes K} (A \otimes K)] \otimes_A B \\ &\cong C \otimes_{A \otimes K} [(A \otimes K) \otimes_A B] \\ &\cong C \otimes_{A \otimes K} [(A \otimes_A B) \otimes K] \\ &\cong C \otimes_{A \otimes K} (B \otimes K) \end{aligned}$$

as abelian groups. Thus applying the functor $- \otimes_{A \otimes_k K} B \otimes_k K$ to a right $A \otimes_k K$-module is equivalent to applying the functor $- \otimes_A B$. Hence ${}_AB$ is faithfully flat iff $B \otimes K$ is a faithfully flat left $A \otimes K$-module. □

Proposition 2.43 is usually applied when B is actually a free left A-module, in which case the proof simplifies immensely. For if $\{ x_i \mid i \in I \}$ is a set of free generators for B as a left A-module, then

$$B \otimes_k K \cong \oplus_i (Ax_i \otimes_k K)$$

as left $A \otimes_k K$-modules, with $\{ x_i \otimes 1 \mid i \in I \}$ a set of free generators for $B \otimes_k K$ as a left $A \otimes_k K$-module.

As an application of this, we see that every subfield of a stably right noetherian algebra must be a finitely generated field extension of k by Vámos' theorem. In particular, if D is a stably right noetherian division algebra over k, then all the maximal subfields of D are finitely generated. We conjecture that this condition is actually enough to characterize stably right noetherian division algebras.

Conjecture 2.44. *Let D be a division algebra over k. Then D is stably right noetherian over k iff every maximal subfield of D is finitely generated.*

A proof of Conjecture 2.44 would be interesting not only because it would yield an internal characterization of stably right noetherian division algebras generalizing Vámos' theorem, but also because it would imply that all *doubly noetherian* division algebras are stably right noetherian. A k-algebra A is called *doubly noetherian* provided that $A \otimes_k A^{op}$ is noetherian, and Resco, Small, and Wadsworth [48] have shown that doubly noetherian division algebras have the property that all commutative subfields are finitely generated field extensions of k. In fact, a doubly noetherian division algebra satisfies the stronger property that all subdivision algebras are finitely generated as division algebras, as Sweedler shows in [59]. An interesting open problem is whether the converse of Sweedler's result holds:

Question 2.45. *If D is a division algebra such that D and all subdivision rings are finitely generated as division algebras, is $D \otimes_k D^{op}$ noetherian?*

The stably right noetherian property also passes to subalgebras of finite codimension.

Proposition 2.46. *Let $A \subseteq B$ be k-algebras with $\dim_k B/A < \infty$. Then if B is stably right noetherian, so too is A.*

Proof. We have that $(B \otimes_k K)/(A \otimes_k K) \cong B/A \otimes_k K$ as right $A \otimes_k K$-modules. Since B/A is finite dimensional over k, $B/A \otimes_k K$ is finite dimensional over K, and thus noetherian as a right K-module. Since K is a subring of $A \otimes_k K$, we see that $B/A \otimes_k K$ is also noetherian as a right $A \otimes_k K$-module.

Now let I be a right ideal of $A \otimes_k K$. Since $B \otimes_k K$ is right noetherian, $I(B \otimes_k K)$ is a finitely generated right $B \otimes_k K$-module, with generators $y_1, \ldots, y_n$, say. Define $J = \sum_i y_i(A \otimes_k K)$, and note that $I(B \otimes_k K)/J$ is a quotient of the noetherian right $A \otimes_k K$-module

$$\underbrace{B/A \oplus \ldots \oplus B/A}_{n \text{ times}}$$

Also, I/J is an $A \otimes_k K$-submodule of $I(B \otimes_k K)/J$. Thus I/J is a finitely generated right $A \otimes_k K$-modules, and since J is also finitely generated by construction, we see that I is finitely generated, and so $A \otimes_k K$ is right noetherian. □

2.5 Further Results on Stably Noetherian Rings

We next review an interesting theorem, due to Small [57], which shows that every affine right noetherian PI algebra is stably right noetherian. The proof relies on a construction of Schelter, and to describe it we need to remind the reader of some of the structure theory for prime PI rings. For the precise definition of PI *algebra* we refer the reader to Chapter 3. The next remark is well known, and can be found, for example, in [53].

Remark 2.47. Let A be a prime PI ring with center Z, an integral domain, and set $S = Z - \{0\}$. Then

1. $Q(A) = AS^{-1}$.

2. The center of AS^{-1} is precisely ZS^{-1}, a field.

3. AS^{-1} is a finite dimensional central simple algebra over ZS^{-1}.

Now let A be a prime PI algebra and set $Q = AS^{-1}$ where $S = Z - \{0\}$. Schelter's idea is to expand A to a certain subalgebra $T(A)$ of Q. Since Q naturally contains A, each element $a \in A$ can be viewed, via left multiplication on Q, as a linear transformation of Q over ZS^{-1}. Thus a has a characteristic polynomial, say $f(x)$, with coefficients in the field ZS^{-1}, and it is these coefficients that we are interested in. We let $T(A)$ denote the subalgebra of Q generated by Z and all coefficients of all characteristic polynomials of elements $a \in A$. This ring $T(A)$ is called the *trace ring* of A. It is closely related to A, as the next proposition shows.

Proposition 2.48 (Schelter's Trace Ring)**.** *Let A be a prime* PI *algebra and let $T(A)$ denote the trace ring of A. Then*

1. *$T(A)$ is integral over A.*
2. *If A is right noetherian, then $T(A)$ is right noetherian, and moreover $T(A)$ is a finitely generated right A-module, generated by central elements.*
3. *If A is affine, then $T(A)$ is a finitely generated right module over its center, and its center is an affine k-algebra.*

Proof. See eg. [53, Construction 6.3.28]. □

Definition 2.49. Let $A \subseteq B$ be rings with B_A a finitely generated right A-module, generated by elements $x_1, \ldots, x_n$ such that $Bx_i = x_iB$ for all i. Then B is called an *finite normalizing extension* of A.

Remark 2.50. One should be careful not to confuse the above definition with *finite almost normalizing extensions.* The former are generated *as modules* by normalizing elements, whereas the latter are generated *as rings* by normalizing elements.

By [40, Corollary 10.1.10], if B is a finite normalizing extension of A, with B right noetherian, then A is right noetherian. With this in hand we can now prove Small's theorem about affine right noetherian PI algebras.

Proposition 2.51 (Small)**.** *Let A be an affine right noetherian* PI *algebra, then A is stably right noetherian.*

Proof. By Remark 2.35 we can reduce to the prime case, so assume that A is prime, and let T denote the trace ring of A. Since A is right noetherian, by Proposition 2.48 (2), T is a finitely generated right A-module, generated by central elements. Thus $T \otimes_k K$ is a finitely generated right module over $A \otimes_k K$ generated by central elements, and by [40, Corollary 10.1.10] it is enough to show that $T \otimes_k K$ is noetherian. But since A is affine, T is a finite right module over its center, which is a commutative affine k-algebra. Say $Z(T) = k[x_1, \ldots, x_n]$. Then $T \otimes_k K$ is a finitely generated right module over $k[x_1, \ldots, x_n] \otimes_k K \cong K[x_1, \ldots, x_n]$, and it follows that $T \otimes_k K$ is right noetherian. □

2.5.1 Negative Results

According to Vámos' theorem, an infinitely generated field extension of k is not stably noetherian. This, in combination with Proposition 2.43, provides a useful tool for constructing counterexamples.

We know from Theorem 2.51 that affine right noetherian PI algebras are stably right noetherian, but without the PI hypothesis, one can find affine right noetherian rings which fail to be stably right noetherian. The first (and so far only) such was found by Resco and Small [47]. Their idea is to work over a non-algebraically closed field and to construct an affine, simple domain which contains an infinitely generated (non-central) subfield.

Example 2.52 (Resco-Small)**.** Let F be any field of characteristic $p > 0$, and let $E = k(t_1, t_2, \ldots)$ be a rational function field over F in countably many indeterminates. There is an F-derivation on E defined by $\delta(t_i) = t_{i+1}$, and so we may form the skew polynomial ring $A := E[x; \delta]$. The following facts may be readily checked:

1. The center of A is the subfield k of E generated by F and the set $\{ t_i^p \mid i \in \mathbb{N} \}$.

2. A is a simple noetherian domain, generated as a k-algebra by t_1 and x.

3. A is not stably right noetherian over k.

In fact, as E is an infinitely generated algebraic field extension of k, if $A \otimes_k E$ were right noetherian, then by faithful flatness, $E \otimes_k E$ would be right noetherian as well, which contradicts Vámos' theorem. Note that the same argument shows that A is not stably left noetherian either.

Bell [8] has shown that, in case k is uncountable and algebraically closed, every countably generated right noetherian k-algebra is stably right noetherian. In this generality, both hypotheses on k are necessary, which leads to the following

Question 2.53. *If k is an algebraically closed field, and A is an affine right noetherian k-algebra, must A be stably right noetherian?*

As an additional example of the sorts of pathology which can occur, we note that the class of stably right noetherian rings is not closed under tensor products, even if we restrict our attention to the affine case. In [50, Theorem 7.3 (2)] Rogalski constructs an affine, $\mathbb{N}$-graded algebra T over any algebraically closed field k, which is stably noetherian (on both sides) and for which $T \otimes_k T$ fails to be right noetherian.

2.5.2 An Example of Wadsworth

Since localizations of stably right noetherian rings are again stably right noetherian, one might wonder about the converse: whether or not (right) orders in stably right noetherian rings must themselves be stably right noetherian. The answer is no, even in the commutative case. We present now an example of Wadsworth of a commutative noetherian integral domain which is not stably noetherian over k, but whose quotient field is a finitely generated field extension of k. This example is taken from [62].

Let F be any field of characteristic $p > 0$. Set $k = F(t)$, and $V = \overline{k}[[x]]$, where $\overline{k}$ denotes the algebraic closure of k. Define an element $y \in V$ by

$$y = \sum_{i \geq 1} t^{c_i} x^{i!}, \qquad \text{where } c_i = p^{-i(i+1)/2}.$$

Claim 1. y is transcendental over $\overline{k}(x)$.

Proof of Claim 1. Suppose that y is algebraic over $\overline{k}(x)$. Then we can write

$$f_0(x) + f_1(x)y + \ldots + f_r(x)y^r = 0$$

where $f_r \neq 0$ and all $f_i(x) \in \overline{k}(x)$. For each $n \in \mathbb{N}$, write $y = a_n + b_n$ where

$$a_n = \sum_{i=1}^{n-1} t^{c_i} x^{i!}, \quad \text{and} \quad b_n = \sum_{i \geq n} t^{c_i} x^{i!}.$$

We then have

$$f_0(x) + f_1(x)(a_n + b_n) + \ldots + f_r(x)(a_n + b_n)^r = 0.$$

If we gather together all terms not involving powers of b_n, we see that b_n is also algebraic over $\overline{k}[x]$, and it satisfies an equation whose constant term is

$$g_n(x) := f_0(x) + f_1(x)a_n + \ldots + f_r(x)a_n^r.$$

Since $\deg(a_n) = (n-1)!$, we take degrees on both sides to get

$$\deg(g_n) = \max_i \{\deg(f_i) + i(n-1)!\} \leq \max_i \{\deg(f_i)\} + r(n-1)!$$

We can then find n large enough so that $\deg(g_n) < n!$, which is a contradiction because all other terms in the equation for b_n involve only powers x^q for $q \geq n!$. $\square$

Next, set $R = V \cap k(x, y)$ and define a map $\varphi \colon R \to \overline{k}$ by sending $r \in R$ to its constant term. (Here we are thinking of r as a power series in x with coefficients in $\overline{k}$.) Note that $k \subseteq R/\ker(\varphi) \subseteq \overline{k}$, so $R/\ker(\varphi)$ is an algebraic integral domain, and hence a field. This shows that $\ker(\varphi)$ is a maximal ideal in R. In fact, we see that $\ker(\varphi) = xV \cap R$. V is a discrete valuation ring with quotient field $Q(V) = \overline{k}((x))$, a Laurent series ring, and $k(x, y)$ is a subfield of $Q(V)$, so by [42, 33.7], R is itself a discrete valuation ring. In particular, R is noetherian. Clearly we have $Q(R) \subseteq k(x, y)$ (in fact, $Q(R) = k(x, y)$), which shows that $Q(R)$ is a finitely generated field extension of k, and hence $Q(R)$ is stably noetherian over k.

Next we show that R is not stably noetherian by eliciting an infinitely generated field extension of k inside $R/\ker(\varphi)$. Since homomorphic images of stably noetherian rings are again stably noetherian, this is sufficient by Vámos' theorem.

Claim 2. The field $R/\ker(\varphi)$ contains a copy of $k(t^{1/p}, t^{1/p^2}, t^{1/p^3} \ldots)$.

Proof of Claim 2. Similar to the proof of Claim 1, fix $n \in \mathbb{N}$ and write

$$y = a_n + t^{c_n} x^{n!} + b_n$$

where

$$a_n = \sum_{i=1}^{n-1} t^{c_i} x^{i!}, \text{ and } b_n = \sum_{i>n} t^{c_i} x^{i!}.$$

Since F has characteristic p, the p^l-th power map is additive, and so

$$y^{p^l} = a_n^{p^l} + t^{c_n \cdot p^l} x^{p^l n!} + b_n^{p^l}.$$

We want to choose l large enough so that $a_n^{p^l} \in k[x]$ (a priori it is only an element of $\overline{k}[x]$). Set $l = n(n-1)/2$ and note the following:

1. $c_i \cdot p^l > 1$ for $i \leq n-1$
2. $c_n \cdot p^l = p^{-n}$

Thus we get

$$(y^{p^l} - a_n^{p^l}) x^{-n! \cdot p^l} = t^{p^{-n}} + b_n^{p^l}.$$

The left hand side of this equation is in $k(x, y)$ (actually, in $k[1/x, y]$), and the right hand side is in V. Moreover, it is clear that this element gets sent to $t^{p^{-n}}$ under the homomorphism φ. We have thus shown that $R/\ker(\varphi)$ contains the field generated by $\{ t^{p^{-n}} \mid n \in \mathbb{N} \}$.

□

Remark 2.54. By employing [23, Theorem 9], one could alternatively show that the ring $V \cap k[1/x, y]$ is noetherian, but not stably noetherian. This example has the slight added advantage of being a subring of an affine commutative ring, whereas the ring R constructed above is not.

Wadsworth's example leads to the following open question.

Question 2.55. *Let C be a commutative noetherian k-algebra which is a domain and suppose C has the property that all quotient fields of all prime homomorphic images of C are finitely generated field extensions of k. Must C be stably noetherian?*

2.5.3 Some Positive Results

Next we turn to algebraic stably right noetherian algebras, and we prove the somewhat surprising result that such rings must be finite dimensional over the ground field.

Theorem 2.56. *Let A be an algebraic stably right noetherian k-algebra. Then $\dim_k A < \infty$.*

Proof. We first consider the case where $A = D$ is a division algebra. Suppose that $\dim_k D = \infty$, and let K be any maximal subfield of D. Since D is a free, hence faithfully flat K-module on either side, we see that K must be stably noetherian over k. But then K is a finitely generated algebraic field extension of k, so $\dim_k K < \infty$. $D \otimes_k K$ is a finite dimensional left D-vector space and thus $D \otimes_k K$ is an artinian left D-module and as $D \subseteq D \otimes_k K$, $D \otimes_k K$ is also a left artinian ring. Next, by [24, Theorem 4.2.1], $D \otimes_k K$ is a dense ring of endomorphisms on D considered as a right vector space over K. So by Jacobson's density theorem, either $D \otimes_k K \cong M_n(K)$ and we're done, or else $\dim_K(D \otimes_k K) = \infty$. So suppose that $\{x_1, x_2, \cdots\}$ is a K-linearly independent set of elements of $D \otimes_k K$. For each $n \in \mathbb{N}$, set

$$I_n := \bigcap_{i=1}^{n} \operatorname{l.ann}_{D \otimes_k K}(x_i).$$

By the density theorem, $I_1 \supsetneq I_2 \supsetneq \cdots$ is a strictly descending chain of left ideals in $D \otimes_k K$, which is impossible since D is left artinian. Thus $D \otimes_k K$ is finite dimensional over K, and since $\dim_k D = \dim_K D \otimes_k K$, we are done.

