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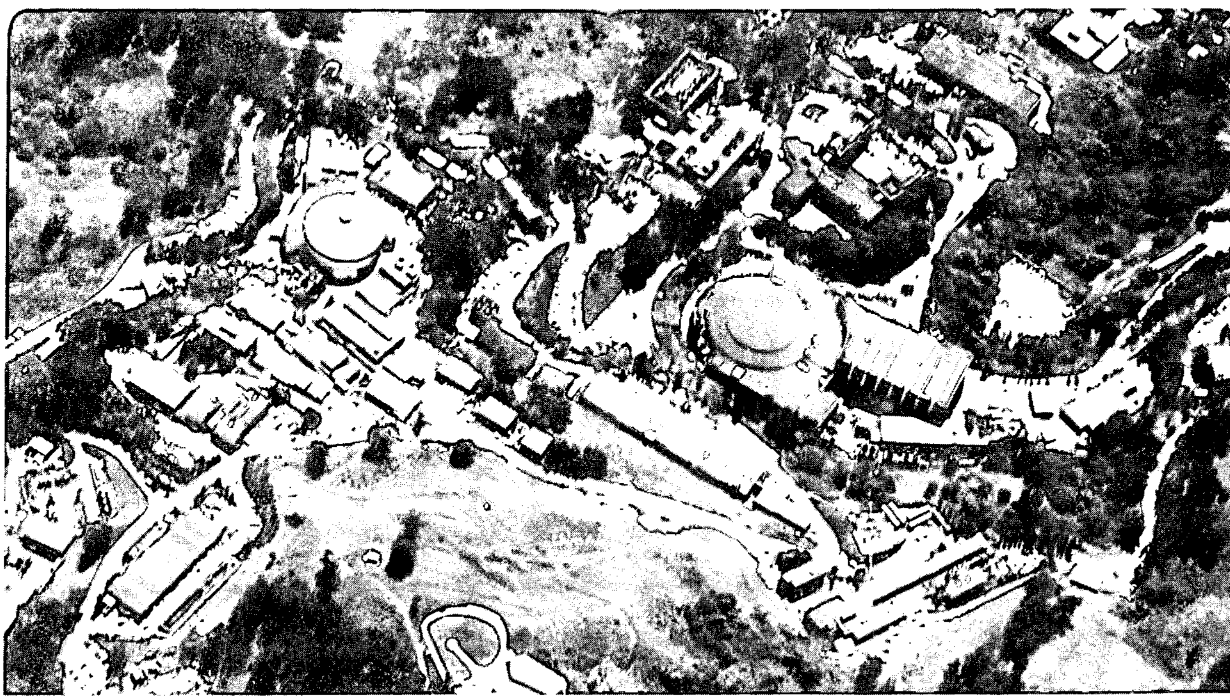

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A CLASS OF BASES IN $\mathcal{L}^2$ FOR THE SPARSE REPRESENTATION OF INTEGRAL OPERATORS*

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A Class of Bases in $\mathcal{L}^2$ for the Sparse Representation of Integral Operators

Bradley K. Alpert

Abstract

A class of *multi-wavelet* bases for $\mathcal{L}^2$ is constructed with the property that a variety of integral operators are represented in these bases as sparse matrices, to high precision. In particular, an integral operator $\mathcal{K}$ whose kernel is smooth except along a finite number of singular bands has a sparse representation. In addition, the inverse operator $(I - \mathcal{K})^{-1}$ appearing in the solution of a second-kind integral equation involving $\mathcal{K}$ is also sparse in the new bases. The result is an order $O(n \log^2 n)$ algorithm for numerical solution of a large class of second-kind integral equations.

Key Words. wavelets, integral equations, sparse matrices

AMS(MOS) subject classifications. 42C15, 45L10, 65R10, 65R20

Families of functions $h_{a,b}$,

$$h_{a,b}(x) = |a|^{-1/2}\, h\left(\frac{x-b}{a}\right), \qquad a, b \in \mathcal{R},\ a \neq 0,$$

derived from a single function h by dilation and translation, which form a basis for $\mathcal{L}^2(\mathcal{R})$, are known as *wavelets* (Grossman and Morlet [7]). In recent years, these families have received study by many authors, resulting in constructions with a variety of properties. Meyer [9] constructed orthonormal wavelets for which $h \in C^\infty(\mathcal{R})$. Daubechies [5] constructed compactly supported wavelets with $h \in C^k(\mathcal{R})$ for arbitrary k, and [5] gives an overview and synthesis of the field. The dissertation [2] of the present author gives an earlier report of the present work.

In this paper we construct a somewhat different type of basis for $\mathcal{L}^2(\mathcal{R})$ that can be readily revised to a basis for $\mathcal{L}^2[0,1]$. Each basis, which we call a *multi-wavelet* basis, is comprised of dilates and translates of a finite set of functions $h_1, \ldots, h_k$. In particular, our bases consist of orthonormal systems

$$h^n_{j,m}(x) = 2^{m/2}\, h_j(2^m x - n), \qquad j = 1, \ldots, k;\ m, n \in \mathcal{Z}, \tag{1}$$

where the functions $h_1, \ldots, h_k$ are piecewise polynomial, vanish outside the interval $[0,1]$, and are orthogonal to low-order polynomials (have vanishing moments),

$$\int_0^1 h_j(x)\, x^i\, dx = 0, \qquad i = 0, 1, \ldots, k-1. \tag{2}$$

The properties of compact support and vanishing moments lead to bases in which a variety of integral operators are represented as sparse matrices. In particular, an integral operator whose kernel is non-oscillatory and analytic except along a finite set of curves, when expanded in one of these bases, is sparse.

In §1, we construct multi-wavelet bases in one and several dimensions and in §2, we prove their rate of convergence for suitably differentiable functions. Second-kind integral equations are introduced in §3 and a generic method for their numerical solution is presented. In §4 we prove that the representations in the multi-wavelet bases of certain integral operators and their inverses are sparse, to high precision. In §5 we give several numerical examples of the bases and the solution of second-kind integral equations and conclude in §6 with a discussion.

