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by

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Abstract

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Catabolism is an operation on tableaux originally defined by Lascoux. Though the operation is not difficult to compute, many of its properties remain elusive. The combinatorics of catabolism has deep ties to various generalizations and subspaces of the Type A coinvariant ring and their structures as graded symmetric group representations. This in turn gives connections to the corresponding q-symmetric functions that give the graded Frobenius characters of these spaces.

In this dissertation, we explore these connections from both the combinatorial and algebraic perspective. We use catabolizability to construct bases for different families of generalized coinvariant rings. Our constructions give bridges between the combinatorics of catabolism and the representation theoretic structures: in particular, we recover combinatorial formulas for the Schur expansions of their graded Frobenius characters. We also study the combinatorics of catabolism directly in an attempt to identify submodules of the coinvariant ring with Frobenius character given by k-Schur functions.

Contents

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Chapter 1

Introduction

The *Schur functions* s_λ form an orthogonal basis of the ring of symmetric functions. It is of interest to study whether a given symmetric function is *Schur-positive*, meaning the coefficients of the Schur expansion are all nonnegative integers. Families of Schur-positive symmetric functions often correspond to interesting geometric or representation theoretic structures: in particular, the Schur functions correspond to the Frobenius characters of irreducible $\mathfrak{S}_n$-representations.

Deepening the connection between symmetric functions and representation theory has been a long ongoing field of research. A leading example of this is the study of *modified Macdonald polynomials* $\widetilde{H}_\mu[X;q,t]$. Macdonald polynomials [42] have a rich connection to the affine Hecke algebra and are simultaneous generalizations of several important families of polynomials, such as Jack, q-Whittaker, and Askey-Wilson polynomials. In type A, they can be transformed to give the modified Macdonald polynomial $\widetilde{H}_\mu[X;q,t]$.

The *Macdonald positivity conjecture*, which asked whether $\widetilde{H}_\mu[X;q,t]$ was Schur-positive, was proven using the geometry of the Hilbert scheme of n points in the plane by realizing $\widetilde{H}_\mu[X;q,t]$ as the Frobenius character of a bigraded $\mathfrak{S}_n$-module [32]. However, there is still no straightforward combinatorial formula for the coefficients in the Schur expansion. This is one of the most influential open problems in algebraic combinatorics and has led to many further questions in the field of Schur positivity of symmetric functions [28, 29, 27].

This dissertation focuses on symmetric group representations which arise as subspaces or generalizations of the coinvariant ring $R_n = \mathbb{C}[\mathbf{x}]/I_n$, where the ideal I_n is generated by homogeneous symmetric polynomials with no constant term. The representations in question are tied to Macdonald positivity or other questions on symmetric functions.

The coinvariant ring is well-studied due to its rich combinatorial, geometric, and representation theoretic significance. As an $\mathfrak{S}_n$-module, it can be thought of as the cohomology ring of the flag variety [8] or a graded version of the regular representation of $\mathfrak{S}_n$ [14]. Its graded Frobenius can expressed using the *cocharge* statistic on standard tableaux [35, 50].

$$\mathrm{Frob}_q(R_n) = \sum_{S \in \mathrm{SYT}_n} q^{\mathrm{cocharge}(S)} s_{\mathrm{sh}(S)}.$$

This work studies the structure of these representations through an operation on tableaux called *catabolism.* Since its introduction in [38], many different generalizations of catabolism have been introduced, with ties to mathematical physics through Kirillov-Reshetikhin crystals [36, 45, 47], as well as to Demazure crystals [5] and GL_ℓ-equivariant Euler characteristics of vector bundles on the flag variety. However, the consistent theme is that though the operation itself is easy to compute, many seemingly evident properties of catabolism remain unproven. Furthermore, its connection to the algebraic structures of $\mathfrak{S}_n$-modules was unexplored.

This dissertation consists of four different papers. Chapter 5 studies the *Garsia–Procesi rings*, a family of $\mathfrak{S}_n$-representations indexed by partitions $\mu \vdash n$ that are further quotients of the coinvariant ring R_n. These quotients carry geometric, representation theoretic, and combinatorial significance, analogous to the coinvariant ring. In particular, they realize the cohomology ring of the Springer fiber indexed by μ [48]. Furthermore, the graded Frobenius character is given by the modified Hall-Littlewood polynomial [34], which can be expressed combinatorially in terms of tableaux that satisfy a certain catabolizability condition. In this chapter, we use combinatorial properties of catabolizability to construct a monomial basis of R_μ. The basis can be used to study the Schur expansion of the graded Frobenius character of the space, bridging the algebraic structure of the spaces as rings and their representation theoretic structure as $\mathfrak{S}_n$-modules. This also gives insight to the mysteries of catabolisms and catabolizability: in particular, we give a lower bound on the catabolizability type of certain permutations constructed from shuffles.

Chapter 6 studies the *Delta-Springer rings*, a larger family of generalized coinvariant rings due to Griffin [25]. This family is a simultaneous generalization of the Garsia–Procesi rings and the Haglund–Rhoades–Shimozono generalized coinvariant ring $R_{n,k}$[30], which gives an algebraic realization of the Delta conjecture [29] at $t = 0$ up to a twist. The Δ-Springer ring $R_{n,\lambda,s}$ can be realized as cohomology ring of a generalization of Type A Springer fibers [26]. In Chapter 6, we extend the work in Chapter 5 to give a monomial basis for $R_{n,\lambda,s}$ in terms of catabolizability. At the appropriate specializations, this reduces to our basis given in Chapter 5, as well as the generalized descent basis of $R_{n,k}$ due to Haglund–Rhoades–Shimozono [29]. In addition, we can use this basis to recover an explicit Schur expansion formula for

$\mathrm{Frob}_q(R_{n,\lambda,s})$. This is equivalent to a previously known combinatorial formula for $\mathrm{Frob}_q(R_{n,\lambda,s})$ due to Gillespie–Griffin [23] up to a combinatorial bijection. Though the Gillespie–Griffin formula is combinatorial, both the motivation and the proof arise from the geometry of generalized Springer fibers. While recent results have provided algebraic explanations for the $R_{n,k}$ case [22], it has been proven difficult to relate this expression to the actual algebraic or combinatorial structure of $R_{n,\lambda,s}$ in general.We show that the Gillespie–Griffin formula arising from geometry is equivalent to ours, which comes from the combinatorics of $R_{n,\lambda,s}$, bridging the known algebraic, combinatorial, and geometric pictures of $R_{n,\lambda,s}$.

Chapter 7 studies higher Specht bases for generalized coinvariant rings. For any partition λ of n, we can explicitly construct the irreducible $\mathfrak{S}_n$-module V_λ using Specht polynomials g_T, indexed by standard tableaux of shape λ. The *higher Specht polynomials* F_S^T, indexed by pairs (S, T) of standard tableaux of the same shape, are a generalization of the Specht polynomials. For certain choices of S, we recover the classical Specht polynomials. The collection of higher Specht polyomials forms a basis of the coinvariant ring R_n [2]. The higher Specht basis directly witnesses the decomposition of R_n into irreducibles. There is an analogous construction for $R_{n,k}$ and R_μ for two row shape as well as conjectured formulas for general R_μ [24]. In Chapter 7, we use the catabolizability conditions that appeared in Chapter 5 and 6 to give a conjectured higher Specht basis for R_μ. We prove our conjecture for certain partition shapes.

Chapter 8 studies the combinatorics of $\mathbf{R}$-catabolizability, introduced in [47]. Originally introduced to give a conjectured combinatorial formula for generalized q-Kostka coefficients, this notion of catabolism has close ties to the *k-Schur functions*. Work of Chen [13] gives a conjectured representation theoretic model for certain Catalan functions, a larger family of q-symmetric functions that contain the k-Schur functions. The first half of this chapter explores the connections between the *skew-linked modules* and the notion of $\mathbf{R}$-catabolizability. In particular, we give evidence towards a conjectured combinatorial formula for the skew-linked modules in terms of $\mathbf{R}$-catabolizability. The second half of the chapter extends the notion of $\mathbf{R}$-catabolizability, which was originally only defined on semistandard tableaux of a specific weight, to a larger family of tableaux that contains the standard tableaux. The upshot to this construction is that, in general, the statistic cocharge on permutations and standard tableaux is nicer than on the corresponding semistandard objects. The hope is that embedding the combinatorics of $\mathbf{R}$-catabolizability into the combinatorics of standard objects will illuminate new properties of this operation.

Chapter 2

Combinatorial Background

2.1 Partitions and Tableaux

A *composition* α of n, denoted $\alpha \models n$, is a sequence $(\alpha_1, \alpha_2, \dots, \alpha_l)$ where $\alpha_i \geqslant 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_l = n$. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ of n, denoted $\lambda \vdash n$, is a composition satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\ell(\lambda)} > 0$, where $\ell(\lambda)$ denotes the length of the partition. We denote the *transpose* of λ as λ'. Throughout, we write Young diagrams of partitions in French notation, meaning that the first part corresponds to the bottom row. We set $n(\lambda) := \sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i = \sum_{i=1}^{\ell(\lambda')}\binom{\lambda'_i}{2}$.

There is a partial ordering on the set of all the partitions of size n called the *dominance* ordering, denoted $\trianglerighteq$, defined by

$$\mu \trianglerighteq \lambda \quad \Leftrightarrow \quad \mu_1 + \cdots \mu_k \geqslant \lambda_1 + \cdots \lambda_k \text{ for all } k$$

It is well known that $\mu \trianglerighteq \lambda \quad \Leftrightarrow \quad \mu' \trianglelefteq \lambda'$. If we move a box of λ to a lower row so that the resulting shape μ is a partition, we have $\mu \trianglerighteq \lambda$. The dominance order is the transitive closure of moving boxes down in the Ferres diagram.

A *semistandard (Young) tableau* of shape λ is a filling of λ such that the rows are weakly increasing from left to right and the columns are strictly increasing from bottom to top. The *weight* of a semistandard tableau T is the tuple $(m_1, m_2, \dots)$ where m_i is the number of times i appears in T. The reading word $\mathrm{rw}(T)$ of a tableau T is the word we get by concatenating the row words from top to bottom.

We denote the shape of T by $\mathrm{sh}(T)$. Let $\mathrm{SSYT}(\mu)$ denote the set of semistandard tableaux of weight μ and $\mathrm{SSYT}(\lambda, \mu)$ denote the set of semistandard tableaux of shape λ, weight μ. The *Kostka number* $K_{\lambda,\mu}$ counts such tableaux: we have that $K_{\lambda,\mu} = |\,\mathrm{SSYT}(\lambda, \mu)|$.

A *standard (Young) tableau* of size n is a semistandard tableau of weight (1^n). For a given shape, we denote the set of standard Young tableaux of shape λ by

$\mathrm{SYT}(\lambda)$. We denote the set of all standard Young tableaux of partition shape and size n by SYT_n. Denote the restriction of a standard Young tableau S to the entries $[k] := \{1, \ldots, k\}$ by $S|_k$. Note that $S|_k \in \mathrm{SYT}_k$.

Example 2.1.1. Consider

$$S = \begin{array}{|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 3 & 5 \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array} \qquad\qquad S|_4 = \begin{array}{|c|c|c|} \cline{1-1} 3 \\ \hline 1 & 2 & 4 \\ \hline \end{array}.$$

We have that $\mathrm{sh}(S) = (4, 2, 1)$ and $\mathrm{rw}(S) = 6351247$.

We can describe the Kostka numbers $K_{\lambda,\mu}$ using standard tableaux in the following way:

Proposition 2.1.2. *For two partitions* $\lambda, \mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash n$, *we have*

$$K_{\lambda,\mu} = |\{S \in \mathrm{SYT}(\lambda) \mid \mathrm{Des}(S) \subset \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{l-1}\}\}| \tag{2.1}$$

where $\mathrm{Des}(S) = \{i : i$ *appears in a lower row than* $i + 1$ *in* $S\}$.

Proof. We construct an explicit bijection

$$\mathrm{SSYT}(\lambda, \mu) \to \{S \in \mathrm{SYT}(\lambda) \mid \mathrm{Des}(S) \subset \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{l-1}\}\}.$$

For any $T \in \mathrm{SSYT}(\lambda, \mu)$, we can standardize T in the naive way: replace the 1s that appear in T from left to right with $1, 2, \ldots, \mu_1$. It is clear that the only possible descent here is μ_1, which is a descent if the leftmost 2 in T is above the rightmost 1. Continuing to standardize in this way gives a standard tableau with the correct descent set. It is also clear that we can reverse this process to give an inverse map. □

2.2 Permutations and the Schensted correspondence

For any word $w = w_1 w_2 \ldots w_n$ we write $\mathrm{rev}(w) := w_n w_{n-1} \ldots w_1$ for the reverse word. A permutation $w \in \mathfrak{S}_n$ is a word of length n where each element in $\{1, \ldots, n\}$ appears exactly once. For any subset $U \subset [n] = \{1, \ldots, n\}$, let $w|_U$ denote the subword of w consisting of w_i such that $w_i \in U$.

The *inversion set* of $w \in \mathfrak{S}_n$ is defined to be $\mathrm{Inv}(w) := \{(i, j) \mid i < j, w_i > w_j\}$. The statistic *inversion* (inv) is defined to be $\mathrm{inv}(w) := |\mathrm{Inv}(w)|$. Similarly, the *descent set* of $w \in \mathfrak{S}_n$ is defined to be $\operatorname{des}(w) := \{i \mid w_i > w_{i+1}\}$. The statistic *major index* (maj) is defined to be $\operatorname{maj}(w) := \sum_{i \in \operatorname{des}(w)} i$.

It turns out that the two statistics are equidistributed in $\mathfrak{S}_n$. That is, we have:

$$\sum_{w \in \mathfrak{S}_n} q^{\operatorname{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\operatorname{maj}(w)} = [n]_q!,$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$ with $[k]_q = 1 + q + \cdots + q^{k-1}$.

Permutations and standard tableaux are related through the *Schensted correspondence*, which gives a bijection from permutations $w \in \mathfrak{S}_n$ to pairs $(P(w), Q(w))$ of standard Young tableau of size n of the same shape. We say $P(w)$ is the *insertion tableau* and $Q(w)$ is the *recording tableau* of w. Two permutations w, w' are *Knuth equivalent* if $P(w) = P(w')$. Alternatively, w and w' are equivalent under the following equivalence relations:

$$xzy \equiv zxy \quad (x \leqslant y < z),$$

$$yzx \equiv yxz \quad (x < y \leqslant z).$$

We also know that for a fixed $S \in \operatorname{SYT}(\lambda)$ we have that

$$|\{w \in \mathfrak{S}_n | P(w) = S\}| = |\operatorname{SYT}(\lambda)| = K_{\lambda,1^n}.$$

Proposition 2.2.1. *The following properties hold:*

(a) For any $w \in \mathfrak{S}_n$ and $k \leqslant n$, we have that $P(w|_k) = P(w)|_k$. ([49, Corollary of Lemma 7.11.2])

(b) For any $w \in \mathfrak{S}_n$, we have $\operatorname{Des}(w) = \operatorname{Des}(Q(w))$. ([49, Lemma 7.23.11])

(c) For any $w \in \mathfrak{S}_n$, we have $P(\operatorname{rev}(w)) = (P(w))'$.([49, Corollary A1.2.11])

We can also rewrite Proposition 2.1.2 using some fixed $S \in \operatorname{SYT}(\lambda)$ in the following way:

Corollary 2.2.2. *For two partitions $\lambda, \mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash n$ and some fixed $S \in \operatorname{SYT}(\lambda)$, we have*

$$K_{\lambda,\mu} = |\{w \in \mathfrak{S}_n, P(w) = S \mid \operatorname{Des}(w) \subset \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{l-1}\}\}|. \quad (2.2)$$

Proof. From Proposition 2.2.1(b), we have a bijection between the $T \in \operatorname{SYT}(\operatorname{sh}(S))$ with $\operatorname{Des}(T) \subset \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{l-1}\}\}$ and permutations $w \in \mathfrak{S}_n$ satisfying $P(w) = S, \operatorname{Des}(w) \subset \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{l-1}\}\}$ which sends T to the permutation w with $P(w) = S$, $Q(w) = T$. □

2.3 Specht modules

The irreducible representations of $\mathfrak{S}_n$ are indexed by its conjugacy classes, which are in bijection with partitions of size n. We denote the irreducible representation of $\mathfrak{S}_n$ indexed by partition $\lambda \vdash n$ by V_λ. Note that the cases $\lambda = (n)$ and 1^n correspond to the trivial ($\mathbb{1}$) and sign ($\mathcal{E}$) representation.

For any $\lambda \vdash n$, we can construct a copy of V_λ in the polynomial ring $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_n]$ in the following way.

Definition 2.3.1. Let $T \in \mathrm{SYT}_n$ of shape λ. Define the *Garnir polynomial* $g_T(\mathbf{x})$ to be

$$g_T(\mathbf{x}) = \prod_{c=1}^{\lambda_1} \prod_{\substack{i<j \\ i,j \text{ in column } c}} (x_j - x_i).$$

Equivalently $g_T(\mathbf{x})$ is the polynomial given by the product of Vandermonde determinants, one for each column of T.

We can also describe $g_T(\mathbf{x})$ in the following way. Define the monomial

$$\mathbf{x}_T := \prod_{i=1}^{n} x_i^{\mathrm{height}(i)-1} \tag{2.3}$$

where height(i) is the row in which i appears in T.

Then we have that

$$g_T(\mathbf{x}) = \sum_{\tau \in C(T)} \mathrm{sgn}(\tau)\tau(\mathbf{x}_T) \tag{2.4}$$

where $C(T))$ denote the row (column) group of T and τ acts on the monomial by permuting the indices.

Example 2.3.2. Let $T = \begin{array}{|c|cc} \hline 6 & & \\ \hline 2 & \multicolumn{1}{|c|}{4} & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$. Using Definition 2.3.1, we have that

$$g_T(\mathbf{x}) = (x_6 - x_1)(x_6 - x_2)(x_2 - x_1)(x_4 - x_3).$$

Equivalently, we can see that

$$g_T(\mathbf{x}) = (\mathrm{id} - (12) - (16) - (26) + (126) + (162))(\mathrm{id} - (34)) \;\; (x_2 x_4 x_6^2)$$

Proposition 2.3.3. *We have that span*$\{g_T(\mathbf{x}) : T \in \mathrm{SYT}(\lambda)\}$ *is a* $\mathbb{C}\mathfrak{S}_n$*- module that is isomorphic to the irreducible representation* V_λ*.*

2.4 Symmetric Functions and the Frobenius characteristic map

For a detailed treatment of symmetric functions, see [41]. Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be an infinite set of variables, and let Λ_F denote the *ring of symmetric functions* in these variables with coefficients in a field F. We denote by $e_\lambda(\mathbf{x}), h_\lambda(\mathbf{x}), s_\lambda(\mathbf{x}), m_\lambda(\mathbf{x})$ the *elementary*, *complete homogeneous*, *Schur*, and *monomial* symmetric functions respectively. We omit the $\mathbf{x}$-variables when it does not cause confusion.

All four of these are bases of Λ_F, meaning we can rewrite any symmetric function f using one of the four families. In particular, we say a symmetric function f is *Schur-positive* if all the coefficients in the Schur expansion of f are nonnegative integers. Note that the Kostka numbers appear as the coefficients of the Schur expansion of h_μ: that is, we have

$$h_\mu = \sum_{\lambda \vdash |\mu|} K_{\lambda,\mu}\, s_\lambda \tag{2.5}$$

The *Hall inner product* is the symmetric inner product defined on Λ by the relation $\langle m_\lambda, h_\gamma \rangle = \delta_{\lambda,\gamma}$. The Schur functions are an orthonormal basis with respect to this inner product. We also have an involution ω on Λ, defined by $\omega e_\lambda = h_\lambda$. The map ω is an isometry: that is, we have $\langle f, g \rangle = \langle \omega f, \omega g \rangle$ for any $f, g \in \Lambda$. We also have $\omega s_\lambda = s_{\lambda'}$ for any λ.

Symmetric functions are deeply connected with the study of $\mathfrak{S}_n$-representations through the Frobenius characteristic map. The *Frobenius characteristic map* is a map F from (virtual) characters of $\mathfrak{S}_n$ to symmetric functions of degree n that takes the irreducible character indexed by $\lambda \vdash n$ to the Schur function s_λ. This map encodes the character of an $\mathfrak{S}_n$ representation V into a Schur-positive symmetric function.

The *Frobenius character* of V, denoted $\mathrm{Frob}(V)$, is defined to be $F(\chi_V)$, where χ_V is the character of V. We have $\mathrm{Frob}(V_\lambda) = F(\chi_{V_\lambda}) = s_\lambda$, where V_λ is the irreducible $\mathfrak{S}_n$ representation indexed by λ.

For any $\mathbb{C}\mathfrak{S}_n$-module V, we can recover $\dim(V)$ from the Schur expansion of $\mathrm{Frob}(V)$ by replacing each s_λ with $\dim(V_\lambda) = K_{\lambda,1^n}$. We can also use $\mathrm{Frob}(V)$ to study dimensions of certain subspaces of V. For the Young subgroup $\mathfrak{S}_\gamma = \mathfrak{S}_{\gamma_1} \times \cdots \times \mathfrak{S}_{\gamma_l} \subset \mathfrak{S}_n$ corresponding to $\gamma \models n$, we define $N_\gamma = \sum_{\sigma \in \mathfrak{S}_\gamma} \mathrm{sgn}(\sigma)\sigma$ to be the antisymmetrizer with respect to γ. The vector space $N_\gamma V$ is the subspace of elements of V that are antisymmetric with respect to $\mathfrak{S}_\gamma$. Let $\mathcal{E} \uparrow_{\mathfrak{S}_\gamma}^{\mathfrak{S}_n}$ denote the induction of the sign representation $\mathcal{E}$ of $\mathfrak{S}_\gamma$ to $\mathfrak{S}_n$. It is well known that $\mathrm{Frob}(\mathcal{E} \uparrow_{\mathfrak{S}_\gamma}^{\mathfrak{S}_n}) = e_\gamma$. Using Frobenius reciprocity (see [18, Chapter 3.3]), we get the following result:

Proposition 2.4.1. *For any $\mathfrak{S}_n$-representation V and Young subgroup $\mathfrak{S}_\gamma \subset \mathfrak{S}_n$, we have*

$$\dim(N_\gamma V) = \langle e_\gamma, \mathrm{Frob}(V)\rangle.$$

For a graded vector space $V = \oplus_{d\geqslant 0} V_d$, the *Hilbert series* $\mathrm{Hilb}_q(V)$ is

$$\mathrm{Hilb}_q(V) = \sum_{d\geqslant 0} q^d \dim(V_d).$$

For a graded $\mathbb{C}\mathfrak{S}_n$-module $V = \oplus_{d\geqslant 0} V_d$, the *graded Frobenius character* $\mathrm{Frob}_q(V)$ is

$$\mathrm{Frob}_q(V) = \sum_{d\geqslant 0} q^d\, \mathrm{Frob}(V_d).$$

The analogous statement for graded modules holds if we replace $\dim(N_\gamma V)$ (resp. $\mathrm{Frob}(V)$) with $\mathrm{Hilb}_q(N_\gamma V)$ (resp. $\mathrm{Frob}_q(V)$). Since we know that any homogeneous symmetric function f of degree n is uniquely determined by the values $\langle e_\gamma, f\rangle$ for all $\gamma \vdash n$, we see that knowing $\mathrm{Hilb}_q(N_\gamma V)$ for all $\gamma \models n$ uniquely defines $\mathrm{Frob}_q(V)$.

2.5 Hall-Littlewood polynomials

The *transformed Hall-Littlewood polynomials* $H_\mu[X;q]$ are defined to be

$$H_\mu[X;q] = \sum K_{\lambda,\mu}(q) s_\lambda \tag{2.6}$$

where $K_{\lambda,\mu}(q)$ is the q-Kostka polynomial (see [41, Chapter III.6] for definition). There is a combinatorial formula for $K_{\lambda,\mu}(q)$ due to Lascoux-Schützenberger [40] using a statistic defined on semistandard Young tableaux called *charge*:

$$K_{\lambda,\mu}(q) = \sum_{T\in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{charge}(T)} \tag{2.7}$$

The sum is over all semistandard tableaux of shape λ and weight μ. In particular, $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$. From this, we can see that setting $q = 1$ in (2.6) recovers (2.5): that is, we have $H_\mu[X;1] = h_\mu$.

The *modified Hall-Littlewood polynomials* $\widetilde{H}_\mu[X;q]$ are defined by:

$$\widetilde{H}_\mu[X;q] = q^{n(\mu)} H_\mu[X;q^{-1}]$$

where $n(\mu) = \sum(i-1)\mu_i$. Then we can write

$$\widetilde{H}_\mu[X;q] = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q) s_\lambda(X)$$

when $\widetilde{K}_{\lambda,\mu}(q) := q^{n(\mu)}K_{\lambda,\mu}(q^{-1})$ is the modified q-Kostka polynomial. The polynomial $\widetilde{K}_{\lambda,\mu}(q)$ can be expressed by the statistic *cocharge* :

$$\widetilde{K}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{cocharge}(T)}$$

where for a semistandard tableau of weight μ we have $\mathrm{cocharge}(T) = n(\mu) - \mathrm{charge}(T)$.

2.6 Cocharge

Definition 2.6.1. The *cocharge word* of $w \in \mathfrak{S}_n$ (denoted $\mathrm{cc}(w)$) is a word of length n consisting of the labeling of w that we obtain in the following way. Label the 1 of w with 0. Assume we labeled the letter i with k. Label $(i+1)$ with a k if it is to the right of i. We label $(i+1)$ with a $(k+1)$ if it is to the left of i.

This process is equivalent to scanning w from left to right, labeling entries in increasing order. The label of i corresponds to the number of times we need to return to the left end of w before we label i.

Definition 2.6.2. The statistic *cocharge*(w) is the sum of the letters in $\mathrm{cc}(w)$.

Example 2.6.3. Let $w = 3\ \ 5\ \ 1\ \ 6\ \ 2\ \ 4\ \ 7$. The corresponding cocharge word is $\mathrm{cc}(w) = 1\ \ 2\ \ 0\ \ 2\ \ 0\ \ 1\ \ 2$, hence $\mathrm{cocharge}(w) = 1 + 2 + 0 + 2 + 0 + 1 + 2 = 8$.

Now, we extend the definitions to standard tableaux. Define $\mathrm{cocharge}(S) := \mathrm{cocharge}(\mathrm{rw}(S))$ for any standard tableau S. This definition is compatible with the Schensted correspondence due to the following fact.

Theorem 2.6.4 (Lascoux–Schützenberger [40])**.** *If* $w, w' \in \mathfrak{S}_n$ *are such that* $P(w) = P(w')$*, then* $\mathrm{cocharge}(w) = \mathrm{cocharge}(w')$.

Since cocharge is constant on Knuth equivalence classes, we have that for any $w \in \mathfrak{S}_n$ we have $\mathrm{cocharge}(w) = \mathrm{cocharge}(P(w))$.

Analogous to how we construct the cocharge word of a permutation, we can construct a cocharge tableau $\mathrm{cc}(S)$ for each standard tableau S by replacing i in S with the corresponding cocharge label of i in $\mathrm{rw}(S)$. Let $\mathrm{cc}(S)_i$ denote the entry in $\mathrm{cc}(S)$ corresponding to i in S.

Alternatively, we can define $\mathrm{cc}(S)$ directly on tableaux:

Definition 2.6.5. The *cocharge tableau* of a standard tableau S (denoted $\mathrm{cc}(S)$) is a filling of the same shape consisting of the labeling of S that we obtain in the following way. Label the 1 of S with 0. Assume we labeled the letter i with k. Label $(i+1)$ with a k if is in the same row or lower than i. We label $(i+1)$ with a $(k+1)$ if it is in a higher row than i.

It is a simple exercise to check that these two definitions are equivalent. Note that for Definition 2.6.5, we do not specify that S must be of partition shape.

Example 2.6.6. Let $\begin{array}{|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 3 & 5 \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array}$. Then $\mathrm{cc}(S) = \begin{array}{|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 1 & 2 \\ \hline 0 & 0 & 1 & 3 \\ \hline \end{array}$. In this case, $\mathrm{cc}(S)_4 = 1$.

We record the following observation, which follows immediately from Definition 2.6.5.

Lemma 2.6.7. *For any $S \in \mathrm{SYT}_n$, we have $\mathrm{cc}(S)_i \leqslant \mathrm{des}(S)$ for any i.*

The charge statistic on permutations can be derived from the cocharge statistic using the fact that $\mathrm{charge}(w) = \binom{n}{2} - \mathrm{cocharge}(w)$.

Proposition 2.6.8. *Charge and cocharge are related in the following way.*

1. *For any permutation $w \in \mathfrak{S}_n$, we have that $\mathrm{charge}(w) = \mathrm{cocharge}(\mathrm{rev}(w))$.*
2. *For any $S \in \mathrm{SYT}$, we have that $\mathrm{charge}(S) = \mathrm{cocharge}(S')$.*

Alternatively, we can define $\mathrm{charge}(w) = \mathrm{maj}(\mathrm{rev}(w^{-1}))$. From this, we can see that the maj and cocharge are closely related statistics. Furthermore, we have the following fact for maj and cocharge of standard tableaux.

Theorem 2.6.9 (Killpatrick [35])**.** *For any partition λ, we have*

$$\sum_{S \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(S)} = \sum_{S \in \mathrm{SYT}(\lambda)} q^{\mathrm{cocharge}(S)}.$$

We have a stronger version of this equidistribution which includes the descent statistic on tableaux. This was a folklore result, with a recent proof due to Chan [12].

Proposition 2.6.10 (Chan [12])**.** *For any partition λ, we have*

$$\sum_{S \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(S)} t^{\mathrm{des}(S)} = \sum_{S \in \mathrm{SYT}(\lambda)} q^{\mathrm{cocharge}(S)} t^{\mathrm{des}(S)}.$$

2.7 Cyclage and Cyclage posets

For any partition μ, we can define a partial order on the set $\mathrm{SSYT}(\mu)$ using rotations of the underlying words.

For two semistandard tableaux $S, T \in \mathrm{SSYT}(\mu)$, we say T *covers* S ($S \lessdot_{\mathrm{cyc}} T$) if there exist a letter $a > 1$ and word x consisting of positive integers such that $P(ax) = T$ and $P(xa) = S$ where $P(x)$ denotes the insertion tableau of the word x with respect to the RSK algorithm.

Equivalently, we have $S \lessdot_{\mathrm{cyc}} T$ if there is a corner b of T such that reverse column inserting at b in T, which results in some letter a and tableau U, and then row inserting a into U produces the tableau S.

Example 2.7.1. Consider $T = \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 3 \\ \cline{1-2} 1 & 1 & 2 & 4 \\ \hline \end{array}$. We can find all S that are covered by T by taking off corners of T :

$$\begin{array}{|c|c|c|c|} \hline \boxed{3} \\ \cline{1-1} 2 & 3 \\ \cline{1-2} 1 & 1 & 2 & 4 \\ \hline \end{array} \to U_1 = \begin{array}{|c|c|c|c|} \hline 2 & 3 \\ \cline{1-2} 1 & 1 & 2 & 4 \\ \hline \end{array}, a_1 = 3 \to S_1 = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 \\ \cline{1-3} 1 & 1 & 2 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & \boxed{3} \\ \cline{1-2} 1 & 1 & 2 & 4 \\ \hline \end{array} \to U_2 = \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 & 1 & 2 & 4 \\ \hline \end{array}, a_2 = 3 \to S_2 = \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 4 \\ \cline{1-2} 1 & 1 & 2 & 3 \\ \hline \end{array}$$

Alternatively, we can see that

$$T = P(3231124) \to S_1 = P(2311243),$$

$$T = P(3321124) \to S_2 = P(3211243).$$

For composition α such that $\mathrm{sort}(\alpha) = \mu$ (where $\mathrm{sort}(\alpha)$ denote the partition we get by reordering the parts of α in weakly decreasing order) there exists a permutation w such that $w\alpha = \mu$. Then $w : T \mapsto wT$ defines a shape-preserving bijection $\mathrm{SSYT}(\alpha) \to \mathrm{SSYT}(\mu)$. For $S, T \in \mathrm{SSYT}(\alpha)$, we say T *covers* S ($S \lessdot_{\mathrm{cyc}} T$) if wT covers wS in $T(\mu)$. The resulting poset $T(\alpha)$ is isomorphic to $\mathcal{T}(\mu)$, where w is the poset isomorphism.

We define *cyclage* to be this covering relation and the *cyclage poset* to be the resulting poset $\mathcal{T}(\alpha)$.

Theorem 2.7.2 (Lascoux–Schützenberger [39]). *The cyclage poset $\mathcal{T}(\mu)$ is ranked, where the rank function is given by cocharge. The unique bottom element of $\mathcal{T}(\mu)$ is the one-row tableau of weight μ.*

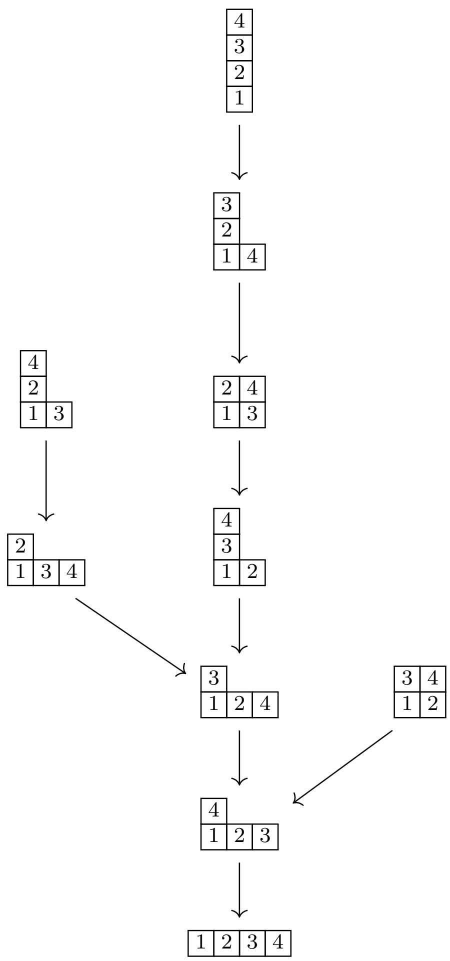


Figure 2.1: Cyclage poset on SYT_4

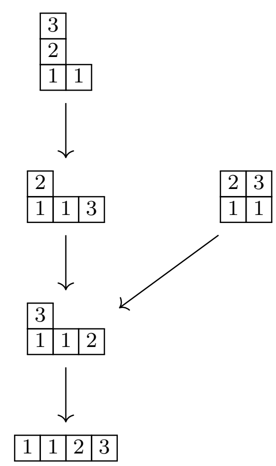


Figure 2.2: Cyclage poset on $\mathrm{SSYT}(2, 1, 1)$

In fact, we can see that if $(S \leqslant_{\text{cyc}} T)$ for $S, T \in \text{SSYT}(\mu)$, we have $\text{cocharge}(T) = \text{cocharge}(S) + 1$.

Furthermore, the family of posets $\{\mathcal{T}(\alpha)\}_{\alpha \models n}$ is equipped with functorial rank-preserving embeddings. For two compositions $\alpha, \beta \models n$ such that $\text{sort}(\alpha) \trianglerighteq \text{sort}(\beta)$, we have an embedding $\theta_\alpha^\beta : \mathcal{T}(\alpha) \hookrightarrow \mathcal{T}(\beta)$. When $\beta = 1^n$, we write $\text{std}_\alpha := \theta_\alpha^{1^n}$.

We have a concrete way of describing these embeddings using operators defined on words which arise in the theory of crystals. Let w be a word consisting of positive integers. For a positive integer r, consider the subword of letters $r, (r+1)$. We replace each r (resp $r+1$) with a right (resp. left) parenthesis and match the parentheses. A letter is *unpaired* if it is not part of a pair of matched parentheses. The subword of w of unpaired $r, r+1$ is of the form $r^p(r+1)^q$.

Definition 2.7.3. The crystal reflection operator s_r and lowering operator f_r act on the unpaired subword $r^p(r+1)^q$ of w in the following way:

1. For $s_r w$, replace the unpaired subword $r^p(r+1)^q$ with $r^q(r+1)^p$.

2. For $f_r w$, replace the unpaired subword $r^p(r+1)^q$ with $r^{p-1}(r+1)^{q+1}$. (Note: in this case, we must have $p > 0$.)

Furthermore, for any permutation τ, we can define $\tau u = s_{i_1} s_{i_2} \cdots s_{i_p} u$ where $s_{i_1} s_{i_2} \cdots s_{i_p}$ is a reduced expression of τ.

Example 2.7.4. Consider $w = 32124112322$. The subword of w consisting of unpaired $1, 2$ is 1222. We have that $s_1 w = 3212411\mathbf{1}3\mathbf{1}2$, $f_1 w = 321241\mathbf{2}2322$.

Note that we can define these operators on any semistandard Young tableau (of potentially skew shape) by acting on the row reading word of the tableau from the following fact. (For a proof of this, see [Proposition 2.76 [46]]).

Proposition 2.7.5. *If w and w' are Knuth equivalent, then so are $s_i(w)$ and $s_i(w')$, as well as $f_i(w)$ and $f_i(w')$.*

Now, we explicitly describe the embedding $\theta_\alpha^\beta : \mathcal{T}(\alpha) \hookrightarrow \mathcal{T}(\beta)$ using these operators.

Definition 2.7.6. The embedding of cyclage posets $\theta_\alpha^\beta : \mathcal{T}(\alpha) \hookrightarrow \mathcal{T}(\beta)$ is given by the following.

1. If $\text{sort}(\beta) = \text{sort}(\alpha)$ and $\beta = \tau\alpha$ is the composition we get from applying τ to permute the parts of α, we have $\theta_\alpha^\beta = \tau$.

2. If $\beta = (\alpha_1 - 1, \alpha_2 + 1, \alpha_3, \ldots)$, then $\theta_\alpha^\beta = f_1$.

If $\mathrm{sort}(\alpha) \trianglerighteq \mathrm{sort}(\beta)$, there exists a sequence $\gamma_0 = \alpha, \gamma^{(1)}, \ldots, \gamma^{(p)} = \beta$ of compositions where $\gamma^{(i)}$ and $\gamma^{(i+1)}$ differ by (1) or (2) above. Then we can define

$$\theta_\alpha^\beta = \theta_{\gamma^{(p-1)}}^\beta \circ \theta_{\gamma^{(p-2)}}^{\gamma^{(p-1)}} \circ \cdots \circ \theta_\alpha^{\gamma^{(1)}}.$$

Note that this sequence γ may not be unique. However, the map θ_α^β is independent of which sequence we choose.

Theorem 2.7.7 (Lascoux [38])**.** *If* $\mathrm{sort}(\alpha) \trianglerighteq \mathrm{sort}(\beta)$*, the map* θ_α^β *is independent of the sequence of compositions* $\{\gamma^{(i)}\}$ *we use to get from* α *to* β*.*

Remark 2.7.8. If we look at Figures 2.1 and 2.2, we can see that the cyclage poset on $\mathrm{SSYT}(2,1,1)$ embeds in SYT_4 by mapping $11 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4$.

Example 2.7.9. Consider $T = \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 2 \\ \hline 1&1&2&3&3&4&4 \\ \hline\end{array} \in \mathrm{SSYT}(2,2,2,2)$. We can compute $\theta_{(2,2,2,2)}^{1^n}(T)$ in the following way.

First, apply $\tau_2\tau_3\tau_4$ to get a tableau of weight $(2,0,2,2,2)$.

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 2 \\ \hline 1&1&2&3&3&4&4 \\ \hline\end{array} \xrightarrow{\tau_4} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 2 \\ \hline 1&1&2&3&3&5&5 \\ \hline\end{array} \xrightarrow{\tau_3} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 2 \\ \hline 1&1&2&4&4&5&5 \\ \hline\end{array} \xrightarrow{\tau_2} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&1&3&4&4&5&5 \\ \hline\end{array}.$$

Now, we apply f_1 to get a tableau of weight $(1,1,2,2,2)$

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&1&3&4&4&5&5 \\ \hline\end{array} \xrightarrow{f_1} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&2&3&4&4&5&5 \\ \hline\end{array}.$$

Apply $\tau_2\tau_1$ to get a tableau of weight $(2,1,1,2,2)$.

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&2&3&4&4&5&5 \\ \hline\end{array} \xrightarrow{\tau_2} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&2&2&4&4&5&5 \\ \hline\end{array} \xrightarrow{\tau_1} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&1&2&4&4&5&5 \\ \hline\end{array}$$

Apply $\tau_2\tau_3\tau_4\tau_5$ to get a tableau of weight $(2,0,1,1,2,2)$

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&1&2&4&4&5&5 \\ \hline\end{array} \xrightarrow{\tau_5} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&1&2&4&4&6&6 \\ \hline\end{array} \xrightarrow{\tau_4} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 3 \\ \hline 1&1&2&5&5&6&6 \\ \hline\end{array} \xrightarrow{\tau_3} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&1&2&5&5&6&6 \\ \hline\end{array} \xrightarrow{\tau_2} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&1&3&5&5&6&6 \\ \hline\end{array}.$$

Now apply f_1 to get get a tableau of weight $(1,1,1,1,2,2)$

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&1&3&5&5&6&6 \\ \hline\end{array} \xrightarrow{f_1} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&2&3&5&5&6&6 \\ \hline\end{array}.$$

Repeat. First apply $\tau_1\tau_2\tau_3\tau_4$ to get $(2,1,1,1,1,2)$

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&2&3&5&5&6&6 \\ \hline\end{array} \xrightarrow{\tau_4} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&2&3&4&5&6&6 \\ \hline\end{array} \xrightarrow{\tau_3} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&2&3&3&5&6&6 \\ \hline\end{array} \xrightarrow{\tau_1\tau_2} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&1&2&3&5&6&6 \\ \hline\end{array}.$$

Then apply $\tau_2\tau_3\tau_4\tau_5\tau_6$ to get $(2,0,1,1,1,1,2)$

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 4 \\ \hline 1&1&2&3&5&6&6 \\ \hline\end{array} \xrightarrow{\tau_2\tau_3\tau_4\tau_5\tau_6} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&1&3&4&6&7&7 \\ \hline\end{array}.$$

Apply f_1 to get $(1,1,1,1,1,1,2)$.

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&1&3&4&6&7&7 \\ \hline\end{array} \xrightarrow{f_1} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&2&3&4&6&7&7 \\ \hline\end{array}.$$

Apply $\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6$ to get $(2,1,1,1,1,1,1)$.

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&1&3&4&6&7&7 \\ \hline\end{array} \xrightarrow{\tau_5\tau_6} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&2&3&4&5&6&7 \\ \hline\end{array} \xrightarrow{\tau_4} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&2&3&4&4&6&7 \\ \hline\end{array} \xrightarrow{\tau_1\tau_2\tau_3} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&1&2&3&4&6&7 \\ \hline\end{array}.$$

Apply $\tau_2\tau_3\tau_4\tau_5\tau_6\tau_7$ to get $(2,0,1,1,1,1,1,1)$.

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 5 \\ \hline 1&1&2&3&4&6&7 \\ \hline\end{array} \xrightarrow{\tau_2\tau_3\tau_4\tau_5\tau_6\tau_7} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 6 \\ \hline 1&1&3&4&5&7&8 \\ \hline\end{array}.$$

Finally, apply f_1 to get a standard tableau.

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 6 \\ \hline 1&1&3&4&5&7&8 \\ \hline\end{array} \xrightarrow{f_1} \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 6 \\ \hline 1&2&3&4&5&7&8 \\ \hline\end{array}.$$

Chapter 3

The Type A Coinvariant ring

The polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ has a natural $\mathfrak{S}_n$ action permuting the variables. Taking the quotient of this polynomial ring by the ideal generated by $\mathfrak{S}_n$-invariant polynomials with no constant term defines the classical *coinvariant ring*:

$$R_n = \mathbb{C}[\mathbf{x}]/\langle e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\rangle,$$

where $e_d(\mathbf{x})$ is the dth elementary symmetric function. We can see that the $\mathfrak{S}_n$-action that permutes the variables is still well-defined on R_n. In fact, this action on R_n gives a graded version of the regular representation of $\mathfrak{S}_n$[14]. As a graded $\mathbb{C}\mathfrak{S}_n$-module, where the grading comes from the natural grading on the polynomial ring given by degrees, it is isomorphic to the cohomology ring of the complex flag variety $\mathcal{F}_n$ [8]. In particular, we have $\mathrm{Hilb}_q(R_n) = [n]_q!$.

The graded Frobenius character of R_n is given by the following formula in terms of the major index statistic $\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i$ on standard Young tableaux [50]:

$$\mathrm{Frob}_q(R_n) = \sum_{T \in \mathrm{SYT}_n} q^{\mathrm{maj}(T)} s_{\mathrm{sh}(T)} \tag{3.1}$$

Using Theorem 2.6.9, we can rewrite $\mathrm{Frob}_q(R_n)$ using the cocharge statistic on standard tableaux:

$$\mathrm{Frob}_q(R_n) = \sum_{S \in \mathrm{SYT}_n} q^{\mathrm{cocharge}(S)} s_{\mathrm{sh}(S)}. \tag{3.2}$$

Example 3.0.1. We can compute $\mathrm{Frob}_q(R_4)$ using the cyclage poset SYT_4 given in Figure 2.1:

$$\mathrm{Frob}_q(R_4) = s_4 + (q + q^2 + q^3)s_{3,1} + (q^2 + q^4)s_{2,2} + (q^3 + q^4 + q^5)s_{2,1,1} + q^6 s_{1,1,1,1}.$$

3.1 Monomial bases arising from permutation statistics

There are two well-known monomial bases of R_n, both arising from permutation statistics. The *Artin basis* is given by the following set of monomials:

$$\{f_\sigma(\mathbf{x}) = \prod_{i<j, \sigma_i > \sigma_j} x_{\sigma_i} \mid \sigma \in S_n\} = \{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \mid a_i < i\}.$$

The *Garsia-Stanton descent basis* is given by the following set of monomials:

$$\{g_w(\mathbf{x}) = \prod_{i : w_i > w_{i+1}} x_{w_1} \cdots x_{w_i} \mid w \in S_n\}.$$

We refer to the exponents of descent monomials, given by the set $\mathcal{D}_n = \{\mathbf{a} \mid \mathbf{x}^{\mathbf{a}} = g_w(\mathbf{x}) \text{ for } \sigma \in \mathfrak{S}_n\}$, as *descent words.*

We see that $\deg(f_w(\mathbf{x})) = \mathrm{inv}(w)$ and $\deg(g_w(\mathbf{x})) = \mathrm{maj}(w)$. From this, we have

$$\sum_{w \in S_n} q^{\deg(f_w(\mathbf{x}))} = \sum_{w \in S_n} q^{\deg(g_w(\mathbf{x}))} = [n]_q! = \mathrm{Hilb}_q(R_n).$$

Thus, either basis gives a combinatorial explanation of $\mathrm{Hilb}_q(R_n) = [n]_q!$.

The Artin basis of R_n arises naturally from Gröbner theory, as they are the nonleading terms with respect to the lexicographical order. The Garsia-Stanton descent basis comes from considering a different order, which we define here.

Definition 3.1.1. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$. Define $\mathrm{sort}(\mathbf{a}) := (a_{i_1} \geqslant \cdots \geqslant a_{i_n})$ to be the unique way to write the entries of $\mathbf{a}$ in weakly decreasing order. Then, we say that $\mathbf{a} <_{\mathrm{des}} \mathbf{b}$ if:

$$\begin{cases} \mathrm{sort}(\mathbf{a}) <_{\mathrm{lex}} \mathrm{sort}(\mathbf{b}) \text{ or,} \\ \mathrm{sort}(\mathbf{a}) = \mathrm{sort}(\mathbf{b}) \text{ and } \mathbf{a} <_{\mathrm{lex}} \mathbf{b} \end{cases}$$

The descent monomials can be characterized as being the nonleading monomials of R_n with respect to the descent order. Note that des does *not* induce a monomial order on $\mathbb{C}[\mathbf{x}_n]$, as we do not have $\mathbf{a} <_{\mathrm{des}} \mathbf{b} \implies \mathbf{a} + \mathbf{c} <_{\mathrm{des}} \mathbf{b} + \mathbf{c}$, but is only a total order on monomials of $\mathbb{C}[\mathbf{x}_n]$. The descent order, however, satisfies the following useful lemma. A proof of this lemma can be found in [11].

Lemma 3.1.2. *Let $A \subset [n]$ with complement A'. If α is a composition, denote $\alpha|_A := (\alpha_{i_1}, \ldots, \alpha_{i_s})$, where $A = \{i_1 < \cdots < i_s\}$. Then, we have*

$$\alpha|_A \leqslant_{\mathrm{des}} \beta|_A, \quad \alpha|_{A'} \leqslant_{\mathrm{des}} \beta|_{A'} \implies \alpha \leqslant_{\mathrm{des}} \beta$$

3.2 Higher Specht basis

From Proposition 2.3.3, we know that we can construct an explicit copy of V_λ in the polynomial ring $\mathbb{C}[\mathbf{x}]$ using Garnir polynomials $g_T(\mathbf{x})$, which are indexed by $T \in SYT(\lambda)$.

We can extend this definition to consider polynomials indexed by pairs (S, T) of standard tableaux of the same shape. This construction is due to Ariki-Terasoma-Yamada [2].

Definition 3.2.1. Consider $S, T \in \mathrm{SYT}_n$ of shape λ. We define the monomial $\mathbf{x}_T^S$ to be given by

$$\mathbf{x}_T^S := \prod_{b \in \lambda} x_{T(b)}^{\mathrm{cc}(S)(b)} \tag{3.3}$$

where $\mathrm{cc}(S)$ denotes the cocharge labeling of S and for any box b in shape λ, we have that $T(b), \mathrm{cc}(S)(b)$ denote the entries in box b in $T, \mathrm{cc}(S)$ respectively.