Next suppose that A is a prime algebraic stably right noetherian algebra. Since A is algebraic and right noetherian, A is actually right artinian. To see this, note that since A is algebraic, every regular element is a unit, and a right noetherian ring in which every regular element is invertible is right artinian. Now, by Wedderburn's theorem $A \cong M_n(D)$ for a suitable division ring D. Moreover, Since A is a free left D-module, D must be stably right noetherian, and so $\dim_k D < \infty$, and thus A is finite dimensional over k as well.

In the general case, let N denote the nilradical of A. Since A is right noetherian, N is nilpotent, and since A is right artinian, A/N is semiprime right artinian and so

$$A/N \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i)$$

for suitable division rings D_i. Moreover, as we have seen, each D_i must be finite dimensional over k, and so A/N is finite dimensional as well.

The last step is to filter A by the powers of N:

$$A \supseteq N \supseteq N^2 \supseteq \cdots \supseteq N^s = 0.$$

As we have seen, each N^i/N^{i+1} is a finitely generated right A/N module, and since A/N is actually finite dimensional, so too is N^i/N^{i+1}. It then follows that A is finite dimensional as well. □

Note that as a result of the above proposition, the so-called Kurosh problem ("is every affine, algebraic division algebra finite dimensional?") has a positive solution for stably right noetherian algebras. There is an additional variant of the Kurosh problem, namely: "is every affine division algebra finite dimensional?" which remains open for stably right noetherian rings, though we suspect that an affine stably right noetherian division algebra is algebraic, from which a positive solution would follow.

As another corollary to Theorem 2.56, we obtain something of a classification theorem for when artinian PI algebras are stably right noetherian. If A is a right artinian PI k-algebra with nilradical N, then A/N is semisimple right artinian PI, and thus a finite direct sum of matrix rings over division rings. By Kaplansky's theorem [29], these division rings are finite modules over their centers, and it follows that A/N, and hence A, is stably right noetherian iff each of these centers is a finitely generated field extension of k. We sum this up in the following

Proposition 2.57. *Let A be a right artinian* PI *k-algebra with nilradical N and write $A/N \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i)$ for suitable division rings D_i. Then A is stably right noetherian iff $Z(D_i)$ is a finitely generated field extension of k, for all i.*

We next turn to endomorphism rings of modules over stably right noetherian rings. If A is stably right noetherian and M is a finitely generated right A-module, then we have seen that M is a stably noetherian module. We will show that in case

M is simple, the ring of A-module endomorphisms of M also inherits the stably right noetherian property.

Lemma 2.58 (McConnell & Robson 9.3.8)**.** *Let A and B be k-algebras and let M be an (A, B)-bimodule with M faithfully flat over A. Then there is an inclusion preserving injection from the set of right ideals of A to the set of B-submodules of M.*

Proof. Consider the map defined by $I \mapsto IM$, where $I \lhd_r A$. Since M is faithfully flat over A, we have that $I \otimes_A M \cong IM$ as right B-modules from Remark 2.40. Moreover, if $I \otimes_A M = 0$ then $I = 0$ by the faithful part of faithful flatness. We thus see that the map in question is injective. To see that it also preserves inclusions, suppose that $I \subsetneq J$ are right ideals of A. It is clear that $IM \subseteq JM$, so we only need to show that $IM \neq JM$. If not, then since $IM/JM \cong (I \otimes_A M)/(J \otimes_A M) \cong I/J \otimes_A M$, and since ${}_AM$ is faithfully flat, we would have $I/J = 0$, i.e. $I = J$. □

Proposition 2.59. *Let A be stably right noetherian and let M be a simple right A-module. Then* $\mathrm{End}_A(M)$ *is stably right noetherian.*

Proof. Set $D = \mathrm{End}_A(M)$ and note that D is a division algebra by Schur's Lemma. $M \otimes_k K$ is a finitely generated right module over the right noetherian ring $A \otimes_k K$, and so it is a noetherian right $A \otimes_k K$-module. On the other hand, $M \otimes_k K$ is a left $D \otimes_k K$-module, and since M is a free left D-module, it follows that $M \otimes_k K$ is a free left $D \otimes_k K$ module. By Lemma 2.58 we then have an inclusion preserving injection

$$\{\text{right ideals of } D \otimes_k K\} \hookrightarrow \{A \otimes_k K\text{-submodules of } M \otimes_k K\}$$

and hence $D \otimes_k K$ is right noetherian. □

Now suppose that A is stably right noetherian and $M = \oplus M_i$ is a finite direct sum of simple right A-modules. Since Hom commutes with direct sums, we have

$$\mathrm{End}_A(\oplus M_i) \cong \oplus_{i,j} \mathrm{Hom}_A(M_i, M_j).$$

Thus we can think of $\mathrm{End}_A(M)$ as a formal matrix ring whose (i,j)-entry is given by $\mathrm{Hom}_A(M_i, M_j)$. By Schur's lemma, we see that

$$\mathrm{Hom}_A(M_i, M_j) \cong \begin{cases} \mathrm{End}_A(M_i), & \text{if } M_i \cong M_j \\ 0, & \text{else.} \end{cases}$$

Thus the (i,j)-entry of $\mathrm{End}_A(\oplus M_i)$ is either 0, or else a free $\mathrm{End}_A(M_i)$-module of rank 1. The upshot of all this is that $\mathrm{End}_A(\oplus M_i)$ is a finitely generated right module over its diagonal subring, and the latter is isomorphic to $\oplus_i \mathrm{End}_A(M_i)$ and hence stably right noetherian by Propositions 2.18 and 2.59. In short, we have proved

Corollary 2.60. *If A is stably right noetherian and M is a completely reducible right A-module, then* $\mathrm{End}_A(M)$ *is stably right noetherian.*

Definition 2.61. A k-algebra A is said to satisfy the *Nullstellensatz* if $\mathrm{End}_A(M)$ is algebraic over k for every simple right A-module M.

Corollary 2.62. *Let A be a stably right noetherian algebra which satisfies the Nullstellensatz and let M be a completely reducible right A-module. Then* $\mathrm{End}_A(M)$ *is finite dimensional.*

Proof. Clear from the above remarks and Propositions 2.56 and 2.59. □

One may wish to generalize these results to a module M of finite length over a stably right noetherian algebra A, but here the situation is less clear. For one, we have the following result of Connell [13]: If M_A is a right module of finite length over any algebra A, then $\mathrm{End}_A(M)$ is semiprimary: That is, the Jacobson radical of $\mathrm{End}_R(M)$ is nilpotent, and modulo the radical we get a semisimple artinian ring. Thus if $\mathrm{End}_R(M)$ were right noetherian, it would also be right artinian. Nevertheless, Proposition 2.14 is at least suggestive of a potential result in this direction, and we are lead to pose the following question.

Question 2.63. *Let A be a stably right noetherian k-algebra and let M_A be a right A module of finite length. If* $\mathrm{End}_A(M)$ *is right noetherian, is it true that* $\mathrm{End}_A(M)$ *is also stably right noetherian?*

2.5.4 An Example of Irving

Strongly right noetherian rings satisfy the Nullstellensatz (see [5]). We noted in Proposition 2.59 that the endomorphism ring of a simple module over a stably right noetherian ring is itself stably right noetherian, so one may wonder whether or not stably right noetherian rings must satisfy the Nullstellensatz. The answer is no. In [26], Irving constructs an example of an affine, primitive, stably right noetherian algebra over an algebraic closure of $\mathbb{Z}_p$, and an explicit simple module V whose endomorphism ring is a rational function field in one variable.[1] We review his construction now.

Let k denote an algebraic closure of the finite field $\mathbb{Z}_p$ and set $C = k[t, t^{-1}]$ the Laurent polynomial ring over k. Fix a nonzero element $b \in k$, $b \neq 1$ and consider the ring automorphism $\varphi\colon C[y] \to C[y]$ defined by $\varphi(y) = ty + b$ and $\varphi|_C = \mathrm{id}_C$. Then the inverse automorphism is given by $\varphi^{-1}(y) = t^{-1}(y - b)$. If $n \in \mathbb{Z}_{>0}$, then

$$\varphi^n(y) = t^n y + b(t^{n-1} + t^{n-2} + \ldots + t + 1)$$

and

$$\varphi^{-n}(y) = t^{-n} y - b(t^{-n} + t^{-n+1} + \ldots + t^{-1}).$$

We have a natural projection map $\pi\colon C[y] \to C$ given by sending y to 1, and we let c_n denote the image of $\varphi^n(y)$ under π. Thus $c_0 = 1$ and for $n \in \mathbb{Z}_{>0}$,

$$c_n = t^n + b(t^{n-1} + t^{n-2} + \ldots + t + 1)$$

and

$$c_{-n} = t^{-n} - b(t^{-n} + t^{-n+1} + \ldots + t^{-1}).$$

Note in particular that by our choice of b, $c_n \neq 0$ for all $n \in \mathbb{Z}$. We let T denote the multiplicatively closed subset of $C[y]$ generated by the elements $\{\, \varphi^n(y) \mid n \in \mathbb{Z} \,\}$. Irving shows that under these conditions the elements $\{\, c_n^{-1} \mid n \in \mathbb{Z} \,\}$, together with $k[t, t^{-1}]$, generate $k(t)$ as a k-algebra. Set $R = C[y]T^{-1}$, the localization of $C[y]$ obtained by inverting all the elements of T. Then φ extends to an automorphism

[1] Note that this gives another proof that the class of stably right noetherian rings is properly larger than the class of strongly right noetherian rings.

of R by defining

$$\varphi^n(y^{-1}) = (\varphi^n(y))^{-1}.$$

Finally, let $A = R[x, x^{-1}; \varphi]$ denote the twisted Laurent polynomial ring. By [26, Proposition 2.3], the algebra A is primitive, with a faithful simple right module V such that $\mathrm{End}_A(V) = k(t)$. In particular, A does not satisfy the Nullstellensatz. That A is stably right noetherian (actually stably noetherian) follows from Propositions 2.26 and 2.28.

We conjecture that this sort of pathology cannot occur in characteristic 0.

Conjecture 2.64. *Let k be an algebraically closed field of characteristic 0 and let A be an affine stably right noetherian k-algebra. Then A satisfies the Nullstellensatz.*

2.6 An Application to Group Algebras

Notation. In this section only, we use $[\,,]$ to denote multiplicative commutators: $[a, b] = aba^{-1}b^{-1}$.

An unsolved problem in the theory of group algebras is to determine when a group algebra $k[G]$ is noetherian. (Since $k[G]$ has an involution induced by $g \mapsto g^{-1}$, $k[G]$ is right noetherian iff it is left noetherian, and so in this section we may safely drop the modifier "right"). One case is well known.

Definition 2.65. G is called *polycyclic-by-finite* if there is a chain of subgroups

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

where each G_{i+1}/G_i is infinite cyclic for $i = 1, \ldots, n-1$, and G/G_{n-1} is finite.

It is known that $k[G]$ is noetherian when G is polycyclic-by-finite (see eg. [53, Theorem 8.2.2]), but whether this condition is necessary remains an important open problem, as there are, as yet, no known examples of noetherian group algebras $k[G]$ where G is not polycyclic-by-finite.

Question 2.66. *Determine necessary and sufficient conditions for $k[G]$ to be noetherian.*

An obvious related question to ask is whether every noetherian group algebra $k[G]$ is stably noetherian. Since it is clear that

$$k[G] \otimes_k K \cong K[G],$$

we immediately see that every group algebra of a polycyclic-by-finite group is stably noetherian, but a general answer appears to require a solution to Question 2.66. A counterexample, that is, a noetherian group algebra which fails to be stably noetherian, would indeed be interesting.

On the other hand, suppose that G is a torsion-free nilpotent group. It is known that the group algebra $k[G]$ is a (right and left) Ore domain [58], and so $k[G]$ has a quotient division ring D. We next show that in this case D is stably noetherian over k iff G is a finitely generated group. Note that if G is a finitely generated torsion-free nilpotent group, then G is polycyclic-by-finite, so one direction of this claim is obvious. $k[G]$ is an Ore domain irrespective of the number of generators of G, (though it is not known whether $k[G]$ is always noetherian).

Recall that if G is a group with a normal subgroup $N \triangleleft G$, then G is finitely generated iff N and G/N are finitely generated. A special case of this is the following

Lemma 2.67. *Let G be a group, $H \leq G$ a subgroup, $N \triangleleft G$ and suppose that $H \cap N$ and HN/N are finitely generated. Then H is finitely generated.*

The next lemma is also probably well known, but since we are unable to find an adequate citation, we offer a proof.

Lemma 2.68. *If G is a finitely generated nilpotent group, then every subgroup of G is finitely generated.*

Proof. The proof is by induction on the index of nilpotence of G. If G is abelian then all subgroups are finitely generated, so there is nothing to show. Since G is nilpotent, we have a chain of subgroups

$$G = G_0 \geq \cdots \geq G_{n-1} \geq G_n = (1),$$

where $G_{i+1} = [G, G_i]$. This implies that $G_{n-1} \subseteq Z(G)$. The first step is to show that all the G_i are finitely generated, and for this we will need the following two relations, which hold for any $a, b, c \in G$:

$$[a, bc] = [a, b]b[a, c]b^{-1} \text{ and } [ab, c] = a[b, c]a^{-1}[a, c]. \qquad (\dagger)$$

Note that G/G_n is a finitely generated nilpotent group with index of nilpotence strictly less than n, so by induction every subgroup of G/G_n is finitely generated. In particular, G_{n-2}/G_{n-1} is finitely generated, say by $\{b_j G_{n-1}\}$. Let $\{a_i\}$ be a (finite) generating set for G. Choose $x \in G_{n-1}$. Since $G_{n-1} = [G, G_{n-2}]$, we may write

$$x = [g_1, y_1][g_2, y_2] \cdots [g_r, y_r], \text{ for some } g_i \in G, y_i \in G_{n-2}.$$

Consider the first commutator $[g_1, y_1]$. We may write $g_1 = a_{1_1} \cdots a_{t_1}$, and $y_1 = b_{1_1} \cdots b_{s_1} h$ for some $h \in G_{n-1}$. Since $[G, G_{n-2}] \subseteq Z(G)$, ($\dagger$) shows that

$$\begin{aligned}[g_1, y_1] &= [a_{1_1} \cdots a_{t_1}, b_{1_1} \cdots b_{s_1} h] \\ &= \prod_{i=1}^{t} \prod_{j=1}^{s} [a_{1_i}, b_{1_j}].\end{aligned}$$

It follows that the (finite) set $\{[a_i, b_j]\}$ generates G_{n-1}.

Finally, let H be an arbitrary subgroup of G. $H \cap G_{n-1}$ is finitely generated since G_{n-1} is finitely generated abelian. On the other hand, G/G_{n-1} is a finitely generated nilpotent group of strictly lower index of nilpotence than that of G. By induction, every subgroup of G/G_{n-1} is finitely generated, and so HG_{n-1}/G_{n-1} is finitely generated. Now Lemma 2.67 applies and we see that H is finitely generated. □

Lemma 2.69. *Let G Let $N \leq Z(G)$. If G/N has an abelian subgroup which is not finitely generated, then so does G.*

Proof. Without loss, we may assume that $Z(G)$ is finitely generated, for otherwise there is nothing to prove. Choose an abelian subgroup of G/N which is not finitely generated. By moving to a smaller subgroup if necessary, we may assume this subgroup is generated by the images of $\{a_1, a_2, \ldots\}$ in G/N. Set $H = \langle a_1, a_2, \ldots \rangle \leq$

G. The idea is to construct a strictly increasing chain of abelian subgroups $A_1 \lneq A_2 \lneq A_3 \lneq \cdots$ of G. The union $\bigcup A_n$ will then be an abelian subgroup of G which is not finitely generated.

To begin, set $A_1 = \langle a_1 \rangle$. We would like to next construct an abelian subgroup A_2 of G which properly contains A_1, and so we need to find an element $b_2 \in G$ such that b_2 commutes with a_1, but $b_2 \notin A_1$.