1 Multi-Wavelet Bases

1.1 The One-Dimensional Construction

We first restrict our attention to the finite interval $[0,1] \subset \mathcal{R}$ and we construct a basis for $\mathcal{L}^2[0,1]$. We employ the multi-resolution analysis framework developed by Mallat [8] and Meyer [10], and discussed at length by Daubechies [5]. We suppose that k is a positive integer and for $m = 0, 1, 2, \ldots$ we define a space S_m^k of piecewise polynomial functions,

$$S_m^k = \{f : \text{ the restriction of } f \text{ to the interval } (2^{-m}n, 2^{-m}(n+1)) \text{ is a polynomial of degree less than } k, \text{ for } n = 0, \ldots, 2^m - 1, \text{ and } f \text{ vanishes elsewhere}\}. \tag{3}$$

It is apparent that the space S_m^k has dimension $2^m k$ and

$$S_0^k \subset S_1^k \subset \cdots \subset S_m^k \subset \cdots.$$

For $m = 0, 1, 2, \ldots$ we define the $2^m k$-dimensional space R_m^k to be the orthogonal complement of S_m^k in S_{m+1}^k,

$$S_m^k \oplus R_m^k = S_{m+1}^k, \qquad R_m^k \perp S_m^k,$$

so we inductively obtain the decomposition

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k. \tag{4}$$

Suppose that functions $h_1, \ldots, h_k : \mathcal{R} \to \mathcal{R}$ form an orthogonal basis for R_0^k. Since R_0^k is orthogonal to S_0^k, the first k moments of $h_1, \ldots, h_k$ vanish,

$$\int_0^1 h_j(x)\, x^i\, dx = 0, \qquad i = 0, 1, \ldots, k-1.$$

The $2k$-dimensional space R_1^k is spanned by the $2k$ orthogonal functions $h_1(2x), \ldots, h_k(2x)$, $h_1(2x-1), \ldots, h_k(2x-1)$, of which k are supported on the interval $[0, \frac{1}{2}]$ and k on $[\frac{1}{2}, 1]$. In general, the space R_m^k is spanned by $2^m k$ functions obtained from $h_1, \ldots, h_k$ by translation and dilation. There is some freedom in choosing the functions $h_1, \ldots, h_k$ within the constraint that they be orthogonal; by requiring normality and additional vanishing moments, we specify them uniquely, up to sign. The remainder of this subsection is devoted to the explicit construction of $h_1, \ldots, h_k$; in the following sections we exploit only the property that $h_1, \ldots, h_k$ form an orthonormal basis for R_0^k.

In preparation for the definition of $h_1, \ldots, h_k$, we construct the k functions $f_1, \ldots, f_k : \mathcal{R} \to \mathcal{R}$, supported on the interval $[-1, 1]$, with the following properties:

1. The restriction of f_i to the interval $(0, 1)$ is a polynomial of degree $k-1$.

2. The function f_i is extended to the interval $(-1, 0)$ as an even or odd function according to the parity of $i + k - 1$.

3. The functions $f_1, \ldots, f_k$ satisfy the following orthogonality and normality conditions:

$$\int_{-1}^1 f_i(x)\, f_j(x)\, dx \equiv \langle f_i, f_j \rangle = \delta_{ij}, \qquad i, j = 1, \ldots, k.$$

4. The function f_j has vanishing moments,

$$\int_{-1}^1 f_j(x)\, x^i\, dx = 0, \qquad i = 0, 1, \ldots, j + k - 2.$$

Properties 1 and 2 imply that there are k^2 polynomial coefficients that determine the functions $f_1, \ldots, f_k$, while properties 3 and 4 provide k^2 (non-trivial) constraints. It turns out that the equations uncouple to give k nonsingular linear systems that may be solved to obtain the coefficients, yielding the functions uniquely (up to sign). Rather than prove that these systems are nonsingular, however, we now determine $f_1, \ldots, f_k$ constructively.

We start with $2k$ functions which span the space of functions that are polynomials of degree less than k on the interval $(0, 1)$ and on $(-1, 0)$, then orthogonalize k of them, first to the functions $1, x, \ldots, x^{k-1}$, then to the functions

$x^k, x^{k+1}, \ldots, x^{2k-1}$, and finally among themselves. We define $f_1^1, f_2^1, \ldots, f_k^1$ by the formula

$$f_j^1(x) = \begin{cases} x^{j-1}, & x \in (0,1), \\ -x^{j-1}, & x \in (-1,0), \\ 0, & \text{otherwise}, \end{cases}$$

and note that the $2k$ functions $1, x, \ldots, x^{k-1}, f_1^1, f_2^1, \ldots, f_k^1$ are linearly independent, hence span the space of functions that are polynomials of degree less than k on $(0,1)$ and on $(-1,0)$.

1. By the Gram-Schmidt process we orthogonalize f_j^1 with respect to $1, x, \ldots, x^{k-1}$, to obtain f_j^2, for $j = 1, \ldots, k$. This orthogonality is preserved by the remaining orthogonalizations, which only produce linear combinations of the f_j^2.

2. The next sequence of steps yields $k-1$ functions orthogonal to x^k, of which $k-2$ functions are orthogonal to x^{k+1}, and so forth, down to 1 function which is orthogonal to x^{2k-2}. First, if at least one of f_j^2 is not orthogonal to x^k, we reorder the functions so that it appears first, $\langle f_1^2, x^k \rangle \neq 0$. We then define $f_j^3 = f_j^2 - a_j \cdot f_0^2$ where a_j is chosen so $\langle f_j^3, x^k \rangle = 0$ for $j = 2, \ldots, k$, achieving the desired orthogonality to x^k. Similarly, we orthogonalize to $x^{k+1}, \ldots, x^{2k-2}$, each in turn, to obtain $f_1^2, f_2^3, f_3^4, \ldots, f_k^{k+1}$ such that $\langle f_j^{j+1}, x^i \rangle = 0$ for $i \leq j + k - 2$.

3. Finally, we do Gram-Schmidt orthogonalization on $f_k^{k+1}, f_{k-1}^k, \ldots, f_1^2$, in that order, and normalize to obtain $f_k, f_{k-1}, \ldots, f_1$.

It is readily seen that the f_j satisfy properties 1-4 of the previous paragraph. Defining $h_1, \ldots, h_k : \mathcal{R} \to \mathcal{R}$ by the formula

$$h_i(x) = 2^{1/2}\, f_i(2x - 1), \qquad i = 1, \ldots, k,$$

we obtain the equality

$$R_0^k = \text{linear span}\, \{h_i : \ i = 1, \ldots, k\},$$

and, more generally,

$$R_m^k = \text{linear span}\, \{h_{j,m}^n : \ h_{j,m}^n(x) = 2^{m/2}\, h_j(2^m x - n), \quad j = 1, \ldots, k;\ n = 0, \ldots, 2^m - 1\}. \tag{5}$$

We will show next that dilates and translates of the piecewise polynomial functions $h_1, \ldots, h_k$ form an orthonormal basis for $\mathcal{L}^2(\mathcal{R})$. Furthermore, a subset of these dilates and translates, combined with a basis for S_0^k, forms a basis for $\mathcal{L}^2[0,1]$.

1.2 Completeness of One-Dimensional Construction

We define the space S^k to be the union of the S^k_m, given by the formula

$$S^k = \bigcup_{m=0}^{\infty} S^k_m, \tag{6}$$

and observe that $\overline{S^k} = \mathcal{L}^2[0,1]$. In particular, S^k contains the Haar basis for $\mathcal{L}^2[0,1]$, consisting of functions piecewise constant on each of the subintervals $(2^{-m}n, 2^{-m}(n+1))$. Here the closure $\overline{S^k}$ is defined with respect to the $\mathcal{L}^2$-norm,

$$\|f\| = \langle f, f\rangle^{1/2},$$

where the inner product $\langle f, g\rangle$ is defined by the formula

$$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx.$$

We let $\{u_1, \ldots, u_k\}$ denote an orthonormal basis for S^k_0; in view of Eqs. (4), (5), and (6), the orthonormal system

$$\begin{aligned} B_k = \quad & \{u_j : \ j = 1, \ldots, k\} \\ \cup\, & \{h^n_{j,m} : \ j = 1, \ldots, k;\ m = 0, 1, 2, \ldots;\ n = 0, \ldots, 2^m - 1\} \end{aligned}$$

spans $\mathcal{L}^2[0,1]$; we refer to B_k as the *multi-wavelet basis of order* k for $\mathcal{L}^2[0,1]$.