Example 3.2.2. Consider

$$S = \begin{array}{|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 3 & 5 \\ \hline \end{array} \qquad T = \begin{array}{|c|c|c|} \cline{1-1} 4 \\ \cline{1-2} 3 & 6 \\ \hline 1 & 2 & 5 \\ \hline \end{array}.$$

We can see that $\mathrm{cc}(S) = \begin{array}{|c|c|c|} \cline{1-1} 3 \\ \cline{1-2} 1 & 2 \\ \hline 0 & 1 & 2 \\ \hline \end{array}$. Then $\mathbf{x}_T^S = x_1^0 x_2^1 x_3^1 x_4^3 x_5^2 x_6^2$

Remark 3.2.3. Note that if we set S to be the row-filled standard Young tableau of shape λ (ie. the bottom row consists of $1, \ldots, \lambda_1$, the second row consists of $(\lambda_1 + 1), \ldots, (\lambda_1 + \lambda_2)$ and so on), we have that $\mathbf{x}_T^S = \mathbf{x}_T$, where $\mathbf{x}_T$ is defined in (2.3).

Definition 3.2.4. For $T \in SYT(\lambda)$, we define $C(T)$ (resp: $R(T)$) to be the group of column (row) permutations, or $\tau \in \mathfrak{S}_n$ (resp: $\sigma \in \mathfrak{S}_n$) which preserves the columns (rows) of T. Define the *Young idempotent* to be:

$$\varepsilon_T := \sum_{\tau \in C(T)} \sum_{\sigma \in R(T)} (\operatorname{sgn} \tau) \tau \sigma$$

Given $S, T \in \mathrm{SYT}(\lambda)$, the *higher Specht polynomial* F_T^S is defined to be

$$F_T^S := \varepsilon \cdot \mathbf{x}_T^{\mathrm{cc}(S)} \tag{3.4}$$

By definition, we can see that the degree of F_T^S is given by $\mathrm{cocharge}(S)$.

Remark 3.2.5. Note again that if we set S to be the row-filled tableau of shape λ, we have that (3.4) coincides with the alternative definition for the Garnir polynomial $g_T(\mathbf{x})$ given in (2.4) up to a scalar factor $|R(T)|$.

Now we have a collection of $n!$ polynomials, each indexed by pairs of tableaux of the same shape.

Theorem 3.2.6 (Ariki–Terasoma–Yamada [2])**.** *The collection of polynomials $\{F_T^S : S, T \in \mathrm{SYT}_n, \mathrm{Sh}(S) = \mathrm{Sh}(T)\}$ is a basis of R_n such that for a fixed S of shape λ, the span of $\{F_T^S : T \in \mathrm{SYT}(\lambda)\}$ in R_n is isomorphic to the irreducible representation V_λ.*

In particular, the higher Specht basis explicitly witnesses the decomposition of R_n into irreducibles given by (3.2). If we set S to be the row-filled tableau, the subspace corresponding higher Specht polynomials is exactly image of the the Specht module given in Proposition 2.3.3 under the projection $\mathbb{C}[\mathbf{x}] \to R_n$.

Remark 3.2.7. Note that for any shape λ, the row-filled standard tableau is the unique standard tableau of that shape with minimal cocharge, given by $n(\lambda)$. From this, for any shape λ we can see that the projection of Specht module given in Proposition 2.3.3 into R_n is the lowest degree copy of V_λ in R_n.

Example 3.2.8. If we look at Figure 2.1, we can see that there are three different standard tableau of shape (2,1,1):

$$\begin{array}{|c|c}\cline{1-1} 4 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \qquad \begin{array}{|c|c}\cline{1-1} 4 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \qquad \begin{array}{|c|c}\cline{1-1} 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline\end{array}.$$

The row-filled tableau $\begin{array}{|c|c}\cline{1-1} 4 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$ is the one with minimal cocharge. This corresponds to the copy of $V_{2,1,1}$ in degree 2 of R_n.

3.3 Harmonics presentation

There is an alternative presentation of the coinvariant ring in terms of harmonics. Given a polynomial $f \in \mathbb{C}[\mathbf{x}]$, we can let $f(\partial)$ be the differential operator on $\mathbb{C}[\mathbf{x}]$ we get by replacing each $x_i \mapsto \partial/\partial x_i$ in f. Using this, we have that $\mathbb{C}[\mathbf{x}]$ acts on itself by

$$f \odot g = f(\partial)g(\mathbf{x}) \quad \text{for all } f, g \in \mathbb{C}[\mathbf{x}].$$

Using this, we can define an inner product on $\mathbb{C}[\mathbf{x}]$ by

$$\langle f, g \rangle := \text{ constant term of } f \odot g.$$

This inner product is $\mathfrak{S}_n$-invariant, meaning that $\langle f, g\rangle = \langle \sigma f, \sigma g\rangle$ for any $\sigma \in \mathfrak{S}_n$.

Given a homogenous ideal $I \in \mathbb{C}[\mathbf{x}]$, the *harmonic space* V associated to I is given by

$$V = I^{\perp} = \{g \in \mathbb{C}[\mathbf{x}], \langle f, g\rangle = 0 \text{ for all } f \in I\}.$$

In the case where I is a $\mathfrak{S}_n$-invariant ideal, we have that the $\mathfrak{S}_n$-invariant inner product gives us $\mathbb{C}[\mathbf{x}]/I \cong_{\mathfrak{S}_n} V$ as graded $\mathbb{C}\mathfrak{S}_n$-modules.

For $I_n = \langle e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})\rangle$, we have that $V_n = I_n^{\perp}$ is the smallest subspace of $\mathbb{C}[\mathbf{x}]$ that contains the *Vandermonde determinant* δ_n, defined by

$$\delta_n := \prod_{i<j}(x_j - x_i),$$

and is closed under partial derivatives $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$.

In this case, an alternative description of I_n is that it is the annhilator of δ_n:

$$.I_n = \{f \in \mathbb{C}[\mathbf{x}] : f(\partial) \odot \delta_n = 0\}. \tag{3.5}$$

Chapter 4

Catabolism and Catabolizability

4.1 Catabolism and catabolizability type

A central topic of this dissertation is *catabolism*, an operation on tableaux that comes from the cyclage poset. Originally introduced by Lascoux [38], this operation gives a way to describe the image of certain cyclage poset embeddings. In general, many things are unknown about catabolism and catabolizability, though the operation of catabolism itself is very easy to compute.

Definition 4.1.1. Let T be a standard tableau with $\mathrm{rw}(T) = ww'$ where w' is the first row of T. *Catabolism* is the operation that takes $T = P(ww')$ to $\mathrm{cat}(T) := P(w'w)$.

It is known that $\mathrm{cocharge}(\mathrm{cat}(T)) = \mathrm{cocharge}(T) - (|\lambda| - \lambda_1)$ where $\lambda = \mathrm{sh}(T)$. Repeated applications of catabolism eventually produce a one row tableau with zero cocharge.

We can see that applying catabolism to T is equivalent to sliding the first row of T to the right until it detaches from the higher rows, and then swapping the two pieces and applying jeu-de-taquin to the resulting skew shape until we get a partition shape tableau. This is illustrated in the example below.

Example 4.1.2. Consider the following tableau of shape $\lambda = (3, 2, 1)$:

$$T = \begin{array}{|c|cc} \hline 6 & & \\ \hline 2 & \multicolumn{1}{|c|}{5} & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}.$$

We can see that $\mathrm{rw}(T) = 625134$ where $w = 625$ and $w' = 134$. Observe that

$$\mathrm{cat}(T) = P(w'w) = P(134625) = \begin{array}{|c|c|cc} \hline 3 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array}.$$

Alternatively, we can use jeu-de-taquin to see

$$T = \begin{ytableau} 6 \\ 2 & 5 \\ 1 & 3 & 4 \end{ytableau} \to \begin{ytableau} 1 & 3 & 4 \\ \none & \none & \none & 6 \\ \none & \none & \none & 2 & 5 \end{ytableau} \xrightarrow[\text{jeu-de-taquin}]{} \begin{ytableau} 3 & 6 \\ 1 & 2 & 4 & 5 \end{ytableau} = \mathrm{cat}(T).$$

Note that $\mathrm{cc}(T) = 8$ and $\mathrm{cc}(\mathrm{cat}(T)) = 5 = 8 - (6 - 3)$, where $|\lambda| = 6, \lambda_1 = 3$.

We use catabolism to define catabolizability type.

Definition 4.1.3. For a standard tableau T, let $d(T)$ be the largest integer m such that the first row of T contains $1, 2, \ldots, m$. The *catabolizability type* of T (denoted $\mathrm{ctype}(T)$) is given by the sequence

$$\mathrm{ctype}(T) = (d(T), d(\mathrm{cat}(T)) - d(T), \ldots, d(\mathrm{cat}^i(T)) - d(\mathrm{cat}^{i-1}(T)), \ldots). \tag{4.1}$$

Example 4.1.4. Consider $T = \begin{ytableau} 6 & 8 \\ 4 & 5 \\ 1 & 2 & 3 & 7 \end{ytableau}$. Since $d(T) = 3$, we have $\mathrm{ctype}(T)_1 = 3$. The result of applying catabolism is

$$\mathrm{cat}(T) = P(12376845) = \begin{ytableau} 7 \\ 6 & 8 \\ 1 & 2 & 3 & 4 & 5 \end{ytableau}.$$

Since $d(\mathrm{cat}(T)) = 5$, we have $\mathrm{ctype}(T)_2 = d(\mathrm{cat}(T)) - d(T) = 2$. Repeating this process, we can compute $\mathrm{ctype}(T) = (3, 2, 1, 1, 1)$.

We can think of $\mathrm{ctype}(T)$ as encoding the sequence of catabolisms we use to go from T to the one row tableau.

Proposition 4.1.5 (Lascoux [38, Lemma 9.6])**.** *For standard tableau T, the sequence* $\mathrm{ctype}(T)$ *is a partition.*

An alternative way to view $\mathrm{ctype}(T)$ is using m-catabolizability (see [47] for details). Let (m) denote the one row shape of length m. If $(m) \subset \mathrm{sh}(T)$ and the first row of T contains the smallest m letters in T, we define the *m-catabolism* of T to be deleting the first m entries of the first row of T and then proceeding as we did for standard catabolism, where we denote the resulting tableau $\mathrm{cat}_m(T)$. We say that T of size n is *λ-catabolizable* for partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ if the first row of T contains the smallest λ_1 many entries of T and $\mathrm{cat}_{\lambda_1}(T)$ is $(\lambda_2, \ldots, \lambda_l)$-catabolizable. Catabolizability type of a tableau and λ-catabolizability are related in the following way.

Proposition 4.1.6 (Shimozono-Weyman [47, Proposition 46])**.** *A standard tableau T is λ-catabolizable if and only if* $\mathrm{ctype}(T) \trianglerighteq \lambda$.

From this, we can compute $\mathrm{ctype}(T)$ by repeatedly applying maximal m-catabolisms, where at each step we have

$$\mathrm{ctype}(T)_i = d(\mathrm{cat}_{\mathrm{ctype}(T)_{i-1}} \cdots \mathrm{cat}_{\mathrm{ctype}(T)_1}(T)).$$

This simplifies our process, since at each step we are dealing with smaller tableaux.

Example 4.1.7. Consider $T = \begin{array}{|c|c|c|c|} \cline{1-2} 6 & 8 \\ \cline{1-2} 4 & 5 \\ \hline 1 & 2 & 3 & 7 \\ \hline \end{array}$. We know $d(T) = 3$ and can compute $\mathrm{cat}_3(T)$:

$$\begin{array}{|c|c|c|c|} \cline{1-2} 6 & 8 \\ \cline{1-2} 4 & 5 \\ \hline 1 & 2 & 3 & 7 \\ \hline \end{array} \to \begin{array}{|c|cc} \cline{1-1} 7 & & \\ \cline{1-1} & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{8} \\ \cline{2-3} & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} \\ \cline{2-3} \end{array} \xrightarrow[\text{jeu-de-taquin}]{} \begin{array}{|c|c|} \cline{1-1} 7 \\ \hline 6 & 8 \\ \hline 4 & 5 \\ \hline \end{array} = \mathrm{cat}_3(T).$$

We see that $\mathrm{ctype}(T)_2 = d(\mathrm{cat}_3(T)) = 2$. Repeating this process, we can compute $\mathrm{ctype}(T) = (3, 2, 1, 1, 1)$, which is the same as what we get in Example 4.1.4.

We can use catabolizability type and dominance order on partitions to identify subsets of SYT_n.

Definition 4.1.8. For $\lambda \vdash n$, let $\Sigma_\lambda := \{S \in \mathrm{SYT}_n, \mathrm{ctype}(S) \trianglerighteq \lambda\}$.

Lascoux recognizes Σ_λ as the image of embedding the cyclage poset on $\mathrm{SSYT}(\lambda)$ into the cyclage poset of SYT_n.

Proposition 4.1.9 (Lascoux [38])**.** *For any $\lambda \vdash n$, we have a bijection* std : $\mathrm{SSYT}(\lambda) \to \Sigma_\lambda \subset \mathrm{SYT}_n$ *which preserves cocharge and shape.*

The bijection std is exactly the cyclage poset embedding $\theta_\lambda^{1^n} : \mathcal{T}(\lambda) \hookrightarrow \mathcal{T}(1^n)$ described in Section 2.7. We refer to this map as the *Lascoux standardization map* on semistandard tableaux.

Example 4.1.10. If $\lambda = (2, 1)$, we have $\Sigma_\lambda = \left\{ \begin{array}{|c|c|} \cline{1-1} 3 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \right\}$. We can see the explicit cocharge/shape bijection between Σ_λ and $\mathrm{SSYT}(\lambda) = \left\{ \begin{array}{|c|c|} \cline{1-1} 2 \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline \end{array} \right\}$.

Remark 4.1.11. Though the map std preserves shapes and the cocharge statistic, it does not preserve the specific cocharge labeling on the tableaux. For example, consider

$$S = \begin{array}{|c|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline \end{array} \qquad \mathrm{std}(S) = \begin{array}{|c|c|c|c|c|} \cline{1-1} 7 \\ \cline{1-2} 6 & 8 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$$

The corresponding cocharge tableaux are

$$\mathrm{cc}(S) = \begin{array}{|c|c|c|c|c|} \cline{1-1} 1 \\ \cline{1-2} 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} \qquad \mathrm{cc}(\mathrm{std}(S)) = \begin{array}{|c|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-2} 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

Using Proposition 4.1.9, we can rewrite the Lascoux–Schützenberger semistandard cocharge formula for the modified Hall-Littlewood polynomials $\widetilde{H}_\lambda[X;q]$ in terms of $\Sigma_\lambda \subset \mathrm{SYT}_n$.

$$\widetilde{H}_\mu[X;q] = \sum_{\substack{S\in\mathrm{SYT}_n \\ \mathrm{ctype}(S)\trianglerighteq\mu}} q^{\mathrm{cocharge}(S)} s_{\mathrm{sh}(S)}(X). \tag{4.2}$$

Remark 4.1.12. This presentation is nice when we think of the context of the coinvariant ring. In Chapter 5, we will discuss a quotient of R_n with graded Frobenius character given by $\widetilde{H}_\mu[X;q]$. The combinatorial formula (5.2) highlights that the corresponding representation is a subspace of R_n.

In general it is hard to check whether something is catabolizable without actually computing catabolism. However, for this notion of catabolism, there is a concrete algorithm, due to Blasiak [4], called the *catabolism insertion algorithm*, which computes the catabolizability type of a permutation w, where we define $\mathrm{ctype}(w) := \mathrm{ctype}(P(w))$.

Algorithm 4.1.13 (Blasiak [4, Algorithm 3.2])**.** We define a function f on pairs (x,ν), where $x = ya$ is a word (a is the last letter of x) and ν is a partition, to be

$$f(x,\nu) = \begin{cases} (y, \nu + \epsilon_{a+1}) & \text{if } \nu + \epsilon_{a+1} \text{ is a partition,} \\ ((a+1)y, \nu) & \text{otherwise.} \end{cases}$$

where ϵ_{a+1} is the composition $(0,\dots,0,1,0,\dots)$ with 1 in the $(a+1)$th coordinate.

Let w be a permutation of length n with cocharge word $\mathrm{cc}(w)$. We apply f to $(\mathrm{cc}(w), \varnothing)$ repeatedly until we get $(\varnothing, \lambda)$, where $\lambda \vdash n$.

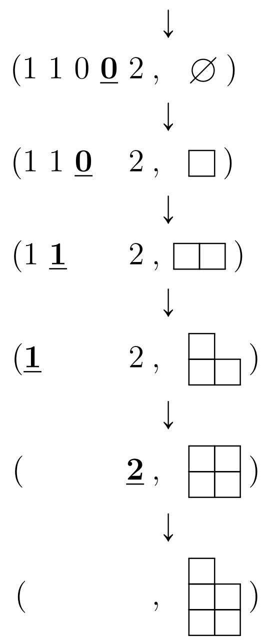
The resulting shape λ is $\mathrm{ctype}(w)$. Furthermore, we have that if $P(w) = P(w')$, then $\mathrm{ctype}(w) = \mathrm{ctype}(w') = \mathrm{ctype}(P(w))$ [4].

Example 4.1.14. Consider

$$w = 3\ 4\ 1\ 2\ 5$$
$$\mathrm{cc}(w) = 1\ 1\ 0\ 0\ 1.$$

We will apply f to $(\mathrm{cc}(w), \varnothing)$ repeatedly until we get an empty word in the first coordinate. In order to keep track of the indices, we do not rotate $\mathrm{cc}(w)$: instead, we read the word from right to left. The position we are reading at each step of the algorithm is underlined.

$$(1\ 1\ 0\ 0\ \underline{\mathbf{1}},\quad \varnothing)$$



From this, we conclude ctype(34125) = $(2, 2, 1)$.

Note that since the cyclage poset embeddings are defined using crystal operators, which are preserved under Knuth equivalence (Proposition 2.7.5), we can define standardization maps of semistandard tableaux of skew shapes as well.

Proposition 4.1.15. *For any skew shape μ/γ of size n and weight $\lambda \vdash n$, there exists a bijection*

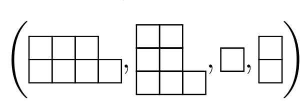
$$\mathrm{std} : \mathrm{SSYT}(\mu/\gamma, \lambda) \to \{S \in \mathrm{SYT}(\mu/\gamma), \mathrm{ctype}(\mathrm{rw}(S)) \trianglerighteq \lambda\}.$$

4.2 R-catabolism

Since Lascoux's original paper, various generalizations have been introduced [5, 13, 47] in connections with families of Schur-positive symmetric functions. Here we consider **R**-catabolizability, introduced by Shimozono–Weyman [47].

Definition 4.2.1. Let $\lambda \vdash n$ be a partition and $\eta = (\eta_1, \ldots, \eta_t)$ be a composition of size $\ell(\lambda)$. Then $R(\eta, \lambda) = \mathbf{R} = (R_1, R_2, \ldots, R_t)$ is a sequence of partitions where R_1 consists of the first η_1 parts of λ, R_2 consists of the next η_2 parts of λ, etc.

Example 4.2.2. If $\lambda = (4, 3, 3, 2, 2, 1, 1, 1)$ and $\eta = (2, 3, 1, 2)$, then we have

$$\mathbf{R} = (R_1, R_2, R_3, R_4) =$$  .

For any semistandard tableau S (potentially of skew shape) and index r, we define $H_r(S) = P(VU)$ where U, V are the north and south tableau obtained by slicing T between rows r and $(r+1)$. For any shape λ, let Z_λ be the unique semistandard tableau of shape and weight λ.

Definition 4.2.3. Consider $\lambda \vdash n$ and $\eta \models \ell(\lambda)$. We say $T \in \mathrm{SSYT}(\lambda)$ is R_1-*catabolizable* if $T|_{[\eta_1]} = Z_{R_1}$. In this case, the R_1-catabolism of T is defined to be $\mathrm{cat}_{R_1}(T) := H_{\eta_1}(T - Z_{R_1})$.

Example 4.2.4. Consider $\mathbf{R} = (\ydiagram{2,3}, \ydiagram{1,2}) = R(\lambda, \eta)$ for $\lambda = (3,2,2,1)$ and $\eta = (2,2)$.
Consider

$$T = \begin{ytableau} 3 \\ 2 & 2 \\ 1 & 1 & 1 & 3 & 4 \end{ytableau} \qquad \text{with} \qquad T|_{[2]} = \begin{ytableau} 2 & 2 \\ 1 & 1 & 1 \end{ytableau} = Z_{R_1}.$$

we have that T is R_1-catabolizable. Furthermore, we can compute

$$\mathrm{cat}_{R_1}(T) = H_2\left(\begin{ytableau} 3 & \none & \none \\ \none & 3 & 4 \end{ytableau} \right) = P\left(\begin{ytableau} 3 & 4 \\ \none & 3 \end{ytableau} \right) = \begin{ytableau} 4 \\ 3 & 3 \end{ytableau}.$$

$\mathbf{R}$-catabolizability is defined recursively using Definition 4.2.3.

Definition 4.2.5. Consider $\lambda \vdash n$, $\eta \models \ell(\lambda)$. For $T \in \mathrm{SSYT}(\lambda)$ and $\mathbf{R} = R(\eta, \lambda)$, the notion of $\mathbf{R}$-catabolizable is uniquely determined by the following rules:

1. If $\mathbf{R}$ is the empty sequence, the unique $\mathbf{R}$-catabolizable tableau is the empty tableau.

2. Otherwise, T is $\mathbf{R}$-catabolizable if and only if $T|_{[\eta_1]} = Z_{R_1}$ and $\mathrm{cat}_{R_1}(T)$ is $\widehat{\mathbf{R}} = (R_2, \ldots, R_t)$-catabolizable in the alphabet $[\eta_1 + 1, n]$.

Example 4.2.6. Let $\mathbf{R} = (\ydiagram{2,3}, \ydiagram{1,2})$. We can see that $T = \begin{ytableau} 3 \\ 2 & 2 \\ 1 & 1 & 1 & 3 & 4 \end{ytableau}$ is $\mathbf{R}$-catabolizable, since $\mathrm{cat}_{R_1}(T) = \begin{ytableau} 4 \\ 3 & 3 \end{ytableau}$ is $(2,1)$-catabolizable in the alphabet $[3,4]$.

Remark 4.2.7. Here are some observations regarding $\mathbf{R}$-catabolizability for $\mathrm{SSYT}(\lambda)$.

1. If $\eta = \ell(\lambda)$ (ie., η is a composition consisting of one part), then $\mathbf{R} = (\lambda)$. In this case, the only $\mathbf{R}$-catabolizable SSYT of weight λ is Z_λ.

2. If $\eta = 1^{\ell(\lambda)}$, then $\mathbf{R} = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ consists of the rows of λ. All SSYT of weight λ are $\mathbf{R}$-catabolizable.

3. For any partition μ, a standard tableau S is $\mathbf{R}$-catabolizable for $\mathbf{R} = R(\mu, 1^n)$ if and only if $\text{ctype}(S^t) \trianglerighteq \mu$, where S^t denotes the transpose of S.

Shimozono–Weyman defined $\mathbf{R}$-catabolizability to give a combinatorial expression of the q-analogue of Littlewood–Richardson coefficients $K_{\mu;\mathbf{R}}(q)$: that is, we have

$$K_{\mu;\mathbf{R}}(1) = \langle s_\mu(x), s_{R_1}(x)s_{R_2}(x)\cdots s_{R_t}(x)\rangle.$$

The polynomial $K_{\mu;R(\eta,\lambda)}$ is defined geometrically to be the Poincaré polynomial of the isotypic component indexed by μ of certain graded GL_n-modules supported in the closure of the nilpotent conjugacy class corresponding to $\text{sort}(\eta)$. These polynomials generalize the q-Kostka coefficients that appear in the Schur expansion of the transformed Hall-Littlewood polynomials given in (2.7).

For specific cases (i.e, when the sequence $\mathbf{R}$ is a sequence of *dominant rectangles*), these polynomials can be interpreted using Kirillov-Reshetikhin crystals and rigged configurations [36, 45].

In [47], the authors conjectured the following tableaux formula for $K_{\mu;\mathbf{R}}(q)$ and prove it in the case where η is a hook partition or is a two-row partition. The full conjecture was resolved in [5].

Conjecture 4.2.8 (Conjecture 27 [47])**.** Consider $\mathbf{R} = R(\eta, \lambda)$. Then

$$K_{\mu;\mathbf{R}}(q) = \sum_S q^{\text{charge}(S)}$$

where S runs over all semistandard tableaux $S \in \text{SSYT}(\lambda)$ of shape μ that are $\mathbf{R}$-catabolizable.

Remark 4.2.9. Here are some special cases of these polynomials:

1. As mentioned above, if $\mathbf{R} = R(\lambda, 1^{\ell(\lambda)})$, we have that $K_{\mu;\mathbf{R}}(q)$ is exactly the q-Kostka polynomial $K_{\mu,\lambda}(q)$ given in (2.7):

$$K_{\mu;\mathbf{R}}(q) = \sum_{T \in \text{SSYT}(\mu,\lambda)} q^{\text{cocharge}(T)}.$$

2. If $\mathbf{R} = R(1^n, \lambda)$, then $K_{\mu;\mathbf{R}}(q) = \widetilde{K}_{\mu',\lambda}(q)$ which is given by

$$\widetilde{K}_{\mu',\lambda}(q) = \sum_{\substack{S \in \text{SYT}(\mu'),\\ \text{ctype}(S) \trianglerighteq \lambda}} q^{\text{cocharge}(T)}.$$

Though it is not difficult to compute **R**-catabolisms, understanding its properties remains an elusive problem. For example, the following are some conjectures about **R**-catabolizability which seem apparent from examples yet are still open.

Conjecture 4.2.10. The following are true:

(a) (Refinement) For partition λ and compositions $\eta, \alpha \models \ell(\lambda)$ such that $\eta \preceq \alpha$, if T is $R(\alpha, \lambda)$-catabolizable then it must be $R(\eta, \lambda)$-catabolizable.

(b) (Swapping rectangles) If there exists R_i, R_{i+1} in $\mathbf{R} = R(\lambda, \eta)$ such that R_i and R_{i+1} are the same width, then T is **R**-catabolizable if and only if it is $R(\lambda, s_i\eta)$-catabolizable.

Remark 4.2.11. Though we have that [Proposition 11 [47]] provides evidence for Conjecture 4.2.10(b), since it proves that $K_{\mu,\mathbf{R}}(q) = K_{\mu,\mathbf{R}(\lambda,s_i\eta)}(q)$ as polynomials, this does not prove the equality on the level of tableaux.

4.3 Catalan functions

The Catalan functions, originally defined in [13, 43], extend the construction given in Shimozono–Weyman [47], as well as Broer [9].

Before we give an explicit definition of Catalan functions, we must provide the appropriate setup. For any $\gamma \in \mathbb{Z}^l$, we can define the Schur functions s_γ to be $s_\gamma := \det(h_{\gamma_i+j-i})$. This is the natural extension of the Jacobi-Trudi formula for s_λ.

Now, we can express $s_\gamma \in \Lambda$ in terms of the Schur functions s_λ indexed by partitions using the following straightening algorithm.

Proposition 4.3.1. *For any $\gamma \in \mathbb{Z}^l$ we have that*

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma+\rho) s_{\operatorname{sort}(\gamma+\rho)-\rho} & \text{if } (\gamma+\rho) \text{ has distinct parts} \\ 0 & \text{otherwise.} \end{cases}$$

where $\rho = (l-1, l-2, \ldots, 1, 0)$ and $\operatorname{sgn}(\beta)$ is the sign of the shortest permutation that takes β to $\operatorname{sort}(\beta)$.

Define Δ_l^+ to be the collection of positive roots:

$$\Delta_l^+ = \{(i,j) \mid 1 \leqslant i < j \leqslant l\}.$$

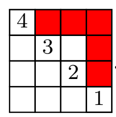
A *root ideal* is an upper order ideal of the poset on Δ_l^+ defined by $(a,b) \leqslant (c,d)$ iff $a \geqslant c$ and $b \leqslant d$. We can visualize Δ_l^+ as all the entries above the main diagonal in a $l \times l$ matrix.

Remark 4.3.2. Pictorially, we can describe a pair (Ψ, γ) in the following way. We let a $l \times l$ grid denote all the possible roots in Δ_l^+: we color the roots that are contained in Ψ, where the coordinate of the boxes are given in matrix coordinates (i.e, the top left corner is (1,1)). We put the entries of γ along the diagonal.

For example, for $\gamma = (4, 3, 2, 1)$, the root ideal

$$\Psi = \{(1,2), (1,3), (1,4), (2,4), (3,4)\} \subset \Delta_4^+$$

is denoted by the following diagram



.

Definition 4.3.3. Given $\gamma \in \mathbb{Z}^l$ and root ideal $\Psi \subseteq \Delta_l^+$, the *Catalan function* $H(\Psi;\gamma)(\mathbf{x};q)$ is defined to be

$$H(\Psi;\gamma)(\mathbf{x};q) := \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\gamma(\mathbf{x})$$

where $R_{ij} s_\gamma := s_{\gamma+\epsilon_i-\epsilon_j}$ where $\epsilon_i \in \mathbb{Z}^l$ is the indicator vector with 1 in the ith coordinate and 0s everywhere else.

Geometrically, they are the graded Euler characteristics of certain vector bundles on the flag variety. They generalize many different well-studied families of symmetric functions. In particular, we have the following.

Example 4.3.4. (i) For partition λ, if the root ideal Φ is empty we have that $H(\varnothing, \lambda) = s_\lambda$.

(ii) For partition λ and root ideal $\Psi = \Delta^+_{\ell(\lambda)}$, we have that

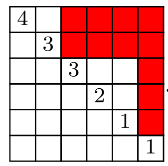
$$H(\Delta^+_{\ell(\lambda)};\lambda)(\mathbf{x};q) = \sum_{T\in \mathrm{SSYT}(\lambda)} q^{\mathrm{charge}(T)} s_{\mathrm{sh}(T)} = H_\lambda[X;q]$$

where $H_\lambda[X;q]$ is the *transformed Hall-Littlewood* polynomial.

(iii) Given $\lambda \vdash n$ and $\eta \models \ell(\lambda)$, the *parabolic root ideal* $\Delta(\eta)$ is defined to be

$$\Delta(\eta) := \{(i,j) \in \Delta^+_{\ell(\lambda)} \text{ above the block diagonal with block sizes } \eta_1, \eta_2, \ldots, \eta_{\ell(\lambda)}\}.$$

For example, if $\lambda = (4,3,3,2,1,1)$ and $\eta = (2,3,1)$, the corresponding root ideal looks like



.

We have that

$$H(\Delta(\eta);\lambda)(\mathbf{x};q) = \sum_{\mu \vdash n} K_{\mu;R(\eta,\lambda)}(q) \quad s_\mu.$$

where $K_{\lambda,\mu}(q)$ are the polynomials defined in Section 4.2.

(iv) For a partition μ satisfying $\mu_1 \leqslant k$ for some positive integer k, define

$$\Delta^k(\mu) := \{(i,j) \in \Delta^+_{\ell(\mu)} : k - \mu_i + i < j\}.$$

In this case we have that the Catalan function $H(\Delta^k(\mu);\mu)(\mathbf{x};q)$ is the *k-Schur function* $\mathfrak{s}^{(k)}_\mu$.

Though Catalan functions have been well-studied [6, 7], Schur-positivity remained an open problem for a decade. Work of Blasiak–Morse–Pun, which first extends the Catalan function framework by defining a nonsymmetric version, gives a combinatorial formula for the Schur expansion in the case where the Catalan function is indexed by a partition.

Theorem 4.3.5 (Blasiak–Morse–Pun [5])**.** *Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_\ell \geqslant 0) \vdash n$ and $\Phi \subseteq \Delta^+_l$ be a root ideal. We have that*

$$H(\Phi;\lambda)(\mathbf{x};q) = \sum_{\substack{T \in \mathrm{SSYT}_\ell(\lambda) \\ U \textit{ is } \mathbf{n}(\Phi)\textit{-catabolizable}}} q^{\mathrm{charge}(T)} s_{\mathrm{sh}(T)} \tag{4.3}$$

where $\mathrm{SSYT}_\ell(\mu)$ denotes the set of semistandard tableaux of weight μ with at most ℓ rows and $\mathbf{n}(\Phi)$-catabolism is a generalization of catabolism which depends on the pair $(\Phi;\lambda)$.

The combinatorial formula (4.3) is over a sum of semistandard tableaux that satisfy some catabolizability criteria in terms of $\mathbf{n}(\Phi)$-catabolism, a notion of catabolism introduced in [5]. We will not discuss this notion of catabolism in this thesis.

Proposition 4.3.6 (Blasiak–Morse–Pun [5])**.** *If $\Phi = \Delta(\eta)$ for some $\eta \models \ell(\lambda)$, we have that $T \in \mathrm{SSYT}(\lambda)$ is $\mathbf{n}(\Phi)$-catabolizable if and only if it is $\mathbf{R} = R(\eta,\lambda)$-catabolizable.*

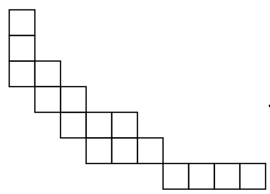
Thus, Theorem 4.3.5 and Proposition 4.3.6 imply Conjecture 4.2.8 of Shimozono–Weyman.

4.4 Skew-linked modules

A certain family of Catalan functions have a representation theoretic model called *skew-linked modules*, introduced by Chen [13].

We say two partitions $\lambda, \mu \vdash n$ are *skew-linked* (denoted $\lambda \xrightarrow{\theta} \mu$ if there exists a skew-shape θ with the same row lengths as λ and the same column lengths as μ.

Example 4.4.1. Consider $\lambda = (4, 3, 3, 2, 2, 1, 1)$ and $\mu = (10, 5, 1)$. Then $\lambda \xrightarrow{\theta} \mu$ for θ given by

$\theta =$  .

Remark 4.4.2. We have the following facts about skew-linked pairs.

(a) Every λ is skew-linked to itself.

(b) For any $\lambda \vdash n$, we have $\lambda \to (n)$ and $1^n \to \lambda$ (i.e. every partition is skew-linked to the one row partition and the column).

(c) If $\lambda \xrightarrow{\theta} \mu$ then $\mu \trianglerighteq \lambda$.

Now, we can construct a module for each pair of skew-linked partitions. Consider $\mathcal{E} \uparrow_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n} \otimes \mathbb{C}[\mathbf{x}_n]$, where $\mathcal{E}$ is the sign representation. Given a skew-linked pair $\lambda \to \mu$, let $d(\lambda, \mu)$ be the smallest degree d such that we have a nontrivial $\mathfrak{S}_n$-module homomorphism

$$\mathcal{E} \uparrow_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n} \otimes \mathbb{C}[\mathbf{x}_n]_d \to \mathbb{1} \uparrow_{\mathfrak{S}_\mu}^{\mathfrak{S}_n} .$$

We have that there exists a unique nonzero $\mathbb{C}[\mathbf{x}_n] * \mathfrak{S}_n$ homomorphism

$$\phi : \mathcal{E} \uparrow_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n} \otimes \mathbb{C}[\mathbf{x}_n]_d \to \mathbb{1} \uparrow_{\mathfrak{S}_\mu}^{\mathfrak{S}_n} \otimes \mathbb{C}[\mathbf{x}_n]^*[-d(\lambda, \mu)] \tag{4.4}$$

where $\mathbb{C}[\mathbf{x}_n]^*$ denotes the dual of $\mathbb{C}[\mathbf{x}_n]$ and $\mathbb{C}[\mathbf{x}_n] * \mathfrak{S}_n$ is the semidirect product of the two groups.

Definition 4.4.3. The skew-linked module $M_{\lambda,\mu}$ is the image of ϕ given in (4.4).

Theorem 4.4.4 (Chen [13]). *The following are properties of skew-linked modules.*

(a) Given a skew-linked pair $\lambda \to \mu$, let $\theta = \alpha/\beta$ be the corresponding skew shape. Then $d(\lambda, \mu) = |\beta|$.

(b) The degree 0 component of $M_{\lambda,\mu}$ is isomorphic to V_λ. The top degree component (degree $d(\lambda, \mu)$) component of $M_{\lambda,\mu}$ is isomorphic to V_μ.

(c) We have that $M_{\mu',\lambda'} \cong \mathrm{rev}(\mathcal{E} \otimes M_{\lambda,\mu})$, where rev *denotes that the grading is reversed.*

In particular, we have the following.

(a) $M_{\lambda,\lambda} \cong V^\lambda$.

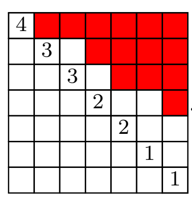
(b) $M_{1^n,\lambda'} \cong \mathcal{E} \otimes R_\lambda$, where R_λ is the Garsia–Procesi ring.

The skew-linked modules are conjectured to give a representation theoretic model for certain Catalan functions.

Definition 4.4.5. Given a skew-linked pair $\lambda \xrightarrow{\theta=\alpha/\beta} \mu$, define the root ideal $\Psi(\lambda \to \mu) \subseteq \Delta^+_{\ell(\lambda)}$ to be

$$\Psi(\lambda \to \mu) = \{(i, j) : i \leqslant \ell(\beta) \text{ and } j \geqslant \mu'_{\beta_i} + i\}. \tag{4.5}$$

Example 4.4.6. The skew-linked pair $\lambda \to \mu$ corresponds to the root ideal



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Conjecture 4.4.7 (Chen [13]). We have that $\mathrm{Frob}_q(M_{\lambda,\mu}) = H(\Psi(\lambda \to \mu); \lambda)(\mathbf{x}; q)$.

Recall from Example 4.3.4, we have that the k-Schur functions are a subfamily of the Catalan functions. In fact, we have that certain skew-linked pairs correspond to k-Schur functions.

We say a partition is *k-bounded* if the longest part is at most length k. Given a k-bounded partition λ, there is a unique skew diagram α/β such that α is a $(k+1)$-core (i.e, no boxes with hook length equal to $k+1$) and β consists of all boxes in α with hook length $< (k+1)$. The columns of this skew-diagram α/β give the k-conjugate λ^{ω_k}. In this case, the Catalan function $H(\Psi(\lambda \xrightarrow{\alpha/\beta} \lambda^{\omega_k}), \lambda)(\mathbf{x}; q)$ is exactly the k-Schur function $\mathfrak{s}^k_\lambda(\mathbf{x}; q)$.

The k-Schur functions were introduced over twenty years ago [37] to refine the Macdonald positivity conjecture. Recent work have showed that the modified Macdonald polynomials are indeed k-Schur positive. Having a representation theoretic model for the k-Schur functions would give insight into understanding the combinatorics of Macdonald positivity through representation theory.

Chapter 5

A cocharge monomial basis for the Garsia-Procesi Rings

In this chapter, we study the *Garsia-Procesi ring* R_μ, a quotient of the coinvariant ring R_n indexed by $\mu \vdash n$. We construct a basis of R_μ using the catabolizability type of standard Young tableaux and the cocharge statistic. This basis turns out to be equal to the descent basis defined in [11]. Our new construction connects the combinatorics of the basis with the well-known combinatorial formula for the modified Hall-Littlewood polynomials $\widetilde{H}_\mu[X;q]$, due to Lascoux. We also use this basis to give an elementary proof of the fact that the graded Frobenius character of R_μ is the modified Hall-Littlewood polynomial $\widetilde{H}_\mu[X;q]$.

Inspired by the connections between shuffles and catabolizability, we prove a result that gives a lower bound for the catabolizability type of a permutation constructed by shuffling cocharge words. The first six sections of this chapter is contained in [33].

5.1 Introduction

Certain quotients of R_n correspond to $\mathbb{C}\mathfrak{S}_n$-modules that come from Springer fibers. Given a partition μ of n, the Springer fiber $\mathcal{F}_\mu \subset \mathcal{F}_n$ is the set of flags that are fixed by a unipotent matrix with Jordan type μ. The cohomology ring $H^*(\mathcal{F}_\mu)$ of the Springer fiber has an $\mathfrak{S}_n$ action which is compatible with that on $H^*(\mathcal{F}_n)$ [48].

Though the definitions above can be extended to different Lie types, there are certain nice properties when we consider Type A. In particular, the map $H^*(\mathcal{F}_n) \to H^*(\mathcal{F}_\mu)$ is a surjection. From this, we have a concrete realization of $H^*(\mathcal{F}_\mu)$ as a quotient of R_n by an $\mathfrak{S}_n$ invariant ideal. The following presentation is due to

DeConcini–Procesi [17] and Tanisaki [51].

Let $p_k(\mu)$ be the number of boxes of the Young diagram of μ that are not in the first k columns. Given a subset $S \subset \mathbf{x}$, we define $e_d(S)$ to be the dth elementary symmetric function in the set of variables S. The *Tanisaki ideal* I_μ is defined to be

$$I_\mu = \langle e_d(S) \mid S \subset \mathbf{x}, d > |S| - p_{n-|S|}(\mu) \rangle.$$

When $\mu = 1^n$, we have $I_{1^n} = \langle e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}) \rangle$, hence the Garsia-Procesi ring indexed by 1^n is the full coinvariant ring R_n. Using this ideal, we have that $H^*(\mathcal{F}_\mu) \cong R_\mu := \mathbb{C}[\mathbf{x}]/I_\mu$.

Example 5.1.1. Consider $\mu = (2,1,1)$. Then we have that $p_1(\mu) = 1, p_2(\mu) = p_3(\mu) = p_4(\mu) = 0$. From this, we have

$$I_\mu = \langle e_3(x_1,x_2,x_3), e_3(x_1,x_2,x_4), e_3(x_1,x_3,x_4), e_3(x_2,x_3,x_4), e_1(\mathbf{x}), e_2(\mathbf{x}), e_3(\mathbf{x}), e_4(\mathbf{x}) \rangle$$

Garsia and Procesi [20] constructed a monomial basis of R_μ that is a subset of the Artin monomials. In light of their work, we refer to R_μ as the *Garsia-Procesi ring* indexed by μ. The question of finding a subset of the Garsia-Stanton descent monomials that is a monomial basis of R_μ was unresolved until recently, when Carlsson and Chou gave a construction using shuffles of descent compositions [11].

The graded Frobenius character of the Garsia-Procesi ring R_μ is given by the modified Hall-Littlewood polynomials $\widetilde{H}_\mu[X;q]$ [34]. These polynomials have a combinatorial formula due to Lascoux [38] and Butler [10]:

$$\widetilde{H}_\mu[X;q] = \sum_{T \in \mathrm{SSYT}(\mu)} q^{\mathrm{cocharge}(T)} s_{\mathrm{sh}(T)}(X) \tag{5.1}$$

$$= \sum_{\substack{S \in \mathrm{SYT}_n \\ \mathrm{ctype}(S) \trianglerighteq \mu}} q^{\mathrm{cocharge}(S)} s_{\mathrm{sh}(S)}(X). \tag{5.2}$$

The first formula is the we–known combinatorial formula for the modified Hall-Littlewood polynomials. The second is an equivalent statement which follows from Proposition 4.1.9.

Example 5.1.2. Looking at Figure 2.2, we can see that

$$\widetilde{H}_{2,1,1}[X;q] = s_4 + (q+q^2)s_{3,1} + q^2\ s_{2,2} + q^3\ s_{2,1,1}.$$

In this paper, we give a monomial basis of R_μ, where the monomials are indexed by standard tableaux that satisfy a catabolizability type condition. The set we construct is the natural subset of the descent basis of R_n to consider in light of the combinatorial formula (5.2) for the graded Frobenius series.

For any permutation w, we associate a word $\mathrm{cc}(w)$ of the same length called the *cocharge word* of w such that the sum of the entries of $\mathrm{cc}(w)$ is $\mathrm{cocharge}(w)$. We refer to the monomial $\mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w))}$, where the exponents are given by the charge word of w, as the *cocharge monomial* corresponding to w.

Theorem 5.A. *Let μ be a partition of n. The set of cocharge monomials*

$$\{\mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w))} \mid w \in \mathfrak{S}_n,\ \mathrm{ctype}(w) \trianglerighteq \mu\}$$

is a monomial basis of R_μ. In fact, it coincides with the basis given by Carlsson–Chou [11].

Our basis consists of cocharge monomials corresponding to certain permutations in $\mathfrak{S}_n$. We later see that the set of cocharge monomials for all permutations in $\mathfrak{S}_n$ is equal to the Garsia-Stanton descent basis of R_n. Thus our set of monomials is indeed a subset of the descent basis.

Although our basis is equal to that of Carlsson–Chou, our construction of it is different and provides the first direct connection between the structure of the Garsia-Procesi ring and the catabolizability formula for the modified Hall-Littlewood polynomial that gives its graded Frobenius character. It is immediate that our basis is of the correct size in each dimension, a fact that was nontrivial from the construction of Carlsson–Chou. The construction of our basis is clearly compatible with the Hilbert series of R_μ that we compute from (5.2):

$$\mathrm{Hilb}_q(R_\mu) = \sum_{\substack{T \in \mathrm{SYT}_n \\ \mathrm{ctype}(T) \trianglerighteq \mu}} q^{\mathrm{cocharge}(T)} |\,\mathrm{SYT}(\mathrm{sh}(T))| \tag{5.3}$$

where $\mathrm{SYT}(\lambda)$ denotes the set of standard Young tableaux of shape λ.

Furthermore, the relation between the cocharge monomial construction and the construction of Carlsson–Chou gives insight to how catabolizability type behaves under shuffling cocharge words. In particular, we can show that there is a lower bound on the catabolizability type of the permutation we get from the shuffles.

Theorem 5.B. *Let $u^{(1)}, u^{(2)}, \ldots, u^{(l)}, w$ be permutations such that $\mathrm{cc}(w)$ is a shuffle of $\mathrm{cc}(u^{(1)}), \ldots, \mathrm{cc}(u^{(l)})$. We have*

$$\mathrm{ctype}(w) \trianglerighteq \mathrm{ctype}(u^{(1)}) + \mathrm{ctype}(u^{(2)}) + \cdots + \mathrm{ctype}(u^{(l)}),$$

where $\mathrm{ctype}(u^{(1)}) + \mathrm{ctype}(u^{(2)}) + \cdots + \mathrm{ctype}(u^{(l)})$ is the partition given by the partwise sum of $\mathrm{ctype}(u^{(1)}), \mathrm{ctype}(u^{(2)}), \ldots, \mathrm{ctype}(u^{(l)})$.

Though it is easy to identify the ungraded Frobenius character of R_μ directly from its structure, as exhibited in Garsia–Procesi [20], it is challenging to do the same for the graded character. This often requires heavy geometric machinery ([17]) or further combinatorial properties of the ring and character ([20]). However, our basis construction gives a way to easily identify the graded Frobenius character of R_μ as the modified Hall-Littlewood polynomial $\widetilde{H}_\mu[X;q]$ that only depends on (5.2) and the ungraded Frobenius character. We show that $\mathrm{Hilb}_q(N_\gamma R_\mu)$, where $N_\gamma = \sum_{\sigma\in\mathfrak{S}_\gamma} \mathrm{sgn}(\sigma)\sigma$ is the antisymmetrizer with respect to the Young subgroup $\mathfrak{S}_\gamma$, is easily determined using our basis construction and the ungraded Frobenius character of R_μ, by proving that the natural subset of our basis enumerated by $\langle e_\gamma, \mathrm{Frob}_q(R_\mu)\rangle$ has the property that applying N_γ yields a basis for $N_\gamma R_\mu$.

Proposition 5.C. *Let μ be a partition of n and $\gamma = (\gamma_1, \dots, \gamma_l)$ be a composition of n. The set*

$$\{N_\gamma \mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w))} \mid w \in \mathfrak{S}_n,\ \mathrm{ctype}(w) \trianglerighteq \mu,\ \mathrm{Asc}(w) \subset \{\gamma_1, \gamma_1+\gamma_2, \dots, \gamma_1+\cdots+\gamma_{l-1}\}\}$$

is a basis of $N_\gamma R_\mu$, where $\mathrm{Asc}(w) := \{i : w_i < w_{i+1}\}$.

The following corollary is immediate from Proposition 5.C and the catabolizability formula (5.2).

Corollary 5.D. *For any partition μ of n we have* $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X;q]$.

This chapter is organized as follows. In Section 5.2, we recall construction of the descent monomial basis of R_μ due to Carlsson–Chou [11]. In Section 5.3 give the construction of our set and prove that it agrees with the Carlsson–Chou basis, which shows that our set is indeed a basis due to their results. In particular, we show that our set of monomials is contained in their basis and then conclude they are the same set since they have the same cardinality. In Section 5.4, we introduce an algorithm that can be used to prove the other direction of containment, as well as other results about catabolizability types. In Section 5.6, we use a result from the previous section to show that this basis gives us a simple description of the Hilbert series of $N_\gamma R_\mu$. From this, we show $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X;q]$.