Define a map $\varphi\colon H \to Z(G)$ by $\varphi_1(h) = [a_1, h]$. Note that for all i, $[a_1, a_i] \in N \subseteq Z(G)$, from which it follows that $[a_1, h] \in Z(G)$ for all $h \in H$. We claim that φ_1 is actually a group homomorphism. A direct computation shows that for any $a, b, c \in G$,

$$[a, bc] = [a, b]b[a, c]b^{-1},$$

and so

$$\varphi_1(xy) = [a_1, xy] = [a_1, x]x[a_1, y]x^{-1}$$

and

$$\varphi_1(x)\varphi_1(y) = [a_1, x][a_1, y].$$

However, since $[a_1, y] \in Z(G)$, these two expressions are equal, and so φ_1 is a group homomorphism. Now, since $Z(G)$ is a finitely generated abelian group, the image of H under φ_1 is also finitely generated. Since H is not finitely generated, Lemma 2.67 assures us that $\ker(\varphi_1)$ is also not finitely generated, and in particular $\ker(\varphi_1) - A_1 \neq \emptyset$. Choose any element $b_2 \in \ker(\varphi_1) - A_1$, and set $A_2 = \langle a_1, b_2 \rangle$.

Next we define $\varphi_2\colon H \to Z(G) \times Z(G)$ via

$$\varphi_2(h) = ([a_1, h], [b_2, h]).$$

That φ_2 is a group homomorphism follows as before, and since $Z(G) \times Z(G)$ is also a finitely generated abelian group, we are again able to find an element $b_3 \in \ker(\varphi_2) - A_2$, and we let A_3 denote the subgroup of G generated by A_1 and b_3. Continuing in this fashion, at the n^{th} stage we have constructed a chain $A_1 \lneq A_2 \lneq \cdots \lneq A_n$, where $A_n = \langle a_1, b_2, \ldots, b_n \rangle$. We have a group homomorphism $\varphi_n\colon H \to Z(G)^n$ given by

$$\varphi_n(h) = ([a_1, h], [b_2, h], \ldots, [b_n, h]).$$

For b_{n+1}, choose any element of $\ker(\varphi_n) - A_n$, and set $A_{n+1} = \langle a_1, b_2, \ldots, b_n, b_{n+1}\rangle$. The union $\bigcup_n A_n$ is then an abelian subgroup of G which is not finitely generated and the proof is complete. □

Theorem 2.70. *Let G be a torsion-free nilpotent group, and let D be the quotient division ring of the group algebra $k[G]$. Then D is stably noetherian over k iff G is finitely generated.*

Proof. If G is finitely generated, then G is polycyclic-by-finite, and in this case it is known that $k[G]$ is stably noetherian. As D is a localization of $k[G]$, it too is stably noetherian.

The other direction is in fact the content of the theorem, and to prove it, suppose that G is not finitely generated. We claim that G contains a non finitely generated abelian subgroup. The proof is again by induction on the index of nilpotence of G. If G is abelian then there is nothing to prove. More generally, we may assume that $Z(G)$ is finitely generated, for otherwise $Z(G)$ is the subgroup we seek. Now, $G/Z(G)$ has index of nilpotence strictly smaller than the index of nilpotence of G, and so by induction $G/Z(G)$ has an abelian subgroup which is not finitely generated. This subgroup can then be lifted by Lemma 2.69 to an abelian subgroup A of G which is not finitely generated.

D contains the quotient field $Q(k[A])$, which we claim is not a finitely generated field extension of k. Every element of $Q(k[A])$ is of the form x/y where $x, y \in k[A]$, so suppose that $Q(k[A])$ is generated, as a field extension of k, by $\{x_1/y_1, \ldots, x_t/y_t\}$. Let H denote the subgroup of A generated by all group elements in the support of all $x_i's$ and all $y_i's$. Since A is not finitely generated, we can find an element a of A which is not in H. But $a \in Q(k[A])$, and so we may write

$$a = fg^{-1}$$

where f and g are elements of the ring $k[\{x_i/y_i\}]$. Thus we have $ag = f$. Now, we may find a word w in the alphabet $\{y_1, \ldots, y_n\}$ such that fw is an element of $k[\{x_i, y_i\}]$. But notice that all group elements in the support of agw are in aH, whereas all group elements in the support of fw lie in H. Since the cosets aH and H are distinct by our choice of a, this is a contradiction. □

3 Embedding Problems

This chapter focuses on a few outstanding problems in the theory of algebras satisfying a polynomial identity. Commutativity in an algebra can be expressed by saying that the relation $xy - yx = 0$ holds identically in the algebra. Other sorts of identities sometimes hold in noncommutative algebras. For example, if x and y are any two matrices in $M_2(k)$, then the (additive) commutator $[x, y] := xy - yx$ has trace 0. By the Cayley-Hamilton theorem $[x, y]^2$ is a scalar matrix, so $[x, y]^2$ is a central element, and hence

$$[[x, y]^2, z] = 0$$

for all choices of 2×2 matrices x, y, and z. A deep theorem of Amitsur and Levitzki asserts that for all choices of $2n$ matrices $x_1, \ldots, x_{2n}$ from $M_n(k)$,

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)} = 0,$$

where S_{2n} is the symmetric group on $2n$ letters, and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. The polynomial appearing above is called the $2n^{\text{th}}$ *standard identity* and is denoted s_{2n}, or $s_{2n}(x_1, \ldots, x_{2n})$ if we need to reference the indeterminates.

Such relations which hold in an algebra can be thought of as saying that a certain polynomial (in noncommuting variables) vanishes upon substituting arbitrary elements of the algebra for the indeterminates. We formalize this notion in the following

Definition 3.1. We say that an algebra A satisfies a *polynomial identity*, or is a *PI algebra* for short, if there is some nonzero element of the free algebra $f(x_1, \ldots, x_n) \in$

$k\langle x_1, \ldots, x_n\rangle$ such that

$$f(a_1, \ldots, a_n) = 0$$

for all choices of $a_1, \ldots, a_n$ from A.

Remark 3.2. Given a PI algebra A, it is a difficult problem to determine all the identities satisfied by A. What is known is that if the ground field has characteristic 0, then there is a finite set of identities for which all others satisfied by A are consequences. (This was the Specht problem, and was solved by Kemer [31]). However, actually finding such a "basis" for the identities of a PI algebra is usually impossible. What is known is that $[x, y] = 0$ is such a basis for commutative algebras. In fact, the problem is even hard for 2×2 matrices. Razmyslov [45] first solved this problem in 1973, and Drenski [14] later showed that the identities $[[x, y]^2, z] = 0$ and s_4 suffice.

One can easily check that the n^{th} standard identity is multilinear and alternating (ie. a transposition applied to the indeterminates results in a sign change), and hence behaves much like a determinant. In particular, if A is an algebra of dimension $< n$, then A satisfies s_n. More generally, if C is a commutative k-algebra and A is an algebra which is generated, as a C-module, by fewer than n elements, then A also satisfies s_n. As a specific case, we see that algebras which are finitely generated modules over their centers are PI. (See Proposition 4.7 for an example where the converse also holds).

Any polynomial identity satisfied by an algebra obviously passes to subalgebras and homomorphic images. It follows that if A is a subalgebra of a matrix ring over a field, then A is PI. The converse is easily seen to be false as follows: Let k be any field of characteristic 0 and let V be a countably infinite dimensional vector space over k with basis $\{e_1, e_2, \ldots\}$. Let A denote the exterior algebra of V, which is generated by the elements $\{\, e_i \mid i \in \mathbb{N}\,\}$ with relations $e_i e_j = -e_j e_i$. By checking monomials, we can see that A satisfies the identity $[[x, y], z] = 0$. However, A is not a subalgebra of a matrix ring because A satisfies no standard identity. To see this, note that given given $\sigma \in S_n$ we have

$$e_{\sigma(1)} e_{\sigma(2)} \cdots e_{\sigma(n)} = \operatorname{sgn}(\sigma) e_1 e_2 \cdots e_n$$

and so

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_{\sigma(1)} e_{\sigma(2)} \cdots e_{\sigma(n)} = n!(e_1 e_2 \cdots e_n) \neq 0.$$

An old (hard) problem in this area is to try to characterize those PI algebras which can appear as subalgebras of matrix algebras, which suggests the following

Definition 3.3. An algebra A over a field k is called *embeddable* if there is a positive integer r, a field extension K/k, and an injective k-algebra homomorphism $\varphi\colon A \hookrightarrow M_r(K)$.

Remark 3.4. There is a hidden unnecessary hypothesis in the above definition: we needn't insist that the homomorphism be unital, as a nonunital embedding can always be adjusted to give a unital one. To see this let $\varphi\colon A \hookrightarrow M_r(K)$ denote a nonunital k-algebra embedding, and set $e = \varphi(1)$, so that $\varphi(A) \subseteq eM_r(K)e$. Since e is idempotent, its only eigenvalues are 0 and 1. Let s denote the algebraic multiplicity of 1 as an eigenvalue of e. By considering the Jordan decomposition of e, we see that e is in fact diagonalizable by some invertible matrix $x \in M_r(K)$. (If e weren't diagonalizable, its Jordan form wouldn't be idempotent). By permuting the Jordan blocks if necessary, we may further assume that xex^{-1} has the form

$$xex^{-1} = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix},$$

where I_s denotes the $s \times s$ identity matrix. Set $\hat{e} = xex^{-1}$. The automorphism of $M_r(K)$ given by conjugation by x induces an isomorphism $eM_r(K)e \cong \hat{e}M_r(K)\hat{e}$, and the latter ring is easily seen to be isomorphic to $M_s(K)$. Putting it all together, we have

$$A \hookrightarrow eM_r(K)e \cong M_s(K),$$

with $1 \mapsto e \mapsto I_s$.

Let us review a few examples of embeddable algebras. If A is a k-algebra with a subfield L (not necessarily central!) such that A is finite dimensional over L, then A is embeddable via the (left) regular representation. A result of Kaplansky shows that any affine PI k-algebra which is also algebraic over k is necessarily finite dimensional, and hence embeddable. Another case which can be easily dispatched is that of commutative noetherian algebras: such rings are all embeddable, and we devote the next few paragraphs to a proof of this fact.

Definition 3.5. An ideal $I \lhd A$ is called *irreducible* if whenever I_1, I_2 are ideals of A properly containing I, then $I_1 \cap I_2 \supsetneq I$ as well. An algebra A is called *irreducible* of 0 is an irreducible ideal, ie. if the intersection of two nonzero ideals is again nonzero.

Remark 3.6. If an algebra A satisfies the ascending chain condition on two-sided ideals, then every ideal of A is a finite intersection of irreducible ideals. (Otherwise, choose an ideal $I \lhd A$ maximal with respect to not being irreducible. Then by definition we can find ideals $I_1, I_2 \supsetneq I$ with $I_1 \cap I_2 = I$. But by construction each I_j is a finite intersection of irreducible ideals, and then so too is I.) In particular, we see that a commutative noetherian algebra is a finite subdirect product of irreducible rings.

The next lemma is [52, Proposition 3.2.53].

Lemma 3.7. *If C is a commutative noetherian ring, then C embeds in a commutative artinian ring.*[1]

Proof. By the preceding remarks, C is a finite subdirect product of irreducible commutative noetherian rings, so we may assume that C is irreducible. Then every element of C must then be either regular or nilpotent. For suppose that $c \in C$ is not nilpotent. Since C is noetherian, the chain

$$\operatorname{ann}(c) \subseteq \operatorname{ann}(c^2) \subseteq \operatorname{ann}(c^3) \cdots$$

must terminate, with say $\operatorname{ann}(c^n) = \operatorname{ann}(c^{n+1})$. By replacing c by c^n, we may assume that $\operatorname{ann}(c) = \operatorname{ann}(c^2)$. If $cx = 0$ with $x \neq 0$, then since C is irreducible, we can find some nonzero element $y \in cC \cap xC$. Thus $y = cb$ for some $b \in C$, and then $0 = cy = c^2 b$ and since $\operatorname{ann}(c^2) = \operatorname{ann}(c)$, we have $0 = cb = y$. Thus c is not regular.

[1] In fact, Gordon [21] has shown that every right noetherian PI ring embeds in a right artinian PI ring.

Now, if $S = \{\text{regular elements of } C\}$, then CS^{-1} is a commutative noetherian ring with the property that every element of CS^{-1} is either nilpotent or a unit. We claim then that CS^{-1} is artinian. Let $N = \{\text{nilpotent elements of } CS^{-1}\}$ denote the nilradical of CS^{-1}, and note that since CS^{-1} is noetherian, N is nilpotent, with index of nilpotence n say. We filter CS^{-1} by powers of N, and note that N^i/N^{i+1} is a finitely generated CS^{-1}/N-module. We thus see that CS^{-1} is a finitely generated module over CS^{-1}/N, hence is artinian. □

Lemma 3.8. *Let B be a commutative local artinian k-algebra, then B is embeddable in $M_n(K)$ for some field extension K/k.*

Proof. Let $\mathfrak{m}$ denote the unique maximal ideal in B. Since B is artinian, we can apply a theorem of Cohen [12] to show that B contains a subfield K which is isomorphic to the residue field $B/\mathfrak{m}$. Note that $\mathfrak{m} = \text{Nil}(B)$ is the nilradical of B, and since B is noetherian, $\mathfrak{m}$ is nilpotent, with index of nilpotence r say. Filter B by powers of of $\mathfrak{m}$,

$$B \supsetneq \mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots \supsetneq \mathfrak{m}^r = 0,$$

and fix $i \geq 0$. Since B is artinian, $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finitely generated B-module which is annihilated by $\mathfrak{m}$, hence it is also a finitely generated module over $B/\mathfrak{m} \cong K$. But then we see that B is in fact a finite dimensional K-algebra, so it embeds in $M_n(K)$ via the left regular representation. □

Proposition 3.9. *Let C be a commutative noetherian k-algebra. Then C embeds in matrices over a field extension K/k.*

Proof. By Lemma 3.7, we can embed C into a commutative artinian k-algebra B. By [15, Corollary 2.16 p 76], B is a finite direct sum of commutative local artinian rings B_i, $i = 1, \ldots, s$. By Lemma 3.8, we have $B_i \subseteq M_{n_i}(K_i)$ for some field extensions K_i/k. Finally, let K be any field extension of k large enough so that each K_i embeds in K. Then

$$B_1 \oplus \cdots \oplus B_s \hookrightarrow M_{n_1}(K_1) \oplus \ldots \oplus M_{n_s}(K_s) \hookrightarrow M_n(K)$$

where $n = \sum_i n_i$, and where the last embedding comes from thinking of an element in the direct sum $\oplus_i M_{n_i}(K_i)$ as a block diagonal matrix in $M_n(K)$. □

3.1 Various Embedding Theorems

3.1.1 Amitsur-Small's Embedding

We show that if A is a right artinian PI k-algebra with $N^2 = 0$, where N is the nilradical of A (= Jacobson radical since A is right artinian), then A embeds in matrices over a field extension of K. This is an old (unpublished) result of Amitsur and Small, and as it is tangentially related to the work in this thesis, it seems fitting to review it now for the interested reader.