Now we construct a basis for $\mathcal{L}^2(\mathcal{R})$ by defining, for $m \in \mathcal{Z}$, the space $\tilde{S}^k_m$ by the formula

$$\begin{aligned} \tilde{S}^k_m = \{f : \ & \text{the restriction of } f \text{ to the interval } (2^{-m}n, 2^{-m}(n+1)) \text{ is} \\ & \text{a polynomial of degree less than } k, \text{ for } n \in \mathcal{Z}\} \end{aligned}$$

and observing that the space $\tilde{S}^k_{m+1} \backslash \tilde{S}^k_m$ is spanned by the orthonormal set

$$\{h^n_{j,m} : \ h^n_{j,m}(x) = 2^{m/2}\, h_j(2^m x - n),\ j = 1, \ldots, k;\ n \in \mathcal{Z}\}.$$

Thus $\mathcal{L}^2(\mathcal{R})$, which is contained in $\overline{\bigcup_m \tilde{S}^k_m}$, has orthonormal basis

$$\{h^n_{j,m} : \ j = 1, \ldots, k;\ m, n \in \mathcal{Z}\}.$$

1.3 Construction in Multiple Dimensions

The construction of our bases for $\mathcal{L}^2[0,1]$ and $\mathcal{L}^2(\mathcal{R})$ can be extended to certain other function spaces, including $\mathcal{L}^2[a,b]^d$ and $\mathcal{L}^2(\mathcal{R}^d)$, for any positive integer d. We now outline this extension by giving the basis for $\mathcal{L}^2[0,1]^2$, which is illustrative of the construction for any finite-dimensional space. We define the space $S_m^{k,2}$ by the formula

$$S_m^{k,2} = S_m^k \times S_m^k, \qquad m = 0, 1, 2, \ldots,$$

where S_m^k is defined by Eq. (3). We further define $R_m^{k,2}$ to be the orthogonal complement of $S_m^{k,2}$ in $S_{m+1}^{k,2}$,

$$S_m^{k,2} \oplus R_m^{k,2} = S_{m+1}^{k,2}, \qquad R_m^{k,2} \perp S_m^{k,2}.$$

Then $R_0^{k,2}$ is the space spanned by the orthonormal basis

$$\{u_i(x)h_j(y),\ h_i(x)u_j(y),\ h_i(x)h_j(y) : \ i, j = 1, \ldots, k\}.$$

Among these $3k^2$ basis elements each element $v(x, y)$ has no projection on low-order polynomials,

$$\int_0^1 \int_0^1 v(x, y)\, x^i\, y^j\, dx\, dy = 0, \qquad i, j = 0, 1, \ldots, k-1.$$

The space $R_m^{k,2}$ is spanned by dilations and translations of the $v(x, y)$ and the basis of $\mathcal{L}^2[0,1]^2$ consists of these functions and the low-order polynomials $\{u_i(x)u_j(y) : i, j = 1, \ldots, k\}$.

2 Convergence of the Multi-Wavelet Bases

For a function $f \in \mathcal{L}^2[0,1]$, a positive integer k, and $m = 0, 1, 2 \ldots$, we define the orthogonal projection $Q_m^k f$ of f onto S_m^k by the formula

$$(Q_m^k f)(x) = \sum_{j,n} \langle f, u_{j,m}^n \rangle \cdot u_{j,m}^n(x),$$

where $\{u_{j,m}^n\}$ is an orthonormal basis for S_m^k. The projections $Q_m^k f$ converge (in the mean) to f as $m \to \infty$. If the function f is several times differentiable, we can bound the error, as established by the following lemma.

Lemma 2.1 *Suppose that the function $f : [0,1] \to \mathcal{R}$ is k times continuously differentiable, $f \in C^k[0,1]$. Then $Q_m^k f$ approximates f with mean error bounded as follows:*

$$\|Q_m^k f - f\| \leq 2^{-mk} \frac{2}{4^k k!} \sup_{x \in [0,1]} |f^{(k)}(x)| \tag{7}$$

Proof. We divide the interval $[0,1]$ into subintervals on which $Q_m^k f$ is a polynomial; the restriction of $Q_m^k f$ to one such subinterval $I_{m,n}$ is the polynomial of degree less than k that approximates f with minimum mean error. We then use the maximum error estimate for the polynomial which interpolates f at Chebyshev nodes of order k on $I_{m,n}$.

We define $I_{m,n} = [2^{-m}n, 2^{-m}(n+1)]$ for $n = 0, 1, \ldots, 2^m - 1$, and obtain

$$\begin{aligned}
\|Q_m^k f - f\|^2 &= \int_0^1 \left[(Q_m^k f)(x) - f(x)\right]^2 dx \\
&= \sum_n \int_{I_{m,n}} \left[(Q_m^k f)(x) - f(x)\right]^2 dx \\
&\le \sum_n \int_{I_{m,n}} \left[(C_{m,n}^k f)(x) - f(x)\right]^2 dx \\
&\le \sum_n \int_{I_{m,n}} \left(\frac{2^{1-mk}}{4^k k!} \sup_{x \in I_{m,n}} |f^{(k)}(x)|\right)^2 dx \\
&\le \left(\frac{2^{1-mk}}{4^k k!} \sup_{x \in [0,1]} |f^{(k)}(x)|\right)^2,
\end{aligned}$$

and by taking square roots we have bound (7). Here $C_{m,n}^k f$ denotes the polynomial of degree k which agrees with f at the Chebyshev nodes of order k on $I_{m,n}$, and we have used the well-known maximum error bound for Chebyshev interpolation (see, *e.g.*, [4]). □

The error of the approximation $Q_m^k f$ of f therefore decays like 2^{-mk} and, since S_m^k has a basis of $2^m k$ elements, we have convergence of order k. For the generalization to d dimensions, a similar argument shows that the rate of convergence is of order k/d.