5.2 Carlsson–Chou construction of the descent basis of R_μ

Recall that the Artin basis $\{f_w(\mathbf{x}) : w \in \mathfrak{S}_n\}$ and the descent basis $\{g_w(\mathbf{x}) : w \in \mathfrak{S}_n\}$ of the coinvariant ring R_n both give give a combinatorial explanation of $\mathrm{Hilb}_q(R_n) =$

$[n]_q!$:

$$\sum_{w \in \mathfrak{S}_n} q^{\deg(f_w(\mathbf{x}))} = \sum_{w \in \mathfrak{S}_n} q^{\deg(g_w(\mathbf{x}))} = [n]_q! = \mathrm{Hilb}_q(R_n) \tag{5.4}$$

Since R_μ is a quotient of R_n, the natural question to ask if whether there exists a subset of the Artin monomials or the descent monomials which is a monomial basis of R_μ. Garsia-Procesi [20] constructed a monomial basis of R_μ which is a subset of the Artin basis of R_n. However, this basis does not give an analogous combinatorial explanation of the expression

$$\mathrm{Hilb}_q(R_\mu) = \sum_{\substack{T \in \mathrm{SYT} \\ \mathrm{ctype}(P(w)) \trianglerighteq \mu}} q^{\mathrm{cocharge}(w)}. \tag{5.5}$$

The problem of finding a subset of the descent basis that gives a monomial basis of R_μ was recently solved by Carlsson–Chou [11]. For two words $z^{(1)}, z^{(2)}$ of length l_1, l_2 we define $\mathrm{Sh}(z^{(1)}, z^{(2)})$ to be the collection of all words u of length $(l_1 + l_2)$ such that $z^{(1)}$ and $z^{(2)}$ are two disjoint subwords of u. We construct such u by interleaving $z^{(1)}$ and $z^{(2)}$. An element u of $\mathrm{Sh}(z^{(1)}, z^{(2)})$ is a *shuffle* of $z^{(1)}$ and $z^{(2)}$. We can extend this definition and let $\mathrm{Sh}(z^{(1)}, \ldots, z^{(l)})$ denote the set of all shuffles of $z^{(1)}, \ldots, z^{(l)}$. We also define the set of *reverse shuffles* of $z^{(1)}, \ldots, z^{(l)}$, denoted $\mathrm{Sh}'(z^{(1)}, \ldots, z^{(l)})$, to be equal to $\mathrm{Sh}(\mathrm{rev}(z^{(1)}), \ldots, \mathrm{rev}(z^{(l)}))$.

The Carlsson–Chou descent basis of the Garsia-Procesi ring is defined as follows. For $\mu \vdash n$, define $\mathcal{D}_\mu$ to be

$$\mathcal{D}_\mu = \bigcup_{(z^{(1)}, \ldots, z^{(l)})} \mathrm{Sh}(z^{(1)}, \ldots, z^{(l)}) \tag{5.6}$$

where $l = \mu_1$ and $(z^{(1)}, \ldots, z^{(l)})$ ranges over all l-tuples in $D_{\mu'_1} \times \cdots \times D_{\mu'_l}$

Remark 5.2.1. Note that this is different from the original notation used in [11], up to transposing the partitions.

Alternatively, they have a presentation in terms of ordered set partitions. Define $\mathcal{OSP}_\lambda$ to be the collection of ordered set partitions $(A_1 | \ldots | A_{\ell(\lambda)})$ of $[n]$ where $|A_i| = \lambda_i$. The set $\mathcal{D}_\mu$ can be described using $\mathcal{OSP}_{\mu'}$

$$\mathcal{D}_\mu := \{\mathbf{a} : \mathbf{a}|_{A_i} \in \mathcal{D}_{\mu'_i} \text{ for } (A_1 | \ldots | A_{\mu_1}) \in \mathcal{OSP}_{\mu'}\}. \tag{5.7}$$

Note that we take $\mathcal{OSP}_{\mu'}$, so the sizes of the parts correspond to the lengths of the columns.

Theorem 5.2.2 (Carlsson–Chou [11]). *The set* $\mathbf{x}^{\mathcal{D}_\mu} = \{\mathbf{x}^\alpha \mid \alpha \in \mathcal{D}_\mu\}$ *is a monomial basis of* R_μ*. Furthermore, they are the non-leading terms with respect to the* des*-order on monomials.*

Example 5.2.3. Consider $\mu = (2,1,1)$. Note that μ consists of two columns, one of length 3 and one of length 1. We have that

$$D_3 = \{012, 011, 101, 001, 010, 000\},$$
$$D_1 = \{0\}.$$

Then we have

$$\mathcal{D}_{2,1,1} = \{0012,\ 0102,\ 0120,\ 0011,\ 0101,\ 1001,\ 1010,\ 0110,\ 0001,\ 0010,\ 0100,\ 0000\}.$$

The corresponding set of monomials is a basis of $R_{(2,1,1)}$:

$$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2,\ x_3x_4,\ x_2x_4,\ x_1x_4,\ x_1x_3,\ x_2x_3,\ x_4,\ x_3,\ x_2,\ 1\},$$

Remark 5.2.4. Carlsson–Chou show that their basis is a subset of the Garsia-Stanton descent basis. However, it is not obvious how to see directly for which $w \in \mathfrak{S}_n$ we have $g_w(\mathbf{x}) \in \mathbf{x}^{\mathcal{D}_\mu}$ without computing $\mathcal{D}_\mu$. It is also not obvious that $|\mathcal{D}_\mu| = \dim(R_\mu)$, since multiple shuffles can correspond to the same descent word. For example, $\mathbf{0}101 \in \mathrm{Sh}(101, \mathbf{0})$ and $01\mathbf{0}1 \in \mathrm{Sh}(011, \mathbf{0})$ both correspond to the same element in $\mathcal{D}_\mu$.

5.3 Construction of cocharge monomial basis of R_μ

We can reformulate the descent monomials to be in terms of cocharge words.

Lemma 5.3.1. *We have* $D_n = \{\mathrm{rev}(\mathrm{cc}(w)) \mid w \in \mathfrak{S}_n\}$.

Proof. Note that the proof in [33] does this in terms of charge words. Here, we show it without using the language of charge. Note that for any $w \in \mathfrak{S}_n$ we can see that

$$\mathbf{x}^{\mathrm{rev}(cc(w))} = \prod_{j:(w^{-1})_{j+1} < (w^{-1})_j} x_{n-(w^{-1})_{j+1}+1} \cdots x_{n-(w^{-1})_n+1}.$$

Note that the condition $(w^{-1})_{j+1} < (w^{-1})_j$ is exactly what it means for $j+1$ to be on the left of j in w.

We can also see that given some $\pi \in \mathfrak{S}_n$, we have that

$$g_{\mathrm{rev}(\pi^{-1})}(\mathbf{x}) = \prod_{\pi_j^{-1} < \pi_{j+1}^{-1}} x_{\pi_{j+1}^{-1}} \dots x_{\pi_n^{-1}}.$$

Thus, if we set $w = \mathrm{rev}(\pi)$, we have

$$g_{\mathrm{rev}((\mathrm{rev}(w))^{-1})}(\mathbf{x}) = \prod_{w_j^{-1} > \pi_{j+1}^{-1}} w_{n-w_{j+1}^{-1}+1} \dots x_{n-w_n^{-1}+1}.$$

Thus $\mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w))} = g_\tau(\mathbf{x})$ when $\tau = \mathrm{rev}(\mathrm{rev}(w)^{-1})$. □

Using this, we can describe the descent basis of R_n in the following way using reverse cocharge words:

$$\{\mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w))} : w \in \mathfrak{S}_n\}.$$

We refer to these monomials as *cocharge monomials*. We now define sets of cocharge monomials that will give monomial bases of Garsia-Procesi rings. Our construction involves cocharge words of permutations whose insertion tableaux satisfy a catabolizability condition.

Definition 5.3.2. The set $\mathcal{C}_\mu$ is defined to be

$$\mathcal{C}_\mu := \{\mathrm{rev}(\mathrm{cc}(w)) \mid w \in \mathfrak{S}_n,\ \mathrm{ctype}(w) \trianglerighteq \mu\}. \tag{5.8}$$

From Lemma 5.3.1, we know $\mathcal{C}_\mu \subset D_n$. The main theorem of this chapter is that this subset of descent monomials gives a basis of R_μ.

Theorem 5.A. *The set*

$$\mathbf{x}^{\mathcal{C}_\mu} = \{\mathbf{x}^\alpha \mid \alpha \in \mathcal{C}_\mu\} \tag{5.9}$$

is a monomial basis of R_μ. In fact, it coincides with the basis $\mathbf{x}^{\mathcal{D}_\mu}$ given by Carlsson–Chou [11], ie., $\mathcal{C}_\mu = \mathcal{D}_\mu$.

Before we prove Theorem 5.A, we point out the connections between this basis and our known properties of R_μ. Comparing with the combinatorial formula (5.5) for $\mathrm{Hilb}_q(R_\mu)$, we can see that the classification of insertion tableaux that appears in (5.8) is the natural one to consider when looking at subsets of descent monomials.

In fact, it is evident from our construction that the degrees of the monomials in our set match what we expect from $\mathrm{Hilb}_q(R_\mu)$. In particular, it is apparent that $|\mathcal{C}_\mu| = \dim(R_\mu)$, which was not obvious from the definition of $\mathcal{D}_\mu$. Furthermore, it is clear that our set has the correct number of monomials of each degree to be a basis of R_μ by construction. That is, for any nonnegative d, we have

$$|\{w \in \mathfrak{S}_n, \mathrm{ctype}(w) \trianglerighteq \mu, \mathrm{cocharge}(w) = d\}| = \dim((R_\mu)_d),$$

where $(R_\mu)_d$ is the degree d component of R_μ. This construction also gives us a direct way to determine for which $w \in \mathfrak{S}_n$ we have $g_w(\mathbf{x}) \in \mathbf{x}^{\mathcal{C}_\mu}$.In particular, we have that $g_\tau(\mathbf{x}) \in \mathbf{x}^{\mathcal{C}_\mu}$ for $\tau = \mathrm{rev}(\mathrm{rev}(w)^{-1})$ and $\mathrm{ctype}(w) \trianglerighteq \mu$. There is a way to interpret this using the recording tableau of τ, along with evacuation on tableaux .

Example 5.3.3. Consider $\mu = (2,1,1)$. There are five standard tableaux S such that $\mathrm{ctype}(S) \trianglerighteq (2,1,1) = \mu$. We list them, along with all words w such that $P(w) = S$ and their cocharge monomials:

S	$\{w \mid P(w) = S\}$	$\{x^{\mathrm{rev}(\mathrm{cc}(w))} \mid P(w) = S\}$
$\begin{array}{ll} 4 & \\ 3 & \\ 1 & 2 \end{array}$	$\{4312,\ 4132,\ 1432\}$	$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2\}$
$\begin{array}{ll} 3 & 4 \\ 1 & 2 \end{array}$	$\{3412\ ,\ 3142\}$	$\{x_3x_4,\ x_2x_4\}$
$\begin{array}{lll} 3 & & \\ 1 & 2 & 4 \end{array}$	$\{3124,\ 1324,\ 1342\}$	$\{x_1x_4,\ x_1x_3,\ x_2x_3\}$
$\begin{array}{lll} 4 & & \\ 1 & 2 & 3 \end{array}$	$\{4123,\ 1423\ ,\ 1243\}$	$\{x_4,\ x_3,\ x_2\}$
$\begin{array}{llll} 1 & 2 & 3 & 4 \end{array}$	$\{1234\}$	$\{1\}$

Note that all cocharge monomials for words with the same insertion tableau S have degree equal to $\mathrm{cocharge}(S)$. The resulting set of cocharge monomials is

$$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2,\ x_3x_4,\ x_2x_4,\ x_1x_4,\ x_1x_3,\ x_2x_3,\ x_4,\ x_3,\ x_2,\ 1\},$$

which is the same as in Example 5.2.3.

We prove Theorem 5.A by showing $\mathcal{C}_\mu = \mathcal{D}_\mu$, which implies that our set is a basis of R_μ by the work of Carlsson–Chou. Since we know that $\mathcal{D}_\mu$ is a basis of R_μ and that $\mathcal{C}_\mu$ has the correct cardinality to be a basis, it suffices to show that $\mathcal{C}_\mu \subset \mathcal{D}_\mu$.

First, we prove some nice properties about cocharge words that we will use. Using the algorithm to construct $\mathrm{cc}(w)$ described in Section 2.6, we can find the following criteria for a word $\mathbf{z}$ to be a valid cocharge word of a permutation.

Lemma 5.3.4. *Let* $\mathbf{z}$ *be a word of nonnegative integers of length* n *containing a 0. We have* $\mathbf{z} = \mathrm{cc}(w)$ *for some* $w \in \mathfrak{S}_n$ *if and only if for any nonnegative* c*, one of the following holds:*

(a) *there exists a* $(c+1)$ *to the left of the rightmost* c *in* $\mathbf{z}$.

(b) c *is the largest letter in* $\mathbf{z}$.

Proof. It is clear that any cocharge word $\mathrm{cc}(w)$ satisfies the condition above, since the cocharge goes up whenever we encounter an $(i+1)$ in w such that $(i+1)$ to the left of i.

We can recover the permutation w from $\mathbf{z}$ by reversing the process. That is: we can construct an ordered set partition $(A_0|\dots|A_k)$ of n, where $\mathbf{z}|_{A_i} = \underbrace{i\dots i}_{|A_i| \text{ many}}$.

Define w so that $w|_{A_i} = (a_i+1)\dots a_j$ where $a_i = \sum_{j=0}^{i} |A_j|$. The resulting word w is a permutation by construction. The fact that $\mathrm{cc}(w) = \mathbf{z}$ follows from the fact that the cocharge goes up when $a+1$ appears to the left of a: from (a), we know that (a_j+1) always appears to the left of a_j, hence the cocharge value will go up. □

Remark 5.3.5. Note that we can make the analogous statement for cocharge fillings corresponding to standard tableaux.

We can modify cocharge words, by inserting or removing certain letters, without changing the fact that it is a cocharge word.

Lemma 5.3.6. *Let* $\mathbf{z} = \mathrm{cc}(w)$ *for* $w \in \mathfrak{S}_n$.

(a) *For a nonnegative integer* d *such that* $d = z_i$ *for some* i *and any word* $\mathbf{u}$ *of length* $n+1$ *such that* $\mathbf{u}_j = d$, $\mathbf{u}|_{[n+1]-\{j\}} = \mathbf{z}$, *we have* $\mathbf{u} = \mathrm{cc}(\tilde{w})$ *for some* $\tilde{w} \in \mathfrak{S}_{n+1}$.

(b) *For index* m *such that* z_m *is the rightmost, largest entry in* $\mathbf{z}$, *we have* $\mathrm{rev}(\mathbf{z}|_{[n]-\{m\}}) \in \mathcal{D}_{n-1}$. *In particular, we have* $\mathbf{z}|_{[n]-\{m\}} = \mathrm{cc}(w|_{n-1})$.

Proof. (a) Note that such $\mathbf{u}$ is equivalent to inserting d into $\mathbf{z}$. Any word we obtain in this way still satisfies the conditions in Lemma 6.4 since $\mathbf{z}$ does.

(b) Since we label the entries by reading from left to right, we know that the rightmost, largest entry in $\mathbf{z}$ must be the last entry in w that we label. Thus $w_m = n$. Thus deleting z_m from $\mathbf{z}$ is equivalent to labeling all but the largest entry in w.

□

As a slight modification to Algorithm 4.1.13, we record the entries of $\mathrm{cc}(w)$ that correspond to the boxes in λ as we build the partition. When reading i results in adding a box to the partition ν, we fill the new box with i. This gives us a standard filling (but not a SYT) of shape ctype(w), meaning $\{1, \ldots, n\}$ appear exactly once. We denote this filling as T_w. For the modified algorithm, the final tuple is $(\varnothing, T_w)$.

Example 5.3.7. Let $w = 634125$. Then we have that $T_w = \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 3 & 2 \\ \hline 5 & 4 \\ \hline \end{array}$ and ctype$(w) = (2, 2, 2)$

We also note some properties of Algorithm 4.1.13.

Lemma 5.3.8. *If we add i to row r of T_w when reading i for the kth time, we have*

$$k + cc(w)_i = r.$$

Proof. Each time we read i without adding it to T_w, we increase the ith letter of the input word by 1. After reading through the word $(k-1)$ times without adding i to the filling, the ith letter in the input word is $\mathrm{cc}(w)_i + (k-1)$. Since we add i the next time we read the ith letter, we know we add it to the $(\mathrm{cc}(w)_i + (k-1) + 1)$th row, where the last 1 is coming from the fact that $f(x, \nu) = (y, \nu + \epsilon_{a+1})$. □

We look at subwords of $\mathrm{cc}(w)$ determined by the columns of T_w, the filling of shape $\mu = \mathrm{ctype}(P(w))$ we get by applying Algorithm 4.1.13 to w. For $1 \leqslant j \leqslant \mu_1$ and $1 \leqslant r \leqslant \mu'_j$, let

$$\{i \mid i \text{ is in row } r',\ \text{column } j \text{ of } T_w,\ 1 \leqslant r' \leqslant r\} = \{j_1^{(r)} < j_2^{(r)} < \cdots < j_r^{(r)}\}.$$

We are numbering the entries in column j of T_w in increasing order, though that order may not match the order in which they appear in the column.

We define the subword $\mathrm{cc}(w)^{(j,r)}$ of $\mathrm{cc}(w)$ to be

$$\mathrm{cc}(w)^{(j,r)} = \mathrm{cc}(w)_{j_1^{(r)}}\, \mathrm{cc}(w)_{j_2^{(r)}} \ldots \mathrm{cc}(w)_{j_r^{(r)}}.$$

In this case, $\mathrm{cc}(w)^{(j,r)}$ is the subword corresponding to the first r rows of column j in T_w. We write $\mathrm{cc}(w)^{(j)} = \mathrm{cc}(w)^{(j,\mu_j)}$ when r is equal to the height of column j.

Example 5.3.9. Using $w = 634125$ and $\mathrm{cc}(w) = 211001$ from Example 5.3.7, we can see that

$$\mathrm{cc}(w)^{(1,3)} = \mathrm{cc}(w)_1\, \mathrm{cc}(w)_3\, \mathrm{cc}(w)_5 = 210.$$

We can also see that $j_1^{(3)} = 1$ but 1 is not in the first row of T_w.

Proposition 5.3.10. *For $w \in \mathfrak{S}_n$ such that* $\mathrm{ctype}(w) = \mu$, *consider j, r such that $1 \leqslant j \leqslant \mu_1$ and $1 \leqslant r \leqslant \mu'_j$. Then* $\mathrm{cc}(w)^{(j,r)} = \mathrm{cc}(\sigma)$ *for some $\sigma \in S_r$.*

Proof. We show this by induction on r. When $r = 1$, we know $\mathrm{cc}(w)^{(j,1)} = 0$, since for any i that appears in the first row of T_w we must have $\mathrm{cc}(w)_i = 0$. Hence the claim holds.

Assume the claim holds for $(r-1)$. We know that we obtain $\mathrm{cc}(w)^{(j,r)}$ from $\mathrm{cc}(w)^{(j,r-1)}$ by adding $\mathrm{cc}(w)_a$ into $\mathrm{cc}(w)^{(j,r-1)}$ in the correct position, where a is the entry in row r, column j of T_w. Let b denote the entry directly below a in T_w. Since b appears below a, we know that when we constructed T_w using Algorithm 4.1.13 to $\mathrm{cc}(w)$ we first added b, and then a, to the filling. If we let k_a, k_b denote the number of times we read a, b before adding them to T_w, we must have $k_b \leqslant k_a$. If $k_a = k_b$, then since we add b before a, we must have $a < b$, since $\mathrm{cc}(w)_b$ must be to the right of $\mathrm{cc}(w)_a$ in $\mathrm{cc}(w)$. We also have $\mathrm{cc}(w)_a = \mathrm{cc}(w)_b + 1$ from Lemma 5.3.8 since a is one row above b. If $k_b < k_a$, then $\mathrm{cc}(w)_b \geqslant \mathrm{cc}(w)_a$.

From this, we know that if $\mathrm{cc}(w)_a$ is larger than every letter in $\mathrm{cc}(w)^{(j,r-1)}$, we have $\mathrm{cc}(w)_a = \mathrm{cc}(w)_b + 1$ where $b > a$ is the entry directly below a in T_w and $\mathrm{cc}(w)_b$ is the largest letter in $\mathrm{cc}(w)^{(j,r-1)}$.

By Lemma 5.3.6, we conclude $\mathrm{cc}(w)^{(j,r)}$ is a cocharge word by induction. □

Remark 5.3.11. The fact that $\mathrm{cc}(w)^{(j,r)}$ is a cocharge word is not obvious a priori, since it is not true that any subword of a cocharge word is a cocharge word. For example, consider $w = 21$, which has $\mathrm{cc}(w) = 10$. Though 1 is a subword of $\mathrm{cc}(w)$, it is not a cocharge word.

Now we show that $\mathcal{C}_\mu \subset \mathcal{D}_\mu$. We first consider the case where $\mathrm{ctype}(w) = \mu$.

Proposition 5.3.12. *Let μ be a partition of n of length l and w be a permutation of n with $\mathrm{ctype}(w) = \mu$. Then $\mathrm{cc}(w) \in \mathrm{Sh}(\mathrm{rev}(z^{(1)}), \ldots, \mathrm{rev}(z^{(l)}))$ for some $(z^{(1)}, \ldots, z^{(l)}) \in D_{\mu'_1} \times \cdots \times D_{\mu'_l}$.*

Proof. We obtain a filling T_w of shape $\mathrm{ctype}(w) = \mu$ by applying Algorithm 4.1.13 to $\mathrm{cc}(w)$. It is clear that $\mathrm{cc}(w) \in \mathrm{Sh}(\mathrm{cc}(w)^{(1)}, \mathrm{cc}(w)^{(2)}, \ldots, \mathrm{cc}(w)^{(l)})$, where $\mathrm{cc}(w)^{(j)}$ is the subword of $\mathrm{cc}(w)$ corresponding to column j in T_w. We know $\mathrm{cc}(w)^{(j)}$ has length μ_j, which is the height of column j. Thus, from Proposition 5.3.10, we know $\mathrm{cc}(w)^{(j)} = \mathrm{cc}(\sigma)$ for $\sigma \in S_{\mu'_j}$, where $\mathrm{rev}(\mathrm{cc}(w)^{(j)}) \in D_{\mu'_j}$ by Lemma 5.3.1. □

Example 5.3.13. Continuing with Example 5.3.9 we construct two subwords of $\mathrm{cc}(w) = 211001$, both of length 3, based on the columns of T_w:

$$\mathrm{cc}(w)^{(1)} = \mathrm{cc}(w)_1\,\mathrm{cc}(w)_3\,\mathrm{cc}(w)_5 = 210$$
$$\mathrm{cc}(w)^{(2)} = \mathrm{cc}(w)_2\,\mathrm{cc}(w)_4\,\mathrm{cc}(w)_6 = 101$$

We can see that $\mathrm{cc}(w)^{(1)} = \mathrm{cw}(321)$ and $\mathrm{cc}(w)^{(2)} = \mathrm{cc}(213)$. Thus $\mathrm{cc}(w) \in \mathrm{Sh}(\mathrm{cc}(321), \mathrm{cc}(213)) = \mathrm{Sh}'(\mathrm{rev}(\mathrm{cc}(321), \mathrm{rev}(\mathrm{cc}(213))$, where $(\mathrm{rev}(\mathrm{cc}(321)), \mathrm{rev}(\mathrm{cc}(213))) \in D_3 \times D_3$.

Note that Proposition 5.3.12 only considers permutations w with $\mathrm{ctype}(P(w))) = \mu$. To prove the inclusion $\mathcal{C}_\mu \subset \mathcal{D}_\mu$, we extend this argument to consider $\mathrm{ctype}(P(w))) \rhd \mu$.

Proof of Theorem 5.A. We know $|\mathcal{C}_\mu| = |\mathcal{D}_\mu|$, since $|\mathcal{C}_\mu| = \dim(R_\mu)$ and $\mathcal{D}_\mu$ is a basis of R_μ. Hence to show equality of the two sets, we will show $\mathcal{C}_\mu \subset \mathcal{D}_\mu$. From Proposition 5.3.12, it suffices to consider $w \in \mathfrak{S}_n$ where $\mathrm{ctype}(P(w)) \rhd \mu$. First, consider $w \in \mathfrak{S}_n$ such that $\mathrm{ctype}(P(w)) = \lambda \trianglerighteq \mu$ where λ and μ differ by moving one box in the Young diagram. Assume that we construct μ from λ by taking the last box in some column j_2 of λ and moving it to the top of column j_1, where $j_1 < j_2$.

Wee consider the filling T_w of λ that we get from Algorithm 4.1.13. Let a be the entry in the last box of column j_2. Take the box containing a and append it into the end of column j_1.

By assumption, we know the resulting shape is μ. Denote the resulting filling of shape μ by $T_w^{(\mu)}$. We claim that each subword of $\mathrm{cc}(w)$ that corresponds to a column of $T_w^{(\mu)}$ is still a valid cocharge word. The only columns we need to consider are columns j_1 and j_2. Since the only modification to column j_2 is taking off the last entry we added, we see that the resulting word is still a valid cocharge word from Proposition 5.3.10.

Since $j_1 < j_2$ and T_w is partition shape, there exists an entry b in column j_1 which is in the same row as a in T_w. Since $j_1 < j_2$, we know we first added b, and then a, to the filling T_w. If we let k_a, k_b denote the number of times we read a, b before adding them to T_w, we must have $k_b \leqslant k_a$. Thus, we know that $\mathrm{cc}(w)_a \leqslant \mathrm{cc}(w)_b$ from Lemma 5.3.8.

Thus adding a to $\mathrm{cc}(w)^{(j_1)}$ does not introduce a new cocharge value to this word since $\mathrm{cc}(w)_b$ is in $\mathrm{cc}(w)^{(j_1)}$. From Lemma 5.3.6(i), we conclude the resulting word is a cocharge word of a permutation.

If $\mathrm{ctype}(P(w)) \trianglerighteq \mu$ differ by moving multiple boxes, we repeat this process until we get a filling of μ where each of the columns correspond to valid cocharge words. It is clear that $\mathrm{cc}(w)$ is a shuffle of these cocharge words. Hence $\mathrm{rev}(\mathrm{cc}(w)) \in D_\mu$. From this, we have $\mathcal{C}_\mu = \mathcal{D}_\mu$. Since $\mathbf{x}^{\mathcal{D}_\mu}$ is a basis of R_μ, we conclude that our set $\mathbf{x}^{\mathcal{C}_\mu}$ is a basis as well. $\square$

Example 5.3.14. We use the same w, T_w as in Example 5.3.7. For $\mu = (2,1,1,1) \trianglelefteq \mathrm{ctype}(w)$, we can create a new filling T_w^μ of shape μ by moving the last box in column 2 of T_w to the end of column 1. This gives us

$$T_w^\mu = \begin{array}{|c|c} \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 4 & \multicolumn{1}{c|}{3} \\ \hline \end{array} .$$

The two new subwords coming from the columns of T_w^μ are $\mathrm{cc}(w)_1\,\mathrm{cc}(w)_2\,\mathrm{cc}(w)_4\,\mathrm{cc}(w)_5 = 1101$ and $\mathrm{cc}(w)_3 = 0$, both of which are still cocharge words.

Remark 5.3.15. Note that [11, Algorithm 1] gives a way to take a descent word $\mathbf{a} \in \mathcal{D}_\mu$ and recover an ordered set partition $(A_1|\dots|A_l)$ of n such that $|A_i| = \mu_i$ and $\mathbf{a}|_{A_i} \in D_{\mu_i}$. That is: the ordered set partition identifies a canonical way to see $\mathbf{a}$ as a shuffle of descent words. This algorithm does not necessarily agree with our construction of a canonical decomposition of the cocharge word $\mathrm{cc}(w)$ outlined in Proposition 5.3.10. For example: if we consider $01011 \in \mathcal{C}_{2,2,1} = \mathcal{D}_{3,2}$, the Carlsson–Chou construction decomposes this into $\mathbf{0}\mathbf{1}0\mathbf{1}1$, while our construction decomposes it into $\mathbf{01}01\mathbf{1}$.

5.4 New properties of catabolizability type

Theorem 5.A says that two conditions, one involving catabolizability type and the other involving shuffles, are equivalent. In particular, we get the following corollary.

Corollary 5.4.1. *Let $u^{(1)}, u^{(2)}, \dots, u^{(l)}$ be permutations where $u^{(j)}$ is of length μ'_j. Then we have*

$$\mathrm{Sh}(\mathrm{cc}(u^{(1)}), \dots, \mathrm{cc}(u^{(l)})) \subset \{\mathrm{cc}(w) \mid w \in \mathfrak{S}_n,\ \mathrm{ctype}(w) \trianglerighteq \mu\}.$$

Proof. This follows from rephrasing the relation $\mathcal{D}_\mu \subset \mathcal{C}_\mu$. □

However, we can show the following stronger result using properties of Algorithm 4.1.13.

Theorem 5.B. *Let $u^{(1)}, u^{(2)}, \dots, u^{(l)}, w$ be permutations such that $\mathrm{cc}(w) \in \mathrm{Sh}(\mathrm{cc}(u^{(1)}), \dots, \mathrm{cc}(u^{(l)}))$. We have*

$$\mathrm{ctype}(w) \trianglerighteq \mathrm{ctype}(u^{(1)}) + \mathrm{ctype}(u^{(2)}) + \dots + \mathrm{ctype}(u^{(l)}),$$

where $\mathrm{ctype}(u^{(1)}) + \mathrm{ctype}(u^{(2)}) + \dots + \mathrm{ctype}(u^{(l)})$ is the partition given by the partwise sum of $\mathrm{ctype}(u^{(1)}), \mathrm{ctype}(u^{(2)}), \dots, \mathrm{ctype}(u^{(l)})$.

We can see that Corollary 5.4.1 follows from Theorem 5.4. For any $u \in \mathfrak{S}_n$, we know $\mathrm{ctype}(u) \trianglerighteq (1)^n$, since $(1)^n$ is the unique smallest partition of size n with respect to dominance order. Hence, for permutations $u^{(1)}, u^{(2)}, \dots, u^{(l)}, w$ where $u^{(j)} \in S_{\mu'_j}$ and $\mathrm{cc}(w) \in \mathrm{Sh}(\mathrm{cc}(u^{(1)}), \dots, \mathrm{cc}(u^{(l)}))$, we have

$$\mathrm{ctype}(w) \trianglerighteq \mathrm{ctype}(u^{(1)}) + \mathrm{ctype}(u^{(2)}) + \dots + \mathrm{ctype}(u^{(l)}) \trianglerighteq (1)^{\mu'_1} + (1)^{\mu'_2} + \dots + (1)^{\mu'_j} = \mu.$$

$$T^w_{u^{(1)}} = \begin{array}{|c|c} \cline{1-1} (3,4) & \\ \cline{1-1} (2,9) & \\ \cline{1-1} (2,5) & \\ \cline{1-1} (1,10) & \\ \hline (1,7) & \multicolumn{1}{c|}{(1,8)} \\ \hline \end{array} \quad , \quad T^w_{u^{(2)}} = \begin{array}{|c|c|} \hline (1,2) & (1,6) \\ \hline (1,1) & (1,3) \\ \hline \end{array}.$$

Figure 5.1: Fillings $T^w_{u^{(1)}}, T^w_{u^{(2)}}$

Hence Theorem 5.4 gives an improved lower bound for the catabolizability type of a shuffle.

We prove Theorem 5.4 by introducing a new modification of Algorithm 4.1.13 that keeps track of the difference between $\text{ctype}(w)$ and $\text{ctype}(u^{(1)}) + \cdots + \text{ctype}(u^{(l)})$. We first construct our inputs for the algorithm. We restrict to the case where $l = 2$, but the same arguments generalize to larger l. Let $u^{(1)}, u^{(2)}, w$ be permutations such that $\text{cc}(w) \in \text{Sh}(\text{cc}(u^{(1)}), \text{cc}(u^{(2)}))$. If we specify a shuffle of $\text{cc}(u^{(1)})$ and $\text{cc}(u^{(2)})$ that is equal to $\text{cc}(w)$, we have that each letter in $\text{cc}(w)$ corresponds uniquely to a letter in $\text{cc}(u^{(1)})$ or $\text{cc}(u^{(2)})$. Fix such a shuffle.

We can create fillings $T^w_{u^{(1)}}, T^w_{u^{(2)}}$ of shapes $\text{ctype}(u^{(1)})$, $\text{ctype}(u^{(2)})$ using Algorithm 4.1.13. Similar to previous modification of Algorithm 4.1.13, we fill the boxes as we add them to the partition. However, we now fill the box with tuples (k, i). Consider the box we add to $T^w_{u^{(1)}}$ when reading the letter $\text{ctype}(u^{(1)})_m$ for the kth time. There exists an i such that $\text{ctype}(u^{(1)})_m$ corresponds to $\text{cc}(w)_{n-i+1}$ in the fixed shuffle. We fill the box with the tuple (k, i).

Note that the convention for the index we put in the box is different from what we used in Section 5.3. The $(n-i+1)$ will be useful when we define a total ordering on these entries. We are abusing notation here, since the fillings depend on how we shuffle the two cocharge words to get $\text{cc}(w)$, not just w.

Example 5.4.2. Let $w = 5\ 9\ 1\ 2\ 6\ 7\ 10\ 3\ 8\ 4$. Then $\text{cc}(w) = \mathbf{1}\ \mathbf{2}\ \mathbf{0}\ \mathbf{0}\ 1\ \mathbf{1}\ \mathbf{2}\ 0\ 1\ 0 \in \text{Sh}(\mathbf{120012}, 1010)$, where we have colored the entries to denote the shuffle we chose. The resulting fillings $T^w_{u^{(1)}}, T^w_{u^{(2)}}$ are depicted in Figure 5.1.

From Lemma 5.3.8, we know if (k, i) appears in row r of $T_{u^{(1)}}, T_{u^{(2)}}$ we must have $k + cc(w)_{n-i+1} = r$. Furthermore, since the second coordinate of an entry records

$$(T_{u^{(1)}} + T_{u^{(2)}})^w = \begin{array}{|c|c|c|c|}\cline{1-1} (3,4) \\ \cline{1-1} (2,9) \\ \cline{1-1} (2,5) \\ \cline{1-3} (1,10) & (1,2) & (1,6) \\ \hline (1,7) & (1,8) & (1,1) & (1,3) \\ \hline \end{array}$$

Figure 5.2: Filling $(T_{u^{(1)}} + T_{u^{(2)}})^w$

which letter in $\mathrm{cc}(w)$ created that box, for each $i \in \{1, \ldots, n\}$, there exists a unique k such that (k, i) appears in $T^w_{u^{(1)}}$ or $T^w_{u^{(2)}}$. This (k, i) appears exactly once.

We combine $T^w_{u^{(1)}}, T^w_{u^{(2)}}$ to get a filling of shape $\mathrm{ctype}(u^{(1)}) + \mathrm{ctype}(u^{(2)})$, denoted $(T_{u^{(1)}} + T_{u^{(2)}})^w$, by taking the row-wise sum of the two diagrams.

Example 5.4.3. We combine the two diagrams we get from Example 5.4.2 to get a filling of $(2+2, 1+2, 1, 1, 1)$, depicted in Figure 5.2.

The lexicographic ordering $(\preceq)$ on tuples $(a, b) \in \mathbb{Z}^2_{>0}$ is defined by:

$$(a, b) \prec (c, d) \Leftrightarrow \begin{cases} a < c & \text{or} \\ a = c, b < d. \end{cases}$$

The lexicographic ordering of the tuples keeps track of the order in which we read the entries in Algorithm 4.1.13. That is: if $(k_1, i_1) \preceq (k_2, i_2)$, then we read $\mathrm{cc}(w)_{n-i_1+1}$ for the k_1th time before we read $\mathrm{cc}(w)_{n-i_2+1}$ for the k_2th time.

Let $T^w_{u^{(1)}+u^{(2)}}$ be the filling obtained by rearranging the entries within the rows of $(T_{u^{(1)}} + T_{u^{(2)}})^w$ so that the entries within the rows are increasing from left to right with respect to $\preceq$. We have the following result:

Lemma 5.4.4. *The filling $T^w_{u^{(1)}+u^{(2)}}$ satisfies the following conditions:*

(i) Each $i \in \{1, 2, \ldots, n\}$ appears in the second coordinate of an entry exactly once

(ii) The entries in each row and column are increasing with respect to the lexicographic ordering on tuples,

(iii) For any entry (k, i) in row r of $T^w_{u^{(1)}+u^{(2)}}$, we have $k + \mathrm{cc}(w)_{n-i+1} = r$.

Proof. The conditions (i) and (iii) hold by construction. It suffices to show the entries within each columns are increasing.

Let (k, i) be the entry in the row r, column j of $T^w_{u^{(1)}+u^{(2)}}$, where $r > 2$. We know (k, i) is the jth smallest entry in row r, since we reordered within the rows so that the entries were increasing. For each entry in row r of $T^w_{u^{(1)}+u^{(2)}}$, there is an entry in row $r - 1$ that appeared directly below it in $T^w_{u^{(1)}}$ or $T^w_{u^{(2)}}$. Since $T^w_{u^{(1)}}, T^w_{u^{(2)}}$ were constructed through the Algorithm 4.1.13, we know that their columns are increasing with respect to the lexicographic ordering.

Thus for each entry (k', i') in row r such that $(k', i') \preceq (k, i)$, there is an entry in the row $(r - 1)$ that was directly below (k', i') in $T^w_{u^{(1)}}$ or $T^w_{u^{(2)}}$ which is smaller than (k', i').

Therefore, if (k, i) is the jth smallest entry in row r, there are at least j many things in row $(r-1)$ that are smaller than (k, i). Hence (k, i) is larger than the entry directly below it. $\square$

Example 5.4.5. Continuing with our example, we have

$$T_{u^{(1)}+u^{(2)}} = \begin{array}{|c|c|c|c|}
\cline{1-1}
(3,4) \\
\cline{1-1}
(2,9) \\
\cline{1-1}
(2,5) \\
\cline{1-3}
(1,2) & (1,6) & (1,10) \\
\hline
(1,1) & (1,3) & (1,7) & (1,8) \\
\hline
\end{array}\,.$$

The entries are increasing within each row and column.

We now define the notion of row insertion when dealing with these fillings. This is a modification of the classical row insertion used in RSK, except we also check the column increasing condition when inserting and we may modify more than one entry in the row, by sliding part of the row over to the right by one.

Algorithm 5.4.6. (Modified Row Insertion) Consider a filling T of a two row partition shape, where each entry is a tuple in $\mathbb{Z}^2_{>0}$ and the rows and columns are increasing with respect to the lexicographic ordering.

Let $(a,b) \in \mathbb{Z}^2_{>0}$. We *insert* (a,b) into the second row of this partition by doing the following. We first find the leftmost (c,d) in the second row that satisfies:

$$(a,b) \preceq (c,d), \tag{5.10}$$

$$(a,b) \succeq (e,f) \tag{5.11}$$

when (e,f) denotes the entry directly below (c,d). If no such (c,d) exists, we say (a,b) *pops out* of this row and we are done.

If such (c,d) exists, replace (c,d) with (a,b). The resulting filling is still increasing in the rows and columns. The only place where we could have a contradiction is directly to the left of (a,b), but this cannot happen since (c,d) is the leftmost entry that satisfies (5.10) and (5.11).

We now repeat the steps above to insert (c,d) into the same row. We continue until an entry pops out or we insert into the last box of the row.

If we insert into the last box of the row, replacing (g,h), we finish our insertion process by appending (g,h) to the end of the second row if the resulting filling has increasing columns and is still a partition shape. This only happens if the second row was strictly shorter than the first. In this case, nothing pops out. Otherwise, (g,h) pops out and the insertion is complete.

Note that once we successfully insert (a,b), the entry (c,d) gets inserted to the direct right of (a,b) or it pops out. From this, we can see that this process is equivalent to determining where we can put (a,b) and then moving the entries larger than it to the right by one until we find a contradiction with the bottom row or we reach the end of the row.

We illustrate this through the following examples. The first one results in an entry popping out.

Example 5.4.7. We insert $(a,b) = (2,6)$ into the second row of the following partition:

(2,4)	(3,2)	(3,3)	(4,1)	
(1,1)	(2,5)	(2,7)	(3,8)	(5,2)

Since $(3,2)$ is the leftmost entry satisfying (5.10) and (5.11), we replace $(3,2)$ with $(2,6)$:

(3,2)

(2,4)	**(2,6)**	(3,3)	(4,1)	
(1,1)	(2,5)	(2,7)	(3,8)	(5,2)

Now we repeat: we replace $(3,3)$ with $(3,2)$:

(3,3)

(2,4)	(2,6)	**(3,2)**	(4,1)	
(1,1)	(2,5)	(2,7)	(3,8)	(5,2)

However, we cannot replace $(4,1)$ with $(3,3)$ since $(3,8) \not\leq (3,3)$. Hence $(3,3)$ pops out of this insertion and we are done.

We also give an example of when nothing pops out of the insertion process.

Example 5.4.8. We insert $(a,b) = (2,6)$ into the second row of the following partition:

(2,4)	(3,2)	
(1,1)	(2,5)	(3,1)

Since $(3,2)$ is the leftmost entry satisfying (5.10) and (5.11), we replace $(3,2)$ with $(2,6)$:

(3,2)

(2,4)	**(2,6)**	
(1,1)	(2,5)	(3,1)

Now, we can append $(3,2)$ to the end of row 2 while maintaining the column increasing condition. Adding a box to the second row does not change the fact that the shape is a two row partition. Hence we insert $(3,2)$ into the empty spot.

(2,4)	(2,6)	**(3,2)**
(1,1)	(2,5)	(3,1)

Note that nothing pops out at the end of this insertion.

We make the following observation.

Lemma 5.4.9. *If we insert (a,b) into row 2 of T and (x,y) pops out, we have $(a,b) \preceq (x,y)$.*

Proof. If we are unable to insert (a, b), then $(x, y) = (a, b)$. Otherwise, the insertion replaces larger entries with smaller ones. $\square$

We can extend this definition of row insertion to any partition shape tableau T filled with tuples in $\mathbb{Z}^2_{>0}$ where the rows and columns are increasing. For any row $r > 1$ of T and $(a, b) \in \mathbb{Z}^2_{>0}$, we define $T \leftarrow_r (a, b)$ to be the result of inserting (a, b) into row r of T. Unlike RSK, we only modify row r; we do not continue inserting into higher rows. It is a simple check to see that the resulting filling $T \leftarrow_r (a, b)$ is still increasing in the rows and columns.

Lemma 5.4.10. *Consider partition shape tableau T filled with tuples in $\mathbb{Z}^2_{>0}$ where the entries within the rows and columns are increasing. Consider two tuples $(a, b) \preceq (a', b')$. Let $T \leftarrow_r (a, b)$ denote the tableau we get by inserting (a, b) into row r of T, where r is not the top row.*

1. *Assume (x, y) pops out when we insert (a, b) into row r of T. If (x', y') pops out when we insert (a', b') into row $(r+1)$ of $T \leftarrow_r (a, b)$, then $(x, y) \preceq (x', y')$.*

2. *If nothing pops out when we insert (a, b) into row r of T, then nothing pops out when we insert (a', b') into row $(r + 1)$ of $T \leftarrow_r (a, b)$.*

Proof. (1) Note that if $(x, y) = (a, b)$, the claim follows immediately. Otherwise, let (p, q) denote the entry in row $(r + 1)$ directly above (x, y) in T. Note that the entry (p, q) may not exist: in that case, assume (p, q) to be the empty box at the end of row $(r+1)$. Let (g, h) denote the entry in row $(r+1)$ directly above (a, b) in $T \leftarrow_r (a, b)$.

We can see that rows $r, r + 1$ of T look like the following:

$$T : \qquad \begin{array}{|c|c|c|c|c|}\hline & (g,h) & \cdots & (p,q) & \\ \hline & & \cdots & (x,y) & \\ \hline \end{array}\,.$$

We can visualize $T \leftarrow_r (a, b)$ in the following way. First, we move the shaded boxes in row r, which are the boxes from the one directly below (g, h) up to (but not including) (x, y), over one to the right. Then we put (a, b) directly below (g, h) which pops out (x, y).

Rows $r, r + 1$ of $T \leftarrow_r (a, b)$ look like the following:

$$T \leftarrow_r (a, b) : \qquad \begin{array}{|c|c|c|c|c|}\hline & (g,h) & \cdots & (p,q) & \\ \hline & (a,b) & \cdots & & \\ \hline \end{array}\,.$$

Now, we insert $(a', b') \succeq (a, b)$ into row $(r+1)$ of $T \leftarrow_r (a, b)$. Let (x', y') be the entry that pops out of this insertion, if such entry exists. To show $(x, y) \preceq (x', y')$, we use the fact that $(x, y) \preceq (p, q)$. It suffices to show $(p, q) \preceq (x', y')$.

We proceed by cases. First, note that if $(p, q) \preceq (a', b')$, the claim automatically holds since $(a', b') \preceq (x', y')$ (with potential equality).

The second case we consider is when $(g, h) \preceq (a', b') \preceq (p, q)$. In this case, we can always insert (a', b') between (g, h) and (p, q). Consider $(g_1, h_1) \preceq (g_2, h_2)$ that appear consecutively in row r of $T \leftarrow_r (a, b)$ such that $(g, h) \preceq (g_1, h_1) \preceq (a', b') \preceq (g_2, h_2) \preceq (p, q)$. This is equivalent to saying that in $T \leftarrow_r (a, b)$, we have the following configuration:

$T \leftarrow_r (a, b)$:

	(g, h)	$\cdots$	(g_1, h_1)	(g_2, h_2)	$\cdots$	(p, q)	
	(a, b)	$\cdots$		(s_1, t_1)	(s_2, t_2)		

.

Since we know that all the shaded entries in row r were shifted 1 to the right when inserting (a, b) into row r of T.we know that the entry (s_1, t_1) that appears directly below (g_2, h_2) was originally below (g_1, h_1) in T. Thus $(s_1, t_1) \preceq (g_1, h_1) \preceq (a', b') \preceq (g_2, h_2)$, which means we can replace (g_2, h_2) with (a', b').

We can also move all the entries from (g_2, h_2) up to (but not including) (p, q) over to the right by one, because this is just equivalent to realigning rows r and $(r+1)$ of $T \leftarrow_r (a, b)$ to how they were in T. In particular, we can place (g_2, h_2) on top of (s_2, t_2), since we know $(s_2, t_2) \preceq (g_2, h_2)$. The result of continuing the insertion up to (p, q) is the following configuration:

(p, q)

	(g, h)	$\cdots$	(g_1, h_1)	(a', b')	(g_2, h_2)	$\cdots$	
	(a, b)	$\cdots$		(s_1, t_1)	(s_2, t_2)		

.

At this point, either we are able to continue by inserting (p, q) into row $(r+1)$, or (p, q) pops out. From this, we know the smallest possible thing that could pop out is (p, q). Thus $(p, q) \preceq (x', y')$.

Now, we consider the case where $(a', b') \preceq (g, h)$. We know (a', b') can be inserted in some spot to the left of (g, h) by just checking (5.10): the column condition follows automatically since $(a,b) \preceq (a',b')$, thus (a',b') is larger than any entry to the left of (a,b) in row r. Let (c,d) be the leftmost entry in row $(r+1)$ such that $(a',b') \preceq (c,d)$.

$$T \leftarrow_r (a,b) : \quad \begin{array}{|c|c|c|c|c|c|c|} \hline & (c,d) & \ldots & (g,h) & \cdots & (p,q) & \\ \hline & & & (a,b) & \cdots & & \\ \hline \end{array}.$$

We replace (c,d) with (a',b'). Now, any entry between (c,d) and (g,h) (which correspond to the boxes colored yellow), must also be larger than (a,b), so we can keep inserting until we have replaced (g,h). This gives us the following configuration:

$$\begin{array}{c} (g,h) \\ \begin{array}{|c|c|c|c|c|c|c|} \hline & (a',b') & \ldots & & \cdots & (p,q) & \\ \hline & & & (a,b) & \cdots & & \\ \hline \end{array}. \end{array}$$

The next step is to insert (g,h) into row $(r+1)$: we can see that this is exactly the second case we considered. Thus, from the argument above, we know the insertion process will continue until we reach (p,q) and the claim holds.

(2) The same argument holds: if such (x,y) does not exist, then (p,q) does not exist. Thus the two cases we consider are $(a',b') \preceq (g,h)$ and $(g,h) \preceq (a',b')$, which correspond to the first two cases in the previous argument. We can keep inserting into row $(r+1)$ until we add a box at the end of row $(r+1)$, which we know we can do since row r of $T \leftarrow_r (a,b)$ is one longer than row r of T. $\square$

Now, we use this new notion of insertion to define a modification of Algorithm 4.1.13 that keeps track of the difference between $\mathrm{ctype}(w)$ and $\mathrm{ctype}(u^{(1)}) + \cdots + \mathrm{ctype}(u^{(l)})$. The input $(\mathrm{cc}(w), \varnothing, T^w_{u^{(1)},\ldots,u^{(l)}})$ depends on the shuffle of $\mathrm{cc}(u^{(1)}), \ldots, \mathrm{cc}(u^{(l)})$ that is equal to $\mathrm{cc}(w)$ that we fix. Throughout this algorithm, we can think of the filling T in the third coordinate of our tuple as keeping track of the "lower bounds" of insertion: that is, if we read a certain entry in T, we must add a box there. However, we may add boxes earlier, which is when we perform the chains of insertions. The final result in the third coordinate will be a tableau T_w with $\mathrm{sh}(T_w) = \mathrm{ctype}(w)$.

The next section will contain an example of this algorithm.

Algorithm 5.4.11. The input is $(\mathrm{cc}(w), \varnothing, S)$, where S is a filling of a partition shape that satisfies the conditions (i) – (iii) in Lemma 5.4.4. For example, we can take S to be $T^w_{u^{(1)},\ldots,u^{(l)}}$.

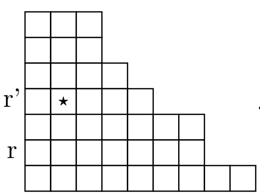
As we did for Algorithm 4.1.13, we apply the function f repeatedly to the first two coordinates until we get $(\varnothing, \mathrm{ctype}(w))$ in the first two coordinates. We modify the shape of the filling in the last coordinate whenever we add a box to the second coordinate, so that the new shape is larger in dominance order.