Lemma 3.10. *Let $A \subseteq B$ be k-algebras with B_A a finitely generated projective right A-module. If A is embeddable in matrices over a field extension of k, then so too is B.*

Proof. Since B_A is finitely generated projective, we can find a right A-module U such that $B \oplus U \cong A^n$ as right A-modules. Next we claim that we can embed B into $\mathrm{End}_A(A^n)$. Given $B \in B$ and $v \in A^n$, decompose v as $v = x + u$ for some $X \in B$ and $u \in U$ and define $\rho_b(v) = bx$. That this gives an A-module action of B on A^n follows from the fact that B is a (B, A)-bimodule. We thus get a (nonunital!) ring homomorphism $B \to \mathrm{End}_A(A^n)$ via $b \mapsto \rho_b$. The kernel of this homomorphism is $\{\, x \in B \mid xB = 0 \,\} = \{0\}$ since B has a 1. We have

$$B \hookrightarrow \mathrm{End}_A(A^n) \cong M_n(A)$$

and since by assumption we have $A \subseteq M_r(K)$ for some field extension K of k, we get $M_n(A) \subseteq M_n(M_r(K)) \cong M_{nr}(K)$. This shows that there is a nonunital embedding of B into $M_{nr}(K)$, and Remark 3.4 then shows that B is embeddable. $\square$

Lemma 3.11. *Let $A \subseteq B$ be k-algebras with B_A finitely generated projective and let M be an (B, B)-bimodule. Then $\left(\begin{smallmatrix} B & M \\ 0 & B \end{smallmatrix}\right)$ is a finitely generated projective right module over $\left(\begin{smallmatrix} A & M \\ 0 & A \end{smallmatrix}\right)$.*

Proof. Let $\{\, b_i \mid i \in I \,\}$ generate B as a right A-module, and without loss $1 = b_i$ for some $i \in I$. Then one checks that $\left(\begin{smallmatrix} B & M \\ 0 & B \end{smallmatrix}\right)$ is generated as a right $\left(\begin{smallmatrix} A & M \\ 0 & A \end{smallmatrix}\right)$-module by

$$\left\{ \begin{pmatrix} b_i & 0 \\ 0 & 0 \end{pmatrix} \mid i \in I \right\}.$$

Since B_A is finitely generated projective, we have $B_A \oplus U \cong A^n$ for some right A-module U. Then

$$\begin{pmatrix} B & M \\ 0 & B \end{pmatrix} \oplus \begin{pmatrix} U & M^{n-1} \\ 0 & U \end{pmatrix} \cong \begin{pmatrix} A^n & M^n \\ 0 & A^n \end{pmatrix} = \begin{pmatrix} A & M \\ 0 & A \end{pmatrix}^n$$

as right $\left(\begin{smallmatrix} A & M \\ 0 & A \end{smallmatrix}\right)$-modules, so $\left(\begin{smallmatrix} B & M \\ 0 & B \end{smallmatrix}\right)$ is projective as well.

□

Lemma 3.12. *If M_A and N_B are finitely generated right projective, then $M \oplus N$ is a finitely generated projective right $A \oplus B$-module.*

Proof. By hypothesis we can find modules U_A and V_B such that

$$M \oplus U \cong A^n \quad \text{and} \quad N \oplus V \cong B^m$$

for some $n, m \in \mathbb{N}$. We may assume that $n \leq m$. Set $\widehat{U} = U \oplus A^{m-n}$ and note then that $M \oplus \widehat{U} \cong A^m$, whence

$$(M \oplus N) \oplus (\widehat{U} \oplus V) \cong (A^m \oplus B^m) \cong (A \oplus B)^m$$

as right $(A \oplus B)$-modules. □

Proposition 3.13 (Amitsur)**.** *Let A be a semisimple right artinian* PI *k-algebra. Then A embeds in matrices over a field extension of k.*

Proof. This is well known. See [1] for the proof. □

Proposition 3.14 (Amitsur-Small)**.** *Let A be a right artinian* PI *k-algebra with nilradical N satisfying $N^2 = 0$. Then A is embeddable in matrices over a field extension of k.*

Proof. We employ the embedding procedure of Lewin from [35]. For a wonderful exposition of this idea, see [53, §6.3]. Lewin describes an algebra embedding

$$A \hookrightarrow \begin{pmatrix} A/N & M \\ 0 & A/N \end{pmatrix}$$

where M is a certain $(A/N, A/N)$-bimodule, the precise nature of which is of no concern to us here. To embed A into matrices it is thus enough to show that the latter formal triangular matrix ring is embeddable.

A/N embeds in $S := M_r(K)$ for some field extension K by Proposition 3.13. Since A/N is semisimple artinian, S is flat as a left and right S-module. Thus we have an $(A/N, A/N)$-bimodule embedding

$$M \hookrightarrow V := S \otimes_{A/N} M \otimes_{A/N} S,$$

which then yields an algebra embedding

$$\begin{pmatrix} A/N & M \\ 0 & A/N \end{pmatrix} \hookrightarrow \begin{pmatrix} S & V \\ 0 & S \end{pmatrix}.$$

The ring $T := \left(\begin{smallmatrix} S & V \\ 0 & S \end{smallmatrix}\right)$ is semiprimary, with Jacobson radical $\mathrm{Jac}(T) = \left(\begin{smallmatrix} 0 & V \\ 0 & 0 \end{smallmatrix}\right)$ satisfying $\mathrm{Jac}(T)^2 = 0$. By [28, Remark 2.9], T can be embedded in a commutative artinian k-algebra, and the latter ring is embeddable by Proposition 3.9. □

Unfortunately, this technique doesn't seem to generalize to the case where $\mathrm{Nil}(A)$ has higher index of nilpotence. If it did, then since every right noetherian PI algebra embeds in a right artinian PI algebra, we would immediately be able to answer the following open question in the affirmative:

Question 3.15. *Is every right noetherian* PI *k-algebra embeddable?*

In Proposition 3.14 it is crucial that A contain the field k, as the following example of Bergman shows [9].

Example 3.16 (Bergman)**.** Let p be a prime number and set $R = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_p \oplus \mathbb{Z}_{p^2})$, which is a *finite* ring, and hence trivially right artinian. Viewing elements of $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ as column vectors, we can think of R as the formal matrix ring

$$R = \begin{pmatrix} \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_p) & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^2}, \mathbb{Z}_p) \\ \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_{p^2}) & \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^2}) \end{pmatrix}.$$

Let e_1, e_2 denote the identity maps on $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ respectively, so that as abelian groups we have

$$\langle e_1 \rangle = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_p) \text{ and } \langle e_2 \rangle = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^2}).$$

Let f denote the natural quotient map $f\colon \mathbb{Z}_{p^2} \to Z_p$, and let g denote the inclusion map from $\mathbb{Z}_p$ to $\mathbb{Z}_{p^2}$. To be precise, if $[a]_n$ denotes an arbitrary element of $\mathbb{Z}_n$, then the map g is given by

$$g([a]_p) = [pa]_{p^2},$$

and so we have the following relations:

$$f \circ g = 0 \ \text{ and } \ g \circ f = pe_2. \tag{†}$$

As an additive group, we then see that

$$R \cong \begin{pmatrix} \langle e_1 \rangle & \langle f \rangle \\ \langle g \rangle & \langle e_2 \rangle \end{pmatrix}.$$

Next, using the relations (†), one checks that the ideal

$$J := \begin{pmatrix} 0 & \langle f \rangle \\ \langle g \rangle & \langle pe_2 \rangle \end{pmatrix}$$

satisfies $J^2 = 0$ and $R/J \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, and so J is the nil radical of R. In [9], Bergman shows that this ring R cannot be embedded in matrices over any commutative ring (here we really do mean "ring", and not "algebra"). Of course, R isn't an algebra over a field because $\mathbb{Z}_{p^2}$ doesn't contain a field.

3.1.2 Embedding Stably Noetherian Rings

In the early 40's, Malcev [37] proved that an affine algebra is embeddable iff it is a subdirect product of finite dimensional algebras whose dimensions are uniformly bounded.[2] As for known sufficient conditions, an old result of Amitsur [1] shows that every semiprime PI algebra is embeddable. Beidar [7] proved that an algebra A which is a finitely generated module over a commutative central noetherian subalgebra is embeddable, and Lewin [35] proved that A is embeddable if A has the property that $[A, A]^2 = 0$, where $[A, A]$ is the ideal generated by all additive commutators in A.

[2]Either hypothesis forces the ring to satisfy a polynomial identity, though Malcev's theorem actually predates the formal definition of a PI algebra.

Clearly, any embeddable algebra must satisfy all identities of $r \times r$ matrices (for some r), but this is not a sufficient condition. Amitsur [3] found the first example of a (non-affine) nonembeddable PI algebra satisfying all identities of $r \times r$ matrices. In the same paper, Amitsur observed that any affine subalgebra of $M_r(K)$ would have a nilpotent Jacobson radical, and he posed the following question:

Question 3.17 (Amitsur)**.** *If an affine algebra satisfies all identities of $r \times r$ matrices, and has a nilpotent Jacobson radical, must it be embeddable?*

(Work of Lewin [34], Kemer [30], and Braun [11] later showed that these two conditions are automatically satisfied by affine PI algebras). In the 1970's and 1980's, a series of counterexamples and successive refinements were made to this question, which we review now for the reader's convenience.

Small [56] found an example of an affine PI algebra satisfying all identities of $r \times r$ matrices but still not embeddable. His example failed to satisfy the ascending chain condition on left or right annihilator ideals. If it were a subalgebra of a matrix algebra, then as Small notes, it would also embed in $M_r(C)$ for some affine commutative algebra C. Since $M_r(C)$ is noetherian, the original algebra would inherit the ascending chain condition on annihilators, and this is how Small concluded that his example was not embeddable.

In [25] Irving went one step further, constructing an affine PI algebra which satisfies the ascending chain condition on annihilators but which is still not embeddable. However, his example had chains of annihilators of arbitrary length. Nonembeddability is a consequence of the fact that every commutative noetherian ring embeds in a commutative artinian ring, and so combined with Small's observation, any embeddable algebra inherits a bound to the lengths of chains of left and right annihilator ideals. The obvious reformulation was then to ask whether Amitsur's conditions, along with a bound to the lengths of chains of annihilator ideals, imply embeddability. Irving and Small [27] showed that the answer is still negative by demonstrating the existence an affine PI algebra for which any chain of annihilators has length at most 4, yet which is not embeddable.

Thus the quest to find a good set of sufficient conditions for embeddability stemming from Amitsur's conditions seemed fruitless, except as a source of good counterexamples. Of course, all of these examples failed to be (right) noetherian, so one might ask whether all affine right noetherian PI algebras are embeddable in matrices. A brilliant result of Ananin [4] answered this question in the affirmative in the early 1990's. Small (unpublished) generalized this slightly to the following: If A is a right noetherian PI k-algebra which is affine as a C-algebra, for some commutative noetherian central subalgebra C, then A is embeddable.

As pleasing as Ananin's result is, one would ideally like to do away with the affine hypothesis. Although we have yet to accomplish this in full generality, we have been able to show that Ananin's result still holds in the case of stably right noetherian algebras (without the affine hypothesis). The proof of this is the main purpose of this section.

Definition 3.18. A prime ring is called *right bounded* if every essential right ideal contains a nonzero two-sided ideal and a ring A is called *right fully bounded* if A/P is right bounded for every prime ideal P of A. If A is right fully bounded and right noetherian, then we will say that A is a *right FBN* ring.

Fully bounded rings are germane to our conversation because every PI ring is right fully bounded [53, Proposition 6.1.48]. In addition, we have the following theorem from [20].

Theorem 3.19 (Goldie). *If A is a right noetherian ring then the following are equivalent:*

1. *A is right fully bounded.*
2. *(Gabriel's H-condition) Given any finitely generated right A-module M there exist elements $m_1, \ldots, m_t \in M$ such that*

$$\operatorname{ann}_A(M) = \bigcap_{i=1}^{t} \operatorname{ann}_A(m_i).$$

We will also need the following lemma, which is probably well known.

Lemma 3.20. *Let A be a right FBN algebra over a field k, and let M be a finitely generated right A-module. Suppose M has an essential submodule N with $\dim_k(N) < \infty$. Then $\dim_k(M) < \infty$.*

Proof. We may assume that M is faithful. Put $I = \operatorname{ann}_A(N)$, and $L = \operatorname{l.ann}(I)$. Note that L is both a left $A/\operatorname{l.ann}(L)$-module and a right A/I-module. Now, $\dim_k(N) < \infty$, and since N is a faithful right A/I-module, we have $\dim_k(A/I) < \infty$. Since $LI = 0$, L is a right A/I-module, and since A/I is right noetherian, L is a finitely generated right A/I-module. So we see that $\dim_k(L) < \infty$. Since L is a faithful left $A/\operatorname{l.ann}(L)$-module, we also see that

$$\dim_k(A/\operatorname{l.ann}(L)) < \infty.$$

Next note that if $x \in A$ and $Mx \subseteq N$, then $(Mx)I = 0$, and so $xI = 0$ since M is faithful over A. That is, $\operatorname{ann}_A(M/N) \subseteq L$. A is right FBN, so A satisfies Gabriel's H-condition and we can find elements $m_1, \ldots, m_t \in M$ with

$$\operatorname{ann}_A(M/N) = \bigcap_{i=1}^{t} \operatorname{ann}_A(m_i + N).$$

Since N is essential in M, each $\operatorname{ann}_A(m_i + N)$ is essential as a right ideal of A. To see this, choose $J \lhd_r A$, $J \neq 0$. We may assume that $m_i J \neq 0$ for otherwise $J \subseteq \operatorname{ann}_A(m_i + N)$ and we're done. Since $m_i J$ is a nonzero submodule of M, and N is essential in M, we have $m_i J \cap N \neq 0$. Thus we can find a nonzero $x \in J$ with $m_i x \in N$, ie

$$(m_i + N)x = N \quad \text{ in } M/N$$

and so $x \in \operatorname{ann}_A(m_i + N) \cap J$, which shows that $\operatorname{ann}_A(m_i + N)$ is essential. Since the intersection of essential submodules is again essential, we see that $\operatorname{ann}_A(M/N)$ is essential, and hence so is L since $L \supseteq \operatorname{ann}_A(M/N)$. It follows that $\operatorname{l.ann}(L)$ is contained in the right singular ideal of A, and hence $\operatorname{l.ann}(L)$ is nilpotent [40, Lemma 2.3.4]. Thus $\operatorname{l.ann}(L)$ is nilpotent and of finite codimension in A, which forces A to be finite dimensional over k, and thus M is finite dimensional over k as well. □

Definition 3.21. If M is a right A-module, we call M *representable* if there is a field extension K/k and a right $A \otimes_k K$-module X such that $\dim_K(X) < \infty$, and $M \subseteq X$ as right A-modules.

Lemma 3.22. *If the right A-module A_A is representable, then A is embeddable.*

Proof. Suppose $A \subseteq X$ for some right $A \otimes_k K$ module X. Then X is a faithful right A-module (for $Xa = 0 \implies Aa = 0$, whence $a = 0$.) Note that since K is commutative, X is also a left K-module in the usual way. We can then think of X as a (K, A)-bimodule, and as such the actions of A and K on X commute. Since X is faithful as A-module, this gives an algebra injection $A \hookrightarrow \mathrm{End}_K(X) \cong M_r(K)$, where $r = \dim_K(X)$. □

In fact the converse is also true, but we will only need this direction.

Theorem 3.23. *Let A be a stably right noetherian* PI *algebra over k, then A is embeddable.*

Notation. For the remainder of this section, A will always denote a stably right noetherian PI algebra over k.

The previous theorem will follow, via Lemma 3.22, once we prove

Theorem 3.24. *Any finitely generated right A-module M is representable.*

Proof. The proof is by contradiction. So suppose that M is a finitely generated right A-module which is not representable. Since M is noetherian, we can find a submodule $N \leq M$ which is maximal with respect to M/N being not representable. Replacing M by M/N, we may assume that M is not representable, but every proper quotient module of M is representable.

The idea of the proof is to show that M has no nonzero representable submodules, and then find a nonzero submodule of M which is representable.

Claim 1. M is uniform.

Proof of Claim 1. If there are nonzero A-submodules $U_1, U_2 \leq M$ such that $U_1 \cap U_2 = 0$, then we get an A-module injection $M \hookrightarrow M/U_1 \oplus M/U_2$. By our assumption on M, each M/U_i is representable. So there are field extensions K_i and right $A \otimes_k K_i$-modules X_i with $\dim_{K_i}(X_i) < \infty$, such that $M/U_i \hookrightarrow X_i$ as right A-modules. Choose a field extension K/k large enough so that each K_i embeds in K (as fields), and set

$$X = (X_1 \otimes_{K_1} K) \oplus (X_2 \otimes_{K_2} K).$$

Then X is a right $A \otimes_k K$-module containing each X_i. Lastly, $\dim_K(X) = \dim_{K_1}(X_1) + \dim_{K_2}(X_2)$, which is finite. □

Claim 2. M has no nonzero representable submodules.