3 Second-Kind Integral Equations

A linear Fredholm integral equation of the second kind is an expression of the form

$$f(x) - \int_a^b K(x,t)\, f(t)\, dt = g(x), \tag{8}$$

where we assume that the kernel K is in $\mathcal{L}^2[a,b]^2$ and the unknown f and right-hand-side g are in $\mathcal{L}^2[a,b]$. For notational simplicity, we restrict our attention to the interval $[a,b] = [0,1]$. We use the symbol $\mathcal{K}$ to denote the integral operator of Eq. (8), given by the formula

$$(\mathcal{K}f)(x) = \int_0^1 K(x,t)\, f(t)\, dt,$$

for all $f \in \mathcal{L}^2[0,1]$ and $x \in [0,1]$. Suppose that $\{b_1, b_2, \ldots\}$ is an orthonormal basis for $\mathcal{L}^2[0,1]$; the expansion of K in this basis is given by the formula

$$K(x,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} K_{ij}\, b_i(x)\, b_j(t), \tag{9}$$

where the coefficient K_{ij} is given by the expression

$$K_{ij} = \int_0^1 \int_0^1 K(x,t)\, b_i(x)\, b_j(t)\, dx\, dt, \qquad i,j = 1,2,\ldots. \tag{10}$$

Similarly, the functions f and g have expansions

$$f(x) = \sum_{i=1}^{\infty} f_i\, b_i(x), \qquad g(x) = \sum_{i=1}^{\infty} g_i\, b_i(x),$$

where the coefficients f_i and g_i are given by the formulae

$$f_i = \int_0^1 f(x)\, b_i(x)\, dx, \qquad g_i = \int_0^1 g(x)\, b_i(x)\, dx, \qquad i = 1,2,\ldots.$$

The integral equation (8) then corresponds to the infinite system of equations

$$f_i - \sum_{j=1}^{\infty} K_{ij}\, f_j = g_i, \qquad i = 1,2,\ldots.$$

The expansion for K may be truncated at a finite number of terms, yielding the integral operator R defined by the formula

$$(Rf)(x) = \int_0^1 \sum_{i=0}^{n}\sum_{j=0}^{n} (K_{ij}\, b_i(x)\, b_j(t))\, f(t)\, dt, \qquad f \in \mathcal{L}^2[0,1],\ x \in [0,1],$$

which approximates $\mathcal{K}$. Integral equation (8) is thereby approximated by the system

$$f_i - \sum_{j=1}^{n} K_{ij}\, f_j = g_i, \qquad i = 1,\ldots,n, \tag{11}$$

which is a system of n equations in n unknowns. Eqs. (11) may be solved numerically to yield an approximate solution to Eq. (8), given by the expression

$$f_R(x) = \sum_{i=1}^{n} f_i\, b_i(x).$$

How large is the error $e_R = f - f_R$ of the approximate solution? We follow the derivation by Delves and Mohamed in [6]. Rewriting Eqs. (8) and (11) in terms of operators $\mathcal{K}$ and R, we have

$$\begin{aligned} (I - \mathcal{K})f &= g \\ (I - R)f_R &= g, \end{aligned}$$

and combining the latter equations yields

$$(I - \mathcal{K})e_R = (\mathcal{K} - R)f_R.$$

Provided that $(I - \mathcal{K})^{-1}$ exists, we obtain the error bound

$$\|e_R\| \leq \|(I - \mathcal{K})^{-1}\| \cdot \|(\mathcal{K} - R)f_R\|. \tag{12}$$

The error depends, therefore, on the conditioning of the original integral equation, as is apparent from the term $\|(I - \mathcal{K})^{-1}\|$, and on the fidelity of the finite-dimensional operator R to the integral operator $\mathcal{K}$.

4 Sparse Representation of Integral Operators and Their Inverses

4.1 Representation in Multi-Wavelet Bases

We consider integral operators $\mathcal{K}$ with kernels that are analytic, except at $x = t$, where they are singular. In particular, we analyze singularities of the form $\log|x-t|$ or the form $|x-t|^\alpha$, with $0 < |\alpha| < 1$. An operator with such a kernel K, expanded in one of the multi-wavelet bases defined above, is represented as a sparse matrix. This sparseness is due to the smoothness of K on rectangles separated from the "diagonal".

Definition 4.1 We say that a rectangular region oriented parallel to the coordinate axes x, t is *separated from the diagonal* if its distance in the horizontal or vertical direction from the line $x = t$ is at least the length of its longer side. In

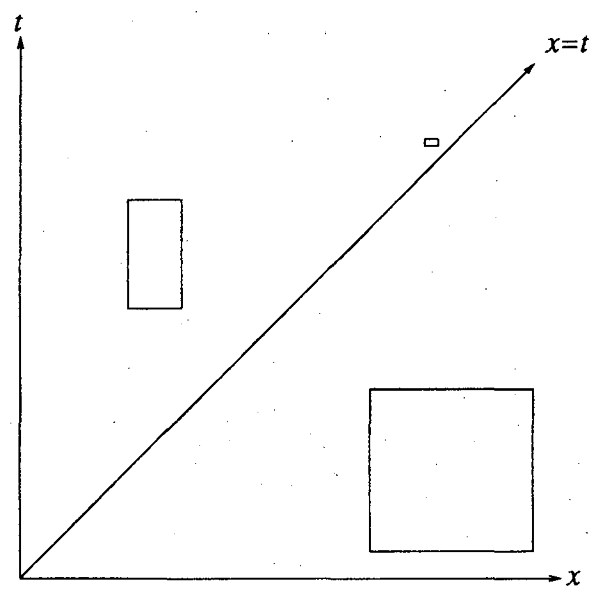

Figure 1: *Rectangular regions (just) separated from the diagonal.*

symbols, a region $[x, x+a] \times [t, t+b] \subset \mathcal{R}^2$ is separated from the diagonal if $a + \max\{a, b\} \le t - x$ or $b + \max\{a, b\} \le x - t$.

This definition is illustrated in Fig. 1.

Suppose that k is a positive integer and that $B_k = \{b_1, b_2, \ldots\}$ is the multi-wavelet basis for $\mathcal{L}^2[0,1]$ of order k, defined in §1. We let I_j denote the interval of support of b_j, and we assume that the sequence of basis functions $b_1, b_2, \ldots$ is ordered so that $I_1, I_2, \ldots$ have non-increasing lengths. For large n, the matrix $\{K_{ij}\}_{i,j=1,\ldots,n}$ is sparse, to high precision, as is proved in the following propositions.

Lemma 4.2 *Suppose that the function $K : [0,1] \times [0,1] \to \mathcal{R}$ is given by the formula $K(x,t) = \log|x-t|$. The expansion (Eq. 9) of K in the multi-wavelet basis B_k of order k has coefficients K_{ij} which satisfy the bound*

$$|K_{ij}| \le \frac{1}{8k \cdot 3^{k-1}} \tag{13}$$

whenever the rectangular region $I_i \times I_j$ is separated from the diagonal.

Proof. Suppose that the intervals I_i and I_j are given by the expressions $I_i = [x_0, x_0 + a]$ and $I_j = [t_0, t_0 + b]$; without loss of generality we assume (as one of two equivalent cases) that $b + \max\{a, b\} \le x_0 - t_0$. It is immediate from this inequality that

$$\left| \frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right| \le \frac{1}{3} \tag{14}$$

for $(x,t) \in I_i \times I_j$.