At each step, we say we *read* (k,i) if we read the letter corresponding to $\mathrm{cc}(w)_{n-i+1}$ for the kth time. After each step of the algorithm, we get a triple (z, ν, T) where z is a word, ν is some partition, and T is a tableau of partition shape.

Consider the step when we read (k,i), with input (z,ν,T), which results in adding a box to row r of ν. Let $(\tilde{z}, \nu + \epsilon_r)$ be the new first and second coordinate of our tuple.

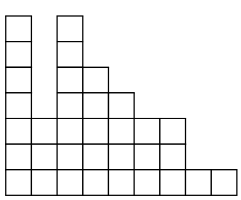
By condition (i), we know there exists a (m,i) that appears in T. If $k = m$, we do not modify the filling T and the new tuple we consider is $(\tilde{z}, \nu + \epsilon_r, T)$. Note that by condition (iii), we know that (k,i) appears in row r of T.

However, if (m,i) is in row r' where $r < r'$, we will create a new tableau $\tilde{T}$ of partition shape in the following way. We first look at T, where $\star$ denotes the box containing (m,i):



.

Delete the box in T containing (m,i) as well as the ones directly above it in T. This results in deleting entries $\{(m,i),(m_1,i_1),\ldots,(m_s,i_s)\}$, where (m_t,i_t) denotes the entry that was t boxes above (m,i) and s is the total number of boxes that appear above (m,i) in T. The resulting diagram looks like this:



removed entries: $\{(m,s),(m_1,i_1),(m_2,i_2),(m_3,i_3)\}$

Note that since the entries are increasing within the columns of T, we have

$$(m,i) \preceq (m_1,i_1) \preceq \cdots \preceq (m_s,i_s) \tag{5.12}$$

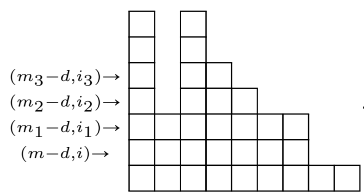
The rows and columns are still increasing with respect to the lexicographic ordering if we ignore the empty spaces in the column that contained $\{(m,i),(m_1,i_1),\ldots,(m_s,i_s)\}$.

We will create a new partition shape filling of n boxes by inserting the entries we removed back into this filling.

Let $d = r' - r$. We first insert $(m-d,i) = (k,i)$ into row r using Algorithm 5.4.6. Note that we use $(m-d,i)$, rather than (m,i), to maintain condition (iii). This change does not change the fact that the entries in the filling satisfy condition (i).

If an entry pops out, we denote that entry by (x_0,y_0). Now, we continue this process, inserting (m_1-d,i_1) into row $r+1$. If some entry (x_1,y_1) pops out, we know $(x_0,y_0) \preceq (x_1,y_1)$ from Lemma 5.4.10 (1) since $(m-d,i) \preceq (m_1-d,i_1)$.

In this way, we insert $(m_1-d,i_1,),\ldots,(m_s-d,i_s)$ into rows $r+1,\ldots,r+s$.



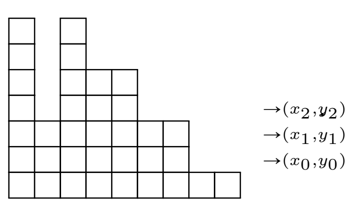
If we insert into a row with a gap in the middle, we ensure that we maintain the empty spot by skipping over it in the insertion process. That is: we never insert in the empty spot in the middle of the row.

By Lemma 5.4.10 (2) and (5.12), we know that once we have an insertion where nothing pops out, we will never have anything pop out for the later insertions as well. Let $j \leqslant s$ be the index such that inserting (m_j-d,i_j) is the last insertion where some entry (x_j,y_j) pops out. The elements that pop out at each step have the following relation:

$$(k,i) \preceq (x_0,y_0) \preceq (x_1,y_1) \preceq \cdots \preceq (x_j,y_j), \tag{5.13}$$

where either $j = s$ or nothing popped out when inserting $(m_{j+1}-d,i_{j+1})$ into row $r+j+1$.

For example, if we assume $j = 2$ in our example, we get:

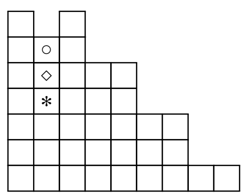


Once we have inserted up to $(m_s - d, i_s)$, we still have an empty column where (m, i) used to be and above.

Since the insertion algorithm replaces entries with smaller ones, we know that the entry below the empty space in row r' must still be smaller than (m, i). If there exists an entry (x_0, y_0) that popped out of row r, we have the following inequalities using (5.13):

$$(k + d, i) = (m, i) \preceq (x_0 + d, y_0) \preceq (x_1 + d, y_1) \preceq \cdots \preceq (x_j + d, y_j). \tag{5.14}$$

Thus we can fill the empty column with the entries $(x_0 + d, y_0), (x_1 + d, y_1), (x_j + d, y_j)$ and preserve the column increasing condition for all the columns.



where $* = (x_0, y_0), \diamond = (x_1, y_1), \circ = (x_2, y_2)$.

Now that we have put all the entries back into the filling, we rearrange the entries within rows r' and above so that the entries within the rows are increasing and we have a partition shape with no gaps. Let the resulting diagram be $\tilde{T}$. We can use the same argument used in the proof of Lemma 5.4.4(ii) to show that the $\tilde{T}$ is increasing within the rows and columns, since the filling we had before rearranging within the rows was column strict. Using this, we know that $\tilde{T}$ still satisfies conditions (i)– (iii). In this case, the new tuple we consider is $(\tilde{z}, \nu + \epsilon_r, \tilde{T})$.

We proceed by reading the next letter in $\tilde{z}$. We continue this process until the first coordinate of our tuple is the empty word.

Algorithm 5.4.11 satisfies the following properties. When we are reading an entry (k, i), we say that smaller than (k, i) in the lexicographic order has *already been read.*

Lemma 5.4.12. *Assume we apply Algorithm 5.4.11 with intial input* $(\mathrm{cc}(w), \varnothing, S)$. *Each intermediate tuple* (z, ν, T) *must satisfy the following conditions:*

(a) As partitions, we have $\mathrm{sh}(T) \trianglerighteq \mathrm{sh}(S)$*, where S is the filling in the original input.*

(b) A tuple (k, i) that appears in T has already been read if and only if it is contained in the subdiagram of T of shape ν.

(c) A tuple (k, i) that appears in T has not been read if and only if the letter corresponding to $\mathrm{cc}(w)_{n-i+1}$ *has not been deleted from the input word.*

Proof. Note that by assumption, we have that T satisfies the conditions (i)–(iii). We proceed by induction. Note that the statement is true for the initial filling S before we read $(1,1)$. We proceed by cases.

Assume we read (k,i) but cannot add a box to ν. By (iii), we know that there exists a tuple $(m,i) \in T$. By induction, we know that (m,i) cannot have been read yet by (c), since it corresponds to an entry that has not been deleted from the input word. Thus $m \geqslant k$. Furthermore, if $m = k$, we would have that the box below (k,i) in T must be contained in the subdiagram of shape ν by (b) and (ii), since it must have already been read. However, this would imply that we would be able to add a box to the diagram ν, which is a contradiction. Thus $k < m$. Since this does not affect any of the conditions, we have that (a)–(c) are true for this tuple.

Now we consider the case where reading (k,i) results in adding a box to row r of ν but does not change the filling T. This does not affect condition (a). In this case, we know (k,i) appears in row r of T. Since we assume that T satisfied conditions (a)–(c) before we read T, we know that (k,i) is the smallest entry in row r of T that is not contained in the subdiagram of shape ν. This implies that the box in T containing (k,i) is exactly the new box in $\nu + \epsilon_r$, thus (b) and (c) hold as well.

Finally, we consider the case where reading (k,i) results in creating a new filling $\tilde{T}$.

In this case, we know that (m,i) appears in row $r' > r$ of T. This means we delete (m,i) (and all the entries above it) and then insert (k,i) into row r. Note that from (b) on T, we know that none of the entries we delete can be contained in the subdiagram of T of shape ν, since all the entries we delete are larger than (m,i).

Let (x,y) be the smallest, unread entry in row r of T. From condition (c) and the fact that reading (k,i) results in adding a box to row r of ν, it follows that the box under (x,y) must be contained in the subdiagram of shape ν. Thus the entry below (x,y) has already been read. Furthermore, since (k,i) is the smallest unread entry, we have that $(k,i) \preceq (x,y)$. Thus inserting (k,i) into row r results in replacing (x,y) with (k,i). Note that this box in $\tilde{T}$ is in the same position as the new box we add to ν.

Note that from equations (5.12), we know the insertion process only involves entries that are larger than (k,i), thus the entries in the subdiagram of shape ν are fixed. Furthermore, we can see that when we rearrange the rows after the insertion so that the final shape is partition, we do not touch any of the entries in the subdiagram of shape $\nu + \epsilon_r$. Hence condition (b), (c) hold.

Finally, we can see this insertion process corresponds to moving entries in T to lower rows. Thus we have condition (1).

$\square$

Corollary 5.4.13. *When applying Algorithm to* $(\mathrm{cc}(w), \varnothing, S)$*, we have that the final output is of the form* $(\varnothing, \mathrm{ctype}(w), T_w)$ *where* $\mathrm{sh}(T_w) = \mathrm{ctype}(w)$*.*

Proof. We know that the final tuple $(\varnothing, \mathrm{ctype}(w), T_w)$ must satisfy Lemma 5.4.12 (a) –(c). By (c), we know that all the entries in T_w must be read. Then, by condition (2), we have $\mathrm{ctype}(w) = \mathrm{sh}(T_w)$. □

Theorem 5.4 immediately follows from this algorithm.

Proof of Theorem 5.4. We apply Algorithm 5.4.11 to the initial input $(\mathrm{cc}(w), \varnothing, T^w_{u^{(1)},\dots,u^{(l)}})$. From Lemma 5.4.12, at any point in the algorithm, where the tuple is of the form (z, ν, T), we have

$$\mathrm{ctype}(u^{(1)}) + \cdots + \mathrm{ctype}(u^{(l)}) = \mathrm{sh}(T^w_{u^{(1)},\dots,u^{(l)}}) \trianglelefteq \mathrm{sh}(T).$$

From Corollary 5.4.13, it follows that $\mathrm{ctype}(u^{(1)}) + \cdots + \mathrm{ctype}(u^{(l)}) \trianglelefteq \mathrm{sh}(T_w) = \mathrm{ctype}(w)$. □

We can also use this modified algorithm to prove the following statement about how certain modifications to permutations raise the catabolizability type.

Proposition 5.4.14. *Consider* $w \in \mathfrak{S}_n$ *and index* i *such that* $w_i + 1 < w_{i+1}$*. If* $\tilde{w} = w_1 \dots w_{i+1} w_i \dots w_n$*, we have* $\mathrm{ctype}(\tilde{w}) \trianglerighteq \mathrm{ctype}(w)$*.*

Proof. We know $\mathrm{cc}(w)_i < \mathrm{cc}(w)_{i+1}$, since they differ by more than one. This swap does not change the cocharge label of any of the letters. Thus $\mathrm{cc}(\tilde{w})$ is the word we get by swapping $\mathrm{cc}(w)_i$ and $\mathrm{cc}(w)_{i+1}$ in $\mathrm{cc}(w)$.

Let T_w be the filling of $\mathrm{ctype}(w)$ we get from Algorithm 4.1.13, where we fill a box with tuple (k, i) if we add the box when reading the letter $\mathrm{cc}(w)_{n-i+1}$ for the kth time. There exist unique k_1, k_2 such that $(k_1, n-i+1), (k_2, n-i)$ appear in T_w. Replace these two tuples with $(k_1, n-i), (k_2, n-i+1)$ to account for the swapping of $\mathrm{cc}(w)_i$ and $\mathrm{cc}(w)_{i+1}$. Denote the resulting filling by $T^{\tilde{w}}_w$.

The filling $T^{\tilde{w}}_w$ satisfies the conditions (1), (3) in Lemma 5.4.4 by construction. We can check that it satisfies (2) as well. Note that the only case where swapping the second coordinates changes the relative order on the elements in $T^{\tilde{w}}_w$ is if $k_1 = k_2$. In this case, the potential issue occurs if $(k_1, n-i), (k_2, n-i+1)$ appear in the same row or column of T_w. However, this never happens. Since $\mathrm{cc}(w)_i \neq \mathrm{cc}(w)_{i+1}$, if the two appeared in the same row we would have $k_1 > k_2$ from Lemma 5.3.8. If $k_1 = k_2$, then $(k_1, n-i)$ appears in a higher row than $(k_2, n-i+1)$. Since we add $(k_1, n-i)$ to T_w before we add $(k_2, n-i+1)$, it is impossible that these two tuples appear in the same column of T_w. Hence $T^{\tilde{w}}_w$ satisfies (2).

Now, we apply Algorithm 5.4.11 to the initial input $(\mathrm{cc}(\tilde{w})), \varnothing, T_w^{\tilde{w}})$. Since $\mathrm{shape}(T_w^{\tilde{w}}) = \mathrm{ctype}(w)$ and the resulting tuple is $(\varnothing, \mathrm{ctype}(\tilde{w}), T_{\tilde{w}})$ with $\mathrm{sh}(T_{\tilde{w}}) = \mathrm{ctype}(\tilde{w})$, it follows that $\mathrm{ctype}(\tilde{w}) \trianglerighteq \mathrm{ctype}(w)$. $\square$

5.5 Example of Algorithm 5.4.11

Let $w = 3\ \ 9\ \ 1\ \ 2\ \ 6\ \ 7\ \ 10\ \ 3\ \ 8\ \ 4$.

Then $\mathrm{cc}(w) = \mathbf{1}\ \ \mathbf{2}\ \ \mathbf{0}\ \ \mathbf{0}\ \ 1\ \ \mathbf{1}\ \ \mathbf{2}\ \ 0\ \ 1\ \ 0 \in \mathrm{Sh}(\mathbf{120012}, 1010)$, where we have colored the entries to denote how the two words are shuffled.

The initial input when applying Algorithm 5.4.11 is given by:

$$\left(1\ \ 2\ \ 0\ \ 0\ \ 1\ \ 1\ \ 2\ \ 0\ \ 1\ \ 0,\ \varnothing\ ,\ \begin{array}{|c|cccc} \hline (3,4) & & & \\ \cline{1-1} (2,9) & & & \\ \cline{1-1} (2,5) & & & \\ \hline (1,2) & \multicolumn{1}{|c|}{(1,6)} & \multicolumn{1}{c|}{(1,10)} & \\ \hline (1,1) & \multicolumn{1}{|c|}{(1,3)} & \multicolumn{1}{c|}{(1,7)} & \multicolumn{1}{c|}{(1,8)} \\ \hline \end{array} \right).$$

We can see that the first three steps of algorithm correspond to adding boxes $(1,1),(1,2),(1,3)$ to the partition in the second coordinate. We do not modify the filling in the last coordinate. The resulting triple we get is:

$$\left(1\ \ 2\ \ 0\ \ 0\ \ 1\ \ 1\ \ \underline{\mathbf{2}}\ \ \_\ \ \_\ \ \_\ ,\ \begin{array}{|c|c} \cline{1-1} \ & \\ \hline \ & \multicolumn{1}{c|}{\ } \\ \hline \end{array}\ ,\ \begin{array}{|c|cccc} \hline (3,4) & & & \\ \cline{1-1} (2,9) & & & \\ \cline{1-1} (2,5) & & & \\ \hline (1,2) & \multicolumn{1}{|c|}{(1,6)} & \multicolumn{1}{c|}{(1,10)} & \\ \hline (1,1) & \multicolumn{1}{|c|}{(1,3)} & \multicolumn{1}{c|}{(1,7)} & \multicolumn{1}{c|}{(1,8)} \\ \hline \end{array} \right). \tag{5.15}$$

The entries have been read up to this point are exactly those in the shape $\nu = \begin{array}{|c|c} \cline{1-1} \ & \\ \hline \ & \multicolumn{1}{c|}{\ } \\ \hline \end{array}$. These boxes are shaded.

At the fourth step (which corresponds to reading $\underline{\mathbf{2}}$), we can see that we will add a box to the partition in the second coordinate. However, we can see that $(3,4)$ is in the fifth row. Thus, we delete $(3,4)$ and insert $(1,4)$ into the third row of our filling:

$$
(1,4) \rightarrow \quad \begin{array}{llll} (2,9) & & & \\ (2,5) & & & \\ (1,2) & (1,6) & (1,10) & \\ (1,1) & (1,3) & (1,7) & (1,8) \end{array} \quad \Rightarrow \quad \begin{array}{llll} (2,9) & & & \\ \mathbf{(1,4)} & (2,5) & & \\ (1,2) & (1,6) & (1,10) & \\ (1,1) & (1,3) & (1,7) & (1,8) \end{array} .
$$

The resulting triple after the fourth step is:

$$
\left(1\ \ 2\ \ 0\ \ 0\ \ 1\ \ \underline{\mathbf{1}}\ \ \_\ \ \_\ \ \_\ \ \_\ ,\quad \begin{array}{ll} \square & \\ \square & \\ \square & \square \end{array} ,\quad \begin{array}{llll} (2,9) & & & \\ (1,4) & (2,5) & & \\ (1,2) & (1,6) & (1,10) & \\ (1,1) & (1,3) & (1,7) & (1,8) \end{array} \right).
$$

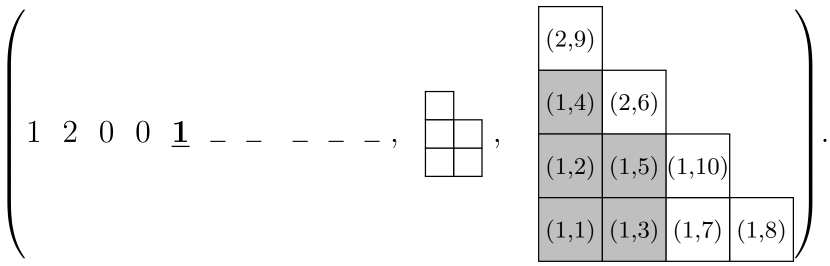
At the fifth step (corresponding to reading $\underline{\mathbf{1}}$), we add a box to the partition in the second coordinate, as well as modify the filling in the third coordinate by deleting (2,5) and inserting (1,5) into the second row:

$$
(1,5) \rightarrow \quad \begin{array}{llll} (2,9) & & & \\ (1,4) & & & \\ (1,2) & (1,6) & (1,10) & \\ (1,1) & (1,3) & (1,7) & (1,8) \end{array} \quad \Rightarrow \quad \begin{array}{llll} (2,9) & & & \\ (1,4) & & & \\ (1,2) & \mathbf{(1,5)} & (1,10) & \\ (1,1) & (1,3) & (1,7) & (1,8) \end{array} \quad \rightarrow \mathbf{(1,6)}.
$$

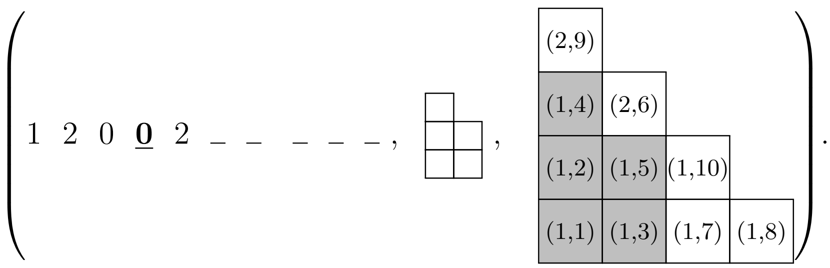
We see that $(1,6)$ pops out of the second row. We then place $(2,6)$ into the empty spot in the third row, where $(2,5)$ was.

$$
(2,6) \rightarrow \quad \begin{array}{llll} (2,9) & & & \\ (1,4) & & & \\ (1,2) & (1,5) & (1,10) & \\ (1,1) & (1,3) & (1,7) & (1,8) \end{array} \quad \Rightarrow \quad \begin{array}{llll} (2,9) & & & \\ (1,4) & \mathbf{(2,6)} & & \\ (1,2) & (1,5) & (1,10) & \\ (1,1) & (1,3) & (1,7) & (1,8) \end{array} .
$$

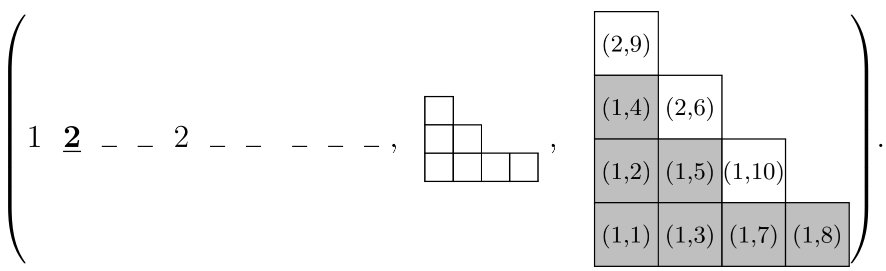
The resulting triple is



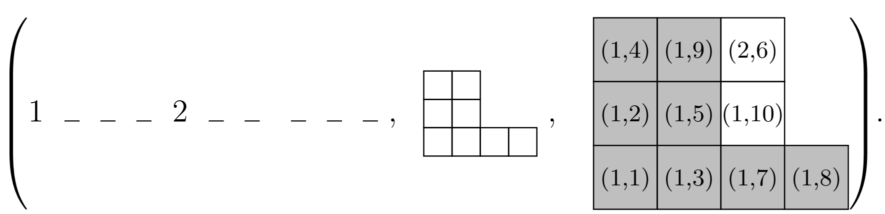
At the next step, we cannot add anything to the second row. So we add one to the element in the word without changing either of the shapes:



The next two steps just add boxes to the first row of the partition in the second coordinate without changing the filling. We get



At the next step, we delete $(2,9)$ and insert (1,9) into the third row of the filling to get:



Now, we can continue to cycle through the word and add boxes to the partition until we get

$$\left(\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \ \_\ \_\ \_\ \_ \ , \ \begin{array}{|c|c|c|c}\hline \ & \ & \ & \\ \hline \ & \ & \ & \\ \hline \ & \ & \ & \multicolumn{1}{c|}{\ } \\ \hline \end{array} \ , \ \begin{array}{|c|c|c|c}\hline (1,4) & (1,9) & (2,6) & \\ \hline (1,2) & (1,5) & (1,10) & \\ \hline (1,1) & (1,3) & (1,7) & \multicolumn{1}{c|}{(1,8)} \\ \hline \end{array} \right).$$

5.6 Identifying the graded Frobenius character

In the work of Garsia–Procesi [20, Section 3], the ungraded Frobenius character of R_μ is identified to be the complete homogeneous symmetric function h_μ directly from the structure of R_μ. However, they require further properties of R_μ, as well as the modified Hall-Littlewood polynomials, to identify the graded Frobenius character.

In particular, using the theory of orbit harmonics, we can show the following:

Theorem 5.6.1 (Garsia-Procesi [20])**.** *$R_\mu \cong_{\mathfrak{S}_n} \mathbb{C}[\mathcal{OSP}_\mu]$ as ungraded $\mathfrak{S}_n$-modules.*

Set T_μ to be the ideal consising of functions in $\mathbb{C}[\mathbf{x}]$ that vanish on the set

$$X_\mu = \{\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{C}^n : \text{multiplicity of } b_i \text{ in } \mathbf{p} \text{ is } \mu_i\}$$

where $b_1, \dots, b_{\ell(\mu)}$ are distinct scalars in $\mathbb{C}$. Theorem 5.6.1 follows from the fact that we have $I_\mu = \mathrm{gr}(T_\mu)$, or the associated graded of T_μ.

An immediate corollary of Theorem 5.6.1 is the following, which uses the observation that $\mathbb{C}[\mathcal{OSP}_\mu] \cong \mathbb{1}\uparrow_{\mathfrak{S}_\mu}^{\mathfrak{S}_n}$.

Corollary 5.6.2. $\mathrm{Frob}(R_\mu) = h_\mu$.

In this section, we use properties of our cocharge monomial basis to give a direct, elementary proof of the fact that $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X;q]$ that only depends on the well-known combinatorial formula (5.2) and the ungraded character given in Corollary 5.6.2.

We do this by showing that for any composition $\gamma = (\gamma_1, \dots, \gamma_l) \models n$, we have $\mathrm{Hilb}(N_\gamma R_\mu) = \langle e_\gamma, \widetilde{H}_\mu[X;q]\rangle$, where N_γ is the antisymmetrizer with respect to the Young subgroup $\mathfrak{S}_\gamma = \mathfrak{S}_{\gamma_1} \times \cdots \times \mathfrak{S}_{\gamma_l} \subset \mathfrak{S}_n$:

$$N_\gamma = \sum_{\sigma \in \mathfrak{S}_\gamma} \mathrm{sgn}(\sigma)\sigma.$$

Using that $\{m_\lambda\}, \{h_\gamma\}$ are dual with respect to the inner product, we get

$$\langle e_\gamma,\ \widetilde{H}_\mu[X;q]\rangle = \langle \omega e_\gamma,\ \omega\widetilde{H}_\mu[X;q]\rangle = \langle h_\gamma,\ \omega\widetilde{H}_\mu[X;q]\rangle = \langle m_\gamma\rangle\, \omega\widetilde{H}_\mu[X;q],$$

where $\langle m_\gamma \rangle \, \omega\widetilde{H}_\mu[X;q]$ denotes the coefficient of m_γ in the monomial symmetric function expansion of $\omega\widetilde{H}_\mu[X;q]$. We compute $\omega\widetilde{H}_\mu[X;q]$ using $\omega s_\lambda = s_{\lambda'}$ for any partition λ and the combinatorial formula (5.2):

$$\omega\widetilde{H}_\mu[X;q] = \sum_{\substack{S\in\mathrm{SYT}_n\\ \mathrm{ctype}(S)\trianglerighteq\mu}} q^{\mathrm{cocharge}(S)} s_{\mathrm{sh}(S')}.$$

From the above, we can see

$$\begin{aligned}\langle e_\gamma,\ \widetilde{H}_\mu[X;q]\rangle &= \sum_{\substack{S\in\mathrm{SYT}_n\\ \mathrm{ctype}(S)\trianglerighteq\mu}} q^{\mathrm{cocharge}(S)} \left(\langle m_\gamma\rangle \, s_{\mathrm{sh}(S')}\right)\\ &= \sum_{\substack{S\in\mathrm{SYT}_n\\ \mathrm{ctype}(S)\trianglerighteq\mu}} q^{\mathrm{cocharge}(S)} K_{\mathrm{sh}(S'),\gamma}.\end{aligned}$$

where $K_{\mathrm{sh}(S'),\gamma}$ counts the number of semistandard Young tableaux of shape $\mathrm{sh}(S')$, weight γ.

Using Proposition 2.1.2, we can describe $K_{\mathrm{sh}(S'),\gamma}$ in terms of standard tableaux:

$$K_{\mathrm{sh}(S'),\gamma} = |\{T\in\mathrm{SYT}_n, \mathrm{sh}(T)=\mathrm{sh}(S') \mid \mathrm{Des}(T)\subset\{\gamma_1,\gamma_1+\gamma_2,\ldots,\gamma_1+\cdots+\gamma_{l-1}\}\}|.$$

We can use that $\mathrm{Des}(T') = [n-1]-\mathrm{Des}(T) := \mathrm{Asc}(T)$, where T' denotes the SYT we get by taking the transpose of T, to rewrite this as:

$$K_{\mathrm{sh}(S'),\gamma} = |\{T\in\mathrm{SYT}_n, \mathrm{sh}(T)=\mathrm{sh}(S) \mid \mathrm{Asc}(T)\subset\{\gamma_l,\gamma_l+\gamma_{l-1},\ldots,\gamma_l+\cdots+\gamma_2\}\}|.$$

Furthermore, we can rewrite the expression above into a statement about permutations using the fact that $\mathrm{Des}(Q(w)) = \mathrm{Des}(w)$ (Proposition 2.2.1(b))

$$K_{\mathrm{sh}(S'),\gamma} = |\{w\in\mathfrak{S}_n \mid P(w)=S, \mathrm{Asc}(w)\subset\{\gamma_l,\gamma_l+\gamma_{l-1},\ldots,\gamma_l+\cdots+\gamma_2\}\}|.$$

where $\mathrm{Asc}(w) := \{i : w_i < w_{i+1}\}$.

Combining these observations, we have the following equality:

$$\langle e_\gamma,\ \widetilde{H}_\mu[X;q]\rangle = \sum_{\substack{w\in\mathfrak{S}_n\\ \mathrm{ctype}(P(w))\trianglerighteq\mu\\ \mathrm{Asc}(w)\subset\{\gamma_l,\gamma_l+\gamma_{l-1},\ldots,\gamma_l+\cdots+\gamma_2\}}} q^{\mathrm{cocharge}(w)}. \tag{5.16}$$

From this, proving $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X;q]$ is equivalent to showing that for any $\gamma \models n$, the Hilbert series $\mathrm{Hilb}(N_\gamma R_\mu)$ is equal to the right hand expression of (5.16). We show this by proving that the natural subset of charge monomials to expect from Equation (5.16) give a basis of $N_\gamma R_\mu$ by antisymmetrization.

Using the ungraded character of R_μ, given by h_μ, along with Proposition 2.4.1, we can make the following observation:

Lemma 5.6.3. *The dimension of $N_\gamma R_\mu$ is*

$$|\{w \in \mathfrak{S}_n,\ \operatorname{ctype}(P(w)) \trianglerighteq \mu, \operatorname{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \dots, \gamma_l + \cdots + \gamma_2\}|\}.$$

Proof. Since the ungraded Frobenius character of R_μ is h_μ, we know that $\dim(N_\gamma R_\mu) = \langle e_\gamma,\ h_\mu\rangle$. Using the fact that $\widetilde{H}_\mu[X;1] = h_\mu$, we have:

$$\dim(N_\gamma R_\mu) = \langle e_\gamma,\ h_\mu\rangle = \langle e_\gamma, \widetilde{H}_\mu[X;1]\rangle = \sum_{\substack{w \in \mathfrak{S}_n \\ \operatorname{ctype}(P(w)) \trianglerighteq \mu \\ \operatorname{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \dots, \gamma_l + \cdots + \gamma_2\}}} 1.$$

□

We now use this fact to determine a basis of $N_\gamma R_\mu$.

Proposition 5.C. *Let $\mu \vdash n$ and $\gamma = (\gamma_1, \dots, \gamma_l) \models n$. The set*

$$\{N_\gamma \mathbf{x}^{\operatorname{rev}\operatorname{cc}(w)} \mid w \in \mathfrak{S}_n,\ \operatorname{ctype}(P(w)) \trianglerighteq \mu, \operatorname{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \dots, \gamma_l + \cdots + \gamma_2\}\}. \tag{5.17}$$

is a basis of $N_\gamma R_\mu$.

Proof. We first show that the terms in (5.17) are nonzero. Consider $w \in \mathfrak{S}_n$ such that $\operatorname{ctype}(P(w)) \trianglerighteq \mu$, and $\operatorname{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \dots, \gamma_l + \cdots + \gamma_2\}$. We divide w into blocks of size $\gamma_l, \gamma_{l-1}, \dots, \gamma_1$ where the entries within each block are decreasing. This implies that no two entries in the same block have the same cocharge value. Then, when we look at $\operatorname{rev}(\operatorname{cc}(w))$, we can see that $\operatorname{rev}(\operatorname{cc}(w))$ can be divided into blocks of size $\gamma_1, \gamma_2, \dots, \gamma_l$, where the entries are strictly increasing within the blocks.

Note that $\mathfrak{S}_\gamma$ permutes indices within the blocks, thus each monomial term of $N_\gamma \mathbf{x}^{\operatorname{rev}(\operatorname{cc}(w))}$ is distinct. Hence $N_\gamma \mathbf{x}^{\operatorname{rev}(\operatorname{cc}(w))} \neq 0$. From Lemma 5.6.3, this shows that our set has the correct cardinality to be a basis of $N_\gamma R_\mu$.

Now, we show that (5.17) spans $N_\gamma R_\mu$. The set $\{N_\gamma \mathbf{x}^\alpha \mid \alpha \in \mathcal{C}_\mu\}$ clearly spans $N_\gamma R_\mu$. Consider $w \in \mathfrak{S}_n$ such that $\operatorname{ctype}(P(w)) \trianglerighteq \mu$, $N_\gamma \mathbf{x}^{\operatorname{rev}(\operatorname{cc}(w))} \neq 0$ but w does not have the correct descent set. We will show that $N_\gamma \mathbf{x}^{\operatorname{rev}(\operatorname{cc}(w))} = N_\gamma \mathbf{x}^{\operatorname{rev}(\operatorname{cc}(\operatorname{sort}_\gamma(w)))}$ for some $\operatorname{sort}_\gamma(w) \in \mathfrak{S}_n$ with the correct ascent set and catabolizability type.

If we divide $\operatorname{rev}(\operatorname{cc}(w))$ into blocks of size $\gamma_1, \gamma_2, \dots, \gamma_l$, the entries within a block all have distinct charge labels since $N_\gamma x^{\operatorname{rev}(\operatorname{cc}(w))} \neq 0$. From this, if we look at the corresponding blocks of sizes $\gamma_l, \dots, \gamma_1$ in w, we can reorder the entries within the blocks to be in decreasing order without changing the cocharge labels of the respective entries. That is: if $w_i < w_{i+1}$ with $\operatorname{cc}(w)_i \neq \operatorname{cc}(w)_{i+1}$, we know $w_i \neq (w_{i+1} +$

1)), hence we can swap w_i and w_{i+1} without changing the respective charge labels. Let $\mathrm{sort}_\gamma(w)$ be the resulting permutation. We see that $\mathrm{Asc}(\mathrm{sort}_\gamma(w)) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \dots, \gamma_l + \cdots \gamma_2\}$. By Proposition 5.4.14, we know $\mathrm{ctype}(\mathrm{sort}_\gamma(w)) \trianglerighteq \mathrm{ctype}(w) \trianglerighteq \mu$, hence $N_\gamma \mathbf{x}^{\mathrm{rev\,cc}(\mathrm{sort}_\gamma(w))}$ is in (5.17). □

Using the construction of our basis, we can translate questions about the structure of R_μ into questions about conditions on tableaux. In this case, choosing the elements that are a basis of $N_\gamma R_\mu$ is equivalent to looking at pairs of standard tableaux (P, Q) of the same shape with conditions on both P and Q. We illustrate this with an example:

Example 5.6.4. Consider $\mu = (2, 1, 1)$ and $\gamma = (2, 2)$. All standard tableaux P that satisfy $\mathrm{ctype}(P) \trianglerighteq \mu$ are listed below:

$$\ytableaushort{4,3,12}\ , \quad \ytableaushort{34,12}\ , \quad \ytableaushort{3,124}\ , \quad \ytableaushort{4,123}\ , \quad \ytableaushort{1234}.$$

We also list all standard tableaux Q such that $\mathrm{Asc}(Q) \subset \{2\}$:

$$\ytableaushort{4,3,2,1}\ , \quad \ytableaushort{24,13}, \quad \ytableaushort{4,2,13}.$$

Thus the pairs (P, Q) of standard tableaux of the same shape that satisfy $\mathrm{ctype}(P) \trianglerighteq \mu$ and $\mathrm{Asc}(Q) = \{\gamma_1\}$ are the pairs $\left(\ytableaushort{4,3,12}\ , \ytableaushort{4,2,13}\ \right)$ and $\left(\ytableaushort{34,12}, \ytableaushort{24,13}\right)$. These pairs corresponds to the words $4132, 3142$, which give us the cocharge monomials $x_2x_4^2, x_2x_4$. From this, we know the basis of $N_\gamma R_\mu$ is given by the polynomials

$$\{x_2x_4^2 - x_1x_4^2 - x_2x_3^2 + x_1x_3^2, x_2x_4 - x_1x_4 - x_2x_3 + x_1x_3\}.$$

The following corollary is immediate from Proposition 3.C.

Corollary 5.6.5. *For any partition μ of n we have* $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X; q]$.

Proof. The graded Frobenius character of a $\mathbb{C}\mathfrak{S}_n$-module V is uniquely determined by the values $\langle e_\gamma, \mathrm{Frob}_q(V)\rangle$ for all $\gamma \vdash n$. From Proposition 3.C, we have

$$\mathrm{Hilb}(N_\gamma R_\mu) = \sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{ctype}(w) \trianglerighteq \mu \\ \mathrm{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \dots, \gamma_l + \cdots + \gamma_2\}}} q^{\mathrm{cocharge}(w)}$$

which is equal to $\langle e_\gamma, \mathrm{Frob}_q(R_\mu)\rangle$ by Equation (5.16). Thus $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X; q]$. □

Remark 5.6.6. Note that to avoid circular reasoning, we must ensure that we can show (5.9) is a basis of R_μ without using the fact that $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X;q]$. Using the fact that $\mathcal{C}_\mu \subset \mathcal{D}_\mu$ and that $\mathbf{x}^{\mathcal{D}_\mu}$ is linearly independent in R_μ [11, Corollary 3.14], we have that (5.9) is linearly independent in R_μ. Furthermore, using the ungraded character, we have

$$\dim(R_\mu) = \langle h_{1^n}, \mathrm{Frob}(R_\mu)\rangle = \langle h_{1^n}, h_\mu\rangle = \langle h_{1^n}, \tilde{H}_\mu[X;1]\rangle = |\mathcal{C}_\mu|.$$

Thus we can show our set is a basis using only the ungraded character of R_μ.

Note that Carlsson-Chou have a similar result on the antisymmetric component with respect to a certain Young subgroup. We define Sh_γ (resp. Sh'_γ) to be the set of all permutations such that $\{1,2,\ldots,\gamma_1\}, \{\gamma_1+1,\ldots,\gamma_1+\gamma_2\},\ldots,\{\gamma_1+\cdots+\gamma_{l-1}+1,\ldots,n\}$ appear in increasing (resp. decreasing) order.

Theorem 5.6.7 (Theorem 3.8.2, Carlsson-Chou [11])**.** *The set* $\{N_\gamma g_\sigma(\mathbf{x}) \mid g_\sigma(\mathbf{x}) = \mathbf{x}^\alpha$ *for* $\alpha \in \mathcal{D}_\mu, \sigma \in \mathrm{Sh}'_\gamma\}$ *is a basis of* $N_\gamma R_\mu$*.*

Similar to what we do above, they use their result to show that $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu$ by deriving a combinatorial formula for $\langle m_\gamma\rangle\, \omega \widetilde{H}_\mu$ using parking functions and shuffle combinatorics which matches their description of the basis.

In contrast to their result, ours arises naturally from the catabolizability formula (5.2) for $\widetilde{H}_\mu$. Our proof of the result is also independent of theirs: we do not relate their classification to ours when we show Proposition 3.C, though they use a similar argument to show that their basis is linearly independent. However, it is easy to see that the two sets coincide by using $x^{\mathrm{rev}(\mathrm{cc}(w))} = g_{\mathrm{rev}(\mathrm{rev}(w)^{-1})}(\mathbf{x})$ to show that the two conditions are equivalent:

$$\begin{aligned}\mathrm{Asc}(w) \subset \{\gamma_l, \gamma_l+\gamma_{l-1},\ldots,\gamma_l+\cdots+\gamma_2\} &\Leftrightarrow \mathrm{Des}(\mathrm{rev}(w)) \subset \{\gamma_1,\gamma_1+\gamma_2,\ldots,\gamma_1+\cdots+\gamma_{l-1}\}\\ &\Leftrightarrow \mathrm{rev}(w)^{-1} \in \mathrm{Sh}_\gamma\\ &\Leftrightarrow \mathrm{rev}(\mathrm{rev}(w)^{-1}) \in \mathrm{Sh}_\gamma\,.\end{aligned}$$

5.7 Connections to modified Macdonald polynomials and Garsia–Haiman modules

The *modified Macdonald polynomials* are a family of symmetric functions, indexed by partitions λ, with coefficients in $\mathbb{Q}(q,t)$. They are uniquely defined by the following conditions:

i $\widetilde{H}_\mu[X(1-q);q,t] = \sum_{\lambda \trianglerighteq \mu} c_{\lambda,\mu}(q,t)s_\lambda(\mathbf{x})$,

ii $\widetilde{H}_\mu[X(1-t);q,t] = \sum_{\lambda \trianglerighteq \mu'} d_{\lambda,\mu}(q,t)s_\lambda(\mathbf{x})$,

iii $\langle \widetilde{H}_\mu[X;q,t], s_\mu \rangle = 1$.

There is a combinatorial formula for $\widetilde{H}_\mu[X;q,t]$ due to Haglund–Haiman–Loehr [31] in terms of fillings of the shape μ. Let $\sigma : \mu \to \mathbb{Z}_+$ be a filling of μ with positive integers. Given a box $u \in \mu$, the *arm* (resp.*leg*) of u is the number of boxes strictly to the right (resp. above) u in the same row (resp. column). A *descent* of σ is a box u where $\sigma(u) > \sigma(v)$ for v directly below u in μ. Let $\mathrm{Des}(\sigma) = \{u \in \mu : u \text{ is a descent}\}$.

Definition 5.7.1. The *maj* statistic on a filling σ is defined to be

$$\mathrm{maj}(\sigma) := \sum_{u \in \mathrm{Des}(\sigma)} (\mathrm{leg}(u) + 1).$$

Definition 5.7.2. Two boxes u, v in μ are an *attacking pair* if one of the following two hold:

(i) u and v are in the same row of μ (u is to the left)

(ii) u and v are in consecutive rows, where u in the higher row and is strictly to the right of the v.

Diagrammatically, this looks like one of the following:

$$(i)\quad \begin{array}{|c|c|c|} \cline{1-2} & & \multicolumn{1}{c}{} \\ \hline u & & v \\ \hline & & \\ \hline \end{array} \qquad\qquad (ii)\quad \begin{array}{|c|c|c|} \cline{1-2} & & \multicolumn{1}{c}{} \\ \hline & & u \\ \hline v & & \\ \hline \end{array}\,.$$

Definition 5.7.3. An attacking pair (u, v) is an *inversion* in σ if $\sigma(u) > \sigma(v)$. The *inv* statistic on a filling σ is defined to be

$$\mathrm{inv}(\sigma) := |\{(u,v) : \sigma(u) > \sigma(v) \text{ is an inversion}\}| - \sum_{u \in \mathrm{Des}(\sigma)} \mathrm{arm}(u).$$

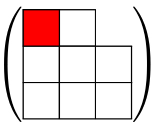
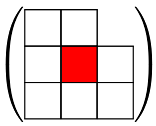
Theorem 5.7.4 (Haglund–Haiman–Loehr [31])**.**

$$\widetilde{H}_\mu[X;q,t] = \sum_{\sigma:\mu \to \mathbb{Z}+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma.$$

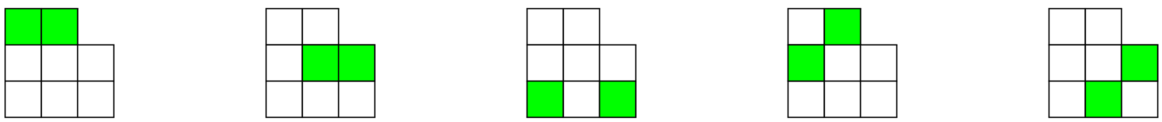
Example 5.7.5. Consider $\mu = (3, 2, 2)$ and filling σ of μ:

$$\sigma = \begin{array}{|c|c|c|} \hline 5 & 3 & \\ \cline{1-2} 2 & 4 & 2 \\ \hline 6 & 1 & 3 \\ \hline \end{array}.$$

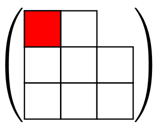
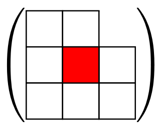
The descents are highlighted in red. From this, we can see that

$\text{maj}(\sigma) = \text{leg}$  $+ 1 + \text{leg}$  $+ 1 = 0 + 1 + 1 + 1 = 3.$

The inversion pairs are given by



Thus we have

$\text{inv}(\sigma) = 5 - \text{arm}$  $- \text{arm}$  $= 5 - 1 - 1 = 3.$

Thus this filling contributes $q^3t^3x_1x_2^2x_3^2x_4x_5x_6$ to $\widetilde{H}_{3,3,2}[X; q, t]$.

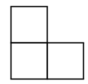
The fact that the modified Macdonald polynomials are Schur positive was proven by Haiman [32] by showing that $\widetilde{H}_\mu[X; q, t]$ is the bigraded Frobenius character of a $\mathbb{C}\mathfrak{S}_n$-module constructed in the following way. This construction is analogous to the harmonics presentation of R_n given in Section 3.3.

Given $\mu \vdash n$, we can think of each box in μ as corresponding to some point in $\mathbb{N} \times \mathbb{N}$, where the box in row i, column j corresponds to the point (i, j). Note that we index the rows and columns starting from 0.

For $\mu \vdash n$, let $\{(p_1, q_1), \ldots, (p_n, q_n)\}$ denote the points in $\mathbb{N}^2$ corresponding to the boxes in μ. Define $\Delta_\mu := \det(x_i^{p_j} y_i^{q_j})_{1 \leqslant i,j \leqslant n}$.

Definition 5.7.6. The *Garsia-Haiman module* $\mathcal{DR}_\mu$ is defined to be $\mathbb{C}[\partial \mathbf{x}, \partial \mathbf{y}] \cdot \Delta_\mu$. That is: it is the $\mathbb{C}$-span of Δ_μ, along with all partial derivatives of Δ_μ.

We can see that $\mathfrak{S}_n$ acts diagonally on the $\mathbf{x}, \mathbf{y}$ alphabets by permuting the indices within each alphabet. The Garsia-Haiman module was originally introduced in [19] to give a representation theoretic model for the modified Maconald polynomials. In particular, they showed that if $\dim \mathcal{DR}_\mu = n!$, then $\text{Frob}_{q,t}(\mathcal{DR}_\mu) = \widetilde{H}_\mu[X; q, t]$. Haiman proved that dimension was $n!$ using the geometry of Hilbert schemes [32].

Example 5.7.7. The partition $\mu =$  corresponds to the collection of points $\{(0,0),(1,0),(0,1)\}$. In this case, we have

$$\Delta_\mu = \det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_1 & y_1 \\ 1 & x_1 & y_1 \end{pmatrix} = x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2.$$

We can compute all the partials to see that

$$\begin{aligned} \mathcal{DR}_\mu &= \operatorname{span}\{\Delta_\mu, \partial/\partial y_1 \Delta_\mu, \partial/\partial y_2 \Delta_\mu, \partial/\partial x_1 \Delta_\mu, \partial/\partial x_2 \Delta_\mu, \partial/\partial x_1 \partial/\partial y_2 \Delta_\mu\} \\ &= \operatorname{span}\{\Delta_\mu, x_3 - x_2, x_1 - x_3, y_2 - y_3, y_3 - y_1, 1\}. \end{aligned}$$

We can see that $\mathrm{Frob}_{q,t}(\mathcal{DR}_\mu) = s_3 + (q+t)s_{2,1} + qt\, s_{1,1,1}$, which is equal to $\widetilde{H}_{2,1}[X;q,t]$.

However, though it is known that $\widetilde{H}_\mu[X;q,t]$ is Schur-positive, there is no combinatorial interpretation of the coefficients $\widetilde{K}_\lambda(q,t)$ that appear in the Schur expansion of $\widetilde{H}_\mu[X;q,t]$.

The Garsia-Procesi ring, and its graded Frobenius character $\widetilde{H}_\mu[X;q]$, are closely related to the modified Macdonald polynomials $\widetilde{H}_\mu[X;q,t]$. In particular, if we set $q = 0$, we recover the modified Hall-Littlewood polynomial $\widetilde{H}_\mu[X;t]$. On the level of the Garsia-Haiman module $\mathcal{DR}_\mu$, this is equivalent to setting $\mathbf{x} = 0$. From this, we can see that studying R_μ can be thought of as looking at a sliver of the Garsia-Haiman module $\mathcal{DR}_\mu$.

Our work in Chapter 5 gives a natural connection between a monomial basis of R_μ and the combinatorial formula for $\mathrm{Frob}_q(R_\mu)$ given by (5.2). A natural question to ask is if we can use similar methods to understand the Schur positivity of $\widetilde{H}_\mu[X;q,t]$ combinatorially. In particular, we can ask the following question.

Question 1. Can we extend a basis of R_μ to get a basis for $\mathcal{DR}_\mu$?

Note that this is not the most well-posed question at this moment, since we have no concrete description of $\mathcal{DR}_\mu$ as a polynomial quotient. In particular, in general there is no description for the generators of a $\mathfrak{S}_n$-invariant ideal $\mathcal{I}_\mu \subset \mathbb{C}[\mathbf{x},\mathbf{y}]$ such that $\mathcal{DR}_\mu \cong \mathbb{C}[\mathbf{x},\mathbf{y}]$ as for the coinvariant ring in Section 3.3.

However, we can still study connections between our work and known results for $\mathcal{DR}_\mu$. By looking at the $q = 0$ specialization of combinatorial formula of $\widetilde{H}_\mu[X;q,t]$ given in [31], we can recover the following.

Proposition 5.7.8 (Haglund–Haiman–Loehr [31]). *We have that*

$$\widetilde{H}_\mu[X;0,q] = \sum_{T\in \mathrm{SSYT}(\mu)} q^{\mathrm{cocharge}(T)} s_{\mathrm{sh}(T)} \tag{5.18}$$

where $\mathrm{SSYT}(\mu)$ *is the set of all semistandard Young tableaux of weight* μ.