Proof of Claim 2. Suppose $B \leq M$ is a nonzero representable submodule of M. Then $B \subseteq X$ for some right $A \otimes_k K$ module X with $\dim_K(X) < \infty$. Extend the A-module embedding to an $A \otimes_k K$-module map $\varphi\colon B \otimes_k K \to X$ by setting $\varphi(b \otimes \alpha) = b\alpha$ and extending linearly. Let $H \leq B \otimes_k K$ denote the kernel of φ, and note that $H \cap B = 0$. Set $\overline{M} = (M \otimes_k K)/H$, and $\overline{B} = (B \otimes_k K)/H$. Since $\overline{B} \cong \varphi(B \otimes_k K) \subseteq X$, we see that $\overline{B}$ is a $A \otimes_k K$-submodule of $\overline{M}$ which is finite dimensional over K.

Next, since $A \otimes_k K$ is noetherian, and $M \otimes_k K$ is a finitely generated right $A \otimes_k K$-module, $\overline{M}$ is a noetherian $A \otimes_k K$-module. Thus we can choose a submodule $\overline{U} \leq \overline{M}$ maximal with respect to $\overline{U} \cap \overline{B} = 0$. This gives an $A \otimes_k K$-module embedding $\overline{B} \hookrightarrow \overline{M}/\overline{U}$. By our choice of $\overline{U}$ the image of $\overline{B}$ in $\overline{M}/\overline{U}$ intersects every nonzero submodule of $\overline{M}/\overline{U}$, which shows that the image of $\overline{B}$ is an essential submodule of $\overline{M}/\overline{U}$.

Since $A \otimes_k K$ is right noetherian PI and hence right FBN, and $\overline{M}/\overline{U}$ is a finitely generated module over $A \otimes_k K$, we can apply Lemma 3.20 to conclude that $\dim_K \overline{M}/\overline{U} < \infty$. In particular, since M is not representable, M does not embed in $\overline{M}/\overline{U}$, so if we let U denote the preimage in $M \otimes_k K$ of $\overline{U}$, then $U \cap M \neq 0$.

Next, since $\overline{B} \cap \overline{U} = 0$, we have

$$(B \otimes_k K)/H \bigcap U/H = 0 \quad \text{in } (M \otimes_k K)/H, \text{ so}$$
$$(B \otimes_k K) \bigcap U \subseteq H.$$

In particular, $U \cap B \subseteq H \cap B$, so we have $(U \cap M) \cap B = U \cap B \subseteq H \cap B = 0$. Since $U \cap M$ and B are nonzero submodules of M, this contradicts Claim 1. □

Now that we have shown that M has no nonzero representable submodules, we will finish the proof of Theorem 3.24 by finding a submodule of M which is representable. To that end, choose an ideal $P \lhd A$ maximal among the annihilators of the nonzero submodules of M, and set $V = \operatorname{ann}_M(P) = \{\, m \in M \mid mP = 0 \,\}$

a nonzero submodule of M. We first claim that P is a prime ideal. Otherwise, we can find ideals I, J with $I, J \supsetneq P$ and $IJ \subseteq P$. Note that $VI \neq 0$ by our choice of P. But then VI is a nonzero submodule of M annihilated by J, again contradicting our choice of P.

Moreover, if $0 \neq U \leq V$, then $UP = 0$, so $P = \mathrm{ann}_A(U)$ by maximality. It follows that V is a torsion-free A/P-module. For if $vc = 0$ for some $0 \neq v \in V$ and some regular $c \in A/P$, then since c generates an essential right ideal of A/P and A is FBN, cA contains some ideal $I \supsetneq P$ of A. Thus $0 = vcA \supseteq vI = (vA)I$, again contradicting our choice of P. So we conclude that V is a torsion-free right A/P-module, and so V embeds (as A-module) into $V \otimes_{A/P} Q$ where Q is the classical right ring of quotients of the prime PI ring A/P.

Let K denote the center of Q and recall that $\dim_K(Q) < \infty$. Since V is a submodule of the noetherian A-module M, V is finitely generated as right A-module, and since $P = \mathrm{ann}_A(V)$, V is also finitely generated as an A/P-module. It then follows that $V \otimes_{A/P} Q$ is finitely generated as a right $A/P \otimes_{A/P} Q$ module, and since $A/P \otimes_{A/P} Q \cong Q$, we see that $V \otimes_{A/P} Q$ is a finitely generated module over the finite dimensional K-algebra Q, hence $\dim_K(V \otimes_{A/P} Q) < \infty$.

This shows that V is representable, contradicting Claim 2. □

A careful reading of the proof of Claim 2 above yields the following

Corollary 3.25. *Let A be a stably right noetherian k-algebra and let M be a finitely generated right A-module. If $N \leq M$ is a submodule such that N and M/N are representable, then M is representable.*

3.2 Universal Constructions

3.2.1 Generic Triangular Matrices

In this section we take a rather different approach, and look for conditions for embedding algebras into upper triangular matrix rings. The material in this section closely resembles work of Procesi [44] and Amitsur [3]. Given a commutative k-

algebra C, we let $T_r(C)$ denote the ring of $r \times r$ upper triangular matrices with entries in C.

Let $\mathcal{I}$ be an arbitrary index set, $X = \{\, x_i \mid i \in \mathcal{I} \,\}$ a collection of noncommuting indeterminates, and let $k\langle X\rangle$ denote the free algebra on the set X. Let $\mathcal{M}_r$ denote the ideal of $k\langle X\rangle$ generated by all polynomial identities for $T_r(C)$, where C is any commutative k-algebra. If A is a k-algebra with the property that, given any $f \in \mathcal{M}_r$, f vanishes on A, we say that A *satisfies all identities of* $r \times r$ *triangular matrices.*

Definition 3.26. The algebra $k\langle X\rangle/\mathcal{M}_r$ is called the *relatively free algebra* in the category of algebras satisfying all identities of $r \times r$ triangular matrices.

The terminology is justified by the following easy remark.

Remark 3.27. If A is a k-algebra (generated by at most $|\mathcal{I}|$ elements) then there is a surjective algebra homomorphism $k\langle X\rangle \twoheadrightarrow A$. If in addition A satisfies all identities of $r \times r$ triangular matrices then this map factors through $k\langle X\rangle/\mathcal{M}_r$. Said another way, $k\langle X\rangle/\mathcal{M}_r$ is the universal object in the category of algebras satisfying all identities of $r \times r$ triangular matrices.

Definition 3.28. Let $\Lambda = \{\, \lambda_{ij}^k \mid 1 \leq i \leq j \leq r,\, k \in \mathcal{I} \,\}$ be a set of commuting indeterminates, and denote by $k[\Lambda]$ the commutative algebra generated by the set Λ. Let $\{e_{ij}\}$ be the standard matrix units, and let

$$M_k = \sum_{i,j} \lambda_{ij}^k e_{ij}$$

denote the matrix whose (i,j)-entry is λ_{ij}^k. Let $\mathrm{M} = \{\, M_k \mid k \in \mathcal{I} \,\}$ and let $k\{\mathrm{M}\}$ denote the subalgebra of $T_r(k[\Lambda])$ generated by the set M. $k\{\mathrm{M}\}$ is called the *algebra of generic* $(r \times r)$ *triangular matrices.*

The idea now is to show that $k\{\mathrm{M}\}$ is a concrete realization of the relatively free algebra $k\langle X\rangle/\mathcal{M}_r$. More precisely, we wish to prove the following

Proposition 3.29. *The kernel of the natural map* $\varphi\colon k\langle X\rangle \to k\{\mathrm{M}\}$ *given by* $\varphi(x_k) = M_k$ *is precisely* $\mathcal{M}_r$.

Proof. Since $k\{\mathrm{M}\}$ is a subalgebra of $T_r(k[\lambda])$, $k\{\mathrm{M}\}$ certainly satisfies all identities of $T_r(k[\lambda])$. Given $f \in \mathcal{M}_r$, f is a polynomial identity for $T_r(k[\lambda])$, and hence also for $k\{\mathrm{M}\}$. Thus $\mathcal{M}_r \subseteq \ker(\varphi)$.

For the other inclusion, choose $f = f(x_1, \dots, x_s) \in \ker(\varphi)$. Let B be an arbitrary commutative k-algebra, and let $b_1, \dots, b_s$ be arbitrary elements of $T_r(B)$. We need to show that $f(b_1, \dots, b_s) = 0$.

Define a map $\rho\colon k\{\mathrm{M}\} \to T_r(B)$ by choosing s elements, say $M_1, \dots, M_s$ from $k\{\mathrm{M}\}$ and defining

$$\rho(M_k) = \begin{cases} b_k, & \text{if } 1 \le k \le s \\ 0 & \text{for all other } k \in \mathcal{I}. \end{cases}$$

The map ρ is merely "specialization," obtained by specifying values for the indeterminates λ_{ij}^k, and as such it is clear that it is a homomorphism. (Of course, ρ need not be surjective, but what matters is that the subalgebra of $T_r(B)$ generated by $\{b_1, \dots, b_s\}$ is contained in the image of ρ.) Now, since $f \in \ker(\varphi)$, f is an identity for $k\{\mathrm{M}\}$, and hence $f(M_1, \dots, M_s) = 0$. But then

$$0 = \rho(f(M_1, \dots, M_s)) = f(\rho(M_1), \dots, \rho(M_s)) = f(b_1, \dots, b_s).$$

□

In contrast to the full ring of generic matrices, which Procesi [44] showed is a prime ring, the ring $k\{\mathrm{M}\}$ isn't even semiprime. One might like to compute the Jacobson radical of the generic triangular matrix ring, and this is what we turn to next. View $k\{\mathrm{M}\}$ as a subalgebra of $T_r([k\Lambda])$ and recall that the Jacobson radical of $T_r(k[\Lambda])$ is given by $U_r(k[\Lambda])$, the ring of strictly upper triangular matrices with entries in $k[\Lambda]$. Moreover, $\mathrm{Jac}(T_r(k[\Lambda])) = [T_r(k[\Lambda]), T_r(k[\Lambda])]$, the ideal of $T_r(k[\Lambda])$ generated by all (additive) commutators. As it turns out, the Jacobson radical of $k\{\mathrm{M}\}$ is similar. Specifically, we have the following

Proposition 3.30. $\mathrm{Jac}(k\{\mathrm{M}\}) = [k\{\mathrm{M}\}, k\{\mathrm{M}\}]$, *the commutator ideal of* $k\{\mathrm{M}\}$.

Proof. Since $\mathrm{Jac}(T_r(k[\Lambda])) = U_r(k[\Lambda])$, we see that $k\{\mathrm{M}\} \bigcap \mathrm{Jac}(T_r(k[\Lambda]))$ is a nilpotent ideal of $k\{\mathrm{M}\}$ and hence is contained in $\mathrm{Jac}(k\{\mathrm{M}\})$. Next, $[k\{\mathrm{M}\}, k\{\mathrm{M}\}]$

is clearly contained in $k\{\mathrm{M}\} \bigcap U_r(k[\Lambda])$, and so we only need to check that

$$\mathrm{Jac}(k\{\mathrm{M}\}) \subseteq [k\{\mathrm{M}\}, k\{\mathrm{M}\}].$$

To that end, choose $a \in \mathrm{Jac}(k\{\mathrm{M}\})$ and observe that a must have zeros along its diagonal since, in particular, $a \in U_r(k[\Lambda])$. Consider the homomorphism $\pi\colon k\{\mathrm{M}\} \to T_r(k[\Lambda])$ defined on generators by

$$\pi(M_k) = D_k, \text{ where } D_k = \sum_i \lambda_{ii}^k e_{ii}.$$

The D_k are algebraically independent, and they commute, so $\pi(k\{\mathrm{M}\})$ is just a polynomial ring in the variables $\{\, D_k \mid k \in \mathcal{I}\,\}$. Identifying $\pi(k\{\mathrm{M}\})$ with $k[X]$, where $X = \{\, x_k \mid k \in \mathcal{I}\,\}$ is a set of commuting indeterminates yields the commutative diagram

$$\begin{array}{ccc} k\langle X\rangle & \xrightarrow{\varphi} & k\{\mathrm{M}\} \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ k[X] & \xrightarrow{\cong} & \pi(k\{\mathrm{M}\}) \end{array}$$

Here the maps are the obvious ones: the bottom isomorphism comes from identifying $\pi(k\{\mathrm{M}\})$ with $k[X]$, φ is the defining homomorphism for $k\{\mathrm{M}\}$, and π' is the homomorphism whose kernel is $[k\langle X\rangle, k\langle X\rangle]$, the commutator ideal in the free algebra.

Now, since $a \in k\{\mathrm{M}\}$ was chosen to have zeros along its diagonal, $\pi(a) = 0$. Lift a to an element $a' \in k\langle X\rangle$. Since the bottom map is an isomorphism, we have $\pi'(a') = 0$, whence $a' \in [k\langle X\rangle, k\langle X\rangle]$. The image of $[k\langle X\rangle, k\langle X\rangle]$ under φ is contained in $[k\{\mathrm{M}\}, k\{\mathrm{M}\}]$, and so $a = \varphi(a') \in [k\{\mathrm{M}\}, k\{\mathrm{M}\}]$ as desired. □

We next characterize the two-sided ideals of $T_r(S)$, where S is any ring. Of particular interest is the case when $S = k[\Lambda]$.

Lemma 3.31. *Let S be any ring. A subring L of $T_r(S)$ is an ideal of $T_r(S)$ iff L is of the form*

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1r} \\ 0 & L_{22} & \cdots & L_{2r} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & L_{rr} \end{pmatrix}$$

where each L_{ij} is an ideal of S and moreover we have $L_{ij} \subseteq L_{kl}$ whenever $k \leq i$ and $l \geq j$. That is, $L_{ij} \subseteq L_{kl}$ whenever L_{kl} lies "above and to the right" of L_{ij}.

Proof. That a subring of the above form is actually an ideal is easily checked. The actual content of the theorem is the reverse implication. Let $L \lhd T_r(S)$. Clearly L decomposes as above, where a priori each L_{ij} is a subgroup of S. Note that the (i, j)-entry in the product $LT_r(S)$ is given by

$$L_{ii}S + L_{i,i+1}S + \ldots + L_{ij}S = \sum_{k=i}^{j} L_{ik}S,$$

while the (i, j)-entry in the product $T_r(S)L$ is given by

$$SL_{ij} + SL_{i+1,j} + \ldots + SL_{jj} = \sum_{k=i}^{j} SL_{kj}.$$

Clearly we must have $\sum_{k=i}^{j} L_{ik}S \subseteq L_{ij}$ and $\sum_{k=i}^{j} SL_{kj} \subseteq L_{ij}$. These together imply that L_{ij} is an ideal of S. As a consequence, we see that

$$\sum_{k=i}^{j} L_{ik} \subseteq L_{ij} \quad \text{and} \quad \sum_{k=i}^{j} L_{kj} \subseteq L_{ij}$$

and so

$$L_{ik} \subseteq L_{ij} \text{ whenever } i \leq k \leq j,$$

and also

$$L_{kj} \subseteq L_{ij} \text{ whenever } i \leq k \leq j$$

which is what we set out to show. □

This lemma has the following pleasant (though probably not useful) corollary.

Corollary 3.32. *Let B be a commutative semiprimary k-algebra and let I be an ideal of B which contains* Jac(B)*. If L is an ideal of $T_r(B)$ of the form*

$$L = \begin{pmatrix} I & * & \cdots & * \\ & I & \ddots & \vdots \\ & & \ddots & * \\ & & & I \end{pmatrix}$$

then $T_r(B)/L$ embeds in $T_r(B/I)$.

Proof. First note that B/I is a commutative semisimple artinian ring, hence B/I is a finite direct product of fields. Since L has the above form, $T_r(B)/L$ has the form

$$T_r(B)/L = \begin{pmatrix} B/I & B_{12} & \cdots & B_{1r} \\ & B/I & \ddots & \vdots \\ & & \ddots & B_{r-1\,r} \\ & & & B/I \end{pmatrix}$$

where, by Lemma 3.31, each B_{ij} is a homomorphic image of B/I, and as such is just a shorter direct product of fields, which in turn naturally embed into B/I. Hence we see that $T_r(B)/L \hookrightarrow T_r(B/I)$. □

As another application of Lemma 3.31, we can show that homomorphic images of triangular matrix rings over commutative rings are again embeddable in triangular matrices (of the same size even). The idea of the proof is due to Bergman (private communication).