We use the Taylor expansion for the natural logarithm about $c > 0$,

$$\log(c + y) = \log(c) + (y/c) - (y/c)^2/2 + (y/c)^3/3 - (y/c)^4/4 + \cdots,$$

for $|y| < c$. We now let $c = x_0 + a/2 - t$ and $y = x - x_0 - a/2$ and for $(x,t) \in I_i \times I_j$ we obtain the formula

$$\log|x - t| = \log(x_0 + a/2 - t) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right)^m . \tag{15}$$

We now apply Eqs. (10), (15), (2), and (14), each in turn, to obtain

$$\begin{aligned} |K_{ij}| &= \left| \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} K(x,t)\, b_i(x)\, b_j(t)\, dx\, dt \right| \\ &\le \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} \log|x-t|\, b_i(x)\, dx \right| |b_j(t)|\, dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} \left[\log(x_0 + \frac{a}{2} - t) \right.\right. \\
&\qquad \left.\left. - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right)^m \right] b_i(x)\, dx \right| |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} \sum_{m=k}^{\infty} \frac{1}{m} \left(\frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right)^m b_i(x)\, dx \right| |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} \frac{1}{k} \sum_{m=k}^{\infty} \left(\frac{1}{3}\right)^m |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} \frac{1}{2k \cdot 3^{k-1}} |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\le \frac{1}{2k \cdot 3^{k-1}} \int_{t_0}^{t_0+b} \sqrt{\left(\int_{x_0}^{x_0+a} b_i^{\,2}(x)\, dx \right) \left(\int_{x_0}^{x_0+a} 1\, dx \right)}\, |b_j(t)|\, dt \\
&\le \frac{\sqrt{ab}}{2k \cdot 3^{k-1}} \le \frac{1}{8k \cdot 3^{k-1}},
\end{aligned}
$$

as was to be proved. □

Lemma 4.3 *Suppose that the function $L:\ D \times D \to \mathcal{C}$ is given by the formula $L(z,w) = f(z,w)\log|z-w|$, where D is the closed disk of radius $\frac{3}{2}$ centered at $z = \frac{1}{2}$ and f is analytic in a domain containing $D \times D \subset \mathcal{C}^2$. Suppose further that the function K is the restriction of L to $[0,1] \times [0,1]$. The expansion of K in the basis B_k has coefficients K_{ij} which satisfy the bound*

$$
|K_{ij}| \le \left(\frac{k}{8} + \frac{3}{16} \right) \frac{1}{3^{k-1}} \sup_{z,w \in \partial D} |f(z,w)|, \tag{16}
$$

whenever the rectangular region $I_i \times I_j$ is separated from the diagonal.

Proof. We combine the method of proof used in Lemma 4.2 with the formula for the derivative of a product,

$$
\frac{\partial^m K(x,t)}{\partial x^m} = \sum_{r=0}^{m} \binom{m}{r} \frac{\partial^r f(x,t)}{\partial x^r} \cdot \frac{\partial^{m-r} \log|x-t|}{\partial x^{m-r}}. \tag{17}
$$

By the Cauchy integral formula we obtain

$$
\left| \frac{\partial^r f(x,t)}{\partial x^r} \right| \le r! \sup_{z,w \in \partial D} |f(z,w)| \tag{18}
$$

for $(x,t) \in [0,1] \times [0,1]$. For the logarithm, differentiation yields the formula

$$
\frac{\partial^{m-r} \log|x-t|}{\partial x^{m-r}} = \frac{(-1)^{m-r-1}(m-r-1)!}{(x-t)^{m-r}}, \tag{19}
$$

for $r < m$. Combining (17), (18), and (19), we obtain

$$\begin{aligned}
\left|\frac{\partial^m K(x,t)}{\partial x^m}\right| &\leq \sum_{r=0}^{m} \binom{m}{r} \left|\frac{\partial^r f(x,t)}{\partial x^r}\right| \cdot \left|\frac{\partial^{m-r} \log|x-t|}{\partial x^{m-r}}\right| \\
&\leq \sup_{z,w\in\partial D} |f(z,w)| \left(\sum_{r=0}^{m-1} \binom{m}{r} r! \frac{(m-r-1)!}{|x-t|^{m-r}} + m! \left|\log|x-t|\right| \right) \\
&\leq S_f \cdot \left(m! \frac{2+\log m}{|x-t|^m} \right) \qquad (20)
\end{aligned}$$

for $|x-t| \leq 1$ and $m \geq 1$, where $S_f = \sup_{z,w\in\partial D} |f(z,w)|$.

Suppose that the intervals I_i and I_j are given by the expressions $I_i = [x_0, x_0 + a]$ and $I_j = [t_0, t_0 + b]$; we assume without loss of generality that $b + \max\{a, b\} \leq x_0 - t_0$. It follows directly from this inequality that

$$\left|\frac{x_0 + a/2 - x}{x_0 + a/2 - t}\right| \leq \frac{1}{3} \qquad (21)$$

for $(x,t) \in I_i \times I_j$. We now apply Eqs. (10), (2), (20), and (21), to obtain

$$\begin{aligned}
|K_{ij}| &= \left| \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} K(x,t)\, b_i(x)\, b_j(t)\, dx\, dt \right| \\
&\leq \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} \sum_{m=0}^{\infty} \frac{(x_0 + a/2 - x)^m}{m!} \frac{\partial^m K(x_0 + a/2, t)}{\partial {x_0}^m} b_i(x)\, dx \right| |b_j(t)|\, dt \\
&\leq \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} \sum_{m=k}^{\infty} \left| \frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right|^m S_f\, (2 + \log m)\, |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\leq \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} S_f \sum_{m=k}^{\infty} \left(\frac{1}{3}\right)^m (m+1)\, |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\leq \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} S_f \left(\frac{k}{2} + \frac{3}{4} \right) \frac{1}{3^{k-1}} |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\leq S_f \left(\frac{k}{2} + \frac{3}{4} \right) \frac{1}{3^{k-1}} \int_{t_0}^{t_0+b} \sqrt{\left(\int_{x_0}^{x_0+a} {b_i}^2(x)\, dx \right) \left(\int_{x_0}^{x_0+a} 1\, dx \right)}\, |b_j(t)|\, dt \\
&\leq S_f \left(\frac{k}{2} + \frac{3}{4} \right) \frac{\sqrt{ab}}{3^{k-1}} \\
&\leq S_f \left(\frac{k}{8} + \frac{3}{16} \right) \frac{1}{3^{k-1}},
\end{aligned}$$

which was to be proved. □

The proofs of the following two lemmas closely resemble those of Lemma 4.2 and Lemma 4.3, and are omitted.

Lemma 4.4 *Suppose that the function $K:\ [0,1]\times[0,1]\to\mathcal{R}$ is given by the formula $K(x,t)=|x-t|^\alpha$ with $0<|\alpha|<1$. Then the expansion coefficient K_{ij} of the function K in the basis B_k satisfies the bound*

$$|K_{ij}|\le\frac{1}{2\cdot 3^{k-1}} \tag{22}$$

whenever the rectangular region $I_i\times I_j$ is separated from the diagonal.

Lemma 4.5 *Suppose that the function $L:\ D\times D\to\mathcal{C}$ is given by the formula $L(z,w)=f(z,w)|z-w|^\alpha$, with $0<|\alpha|<1$, where D is the closed disk of radius $\frac{3}{2}$ centered at $z=\frac{1}{2}$ and f is analytic in a domain containing $D\times D\subset\mathcal{C}^2$. Suppose further that the function K is the restriction of L to $[0,1]\times[0,1]$. The expansion of K in the basis B_k has coefficients K_{ij} which satisfy the bound*

$$|K_{ij}|\le\left(\frac{k}{2}+\frac{3}{4}\right)\frac{1}{3^{k-1}}\sup_{z,w\in\partial D}|f(z,w)|, \tag{23}$$

whenever the rectangular region $I_i\times I_j$ is separated from the diagonal.