The proof involves a bijection between all fillings $\sigma : \mu \to \mathbb{Z}^+$ where $\mathrm{inv}(\sigma) = 0$ and pairs (P, Q) where P, Q are SSYT of the same shape, the first tableau P has weight μ and the second tableau Q has weight equal to the weight of the filling σ. This bijection takes $\mathrm{maj}(\sigma)$ to $\mathrm{cocharge}(P)$. Composing this bijection with Proposition 4.1.9, we get

$$\{\sigma : \mu \to \mathbb{Z}_+ |\ \mathrm{inv}(\sigma) = 0\} \xrightarrow{\cong} \left\{ (P, Q) \;\middle|\; \begin{array}{l} P \in \mathrm{SYT}_n, Q \in \mathrm{SSYT} \\ \mathrm{sh}(P) = \mathrm{sh}(Q), \mathrm{ctype}(P) \trianglerighteq \mu \end{array} \right\}. \tag{5.19}$$

A first step in studying a potential basis of $\mathcal{DR}_\mu$ is look at $\langle \widetilde{H}_\mu[X; 1, t], h_{1^n} \rangle$, which is equivalent to looking at the coefficient of m_{1^n} of $\widetilde{H}_\mu[X; q, t]$. On the level of fillings, this means we want to study all standard fillings (i.e, where the numbers $1, \ldots, n$ appear exactly once.)

In particular, we want to use the bijection (5.19) to construct a bijection between standard fillings of μ and pairs of tableaux.

Lemma 5.7.9. *There exists a bijection*

$$\Psi : \{\sigma : \mu \to \mathbb{Z}_+ |\ \sigma \textit{ standard}\} \xrightarrow{\cong} \left\{ (P, Q) \;\middle|\; \begin{array}{l} P, Q \in \mathrm{SYT}_n, \mathrm{sh}(P) = \mathrm{sh}(Q) \\ \mathrm{ctype}(P) \trianglerighteq \mu \end{array} \right\} \times S_\mu.$$

where $\mathrm{maj}(\sigma) = \mathrm{cocharge}(P)$ *if* $\mathrm{inv}(\sigma) = 0$.

Proof. We can restrict the bijection (5.19) to get a bijection

$$\widetilde{\Psi} : \left\{ \sigma : \mu \to \mathbb{Z}_+ \;\middle|\; \begin{array}{l} \sigma \text{ standard} \\ \mathrm{inv}(\sigma) = 0 \end{array} \right\} \xrightarrow{\cong} \left\{ (P, Q) \;\middle|\; \begin{array}{l} P, Q \in \mathrm{SYT}_n, \mathrm{sh}(P) = \mathrm{sh}(Q) \\ \mathrm{ctype}(P) \trianglerighteq \mu \end{array} \right\} \tag{5.20}$$

that takes $\mathrm{maj}(\sigma)$ to $\mathrm{cocharge}(P)$. We can extend $\widetilde{\Psi}$ to define Ψ. Consider a standard filling σ of μ. If $\mathrm{inv}(\sigma) = 0$, let $\Psi(\sigma) = (\widetilde{\Psi}(\sigma), e)$ where e is the identity permutation in S_μ. If not, we know there is a unique way to reorder the entries within the rows of μ to get a filling $\tilde{\sigma}$ such that $\mathrm{inv}(\tilde{\sigma}) = 0$ (see Haglund [27]). Since the filling is standard, there exists a unique $\tau \in S_\mu$ such that $\tau\sigma = \tilde{\sigma}$, where a transposition $(i_1, i_2) \in S_{\mu_j}$ acts on a filling σ by switching boxes i_1 and i_2 in row j of σ. Set $\Psi(\sigma) = (\widetilde{\Psi}(\tilde{\sigma}), \tau)$. □

Remark 5.7.10. Note that the existence of the bijection between inv-less, standard fillings of μ and the elements of the basis given in Theorem **??** is hinted at in [11].

However, the authors say this bijection appears to be complicated. Our new description of the basis given in Theorem 5.A significantly simplifies this bijection, since the left-hand side of (5.20) is clearly in bijection with $\{w \in \mathfrak{S}_n \mid \text{ctype}(w) \trianglerighteq \mu\}$, which indexes the basis elements.

Question 2. Does there exist a bijection

$$\Phi : \{\sigma : \mu \to \mathbb{Z}^+ \mid \text{standard}\} \to D_{\mu'_1} \times D_{\mu'_2} \times \cdots \times D_{\mu'_l} \times \mathcal{OSP}_{\mu'} \tag{5.21}$$

such that the following hold?

(i) $\text{maj}(\sigma)$ is related to $\sum_{i=1}^{l} |\mathbf{a_i}|$ where $\Phi(\sigma) = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_l}, \pi)$.

(ii) $\Phi|_{\text{inv}\,\sigma=0}$ restricts to a bijection to the basis $\mathcal{C}_\mu$ given in Theorem 5.A.

Note that Chou has a similar result:

Proposition 5.7.11 (Chou [15])**.**

$$\sum_{(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_l},\pi)} t^{\sum|\mathbf{a}_i|} = \sum_{\sigma:\mu\to\mathbb{Z}^+,\textit{standard}} t^{\text{maj}(\sigma)}.$$

where $(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_l}, \pi) \in D_{\mu'_1} \times D_{\mu'_2} \times \cdots \times D_{\mu'_l} \times \mathcal{OSP}(\mu')$.

Chou proves this by constructing a weight preserving bijection, similar to Lemma 5.7.9 between the two sets. However, the bijection does not map to $\mathcal{D}_{\mu'}$ when restricted to standard fillings σ with $\text{inv}(\sigma) = 0$. Such bijection would give insight to how we can extend our construction to consider the full Garsia-Haiman module.

Chapter 6

Descent bases for Delta-Springer Rings

In this chapter, we look at the Δ-Springer modules $R_{n,\lambda,s}$, a family of generalized coinvariant rings indexed by tuples (n, λ, s). We give a descent monomial basis of Δ-Springer modules $R_{n,\lambda,s}$, first defined by Griffin in [25]. Our construction simultaneously generalizes the descent basis for the Garsia-Procesi module R_λ given in Chapter 5, as well as the descent basis for the generalized coinvariant algebras $R_{n,k}$ studied in [30]. This basis is deeply connected with a combinatorial object called battery-powered tableaux, introduced by Gillespie–Griffin [26]. We highlight the representation theoretic properties of this monomial basis by using it to give a direct combinatorial proof of the graded Frobenius character of $R_{n,\lambda,s}$ in terms of battery-powered tableaux, a fact which had only the geometric proof of Gillespie-Griffin. This chapter is based on material in [16] and which is joint work with Raymond Chou.

6.1 Introduction

In Chapter 5, we discussed the Garsia-Procesi ring R_μ, which generalizes the story of the classical Type A coinvariant ring R_n.

Another well-known generalization of R_n comes from the famous *Delta conjecture* in symmetric function theory. Haglund-Rhoades-Shimozono [30] constructed a partial representation theoretic model for $\Delta'_{e_{k-1}} e_n$ (up to a twist) where Δ' is a certain operator on q, t-symmetric functions $\Lambda_{q,t}$, first defined by F. Bergeron-Garsia-Haiman-Tesler [3]. The model depends on two parameters $k \leqslant n$. We first define the ideal

$$I_{n,k} = (x_i^k : 1 \leqslant i \leqslant n) + (e_n(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n))$$

and then set $R_{n,k} := \mathbb{C}[\mathbf{x}_n]/I_{n,k}$. It is clear that when we specialize to $k = n$ we recover the coinvariant ring R_n.

Analogous to how R_λ is closely related to $\mathcal{OSP}_\lambda$ (Theorem 5.6.1), we have that $R_{n,k}$ is closely related to the set $\mathcal{OSP}_{n,k}$, the collection of all ordered set partitions of $[n]$ consisting of k nonempty sets. In particular, we have that $R_{n,k} \cong_{\mathfrak{S}_n} \mathbb{C}[\mathcal{OSP}_{n,k}]$ as ungrades $\mathfrak{S}_n$-modules. There also exist generalized Artin and descent bases for $R_{n,k}$ corresponding to (co)inv and (co)maj on $\mathcal{OSP}_{n,k}$ due to Haglund–Rhoades–Shimozono [30]. Here we recall the definition for the descent basis.

Theorem 6.1.1 (Haglund–Rhoades–Shimozono [30])**.** *The following set $\mathcal{GS}_{n,k}$ descends to a basis of $R_{n,k}$. Furthermore, they are the non-leading terms with respect to the* des-*order.*

$$\mathcal{GS}_{n,k} = \left\{ g_\sigma(\mathbf{x}) \prod_{j=1}^{n-k} (x_{\sigma_1} \ldots x_{\sigma_j})^{i_j} \mid \sigma \in \mathfrak{S}_n, 0 \leqslant i_1 + \ldots i_{n-k} < k - \mathrm{des}(\sigma) \right\} \tag{6.1}$$

.

The ring $R_{n,k}$ is still a $\mathbb{C}\mathfrak{S}_n$-module under the action that permutes the variables. As an ungraded $\mathfrak{S}_n$-representation, we have that $R_{n,k} \simeq_{\mathfrak{S}_n} \mathbb{C}[\mathcal{OSP}_{n,k}]$. The authors in [30] computed the graded Frobenius character of $R_{n,k}$:

$$\mathrm{Frob}_q(R_{n,k}) = \mathrm{rev}_q \circ \omega\big(\Delta'_{e_{k-1}} e_n\big|_{t=0}\big).$$

This gives a representation theoretic model for the combinatorial expression given by the Delta conjecture at $t = 0$ (up to a sign and degree reversing twist). There is also a concrete presentation of $R_{n,k}$ in terms of cohomology rings, given by Pawlowski-Rhoades [44].

We can combine these two generalizations to give a broader family of rings which contains both R_λ and $R_{n,k}$. Given a tuple (n, λ, s), where $\lambda \vdash k \leqslant n, s \geqslant \ell(\lambda)$. Define

$$I_{n,\lambda,s} := \langle x_i^s : 1 \leqslant i \leqslant n \rangle + \langle e_d(S) : d > |S| - p_{n-|S|}(\mu) \rangle$$

and set $R_{n,\lambda,s} := \mathbb{C}[\mathbf{x}_n]/I_{n,\lambda,s}$. We refer to $R_{n,\lambda,s}$ as the Δ-*Springer module*, due to the specializations $R_{k,\lambda,\ell(\lambda)} = R_\lambda, R_{n,(1^k),k} = R_{n,k}$. In particular, we have that $R_{k,1^k,k}$ is exactly the coinvariant ring R_k. Originally introduced by Griffin [25], they correspond to $\mathcal{OSP}_{n,\lambda,s}$, or the collection of ordered set partitions of $[n]$ into s blocks, where for $i \leqslant \ell(\lambda)$ we have the additional condition that block i has at least λ_i elements. Note that some blocks may be empty.

Theorem 6.1.2 (Griffin [25, Theorem 3.21]). *$R_{n,\lambda,s} \simeq_{\mathfrak{S}_n} \mathbb{C}\mathcal{OSP}_{n,\lambda,s}$ as ungraded $\mathfrak{S}_n$-representations.*

Geometrically, they can be thought of the cohomology ring of *Delta-Springer fibers* $Y_{n,\lambda,s}$ which consists of partial flags $F_\bullet \in \mathcal{F}_{1^n}(\mathbb{C}^K)$ preserved by a certain nilpotent operator depending on λ, and $K = s(n-k)+k$ [26]. There is a generalized Artin basis for $R_{n,\lambda,s}$ due to Griffin [25].

In this chapter, we construct an extension $\mathcal{D}_{n,\lambda,s}$ of the Garsia-Stanton basis for $R_{n,\lambda,s}$, which generalizes the descent (cocharge) basis of R_λ from Chapter 5, as well as the descent basis of $R_{n,k}$ due to Haglund–Rhoades–Shimozono.

Theorem 6.A. *The set of monomials $\mathcal{B}_{n,\lambda,s} := \{\mathbf{x}^{\mathbf{b}} : \mathbf{b} \in \mathcal{D}_{n,\lambda,s}\}$ descend to a monomial basis of $R_{n,\lambda,s}$.*

The proof involves using two different presentations of $\mathcal{D}_{n,\lambda,s}$, one in terms of shuffles, analogous to the Carlsson–Chou construction of the descent basis of R_λ, and the other in terms of cocharge, analogous to the construction given in Chapter 5.

Furthermore, we can use this construction to obtain a new combinatorial formula for the Schur expansion of the graded Frobenius character $\mathrm{Frob}_q(R_{n,\lambda,s})$.

Theorem 6.B. *We have*

$$\mathrm{Frob}_q(R_{n,\lambda,s}) = \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} q^{\mathrm{cocharge}(S)+|\mu|} s_{\mathrm{sh}(S)}. \tag{6.2}$$

where the indexing set $\Sigma_{n,\lambda,s}$ consists of pairs (S,μ) where S is a standard tableau, μ is a partition that satisfy $\mathrm{ctype}(S|_k) \trianglerighteq \lambda$, $\mu \subseteq (n-k) \times (s - \mathrm{des}(S) - 1)$.

This construction is naturally connected to the theory of *battery-powered tableaux*, a combinatorial object originally introduced by Gillespie–Griffin [23]. Gillespie–Griffin have a Schur expansion formula for $\mathrm{Frob}_q(R_{n,\lambda,s})$ in terms of battery-powered tableaux. Though the formula is combinatorial, the motivation and proof of it were geometric. Furthermore, they were unable to connect the formula back to the known combinatorics of $R_{n,\lambda,s}$. Our work bridges this gap: in particular, we show that Theorem 6.B is equivalent to their formula through a cocharge/shape preserving bijection between $\Sigma_{n,\lambda,s}$ and the set of corresponding battery-powered tableaux.

Corollary 6.C. *There is a weight-preserving bijection between the sets $\Sigma_{n,\lambda,s}$ and the battery-powered tableaux $\mathcal{T}^+_{n,\lambda,s}$.*

This chapter is organized as follows. In Section 6.2, we recall the definition of battery-powered tableaux and express the dimension of $R_{n,\lambda,s}$ using these objects. In Section 6.3 we reframe existing results of Haglund–Rhoades–Shimozono using the language of tableaux and cocharge to match our work in Chapter 5. In Section 6.4, we introduce two different constructions of our descent basis of $R_{n,\lambda,s}$. We then show our set has the correct size through a series of combinatorial bijections. In Section 6.5 we prove linear independence of our monomials in $R_{n,\lambda,s}$, which proves Theorem 6.A. In Section 6.6, we use the descent basis to give a combinatorial formula for the graded Frobenius character of $R_{n,\lambda,s}$. We highlight connections between known graded Frobenius characters and specializations of ours, as well as show that our formula coincides with the battery-powered tableaux formula of Gillespie–Griffin.

6.2 Battery-powered tableaux

We first give an expression for $\dim(R_{n,\lambda,s})$. We first use Theorem 6.1.2 to get an expression for the ungraded Frobenius character of $R_{n,\lambda,s}$.

Corollary 6.2.1 (Griffin [25, Theorem 3.21])**.** $\mathrm{Frob}(R_{n,\lambda,s}) = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_s)\models n,\\ \lambda\subset\alpha}} h_\alpha.$

Now we rewrite Corollary 6.2.1 by expanding h_α into Schur functions using the set $\mathrm{SSYT}(\alpha)$, which denotes the collection of all semistandard tableaux of weight α:

$$\mathrm{Frob}(R_{n,\lambda,s}) = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_s)\models n,\\ \lambda\subset\alpha}} \sum_{T\in\mathrm{SSYT}(\alpha)} s_{\mathrm{sh}(T)}. \tag{6.3}$$

Using (6.3) , we can easily get an expression for $\dim(R_{n,\lambda,s})$ in terms of semistandard Young tableaux, by replacing $s_{\mathrm{sh}(T)}$ with $|\,\mathrm{SYT}(\mathrm{sh}(T))|$.

Corollary 6.2.2. $\dim(R_{n,\lambda,s}) = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_s)\models n,\\ \lambda\subset\alpha}} \sum_{T\in\mathrm{SSYT}(\alpha)} |\,\mathrm{SYT}(\mathrm{sh}(T))|$

We can reformulate Corollary 6.2.2 using a combinatorial object called *battery-powered tableaux*, originally defined by Gillespie–Griffin in [23].

Definition 6.2.3. Consider (n, λ, s) where $\lambda \vdash k \leqslant n$, $s \geqslant \ell(\lambda)$. A *battery-powered tableau* of parameters (n, λ, s) consists of a pair $T = (D, B)$ of semistandard Young tableaux, where B is of shape $(n-k)\times(s-1)$ and the total content of (D, B) is given by $\Lambda_{n,\lambda,s} := (n-k)\times s + \lambda = (\lambda_1 + (n-k), \lambda_2 + n - k, \dots, \lambda_s + (n-k))$ where we assume $\lambda_i = 0$ for $i \geqslant \ell(\lambda)$.

We refer to D, B as the *device* and *battery* of T. Note that a battery-powered tableaux is uniquely defined by the device: that is, once we know D, there is at most one B so that the total tableau has the correct weight. Let $\mathcal{T}^{+}_{n,\lambda,s}$ denote the set of all battery-powered tableaux of parameters (n, λ, s).

Example 6.2.4. Let $(n, \lambda, s) = (4, \ydiagram{1,2}, 3)$. We have that $\Lambda_{n,\lambda,s} = (3, 2, 1)$ and the shape of B is $\ydiagram{1,1} = 1 \times 2$. We have $T = \begin{ytableau} 2 \\ 1 & 1 & 2 \\ \none & \none & \none & 3 \\ \none & \none & \none & 1 \end{ytableau} \in \mathcal{T}^{+}_{n,\lambda,s}$, where the device D is $\ytableaushort{2,112}$ and the battery is $\ytableaushort{3,1}$.

Lemma 6.2.5. *The map* $\mathcal{T}^{+}_{n,\lambda,s} \to \bigcup_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\models n \\ \lambda\subset\alpha}} \mathrm{SSYT}(\alpha)$ *that takes* $(D, B) \mapsto D$ *is a bijection.*

Proof. Note that for any battery-powered tableau (D, B), it is clear that $\mathrm{weight}(D) = \alpha$ for composition $(\alpha_1 \ldots, \alpha_s) \models n$ such that $\lambda \subset \alpha$. We can also see that for any $D \in \mathrm{SSYT}(\alpha)$, there is exactly one way to fill in the rectangle shape B so that the skew-shape (D, B) has total weight $\Lambda_{n,\lambda,s}$. $\square$

By applying the bijection in Lemma 6.2.5 to Corollary 6.2.2, we get a new expression for $\dim(R_{n,\lambda,s})$ in terms of these battery-powered tableaux.

Corollary 6.2.6. $\dim(R_{n,\lambda,s}) = \sum_{(D,B)\in\mathcal{T}^{+}_{n,\lambda,s}} |\,\mathrm{SYT}(\mathrm{sh}(D))|.$

We will use Corollary 6.2.6 when showing that our set of monomials has the right count to be a basis of $R_{n,\lambda,s}$. The battery-powered tableaux were originally introduced in [23] to give a combinatorial formula for $\mathrm{Frob}_q(R_{n,\lambda,s})$:

Theorem 6.2.7 (Gillespie–Griffin [23, Theorem 1.6])**.**

$$\mathrm{Frob}_q(R_{n,\lambda,s}) = \frac{1}{q^{\binom{s-1}{2}(n-k)}} \sum_{T=(D,B)\in\mathcal{T}^{+}_{n,\lambda,s}} q^{\mathrm{cocharge}(T)} s_{\mathrm{sh}(D)}. \tag{6.4}$$

Example 6.2.8. For $(n, \lambda, s) = (4, \ydiagram{1,2}, 3)$, the set of battery-powered tableaux are:

$$\begin{array}{ccccc} 1&1&2&3& \\ &&&&2 \\ &&&&1 \end{array}, \quad \begin{array}{ccccc} 1&1&2&2& \\ &&&&3 \\ &&&&1 \end{array}, \quad \begin{array}{ccccc} 1&1&1&2& \\ &&&&3 \\ &&&&2 \end{array},$$

$$\begin{array}{cccc} 3&&& \\ 1&1&2& \\ &&&2 \\ &&&1 \end{array}, \quad \begin{array}{cccc} 2&&& \\ 1&1&2& \\ &&&3 \\ &&&1 \end{array}, \quad \begin{array}{cccc} 2&&& \\ 1&1&1& \\ &&&3 \\ &&&2 \end{array},$$

$$\begin{array}{ccc} 2&3& \\ 1&1& \\ &&2 \\ &&1 \end{array}, \quad \begin{array}{ccc} 2&2& \\ 1&1& \\ &&3 \\ &&1 \end{array} \quad \begin{array}{cccc} 2&&& \\ 1&1&3& \\ &&&2 \\ &&&1 \end{array}, \quad \begin{array}{ccc} 3&& \\ 2&& \\ 1&1& \\ &&2 \\ &&1 \end{array}.$$

From this, we know that $\mathrm{Frob}_q(R_{n,\lambda,s}) = (1+q+q^2)s_4 + (q+2q^2+q^3)s_{3,1} + (q^2+q^3)s_{2,2} + q^3 s_{2,1,1}$.

6.3 Tableaux Construction for $R_{n,k}$

In this section, we reformulate the generalized descent basis of $R_{n,k}$ given in Theorem 6.1.1 using a generalization of cocharge to match Theorem 5.A (for R_λ).

First, we define the following indexing set:

$$\Sigma_{n,k} = \{(S,\mu) : S \in \mathrm{SYT}_n, \mu \subseteq (n-k) \times (k - \mathrm{des}(S) - 1)\}. \tag{6.5}$$

Now, we define a map on the set $\{(w,\mu) \mid w \in S_n, (P(w),\mu) \in \Sigma_{n,k}\}$ for any $k \leqslant n$ which gives us a *boosted cocharge word*: that is, it will be the result of increasing some values in $\mathrm{cc}(w)$ according to the partition μ.

Given (w,μ), let $\mathrm{cc}(w,\mu)$ be the word of length n defined by

$$\mathrm{cc}(w,\mu)_i = \begin{cases} \mathrm{cc}(w)_i + \mu_{n-w_i+1} & \text{if } w_i \in \{k+1,\dots,n\} \\ \mathrm{cc}(w)_i & \text{otherwise} \end{cases} \tag{6.6}$$

In other words, we can think of $\mathrm{cc}(w,\mu)$ as the word we get from "boosting" the cocharge of entries $k+1,\dots,n$ in w according to μ.

Example 6.3.1. Consider $(S,\mu) = \left(\begin{array}{ccccc} 6&&&& \\ 4&7&8&& \\ 1&2&3&5&9 \end{array}, (3,1,1) \right) \in \Sigma_{9,6}$. Then $\mathrm{cc}(\mathrm{rw}(S)) = 212200012$ and $\mathrm{cc}(w,\mu) = 213300015$.

We can also define this notion of boosted cocharge directly on $\Sigma_{n,k}$. Given $(S, \mu) \in \Sigma_{n,k}$, let $\mathrm{cc}(w, \mu)$ be a semistandard tableaux of the same shape obtained by taking $\mathrm{cc}(S)$ and adding μ_{n-j+1} to $\mathrm{cc}(S)_j$ for $j \in \{k+1, \ldots, n\}$. We have $\mathrm{cc}(\mathrm{rw}(S), \mu) = \mathrm{rw}(\mathrm{cc}(S, \mu))$ for any $(S, \mu) \in \Sigma_{n,k}$. We can express the set $\mathcal{GS}_{n,k}$ in Theorem 6.1.1 using these boosted cocharge words $\mathrm{cc}(w, \mu)$.

Proposition 6.3.2. *We have that*

$$\mathcal{GS}_{n,k} = \{\mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w,\mu))} : w \in \mathfrak{S}_n, (P(w), \mu) \in \Sigma_{n,k}\}.$$

Proof. Consider (w, μ) where $(P(w), \mu) \in \Sigma_{n,k}$. Note that we set $i_{n-k} = \mu_{n-k}$, $i_j = \mu_j - \mu_{j+1}$ for $j < n-k$. We can see that for $\sigma = \mathrm{rev}((\mathrm{rev}(w))^{-1})$,we have

$$\begin{aligned}
\prod_{j=1}^{n-k} (x_{\sigma_1} \ldots x_{\sigma_j})^{i_j} &= \prod_{j=1}^{n-k} x_{\sigma_j}^{i_j + \cdots + i_{n-k}} \\
&= \prod_{j=1}^{n-k} x_{n-w^{-1}_{n-j+1}+1}^{i_j + \cdots + i_{n-k}}, \\
&= \prod_{j=1}^{n-k} x_{n-w^{-1}_{n-j+1}+1}^{\mu_j}.
\end{aligned}$$

Thus $g_\sigma(\mathbf{x}) \prod_{j=1}^{n-k} (x_{\sigma_1} \ldots x_{\sigma_j})^{i_j} = \mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w,\mu))}$. Furthermore, by definition of $\Sigma_{n,k}$, we know that $\mathrm{des}(P(w)) < k$. From Lemma 2.6.7, this is equivalent to saying the maximal cocharge value in $\mathrm{cc}(w)$ is at most $k-1$. In other words, we have $\mathrm{des}(w^{-1}) < k$. This condition is equivalent to the condition $\mathrm{des}(\sigma) < k$ on σ, since

$$\mathrm{des}(\sigma) < k \iff \mathrm{des}(\pi^{-1}) \geqslant n-k+1 \iff \mathrm{des}(w^{-1}) < k.$$

where $\pi = (\mathrm{rev}(\sigma))^{-1}$. □

6.4 Construction of basis and Combinatorial bijections

We now introduce the construction of our descent basis of $R_{n,\lambda,s}$. First, we give a definition which is a generalization of (5.7). Note that we define $[i]_0 := \{0, 1, \ldots, i\}$ throughout:

Definition 6.4.1. For any $\lambda \vdash k \leqslant n$, define $\mathcal{OSP}_{n,\lambda'}$ to be the collection of ordered set partitions $(A_1| \dots | A_{\lambda_1} \mid B_1 \mid \dots \mid B_{n-k})$ of $[n]$ where $|A_i| = \lambda'_i$ and $|B_j| = 1$ for any i, j. We denote the ordered set partitions as $(\mathbf{A}_\lambda|\mathbf{B})$, and denote $\mathbf{B} = \bigsqcup_{i=1}^{n-k} B_i$. We define the n, λ, s-*generalized descent words* to be

$$\mathcal{D}_{n,\lambda,s} := \{\mathbf{a} : \mathbf{a}|_{A_i} \in \mathcal{D}_{\lambda'_i}, \mathbf{a}|_{B_j} \in [s-1]_0 \text{ for some } (\mathbf{A}_\lambda|\mathbf{B}) \in \mathcal{OSP}_{n,\lambda'}\}. \tag{6.7}$$

Example 6.4.2. Consider $(n, \lambda, s) = (4, (2, 1), 3)$. Recall that $\mathcal{D}_1 = \{0\}, \mathcal{D}_2 = \{00, 01\}$. For $(13|4|2) \in \mathcal{OSP}_{n,\lambda'}$, we can see that

$$\{\mathbf{a} : \mathbf{a}|_{1,3} \in \mathcal{D}_2, \mathbf{a}|_4 \in \mathcal{D}_1, \mathbf{a}|_2 \in \{0, 1, 2\}\} = \{0000, 0100, 0200, 0010, 0110, 0210\}.$$

For example, we have that $0110|_{13} = 01, 0110|_4 = 2, 0110|_4 = 1$. Using this, we have that

$$\begin{aligned}\mathcal{D}_{n,\lambda,s} = \{&0000, 1000, 0100, 0010, 0001, 2000, 0200, 0020, 0002,\\ &0101, 0110, 0011, 1001, 1010, 0102, 0120, 0012, 2001, 2010, 0201, 0210\}.\end{aligned}$$

Using this, we can explicitly state our first theorem.

Theorem 6.A. *The set of monomials $\mathcal{B}_{n,\lambda,s} := \{\mathbf{x}^{\mathbf{b}} : \mathbf{b} \in \mathcal{D}_{n,\lambda,s}\}$ descend to a monomial basis of $R_{n,\lambda,s}$.*

We will show Theorem 6.A by first checking that $|\mathcal{D}_{n,\lambda,s}| = \dim(R_{n,\lambda,s})$ and then showing that the monomials are linearly independent in $R_{n,\lambda,s}$.

However, with the current presentation of $\mathcal{D}_{n,\lambda,s}$ we run into the same issue we mentioned in Remark 5.2.4. Though the construction given in Definition 6.4.1 is straightforward, it is difficult to directly count the cardinality of $\mathcal{D}_{n,\lambda,s}$. In particular, we can have multiple ordered set partitions that satisfy the condition in (6.7) for the same word. For $\mathbf{z} = 0110$ in the example above, we have $(12|4|3)$ and $(13|4|2)$ both work.

To get around this, we introduce a second construction of $\mathcal{D}_{n,\lambda,s}$ using tableaux and cocharge. We first extend (6.5) to consider tuples(n, λ, s). Now, we extend our definition of $\Sigma_{n,k}$ to consider tuples (n, λ, s).

Definition 6.4.3. Given (n, λ, s), define the set $\Sigma_{n,\lambda,s}$ to be

$$\Sigma_{n,\lambda,s} := \{(S, \mu) : S \in \mathrm{SYT}_n, \mathrm{ctype}(S|_k) \trianglerighteq \lambda \ \in \Sigma_\lambda, \mu \subseteq (n-k) \times (s - \mathrm{des}(S) - 1)\}. \tag{6.8}$$

Remark 6.4.4. Note that we do not consider S where $s - \mathrm{des}(S) < 0$.

Example 6.4.5. Consider $(n, \lambda, s) = (4, (2, 1), 3)$. The possible pairings (S, μ) are recorded in the table below:

S	$s - \mathrm{des}(S)$	possible $\mu \subseteq 1 \times (2 - \mathrm{des}(S))$
$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}$	3	$\emptyset, \ydiagram{1}, \ydiagram{2}$
$\begin{ytableau} 4 \\ 1 & 2 & 3 \end{ytableau}$	2	$\emptyset, \ydiagram{1}$
$\begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau}$	2	$\emptyset, \ydiagram{1}$
$\begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau}$	2	$\emptyset, \ydiagram{1}$
$\begin{ytableau} 4 \\ 3 \\ 1 & 2 \end{ytableau}$	1	$\emptyset.$

This indexing set $\Sigma_{n,\lambda,s}$ specializes naturally to the indexing sets of R_λ, given in Theorem 5.A, and $R_{n,k}$ given in Proposition 6.3.2: it is immediate that $\Sigma_{k,\lambda,\ell(\lambda)} = \Sigma_\lambda := \{S \in \mathrm{SYT}_k : \mathrm{ctype}(S) \trianglerighteq \lambda\}$, and $\Sigma_{n,1^k,k} = \Sigma_{n,k}$.

Now, we will show that $|\mathcal{D}_{n,\lambda,s}| = \dim R_{n,\lambda,s}$ using $\Sigma_{n,\lambda,s}$. This is analogous to how the construction in Chapter 5 introduced a way to directly count the monomials constructed in [11].

We will first show that $\mathcal{D}_{n,\lambda,s}$ is in bijection with the boosted cocharge words arising from $\Sigma_{n,\lambda,s}$ (Proposition 6.4.6). We will then show that there is a bijection $\Sigma_{n,\lambda,s} \to \mathcal{T}^+_{n,\lambda,s}$ (Proposition 6.4.13) that preserves properties of the shape and cocharge of the tableaux. Combining these two results will give us that $|\mathcal{D}_{n,\lambda,s}| = \dim(R_{n,\lambda,s})$.

We first construct a bijection between pairs (w, μ) where $(P(w), \mu) \in \Sigma_{n,\lambda,s}$ and our indexing set $\mathcal{D}_{n,\lambda,s}$, using the notion of boosted cocharge words.

Proposition 6.4.6. *There exists a bijection* $\{(w, \mu) \mid w \in \mathfrak{S}_n, (P(w), \mu) \in \Sigma_{n,\lambda,s}\} \to \mathcal{D}_{n,\lambda,s}$.

Proof. Consider the map that takes $(w, \mu) \mapsto \mathrm{rev}(\mathrm{cc}(w, \mu))$. We claim that this gives such a bijection. We first check that $\mathrm{rev}(\mathrm{cc}(w, \mu))) \in \mathcal{D}_{n,\lambda,s}$. Note that it suffices to construct $(\mathbf{A}_\lambda|\mathbf{B}) \in \mathcal{OSP}_{n,\lambda'}$ such that $\mathrm{rev}(\mathrm{cc}(w, \mu)|_{A_i}) \in \mathcal{D}_{\lambda'_i}$ and $\mathrm{rev}(\mathrm{cc}(w, \mu)|_{B_j}) \in [s-1]_0$.

First, note that $\mathrm{cc}(w, \mu)_i = \mathrm{cc}(w)_i$ if $w_i \in [k]$. Let $U = \{i \mid w_i \in [k]\}$. We can see that $\mathrm{cc}(w, \mu)|_U = \mathrm{cc}(w|_k)$. Since $\mathrm{ctype}(w|_k) \trianglerighteq \lambda$, by Theorem 5.A, we know $\mathrm{rev}(\mathrm{cc}(w|_k)) \in \mathcal{D}_\lambda$. Thus, by the definition of $\mathcal{D}_\lambda$, there exists some $\mathbf{A}_\lambda \in \mathcal{OSP}_{\lambda'}(U)$, where $\mathbf{A}_\lambda$ is an ordered set partition of the set U such that $\mathrm{rev}(\mathrm{cc}(w, \mu)|_{A_i}) \in \mathcal{D}_{\lambda'_i}$.

Furthermore, for i such that $w_i \in [k+1,n]$ by Lemma 2.6.7 we have the following relation:

$$0 \leqslant \mathrm{cc}(w,\mu)_i = \mathrm{cc}(w)_i + \mu_{n-w_i+1} < \mathrm{cc}(w)_i + s - \mathrm{des}(P(w)) \leqslant s. \tag{6.9}$$

Thus $\mathrm{cc}(w,\mu)_i \leqslant s-1$. Let $B_j = \{i \mid w_i = k+j\}$. Then $\mathrm{rev}(\mathrm{cc}(w,\mu)|_{B_j}) \in [s-1]_0$ by (6.9). Thus $\mathrm{rev}(\mathrm{cc}(w,\mu)) \in \mathcal{D}_{n,\lambda,s}$.

Now we construct an inverse map. Let $\mathrm{rev}(\mathbf{u}) \in \mathcal{D}_{n,\lambda,s}$. We know there exists $(\mathbf{A}_\lambda|\mathbf{B}) \in \mathcal{OSP}_{n,\lambda'}$ such that $\mathrm{rev}(\mathbf{u}|_{A_i}) \in \mathcal{D}_{\lambda'_i}, \mathbf{u}|_{B_j} \in [s-1]_0$.

Note that we can put a total order (the *reading order*) on the set $[n]$ in the following way:

$$i <_{\mathbf{u}} j \Leftrightarrow \begin{cases} u_i < u_j \\ u_i = u_j, i < j. \end{cases} \tag{6.10}$$

Let $U \subset [n]$ be the subset consisting of the smallest k entries with respect to $<_{\mathbf{u}}$. We want to construct a new $(\mathbf{A}'_\lambda|\mathbf{B}') \in \mathcal{OSP}_{n,\lambda'}$ such that $\mathbf{A'}_\lambda$ is an ordered set partition of U and $\mathrm{rev}(\mathbf{u}|_{A'_i}) \in \mathcal{D}_{\lambda'_i}, \mathbf{u}|_{B'_j} \in [s-1]_0$ still holds.

We do this by modifying the original ordered set partition $(\mathbf{A}_\lambda|\mathbf{B})$. Let m be the largest entry with respect to $<_{\mathbf{u}}$. If $m \notin A_i$ for all i, we do not modify the ordered set partition. Otherwise, if $m \in A_i$ for some i, there exists a c such that $c \in U$ but $B_j = \{c\}$ for some j.

Since $c <_{\mathbf{u}} m$, we know that $u_c \leqslant u_m$ which implies $\mathrm{rev}(\mathbf{u}|_{A_i \cup \{c\}}) \in \mathcal{D}_{\lambda'_i+1}$ from Lemma 5.3.6(1). Furthermore, we know that u_m is the rightmost, largest entry in $\mathbf{u}|_{A_i \cup \{c\}}$. Thus, by Lemma 5.3.6 (2), we have that $\mathrm{rev}(\mathbf{u}|_{(A_i \cup \{c\})\backslash\{m\}}) \in \mathcal{D}_{\lambda'_i}$.

Hence, if we let $(\mathbf{A}'_\lambda|\mathbf{B}')$ be the ordered set partition we get by setting $A'_i = (A_i \cup \{c\})\backslash\{m\}, B'_j = \{m\}$ without changing the rest of $(\mathbf{A}_\lambda|\mathbf{B})$, we still have $\mathrm{rev}(\mathbf{u}|_{A'_i}) \in \mathcal{D}_{\lambda'_i}$. Furthermore, since $u_m \leqslant \lambda'_i < s$, we still have $\mathbf{u}|_{B_j} \in [s-1]_0$ for all choices of j.

We can repeat this process for each entry not in U, until we get $(\mathbf{A}'_\lambda|\mathbf{B}')$ where $\mathbf{A}'_\lambda$ is an ordered set partition of U.

Now, we construct (w,μ) such that $\mathrm{cc}(w,\mu) = \mathbf{u}$. If i is the dth smallest entry with respect to $<_{\mathbf{u}}$, we set $w_i = d$. By construction, we have that $\mathrm{cc}(w|_k) = \mathrm{rev}(\mathbf{u}|_{\mathbf{A}'_\lambda})$, thus $P(w|_k) \in \Sigma_\lambda$ by Proposition 2.2.1 (a).

Set $\mu_{n-k+1} = 0$. For any $j \leqslant n-k$, let $a = w^{-1}_{n-j}$, $b = w^{-1}_{n-j+1}$. Note that a, b are the $(n-j)$th, $(n-j+1)$th smallest letters with respect to $<_{\mathbf{u}}$. We can define μ_j to be

$$\mu_j := \begin{cases} \mu_{j+1} + u_b - u_a & \text{if } a < b \\ \mu_{j+1} + u_b - u_a - 1 & \text{if } a > b. \end{cases} \tag{6.11}$$

The first (resp. second) case is when $(n-j)$ appears to the left (resp.right) of $(n-j+1)$ in w. Note that in either case, we have $\mu_j \geqslant \mu_{j+1} \geqslant 0$. We can also see that $\mathrm{cc}(w,\mu) = \mathbf{u}$ and $\mathbf{u}_m = \mathrm{cc}(w)_m + \mu_1 < s$ implies $\mu_1 < s - \mathrm{des}(S)$, thus $\mu \subseteq (n-k) \times (s - \mathrm{des}(S) - 1)$. □

Thus we have a bijection between our boosted cocharge words coming from $\Sigma_{n,\lambda,s}$ and $\mathcal{D}_{n,\lambda,s}$. Now, we want to relate the set $\Sigma_{n,\lambda,s}$ to the set of battery-powered tableaux $\mathcal{T}^+_{n,\lambda,s}$. First, we define the set of *standardized* battery-powered tableaux, which is just the image of the battery-powered tableaux under the standardization map that preserves cocharge and shape, where std is the map given in Section 2.7.

Definition 6.4.7. Let $\mathrm{std}(\mathcal{T}^+_{n,\lambda,s})$ be the image of the set $\mathcal{T}^+_{n,\lambda,s}$ under Lascoux standardization.

For $T \in \mathcal{T}^+_{n,\lambda,s}$, let $\mathrm{std}(T)|_D, \mathrm{std}(T)|_B$ denote the device and battery part of $\mathrm{std}(T)$. Let $\mathrm{cc}(\mathrm{std}(T))|_D, \mathrm{cc}(\mathrm{std}(T))|_B$ denote the device and battery part of $\mathrm{cc}(\mathrm{std}(T))$.

Example 6.4.8. Continuing with the battery powered tableau T from Example 6.2.4, we have $\mathrm{std}(T) = \begin{array}{cccc} 4 & & & \\ 1 & 2 & 5 & \\ & & & 6 \\ & & & 3 \end{array}$, where $\mathrm{std}(T)|_D = \begin{array}{ccc} 4 & & \\ 1 & 2 & 5 \end{array}$, $\mathrm{std}(T)|_B = \begin{array}{c} 6 \\ 3 \end{array}$.

Lemma 6.4.9. *The tableau* $\mathrm{cc}(\mathrm{std}(T))|_B$ *is the superstandard tableau of shape* $(n-k) \times (s-1)$*: that is, the semistandard tableau with row i filled with* $(i-1)$*.*

Proof. Assume otherwise. Note that since we know $\mathrm{ctype}(\mathrm{std}(T)) \trianglerighteq \mathrm{weight}(T) = (n-k) \times s + \lambda$, by Algorithm 4.1.13, we know that the largest entry in $\mathrm{cc}(\mathrm{std}(T))$ is at most $s-1$. Since $\mathrm{cc}(\mathrm{std}(T))$ is a semistandard Young tableau and the battery has $s-1$ many rows, this implies that the top right corner of the battery contains $s-1$.

Furthermore, row i of the battery contains only $(i-1)$'s and i's. When we apply Algorithm 4.1.13, we see that reading the i's in row i does not result in adding a box to the partition ν. Thus we add 1 to i and move it to the end of the word. From this, we know that the $(s-1)$ in the top right corner of the battery ends up contributing a box to row $s+1$ or higher in the final partition $\mathrm{ctype}(\mathrm{std}(T))$. However, this also contradicts $\mathrm{ctype}(\mathrm{std}(T)) \trianglerighteq \mathrm{weight}(T) = (n-k) \times s + \lambda$. □

Example 6.4.10. Continuing with Example 6.4.8, we can see that $\mathrm{cc}(\mathrm{std}(T)) = \begin{array}{cccc} 1 & & & \\ 0 & 0 & 1 & \\ & & & 1 \\ & & & 0 \end{array}$.

Now, we construct a map Ψ on $\Sigma_{n,\lambda,s}$. Consider $(S, \mu) \in \Sigma_{n,\lambda,s}$. Recall that for any (S, μ) we can construct a boosted cocharge tableau $\mathrm{cc}(S, \mu)$. Let $\tilde{\Psi}(S, \mu)$ be the tableau we get by appending the superstandard tableau of shape $(n-k) \times (s-1)$ to the bottom right corner of $\mathrm{cc}(S, \mu)$.

Example 6.4.11. Consider $\left(\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|}\hline \\ \hline \end{array}\right) \in \Sigma_{4,(2,1),3}$. Then we have

$$\mathrm{cc}(S,\mu) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 0 & 0 \\ \hline \end{array} \qquad \tilde{\Psi}(S,\mu) = \begin{array}{|c|c|c}\hline 1 & 2 & \\ \hline 0 & 0 & \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{0} \\ \cline{3-3} \end{array}.$$

Lemma 6.4.12. *For $(S, \mu) \in \Sigma_{n,\lambda,s}$, we have that $\tilde{\Psi}(S, \mu)$ is equal to $\mathrm{cc}(\mathrm{std}(T))$ for some $T \in \mathcal{T}^+_{n,\lambda,s}$.*

Proof. Note that $\tilde{\Psi}(S, \mu)$ contains at least one of $0, \ldots, s-2$, since they are contained in the superstandard tableau we appended to the bottom right corner. We can see that for any $c \in [s-3]_0$, there exists a $(c+1)$ in some row higher than the lowest c, since we know the lowest c is in row $(c+1)$ of the battery and there is a $(c+1)$ in the row above it. Furthermore, we know the entries in the device part of $\tilde{\Psi}(S, \mu)$ are at most $s-1$, since the largest entry is $\mathrm{des}(S) + \mu_1 \leqslant s-1$. Thus, either $(s-2)$ is the largest entry in $\tilde{\Psi}(S, \mu)$, or there exists a $(s-1)$ in a row higher than the lowest $(s-2)$. Thus $\tilde{\Psi}(S, \mu)$ is the cocharge tableau for some standard tableau by Lemma (and Remark 5.3.5).

Furthermore, set $\mathrm{rw}(\tilde{\Psi}(S, \mu)) = \mathbf{db}$, where $\mathbf{d}$ is the reading word of the top semistandard tableau and $\mathbf{b}$ is the reading word of the superstandard tableau appended at the bottom right. We can see that $\mathrm{rev}(\mathbf{d}) \in \mathcal{D}_{n,\lambda,s}$ by Proposition 6.4.6 since $\mathrm{rev}(\mathbf{d}) = \mathrm{cc}(w, \mu)$. In particular, there exists a subset $U \subset [n]$ such that $\mathrm{rev}(\mathbf{d}|_U) \in \mathcal{D}_\lambda$ where $[n] \setminus U = \{a_1, \ldots, a_{n-k}\}$. Furthermore, we know that $\mathbf{b}$ can be broken up into $n-k$ many disjoint subwords of the form $(s-2)(s-1)\ldots 210$. Then by Lemma 5.3.6, we know that $\mathrm{rev}(d_{a_i}(s-2)(s-1)\ldots 210) \in \mathcal{D}_s$. From this, we know that we can break up $\mathrm{rev}(\mathbf{db})$ into $\mathrm{rev}(\mathbf{d}|_U) \in \mathcal{D}_\lambda$ and $(n-k)$ many words, each in $\mathcal{D}_s$. Thus $\mathrm{rev}(\mathbf{db}) \in \mathcal{D}_{(n-k)^s+\lambda}$. Thus by Theorem 5.A and Proposition 4.1.15, we have $\tilde{\Psi}(S, \mu) = \mathrm{std}(T)$ for some $T \in \mathcal{T}^+_{n,\lambda,s}$. $\square$

Now, we define a map $\Psi : \Sigma_{n,\lambda,s} \to \mathcal{T}^+_{n,\lambda,s}$ in the following way. Let $T \in \mathcal{T}^+_{n,\lambda,s}$ such that $\tilde{\Psi}(S, \mu) = \mathrm{std}(T)$. Define $\Psi(S, \mu) := T \in \mathcal{T}^+_{n,\lambda,s}$. Since std is a bijection (Proposition 4.1.15), we know that $\mathrm{std}(T)$ uniquely determines $T \in \mathcal{T}^+_{n,\lambda,s}$.

Proposition 6.4.13. *We have a bijection*

$$\Psi : \Sigma_{n,\lambda,s} \to \mathrm{std}(\mathcal{T}^+_{n,\lambda,s}) \to \mathcal{T}^+_{n,\lambda,s}$$

such that the device D of $\Psi(S,\mu)$ is the same shape as S and $\operatorname{cocharge}(\Psi(S,\mu)) = \operatorname{cocharge}(S) + |\mu|$.

We prove this by constructing an inverse map Ψ^{-1}. Note that since Ψ is a tableaux analogue of the map $(w,\mu) \mapsto \operatorname{rev}(\operatorname{cc}(w,\mu))$, the inverse map Ψ^{-1} will involve the inverse map constructed in Proposition 6.4.6. For any $\mathbf{d} \in \mathcal{D}_{n,\lambda,s}$, let $\operatorname{cc}^{-1}(\operatorname{rev}(\mathbf{d})) := (w,\mu)$ where $\operatorname{rev}(\operatorname{cc}(w,\mu)) = \mathbf{d}$.

Proposition 6.4.14. *Let* $\operatorname{cc}(\operatorname{rw}(\operatorname{std}(T))) = \mathbf{db}$*, where* $\mathbf{d}, \mathbf{b}$ *are the subwords corresponding to the entries in the device and battery respectively. Then* $\operatorname{rev}(\mathbf{d}) \in \mathcal{D}_{n,\lambda,s}$.

Proof. We know $\operatorname{ctype}(\operatorname{std}(T)) \trianglerighteq (n-k)^s+\lambda$, which implies that $\operatorname{rev}(\mathbf{db}) = \operatorname{rev}(\mathbf{b})\operatorname{rev}(\mathbf{d}) \in \mathcal{D}_{(n-k)^s+\lambda}$. From Theorem 5.A, we know that there exists $(C_1 \mid \ldots \mid C_{n-k} \mid \mathbf{A}_\lambda) \in \mathcal{OSP}_{k+(n-k)s,((n-k)^s+\lambda)'}$ where each restriction satisfies the condition in (6.7).

By the construction used in the proofs of Theorem 5.A and Proposition 5.3.12, we know that $\operatorname{rev}(\mathbf{db}|_{C_i}) = 01\ldots(s-1)m_i$, where $m_i \in [s-1]_0$ and the subword $01\ldots(s-1)$ consists of letters in $\operatorname{rev}(\mathbf{b})$.

If we set $B_i = \{\text{index corresponding to } m_i\}$, we can construct a new ordered set partition $([(n-k)(s-1)]|\mathbf{A}_\lambda|\mathbf{B})$. Since $(\operatorname{rev}(\mathbf{db}))|_{[(n-k)(s-1)]} = \operatorname{rev}(\mathbf{b})$, we see that $(\mathbf{A}_\lambda|\mathbf{B}))$ gives an ordered set partition of the indices of $\operatorname{rev}(\mathbf{d})$ such that (6.7) holds. ∎

Lemma 6.4.15. $\operatorname{cc}^{-1}(\operatorname{rev}(\mathbf{d})) = (\operatorname{rw}(S),\mu)$ *for some* $S \in \Sigma_{n,\lambda,s}$.

Proof. This follows from the fact that we can see $\operatorname{cc}^{-1}(\operatorname{rev}(\mathbf{d}))$ directly on the tableau $\operatorname{cc}(\operatorname{std}(T))|_D$. Note that since $\operatorname{std}(T)$ is a standard tableau, there is a total ordering on the boxes in $\operatorname{cc}(\operatorname{std}(T))|_D$, given by the ordering of the entries in $\operatorname{std}(T)$.