Proposition 3.33. *Let C be a commutative k-algebra, and let R be a homomorphic image of $T_r(C)$. Then R is embeddable in $T_r(B)$ for some commutative k-algebra B.*

Proof. First we need a bit of notation. Since we are dealing with upper triangular matrices, an ordered pair (i,j) will be understood to satisfy $i \leq j$. We impose a partial order $\preceq$ on the set of ordered pairs $\{\,(i,j) \mid 1 \leq i \leq j \leq r\,\}$ by declaring

$$(i,j) \preceq (k,l) \iff k \leq i \text{ and } l \geq j.$$

By Lemma 3.31, R has the form

$$R = \begin{pmatrix} R_{11} & \cdots & R_{1r} \\ & \ddots & \vdots \\ & & R_{rr} \end{pmatrix}$$

with the property that each R_{ij} is a commutative k-algebra, and moreover R_{kl} is a homomorphic image of R_{ij} whenever $(i,j) \preceq (k,l)$.

To begin, let $B = \oplus_{k,l} R_{kl}$ denote the direct sum of the $n(n+1)/2$ algebras R_{kl}, and note that B is a commutative k-algebra. Each R_{kl} is a homomorphic image of

C, and as such B has a natural (C, C)-bimodule structure. Moreover, the left and right actions of C on B are the same: $cb = bc$ for all $c \in C, b \in B$. For each tuple (i, j), let E_{ij} denote that element of B whose R_{kl}-component is 1 if $(k, l) \succeq (i, j)$ and is 0 otherwise. Given $a_{ij} \in R_{ij}$, we let $a_{ij}E_{ij}$ denote the action of a_{ij} on E_{ij}. This makes sense because

$$\bigoplus_{(k,l)\succeq(i,j)} R_{k,l}$$

has the structure of an (R_{ij}, R_{ij})-bimodule inherited from the surjective k-algebra homomorphism from C onto R_{ij}. Since the left and right actions are the same, we have $E_{ij}a_{ij} = a_{ij}E_{ij}$.

To be very precise, $a_{ij}E_{ij}$ is that element of B whose R_{kl}-component is *the image of* a_{ij} in R_{kl} whenever $(i, j) \preceq (k, l)$, and is 0 otherwise.

Now note the following:

$$E_{ij}E_{kl} = E_{\min(i,k)\max(j,l)} \qquad \text{(in } B\text{)}$$

$$(E_{ij}e_{ij})(E_{kl}e_{kl}) = \begin{cases} E_{il}e_{il} & \text{if } j = k \\ 0 & \text{else.} \end{cases} \qquad \text{(in } T_r(B)\text{)}$$

Both of these facts are readily checked. In particular, the second assertion follows from the first.

Now, an element $a \in R$ has the form $a = \sum_{i,j} a_{ij}\overline{e}_{ij}$, where $a_{ij} \in R_{ij}$ and $\overline{e}_{ij}$ denotes the image of the corresponding matrix unit in R. (Hopefully this won't cause confusion, as we have no need to distinguish the matrix units of $T_r(C)$ from their images in R). Let e_{ij} denote the corresponding matrix unit in $T_r(B)$. Define a map $\varphi\colon R \to T_r(B)$ by

$$\sum_{(i,j)} a_{ij}\overline{e}_{ij} \to \sum_{(i,j)} a_{ij}E_{ij}e_{ij}.$$

We next need to check that φ is a k-algebra homomorphism. Let

$$a = \sum_{(i,j)} a_{ij}\overline{e}_{ij}, \quad \text{and} \quad b = \sum_{(k,l)} b_{kl}\overline{e}_{kl}$$

be elements of R, so that by the usual rules of matrix multiplication we have

$$ab = \sum_{(i,l)} (\sum_{j=i}^{l} a_{ij}b_{jl})\overline{e}_{il}.$$

We now compute

$$\begin{aligned}
\varphi(a)\varphi(b) &= \Big(\sum_{(i,j)} a_{ij}E_{ij}e_{ij}\Big)\Big(\sum_{(k,l)} b_{kl}E_{kl}e_{kl}\Big) \\
&= \sum_{(i,j),(k,l)} a_{ij}b_{kl}(E_{ij}e_{ij})(E_{kl}e_{kl}) \\
&= \sum_{\substack{i,l \\ i\le l}} (\sum_{j=i}^{l} a_{ij}b_{jl})E_{il}e_{il} \\
&= \varphi(ab).
\end{aligned}$$

It is clear that φ is injective, but it is not unital, since $\varphi(1) = \sum_i E_{ii}e_{ii}$, which is not the identity element of $T_r(B)$. However, we can remedy this deficiency with a bit of sleight of hand.

We wish to modify φ to a unital k-algebra homomorphism $\theta\colon R \to T_r(B)$. To do this, let the nondiagonal entries of the image of $\sum_{ij} a_{ij}\overline{e}_{ij}$ be defined just as before. For the (i,i)-entry, use that element of B whose R_{kl}-component is still as above when $(k,l) \succeq (i,j)$, but for any other (k,l) is the image of a_{kk} in R_{kl}. (In the original map φ, all of these "out of the box" components were 0.) If we let f_{kl} denote the image of 1_B in R_{kl} under the natural projection, then we see that the (i,j)-entry of $\theta(a)$ is given by

$$(i,j)\text{-entry of } \theta(a) = \begin{cases} a_{ij}E_{ij}, & \text{if } i<j \\ a_{ii}E_{ii} + \displaystyle\sum_{(k,l)\not\succeq(i,i)} a_{kk}f_{kl}, & \text{if } i=j \end{cases}$$

Setting

$$\eta_i(a) := \sum_{(k,l)\not\succeq(i,i)} a_{kk}f_{kl} \;\text{ and }\; \eta(a) := \sum_{i=1}^{r} \eta_i(a)e_{ii},$$

we can write $\theta(a) = \varphi(a)+\eta(a)$. It follows that θ is a ring homomorphism provided that $\eta(ab) = \varphi(a)\eta(b) + \eta(a)\varphi(b) + \eta(a)\eta(b)$, for all $a, b \in R$. In fact we will show that η is a (nonunital) ring homomorphism, and moreover that $\varphi(a)\eta(b) =$

$\eta(a)\varphi(b) = 0$. Let us deal with $\varphi(a)\eta(b)$ first. We compute

$$\begin{aligned}\varphi(a)\eta(b) &= \Big(\sum_{(i,j)} a_{ij}E_{ij}e_{ij}\Big)\Big(\sum_{t=1}^{r}\eta_t(b)e_{tt}\Big)\\ &= \sum_{(i,j)} a_{ij}E_{ij}\eta_j(b)e_{ij}\\ &= \sum_{(i,j)} a_{ij}E_{ij}\Big(\sum_{(k,l)\not\succeq(j,j)} b_{kk}f_{kl}\Big)e_{ij}\\ &= \sum_{(i,j)}\Big(\sum_{(k,l)\not\succeq(j,j)} a_{ij}b_{kk}E_{ij}f_{kl}\Big)e_{ij}. \qquad (\dagger)\end{aligned}$$

Note that

$$E_{ij}f_{kl} = \begin{cases} f_{kl}, & \text{if } (i,j)\preceq(k,l) \\ 0, & \text{else}\end{cases} \qquad (\text{in } B)$$

Fix a tuple (i,j) and note that since $i \le j$, $(j,j) \preceq (i,j)$. So we see that $(k,l) \not\succeq (j,j) \implies (k,l) \not\succeq (i,j)$. It follows that every summand in (†) is zero, and so $\varphi(a)\eta(b) = 0$. Since we alo have $(i,i) \preceq (i,j)$, a similar calculation shows that $\eta(a)\varphi(b) = 0$.

Lastly, we claim that $\eta(ab) = \eta(a)\eta(b)$. To see this, note that since

$$ab = \sum_{(i,j)}(\sum_{t=i}^{l} a_{it}b_{tj})\overline{e}_{ij},$$

we have

$$\eta(ab) = \sum_{i=1}^{r}\eta_i(ab)e_{ii} = \sum_{i=1}^{r}\Big(\sum_{(k,l)\not\succeq(i,i)} a_{kk}b_{kk}f_{kl}\Big)e_{ii}.$$

On the other hand, $f_{kl}f_{rs} = \delta_{kr}\delta_{ls}$, where δ_{ij} is the Kronecker delta function, and so

$$\begin{aligned}\eta_i(a)\eta_i(b)e_{ii} &= \Big(\sum_{(k,l)\not\succeq(i,i)} a_{kk}f_{kl}\Big)\Big(\sum_{(r,s)\not\succeq(i,i)} b_{rr}f_{rs}\Big)e_{ii}\\ &= \Big(\sum_{(k,l)\not\succeq(i,i)} a_{kk}b_{kk}f_{kl}\Big)e_{ii}.\end{aligned}$$

Summing over all i, we see that $\eta(ab) = \eta(a)\eta(b)$. This shows that θ is a homomorphim, and all that remains is to show that θ is unital:

$$\begin{aligned}\theta(1) &= \varphi(1) + \eta(1)\\ &= \sum_{i=1}^{r} E_{ii}e_{ii} + \sum_{(k,l)\not\leq(i,i)} f_{kl}e_{ii}\\ &= \sum_{i=1}^{r} \sum_{(k,l)\not\leq(i,i)} (E_{ii} + f_{kl})e_{ii}\\ &= \sum_{i=1}^{r} 1_B e_{ii}.\end{aligned}$$

This completes the proof. □

We next develop an analog of a theorem of Amitsur [3].

Proposition 3.34. *Let $I \lhd k\{\mathrm{M}\}$ be an ideal in the algebra of generic triangular matrices. Then $k\{\mathrm{M}\}/I$ is embeddable into $T_r(B)$ for some commutative k-algebra B iff I satisfies*

$$T_r(k[\Lambda]) \cdot I \cdot T_r(k[\Lambda]) \bigcap k\{\mathrm{M}\} = I.$$

Proof. Set $A = T_r(k[\Lambda])$ and note that for any ideal $I \lhd A$, we always have $I \subseteq AIA \cap k\{\mathrm{M}\}$, so the content of the result is that the reverse inclusion holds precisely when $k\{\mathrm{M}\}/I$ embeds in $T_r(B)$ for some B. First suppose that $j\colon k\{\mathrm{M}\} \to T_r(B)$ is an embedding. We have a cannonical injection $\tau_0\colon k \hookrightarrow B$ given by $\tau_0(\alpha) = \alpha \cdot 1$. We wish to extend τ_0 to a homomorphism $\tau\colon T_r(k[\Lambda]) \to T_r(B)$ in such a way as to produce the following commutative diagram

$$\begin{array}{ccc} k\{\mathrm{M}\} & \xrightarrow{i} & T_r(k[\Lambda]) \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\tau}\big\downarrow \\ k\{\mathrm{M}\}/I & \xrightarrow{j} & T_r(B) \end{array}$$

where i is the inclusion map $k\{\mathrm{M}\} \subseteq T_r(k[\Lambda])$, and π is the natural projection map.

To construct τ, first extend τ_0 to a homomorphism $\tau_1\colon k[\Lambda] \to B$ in the following way: If $j(\overline{M_k}) = (b_{ij}^k)$ is a matrix in $T_r(B)$, then set $\tau_1(\lambda_{ij}^k) = b_{ij}^k$. This defines

τ_1 on Λ, and we extend it to a map defined on $k[\Lambda]$ by linearity. Next, τ_1 induces a homomorphism $\tau\colon T_r(k[\Lambda]) \to T_r(B)$ in the obvious way. Here one may wish to think of τ as the matrix operator

$$\tau = \begin{pmatrix} \tau_1 & \cdots & \tau_1 \\ & \ddots & \vdots \\ & & \tau_1 \end{pmatrix}$$

which acts on $T_r(k[\Lambda])$ via "left multiplication".

It is clear that τ satisfies $\tau \circ i = j \circ \pi$. Now, since $j \circ \pi(I) = 0$ we have $\tau \circ i(I) = 0$. Note that $i(I) = I$ since i is just the natural inclusion map. This says that $I \subseteq \ker(\tau)$, which in turn implies that $AIA \subseteq \ker(\tau)$ since $\ker(\tau) \lhd A$. Finally, choose $y \in AIA \cap k\{\mathrm{M}\}$, and compute

$$0 = \tau(y) = \tau \circ i(y) = j \circ \pi(y)$$

and so $\pi(y) \in \ker(j) = J$. This shows that $AIA \cap k\{\mathrm{M}\} \subseteq I$.

For the other direction, we suppose that $AIA \cap k\{\mathrm{M}\} \subseteq I$ and we wish to build an injective k-algebra homomorphism $k\{\mathrm{M}\}/I \hookrightarrow T_r(B)$ for some commutative k-algebra B. Let $L = AIA$. By Lemma 3.31, L has the form

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1r} \\ 0 & L_{22} & \cdots & L_{2r} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & L_{rr} \end{pmatrix}$$

where $L_{ik} \subseteq L_{ij}$ for $i \leq k \leq j$ and $L_{kj} \subseteq L_{ij}$ for $i \leq k \leq j$, and where each $L_{ij} \lhd k[\Lambda]$. Since $I \subseteq AIA = L$, the injection i maps I into L and thus induces a homomorphism $\hat{i}\colon k\{\mathrm{M}\}/I \to A/L$. Moreover, $\ker \hat{i} = I$ since $AIA \cap k\{\mathrm{M}\} = I$, and so $\hat{i}$ is an injection. Finally, Proposition 3.33 provides an embedding of A/L into $T_r(B)$ for some commutative k-algebra B and so we have

$$k\{\mathrm{M}\}/I \hookrightarrow A/L \hookrightarrow T_r(B).$$

□

4 Just Infinite Algebras

Simple rings are so-called because their two-sided ideal structure is as simple as possible (of course, this says nothing about the category of modules over a simple ring, which can be quite complicated). One way of generalizing the notion of a simple ring is to allow the presence of two-sided ideals, but to insist that the nonzero ones be very "large." Specifically, we will consider algebras in which all the nonzero ideals have finite codimension (equivalently: all proper homomorphic images are finite dimensional algebras). The nomenclature which has been adopted is to call such rings *just infinite*, or, in case the ring is $\mathbb{N}$-graded, to refer to them as *projectively simple*.

The obvious hope then is to prove theorems analogous to those which are known for simple rings. As one might expect from algebraic geometry, the presence of a grading often allows one to obtain sharper results, many of which can be found in [46]. We are interested primarily in the ungraded case, and we begin, as always, with a definition.

Definition 4.1. A k-algebra is called *just infinite dimensional*, or *just infinite* for short, if $\dim_k(A) = \infty$ and each of its nonzero two-sided ideals has finite codimension.

Just infinite algebras are easy to find. Aside from any infinite dimensional simple ring, the most immediate example is a polynomial ring in one variable $k[x]$. Others include $k[x, x^{-1}]$, $k[[x]]$, and, since being just infinite is a Morita invariant [17], matrix rings over these. As we will see, the theory of just infinite algebras naturally splits into two disjoint cases, depending on whether or not the algebra satisfies a polynomial identity. The PI case turns out to be essentially subsumed by the commutative case, whereas the non-PI case provides plenty of interesting open problems. For examples of non-PI just infinite algebras, we refer the reader to [6], where the author constructs various interesting rings arising from groups acting on infinite trees. One can also find an example of a non-PI just infinite algebra arising from the famous Golod-Shafarevich construction in [18].

In [46], the authors prove that a projectively simple algebra has a unique maximal ideal, namely the augmentation ideal. As a curiosity, we offer the following similar result for just infinite algebras which are not semiprimitive. We need a preliminary lemma, which is probably well known.