The four preceding lemmas show that for a smooth kernel K with logarithm or power singularity at $x=t$, the order k of the multi-wavelet basis B_k in which K is expanded may be chosen large enough that the expansion coefficient K_{ij} is negligible, provided $I_i\times I_j$ is separated from the diagonal. A similar statement can be proven for any kernel of the form $K(x,t)=\phi(x,t)s(|x-t|)+\psi(x,t)$, where ϕ,ψ are entire analytic functions of two variables and s is an analytic function except at the origin (where it has a singularity), provided that s is integrable. We do not prove this statement here.

The next lemma establishes the fact that, asymptotically, most regions $I_i\times I_j$ *are* separated from the diagonal.

Lemma 4.6 *Suppose that $I_1,\ldots,I_n$ are the (non-increasing) intervals of support of the first n functions of the basis B_k. Of the n^2 rectangular regions $I_i\times I_j$, we denote the number separated from the diagonal by $S(n)$ and the number "near" the diagonal by $N(n)=n^2-S(n)$. Then $N(n)$ grows as $\mathrm{O}(n\log n)$; in particular, for $n=2^lk$ with $l>0$, we have the formula*

$$N(n)=6nlk-15nk-6lk^2+16k^2. \tag{24}$$

Proof. The restriction that $n=2^lk$ ensures that the first n basis functions consist of those functions whose intervals of support have length at least 2^{1-l}. We define $S^=(p)$ to be the number of pairs (i,j) such that the rectangular region $I_i\times I_j$ is separated from the diagonal and $|I_i|=|I_j|=2^{-p}$, and we observe that

$S^{=}(p) = (2^p - 1)(2^p - 2)\, k^2$ for $p = 0, 1, 2, \ldots$. We further define $S^{\neq}(p,q)$ to be the number of pairs (i,j) such that $I_i \times I_j$ is separated from the diagonal and $|I_i| = 2^{-p}$, $|I_j| = 2^{-q}$, and we observe that $S^{\neq}(p,q) = S^{=}(\min\{p,q\})\, 2^{|p-q|}$ for $p, q = 0, 1, 2, \ldots$. Finally, we combine these formulae to obtain

$$\begin{aligned}
S(n) &= \sum_{p=0}^{l-1} \left(S^{=}(p) + \sum_{q=p+1}^{l-1} \left(S^{\neq}(p,q) + S^{\neq}(q,p) \right) \right) \\
&= \sum_{p=0}^{l-1} S^{=}(p) \left(1 + 2(2^{l-p} - 2) \right) \\
&= \sum_{p=0}^{l-1} (2^p - 1)(2^p - 2)\, k^2\, (2^{l-p+1} - 3) \\
&= (4^l - 6 \cdot 2^l l + 15 \cdot 2^l + 6l - 16)\, k^2 \\
&= n^2 - 6nlk + 15nk + 6lk^2 - 16k^2,
\end{aligned}$$

from which Eq. (24) follows directly. The assertion that the general growth of $N(n)$ is $O(n \log n)$ follows from Eq. (24) and that fact that N is a monotonic function of n. □

4.2 Products of Integral Operators

The previous subsection established the fact that a wide class of integral operators, when expanded in multi-wavelet coordinates, are represented to high accuracy as sparse matrices. It readily follows that a product of such integral operators can be similarly represented. For if we define integral operators $\mathcal{K}_1, \mathcal{K}_2$ by the formulae

$$\begin{aligned}
(\mathcal{K}_1 f)(x) &= \int_0^1 K_1(x,t)\, f(t)\, dt \\
(\mathcal{K}_2 f)(x) &= \int_0^1 K_2(x,t)\, f(t)\, dt,
\end{aligned}$$

then the product operator $\mathcal{K}_3 = \mathcal{K}_2 \mathcal{K}_1$ is given by the formula

$$\begin{aligned}
(\mathcal{K}_2 \mathcal{K}_1 f)(x) &= \int_0^1 \int_0^1 K_2(x,y)\, K_1(y,t)\, f(t)\, dt\, dy \\
&= \int_0^1 \left(\int_0^1 K_2(x,y)\, K_1(y,t)\, dy \right) f(t)\, dt \\
&= \int_0^1 K_3(x,t)\, f(t)\, dt,
\end{aligned}$$

where the kernel K_3 of the product has the form

$$K_3(x,t) = \int_0^1 K_2(x,y)\, K_1(y,t)\, dy.$$

If kernels K_1 and K_2 are analytic except along the diagonal $x = t$, where they have integrable singularities, then the same is true of the product kernel K_3. As a result, the product $\mathcal{K}_3$ also has a sparse representation in a multi-wavelet basis.

4.3 Schulz Method of Matrix Inversion

Schulz's method [11] is an iterative, quadratically convergent algorithm for computing the inverse of a matrix. Its performance is characterized as follows.

Lemma 4.7 *Suppose that A is an invertible matrix, X_0 is the matrix given by $X_0 = A^H/\|A^H A\|$, and for $m = 0, 1, 2, \ldots$ the matrix X_{m+1} is defined by the recursion*

$$X_{m+1} = 2X_m - X_m A X_m.$$

Then X_{m+1} satisfies the formula

$$I - X_{m+1}A = (I - X_m A)^2. \tag{25}$$

Furthermore, $X_m \to A^{-1}$ as $m \to \infty$ and for any $\epsilon > 0$ we have

$$\|I - X_m A\| < \epsilon \quad \text{provided} \quad m \geq 2\log_2 \kappa(A) + \log_2 \log(1/\epsilon), \tag{26}$$

where $\kappa(A) = \|A\| \cdot \|A^{-1}\|$ is the condition number of A and the norm is given by $\|A\| = (\text{largest eigenvalue of } A^H A)^{1/2}$.

Proof. Eq. (25) is obtained directly from the definition of X_{m+1}. Bound (26) is equally straightforward. Noting that $A^H A$ is symmetric positive-definite and letting λ_0 denote the smallest and λ_1 the largest eigenvalue of $A^H A$ we have

$$\begin{aligned} \|I - X_0 A\| &= \left\| I - \frac{A^H A}{\|A^H A\|} \right\| \\ &= 1 - \lambda_0/\lambda_1 \\ &= 1 - \kappa(A)^{-2}. \end{aligned} \tag{27}$$

From Eq. (25) we obtain $I - X_m A = (I - X_0 A)^{2^m}$, which in combination with Eq. (27) and simple manipulation yields bound (26). □

The Schulz method is a notably simple scheme for matrix inversion and its convergence is extremely rapid. It is rarely used, however, because it involves matrix-matrix multiplications on each iteration; for most problem formulations, this process requires order $O(n^3)$ operations for an $n \times n$ matrix. In [3], on the other hand, it is observed that a sparse matrix, possessing a sparse inverse, whose iterates X_n are also sparse, may be rapidly inverted using the Schulz method. We have seen above that a discretized integral operator A represented in the basis B_k has only order $O(n \log n)$ elements (to finite precision). In addition, $A^H A$ and $(A^H A)^m$ are similarly sparse. This property enables us to employ the Schulz algorithm to compute A^{-1} in order $O(n \log^2 n)$ operations.