This ordering matches the reading order of $\mathbf{d}$ specified in (6.10). ∎

Proof of Proposition 6.4.13. It suffices to construct an inverse map. Take $\Psi^{-1}(T) = (S,\mu)$, where $\operatorname{cc}(\operatorname{std}(T)) = \mathbf{db}$ and $\operatorname{cc}^{-1}(\operatorname{rev}(\mathbf{d})) = (\operatorname{rw}(S),\mu)$ from Lemma 6.4.15. We can check that the composition is the identity using the fact that $(w,\mu) \mapsto \operatorname{rev}(\operatorname{cc}(w,\mu))$ itself is a bijection between the sets $\{(w,\mu) \mid w \in \mathfrak{S}_n, (P(w),\mu) \in \Sigma_{n,\lambda,s}\} \to \mathcal{D}_{n,\lambda,s}$ (Proposition 6.4.6).

The second statement follows from the fact that $\operatorname{cocharge}(\Psi(S,\mu))$ is the sum of the entries in $\tilde{\Psi}(S,\mu)$. From Lemma 6.4.9, we know the sum of the entries in the battery component is $\binom{s-1}{2}(n-k)$. The sum of the entries in the device component is $\operatorname{cocharge}(S) + |\mu|$. ∎

Corollary 6.4.16. *We have that* $\dim(R_{n,\lambda,s}) = \mathcal{D}_{n,\lambda,s}$.

Proof. From Proposition 6.4.13 and using Corollary 6.2.6 and Proposition 6.4.6, we see that

$$|\mathcal{D}_{n,\lambda,s}| = \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} |\operatorname{SYT}(\operatorname{sh}(S))| = \dim(R_{n,\lambda,s}).$$

□

Remark 6.4.17. Note that though the map $(D, B) \mapsto D$ defined in Lemma 6.2.5 is the most straightforward bijection between $\mathcal{T}^{+}_{n,\lambda,s}$ and the collection of semistandard tableaux, it does not behave well with respect to cocharge. In particular, there is no clear relation between the cocharge of the two tableaux like the one in Proposition 6.4.13, which makes it difficult to relate the formula (6.3) to one using battery-powered tableaux. In Section 6.6, we highlight how using the set $\Sigma_{n,\lambda,s}$, which involves standard tableaux, resolves this issue.

Note that using Proposition 6.4.6, we can restate Theorem 6.A:

Theorem 6.A (Cocharge version)**.** *The set* $\mathcal{B}_{n,\lambda,s} = \{\mathbf{x}^{\operatorname{rev}(\operatorname{cc}(w,\mu))} : w \in \mathfrak{S}_n, (P(w), \mu) \in \Sigma_{n,\lambda,s}\}$ *descends to a monomial basis of* $R_{n,\lambda,s}$*.*

Using this formulation and the fact that $\Sigma_{k,\lambda,\ell(\lambda)} = \Sigma_{\lambda}, \Sigma_{n,1^k,k} = \Sigma_{n,k}$, it is evident that our descent basis construction specializes to the known descent bases for $R_{\lambda}, R_{n,k}$.

6.5 Linear independence

We now show that the monomials in Theorem 6.A are linearly independent. We follow the argument in [11], of which we give a brief outline here.

Denote by $\widehat{\lambda}' = (\lambda'_2, \ldots, \lambda'_m)$ the partition obtained by removing the first part of λ', which corresponds to taking off the first column of λ. Given any subset $A \subset [n]$ of size $|A| = \lambda'_1$, we will show that there is a well-defined map

$$\varphi_A : R_{n,\lambda,s} \to R_{\lambda'_1} \otimes R_{n,\widehat{\lambda},s}.$$

Then, we use these maps to inductively show that if there is a relation of the form

$$\sum_{\mathbf{b} \leqslant_{\operatorname{des}} \mathbf{a}} c_{\mathbf{b}} \mathbf{x}^{\mathbf{b}} = 0$$

with $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$, then we must have that $c_{\mathbf{a}} = 0$. This implies the monomials $\mathbf{x}^{\mathbf{a}} \in \mathcal{B}_{n,\lambda,s}$ are linearly independent in $R_{n,\lambda,s}$. Applying Corollary 6.4.16 upgrades $\mathcal{B}_{n,\lambda,s}$ to a basis.

Proposition 6.5.1. *Let $\lambda \vdash k \leqslant n$, $s \geqslant \ell(\lambda)$, and $\widehat{\lambda}$ be as above. Given any subset $A \subset [n]$ of size $|A| = \lambda_1'$, the composite map*

$$\widetilde{\varphi}_A : \mathbb{Q}[\mathbf{x}_n] \to \mathbb{Q}[\mathbf{x}_A] \otimes \mathbb{Q}[\mathbf{x}_{[n]\setminus A}] \to R_{\lambda_1'} \otimes R_{n,\widehat{\lambda},s}$$

given by the canonical projection in both tensor components descends to a well-defined map $\varphi_A : R_n \to R_{\lambda_1'} \otimes R_{n,\widehat{\lambda},s}$.

Proof. It suffices to check that $\widetilde{\varphi}_A(I_{n,\lambda,s}) = 0$ in $R_{\lambda_1'} \otimes R_{n,\widehat{\lambda},s}$ for all $A \subset [n]$ with $|A| = \lambda_1'$. Since $s \geqslant \lambda_1'$, we have that $x_i^s \in I_{\lambda_1'}$ for $i \in A$. Observe $x_j^s \in I_{n,\widehat{\lambda},s}$ for $j \in [n]\setminus A$ by definition. Therefore, we have that

$$\widetilde{\varphi}_A(x_i^s) = \begin{cases} x_i^s \otimes 1 & \text{if } i \in A \\ 1 \otimes x_i^s & \text{if } i \in [n]\setminus A \end{cases}$$

always vanishes in $R_{\lambda_1'} \otimes R_{n,\widehat{\lambda},s}$.

Next, consider $e_d(B) \in I_{n,\lambda,s}$, where $B \subset [n]$ and d satisfies

$$d > |B| - (\lambda'_{n-|B|+1} + \cdots + \lambda'_n) \tag{6.12}$$

Notice we can factor $e_d(B)$ along the sets $A \cap B$ and $B \setminus A$ to yield

$$\widetilde{\varphi}_A(e_d(B)) = \sum_{i=\max(0,d-|B\setminus A|)}^{|A\cap B|} e_i(A \cap B) \otimes e_{d-i}(B\setminus A) \tag{6.13}$$

We show for each summand, either $e_i(A \cap B) \in I_{\lambda_1'}$, or $e_{d-i}(B\setminus A) \in I_{n,\widehat{\lambda},s}$.

First, we consider the case where $A \subset B$. Then we have that $e_i(A \cap B) = e_i(A) \in I_{\lambda_1'}$ whenever $i > 0$. It remains to check the case where $i = 0$, which is only possible if $|B| > |A|$. Making the substitution $\widehat{\lambda}_i' = \lambda_{i+1}'$, we have

$$\begin{aligned} |B\setminus A| - (\widehat{\lambda}'_{n-|A|-|B\setminus A|+1} + \dots) &= |B| - |A| - (\lambda'_{n-|B|+2} + \dots) \\ &\leqslant |B| - (\lambda'_{n-|B|+1} + \dots) < d \end{aligned}$$

where the weak inequality uses the fact that $|A| = \lambda_1' \geqslant \lambda'_{n-|B|+1}$. This implies that $e_d(B\setminus A) \in I_{n,\widehat{\lambda},s}$.

Now consider the case where $A \not\subset B$. We will show that $e_{d-i}(B\setminus A) \in I_{n,\widehat{\lambda},s}$ for all i. We have by (6.13) that $i \leqslant |A \cap B|$. Subtracting this from (6.12), we have

$$|B| - |A \cap B| - (\lambda_{n-|B|+1} + \dots) < d - i \tag{6.14}$$

Then, as before, we have

$$\begin{aligned}|B\setminus A| - (\widehat{\lambda}'_{n-|A|-|B\setminus A|+1} + \dots) &= |B| - |A\cap B| - (\lambda'_{n-|B|+2-(|A|-|A\cap B|)} + \dots)\\ &\leqslant |B| - |A\cap B| - (\lambda'_{n-|B|+1} + \dots) < d - i\end{aligned}$$

where the weak inequality uses $|A| - |A\cap B| > 0$ since $A \not\subseteq B$. This completes the proof. $\square$

Proposition 6.5.2. *Let $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$, and suppose*

$$\sum_{\mathbf{b}\leqslant_{\mathrm{des}}\mathbf{a}} c_{\mathbf{b}} x^{\mathbf{b}} = 0 \tag{6.15}$$

in $R_{n,\lambda,s}$. Then, we have $c_{\mathbf{a}} = 0$. In particular, the monomials $\mathcal{B}_{n,\lambda,s}$ are linearly independent in $R_{n,\lambda,s}$.

Proof. We induct on the number of columns λ_1 of λ, i.e. the parts of λ'. If $\lambda_1 = 0$, i.e. $\lambda = \varnothing$, it is clear that $\mathcal{B}_{n,\varnothing,s} = \{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in [s-1]_0^n\}$ is a basis of $R_{n,\varnothing,s} = \mathbb{Q}[\mathbf{x}_n]/(x_i^s : 1 \leqslant i \leqslant n)$, and are nonleading terms with respect to any total order on monomials.

Now take $\lambda_1 \geqslant 1$, and let $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$, so that there is $\sigma = (\mathbf{A}_\lambda|\mathbf{B}) \in \mathcal{OSP}_{n,\lambda'}$ with $\mathbf{a}|_{A_i} \in \mathcal{D}_{\lambda'_i}$, $\mathbf{a}|_{B_j} \in [s-1]_0$. Choose $A = A_1$ so $B = [n]\setminus A$, and let $\mathbf{a}' = \mathbf{a}|_A, \mathbf{a}'' = \mathbf{a}|_B$ denote the corresponding reverse cocharge words in $\mathcal{D}_{\lambda'_1}, \mathcal{D}_{n-\lambda'_1,\hat{\lambda},s}$, where $\widehat{\lambda}$ is as in the previous proposition. By Proposition 6.5.1, we compute

$$0 = \varphi_A\Big(\sum_{\mathbf{b}\leqslant_{\mathrm{des}}\mathbf{a}} c_{\mathbf{b}} x^{\mathbf{b}}\Big) = \sum_{\mathbf{b}'\in\mathcal{D}_{\lambda'_1}} \mathbf{x}^{\mathbf{b}'} \otimes \Big(\sum_{\mathbf{b}''} d_{\mathbf{b}',\mathbf{b}''}\mathbf{x}^{\mathbf{b}''}\Big) \tag{6.16}$$

where we expand in the first factor using the descent basis, and push all coefficients into the second tensor factor. For each nonzero $d_{\mathbf{b}',\mathbf{b}''}$ in (6.16), there is some $c_{\mathbf{b}} \neq 0$ contributing to equation (6.15) such that

$$\mathbf{b} \leqslant_{\mathrm{des}} \mathbf{a} \quad \mathbf{b}' \leqslant_{\mathrm{des}} \mathbf{b}|_A, \quad \mathbf{b}'' = \mathbf{b}|_B, \tag{6.17}$$

where the second inequality follows from the fact that $\mathbf{b}' \in \mathcal{D}_{\lambda'_1}$ are the nonleading terms with respect to the des-order (for an explicit straightening algorithm, see [1]).

Since the left hand side of (6.15) is 0, we must have

$$\sum_{\mathbf{b}''} d_{\mathbf{a}',\mathbf{b}''}\mathbf{x}^{\mathbf{b}''} = 0 \tag{6.18}$$

in $R_{n-\lambda'_1,\hat{\lambda},s}$. Furthermore, Lemma 3.1.2 implies that $d_{\mathbf{a}',\mathbf{b}''} \neq 0$ only if $\mathbf{b}'' \leqslant_{\mathrm{des}} \mathbf{a}''$, so we see $d_{\mathbf{a}',\mathbf{a}''} = c_{\mathbf{a}}$ and write

$$\sum_{\mathbf{b}'' \leqslant_{\mathrm{des}} \mathbf{a}''} d_{\mathbf{a}',\mathbf{b}''} \mathbf{x}^{\mathbf{b}''} = 0 \tag{6.19}$$

The claim follows from applying the induction hypothesis. □

Proof of Theorem 6.A. Proposition 6.5.2 shows that $\mathcal{B}_{n,\lambda,s}$ are linearly independent, and by Corollary 6.4.16, we have that $|\mathcal{B}_{n,\lambda,s}| = \dim R_{n,\lambda,s}$. □

6.6 Identifying the graded Frobenius character

Now, we use the descent basis of $R_{n,\lambda,s}$ to give a combinatorial formula for the graded Frobenius character $\mathrm{Frob}_q(R_{n,\lambda,s})$ in terms of our indexing set $\Sigma_{n,\lambda,s}$. First, recall the formula for the ungraded Frobenius character (6.3). We can rewrite (6.3) in terms of the set $\Sigma_{n,\lambda,s}$ by composing the bijection in Lemma 6.2.5 with the bijection Ψ in Proposition 6.4.13:

$$\mathrm{Frob}(R_{n,\lambda,s}) = \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} s_{\mathrm{sh}(S)}. \tag{6.20}$$

Now, we can use (6.20) to get $\dim(N_\gamma R_{n,\lambda,s})$, which we will use to determine $\mathrm{Hilb}_q(N_\gamma R_{n,\lambda,s})$.

$$\begin{aligned}
\dim(N_\gamma R_{n,\lambda,s}) &= \langle e_\gamma, \mathrm{Frob}(R_{n,\lambda,s})\rangle \\
&= \Big\langle \omega e_\gamma, \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} \omega s_{\mathrm{sh}(S)} \Big\rangle \\
&= \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} \langle h_\gamma, s_{\mathrm{sh}(S')}\rangle \\
&= \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} K_{\mathrm{sh}(S'),\gamma}
\end{aligned} \tag{6.21}$$

where $K_{\mathrm{sh}(S'),\gamma}$ is the Kostka number. Using Corollary 2.2.2 and Proposition 2.2.1 (c), we can rewrite the last expression to be

$$\begin{aligned}
&\dim(N_\gamma R_{n,\lambda,s}) \\
&\quad= \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} |\{w \in \mathfrak{S}_n, P(w) = S', \mathrm{Des}(w) \subset \{\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\cdots+\gamma_{l-1}\}| \\
&\quad= \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} |\{w \in \mathfrak{S}_n, P(w) = S, \mathrm{Asc}(w) \subset \{\gamma_l, \gamma_l+\gamma_{l-1}, \ldots, \gamma_l+\cdots+\gamma_2\}|.
\end{aligned} \tag{6.22}$$

where $\mathrm{Asc}(w)$ denotes the *ascent set* of w, defined to be $\mathrm{Asc}(w) = \{i : w_i < w_{i+1}\}$.

Now, we construct a basis of $N_\gamma R_{n,\lambda,s}$ using the descent basis. This basis will give us an explicit formula for $\mathrm{Hilb}_q(R_{n,\lambda,s})$, which in turn will uniquely determine $\mathrm{Frob}_q(R_{n,\lambda,s})$.

For any composition $\gamma \models n$ and word $\mathbf{z}$ of length n, we say $\mathbf{z}$ is *strictly increasing on* γ if when we divide $\mathbf{z}$ into blocks of sizes $\gamma_1, \gamma_2, \ldots, \gamma_l$, we have that the entries within the blocks are strictly increasing.

Example 6.6.1. Consider $\mathbf{z} = 2423679$ and $\gamma = (2, 2, 3)$. Then when we divide $\mathbf{z}$ into three blocks $24|23|679$, the entries are strictly increasing. Thus $\mathbf{z}$ is strictly increasing on γ.

We use this to define a subset of $\mathcal{D}_{n,\lambda,s}$.

$$\mathcal{D}^\gamma_{n,\lambda,s} := \{\mathbf{a} \in \mathcal{D}_{n,\lambda,s} : \mathbf{a} \text{ is strictly increasing on } \gamma\}.$$

We will show that this set indexes a basis of $N_\gamma R_{n,\lambda,s}$. We first relate this set to pairs (w, μ) where $(P(w), \mu) \in \Sigma_{n,\lambda,s}$. To do this, we make the following observation.

Lemma 6.6.2. *Consider* $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$ *and* $\mathrm{cc}^{-1}(\mathrm{rev}(\mathbf{a})) = (w, \mu)$. *Then we have* $a_{n-j} < a_{n-j+1}$ *if and only if* $w_j > w_{j+1}$.

Proof. Note that the condition $a_{n-j} < a_{n-j+1}$ is equal to $z_j > z_{j+1}$, where $\mathbf{z} = \mathrm{rev}(\mathbf{a}) = \mathrm{cc}(w, \mu)$. By construction we can see that $\mathrm{cc}(w, \mu)_j > \mathrm{cc}(w, \mu)_{j+1}$ can only happen if $w_j > w_{j+1}$. In particular: if $w_j > w_{j+1}$, it is clear that $\mathrm{cc}(w, \mu)_j > \mathrm{cc}(w, \mu)_{j+1}$ since $\mathrm{cc}(w)_j > \mathrm{cc}(w)_{j+1}$. On the other hand, if $w_j < w_{j+1}$, then we have $\mathrm{cc}(w, \mu)_j \leqslant \mathrm{cc}(w, \mu)_{j+1}$. □

Using Lemma 6.6.2, we can compute $|\mathcal{D}^\gamma_{n,\lambda,s}|$, by looking at the preimage of $\mathcal{D}^\gamma_{n,\lambda,s}$ under the map $(w, \mu) \mapsto \mathrm{rev}(\mathrm{cc}(w, \mu))$.

Corollary 6.6.3. *For* $\gamma \models n$, *we have* $\mathrm{rev}(\mathrm{cc}(w, \mu)) \in \mathcal{D}^\gamma_{n,\lambda,s}$ *if and only if* $\mathrm{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \ldots, \gamma_l + \gamma_{l-1} + \cdots + \gamma_2\}$. *In particular, we have* $|\mathcal{D}^\gamma_{n,\lambda,s}| = \dim(N_\gamma R_{n,\lambda,s})$.

Proof. The first statement follows immediately from Lemma 6.6.2. The second statement follows from (6.22). □

We can also use the following fact about $\mathcal{D}_{n,\lambda,s}$. Note that this is analogous to [11, Lemma 3.4] and the same proof holds.

Lemma 6.6.4. *Consider* $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$ *with* $a_j > a_{j+1}$. *Let* $\tilde{\mathbf{a}}$ *be the word we get by swapping* a_j *and* a_{j+1}. *Then* $\tilde{\mathbf{a}} \in \mathcal{D}_{n,\lambda,s}$.

Remark 6.6.5. The equivalent statement phrased in terms of boosted cocharge words is the following: let $\mathbf{c} = \mathrm{cc}(w, \mu)$ and $c_j < c_{j+1}$. If $\mathbf{c}$ is the word we get by swapping c_j and c_{j+1}, we have that $\mathbf{c} = \mathrm{cc}(\tilde{w}, \mu)$ for some other $\tilde{w}$ satisfying $\mathrm{ctype}(w|_k) \trianglerighteq \lambda$.

Now, we can use these results to construct an explicit basis of $N_\gamma R_{n,\lambda,s}$.

Proposition 6.6.6. *$\mathcal{B}^\gamma_{n,\lambda,s} := \{N_\gamma \mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}^\gamma_{n,\lambda,s}\}$ is a basis of $N_\gamma R_{n,\lambda,s}$.*

Proof. From Theorem 6.A, we know that $\{N_\gamma \mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_{n,\lambda,s}\}$ must span $N_\gamma R_{n,\lambda,s}$.

We show that $\mathcal{B}^\gamma_{n,\lambda,s}$ spans $R_{n,\lambda,s}$. First, consider $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$ such that when we split $\mathbf{a}$ into blocks of size $\gamma_1, \ldots, \gamma_l$, some block contains a repeated entry. Then, we must have $N_\gamma \mathbf{x}^{\mathbf{a}} = 0$, since N_γ antisymmetrizes the exponents within the blocks.

Now, consider $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$ where the blocks consist of distinct entries, but $\mathbf{a} \notin \mathcal{D}_{n,\lambda,s}$. This means there exists an index j such that $a_j > a_{j+1}$ belong to the same block. If $\tilde{ax}$ is the result of swapping a_j and a_{j+1}, we have that $N_\gamma \mathbf{x}^{\mathbf{a}} = -N_\gamma \mathbf{x}^{\tilde{\mathbf{a}}}$. Furthermore, we know $\tilde{\mathbf{a}} \in \mathcal{D}_{n,\lambda,s}$ by Lemma 6.6.4. In this way, for each such $\mathbf{a}$, we can construct a $\mathbf{b} \in \mathcal{D}^\gamma_{n,\lambda,s}$ such that $N_\gamma \mathbf{x}^{\mathbf{a}} = \pm N_\gamma \mathbf{x}^{\mathbf{b}}$. Thus $\mathcal{B}^\gamma_{n,\lambda,s}$ spans $N_\gamma R_{n,\lambda,s}$.

Note that $|\mathcal{B}^\gamma_{n,\lambda,s}| = |\mathcal{D}^\gamma_{n,\lambda,s}|$, thus from Corollary 6.6.3 we have that $\mathcal{B}^\gamma_{n,\lambda,s}$ is a basis of $N_\gamma R_{n,\lambda,s}$. □

Corollary 6.6.7. *We have*

$$\mathrm{Hilb}_q(N_\gamma R_{n,\lambda,s}) = \sum_{\substack{(w,\mu) \\ (P(w),\mu) \in \Sigma_{n,\lambda,s} \\ \mathrm{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \ldots, \gamma_l + \gamma_{l-1} + \cdots + \gamma_2\}}} q^{\mathrm{cocharge}(w) + |\mu|}. \tag{6.23}$$

Proof. This follows from the fact that the degree of $N_\gamma \mathbf{x}^{\mathbf{a}}$ is equal to $\mathrm{cocharge}(w) + |\mu|$, where $\mathrm{rev}(\mathrm{cc}(w, \mu)) = \mathbf{a}$, and Proposition 6.6.6. □

Finally, we get an explicit Schur expansion of $\mathrm{Frob}_q(R_{n,\lambda,s})$ using $\Sigma_{n,\lambda,s}$.

Theorem 6.B.

$$\mathrm{Frob}_q(R_{n,\lambda,s}) = \sum_{(S,\mu) \in \Sigma_{n,\lambda,s}} q^{\mathrm{cocharge}(S) + |\mu|} s_{\mathrm{sh}(S)} \tag{6.24}$$

Proof. Doing the same computation as in (6.21) and (6.22), except now carrying over the power of q, we have

$$\begin{aligned}\langle e_\gamma, \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} q^{\mathrm{cocharge}(S)+|\mu|} s_{\mathrm{sh}(S)}\rangle &= \sum_{(S,\mu)\in\Sigma_{n,\lambda,s}} q^{\mathrm{cocharge}(S)+|\mu|} K_{\mathrm{sh}(S)^t,\gamma} \\ &= \sum_{\substack{(w,\mu) \\ (P(w),\mu)\in\Sigma_{n,\lambda,s} \\ \mathrm{Asc}(w)\subset\{\gamma_l,\gamma_l+\gamma_{l-1},\ldots,\gamma_l+\cdots+\gamma_2\}}} q^{\mathrm{cocharge}(P(w))+|\mu|}.\end{aligned} \tag{6.25}$$

Since $\mathrm{cocharge}(P(w)) = \mathrm{cocharge}(w)$, we can see that (6.25) is equal to (6.23). □

6.7 Connections to known formulas for Frobenius characters

We can see that the correct specializations recover the known Frobenius characters for R_λ and $R_{n,k}$. In particular, when $(n, \lambda, s) = (k, \lambda, \ell(\lambda))$, we recover (5.2), as proven in [33].

When we set $(n, \lambda, s) = (n, 1^k, k)$, we get

$$\mathrm{Frob}_q(R_{n,\lambda,s}) = \sum_{\substack{S\in\mathrm{SYT}_n \\ \mathrm{des}(S)<k}} \left(\sum_{\mu\subset(n-k)\times(k-\mathrm{des}(S)-1)} q^{|\mu|} \right) q^{\mathrm{cocharge}(S)} s_{\mathrm{sh}(S)}.$$

Note that this is equivalent to following expression for $\mathrm{Frob}_q(R_{n,k})$ [30, Corollary 6.13], using properties of the q-binomial coefficients, as well Proposition 2.6.10.

$$\mathrm{Frob}_q(R_{n,k}) = \sum_{\substack{S\in\mathrm{SYT}_n \\ \mathrm{des}(S)<k}} \begin{bmatrix} n-\mathrm{des}(S)-1 \\ n-k \end{bmatrix}_q q^{\mathrm{maj}(S)} s_{\mathrm{sh}(S)}. \tag{6.26}$$

Furthermore, Theorem 6.2.7 gives a combinatorial formula for $\mathrm{Frob}_q(R_{n,\lambda,s})$ in terms of battery-powered tableaux. In [23], the authors give a geometric proof of this and pose the question whether there is more direct algebraic or combinatorial proof. Though recent results [22] give such explanation for $R_{n,k}$, there is still no known such argument for general $R_{n,\lambda,s}$.

Our formula provides an algebraic and combinatorial proof for Theorem 6.2.7 by showing that it is equivalent to Theorem 6.B up to a combinatorial bijection.

Corollary 6.C. *Theorem 6.B is equivalent to Theorem 6.2.7 up to a combinatorial bijection between the sets* $\Sigma_{n,\lambda,s}$ *and* $\mathcal{T}^{+}_{n,\lambda,s}$.

Proof. The equality of (6.4) and (6.24) follows from Proposition 6.4.13. In particular, if we apply Ψ to (6.24), we get (6.4). $\square$

Chapter 7

Higher Specht Bases for generalized coinvariant rings

In this chapter, we give a conjecture for a higher Specht basis for $R_{n,\lambda,s}$. We show how our conjecture relates with other known higher Specht bases, as well as the results in Chapter 6. We also give proofs for certain cases. This chapter is based on material in [16], as well as ongoing joint work with Raymond Chou and Maria Gillespie.

7.1 A conjectured higher Specht basis

The higher Specht basis of R_n, given in Theorem 3.2.6, witnesses the decomposition of R_n into irreducibles. Inspired by this, we can define the notion of a higher Specht basis for any $\mathbb{C}\mathfrak{S}_n$-module M. This definition is due to Gillespie–Rhoades [24].

Definition 7.1.1. A *higher Specht basis* for $\mathbb{C}\mathfrak{S}_n$-module M is a basis $\mathcal{B}$ that admits a set partition $\mathcal{B} = \bigsqcup \mathcal{B}_{\lambda^{(i)}}$ such that each $\mathcal{B}_{\lambda^{(i)}}$ spans a copy of the irreducible V_λ in the decomposition of M into irreducibles.

Gillespie–Rhoades have some results on higher Specht bases for generalized coinvariant rings. In particular, they prove the following.

Theorem 7.1.2 (Gillespie–Rhoades[24])**.** *The following collection of polynomials*

$$\{F_T^S \cdot e_1^{i_1} e_2^{i_2} \cdots e_{n-k}^{i_{n-k}} : S, T \in \mathrm{SYT}_n, \mathrm{Sh}(S) = \mathrm{Sh}(T), i_j \in \mathbb{Z}_{\geqslant 0}, 0 \leqslant i_1 + \cdots i_{n-k} < k - \mathrm{des}(S)\} \tag{7.1}$$

descends to a higher Specht basis of $R_{n,k}$.

We can restate (7.1) using the indexing set $\Sigma_{n,k}$ we introduced in Section 6.3. Note that we can take a tuple of nonnegative integers $(i_1, \ldots, i_{n-k})$ such that $i_1 + \cdots + i_{n-k} < k - d$ and write it as a partition $\mu \subseteq (n-k) \times (k-d-1)$ by setting $\mu_j = i_j + \cdots + i_{n-k}$. In other words, we can consider each polynomial $F_T^S \cdot e_1^{i_1} e_2^{i_2} \cdots e_{n-k}^{i_{n-k}}$ to be indexed by a pair $((S,\mu),T)$ where $(S,\mu) \in \Sigma_{n,k}$ and $\mathrm{sh}(T) = \mathrm{sh}(S)$. It is clear that this witnesses the decomposition of $R_{n,\lambda,s}$ into irreducibles given by the combinatorial formula for $\mathrm{Frob}_q(R_{n,k})$ given by (6.26).

They also a conjectured higher Specht basis for R_μ. To state their conjecture, we need to define the notion of a higher Specht polynomial F_T^S for S semistandard.

Definition 7.1.3. Given a word z of weight μ, we first decompose z into subwords $z^{(1)}, \ldots, z^{(\mu_1)}$, where each subword is a permutation, in the following way. Choose the rightmost 1 to be in $z^{(1)}$. After choosing i to be in $z^{(1)}$, we choose $i+1$ in the following way. If there is a $i+1$ to the left of the i we chose, we choose the rightmost such $i+1$ to be in $z^{(1)}$. Otherwise, we choose the rightmost $i+1$ in w. We continue this process until we cannot add anymore letters.

Now, delete the subword $z^{(1)}$ from z and repeat, continuing until we have fully decomposed z into subwords. The cocharge of z is

$$\mathrm{cocharge}(z) := \sum_{i=1}^{\mu_1} \mathrm{cocharge}(z^{(i)}) \tag{7.2}$$

Remark 7.1.4. We can compute charge of a word z by taking

$$\mathrm{charge}(z) := \sum_{i=1}^{\mu_1} \mathrm{charge}(z^{(i)}) \tag{7.3}$$

where we construct z in the exact same way. Note that we have $\mathrm{charge}(z) = n(\mu) - \mathrm{cocharge}(z)$.

The cocharge word of z is the word we get by replacing $z^{(i)}$ with $\mathrm{cc}(z^{(i)})$ in z. Using this definition of cocharge of words of partition weight, we can also define the notion of $\mathrm{cocharge}(S)$ and $\mathrm{cc}(S)$.

Then, we can define F_T^S for $T \in \mathrm{SYT}_, S \in \mathrm{SSYT}(\mu)$ for T, S of the same shape using Definitions 3.2.1, 3.2.4, except now considering the cocharge label $\mathrm{cc}(S)$ of a semistandard tableau S.

Conjecture 7.1.5 (Gillespie–Rhoades[24])**.** The following set gives a higher Specht basis of R_μ:

$$\bigcup_{\lambda \vdash n} \{F_T^S : (S,T) \in \mathrm{SSYT}(\lambda,\mu) \times \mathrm{SYT}(\lambda)\}. \tag{7.4}$$

In light of the work in Chapter 6, we make the following conjecture for a higher Specht basis for $R_{n,\lambda,s}$.

Conjecture 7.1.6. Let $\lambda \vdash k$, $k \leqslant n$, and $\ell(\lambda) \leqslant s$. The following set gives a higher Specht basis of $R_{n,\lambda,s}$:

$$\{F_T^S e_1^{i_1} \dots e_{n-k}^{i_{n-k}} : (S, \mathbf{i}) \in \Sigma_{n,\lambda,s}, \mathrm{sh}(S) = \mathrm{sh}(T)\}. \tag{7.5}$$

Note that while Conjecture 7.1.6 agrees with Theorem 7.1.2 at the appropriate specializations, it differs from Conjecture 7.1.5. In particular, if we specialize Conjecture 7.1.6, we get the following:

Conjecture 7.1.7. The following set gives a higher Specht basis of R_μ:

$$\{F_T^S : \mathrm{ctype}(S) \trianglerighteq \mu, \mathrm{sh}(S) = \mathrm{sh}(T)\}. \tag{7.6}$$

While the set (7.4) uses cocharge tableaux arising from semistandard tableaux of weight μ, our conjecture (7.6) uses the corresponding subset of standard tableaux that $\mathrm{SSYT}(\mu)$ gets mapped to with respect to std. Though the two formulas seem similar, they yield different sets, since in general, the standardization map in Proposition 4.1.9 only preserves the cocharge statistic, not the cocharge tableaux (see Remark 4.1.11).

However, the following example shows that Conjecture 7.1.5 fails in general.

Example 7.1.8. Consider the following semistandard tableau

$$S = \begin{array}{|c|ccc}\cline{1-1} 5 \\ \cline{1-1} 3 \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \mathrm{cc}(S) = \begin{array}{|c|ccc}\cline{1-1} 2 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline 0 & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{2} \\ \hline \end{array}.$$

For any T of the same shape, we have that $F_T^S = 0$. In particular, if we let

$$T = \begin{array}{|c|ccc}\cline{1-1} 10 \\ \cline{1-1} 9 \\ \hline 5 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

then $\mathbf{x}_T^S = x_4^2 x_5 x_6 x_7 x_8^3 x_9^2 x_{10}^2$. We can see that $(9, 10) \in C(T)$ is a transposition in the column group that fixes $\mathbf{x}_T^S$, thus $\varepsilon_T \mathbf{x}_T^S = 0$.

In other words, we can see that the polynomials in (7.4) can be equal to 0 for certain choices of S. In contrast, the polynomials in (7.6) are never 0 since they are a subset of the higher Specht basis of R_n. However, they may be equivalent to 0 when we quotient out by I_μ.

Though Conjecture 7.1.5 fails for general partition shapes, the authors prove it holds for two-row partitions. However, this is no coincidence: it turns out in this case, our conjecture matches theirs.

Proposition 7.1.9. *If $\mu = (\mu_1, \mu_2)$, then our conjectured higher Specht basis* (7.6) *coincides with the conjectured basis* (7.4) *of Gillespie–Rhoades.*

Proof. We will show that if μ is a two row partition, and $S \in \mathrm{SSYT}(\mu)$, we have that $\mathrm{cc}(S) = \mathrm{cc}(\mathrm{std}(S))$.

To see this, we describe the standardization map $\mathrm{std} : \mathrm{SSYT}(\mu) \to \Sigma_\mu$ given in Definition 2.7.6. When μ is a two-row partition, we know that any $S \in \mathrm{SSYT}(\mu)$ can be at most two rows. If we set $\lambda = (\lambda_1, \lambda_2)$ to be the shape S, the std map takes S to the standard tableau obtained in the following way:

- Replace the 1s in the first row with $12 \dots \mu_1$,
- replace the 2s in the second row with $(\mu_1) \dots (\mu_1 + \lambda_2)$,
- replace the 2s in the second row with $(\mu_1 + \lambda_2) \dots n$.

We see that $\mathrm{cc}(S)$ consists of all 0s in the first row and 1s in the second row. From the description of $\mathrm{std}(S)$, we can see that the cocharge tableau of $\mathrm{cc}(\mathrm{std}(S))$ is also the same. $\square$

Corollary 7.1.10. *Conjecture 7.1.7 holds for two-row partition shapes.*

7.2 Connections to hook-shape Garsia-Haiman modules

The higher Specht construction has been extended to the two-variables case, which looks at $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ under the diagonal $\mathfrak{S}_n$-action. In particular, Gillespie gives a higher Specht basis for hook-shape Garsia-Haiman module.

Definition 7.2.1 (Gillespie [21])**.** Let $\mu = (n-k+1, 1^{k-1})$. We define the μ-cocharge tableau of $S \in \mathrm{SYT}_n$, denoted $\mathrm{ccTab}_\mu(S)$, to be the tableau of the same shape where

- we have 0 in the squares occupied by $1, 2, \dots, n-k+1$ in S
- the rest of the squares are given by the cocharge labeling on $n-k+1, \dots, n$.

Definition 7.2.2. Let $\mu = (n-k+1, 1^{k-1})$. We define the reverse μ-cocharge tableau of $S \in \mathrm{SYT}_n$, denoted $\mathrm{ccTab}'_\mu(S)$, to be the tableau of the same shape where

- we have 0 in the squares occupied by $n-k+1,\ldots,n$ in S
- for $n-k+1,\ldots,1$, we calculate the cocharge in reverse.

Using the two definitions above, we define

$$\mathbf{xy}_T^S = \mathbf{x}_T^{\mathrm{ccTab}_\mu(S)}\mathbf{y}_T^{\mathrm{ccTab}'_\mu(S)}$$

in the same way that we defined $\mathbf{x}_T^S$ in Definition 3.2.1. If we consider $\mathfrak{S}_n$ to act diagonally on this monomial (i.e, it permutes the indices on the x, y variables simultaneously), we can define

$$F_T^S(\mathbf{x},\mathbf{y}) = \varepsilon_T \mathbf{xy}_T^S.$$

This is the two variable analogue of Definition 3.2.4.

Theorem 7.2.3 (Gillespie [21])**.** *Let $\mu = (n-k+1, 1^{k-1})$. Then $\{F_T^S(\mathbf{x},\mathbf{y}) : S, T \in \mathrm{SYT}_n, \mathrm{Sh}(S) = \mathrm{Sh}(T)\}$ is a higher Specht basis for $\mathcal{DR}_\mu$.*

Theorem 7.2.3 is closely related to Conjecture 7.1.7 through the following observation.

Lemma 7.2.4. *Let $\mu = (n-k+1, 1^{k-1})$. We have that $F_T^S(\mathbf{x}, 0) \neq 0$ if and only if $\mathrm{ctype}(S) \trianglerighteq \mu$.*

Proof. Note that $F_T^S(\mathbf{x}, 0) \neq 0$ if and only if $\mathrm{ccTab}'_\mu(S) = 0$, which means $1,\ldots,n-k+1$ all appear in the same row of S. This is equivalent to saying the first row of S contains $1,\ldots,n-k+1$, which implies that S is μ-catabolizable. Thus $\mathrm{ctype}(S) \trianglerighteq \mu$ by Proposition 4.1.6. □

A direct corollary of this is the following:

Corollary 7.2.5. *Conjecture 7.1.7 is true for hook shapes.*

Remark 7.2.6. Note that unsurpisingly, Conjecture 7.1.5 does not coincide with the $\mathbf{y} = 0$ specialization of Theorem 7.2.3.

7.3 Garsia-Procesi for three rows

In future work with Gillespie and Chou, we show that Conjecture 7.1.7 holds for all three-row partitions. In this section, we include some of the main ideas used in our proof.

The advantage that Conjecture 7.1.7 has over the Gillespie–Rhoades conjecture (Conjecture 7.1.5) is that the conjectured basis (7.6) is a subset of the higher Specht basis of R_n. From this, we know immediately that the elements in our conjectured basis will not vanish in the quotient R_n. However, we need to check whether they vanish in the further quotient R_μ.

First, we recall some facts about Garsia–Procesi rings and Garnir polynomial. Note that the definition for the Garnir polynomial, given in Definition 2.3.1, can be extended to any column strict, standard fillings $\tilde{T}$ of the shape μ in the same way. We make a quick observation:

Lemma 7.3.1. *Let $\widetilde{T}$ be a column strict, standard filling of the shape μ. We have $g_{\tilde{T}} \in \operatorname{span}\{g_T \mid T \in \mathrm{SYT}(\mu)\}$.*

Proof. For a column strict filling $\tilde{T}$ of μ, let π be the permutation that sorts the entries within each row so that they are increasing.

We now show that the resulting tableau $\pi(\tilde{T})$ is still column strict. Let a be the jth entry in row i of $\pi(\tilde{T})$ where $i > 0$. For each entry b to the left of a in the row i of $\pi(\tilde{T})$, we know that there exists a unique corresponding entry in row $(i-1)$ that is smaller than b (whatever was below b in $\tilde{T}$.) Thus a is larger than at least j many entries in row $(i-1)$ of $\tilde{T}$, hence it is larger than the entry below it in $\pi(\tilde{T})$.

It is clear that $\pi g_{\tilde{T}} = g_{\pi\tilde{T}}$. Thus $g_{\tilde{T}} = \pi^{-1} g_T$ for $T = \pi\tilde{T} \in \mathrm{SYT}(\mu)$ and the result follows from Proposition 2.3.3. □

We can use the Garnir polynomials to identify whether an arbitrary $f \in \mathbb{C}[\mathbf{x}_n]$ satisfies $f \in I_\mu$. This is equivalent to the description of the coinvariant ideal I_n as the annhilator of the Vandermonde (3.5).

Theorem 7.3.2 (Bergeron-Garsia)**.** *We have $f \in I_\mu$ iff $f \odot g_T = 0$ for all $T \in \mathrm{SYT}(\mu)$.*

In particular, we have the following corollary using Lemma 7.3.1.

Corollary 7.3.3. *If $f \odot g_U \neq 0$ for some column strict standard filling U of μ, then $f \notin I_\mu$.*

Now, we will show that these polynomials are actually nonzero in quotients where they appear as part of the higher Specht basis. To do this, we will show the following:

Proposition 7.3.4. *If $\mathrm{ctype}(S) = \mu$, there exists a column strict filling U of μ such that $F_T^S \odot g_U \neq 0$.*

Assuming Proposition 7.3.4, we can show the following.

Theorem 7.3.5. *If $S, T \in \mathrm{SYT}_n$ with the same shape and* $\mathrm{ctype}(S) \trianglerighteq \mu$*, we have that* $F_T^S \notin I_\mu$.

Proof. Consider $S \in \mathrm{SYT}_n$ with $\mathrm{ctype}(S) = \nu \trianglerighteq \mu$. By Proposition 7.3.4 and Corollary 7.3.3, we have that $F_T^S \notin I_\nu$. Since $\nu \trianglerighteq \mu \Leftrightarrow I_\mu \subset I_\nu$, this implies that $F_T^S \notin I_\mu$. □

We now show Proposition 7.3.4. To do this, we will first show that monomial $\mathbf{x}_T^S$ appears with nonzero coefficient in F_T^S. Then, we will construct a Garnir polynomial g_U for some injective filling U where $\mathbf{x}_T^S \odot g_U \neq 0$. Finally, we show that for that choice of U we have $F_T^S \odot g_U \neq 0$.

We first give some definitions that we will make use of when discussing higher Specht polynomials.

Definition 7.3.6. Let $T \in \mathrm{SYT}(\lambda)$. A *$T$-snaking* is a filling P with shape λ such that if i, j occur in the same column in T, then i, j occur in different rows of P.

We refer to these fillings as *T-snaking* because of the following visualization: we may take the columns of T and "snake" them down different paths (possibly disconnected) as in the following example:

Example 7.3.7. Define S, T, P as follows:

$$S = \begin{array}{|c|c|c}\cline{1-2} 0 & 1 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{0} \\ \hline 2 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad T = \begin{array}{|c|c|c}\cline{1-2} 6 & 8 & \\ \hline 3 & 4 & \multicolumn{1}{c|}{7} \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad P = \begin{array}{|c|c|c}\cline{1-2} 4 & 1 & \\ \hline 5 & 6 & \multicolumn{1}{c|}{2} \\ \hline 8 & 7 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Then P is a T-snaking, and the corresponding monomial is

$$x_P^S = x_1^1 x_2^0 x_3^3 x_4^0 x_5^1 x_6^2 x_7^1 x_8^2 = x_1 x_3^3 x_5 x_6^2 x_7 x_8^2.$$

Note that using the T-snaking terminology, we can write

$$F_T^S = \sum_{P : T\text{-snaking}} (-1)^P \mathbf{x}_P^S.$$

Definition 7.3.8. The **box action** of S_n on the set of all fillings of a partition of λ of size n, denoted $\cdot$, is defined as follows. If b is a box in λ (where we think of b as a number from 1 to n, assigned to the boxes in ascending order up each column from left to right) then $\pi \cdot S(b) = S(\pi^{-1}(b))$.

Definition 7.3.9. The **ρ-group** $H(S)$ of a tableau S is the group of row permutations ρ for which $\rho \cdot S = S$. Note that $H(S)$ is a subgroup of the row group $R(T)$ for the standard filling T of the same shape, and $H(S) \cong H(\pi \cdot S)$ for any row permutation π.

Definition 7.3.10. The **(alternating) τ-set** $A(S)$ of a filling S of a Young diagram λ is the set of column permutations τ of the standard filling T of λ, whose box action on S preserves the content of each row of S.

In future work, we prove the following theorem:

Theorem 7.3.11. *Given a T-snaking P, the coefficient of x_P^S in F_T^S is $\pm|H(S)| \cdot \sum_{\tau\in A(\mathrm{cc}(S)_P)} \mathrm{sgn}(\tau)$.*

Note that in the case $P = T$, we have that $\mathrm{cc}(S)_P = \mathrm{cc}(S)$. Thus the coefficient of $\mathbf{x}_T^S$ in F_T^S is given by

$$|H(S)| \cdot \sum_{\tau\in A(\mathrm{cc}(S))} \mathrm{sgn}(\tau).$$

Throughout the rest of the note, we assume $S, T \in \mathrm{SYT}_n$ with $\mathrm{ctype}(S) = \lambda$. Since S is a SYT, we know the filling $\mathrm{cc}(S)$ is a semistandard tableau (i.e. the columns consist of distinct entries and are increasing).

We will show the following. Note that this implies the result of [21] showing that for $\mathrm{cc}(S)$ being an SSYT, we have F_T^S is nonzero.

Proposition 7.3.12. *Write S for a semistandard cocharge tableau. If $\tau \in A(S)$, then $\tau = id$.*

Proof. Note that $\tau \in A(S)$ means that for all rows k (where the sets below are multisets)

$$\{(\tau \cdot S)(b) : b \text{ in row } k \text{ of } T\} = \{S(b) : b \text{ in row } k \text{ of } T\}. \tag{7.7}$$

By definition of the box action on S, this is equivalent to the following

$$\{S(\tau^{-1}(b)) : b \text{ in row } k \text{ of } T\} = \{S(b) : b \text{ in row } k \text{ of } T\}.$$

We will prove that this implies $\tau = \mathrm{id}$. For sake of contradiction, assume this is not true. Consider the lowest, leftmost box b such that $\tau^{-1}(b) \neq b$. By choice of b (and since τ is a column permutation), we know that $\tau^{-1}(b)$ must be above b. Furthermore, since $\mathrm{cc}(S)$ is a semistandard tableau, we have that $(\tau \cdot S)(b) = S(\tau^{-1}(b)) > S(b)$. From (7.7) and the fact that τ fixes all the entries to the left of b, we know there exists a $b' > b$ in the same row such that $\tau \cdot S(b') = S(\tau^{-1}(b')) = S(b)$. However, this is impossible, since $\tau^{-1}(b')$ would be strictly above and to the right of b in T, meaning $S(\tau^{-1}(b')) > S(b)$. □

Corollary 7.3.13. *The coefficient of $\mathbf{x}_T^S$ in F_T^S is $H(S)$.*

Now, we will construct an injective filling U of shape λ such that $\mathbf{x}_T^S \odot g_U \neq 0$. To do this, we introduce a modification of Algorithm 4.1.13 to form a filling U rather than just a partition shape ν, as follows.

Definition 7.3.14. Let S, T be standard Young tableaux of the same shape. Let $\mathbf{z} = \mathrm{cc}(\mathrm{rw}(S))$, and let $U = \varnothing$ be the empty Young tableau. The **U-tableau** U of the pair (S, T) is formed by applying Algorithm 4.1.13 to where we fill the square added when reading z_i by the label t from T that appears in the same box as z_i occurs in S.

We define U to be the output of this algorithm, and note that it is an injective tableau of shape $\mathrm{ctype}(S)$.

Example 7.3.15. Consider $S = \begin{array}{|c|c|c|}\hline 6 \\ \cline{1-1} 3 & 4 \\ \cline{1-2} 1 & 2 & 5 \\ \hline\end{array}$, $\mathrm{cc}(S) = \begin{array}{|c|c|c|}\hline 2 \\ \cline{1-1} 1 & 1 \\ \cline{1-2} 0 & 0 & 1 \\ \hline\end{array}$, $T = \begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 4 & 6 \\ \hline\end{array}$. Note that this is the same S that appears in Example 4.1.14. Then we have $U = \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 5 & 2 \\ \hline 4 & 1 \\ \hline\end{array}$.

Lemma 7.3.16. *If b appears in row i of U, then $\mathrm{cc}(S)(b) \leqslant i$.*

Proof. This follows immediately from the definition of the U-tableau. $\square$

Lemma 7.3.17. *If b_1, b_2 appear consecutively in a column in U (where b_1 is lower), we must have $\mathrm{cc}(S)(b_2) = \mathrm{cc}(S)(b_1)$ or $\mathrm{cc}(S)(b_1) + 1$.*

Proof. This follows from the property that the collection $\{\mathrm{cc}(S)(b) : b \text{ appears in column } j \text{ of } U\}$ gives the entries of a cocharge word of that length. $\square$

Now, let $\mathbf{x}_U := \prod x_i^{\mathrm{row}(i)}$ where again the rows are 0-indexed.

Lemma 7.3.18. *We have $\mathbf{x}_T^S \odot \mathbf{x}_U \neq 0$.*

Proof. If b appears in row i of U, the exponent of x_b in $\mathbf{x}_U$ is i. By Lemma 7.3.16, we know that $\mathrm{cc}(S)(b) \leqslant i$. Thus ∂_b appears with exponent at most i in $\mathbf{x}_T^S \odot$. Since this holds for any choice of b, the claim holds. $\square$

Corollary 7.3.19. *We have $\mathbf{x}_T^S \odot\ g_U(\mathbf{x}) \neq 0$. Moreover, $\mathbf{x}_T^S \odot \mathbf{x}_U$ appears in $\mathbf{x}_T^S \odot g_U(\mathbf{x})$ with coefficient $\prod_i \lambda_i!$.*

Proof. We know

$$g_U(\mathbf{x}) = \left(\prod_i \lambda_i!\right) \sum_{\tau \in C(U)} \mathrm{sgn}(\tau)\tau(\mathbf{x}_U).$$

It is clear that for $\tau \neq \mathrm{id}$, we have $\tau(\mathbf{x}_U) \neq \mathbf{x}_U$. Thus $\mathbf{x}_T^S \odot \mathbf{x}_U \neq \mathbf{x}_T^S \odot \tau(\mathbf{x}_U)$ for any $\tau \neq \mathrm{id}$ (note that the monomials can only be constant if U is a single row shape, and in that case there is no $\tau \neq \mathrm{id}$). Thus from Lemma 7.3.18, the term $\mathbf{x}_T^S \odot \mathbf{x}_U$ appears in $\mathbf{x}_T^S \odot g_U(\mathbf{x})$ with nonzero coefficient.