Lemma 4.2. *If A is a finite dimensional algebra then A contains only finitely many distinct primitive ideals.*

Proof. Suppose not. Then there is an infinite set, $\{ P_i \mid i \in \mathbb{N} \}$, of distinct primitive ideals. Consider the chain of ideals,

$$P_1 \supseteq P_1 \cap P_2 \supseteq P_1 \cap \ldots \cap P_n \supseteq \ldots.$$

Because A is finite dimensional and hence satisfies the descending chain condition on ideals, this chain must stabilize and so for some n,

$$P_1 \cap \ldots \cap P_n = P_1 \cap \ldots \cap P_n \cap P_{n+1} \subseteq P_{n+1}.$$

In a finite dimensional algebra, primitive implies both prime and maximal. Because P_{n+1} must be prime, there exists some $i \leq n$ such that $P_i \subseteq P_{n+1}$. As P_i is maximal, $P_i = P_{n+1}$, which gives us our contradiction. □

Proposition 4.3. *Let A be a just infinite k-algebra such that $\operatorname{Jac}(A) \neq 0$. Then A has only finitely many nonzero maximal ideals, and so $|\operatorname{spec}(A)| < \infty$.*

Proof. Note that $\operatorname{Jac}(A)$ is the intersection of the primitive ideals of A. Since A is not semiprimitive, each primitive ideal contains $\operatorname{Jac}(A)$, but since $\operatorname{Jac}(A)$ has finite codimension, there can be only finitely many primitives containing it by Lemma 4.2. Lastly, in a just infinite algebra, every nonzero prime ideal (and hence every primitive ideal of A) is maximal. □

The hypothesis that $\text{Jac}(A) \neq 0$ is necessary, as $\mathbb{C}[x]$ demonstrates. In [18] the authors prove that if A is any affine, infinite dimensional k-algebra, then there is a prime ideal $P \in \text{spec}(A)$ such that A/P is just infinite, and they deduce as a consequence that every affine just infinite algebra is prime. Of course, there are plenty of non-affine just infinite algebras, perhaps the simplest example being $k[[x]]$. As it turns out, these infinitely generated just infinite algebras are also prime rings. We will prove this next, but first we need the following

Lemma 4.4. *If A is a just infinite dimensional k-algebra, then A is semiprime.*

Proof. Because A is just infinite it satisfies the ascending chain condition on two-sided ideals. Thus if $I \lhd A$ is a two-sided ideal then I is finitely generated as an (A, A)-bimodule, and thus also as a left $A \otimes_k A^{op}$-module.

Suppose $I \lhd A$ is a nonzero ideal of A such that $I^2 = 0$. Then, since I is contained in both the left and right annihilator of I, I is also finitely generated as an $(A/I, A/I)$-bimodule. This implies that I is finitely generated as a left $(A/I) \otimes_k (A/I)^{op}$-module. But A/I and $(A/I)^{op}$ are finite dimensional over k, so $\dim_k (A/I) \otimes (A/I)^{op} < \infty$. As I is a finitely generated module over a finite dimensional algebra, we see that $\dim_k I < \infty$.

As both $\dim_k(A/I) < \infty$ and $\dim_k(I) < \infty$, we conclude that A is finite dimensional over k, which is a contradiction. □

With this lemma in hand, we can now prove the following

Theorem 4.5. *If A is a just infinite k-algebra, then A is prime.*

Proof. The argument is essentially the same as the previous lemma. Suppose $I, J \lhd A$ are nonzero ideals of A with $IJ = 0$. By Lemma 4.4, A is semiprime. Since $(JI)^2 = (0)$, $JI = 0$ as well. We then have

$$J \subseteq \text{l.}\,\text{ann}_A(I) \bigcap \text{r.}\,\text{ann}_A(I),$$

so I is finitely generated as a left $(A/J) \otimes (A/J)^{op}$-module. Because A is just infinite, $\dim_k(A/J) \otimes (A/J)^{op} < \infty$, and so $\dim_k(I) < \infty$. As in Lemma 4.4, this shows that A is finite dimensional, which is a contradiction. □

Theorem 4.5 is particularly satisfying because prime rings, being in a sense the "correct" generalization of commutative domains, are so fundamental in non-commutative ring theory. As a corollary to Theorem 4.5 we have the following sufficient condition for an affine algebra to be just infinite.

Corollary 4.6. *If A is an affine infinite dimensional algebra with the property that for every nonzero prime ideal P of A, $\dim_k A/P < \infty$, then A is just infinite.*

Proof. Let A be an affine infinite dimensional algebra with A/P finite dimensional for any nonzero prime ideal $P \lhd A$. Were A not just infinite, it would have a nonzero ideal of infinite codimension. Because A is affine, we claim that there is an ideal, $M \lhd A$, maximal with respect to the property that A/M is infinite dimensional. The idea is to use Zorn's Lemma, and so we need to show that Zorn's Lemma actually applies in this situation. Let

$$I_1 \subseteq I_2 \subseteq \cdots$$

be a chain of two-sided ideals in A, each of infinite codimension, and set $I := \bigcup_n I_n$. To apply Zorn's Lemma, we need to know that I is also of infinite codimension, so suppose not. Let $\overline{\theta}_1, \ldots, \overline{\theta}_r$ be a k-spanning set for A/I. Since A is affine, we may assume that $\{\theta_i\}$ generates A as a k-algebra. For each (i, j), we can find elements $\{\alpha_{ij}^k\}$ so that

$$\overline{\theta}_i\overline{\theta}_j - \sum_{k=1}^{r} \alpha_{ij}^k \overline{\theta}_k = 0.$$

Set $x_{ij} := \theta_i\theta_j - \displaystyle\sum_{k=1}^{r}\alpha_{ij}^k\theta_k \in A$, and let J denote the two-sided ideal of A generated by the elements $\{x_{ij}\}$. Each $x_{ij} \in I$, and since there are only finitely many x_{ij}, we see that $\{x_{ij}\} \subseteq I_n$ for some n. Thus $J \subseteq I_n$ and so $\dim_k(A/J) = \infty$. On the other hand, by construction the finite set $\{\theta_i + J\}$ spans A/J, a contradiction, and so we conclude that $\dim_k(A/I) = \infty$.

Thus Zorn's Lemma applies and we can find an ideal $M \lhd A$ maximal with respect to A/M being infinite dimensional. By the correspondence theorem, A/M is just infinite and hence, by Theorem 4.5, prime, which in turn implies that M is a prime ideal of A. By assumption, M prime in A means that $\dim_k(A/M) < \infty$, a contradiction. □

We are now in a position to characterize just infinite algebras which satisfy a polynomial identity. These turn out to be very nice rings, from a structure-theoretic point of view.

Proposition 4.7. *If A is a* PI *just infinite k-algebra then A is a finitely generated module over its center, $Z(A)$. Moreover, $Z(A)$ is itself just infinite.*

Proof. Set $Z = Z(A)$. Let I be a nonzero ideal of Z and fix a nonzero $z \in I$. Since Z is a domain and A is a torsion-free Z-module, $zZ = zA \cap Z$. This implies that we have an inclusion $Z/zZ \hookrightarrow A/zA$. Thus Z/zZ, and hence also Z/I, is finite dimensional, so Z is just infinite.

Also, since $I/zZ \subseteq Z/zZ$, we see that I/zZ is also finite dimensional. In particular, I/zZ is a finitely generated Z-module and since zZ is obviously finitely generated as well, we see that I is finitely generated over Z. This shows that Z is noetherian. By a theorem of Formanek [19], if A is a prime algebra satisfying a polynomial identity whose center Z is noetherian, then A is a finitely generated module over Z. □

Most ring-theoretic properties easily pass from a ring to an overring which is a finitely generated module, and so Proposition 4.7 essentially reduces the study of PI just infinite algebras to the commutative case. On the other hand, in stark contrast to Proposition 4.7, a just infinite algebra which doesn't satisfy a polynomial identity must necessarily have a very small center. In fact, we next show that the center of any non-PI just infinite algebra is a finite dimensional field extension of the ground field. In practice this means that in the non-PI case, one can usually assume that the center is reduced to scalars (ie. $Z(A) = k$). This result is essentially a consequence of two lemmas in [18], which we include for completeness. For a fixed positive integer n, we denote by $I_n(A)$ the two-sided ideal of the algebra A generated by all specializations in A of all polynomial identities of $n \times n$ matrices.

Lemma 4.8. *Let A be a k-algebra. If $a \in A$ satisfies $\dim_k(A/Aa) = n$, then*

$$I_n(A) \subseteq \bigcap_{d \in \mathbb{N}} Aa^d.$$

Proof. See [18, Lemma 1.1]. □

Lemma 4.9. *Let A be a k-algebra. If $a \in A$ is a regular element and*

$$\bigcap_{d \in \mathbb{N}} Aa^d$$

has finite codimension in A, then a is a unit.

Proof. See [18, Lemma 1.2]. □

Proposition 4.10. *Let A be a non-*PI *just infinite k-algebra. Then $Z(A)$ is a finite dimensional field extension of k*

Proof. Since A is just infinite, and thus prime, we know that the center is a domain. Choose $0 \neq z \in Z(A)$, and set $J = \bigcap_{d \in \mathbb{N}} Az^d$, a two-sided ideal of A. A is just infinite, so the two-sided ideal Az has finite codimension, n say, in A. Then by Lemma 4.8 $I_n(A) \subseteq J$. Since A doesn't satisfy any polynomial identity, $I_n(A)$ is a nonzero ideal of A, and so J is nonzero as well. Then Lemma 4.9 shows that z is a unit of A, hence of Z,and so $Z(A)$ is a field extension of k. Next, since $Z(A)$ is a field, we see that $I \cap Z(A) = 0$ for any nonzero proper ideal I of A. The natural projection map $A \to A/I$ then induces an injection from $Z(A)$ to A/I, and the latter is finite dimensional. □

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4.1 Stably Just Infinite Algebras

We turn our attention now to stability properties of just infinite algebras. Of primary concern to us is when a just infinite k-algebra yields a just infinite K-algebra upon extension of scalars by an arbitrary field extension K/k. The following definition should come as no surprise to the reader.

Definition 4.11. A just infinite k-algebra A is called *stably just infinite* if $A \otimes_k K$ is just infinite (over K), for every field extension K/k.

We saw in Chapter 2 that "most" right noetherian rings are in fact stably right noetherian. Just infinite algebras on the other hand, even affine ones, may behave rather poorly upon extension of scalars. For example, $\mathbb{C}[x]$ is an affine just infinite $\mathbb{R}$-algebra. Note that $\mathbb{C}[x] \otimes_{\mathbb{R}} \mathbb{C}$ contains $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, which isn't a domain. Thus $\mathbb{C}[x] \otimes_{\mathbb{R}} \mathbb{C}$ isn't prime and hence fails to be just infinite. This example shows that in case the ground field isn't algebraically closed, things can go horribly awry even upon extending scalars by a finite dimensional field extension! We conjecture that such pathology can be avoided in case the ground field is sufficiently nice:

Conjecture 4.12. *Let k be an uncountable, algebraically closed field, and let A be an affine just infinite k-algebra. Then A is stably just infinite.*

That this conjecture is plausible is evidenced by [46, Lemma 1.8]. There the authors prove that if k is algebraically closed, then every connected, finitely $\mathbb{N}$-graded, just infinite k-algebra is stably just infinite. Moreover, we will see that Conjecture 4.12 holds in case A is PI (Proposition 4.16), and in the non-PI case provided that A is either right Goldie (Proposition 4.23) or semiprimitive (Corollary 4.28).

From [46] and the example above, it seems reasonable to insist that the ground field be algebraically closed. In this way we can, at the very least, avoid the unpleasantness of having a just infinite algebra A for which extension of scalars results in a ring which fails to be prime. The verification of this last fact is the content of the next two propositions (which don't require an affine hypothesis).

Proposition 4.13. *Let k be an algebraically closed field, and let D and L be domains containing k. If L is commutative, then $D \otimes_k L$ is a domain.*

Proof. (Guralnick) Suppose there are $0 \neq \alpha, \beta \in D \otimes_k L$ with $\alpha\beta = 0$. We may write $\alpha = \sum e_i \otimes a_i$ and $\beta = \sum e_i \otimes b_i$ where the set $\{e_i\}$ is linearly independent over k. To see this, write $\alpha = \sum e_i \otimes a_i$ and $\beta = \sum f_i \otimes b_i$. Let K denote the k-subspace of D spanned by $\{e_i\} \cup \{f_i\}$, and let $\{e_i'\}$ be a k-basis for K. Fix an e_i. Then e_i is a k-linear combination of the set $\{e_i'\}$ and so we can write $e_i = \sum c_i e_i'$

where the $c_i \in k$. Whence $e_i \otimes a_i = \sum c_i e_i' \otimes a_i = \sum e_i' \otimes c_i a_i$. So defining $a_i' = \sum a_i c_i$ for each i, and similarly for the b_i establishes the claim.

By replacing L with the subalgebra $k[\{a_i\}, \{b_i\}]$ if necessary, we may assume that L is an affine commutative domain (hence noetherian). Let I and J denote the ideals of L generated by the sets $\{a_i\}$ and $\{b_i\}$ respectively. Any affine commutative domain is semiprimitive, so

$$\bigcap \mathfrak{m} = 0,$$

where the intersection is over all maximal ideals $\mathfrak{m}$ of L. By Hilbert's Nullstellensatz, for each $\mathfrak{m} \in \operatorname{mspec}(L)$, $L/m \cong k$, and so

$$(D \otimes L)/(D \otimes \mathfrak{m}) \cong D \otimes (L/\mathfrak{m}) \cong D.$$

But $\bar{\alpha}\bar{\beta} = \bar{0}$ in $D \otimes (L/\mathfrak{m}) \cong D$, a domain. Thus $\bar{\alpha} = \bar{0}$ or $\bar{\beta} = \bar{0}$, and we see that for each $\mathfrak{m} \in \operatorname{mspec}(L)$, $I \subseteq \mathfrak{m}$ or $J \subseteq \mathfrak{m}$. In particular, $IJ \subseteq \bigcap \mathfrak{m} = 0$. As L is prime, $I = 0$ or $J = 0$. □

Proposition 4.13 fails completely without the assumption that k is algebraically closed, even when both domains are finite dimensional over k, as the example $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ shows. As an aside, the following corollary provides a new construction of division rings, via localization.

Corollary 4.14. *If A is a right Ore domain which is an algebra over an algebraically closed field k, then $A \otimes_k K$ is an Ore domain for every field extension K of k.*

Proof. All that needs to be checked is the right Ore condition. For this we may assume that K is a finitely generated field extension of k. K is then a finite algebraic extension of a purely transcendental extension $k(x_1, \ldots, x_n)$. $A \otimes_k K$ is then a finite module over $A \otimes k(x_1, \ldots, x_n)$ and the latter ring is a localization of $A[x_1, \ldots, x_n]$, which is clearly right Ore since A is. □

With a bit of care, Proposition 4.13 extends to prime rings as well. Bergman [10, Proposition 17.2] actually proved the following: over an algebraically closed field k, the tensor product of any two prime k-algebras is again prime. As we are mostly interested in field extensions, we will content ourselves with a weakened version of Bergman's Theorem.

Proposition 4.15 (Bergman)**.** *Let A be a prime algebra over an algebraically closed field k. Then $A \otimes_k K$ is prime for every field extension K/k.*

Proof. See [10, Proposition 17.2]. □

Recall from Chapter 2 that affine commutative noetherian algebras behave rather well with respect to extension of scalars. As every commutative just infinite algebra is noetherian, one should not be surprised that a similar result holds in this context. Affine commutative just infinite k-algebras are precisely the affine commutative domains of Krull dimension 1. By the Noether Normalization Lemma, if A is an affine just infinite k-algebra, then A is a finitely generated module over $k[x]$, for some transcendental element $x \in A$. If we then take a field extension K/k, then $A \otimes_k K$ is just a finitely generated module over $K[x]$. In particular, $A \otimes_k K$ is an affine K-algebra of Krull dimension 1, so is just infinite over K provided that $A \otimes_k K$ is a domain. If we agree to work over an algebraically closed field k, then Proposition 4.13 implies that every affine commutative just infinite k-algebra is stably just infinite.

Suppose now that A is an affine PI just infinite algebra over an algebraically closed field k. By Proposition 4.7, A is a finitely generated module over $Z(A)$, which is itself just infinite. The Artin-Tate lemma [53, Proposition 6.2.5] then tells us that $Z(A)$ is affine. Thus $Z(A)$ is stably just infinite by the previous remarks.