5 Numerical Examples

5.1 Basis Functions

In this section we give numerical expressions for the multi-wavelet functions $f_0, f_1, \ldots, f_{k-1}$ and show their graphs for several values of k. These functions were obtained using the procedure of §1, implemented in a simple Maple program (available from the author). Table 1 contains, for small k, the polynomials which represent the f_i on the interval $(0,1)$, together with the reflection formula

Table 1: *Expressions for the orthonormal, vanishing-moment functions* $f_1, \ldots, f_k$, *for various* k, *for argument* x *in the interval* $(0,1)$. *The function* f_i *is extended to the interval* $(-1,1)$ *as an odd or even function, according to the formula* $f_i(x) = (-1)^{i+k-1} f_i(-x)$ *for* $x \in (-1,0)$, *and is zero outside* $(-1,1)$.

	$k=1$	
$f_1(x) =$	$\sqrt{\frac{1}{2}}$	
	$k=2$	
$f_1(x) =$	$\sqrt{\frac{3}{2}}$	$(-1+2x)$
$f_2(x) =$	$\sqrt{\frac{1}{2}}$	$(-2+3x)$
	$k=3$	
$f_1(x) =$	$\frac{1}{3}\sqrt{\frac{1}{2}}$	$(1-24x+30x^2)$
$f_2(x) =$	$\frac{1}{2}\sqrt{\frac{3}{2}}$	$(3-16x+15x^2)$
$f_3(x) =$	$\frac{1}{3}\sqrt{\frac{5}{2}}$	$(4-15x+12x^2)$
	$k=4$	
$f_1(x) =$	$\sqrt{\frac{15}{34}}$	$(1+4x-30x^2+28x^3)$
$f_2(x) =$	$\sqrt{\frac{1}{42}}$	$(-4+105x-300x^2+210x^3)$
$f_3(x) =$	$\frac{1}{2}\sqrt{\frac{35}{34}}$	$(-5+48x-105x^2+64x^3)$
$f_4(x) =$	$\frac{1}{2}\sqrt{\frac{5}{42}}$	$(-16+105x-192x^2+105x^3)$
	$k=5$	
$f_1(x) =$	$\sqrt{\frac{1}{186}}$	$(1+30x+210x^2-840x^3+630x^4)$
$f_2(x) =$	$\frac{1}{2}\sqrt{\frac{1}{38}}$	$(-5-144x+1155x^2-2240x^3+1260x^4)$
$f_3(x) =$	$\sqrt{\frac{35}{14694}}$	$(22-735x+3504x^2-5460x^3+2700x^4)$
$f_4(x) =$	$\frac{1}{8}\sqrt{\frac{21}{38}}$	$(35-512x+1890x^2-2560x^3+1155x^4)$
$f_5(x) =$	$\frac{1}{2}\sqrt{\frac{7}{158}}$	$(32-315x+960x^2-1155x^3+480x^4)$

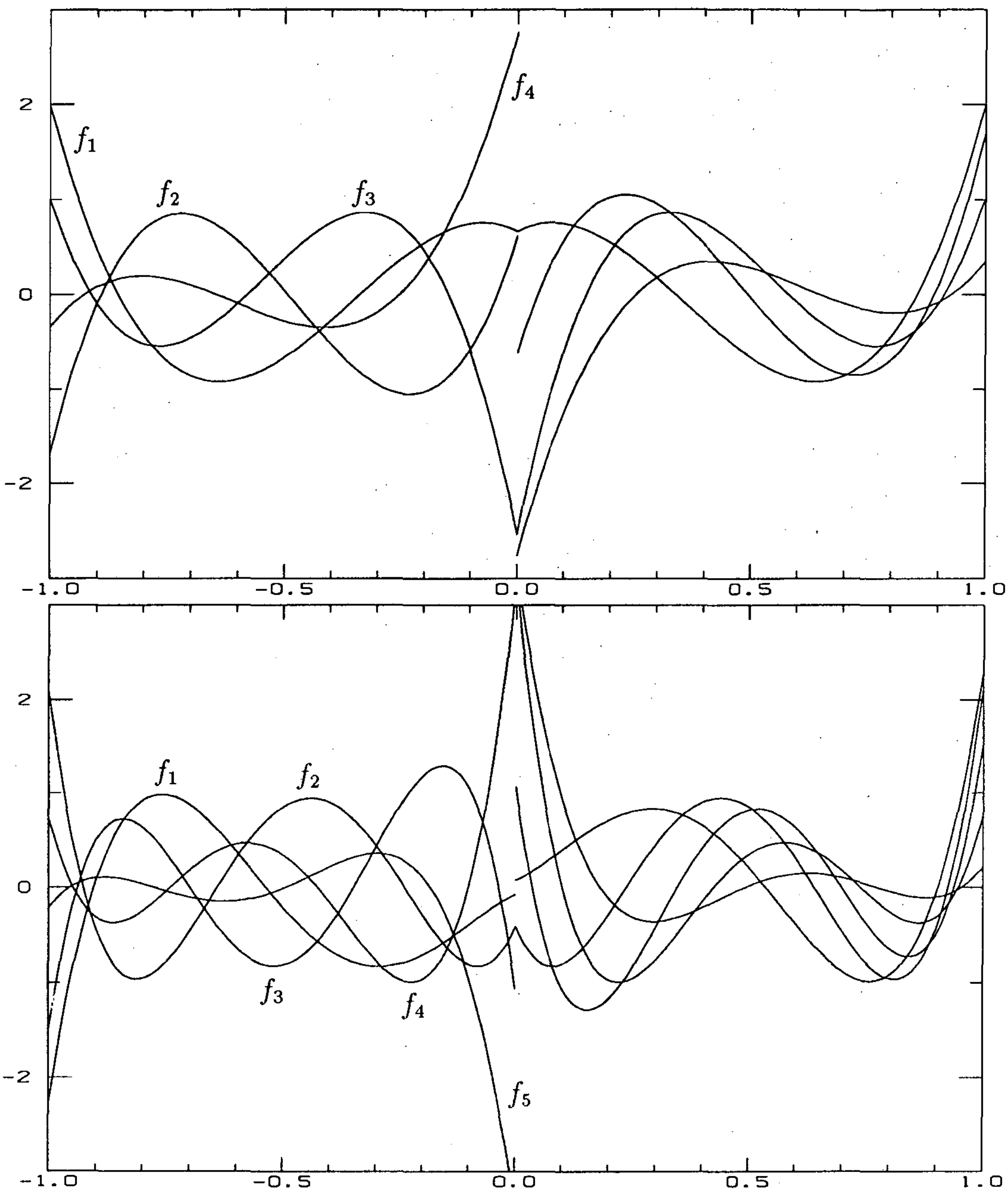

Figure 2: *Functions $f_1, \ldots, f_k$ are graphed for $k = 4$ (top graph) and $k = 5$ (bottom). Each function (given in Table 2.1) is a polynomial on the interval $(0, 1)$, is an odd or even function on $(-1, 1)$, and is zero elsewhere.*

to extend the functions to $(-1, 1)$, which is their interval of support. Fig. 2 shows the graphs of the functions for $k = 4$ and $k = 5$.