The coefficient now follows from the above equation. □

Using the T-snaking terminology, we have

$$\begin{aligned} F_T^S \odot g_U(\mathbf{x}) &= \left(\prod_i \lambda_i!\right) \sum_{P:T\text{-snaking}} (-1)^P \mathbf{x}_P^S \odot \left(\sum_{\tau \in C(U)} (-1)^\tau \tau(\mathbf{x}_U)\right) \\ &= \left(\prod_i \lambda_i!\right) \sum_{P:T\text{-snaking}} \sum_{\tau \in C(U)} (-1)^P (-1)^\tau \mathbf{x}_P^S \odot \tau(\mathbf{x}_U). \end{aligned}$$

We want to look for which combinations of P, τ we have $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$. To do this, we first state the following Lemma.

Lemma 7.3.20. *Let* $\mathbf{b}$ *be a column in* U*. Then the monomial* $\mathbf{x}_T^S$ *has exactly one variable from column* $\mathbf{b}$ *with exponent 0. In particular, if* b_1, b_2 *are the entries in rows 0 and 1 of* $\mathbf{b}$*, then* $(\mathbf{x}_T^S)_{b_1,b_2} = x_{b_1}^0 x_{b_2}^1$.

Proof. Note that we construct U using Definition 7.3.14. We know that all entries of S with cocharge 0 contribute to the first row of U. Thus it is impossible to have two entries with cocharge 0 contribute to the same column of U. The second statement follows from Lemma 7.3.17. □

Lemma 7.3.21. *Consider* (S, T) *where* S, T *are SYT of the same shape and* $\mathrm{ctype}(S) = \mu$*, where* μ *is a three-row partition. Then the following are true.*

(a) $\mathrm{cc}(S)$ *is a semistandard tableaux consisting of 0s,1s and 2s. In particular, all the entries in row 2 must be 2s.*

(b) If c *appears in row 2 of* T*, it appears in row 2 of* U *as well.*

Proof. These are all consequences of Algorithm 4.1.13 □

We now show the conjecture holds for three rows.

Proposition 7.3.22. *Let* μ *be a three-row partition,* S *is a standard tableau satisfying* $\mathrm{ctype}(S) = \mu$ *and* U *is the filling of* μ *constructed by Algorithm* **??**.

Given $\tau \in C(U)$*, if* $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$ *for some* T*-snaking* P*, then* $\tau(\mathbf{x}_T^S \odot (\mathbf{x}_U)) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$ *and* $\tau(\mathbf{x}_T^S) = \mathbf{x}_P^S$.

Lemma 7.3.23. *There is exactly one* 3×3 $\mathbb{N}$*-matrix* A *that satisfies the following:*

(i) $A_{2,3} = A_{3,2} = A_{3,3} = 0$

(ii) $A_{i,1} + A_{i,2} + A_{i,3}$ *is the number of variables in* $\mathbf{x}_T^S \odot \mathbf{x}_U$ *that appear with exponent* $i - 1$

(iii) $A_{1,j} + A_{2,j} + A_{3,j}$ *is the number of values in* $\mathrm{cc}(S)$ *with cocharge* $j - 1$

(iv) $A_{1,1} = \mu_1$, $A_{2,1} + A_{1,2} = \mu_2$, $A_{3,1} + A_{2,2} + A_{1,3} = \mu_3$.

Example 7.3.24. For the setting of Example 4.1.14, the matrix A is

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Indeed, we have $x_U = x_2 x_3^2 x_5 x_6^2$ and $x_T^S = x_2 x_3^2 x_5 x_6$. The monomial $\mathbf{x}_T^S \odot x_U$ is a constant times x_6, so the row sums are $5, 1, 0$ from top to bottom, the column sums are $2, 3, 1$ from left to right, and the diagonal sums are $2, 2, 2$ from top to middle.

Proof. Define A to be a 3×3 $\mathbb{N}$-matrix where

$A_{i,j} = \#$ of variables which appear with exponent $i-1$ in $\mathbf{x}_T^S \odot \mathbf{x}_U$ and exponent $j-1$ in $\mathbf{x}_T^S$.

It is clear that this matrix satisfies conditions (i),(ii),(iv). To see that it satisfies (iv), note that

$$A_{1,1} = \# \text{ of variables which appear with exponent } 0 \text{ in } \mathbf{x}_U = \mu_1$$

$$A_{2,1} + A_{1,2} = \# \text{ of variables which appear with exponent } 1 \text{ in } \mathbf{x}_U = \mu_2$$

$$A_{3,1} + A_{2,2} + A_{1,3} = \# \text{ of variables which appear with exponent } 2 \text{ in } \mathbf{x}_U = \mu_3.$$

Thus A satisfies the conditions.

It is a quick computation to show that this must be unique. We can see that condition (i),(ii) fix entry $A_{3,1}$. Since (iv) fixes entry $A_{1,1}$, we have the entry $A_{2,1}$ must also be fixed by (iii). Thus the first column is fixed. Using similar logic, we can show that the whole matrix is fixed. □

Remark 7.3.25. All this lemma is saying is that for any permutation τ such that $(\tau \mathbf{x}_T^S \odot)(\tau \mathbf{x}_U) = \mathbf{x}_T^S \odot \mathbf{x}_U$, τ must permute the exponents of $\mathbf{x}_T^S \odot$ in a way that is compatible with the action of τ on $\mathbf{x}_U$.

Proof of Proposition 7.3.22. Consider τ such that $\mathbf{x}_T^S \odot \mathbf{x}_U = \mathbf{x}_P^S \odot \tau\mathbf{x}_U$. We can explicitly check how τ acts on each column, reducing the proof to the following claim.

Claim: For each column $\mathbf{b}$ of U, we have that $(\tau\mathbf{x}_T^S \odot \mathbf{x}_U)_{\mathbf{b}} = (\mathbf{x}_T^S \odot \mathbf{x}_U)_{\mathbf{b}}$ and $(\tau\mathbf{x}_T^S)_{\mathbf{b}} = (\mathbf{x}_P^S)_{\mathbf{b}}$.

We prove this claim using several cases.

Case 1. If $\mathbf{b}$ is of height one, we know that the first condition must hold. We also have that $(\mathbf{x}_T^S \odot \mathbf{x}_U)_{\mathbf{b}} = (\mathbf{x}_P^S \odot \tau\mathbf{x}_U)_{\mathbf{b}} = (\mathbf{x}_P^S \odot \mathbf{x}_U)_{\mathbf{b}}$, which implies that $(\mathbf{x}_P^S)_{\mathbf{b}} = (\mathbf{x}_T^S)_{\mathbf{b}} = (\tau\mathbf{x}_T^S)_{\mathbf{b}}$. Thus the claim holds.

Case 2. If $\mathbf{b} = \begin{array}{|c|}\hline b_2 \\ \hline b_1 \\ \hline\end{array}$, then $(\mathbf{x}_U)_{\mathbf{b}} = x_{b_1}^0 x_{b_2}^1$. Furthermore, from Lemma 7.3.20, we know that $(\mathbf{x}_T^S(\partial))_{\mathbf{b}} = \partial_{b_1}^0 \partial_{b_2}^1$. Thus $(\mathbf{x}_T^S \odot \mathbf{x}_U)_{b_1,b_2} = x_{b_1}^0 x_{b_2}^0$, which implies the first claim holds for any τ.

If $\tau(b_1) = b_2$, then $(\partial_P^S \tau\mathbf{x}_U)_{b_1,b_2} = x_{b_1}^0 x_{b_2}^0$ implies that $(\mathbf{x}_P^S)_{\mathbf{b}} = x_{b_1}^1 x_{b_2}^0 = (\tau\mathbf{x}_T^S)_{b_1,b_2}$, thus the second claim holds. Otherwise, τ fixes the column and the second claim holds as well.

Case 3. Now, consider column $\mathbf{b} = \begin{array}{|c|}\hline b_3 \\ \hline b_2 \\ \hline b_1 \\ \hline\end{array}$ of height 3. Then $(\mathbf{x}_U)_{b_1,b_2,b_3} = x_{b_1}^0 x_{b_2}^1 x_{b_3}^2$. We know from Lemmas 7.3.17 and 7.3.20 that $(\mathbf{x}_T^S(\partial))_{\mathbf{b}} = \partial_{b_1}^0 \partial_{b_2}^1 \partial_{b_3}^2$ or $(\mathbf{x}_T^S(\partial))_{\mathbf{b}} = \partial_{b_1}^0 \partial_{b_2}^1 \partial_{b_3}^1$.

If $(\mathbf{x}_T^S(\partial))_{\mathbf{b}} = \partial_{b_1}^0 \partial_{b_2}^1 \partial_{b_3}^2$, then as in Case 2 we have $(\mathbf{x}_T^S \odot \mathbf{x}_U)_{\mathbf{b}} = x_{b_1}^0 x_{b_2}^0 x_{b_3}^0$, and so applying any τ fixes this monomial, and the first claim holds. For the second claim, note that $\mathbf{x}_P^S \odot \tau\mathbf{x}_U = 1$ as well, and so $\mathbf{x}_P^S = \tau\mathbf{x}_U = \tau\mathbf{x}_T^S$ in this case.

Now, if $(\mathbf{x}_T^S(\partial))_{\mathbf{b}} = \partial_{b_1}^0 \partial_{b_2}^1 \partial_{b_3}^1$, we have that $(\mathbf{x}_P^S \odot \tau\mathbf{x}_U)_{\mathbf{b}} = (\mathbf{x}_T^S \odot \mathbf{x}_U)_{\mathbf{b}} = x_{b_1}^0 x_{b_2}^0 x_{b_3}^1$. Thus $\tau(b_1)$ cannot be b_3, since otherwise $\mathbf{x}_{b_3}$ would have an exponent of 0 in $\tau\mathbf{x}_U$ (and hence in $\mathbf{x}_P^S \odot \tau\mathbf{x}_U$).

Thus $\tau(b_1) = b_1$ or b_2. Thus, we have the following possible subcases.

(a) τ fixes $\mathbf{b}$

(b) $\tau(b_1) = b_2$, $\tau(b_3) = b_3$.

(c) $\tau(b_1) = b_1$, $\tau(b_2) = b_3$.

(d) $\tau(b_1) = b_2$, $\tau(b_2) = b_3$.

The claim is clearly true for (a). For (b), we can see that τ fixes $(\mathbf{x}_T^S \odot \mathbf{x}_U)_{\mathbf{b}}$. We also have that $(\mathbf{x}_P^S)_{\mathbf{b}} = x_{b_1}^1 x_{b_2}^0 x_{b_3}^1 = (\tau\mathbf{x}_T^S)_{\mathbf{b}}$. Thus the claims hold.

Now for cases (c),(d), we can see that $(\mathbf{x}_P^S)_{\mathbf{b}} = x_{b_1}^0 x_{b_2}^2 x_{b_3}^0$ (for (c)) and $(\mathbf{x}_P^S)_{\mathbf{b}} = x_{b_1}^1 x_{b_2}^2 x_{b_3}^0$ (for (d)). We claim that cases (c),(d) are impossible. Assume that for our choice of P, τ there is at least one column $\mathbf{b}$ in U that satisfies case (c) or (d).

Define a 3×3 $\mathbb{N}$-matrix $\tilde{A}$ by

$\tilde{A}_{i,j} = \#$ of variables which appear with exponent $i-1$ in $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U)$ and exponent $j-1$ in $\mathbf{x}_P^S$.

Since $\mathbf{x}_T^S \odot \mathbf{x}_U = \mathbf{x}_P^S \odot \tau(\mathbf{x}_U)$ and $\tau \mathbf{x}_P^S = \mathbf{x}_T^S$ for some permutation τ, we know that $\tilde{A}$ both satisfy the conditions outlined in Lemma 7.3.23. However the existence of case (c) or (d) implies that $\tilde{A}_{1,3} > A_{1,3}$ and $\tilde{A}_{1,2} < A_{1,2}$. By Lemma 7.3.23, we know that such $\tilde{A}_{1,3}$ cannot exist. Thus cases (c), (d) cannot happen. □

Lemma 7.3.26. *Let $\tau \in C(U)$ and $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$ for some T-snaking P. Then τ is a product of transpositions. Furthermore, we have the following:*

1. *τ fixes row 2 of T*
2. *If b_1, b_2 appear in rows 0,1 of some column $\mathbf{b}$ of U and τ swaps b_1 and b_2, then b_2 appears in row 1 of T, weakly to the left of b_1.*

Proof. The first statement is a direct corollary of the proof of Proposition 7.3.22: since cases (3)c and (3)d cannot happen, we know that τ acts on columns of U by either fixing it or swapping the first two entries. Thus τ fixes row 2 of U. By Lemma 7.3.21(2), this means τ fixes row 2 of T as well.

To show the second statement, note that b_1 appears in row 0 of T since it corresponds to an entry with cocharge 0 in S. Furthermore, from Lemma 7.3.20 we know that b_2 corresponds to an entry with cocharge 1 in S. By definition of Algorithm **??**, this means the entry corresponding to b_2 appears to the left of the one corresponding to b_1 in the cocharge word of S. This is equivalent to b_2 appears above and weakly to the left of b_1 in T. □

Proposition 7.3.27. *Let $\tau \in C(U)$ and $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$ for some T-snaking P. Then $\tau \in C(T)$. In particular, it swaps entries in rows 0 and 1 of T corresponding to values with cocharge 0 and 1.*

Proof. Note that from Lemma 7.3.26 we know that τ only permutes entries of cocharge 0 and 1. Hence it suffices to check that if τ does not fix the bottom box in a column $\mathbf{c}$ of T, then τ swaps the first two boxes in column $\mathbf{c}$.

We can do this by induction on columns. Look at the entry b in the bottom left corner of T. If τ does not fix b, we have that $\tau(b) = b'$ where b' must appear directly above b in T by Lemma 7.3.26(2). Thus τ preserves the column content of the first column of T.

Now assume $\tau(b) = b$ but $\tau(b') \neq b'$. We know that $\tau(b')$ must correspond to some entry with cocharge 0, thus $\tau(b')$ appears in row 0 of T. Thus $(\mathbf{x}_P^S)_{b,b'} = (\tau \mathbf{x}_T^S)_{b,b'} =$

$x_b^0 x_{b'}^0$. This implies that b, b' appear in row 0 of P. However, since P is some T-snaking and b, b' appear in the same column of T, it is impossible for both b, b' to appear in the same row in P. Thus by contradiction, we have that if $\tau(b) = b$ then $\tau(b') = b'$ as well.

Thus in either case, we have that τ preserves the column content of the first column. By induction, this shows that $\tau \in C(T)$. $\square$

Lemma 7.3.28. *Let $\tau \in C(U)$ and $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$ for some T-snaking P. Then*

$$\{c \text{ in row 0 of } T \text{ with } (\mathbf{x}_T^S)_c = x_c^2\} = \{c \text{ in row 0 of } P \text{ with } (\mathbf{x}_P^S)_c = x_c^2\} \tag{7.8}$$

Proof. Note that if c is in row 0 of T and $(\mathbf{x}_T^S)_c = x_c^2$, then c must be in a column of height 1 by Lemma 7.3.21(1). This means that c must also appear in row 0 of P by the definition of a T-snaking. Furthermore, from Proposition 7.3.27, we know that τ fixes c. Thus the condition $(\mathbf{x}_P^S)_c = (\tau \mathbf{x}_T^S)_c = x_c^2$ implies that the first set in (7.8) is contained in the second. Since the two sets must have the same cardinality, the claim holds. $\square$

Corollary 7.3.29. *For τ, P as in Prop 7.3.22, we have* $\operatorname{sgn}(\tau) = \operatorname{sgn}(P)$.

Proof. From Proposition 7.3.27, we know that $\tau \in C(T)$ such that $\tau(\mathbf{x}_T^S) = \mathbf{x}_P^S$. Note that to show $\operatorname{sgn}(\tau) = \operatorname{sgn}(P)$, it suffices to show that $\tau \cdot T = \pi \cdot P$ for some $\pi \in R(P)$. In particular, we want to show the following:

Claim: For any column $\mathbf{c}$ of T, we have that if c appears in row i of $\tau \cdot T$, then it also appears in row i of P.

For columns of height 1, we are done by Lemma 7.3.28.

Consider a column $\mathbf{c} = \begin{array}{|c|}\hline c_3 \\ \hline c_2 \\ \hline c_1 \\ \hline \end{array}$. From Lemma 7.3.26(1), we know that τ fixes c_3. Thus $(\tau \mathbf{x}_T^S)_{\mathbf{c}} = x^0_{\tau(c_1)} x^1_{\tau(c_2)} x^2_{c_3}$.We know exactly one of c_1, c_2, c_3 must appear in row 2 of P, which means it has exponent 2 in $\mathbf{x}_P^S$. This implies that c_3 is in row 2 of P. A similar argument then shows that c_2 must stay in row 2, and c_1 in row 1. Thus the claim holds for columns of height 3 in T.

Now consider column $\mathbf{c} = \begin{array}{|c|}\hline c_2 \\ \hline c_1 \\ \hline \end{array}$. We have three cases:

Case A: $(\mathbf{x}_T^S)_{\mathbf{c}} = x^0_{c_1} x^2_{c_2}$. In this case, τ fixes $\mathbf{c}$ by Proposition 7.3.27. Then $(\mathbf{x}_T^S)_{\mathbf{c}} = (\mathbf{x}_P^S)_{\mathbf{c}}$ implies that c_1 must appear in row 0 of P, thus c_2 appears in row 1 of P.

Case B: $(\mathbf{x}_T^S)_{\mathbf{c}} = x^0_{c_1} x^1_{c_2}$. Then $(\tau(\mathbf{x}_T^S))_{\mathbf{c}} = x^0_{\tau(c_1)} x^1_{\tau(c_2)} = (\mathbf{x}_P^S)_{\mathbf{c}}$. This implies that $\tau(c_1)$ must appear in row 0 of P, thus $\tau(c_2)$ appears in row 1 of P.

Case C: $(\mathbf{x}_T^S)_{\mathbf{c}} = x_{c_1}^1 x_{c_2}^2$. In this case, τ fixes $\mathbf{c}$ by Proposition 7.3.27. Then $(\mathbf{x}_P^S)_{\mathbf{c}} = x_{c_1}^1 x_{c_2}^2$ as well. If c_2 was in row 0 of P, we would have that c_2 is contained in the second set in (7.8) but not the first, which contradicts the equality in Lemma 7.3.28. Thus c_1 appears in row 0 and c_2 appears in row 1. $\square$

Theorem 7.3.30. *For μ a three-row partition, we have that $\mathbf{x}_T^S \odot \mathbf{x}_U$ appears in $F_T^S \odot g_U(\mathbf{x})$ with positive coefficient.*

Proof. To see that $\mathbf{x}_T^S \odot (\mathbf{x}_U)$ appears in $F_T^S \odot g_U(\mathbf{x})$ with nonzero coefficient, recall that

$$F_T^S \odot g_U(\mathbf{x}) = \left(\prod_i \mu_i!\right) \sum_{P:T\text{-snaking}} \sum_{\tau \in C(U)} (-1)^P (-1)^\tau \mathbf{x}_P^S \odot \tau(\mathbf{x}_U).$$

Thus it suffices to consider all combinations of P, τ such that $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$. For such a pair, $\mathbf{x}_P^S \odot \tau \mathbf{x}_U$ appears with sign $(\operatorname{sgn}(P)\operatorname{sgn}(\tau))$ in $F_T^S \odot g_U(\mathbf{x})$, from Corollary 7.3.29 it follows that $\mathbf{x}_P^S \odot \tau \mathbf{x}_U$ has positive coefficient in this expression. Since there is at least one instance of the monomial by Corollary 7.3.19, the proof is complete. $\square$

Corollary 7.3.31. *We have that if* $\operatorname{ctype}(S) \trianglerighteq \mu$*, then $F_T^S \notin I_\mu$.*

Remark 7.3.32. The issue with extending Theorem 7.3.30 beyond three rows is that Lemma 7.3.23 does not hold for more than 3 rows. However, we still believe the following conjecture holds in general, regardless of the number of rows in μ.

Conjecture 7.3.33. Given $\tau \in C(U)$, if $\mathbf{x}_P^S \odot \tau(\mathbf{x}_U) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$, then $\tau(\mathbf{x}_T^S \odot (\mathbf{x}_U)) = \mathbf{x}_T^S \odot (\mathbf{x}_U)$ and $\tau(\mathbf{x}_T^S) = \mathbf{x}_P^S$.

Now, we prove a lemma classifying the tableaux with catabolizability type $\mu = (n, m, r)$ for any three row shape μ.

Lemma 7.3.34. *There are exactly $r+1$ tableaux S with catabolism type $\mu = (n, m, r)$ for any $n \geqslant m \geqslant r \geqslant 0$. They are precisely those formed by placing $1, \ldots, n$ in the bottom row, $n+1, \ldots, n+m$ in the second row, and then $n+m+1, \ldots, n+m+i$ in the bottom row for some i, and the remaining numbers up in the third row.*

Example 7.3.35. For $\mu = (5, 3, 3)$, there are four tableaux S with catabolism type μ, namely

$$\begin{ytableau} 9 & 10 & 11 \\ 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 \end{ytableau} \qquad \begin{ytableau} 10 & 11 \\ 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 9 \end{ytableau} \qquad \begin{ytableau} 11 \\ 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 9 & 10 \end{ytableau} \qquad \begin{ytableau} 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 9 & 10 & 11 \end{ytableau}.$$

with cocharge tableaux

$$\begin{array}{ccccc} 2&2&2&& \\ 1&1&1&& \\ 0&0&0&0&0 \end{array} \qquad \begin{array}{cccccc} 2&2&&&& \\ 1&1&1&&& \\ 0&0&0&0&0&1 \end{array} \qquad \begin{array}{ccccccc} 2&&&&&& \\ 1&1&1&&&& \\ 0&0&0&0&0&1&1 \end{array} \qquad \begin{array}{cccccccc} 1&1&1&&&&& \\ 0&0&0&0&0&1&1&1 \end{array}.$$

Proof. If $\mathrm{ctype}(S) = \mu = (n, m, r)$, we must have $1, \ldots, n$ on the bottom row, and then $n+1, \ldots, n+r$ must be in the second row in order from left to right, in order for catabolizability type to be (n, r) up to that point. Indeed, if any of $n+1, \ldots, n+r$ were in the third row, then after one step of catabolism we would not have them all in a row. If any were in the bottom row, then if they are larger than any in the second row then after one step of catabolism they would be above, and if they are smaller than all in the second row then one such is $n+1$ and it is therefore $n+1$-catabolizable, a contradiction.

So there are n zeroes in a row on the bottom row of the cocharge tableau and r ones above it, corresponding in our example to the tableau:

$$\begin{array}{ccccc} 6&7&8&& \\ 1&2&3&4&5 \end{array}$$

and the question is where the numbers $n+r+1, \ldots, n+2r$ (that is, $9, 10, 11$ in our example) are positioned and with what cocharge values. We cannot have a cocharge value of 3 by the catabolism type, so the remaining values have cocharge 1 or 2. In particular S has at most three rows. Now consider the position of $n+r+1$ (in the example, 9); if it is in the second row, we can n-catabolize and then $r+1$-catabolize, a contradiction, so it is in the first or third row.

Let i be the smallest number such that $n+r+i+1$ is not in the third row; that is, $n+r+1, \ldots, n+i+1$ are in the third row and have cocharge 1, and then $n+r+i+1$ is in the second or third row. (In our running example suppose $i = 1$ so that 10 is the first number not in row 1 among $9, 10, 11$). If it is in the second row then it has cocharge 2, and by Algorithm 4.1.13 we would encounter that 2 before having any boxes in the second row, which means the catabolizability type has at least 4 parts, a contradiction. Thus it is in the top row, with cocharge 2:

$$\begin{array}{cccccc} 10&&&&& \\ 6&7&8&&& \\ 1&2&3&4&5&9 \end{array} \qquad \begin{array}{cccccc} 2&&&&& \\ 1&1&1&&& \\ 0&0&0&0&0&1 \end{array}$$

If the next number, $n+r+i+2$, was placed in the second or third row, then a catabolism step would put it above $n+r+i+1$, a contradiction, and inductively we see that the remaining entries $n+r+i+2, \ldots, n+2r$ are also all in the top row:

$$\begin{array}{cccccc} 10&11&&&& \\ 6&7&8&&& \\ 1&2&3&4&5&9 \end{array} \qquad \begin{array}{cccccc} 2&2&&&& \\ 1&1&1&&& \\ 0&0&0&0&0&1 \end{array}$$

Thus indeed the tableaux S with catabolism type μ are all of the claimed form. $\square$

Lemma 7.3.36. *Let μ and $\widetilde{\mu}$ be distinct shapes with at most three rows. Then if* $\mathrm{ctype}(S) = \mu$ *and* $\mathrm{ctype}(\widetilde{S}) = \widetilde{\mu}$, *then S and $\widetilde{S}$ cannot have the same shape and also the same total cocharge.*

Proof. This follows from Lemma 7.3.34; any S whose catabolism type has at most three rows has a cocharge tableau with only 0s and 1s in the bottom row, only 1s in the second, and only 2s in the top, and we can reconstruct S uniquely from this tableau. We can also reconstruct its catabolism type uniquely - the first part is the number of 0s, the second part is the length of the second row, and the third part is the total number of remaining squares. Thus we can reconstruct the catabolism type uniquely from the shape and cocharge of S. In other words, we cannot have two with distinct catabolism types agree on shape and cocharge. $\square$

Remark 7.3.37. Note that Lemma 7.3.36 does not hold for general μ. Consider

$$S = \begin{array}{|c|c|c|}\cline{1-1} 5 \\ \cline{1-2} 3 & 6 \\ \hline 1 & 2 & 4 \\ \hline\end{array}\ , \widetilde{S} = \begin{array}{|c|c|c|}\cline{1-1} 5 \\ \cline{1-2} 3 & 4 \\ \hline 1 & 2 & 6 \\ \hline\end{array}\ .$$

We can see that $\mathrm{ctype}(S) = (2,1,1,1,1), \mathrm{ctype}(\widetilde{S}) = (2,2,1,1)$ and both have cocharge 6.

Using the fact that are basis elements are nonzero in R_μ (Corollary 7.3.31), we can prove that Conjecture 7.1.7 holds for $\mu = (\mu_1, \mu_2, \mu_3)$ by induction. The full proof of this result will be contained in forthcoming work.

Theorem 7.3.38. *Conjecture 7.1.7 holds for three-row partition μ.*

Proof Sketch. We prove by induction. First, from Corollary 7.2.5, we know the claim holds for partitions of the form $(n,1,1)$. Using this, we can show that the claim holds for partitions of the form $(n,m,1)$ by induction on m. By induction, we assume the claim holds for $\lambda = (n+1, m-1, 1) \trianglerighteq \mu = (n,m,1)$. Since we have that λ and μ differ by a covering relation in dominance order, we can write our conjectured higher Specht basis (7.6) as

$$\{F_T^S : \mathrm{ctype}(T) \trianglerighteq \lambda\} \cup \{F_T^S : \mathrm{ctype}(T) = \mu\} \tag{7.9}$$

First note that we have the correct cardinality to be a basis since none of the polynomials are 0 in the quotient R_μ. Since R_λ is a further quotient of R_μ, we have that the first set pulls back to a linearly independent set in R_μ. Hence it suffices to show that the second set is linearly independent from the first. This follows from the fact that each collection $\{F_T^S : T \in \mathrm{SYT}(\mathrm{sh}(S))\}$ spans a copy of $V^{\mathrm{sh}(S)}$. Lemma 7.3.36 guarantees that the span of the second set in (7.9) consists of irreducibles that have not appeared in R_λ. $\square$

Chapter 8

R-catabolizability and subspaces of R_n

This chapter studies the notion of **R**-catabolizability, introduced by Shimozono–Weyman [47]. The first half of this chapter studies the combinatorics of the skew-linked module embedded into a Garsia-Procesi ring. We identify tableaux that we conjecture index the top and bottom components of a skew-linked module and then show they satisfy some **R**-catabolizability condition. The second half of the chapter focuses on taking the notion of $R(\eta, \lambda)$-catabolizability of semistandard tableaux of weight λ and extending it to a larger family of tableaux. This translates problems and conjectures regarding **R**-catabolizability into the context of standard tableaux. Using this new framework, we state some conjectures about **R**-catabolizability and skew-linked modules.

8.1 R-catabolism and skew-linked modules

There are many parallels between the skew-linked modules and the Garsia-Procesi rings. For example, the skew-linked modules are defined algebraically in an analogous way to the Garsia-Procesi rings. However, both the algebraic and combinatorial properties of the skew-linked modules remain mysterious. For example, we have no explicit algebraic description of the skew-linked module $M_{\lambda,\mu}$ as a subspace or quotient of the polynomial ring $\mathbb{C}[\mathbf{x}_n]$.

However, using results of Chen [13, Proposition 4.6.1], we can deduce that $M_{,\mu}$ is a quotient of $M_{\lambda,(n)}$. By taking the conjugate and reversing grading, we see that $M_{\mu',\lambda'}$ is a subspace of $M_{1^n,\lambda'} \cong \mathcal{E} \otimes R_\lambda$. Thus we can think of $\mathcal{E} \otimes M_{\mu',\lambda'}$ as a submodule of R_λ. Equivalently, we have that $M_{\lambda,\mu}$ (up to reversing the grading) is a submodule of R_λ.

Thus we can embed any skew-linked module $M_{\lambda,\mu}$ into the Garsia-Procesi module R_λ up to a reversing of the grading. Let $\mathrm{rev}(M_{\lambda,\mu})$ denote this module. In particular, the unique copy of V^λ (which is at the bottom of $M_{\lambda,\mu}$, or the top of $\mathrm{rev}(M_{\lambda,\mu})$) is exactly the unique copy of V^λ at the top degree of R_λ, which is indexed by the semistandard tableau Z_λ of shape and weight λ. In this section, we construct a semistandard tableaux of shape μ weight λ that we conjecture corresponds to the unique copy of V^μ in $M_{\lambda\mu}$.

To do this, we first recall the *row labeling* of a skew-linking shape, introduced in [13]. We do not include the proof that this algorithm is well-defined: for details, see [13, Section 2.2].

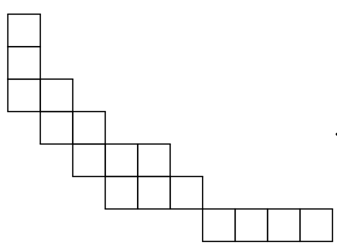
Algorithm 8.1.1. Let $\theta = \alpha/\beta$ be a skew-linking shape of $\lambda \to \mu$. We construct an ordered set partition $(A_1|\ldots|A_{\ell(\mu)})$ of $[\ell(\lambda)]$ such that

$$\sum_{r\in A_j} \lambda_r = \mu_j.$$

First, let $1 \in A_1$. We proceed inductively. Assume that $j \in A_1$. If $\beta_j = 0$, then we are done. Otherwise, let j' be the smallest j such that $\alpha_{j'} = \beta_j$. Then $j' \in A_1$. We continue until we add $p \in A_1$ such that $\beta_p = 0$.

Now, we remove all rows $r \in A_1$ from θ. This gives a smaller skew-linking shape [13, Proposition 2.3.3]. We can now construct A_2 in the same manner. We continue until we have an ordered set partition of $[\ell(\lambda)]$.

Example 8.1.2. We continue with the skew-linking shape from Example 4.4.1:

$\theta =$  .

We see that $A_1 = \{1, 2, 4.6\}, A_2 = \{3, 7\}, A_3 = \{6\}$.

By transposing the diagram, this now gives a *column labeling* of the conjugate skew-linking shape. We now use the column labeling to construct a semistandard tableau $T_{\lambda,\mu}$ of weight λ that satisfies the following:

(a) $\mathrm{sh}(T_{\lambda,\mu}) = \mu$,

(b) $\mathrm{cocharge}(Z_\lambda) - \mathrm{cocharge}(T_{\lambda,\mu}) = d(\lambda,\mu)$ (the top degree of $M_{\lambda,\mu}$).

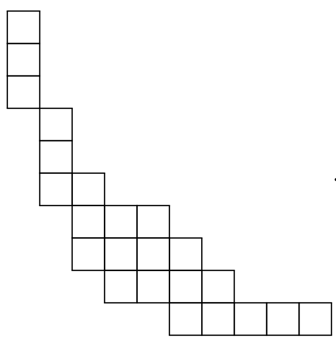
Algorithm 8.1.3. Consider $\lambda \xrightarrow{\theta} \mu$. We construct a filling of θ in the following way. Let $(B_1|\dots|B_{\lambda_1})$ be the ordered set partition of $[\mu_1]$ we get by taking the column labeling of θ.

We construct the filling recursively. First note that $1 \in B_1$. We fill column 1 of θ with $1,\dots,\mu_1'$, so that the entries are increasing as we go up. Now, take the smallest $j \neq 1$ such that $j \in B_1$. Fill in column j with $\mu_1'+1,\dots,\mu_1'+\mu_j'$. Continue until we have filled all columns j' for $j' \in B_1$.

Repeat this for each set B_i until we have filled all of θ. Let $\widetilde{T}_{\lambda,\mu}$ denote the filling of shape μ we get by sliding all the columns of this filling down. Let $T_{\lambda,\mu}$ be the filling we get by rearranging within the rows of $\tilde{T}_{\lambda,\mu}$ so that the entries are weakly increasing.

Example 8.1.4. Consider $\lambda = (5,4,4,3,2,1,1,1,1,1)$ and $\mu = (10,7,6)$. We have that

$\theta =$



.

The column labeling is given by $B_1 = \{1,2,4,9\}, B_2 = \{3,7\}, B_3 = \{5,8\}, B_4 = \{6\}, B_5 = \{10\}$.

The corresponding filling is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
3 &&&&&&&&& \\
2 &&&&&&&&& \\
1 &&&&&&&&& \\
& 6 &&&&&&&& \\
& 5 &&&&&&&& \\
& 4 & 3 &&&&&&& \\
&& 2 & 9 & 3 &&&&& \\
&& 1 & 8 & 2 & 3 &&&& \\
&&& 7 & 1 & 2 & 5 &&& \\
&&&&& 1 & 4 & 4 & 10 & 1
\end{array}$$

By sliding the columns down, we get the filling

$$\widetilde{S}_{\lambda,\mu} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-6} 3&6&3&9&3&3 \\ \cline{1-7} 2&5&2&8&2&2&5 \\ \hline 1&4&1&7&1&1&4&4&10&1 \\ \hline \end{array}$$

Reordering the entries within the rows so that they are weakly increasing gives the semistandard tableaux

$$S_{\lambda,\mu} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-6} 3&3&3&3&6&9 \\ \cline{1-7} 2&2&2&2&5&5&8 \\ \hline 1&1&1&1&1&4&4&4&7&10 \\ \hline \end{array}.$$

It is clear that $S_{\lambda,\mu}$ is a semistandard tableau of shape μ, weight λ.

Now, we want to show that $\text{cocharge}(Z_\lambda) - \text{cocharge}(S_{\lambda,\mu}) = d_{(}\lambda, \mu)$. Recall that we know $\text{cocharge}(Z_\lambda) = n(\lambda)$ and for any $T \in \text{SSYT}(\lambda)$ we have that $\text{charge}(T) = n(\lambda) - \text{cocharge}(T)$. Thus, it suffices to show that $\text{charge}(S_{\lambda,\mu}) = d(\lambda, \mu)$.

Lemma 8.1.5. *Let k be the entry in the highest box of column j in $\widetilde{S}_{\lambda,\mu}$, where $j \in B_1$. Then for any $j' < j$ we have that column j' of $\widetilde{S}_{\lambda,\mu}$ does not contain $(k+1)$.*

Proof. For any $j' < j$, we know that either $j' \in B_1$ or $j' \in B_i$ for some $i \neq 1$. If $j' \in B_1$, we know that all entries in column j' are less than k by construction. If $j' \neq 1$, we know that

$$\sum_{b \in B_i, b<j} \mu'_b \leqslant \sum_{b \in B_1, b \leqslant j} \mu'_b$$

since $\sum_{b \in B_1, b \leqslant j} \mu'_b = \alpha'_1 - \beta'_j$ and $\sum_{b \in B_i, b<j} \mu'_b = \alpha'_{i_1} - \beta'_{i_s}$ where i_1 is the smallest entry in B_i and i_s is the largest element in B_i satisfying $i_s < j$. ☐

Proposition 8.1.6. *For skew-linked pair $\lambda \to \mu$, we have that charge$(T_{\lambda,\mu}) = d(\lambda, \mu)$.*

Proof. Recall that $d(\lambda, \mu) = |\beta|$, where $\theta = \alpha/\beta$.

We construct the first subword used to compute $\text{charge}(S_{\lambda,\mu})$ described in Definition 7.1.3. First, consider $\{1, \ldots, \mu'_1\}$ which appear in the first column of θ. We know there cannot appear any $\mu'_1 + 1$ to the left of μ'_1 in $\text{rw}(S_{\lambda,\mu})$. Thus $(\mu'_1 + 1)$ appears to the right of 1 in $\mathbf{w}$. Repeating this process (using Lemma 8.1.5), we have that $\mathbf{z}^{(1)} = \mathbf{c}^{(j_1)}\mathbf{c}^{(j_2)} \ldots \mathbf{c}^{(j_k)}$ where $B_1 = \{j_1 < j_2 < \cdots < j_k\}$ and $\mathbf{c}^{(j_i)}$ denotes the reading word of column j_i in θ. In particular, $\mathbf{c}^{(j_1)}$ is $\mu'_1 \ldots 21$.

We can see that

$$\text{charge}(\mathbf{z}^{(1)}) = \sum_i (i-1)\mu'_{j_i} = \sum_i (i-1)(\beta'_{j_{i-1}} - \beta'_{j_i}) = \sum_i \beta'_{j_i}. \tag{8.1}$$

The claim follows from induction on λ_1. If $\lambda = 1^n$, then $\mathbf{z}^{(1)} = \mathrm{rw}(S_{1^n,\mu})$, thus

$$\mathrm{charge}(S_{1^n,\mu}) = \sum_j \beta'_j = |\beta|.$$

Now assume the claim is true for partitions ν such that $\nu_1 < \lambda_1$. Deleting all the columns of θ corresponding to entries j in B_1 gives a smaller skew-linked shape $\widetilde{\alpha}/\widetilde{\beta}$ of $(\widetilde{\lambda}, \widetilde{\mu})$ where $\widetilde{\lambda} = (\lambda_1 - 1, \ldots, \lambda_l - 1)$. Then we have that

$$\mathrm{charge}(S_{\lambda,\mu}) = \mathrm{charge}(\mathbf{z}^{(1)}) + \mathrm{charge}(S_{\widetilde{\lambda},\widetilde{\mu}}). \tag{8.2}$$

By induction, we know that

$$\mathrm{charge}(S_{\widetilde{\lambda},\widetilde{\mu}}) = |\widetilde{\beta}| = \sum_{j \notin B_1} \beta_j. \tag{8.3}$$

Thus combining (8.1), (8.2), and (8.3) gives us the claim. □

Using this construction, we conjecture the following.

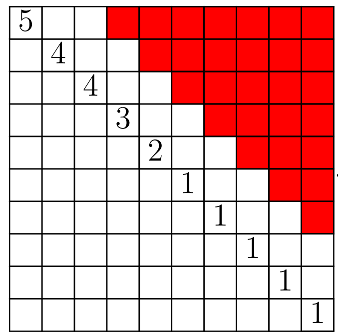
Conjecture 8.1.7. (i) There exists an explicit algebraic description of $\mathrm{rev}(M_{\lambda,\mu})$ as a submodule of R_λ, where the top and bottom degree pieces are indexed by $S_{\lambda,\mu}, Z_\lambda$ respectively.

(ii) There exists a combinatorial formula for the Schur expansion of $H(\Psi(\lambda \to \mu); \lambda)$, where the bottom and top degree pieces are indexed by $S_{\lambda,\mu}, Z_\lambda$ respectively.

Remark 8.1.8. Note that Conjecture 8.1.7 (a) is not the most well-defined: there does not exist a well-defined notion of a submodule of R_λ indexed by a given semistandard tableau. However, we can use constructions like the higher Specht basis in Chapter 7 that explicitly construct irreducible components of R_λ indexed by certain tableaux to make this statement more precise.

Conjecture 8.1.9. Consider $\eta \models \ell(\lambda)$ such that $(\Psi(\lambda \to \mu); \lambda) \subseteq (\Delta(\eta); \lambda)$ as root ideals. Then both Z_λ and $S_{\lambda,\mu}$ are $R(\eta, \lambda)$-catabolizable.

Remark 8.1.10. The condition $(\Psi(\lambda \to \mu); \lambda) \subseteq (\Delta(\eta); \lambda)$ can be described in the following way. Given our skew-linked pair that in Example 8.1.4, the corresponding root ideal is

$(\Psi(\lambda \to \mu); \lambda) =$ 

.

Then want to find $\eta \models 10$ so that the block diagonal matrix given by η (i.e, the block sizes are $\eta_1, \eta_2, \ldots,$) fits into the nonroots in $(\Psi(\lambda \to \mu); \lambda)$. For example, $(3, 3, 3, 1)$ is a valid composition, while $(2, 4, 3, 1)$ is not.

We will give a proof sketch for Conjecture 8.1.9. To do this, we require the following lemma.

Lemma 8.1.11. *Let θ be a skew-linking shape corresponding to the pair (λ, μ). Deleting the first row of θ is still a skew-linking shape.*

Proof. Let $\widehat{\theta}$ denote the skew-shape we get by deleting the first row of θ. We can see that the row lengths of θ correspond to the partition $\widehat{\lambda} = (\lambda_2, \ldots)$. It suffices to show that the sequence of column lengths of $\widehat{\theta}$ still form a partition. We can see that the column lengths of θ are given by $(\mu'_1, \mu'_2, \ldots, \mu'_{\mu_1})$. Since deleting the first row of θ is equivalent to deleting one box from each of the the last λ_1 columns, the column lengths of $\widehat{\theta}$ are given by the sequence $(d_1, \ldots, d_{\mu_1})$ where

$$d_i = \begin{cases} \mu'_i & \text{if } i \leqslant \mu_1 - \lambda_1 \\ \mu'_i - 1 & \text{if } \mu_1 - \lambda_1 < i \leqslant \mu_1 \end{cases}$$

It is clear that $(d_1 \geqslant \cdots \geqslant d_{\mu_1})$ is a partition, hence $\widehat{\theta}$ is a skew-linking shape. $\square$

Proposition 8.1.12. *Consider $\eta \models \ell(\lambda)$ such that $(\Psi(\lambda \to \mu); \lambda) \subseteq (\Delta(\eta); \lambda)$ as root ideals. Then we have that $S_{\lambda,\mu}$ is R_1-catabolizable, where $R_1 = (\lambda_1, \ldots, \lambda_{\eta_1})$.*

Proof. We first show that $S_{\lambda,\mu}$ is R_1-catabolizable. Note that by the definition of $\Psi(\lambda \to \mu)$ (4.5), we have that the root ideal containment condition implies that $\eta_1 \leqslant \mu'_{\beta_1}$. WLOG we assume that $\beta_1 \neq 0$.

We claim that for $i \leqslant \mu'_{\beta_1}$, we have that every i in $S_{\lambda,\mu}$ appears in row i, which implies that $S_{\lambda,\mu}$ is $R_1 = (\lambda_1, \ldots, \lambda_{\eta_1})$-catabolizable for any $\eta_1 \leqslant \mu'_{\beta_1}$.

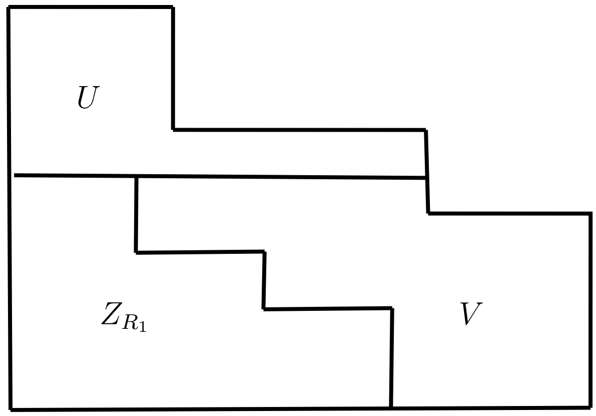


Figure 8.1: Slicing $S_{\lambda,\mu}$ into north and south tableaux

We can see where such i appear in $\widetilde{S}_{\lambda,\mu}$, the filling we have before we reorder within the rows. If i appears in column $j \leqslant \beta_1$, we know that $\mu'_j \geqslant \mu'_{\beta_1}$. Thus by construction of $\widetilde{S}_{\lambda,\mu}$, we know that i must appear in row i, since j is the smallest entry in some B_k. On the other hand, if i appears in column $j' > \beta_1$, we know that j' is the only column in some B_k. Thus i appears in the row i. Since reordering within the rows does not change this property, we have that the claim holds, thus $\mathrm{cat}_{R_1}(S_{\lambda,\mu})$ is defined. □

Conjecture 8.1.13. Furthermore, we have that $\mathrm{cat}_{R_1}(S_{\lambda,\mu}) = S_{\nu,\kappa}$ for some partitions ν, κ.

Remark 8.1.14. From Lemma 8.1.11, we know that deleting the first η_1 rows of $\lambda \to \mu$ is still a skew-linking shape of pair (ν, κ) where $\nu = (\lambda_{\eta_1+1}, \dots)$. We claim that $\mathrm{cat}_{R_1}(S_{\lambda,\mu}) = S_{\nu,\kappa}$ up to shifting the alphabet.

Now, we can give a proof sketch of Conjecture 8.1.9 if we assume Conjecture 8.1.13 is true

Proof of Conjecture 8.1.9. We can see that this follows from induction on the length of η. If $\eta = (\ell(\lambda))$, the root ideal containment condition implies that $\Psi(\lambda \to \mu)$ is empty: in particular, we must have $\mu = \lambda$ and $S_{\lambda,\lambda} = Z_\lambda$, which is clearly $\mathbf{R} = (\lambda)$-catabolizable. Thus the claim holds.

Now assume the claim holds for composition with at most $k-1$ parts. Consider $(\eta_1, \dots, \eta_k) \models \ell(\lambda)$. We have that $\mathrm{cat}_{R_1}(S_{\lambda,\mu}) = S_{\nu,\kappa}$. Note that $(\eta_2, \dots, \eta_k) \models \ell(\nu)$. Furthermore, we have that $\Psi(\nu \to \kappa)$ is contained in $(\Delta((\eta_2, \dots), \nu)$. Thus by induction, we can conclude that $S_{\nu,\kappa} = \mathrm{cat}_{R_1}(S_{\lambda,\mu})$ is $(R_2, \dots)$-catabolizable. Thus $S_{\lambda,\mu}$ is $\mathbf{R}$-catabolizable.

The same argument holds for Z_λ, since $Z_\lambda = S_{\lambda,\lambda}$ and the corresponding root ideal $\Psi(\lambda \to \lambda)$ is empty. Thus we have that $(\Psi(\lambda \to \lambda); \lambda) \subseteq (\Psi(\lambda \to \mu); \lambda) \subseteq (\Delta(\eta); \lambda)$. $\square$

Conjecture 8.1.13 is a weaker version of the following conjecture, which originally appeared in Chen's thesis [13].

Conjecture 8.1.15.

$$H(\Psi(\lambda \to \mu); \lambda) = \sum_{\substack{T \in \mathrm{SSYT}(\lambda) \\ T \text{ is } R(\eta,\lambda)\text{-catabolizable for all } (\Delta(\eta);\lambda) \subseteq (\Psi(\lambda \to \mu);\lambda)}} q^{\mathrm{charge}(T)} s_{\mathrm{sh}(T)}.$$

8.2 Extending R-catabolism

In this section, we define $\mathbf{R} = R(\eta, \lambda)$-catabolizability of standard tableaux for any choice of partition λ. To do this, we first extend the definition $\mathbf{R}$-catabolizability given in Section 4.2 to consider a family of semistandard tableaux that contains both $\mathrm{SSYT}(\lambda)$ and SYT_n. We then show that this notion of $\mathbf{R}$-catabolizablity is preserved under certain cyclage poset embeddings.

Recall that the pair (η, λ) satisfies the conditions $\eta \models \ell(\lambda), \lambda \vdash n$.

Definition 8.2.1. For two compositions $\alpha, \beta \models n$, we say β is a *refinement* of α (denoted $\beta \preceq \alpha$) if β can be obtained by replacing each part α_i with a composition $(\beta_j, \beta_{j+1}, \dots, \beta_k) \models \alpha_i$. We let $\beta_j \subseteq \alpha_j$ denote that β_j is part of the composition that refines α_j.

Consider $\mathbf{R} = R(\eta, \lambda) = (R_1, R_2, \dots)$ and composition $\alpha \models n$ such that α is a refinement of λ. Then α can be written as a concatenation of compositions $\alpha = (A_1, A_2, \dots, A_l)$ where A_i is a refinement of R_i. Note that our compositions A_i may have parts equal to 0. We can assume each A_i starts with a nonzero part.

Example 8.2.2. Consider $\mathbf{R} = (\ydiagram{3,2}, \ydiagram{2,1})$ for $\lambda = (3, 2, 2, 1), \eta = (2, 2)$. We see that $\alpha = (1, 2, 2, 1, 1, 1) \preceq \lambda$. We can write $\mathbf{A} = (A_1, A_2)$ for $A_1 = (1, 2, 2), A_2 = (1, 1, 1)$.

Given a shape λ and $\alpha \preceq \lambda$, we define $Z_\lambda(\alpha)$ to be a semistandard tableau of shape λ weight α obtained by filling rows from bottom to top with the smallest available entries.