If K/k is a field extension, then $Z(A) \otimes_k K$ is a noetherian just infinite K-algebra. Since $A \otimes_k K$ is a finitely generated $Z(A) \otimes_k K$-module, $A \otimes_k K$ is also noetherian. Choose a nonzero two-sided ideal $I \lhd A \otimes_k K$. As $A \otimes_k K$ is prime noetherian, I contains a nonzero regular element x. Since $A \otimes_k K$ is a finite $Z(A) \otimes_k K$-module, x is algebraic over $Z(A) \otimes_k K$. If

$$x^t + z_{t-1}x^{t-1} + \ldots + z_0 = 0, \quad z_j \in Z(A) \otimes_k K$$

has t minimal, then $z_0 \neq 0$. It follows that $z_0 \in I \cap (Z(A) \otimes_k K)$, and so $I \cap (Z(A) \otimes_k K) \neq 0$. Finally, $(A \otimes_k K)/I$ is a finitely generated module over $(Z \otimes_k K)/(I \cap (Z \otimes_k K))$, hence $A \otimes_k K$ is just infinite over K. To summarize, we have proved the following

Proposition 4.16. *Let k be an algebraically closed field, and let A be an affine* PI *just infinite k-algebra. Then A is stably just infinite.*

One can view Proposition 4.16 as an analog of Proposition 2.51. Note that both hypotheses (that A is affine and that k is algebraically closed) are necessary. Indeed, the power series ring $\mathbb{C}[[x]]$ is a just infinite $\mathbb{C}$-algebra which is not stably noetherian, and hence not stably just infinite either. On the other hand, we have already seen that $\mathbb{C}[x] \otimes_{\mathbb{R}} \mathbb{C}$ is not a domain, even though $\mathbb{C}[x]$ is an affine just infinite $\mathbb{R}$-algebra.

Insofar as we are interested in characterizing affine stably just infinite algebras over algebraically closed fields, it is thus reasonable to restrict our attention to the non-PI case. If A is a non-PI just infinite algebra, then we know from Proposition 4.7 that the center of A is a finite dimensional field extension of k, and so we may assume that k coincides with the center of A. Recall that if A is a central simple k-algebra, then $A \otimes_k K$ is simple for every field extension K/k. One may wish to prove an analogous theorem for non-PI just infinite algebras, and although the general situation is still rather murky, some positive results can be obtained using the so-called *extended center* of A, to which we turn now.

4.1.1 Martindale's Ring of Quotients

Let A be a prime ring and consider the set of all right A-module homomorphisms $f\colon I \to A$, where I ranges over all nonzero two-sided ideals of A. Martindale's basic idea is to endow this set of maps with an algebra structure. Since A is prime, the intersection of two nonzero ideals contains their product, and so is again nonzero. If we have right A-module maps

$$f\colon I \to A \quad \text{and} \quad g\colon J \to A,$$

then $f+g$ is a right A-module map defined on $I \cap J$. Also, $f \circ g$ is a right A-module map defined on JI since $g(JI) = g(J)I \subseteq I$, and f is defined on I. For convenience we include the full definition of Martindale's ring of quotients, but we refer the reader to [22], [32], or [52] for the proof that this construction actually yields a ring with the stated properties.

Definition 4.17. Let A be a prime k-algebra. The *(right) Martindale ring of quotients of* A, denoted $Q_r(A)$, consists of equivalence classes of pairs (I, f) where $I \lhd A$, $I \neq 0$, and $f \in \mathrm{Hom}_A(I_A, A_A)$. Here two pairs (I, f), (J, g) are defined to be equivalent if $f = g$ on the intersection $I \cap J$. Addition and multiplication are given by

$$(I, f) + (J, g) = (I \cap J, f + g),$$
$$(I, f) \cdot (J, g) = (JI, f \circ g).$$

(The choice of JI for the domain of $f \circ g$ is somewhat arbitrary. For example, we could instead use $(I \cap J)^2$.)

Given an element $a \in A$, the map $\ell_a \colon A \to A$ given by $\ell_a(x) = ax$ is a right A-module map. Moreover, $\ell_{ab} = \ell_a \circ \ell_b$ since $(ab)x = a(bx)$. Thus the map $a \mapsto (A, \ell_a)$ gives a k-algebra embedding of A into $Q_r(A)$.

Definition 4.18. Let A be a prime k-algebra. The *extended center of* A, written $C(A)$, is defined to be $Z(Q_r(A))$. $C(A)$ is a field extension of k and $C(A) \cap A = Z(A)$. The *central closure of* A, denoted $AC(A)$, is the $C(A)$-linear subspace of $Q_r(A)$ generated by A. Lastly, A is called *centrally closed* if $C(A) = Z(A)$, (equivalently, if $AC(A) = A$). Note that the central closure of any ring is itself centrally closed.

There is an entirely internal characterization of $C(A)$ which bears mentioning.

Remark 4.19. The extended center of A consists precisely of those pairs (I, f) where $f \colon I \to A$ is an (A, A)-bimodule homomorphism:

$$C(A) = \{\, (I, f) \mid 0 \neq I \lhd A, f \in \mathrm{Hom}_A({}_AI_A, {}_AA_A) \,\}.$$

Lemma 4.20 ([16, Theorem 3.1])**.** *If $x_1, \dots, x_n$ are elements of $Q_r(A)$ which are linearly independent over $C(A)$, then there exist elements*

$$a_1, \dots, a_n,\, b_1, \dots, b_n \in A$$

such that

$$\sum_i a_i x_1 b_i \neq 0, \quad \textit{and} \quad \sum_i a_i x_j b_i = 0, \; \textit{for } 2 \leq j \leq n.$$

Centrally closed prime algebras behave particularly well with respect to extension of scalars, and as a result we can show that centrally closed just infinite algebras are stably just infinite.

Lemma 4.21 ([16, Lemma 3.4])**.** *Let A be a centrally closed prime k-algebra and let K/k be an extension field. Then any nonzero ideal of $A \otimes_k K$ has nonzero intersection with A.*

Proof. Suppose I is a nonzero ideal of $A \otimes_k K$ with $I \cap A = 0$. Choose a nonzero element $\sum_{j=1}^{n} x_j \otimes_k \lambda_j$ of minimal length with the set $\{x_j\}$ k-linearly independent. Since I is an ideal, we may assume that $\lambda_1 = 1$. Since $k = C(A)$, Lemma 4.20 shows that we can find elements $\{a_i\}, \{b_i\}$ in A such that

$$\sum_i a_i x_1 b_i \neq 0, \;\; \text{and} \;\; \sum_i a_i x_j b_i = 0, \;\; \text{for } 2 \leq j \leq n.$$

Now note that

$$\sum_{i,j} a_i x_j b_i \otimes_k \lambda_j = (\sum_i a_i x_1 b_i) \otimes_k 1 \in I \cap A = 0$$

and so $\sum_i a_i x_1 b_i = 0$, a contradiction. □

Proposition 4.22. *Any centrally closed just infinite algebra is stably just infinite.*

Proof. Choose a nonzero ideal $I \lhd A \otimes_k K$ and set $J = I \cap A$, so $J \otimes_k K \subseteq I$. By Lemma 4.21, J is a nonzero ideal of A. Lastly,

$$(A \otimes_k K)/(J \otimes_k K) \cong A/J \otimes_k K$$

and the latter is finite dimensional over K since A is just infinite. The correspondence theorem then shows that I has finite codimension in $A \otimes_k K$. □

Other than simple rings, one instance in which the centrally closed hypothesis is satisfied is when A is affine right Goldie (and the field k is sufficiently large).

Proposition 4.23. *Let A be an affine non-*PI *right Goldie just infinite algebra whose center k is an uncountable algebraically closed field. Then A is stably just infinite.*

Proof. The idea is to show that A is centrally closed. Since A is right Goldie, $Q_r(A) \subseteq Q(A)$ (see e.g. [16]). We write $I_n(A)$ for the ideal of A generated by all specializations of all polynomial identities of $n \times n$ matrices. Each $I_n(A)$ is nonzero since A is not PI. Since A is prime, $I_n(A)$ is essential as a right ideal, and thus contains a nonzero regular element a_n.

Set $B := A[a_1^{-1}, a_2^{-1}, \ldots]$, the ring extension generated by the inverses of these regular elements. Since $A \subseteq B \subseteq Q(A)$, we see that $Q(B) = Q(A)$, and thus B is prime right Goldie. Moreover, B is an essential extension of A, and so every nonzero ideal of B intersects A nontrivially. A is just infinite and not PI, and so every nonzero ideal of A contains one of the $I_n(A)$. Since each a_n is a unit in B, B is a simple ring. Note that since A is affine, $\dim_k(A)$ is countable, and since B is a countably generated ring extension of A, $\dim_k(B)$, and thus $\dim_k(Z(B))$, is countable as well. Since $|k|$ is uncountable, $Z(B)$ is an algebraic field extension of k, so $Z(B) = k$ since k is algebraically closed. Also, since B is simple right Goldie, we know that $Z(Q(B)) = Z(B)$. Finally, $C(A) \subseteq Z(Q(A))$, so A is centrally closed. □

There is another possible proof of Proposition 4.23 which we would like to mention. The central idea is an (unpublished) result of Bell and Farina, which may be of independent interest.

Definition 4.24. A multiplicatively closed subset S of a ring R is called *right Ore* if $xS \cap sR \neq \emptyset$ for all $x \in R$ and $s \in S$.

Proposition 4.25 (Bell-Farina)**.** *Let k be a field and let A be an affine k-algebra which is prime right Goldie. If S_0 is a countable set of right regular elements of A, then there exists a countable set of right regular elements $T \supseteq S_0$ which is right Ore.*

Proof. Since A is affine, $\dim_k(A)$ is countable. Let $\mathcal{B} = \{b_1, b_2, \ldots\}$ be a k-vector space basis for A. Write $S_0 = \{\, s_j^{(0)} \mid j \in \mathbb{N}\,\}$ and note that since S_0 is countable, so is the multiplicatively closed subset of A generated by S_0. Therefore, without loss we may assume that S_0 is multiplicatively closed. Let S denote the set of right regular elements of A, which is right Ore by Goldie's theorem. We will use the following well known fact repeatedly:

Fact 4.26. *Given $x_1, x_2, \ldots, x_n \in A$ and $s_1, s_2, \ldots, s_n \in S$, there exists some element $s' \in S$ such that*

$$x_i s' \in s_i A, \text{ for } 1 \le i \le n.$$

Fix $(j, n) \in \mathbb{N}^2$. Then there exists some element $s_{(j,n)}^{(1)} \in S$ such that

$$b_i s_{(j,n)}^{(1)} \in s_j^{(0)} A, \text{ for } 1 \le i \le n.$$

The set

$$X_1 := \{\, s_{(j,n)}^{(1)} \mid (j, n) \in \mathbb{N}^2 \,\} \bigcup S_0$$

is countable, and hence so too is the multiplicatively closed subset S_1 of A generated by X_1. We may therefore write $S_1 = \{\, s_j^{(1)} \mid j \in \mathbb{N}\,\}$.

Again, for fixed $(j, n) \in \mathbb{N}^2$, there exists some element $s_{(j,n)}^{(2)} \in S$ such that

$$b_i s_{(j,n)}^{(2)} \in s_j^{(1)} A, \text{ for } 1 \le i \le n.$$

The set $X_2 := \{\, s_{(j,n)}^{(1)} \mid (j, n) \in \mathbb{N}^2 \,\} \bigcup S_1$ is countable, and hence so too is the multiplicatively closed subset S_2 of A generated by X_2. As before, we write $S_2 = \{\, s_j^{(2)} \mid j \in \mathbb{N}\,\}$. Repeating this process yields an increasing chain of countable, multiplicatively closed subsets

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots,$$

and we may form the union $T := \bigcup_{d \in \mathbb{N}} S_d$. Note that T is multiplicatively closed since each S_d is, and that T is countable by construction.

All that remains to be checked is the right Ore condition. To that end, choose $x \in A$ and $t \in T$. We need to show that there is some element $t' \in T$ such that $xt' \in tA$. We may write $x = \sum_{i=1}^{n} \alpha_i b_i$, with $\alpha_i \in k$. Since $t \in T$, $t \in S_d$ for some minimal d. Thus $t = s_j^{(d)}$ for some $j \in \mathbb{N}$. Note that the element $t' := s_{(j,n)}^{(d+1)} \in X_{d+1} \subseteq T$ satisfies

$$xt' = \sum_{i=1}^{n} \alpha_i b_i s_{(j,n)}^{(d+1)} \in s_j^{(d)} A$$

since $b_i s_{(j,n)}^{(d+1)} \in s_j^{(d)} A$, for $1 \leq i \leq n$ by construction. □

Now suppose that k is an uncountable algebraically closed field and that A is an affine non-PI just infinite right Goldie k-algebra. As in the proof of Proposition 4.23, we find a countable set of regular elements $\{ a_n \mid n \in \mathbb{N} \}$ chosen from the ideals $I_n(A)$. By Proposition 4.25, we can find a countable right Ore set $T \supseteq \{a_n\}$ and we may form the ring $B := AT^{-1}$. It is then clear that B is simple, and hence centrally closed. Moreover, the central closures of A and B obviously coincide, and so A is stably just infinite. This gives an alternate proof of Proposition 4.22.

Returning to the topic at hand, we can obtain further stability results by combining the above ideas with the Nullstellensatz.

Proposition 4.27. *Let A be a primitive non-*PI *just infinite algebra whose center, k, is algebraically closed. If A satisfies the Nullstellensatz, then A is stably just infinite.*

Proof. By a result of Martindale, (see [38]), the extended center $C(A)$ embeds in $\mathrm{End}_A(M)$, for any faithful simple A-module M. Since A satisfies the Nullstellensatz, this implies that $C(A) = k$, so A is centrally closed and the result follows from Proposition 4.22. □

Corollary 4.28. *Let A be an affine semiprimitive non-*PI *just infinite k-algebra whose center k is an uncountable algebraically closed field. Then A is stably just infinite.*

Proof. By [18, Theorem 2.2], A is primitive. It is well known that affine algebras over uncountable fields satisfy the Nullstellensatz ([2]), and so all the hypotheses of Proposition 4.27 are satisfied. □

4.1.2 A Reduction Theorem

A priori, it seems that to decide whether or not a just infinite algebra A is stably just infinite requires consideration of arbitrary field extensions K/k. In fact, there is a certain distinguished field which one can focus on, namely the extended center $C(A)$. To prove this we will first need another

Lemma 4.29. *Let A be a k-algebra and let K/k be a field extension. If $A \otimes_k K$ is just infinite (over K), then A is just infinite (over k).*

Proof. Choose a nonzero ideal $I \lhd A$. Then $I \otimes_k K$ is a nonzero two-sided ideal of $A \otimes_k K$ with $(A \otimes_k K)/(I \otimes_k K) \cong (A/I) \otimes_k K$. Taking dimensions then yields

$$\begin{aligned} \dim_k(A/I) &= \dim_K((A/I) \otimes_k K) \\ &= \dim_K((A \otimes_k K)/(I \otimes_k K)) < \infty. \end{aligned}$$

□

Proposition 4.30. *Let A be a non-*PI *just infinite k-algebra, and let $C(A)$ denote the extended center of A. Then A is stably just infinite iff $A \otimes_k C(A)$ is just infinite over $C(A)$.*

Proof. One direction is trivial. For the other, suppose that $A \otimes_k C(A)$ is just infinite over $C(A)$. Note that we have a surjective k-algebra homomorphism from $A \otimes_k C(A)$ onto the central closure of A (given by multiplication). Since A is just infinite not PI, $\dim_{C(A)} AC(A) = \infty$, and thus the above map is an isomorphism: $A \otimes_k C(A) \cong AC(A)$. We see that $AC(A)$ is just infinite over $C(A)$, and so by Proposition 4.22 $AC(A)$ is stably just infinite over $C(A)$.

Now, let K/k be any field extension, and let L denote a compositum (over k) of K and $C(A)$. We then have that

$$A \otimes_k L \cong (A \otimes_k C(A)) \otimes_{C(A)} L$$

is just infinite over L, and Lemma 4.29 shows that $A \otimes_k K$ is just infinite over K, completing the proof. □

The above result is both pleasing and somewhat misguided. On the plus side, since the extended center is defined in terms of bimodule maps of ideals, Proposition 4.30 gives an entirely internal characterization of stably just infinite algebras. The downside is that in practice it is almost always impossible to actually compute the extended center, and so in a sense this result brings us no closer to understanding the class of stably just infinite algebras.

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