5.2 Integral Operators and Their Inverses

We compute the expansion in multi-wavelet bases of the integral operator $\mathcal{K}$ defined by the formula

$$(\mathcal{K}f)(x) = \int_0^1 \log|x-t|\, f(t)\, dt, \tag{28}$$

which yields the matrix

$$T = \{K_{ij}\}_{i,j=1,\ldots,n}, \tag{29}$$

where

$$K_{ij} = \int_0^1 \int_0^1 K(x,t)\, b_i(x)\, b_j(t)\, dx\, dt$$

and $\{b_1, b_2, \ldots\}$ is a multi-wavelet basis of $\mathcal{L}^2[0,1]$. This computation is done for the multi-wavelet basis of order $k=4$, for various sizes n.

In addition the inverse matrix $(I-T)^{-1}$ is obtained by the Schulz method. Table 2 displays, for various precisions ϵ, the average number of elements per row in the matrices $I-T$ and $(I-T)^{-1}$. Fig. 3 displays the matrices for $n=128$ and $\epsilon = 10^{-3}$.

Table 2: *The average number of elements per row of the matrices $I-T$ and $(I-T)^{-1}$, where T is defined in Eq. (29), is tabulated for various precisions ϵ and various sizes n. Here $k=4$.*

	$\epsilon = 10^{-2}$		$\epsilon = 10^{-3}$		$\epsilon = 10^{-4}$	
n	$I-T$	$(I-T)^{-1}$	$I-T$	$(I-T)^{-1}$	$I-T$	$(I-T)^{-1}$
32	8.8	9.7	19.3	19.6	22.8	23.6
64	9.3	10.0	25.8	26.0	31.9	32.6
128	9.9	10.1	29.2	29.4	38.2	38.8
256	11.8	11.8	30.1	30.3	41.9	42.7

6 Discussion

The results of the previous subsection demonstrate, for a particular integral operator, that the multi-wavelet representations are sparse. The matrix has a peculiar structure in which the non-negligible elements are contained in blocks lying along rays emanating from one corner of the matrix. Furthermore, the inverse matrix shares that structure. This property is a general characteristic of integral operators with non-oscillatory kernels that possess diagonal singularities.

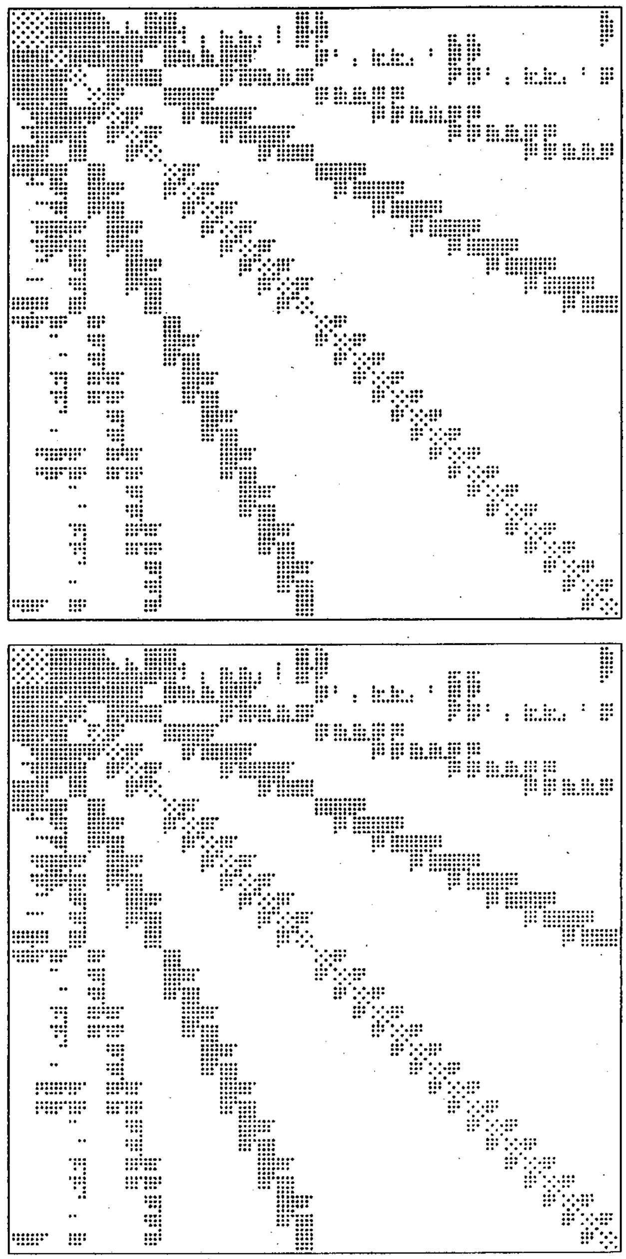

Figure 3: *Matrices representing the operators* $I-\mathcal{K}$ *(top) and* $(I-\mathcal{K})^{-1}$ *(bottom), with* $\mathcal{K}$ *defined by Eq. (28), expanded in the multi-wavelet basis of order* $k = 4$*, for* $n = 128$*. The dots represent elements above a threshold, which is determined so as to bound the relative truncation error at* $\epsilon = 10^{-3}$.

The kernel $K(x,t) = \log|x-t|$ of the previous subsection was chosen, however, because the projections K_{ij} could be computed analytically, thereby avoiding use of quadratures. The difficulty here with quadratures is that they would be required for each element K_{ij}, and would have to cope with the singularity of the logarithm. It was felt that the analytical computation would be more efficient. In fact, the analytical computation, which requires integrating monomials x^j ($0 \leq j < k$) against the logarithm and combining the results with large coefficients, is a very poorly-conditioned procedure. The computations described above required quadruple-precision arithmetic to obtain single-precision accuracy for n as small as 64. This procedure is not recommended.

The fault lies, of course, not with the idea of projection to the multi-wavelet basis, but with the method of projection. The integration should be performed numerically, with quadratures. As mentioned above, such a procedure would require use of quadratures for each matrix element K_{ij}, or potentially order $O(n \log n)$ times. A more efficient procedure is to use the Nyström method, in which only n quadrature applications are required. Numerical quadratures and a vector-space analogue of the multi-wavelet bases are developed in [1],[3]; these tools enable efficient solution of second-kind integral equations using Nyström's method. We believe that the present paper, rather than directly providing numerical tools, offers a particularly simple framework in which to understand the ideas for sparse representation of integral operators.

Acknowledgement

The bases constructed in this paper are the limiting case of the discrete construction in [3]; thanks to R. Coifman for prodding this author to consider the limit, which surprised us with its simplicity.

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