Example 8.2.3. We have for $\lambda = (5,3,2), \alpha = (1,2,2,2,1,2)$, the tableau $Z_\lambda(\alpha)$ is given by

$$Z_\lambda(\alpha) = \begin{array}{|c|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-3} 4 & 4 & 5 \\ \hline 1 & 2 & 2 & 3 & 3 \\ \hline \end{array}.$$

Definition 8.2.4. Consider $\mathbf{R} = R(\eta, \lambda)$ and $\alpha \preceq \lambda$. We say $T \in \mathrm{SSYT}(\alpha)$ is R_1-*catabolizable* if $T|_{[A_1]} = Z_{R_1}(A_1)$. In this case, we define $\mathrm{cat}_{R_1}(T) := H_{\eta_1}(T - Z_{R_1}(A_1))$.

Definition 8.2.5. Consider $\mathbf{R} = R(\eta, \lambda)$ and $\alpha \preceq \lambda$. For $T \in \mathrm{SSYT}(\alpha)$, the notion of $\mathbf{R}$-catabolizable is uniquely determined by the following rules.

1. If $\mathbf{R}$ is the empty sequence, the unique $\mathbf{R}$-catabolizable tableau is the empty tableau.

2. Otherwise, T is $\mathbf{R}$-catabolizable if and only if $T|_{[A_1]} = Z_{R_1}(A_1)$ and $\mathrm{cat}_{R_1}(T)$ is $\hat{\mathbf{R}} = (R_2, \ldots, R_t)$-catabolizable in the alphabet $[\ell(A_1) + 1, n]$.

Example 8.2.6. Continuing with Example 8.2.2, we have $\mathbf{R} = \left(\begin{array}{|c|c|c|}\cline{1-2} \; & \; \\ \hline \; & \; & \; \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} \; \\ \hline \; & \; \\ \hline \end{array}\right)$ and $A_1 = (1,2,2), A_2 = (1,1,1)$. Consider $T = \begin{array}{|c|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 3 & 3 \\ \hline 1 & 2 & 2 & 5 & 7 \\ \hline \end{array}$.

We can see that

$$T|_{[A_1]} = \begin{array}{|c|c|c|}\cline{1-2} 3 & 3 \\ \hline 1 & 2 & 2 \\ \hline \end{array} = Z_{R_1}(A_1),$$

thus T is R_1-catabolizable. We can check that

$$\mathrm{cat}_{R_1}(T) = H_2\left(\begin{array}{ccccc} \boxed{6} & & & & \\ & & & \boxed{5} & \boxed{7} \end{array}\right) = P\left(\begin{array}{ccc} \boxed{5} & \boxed{7} & \\ & & \boxed{6} \end{array}\right) = \begin{array}{|c|c|}\cline{1-1} 7 \\ \hline 5 & 6 \\ \hline \end{array}$$

is R_2-catabolizable in the alphabet $[5,7]$. Thus T is $\mathbf{R}$-catabolizable.

Remark 8.2.7. Here are some observations regarding $\mathbf{R}$-catabolizability for standard Young tableaux.

(a) If $\eta = \ell(\lambda)$, the only $R(\ell(\lambda), \lambda)$-catabolizable SYT is $Z_\lambda(1^n)$.

(b) If $\eta = 1^{\ell(\lambda)}$, then $R(1^{\ell(\lambda)}, \lambda)$-catabolizability is equivalent to λ-catabolizablity on SYT. Thus

$$\{T \in \mathrm{SYT}_n \mid T \text{ is } R(\ell(\lambda), \lambda)\text{-catabolizable}\} = \{T \in \mathrm{SYT}_n \mid \mathrm{ctype}(T) \trianglerighteq \lambda\}.$$

We can now state our main theorem, which is that $\mathbf{R}$-catabolizablity on SSYT(λ) is equivalent to that on SYT_n via the Lascoux standardization map std.

Theorem 8.2.8. *For $\mathbf{R} = R(\eta, \lambda)$, a standard Young tableau T is $\mathbf{R}$-catabolizable if and only if $T = \mathrm{std}(S)$ for some $S \in \mathrm{SSYT}(\lambda)$ where S is $\mathbf{R}$-catabolizable and $\mathrm{SSYT}(\lambda)$ is the collection of all semistandard of weight λ.*

We first prove the following proposition, which is a weaker version of Theorem 8.2.8.

Proposition 8.2.9. *Consider $S \in \mathrm{SSYT}(\lambda)$ and $\mathbf{R} = R(\eta, \lambda)$. We have S is $\mathbf{R}$-catabolizable if and only if $\mathrm{std}(S)$ is $\mathbf{R}$-catabolizable.*

Remark 8.2.10. Proposition 8.2.9 does not consider standard tableau T where $\mathrm{ctype}(T) \ntrianglerighteq \lambda$.

Recall from Section 4.2 that we can define std using a sequence of compositions from λ to 1^n such that each intermediate composition is a refinement of λ. From Theorem 2.7.7, we know that the map std is independent of how we construct this sequence. Hence we can choose a sequence so that at each step we get that the composition is a refinement of λ.

To do this, we first make the following observation. Recall that we give the definitions of the crystal operators in Definition 2.7.3.

Lemma 8.2.11. *Consider a composition α such that $\alpha \preceq \lambda$. Let β be the resulting composition we get from applying one of the four operations below to α. We have that $\beta \preceq \lambda$ as well.*

(i) s_i *where* $\alpha_{i+1} = 0$,

(ii) f_1 *where* $\alpha_2 = 0$,

(iii) s_i *where* $\alpha_i = 1, \alpha_{i+1} > 1$ *and* $\alpha_i, \alpha_{i+1} \subseteq \lambda_k$ *for some* k,

(iv) $\tau = s_{i-t} \ldots s_{i-1}$ *where* $(\alpha_{i-t}, \ldots, \alpha_{i-1}, \alpha_i) = (1, \ldots, 1, t)$ *and* $\alpha_i \subseteq \lambda_k$ *but* $\alpha_{i-1} \nsubseteq \lambda_k$ *for some* k.

Proof. The claim is obvious for the first three cases. For (iv), note that $\alpha_i \leqslant \lambda_k \leqslant \lambda_{k-1}$. Then, since $\alpha_{i-t} + \cdots + \alpha_{i-1} = t$, we know that $\alpha_{i-t}, \ldots, \alpha_{i-1}, \subseteq \lambda_{k-1}$. The operation τ swaps the t in the refinement of λ_k with the same number of 1s in the refinement of λ_{k-1}. □

Using this observation, we can define a sequence of compositions from λ to 1^n such that each step of intermediate compositions differ by one of the four operations listed in Lemma 8.2.11.

Algorithm 8.2.12. Consider a partition $\lambda \vdash n$, where we assume λ has n many 0s at the end. We standardize the parts of λ in order, starting with λ_1. We first apply operation (i) repeatedly until we get the following composition:

$$(\lambda_1, \dots, \lambda_{h(\lambda)}, 0, \dots) \mapsto (\lambda_1, 0^{\lambda_1 - 1}, \lambda_2, \dots, \lambda_{h(\lambda)}, 0, \dots).$$

Each intermediate composition between the two above is still a refinement of λ. Now, apply (ii) to get

$$(\lambda_1, 0, 0, \dots,) \mapsto (\lambda_1 - 1, 1, 0, \dots).$$

Repeatedly apply (i) to get

$$(\lambda_1 - 1, 1, 0^{\lambda_1 - 2}, \dots) \mapsto (\lambda_1 - 1, 0^{\lambda_1 - 2}, 1, \dots).$$

We repeat this process, using (i) and (ii) to get $(1^{\lambda_1}, \lambda_2, \dots,)$.

Once we have replaced λ_1 with all 1s, we do the same to λ_2. We first apply (iv) to get

$$(1^{\lambda_1 - \lambda_2}, 1^{\lambda_2}, \lambda_2, \dots) \mapsto (1^{\lambda_1 - \lambda_2}, \lambda_2, 1^{\lambda_2}, \dots).$$

Then we apply (iii) repeatedly until we get

$$(1^{\lambda_1 - \lambda_2}, \lambda_2, 1^{\lambda_2}, \dots) \mapsto (\lambda_2, 1^{\lambda_1 - \lambda_2}, 1^{\lambda_2} \dots).$$

We repeat the steps above to replace λ_2 with 1s, then do the same for each part of λ until the weight is 1^n.

We illustrate this standardization in Section 8.3.

Now, we want to show **R**-catabolism and the operations listed in Lemma 8.2.11 are compatible. We first prove some lemmas that we will use to show this claim.

Lemma 8.2.13. *Consider words* $\mathbf{u}, \mathbf{v}$ *where* $\mathbf{uv}$ *contains exactly one* i *and at least two* $(i+1)$*. Then the unique* $i+1$ *that is fixed when applying* s_i *to* $\mathbf{uv}$ *is the same as the one we get when applying* s_i *to* $\mathbf{vu}$*.*

Proof. The proof follows from casework. The cases we consider depend on whether the unique i appears in $\mathbf{u}$ or $\mathbf{v}$. Note that determining the unique $(i+1)$ that is fixed by s_i is equivalent to checking whether there exists an $(i+1)$ that is paired with the unique i.

Case 1: i appears in $\mathbf{v}$. If there exists a $(i+1)$ in $\mathbf{v}$ before i, then i is always paired with an $(i+1)$ in $\mathbf{v}$ regardless of whether we consider $\mathbf{uv}$ or $\mathbf{vu}$. If no $(i+1)$ appears in $\mathbf{v}$ before i, we must consider pairings on $\mathbf{uv}$ and $\mathbf{vu}$ separately.

(a) We first consider **uv**. If i is paired with an $(i+1)$ in **u**, then we know i is paired with the last $(i+1)$ in **u**. On the other hand, if we consider **vu**, then i is no longer paired with anything. In either case, applying s_i changes all $(i+1)$ but the last $(i+1)$ in **u** to i.

(b) If i in **uv** is unpaired, then **v** contains no $(i+1)$ before i and **u** contains no $(i+1)$. Applying to s_i changes all but the last $(i+1)$ in **v** to i. The same holds when applying s_i to **vu**.

Case 2: i appears in u. If there exists a $(i+1)$ in **u** before i, then i is always paired with an $(i+1)$ in **u** regardless of whether we consider **uv** or **vu**. If no $(i+1)$ appears in **u** before i, then the i remains unpaired when considering **uv**. We consider the following two subcases.

(a) If there exists a $(i+1)$ in **v**, we know that when applying s_i to **uv**, the only $(i+1)$ that remains is the last one in **v**. On the other hand, if we swap **u** and **v**, then i is now paired with the last $(i+1)$ in **v**. Thus when applying s_i to **vu** the only $(i+1)$ that remains is the last one in **v**.

(b) If there are no $(i+1)$ in **v**, swapping **v** and **u** does not change anything.

□

Lemma 8.2.14. *Consider* $\mathbf{w} = r^{q_1}(r-t)(r-t+1)\dots(r-2)(r-1)r^{q_2}$ *with* $q_1 \leqslant t$ *and permutation* $\tau = s_{r-t}\dots s_{r-1}$. *Then* $\tau\mathbf{w} = (r-q_1+1)\dots(r-1)r(r-t)^{q_1+q_2}(r-t+1)\dots(r-q_1-1)(r-q_1)$.

Proof. This follows from a direct computation. To be precise,applying τ to **w** is equivalent to replacing each subwords of **w** in the following way:

- $r^{q_1} \mapsto (r-q_1+1)\dots(r-1)r$,
- $(r-t)(r-t+1)\dots(r-2)(r-1)r^{q_2} \mapsto (r-t)^t(r-t+1)\dots(r-q_1-1)(r-q_1)$.

Note that for the first q_1 transpositions we will have one pairing appear when applying the operator. □

Now we use these lemmas to show the following.

Proposition 8.2.15. *Consider* $T \in \mathrm{SSYT}(\alpha)$ *such that* $\alpha \preceq \lambda$ *and* T *is* **R**-*catabolizable for* $\mathbf{R} = R(\lambda, \eta)$. *If* $\widetilde{T}$ *is the result of applying one of the four operations in Lemma 8.2.11 to* T, *then* $\widetilde{T}$ *is* **R**-*catabolizable.*

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & i & i+1 & \cdots & \cdots & \underline{i+1} & \cdots \\ \hline \end{array}$$

Figure 8.2: Case 1a

$$\begin{array}{|c|c|c|ccc}\hline \cdots & \underline{i+1} & \cdots & & & \\ \hline \cdots & \cdot & \cdots & \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{i+1} & \multicolumn{1}{c|}{\cdots} \\ \hline \end{array}$$

Figure 8.3: Case 1b

Proof. Let β be the weight of $\widetilde{T}$. From Lemma 8.2.11, we know that $\beta \preceq \lambda$. We can write α, β as sequences of compositions $\alpha = (A_1, A_2, \ldots, A_l)$, $\beta = (B_1, B_2, \ldots, B_l)$, where A_i, B_i are refinements of R_i. Note that by induction on the length of $\mathbf{R}$, it suffices to show that the following facts hold:

(a) $\widetilde{T}$ is R_1-catabolizable: $\widetilde{T}|_{[B_1]} = Z_{R_1}(B_1)$

(b) R_1-catabolism and the operation "commute": $\phi(\mathrm{cat}_{R_1}(T)) = \mathrm{cat}_{R_1}(\widetilde{T})$

where $\phi(\mathrm{cat}_{R_1}(T))$ denotes the result of applying operations in Lemma 8.2.11 so that $\phi(\mathrm{cat}_{R_1}(T))$ and $\mathrm{cat}_{R_1}(\widetilde{T})$ have the same weight.

We check this for each of the four operations. The claim clearly holds for (i).

For (ii), note that $\alpha_2 = 0$. Thus $\widetilde{T}$ is the tableau we get by changing the last 1 in the first row to a 2. This does not change the relative order of the letters in the subtableau of shape R_1 in $\widetilde{T}$, thus (a) holds. It also does not change anything outside of the subtableau of shape R_1, thus (b) holds.

For (iii), since $\alpha_i = 1, \alpha_{i+1} > 1$, we know that applying s_i replaces all but one of the $(i+1)$ in T with i. We assume $\alpha_i, \alpha_{i+1} \subseteq \lambda_k$. We consider two different cases.
Case 1: $1 \leqslant k \leqslant \eta_1$ In this case, $i, i+1$ are part of the subtableau $T|_{[A_1]} = Z_{R_1}(A_1)$. We have either i and $i+1$ all appear in the same row of T (Case 1a in Figure 8.2) or some of the $i+1$ appear in the row above i in T (Case 1b in Figure 8.3). Note that in the figures, the pink boxes denotes those with $i, i+1$.

In Case 1a, the unique i in T is not paired with any $i+1$, hence s_i changes all but the rightmost $i+1$ to i. In Case 1b, i is paired with the last $(i+1)$ in the row above, hence s_i changes all but the the rightmost $i+1$ in the higher row to i. In either case, applying s_i still preserves the relative order of these entries in the subtableau of shape R_1, thus (a) holds. Since it does not change anything outside of R_1, (b) holds as well.
Case 2: $k > \eta_1$ In this case, we have that $B_1 = A_1$ and $\widetilde{T}|_{[B_1]} = T|_{[A_1]}$, thus (a) is true. Now, let U, V denote the north and south tableau obtained by slicing

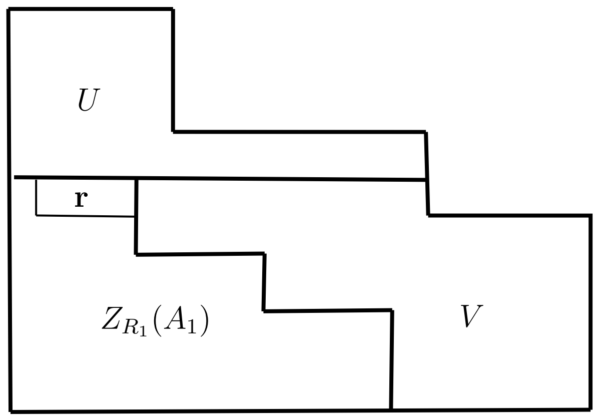


Figure 8.4: Slicing T into north and south tableaux

$T \setminus Z_{R_1}(A_1)$ between rows $\eta_1, \eta_1 + 1$ (see Figure 8.4). Denote the corresponding reading words by $\mathbf{u}, \mathbf{v}$. Note that $\mathrm{cat}_{R_1}(T) = P(\mathbf{vu})$.

In this case, (b) is equivalent to $\mathrm{cat}_{R_1}(s_i(T)) = s_i\ \mathrm{cat}_{R_1}(T)$. Since s_i preserves Knuth equivalence (Proposition 2.7.5), it suffices to check that applying s_i to $\mathbf{vu}$ and then swapping $\mathbf{v}$ and $\mathbf{u}$ is equivalent to applying s_i on $\mathbf{uv}$. This follows from Lemma 8.2.13.

For (iv), note that if $k \neq \eta_1 + 1$ (i.e, α_r is the first part of A_2), the argument is the same as iteratively applying the arguments in (iii). Thus we restrict to the case where $k = \eta_1$ and τ replaces $(1^t, t)$ with $(t, 1^t)$.

Let $r = |R_1| + 1$. We first note that $t \subseteq \lambda_{\eta_1+1}$, hence $t \leqslant \lambda_{\eta_1+1} \leqslant \lambda_{\eta_1}$. Note that the last row of the partition R_1 is of length λ_{η_1}. From this, we know that $\mathbf{r} := (r-t)(r-t+1)\dots(r-2)(r-1)$ appear in the last row of shape R_1 (see Figure 8.4). Thus, the subword of $\mathrm{rw}(T)$ consisting of letters $r, r-1, r-2, \dots, r-t$ is of the form $r^q\ \mathbf{r}\ r^{t-q}$, where r^q is in the reading word $\mathbf{u}$ of U and r^{t-q} is in the reading word $\mathbf{v}$ of V.

Let $\tilde{\mathbf{u}}$ (resp. $\tilde{\mathbf{v}}$) be the word we get by replacing the subword r^q (resp. r^{t-q}) of $\mathbf{u}$ (resp. $\mathbf{v}$) with $(r-q+1)\dots(r-1)r$ (resp. $(r-t+1)\dots(r-q-1)(r-q)$. From Lemma 8.2.14, we know that $\mathrm{rw}(\widetilde{T})$ can be obtained by replacing $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}$ in $\mathrm{rw}(T)$ with $\tilde{\mathbf{u}}$, $\tilde{\mathbf{v}}$, $(r-t)^t$. In particular, we replace $\mathbf{r}$ in T with $(r-t)^t$. Thus we have that $\widetilde{T}|_{[B_1]} = \widetilde{T}|_{[1,r-t]} = Z_{R_1}(B_1)$. Hence $\widetilde{T}$ is R_1-catabolizable, satisfying (a). Furthermore, we can see that $\mathrm{rw}(\mathrm{cat}_{R_1}(\widetilde{T})) \sim \tilde{\mathbf{v}}\tilde{\mathbf{u}}$, thus $cat_{R_1}(\widetilde{T} = P(\tilde{\mathbf{v}}\tilde{\mathbf{u}})$.

Now we check (b). In this case, we have that $B_2 = (1^t, \widehat{A_2})$, if we set $A_2 = (t, \widehat{A_2})$. Then ϕ takes $\mathrm{cat}_{R_1}(T)$, which has weight $(t, \widehat{A_2}, \dots)$ to a tableau of weight

$(1^t, \widehat{A_2}, \dots)$.

Since $\mathrm{rw}(\mathrm{cat}_{R_1}(T)) \sim \mathbf{vu}$ and Proposition 2.7.5 we have that $\mathrm{rw}(\phi(\mathrm{cat}_{R_1}(T))) \sim \phi(\mathbf{vu})$. Thus it suffices to see how ϕ acts on the word $\mathbf{vu}$. We see that $\phi(\mathbf{vu})$ is the word we get by applying (i) to $\mathbf{vu}$ repeatedly to get a word of weight $(t, 0^t, \widehat{A_2}, \dots)$ and then applying (ii) and (i) until we have a word of weight $(1^t, \widehat{A_2}, \dots)$.

In particular, this takes the subword $r^t = r^{t-q}r^q$ of $\mathbf{vu}$ and replaces it with $(r-t+1)\dots(r-1)r$. Thus $\phi(\mathbf{vu}) = \tilde{\mathbf{v}}\tilde{\mathbf{u}}$. Thus $\phi(\mathrm{cat}_{R_1}(T)) = P(\tilde{\mathbf{v}}\tilde{\mathbf{u}}) = \mathrm{cat}_{R_1}(\widetilde{T})$. Thus we must have $\phi(\mathrm{cat}_{R_1}(T)) = \mathrm{cat}_{R_1}(\widetilde{T})$ ☐

Combining Proposition 8.2.15 and Algorithm 8.2.12 results in the following result:

Corollary 8.2.16. *Let $S \in \mathrm{SSYT}(\lambda)$. If S is* **R***-catabolizable, so is* $\mathrm{std}(S)$.

This gives us one direction of Proposition 8.2.9.

To prove the other direction, we require the following lemma.

Lemma 8.2.17. *Consider partitions $\mu = (\mu_1, \dots, \mu_l)$ and $\hat{\mu} = (\mu_1, \dots, \mu_{k-1}, r)$ for $1 \leqslant k < l, 1 \leqslant r \leqslant \mu_k$. Let $S \in \mathrm{SSYT}(\mu)$ such that $\mathrm{std}(S)$ contains $Z_{\hat{\mu}}$. Then*

(a) $\mathrm{std}(S)$ *must contain* $Z^{\mathrm{std}}_{(\mu_1,\dots,\mu_k)}$,

(b) Let $m = \mu_1 + \dots + \mu_k$. If $\mathrm{std}(S)$ does not contain $Z^{\mathrm{std}}_{(\mu_1,\dots,\mu_{k+1})}$, we must have the following for some $0 < d \leqslant \mu_{k+1}$.

  (i) $m+1, \dots, m+d$ appear in row k or below of $\mathrm{std}(S)$,

  (ii) $m+d+1, \dots, m+\mu_{k+1}$ appear in row $k+1$ of $\mathrm{std}(S)$.

Proof. (a) Since $\mathrm{std}(S)$ contains $Z^{\mathrm{std}}_{\hat{\mu}}$, we know that for $1 \leqslant i \leqslant k-1$ there are exactly μ_i many entries in $\mathrm{std}(S)$ with cocharge $i-1$ and they contribute to row $i-1$ of $\mathrm{ctype}(S)$. Thus, $\mathrm{ctype}(\mathrm{std}(S))_i = \mu_i$. Since $\mathrm{ctype}(\mathrm{std}(S)) \trianglerighteq \mu$, we must have $\mathrm{ctype}(\mathrm{std}(S))_k \geqslant \mu_k$. The only way this is possible is if we have at least μ_k many entries added to row k of $\mathrm{ctype}(\mathrm{std}(S))$. This can only happen if there are at least μ_k many entries of cocharge $k-1$ that appear above row $k-1$. This means that $\mathrm{ctype}(\mathrm{std}(S))$ must contain $Z^{\mathrm{std}}_{(\mu_1,\dots,\mu_{k+1})}$.
(b) Since $\mathrm{ctype}(\mathrm{std}(S)) \trianglerighteq \mu$ and $\mathrm{ctype}(S)_i = \mu_i$ for $1 \leqslant i \leqslant k-1$, we must have

$$\mathrm{ctype}(\mathrm{std}(S))_k + \mathrm{ctype}(\mathrm{std}(S))_{k+1} \geqslant \mu_k + \mu_{k+1}.$$

This means we need to have at least $\mu_k + \mu_{k+1}$ entries added to rows $k, k+1$ of $\mathrm{ctype}(\mathrm{std}(S))$ when applying Algorithm 4.1.13. Since all entries of cocharge $i-1$ for $i < k$ contribute to $\mathrm{ctype}(\mathrm{std}(S))_i$, the only entries that can contribute to row $k, k+1$ are the following:

- entries of cocharge $k-1$ that appear in row k of $\mathrm{std}(S)$ can contribute to $\mathrm{ctype}(\mathrm{std}(S))_k$,
- entries of cocharge $k-1$ that appear below row k of $\mathrm{std}(S)$ can contribute to $\mathrm{ctype}(\mathrm{std}(S))_{k+1}$,
- entries of cocharge k that appear in row $k+1$ of $\mathrm{std}(S)$ can contribute to $\mathrm{ctype}(\mathrm{std}(S))_{k+1}$,

Thus we must have that we have at least $\mu_k + \mu_{k+1}$ such entries. This is equivalent to the statement of claim (b). $\square$

Now we prove the other direction.

Proposition 8.2.18. *Consider $S \in \mathrm{SSYT}(\lambda)$. If S is not R_1-catabolizable, then neither is* $\mathrm{std}(S)$*.*

Proof. We want to show that for any intermediate step $T \to \widetilde{T}$ of Algorithm 8.2.12, if T is not R_1-catabolizable then neither is $\widetilde{T}$. We consider the same cases we checked in the proof of Proposition 8.2.15.

For (i) and (iii) Case 2, the claim is immediate. For (ii), the claim follows from the same argument used to prove Proposition 8.2.15.

Consider (iii), Case 1, where we assume T is not **R**-catabolizable but $\widetilde{T}$ is. Since s_i only affects the letters $i, i+1$, the shape of $T|_{[A_1]}$ is preserved under acting by s_i. Thus $\mathrm{sh}(T|_{[A_1]}) = R_1$ and all the entries in $T|_{[A_1]}$ must be in the correct order except i and $(i+1)$. The only way this can happen is if the subword of $\mathrm{rw}(T)$ consisting of $i, i+1$ is of the form $i(i+1)^{q_1}(i+1)^{q_2}$ where the first q_1 many $(i+1)$ appear in a row higher than the last ones, as demonstrated below:

$$\begin{array}{|c|c|c|cc} \hline i & i+1 & \cdots & & \\ \hline \cdots & \cdot & \cdots & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\underline{i+1}} \\ \hline \end{array} \xrightarrow{s_i} \begin{array}{|c|c|c|cc} \hline i & i & \cdots & & \\ \hline \cdots & \cdot & \cdots & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\underline{i+1}} \\ \hline \end{array}$$

Applying s_i changes all but the last $(i+1)$ (underlined above) to an i, which means we still have $(i+1)$ in a row lower than i. Thus $\widetilde{T}|_{[A_1]} \neq Z_{R_1}(A_1)$.

For (iv), we restrict to the case where $k = \eta_1 + 1$. By definition of Algorithm 8.2.12, we know that T has weight given by

$$(1^{|R_1|}, \lambda_p, 1^{\lambda_{\eta_1+1}+\cdots+\lambda_{p-1}}, \dots)$$

for $p > \eta_1$. We also know that $\lambda_p \leqslant \lambda_{\eta_1}$, which is the last row of R_1, thus we can define a new partition $\widetilde{R_1} = (\lambda_1, \dots, \lambda_{\eta_1 - 1}, \lambda_{\eta_1} - \lambda_p)$.

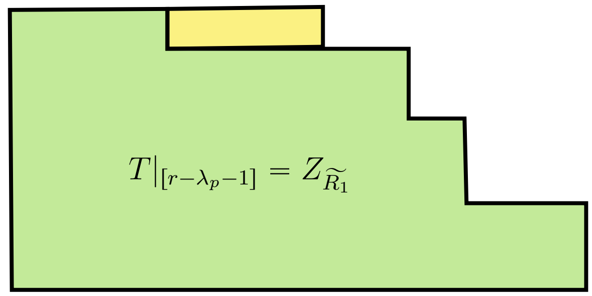
$Z_{R_1} \subsetneq T|_{[r+1]} =$ 

Figure 8.5: The tableau $T|_{[r+1]}$. Note that R_1 is contained in this shape. The yellow portion contains the letters $[r-\lambda_p, r+1]$ and is a subset of the last row of $T|_{[r+1]}$

Let $r = |R_1|+1$. Assume for sake of contradiction that T is not R_1-catabolizable, but $\widetilde{T}$ is. Since (iv) only affects entries $r, r-1, \dots, r-\lambda_p$, it suffices to consider the tableau $T|_{[r+1]}$.

Since T is not R_1-catabolizable, we know T cannot contain Z_{R_1}. However, since $\widetilde{T}$ is R_1-catabolizable and (iv) does not affect entries $1, \dots, r-\lambda_p-1$, we know that $T|_{[r-\lambda_p-1]} = \widetilde{T}|_{[r-\lambda_p-1]} = Z_{\widetilde{R_1}}$ (see Figure 8.5). Furthermore, since (iv) does not change the shape of the tableau, we know $R_1 \subset \mathrm{sh}(T|_{[r+1]}) = \mathrm{sh}(\widetilde{T}|_{[r+1]})$.

Now note that by Algorithm 8.2.12, there exists an intermediate tableau T_0 of the weight

$$(1^P, \lambda_p, \lambda_{p+1}, \dots),$$

where we set $P = \lambda_1 + \dots + \lambda_{p-1} \geqslant |R_1|$, that appeared when applying Algorithm 8.2.12 to the original tableau S. We can see that $T_0|_{[P]} = \mathrm{std}(S|_{[p-1]})$.

We can also see that $T|_{[i]} = T_0|_{[i]}$ for any $i \leqslant |R_1|$ by definition of T_0, thus

$$T|_{[|R_1|]} = T_0|_{[|R_1|]} \subset T_0|_{[P]} = \mathrm{std}(S|_{[p-1]}).$$

Similarly, since $r-\lambda_p-1 = |R_1|-\lambda_p < |R_1| \leqslant P$, we have that

$$Z_{\widetilde{R_1}} = T|_{[r-\lambda_p-1]} = T_0|_{[r-\lambda_p-1]} \subseteq T_0|_{[P]} = \mathrm{std}(S|_{[p-1]}).$$

Thus $\mathrm{std}(S|_{[p-1]})$ contains $Z_{\widetilde{R_1}}$ but not Z_{R_1}. However, by Lemma 8.2.17 (a) this is only possible if $\lambda_{\eta_1} - \lambda_p = 0$. Thus $\widetilde{R_1} = (\lambda_1, \dots, \lambda_{\eta_1-1})$.

Furthermore, from Lemma 8.2.17 (b), we know that

(i) $r-\lambda_{\eta_1}, \dots, r-d-1$ appear in row $\eta_1 - 1$ or lower in $\mathrm{std}(S|_{[p-1]}) = T_0|_{[P]}$

(ii) (if $d>0$) $\quad r-d, \dots, r-1$ appear in row η_1 in $\mathrm{std}(S|_{[p-1]}) = T_0|_{[P]}$, where $0 \leqslant d < \lambda_{\eta_1}$. Since $R_1 \subset \mathrm{sh}(T|_{[r+1]})$, we also know that there are $(\lambda_{\eta_1} - d)$ many rs in row η_1 of $T|_{[|R_1|]} = T_0|_{[|R_1|]}$. Thus we now know what $T|_{[|R_1|]} = T_0|_{[|R_1|]}$ looks like in T.

Now, we want to apply (iv) to T, which is equivalent to $s_{r-\lambda_{\eta_1}} \dots s_{r-1}$. We want to see whether $\widetilde{T}|_{[r-\lambda_{\eta_1}]}$ is equal to $Z_{R_1}(1^{|\widetilde{R_1}|}, \lambda_{\eta_p})$

If $d = 0$, then row η_1 of T contains λ_{η_1} many rs. When applying s_{r-1} to T, we can see that the unique $r-1$ in T appears in a lower rowm thus s_{r-1} fixes the last r in row η_1. Thus the subtableau of $\widetilde{T}$ of shape R_1 contains a r, hence $\widetilde{T}|_{[r-\lambda_{\eta_1}]} \neq Z_{R_1}(1^{|\widetilde{R_1}|}, \lambda_{\eta_p})$. Thus $\widetilde{T}$ is not R_1-catabolizable.

Now assume $d > 0$. Using the second observation above, we know that the subword of $\mathrm{rw}(T)$ consisting of $r-d, \dots, r$ is of the form

$$r^s(r-d)(r-d+1)\dots(r-1)r^{\lambda_{\eta_1}-d}r^{d-s}$$

for some nonnegative integer s. We have that $s \leqslant d$ by semistandardness of T. Note that the subword corresponding to row η_1 of the subtableau of T of shape R_1 is given by

$$(r-d)(r-d+1)\dots(r-1)r^{\lambda_{\eta_1}-d}.$$

Using Lemma 8.2.14 after applying $s_{r-d}\dots s_{r-1}$ we get

$$(r-s+1)\dots r(r-d)^{\lambda_{\eta_1}}(r-d+1)\dots(r-s).$$

Now, note that the subword consisting of the entries in row η_1 of the subtableau of shape R_1 is $(r-d)^{\lambda_{\eta_1}}$. Since we know that $(r-d-1)$ appears in a row lower that row η_1, when we apply s_{r-d-1} the last $(r-d)$ in row η_1 gets paired and remains unchanged. Thus one of the entries in row η_1 remains $(r-d) > r - \lambda_{\eta_1}$, which means $\widetilde{T}|_{[r-\lambda_{\eta_1}]} \neq Z_{R_1}(1^{|\widetilde{R_1}|}, \lambda_{\eta_p})$. Thus $\widetilde{T}$ is not R_1-catabolizable. □

Proposition 8.2.9 follows from Propositions 8.2.15 and 8.2.18.

Now, we want to show Theorem 8.2.8, by considering the case where $\mathrm{ctype}(T) \ntrianglerighteq \lambda$. We first fix some notation. For $T \in \mathrm{SYT}$ that is $R_1 = (\lambda_1, \dots, \lambda_{\eta_1})$-catabolizable, let U, V be the north and south tableaux we get by slicing $T - Z_{R_1}$ between rows $\eta_1, \eta_1 + 1$. (see Figure 8.6). Let $\mathbf{z}_U, \mathbf{z}_V$ denote the subwords of $\mathrm{cc}(\mathrm{rw}(T))$ corresponding to U, V. We also let $\mathbf{w}_U$, (resp. $\mathbf{w}_V$ be the word we get by subtracting η_1 (resp. $(\eta_1 - 1)$) from each entry in $\mathbf{z}_U$ (resp. $\mathbf{z}_V$).

We introduce some lemmas.

Lemma 8.2.19. *Consider $T \in \mathrm{SYT}_n$ such that T is $R_1 = (\lambda_1, \dots, \lambda_{\eta_1})$-catabolizable. We have*

$$\mathrm{cc}(\mathrm{rw}(\mathrm{cat}_{R_1}(T))) \equiv \mathbf{w}_V\mathbf{w}_U$$

.

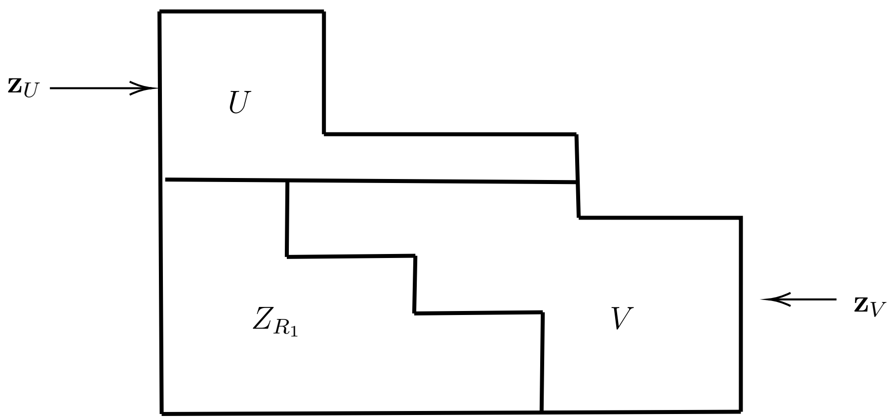


Figure 8.6: T that is $R_1 = (\lambda_1, \dots, \lambda_{\eta_1})$-catabolizable

Proof. Note that $\mathrm{rw}(\mathrm{cat}_{R_1}(T)) \sim \mathrm{rw}(V)\,\mathrm{rw}(U)$ and cocharge labels of permutations are invariant under Knuth moves. It suffices to show that $\mathrm{cc}(\mathrm{rw}(V)\,\mathrm{rw}(U)) = \mathbf{w}_V \mathbf{w}_U$.

We know that if m is the smallest letter that appears in $\mathrm{cat}_{R_1}(T)$, then it either appears with cocharge $\eta_1 - 1$ in $\mathbf{z}_V$ or η_1 in $\mathbf{z}_U$: in either case, the corresponding value in $\mathbf{w}_V, \mathbf{w}_U$ is 0.

Now, assume that the cocharge values in $\mathrm{cc}(\mathrm{rw}(V)\,\mathrm{rw}(U))$ match the values in $\mathbf{w}_V \mathbf{w}_U$ up to some entry i. If $i, i+1$ are in the same subtableau, then it is clear that the cocharge labels of $i+1$ match, since the relative positions of $i, i+1$ are the same between $\mathrm{rw}(U)\,\mathrm{rw}(V)$ and $\mathrm{rw}(V)\,\mathrm{rw}(U)$.

If $i \in U$ and $i+1 \in V$, we can see that they get the same cocharge label in T. Thus the entry corresponding to $i+1$ in $\mathbf{w}_V$ is one larger than the entry corresponding to i in $\mathbf{w}_U$. We can also see that in $\mathrm{rw}(V)\,\mathrm{rw}(U)$, we have that $i+1$ appears to the left of i, hence the cocharge label increases by 1.

A similar argument holds for the case $i \in V, i+1 \in U$. $\square$

Lemma 8.2.20. *Consider $T \in \mathrm{SYT}_n$ such that T is $R_1 = (\lambda_1, \dots, \lambda_{\eta_1})$-catabolizable and* $\mathrm{ctype}(T)_i = \lambda_i$ *for $i \leqslant \eta_1$. We have* $\mathrm{ctype}(T) = (R_1, \mathrm{ctype}(\mathrm{cat}_{R_1}(T)))$.

Proof. Let U, V denote the north and south tableaux we get by slicing $T - Z_{R_1}$ between rows $\eta_1, \eta_1 + 1$. Since $R_1 = (\mathrm{ctype}(T)_1, \dots, \mathrm{ctype}(T)_{\eta_1})$ and T is R_1-catabolizable, all the entries in V have cocharge $\eta_1 - 1$ or larger. Furthermore, when applying Algorithm 4.1.13 to T, we skip over all the entries in V for the first iteration.

Since there are exactly $\text{ctype}(T)_{\eta_1}$ many entries in row η_1 of T with cocharge $\eta_1 - 1$, and $\text{cc}(T)$ is a semistandard tableau, we know there are at most $\text{ctype}(T)_{\eta_1}$ many entries in U of cocharge η_1.

Thus, when applying Algorithm 4.1.13, we know that all entries in U of cocharge η_1 contribute to row $\eta_1 + 1$ of $\text{ctype}(T)$. Thus, $(\text{ctype}(T)_{\eta_1+1}, \dots)$ is the partition we get by applying Algorithm 4.1.13 to $\mathbf{w}_V\mathbf{w}_U$. From Lemma 8.2.19, this is exactly $\text{ctype}(\text{cat}_{R_1}(T))$. □

Remark 8.2.21. Note that though Lemma 8.2.20 seems relatively straightforward, it depends on the fact that R_1 is exactly the first η_1 parts of $\text{ctype}(T)$. If $(\text{ctype}(T)_1, \dots, \text{ctype}(T)_{\eta_1}) \neq R_1$, we could have that row η_1 of T contains another entry of cocharge η_1 that gets added in the first pass. This assumption assures that nothing in V could have actually been deleted in the first iteration of the cat insertion alg.

Furthermore, we could have that the number of entries that contribute to $\text{ctype}(T)_{\eta_1+1}$ in the first iteration is different than if we just run Algorithm 4.1.13 on the tableau U, because we have the condition that the number of things we can add to row $\eta_1 + 1$ in the first pass is at most the number we added to row η_1. In this case, since $\text{ctype}(T)_{\eta_1}$ is exactly the number of $\eta_1 - 1$ in row η_1, we know the number of $\eta_1 + 1$ in U is at most $\text{ctype}(T)_{\eta_1}$, which guarantees this condition holds.

Lemma 8.2.22. *Let $T \in \text{SYT}$ such that $\text{ctype}(T) = \lambda = (\lambda_1, \dots, \lambda_l)$. If T contains Z_γ for some partition $\gamma = (\gamma_1, \dots, \gamma_k)$, then $\gamma = (\lambda_1, \dots, \lambda_{k-1}, r)$ where $1 \leqslant r \leqslant \lambda_k$.*

Proof. If T contains Z_γ, then we know that the only entries in T with cocharge label $0, \dots, k-2$ appear in the subtableau of shape γ. If we apply Algorithm 4.1.13 to T, the entries that contribute to the first $k-1$ rows of $\text{ctype}(T)$ are exactly those in the first $k-1$ rows of the subtableau of shape γ. Thus $\lambda_i = \text{ctype}(T)_i = \gamma_i$ for $1 \leqslant i \leqslant k-1$ and $\lambda_k \geqslant \gamma_k$. □

Now we finish the proof of Theorem 8.2.8 by showing this result.

Proposition 8.2.23. *If $T \in \text{SYT}$ with $\text{ctype}(T) \ntrianglerighteq \lambda$, then T is not $R(\eta, \lambda)$-catabolizable for any choice of η.*

Proof. We proceed by induction on the length of $\mathbf{R}$. The base case ($\mathbf{R} = R(\ell(\lambda), \lambda)$) follows from Proposition 4.1.6.

Now, assume we have $\text{ctype}(T) \ntrianglerighteq \lambda$ and some sequence $\mathbf{R} = R(\eta, \lambda) = (R_1, R_2, \dots, R_l)$. If T does not contain Z_{R_1}, we are done.

Assume otherwise, where $R_1 = (\lambda_1, \dots, \lambda_{\eta_1})$. By Lemma 8.2.22, we know that $\text{ctype}(T)_i = \lambda_i$ for $1 \leqslant i < \eta_1$ and $\text{ctype}(T)_{\eta_1} \geqslant \lambda_{\eta_1}$.

We first consider the case where $\mathrm{ctype}(T)_{\eta_1} = \lambda_{\eta_1}$. Since this implies that the first η_1 parts of λ and $\mathrm{ctype}(T)$ agree, we have that $\mathrm{ctype}(T) \not\trianglerighteq \lambda$ implies that

$$(\mathrm{ctype}(T)_{\eta_1+1}, \dots) \not\trianglerighteq= (\lambda_{\eta_1+1}, \dots).$$

By Lemma 8.2.20, we know that $\mathrm{ctype}(\mathrm{cat}_{R_1}(T)) = (\mathrm{ctype}(T)_{\eta_1+1}, \dots)$. Thus by induction, we know that $\mathrm{cat}_{R_1}(T)$ is not $(R_2, \dots)$-catabolizable and the claim holds. Now, consider the case where $\mathrm{ctype}(T)_{\eta_1} = \lambda_{\eta_1} + m$ for some $m > 0$. Note that there must exist a $r > 1$ such that

$$\lambda_{\eta_1} + m + \sum_{i=1}^{r} \mathrm{ctype}(T)_{\eta_1+i} < \lambda_{\eta_1} + \sum_{i=1}^{r} \lambda_{\eta_1+i}.$$

Equivalently, we have that

$$\sum_{i=1}^{r} \lambda_{\eta_1+i} > m + \sum_{i=1}^{r} \mathrm{ctype}(T)_{\eta_1+i} \tag{8.4}$$

Now, we can write $\mathbf{w}_V = 0^m \mathbf{w}'_V$ for some word $\mathbf{w}'_V$, since we know the first m entries of $\mathbf{z}_V$ must be η_1. Using the same argument as in the proof of Proposition 8.2.19, we see that $\mathrm{cc}(\mathrm{cat}_{R_1}(T)) \sim 0^m \mathbf{w}'_V \mathbf{w}_U$ and that $(\mathrm{ctype}(T)_{\eta_1+1}, \dots)$ is the result of applying the catabolism insertion algorithm to $\mathbf{w}'_V \mathbf{w}_U$.

Thus we have $\mathrm{ctype}(\mathrm{cat}_{R_1}(T)) = (\mathrm{ctype}(T)_{\eta_1+1} + m, \mathrm{ctype}(T)_{\eta_1+2}, \dots)$. From (8.4), this implies that $(\lambda_{\eta_1+1}, \dots) \not\trianglerighteq \mathrm{ctype}(\mathrm{cat}_{R_1}(T))$ Thus by induction we have that $\mathrm{cat}_{R_1}(T)$ is not $(R_2, \dots)$-catabolizable. $\square$

8.3 Example of Algorithm 8.2.12

Example 8.3.1. Consider $T = \begin{array}{l} \boxed{3} \\ \boxed{2}\boxed{2} \\ \boxed{1}\boxed{1}\boxed{1}\boxed{3}\boxed{4} \end{array}$ with weight $\lambda = (3, 2, 2, 1)$. We will standardize T using Algorithm 8.2.12.

We first use (i) to replace all $i > 1$ with $i + 2$ to obtain a tableau of weight $(3, 0, 0, 2, 2, 1)$:

$$\begin{array}{l} \boxed{3} \\ \boxed{2}\boxed{2} \\ \boxed{1}\boxed{1}\boxed{1}\boxed{3}\boxed{4} \end{array} \to \begin{array}{l} \boxed{5} \\ \boxed{4}\boxed{4} \\ \boxed{1}\boxed{1}\boxed{1}\boxed{5}\boxed{6} \end{array}.$$

Now we use (ii) and (i) repeatedly until we get a tableau of weight $(1, 1, 1, 2, 2, 1)$.

$$\begin{array}{c} \begin{array}{l} \boxed{5} \\ \boxed{4}\boxed{4} \\ \boxed{1}\boxed{1}\boxed{1}\boxed{5}\boxed{6} \end{array} \\ (3,0,0,2,2,1) \end{array} \xrightarrow{f_1} \begin{array}{c} \begin{array}{l} \boxed{5} \\ \boxed{4}\boxed{4} \\ \boxed{1}\boxed{1}\boxed{2}\boxed{5}\boxed{6} \end{array} \\ (2,1,0,2,2,1) \end{array} \xrightarrow{s_2} \begin{array}{c} \begin{array}{l} \boxed{5} \\ \boxed{4}\boxed{4} \\ \boxed{1}\boxed{1}\boxed{3}\boxed{5}\boxed{6} \end{array} \\ (2,0,1,2,2,1) \end{array} \xrightarrow{f_1} \begin{array}{c} \begin{array}{l} \boxed{5} \\ \boxed{4}\boxed{4} \\ \boxed{1}\boxed{2}\boxed{3}\boxed{5}\boxed{6} \end{array} \\ (1,1,1,2,2,1). \end{array}$$

We can see that at each step, the composition is a refinement of λ.

Now, we use (iv), which is the permutation operator $\tau = s_1 s_2 s_3$, to obtain a tableau of weight $(2,1,1,1,2,1)$.

$$\begin{ytableau} 5 \\ 4 & 4 \\ 1 & 2 & 3 & 5 & 6 \end{ytableau} \xrightarrow{s_3} \begin{ytableau} 5 \\ 3 & 4 \\ 1 & 2 & 3 & 5 & 6 \end{ytableau} \xrightarrow{s_2} \begin{ytableau} 5 \\ 3 & 4 \\ 1 & 2 & 2 & 5 & 6 \end{ytableau} \xrightarrow{s_1} \begin{ytableau} 5 \\ 3 & 4 \\ 1 & 1 & 2 & 5 & 6 \end{ytableau}$$

$$(1,1,1,2,2,1) \qquad (1,1,2,1,2,1) \qquad (1,2,1,1,2,1) \qquad (2,1,1,1,2,1).$$

Though the intermediate compositions are not refinements of λ, we can see that both the initial and resulting compositions are.

Now use (i) to replace all $i > 1$ with $i+1$ to obtain a tableau of weight $(2,0,1,1,1,2,1)$ and then apply (ii):

$$\begin{ytableau} 6 \\ 4 & 5 \\ 1 & 1 & 3 & 6 & 7 \end{ytableau} \xrightarrow{f_1} \begin{ytableau} 6 \\ 4 & 5 \\ 1 & 2 & 3 & 6 & 7 \end{ytableau}.$$

The resulting weight is $(1,1,1,1,1,2,1)$. We repeat. We apply (iv) twice: $s_4 s_5$ swaps $(1,1,1,1,1,2) \mapsto (1,1,1,2,1,1)$ and $s_2 s_3$ swaps $(1,1,1,2,1,1) \mapsto (1,2,1,1,1)$. We then apply (iii).

$$\begin{ytableau} 6 \\ 4 & 5 \\ 1 & 2 & 3 & 6 & 7 \end{ytableau} \xrightarrow{s_5} \begin{ytableau} 6 \\ 4 & 5 \\ 1 & 2 & 3 & 5 & 7 \end{ytableau} \xrightarrow{s_4} \begin{ytableau} 6 \\ 4 & 4 \\ 1 & 2 & 3 & 5 & 7 \end{ytableau}$$
$$\xrightarrow{s_3} \begin{ytableau} 6 \\ 3 & 4 \\ 1 & 2 & 3 & 5 & 7 \end{ytableau} \xrightarrow{s_2} \begin{ytableau} 6 \\ 3 & 4 \\ 1 & 2 & 2 & 5 & 7 \end{ytableau} \xrightarrow{s_1} \begin{ytableau} 6 \\ 3 & 4 \\ 1 & 1 & 2 & 5 & 7 \end{ytableau}.$$

Now we use (i) to replace all $i > 1$ with $i+1$ to obtain a tableau of weight $(2,0,1,1,1,1,1,1)$ and then apply (ii)

$$\begin{ytableau} 7 \\ 4 & 5 \\ 1 & 1 & 3 & 6 & 8 \end{ytableau} \xrightarrow{f_1} \begin{ytableau} 7 \\ 4 & 5 \\ 1 & 2 & 3 & 6 & 8 \end{ytableau}.$$

Thus we have $\operatorname{std}(T) = \begin{ytableau} 7 \\ 4 & 5 \\ 1 & 2 & 3 & 6 & 8 \end{ytableau}.$

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