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Topics in N=2 Lightcone Supersymmetry*

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Abstract

This thesis contains a calculation of the β function of $N = 2$ Yang-Mills at one and two loops. Instead of using the usual $N = 1$ covariant superfield formalism, we work in an $N = 2$ lightcone superspace. In contrast to the covariant calculation we do not encounter any offshell infrared infinities. We also qualitatively discuss the inclusion of $N = 2$ matter. Most importantly, this work demonstrates the feasibility of doing multiloop computations in lightcone field theories and to this end some practical methods for calculating lightcone integrals are developed.

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Contents

I Introduction

It has been known for some time [1] that N=4 supersymmetric Yang-Mills has a vanishing beta function. This is most directly seen if the theory is written so as to be perturbatively finite, i.e. it requires no ultraviolet counterterms whatsoever. Mandelstam [2] and Brink et.al. [3] accomplished this by first using null plane quantization and then constructing superspace for the light cone field theory. The resulting perturbation theory is then graph by graph finite. N=2 theories are generally thought [4] to be finite above one loop and proofs based on a covariant superspace have been offered [5].

The finiteness properties of N=4 light cone theories are immediately apparent from their superspace Feynman rules. It is the manifest supersymmetry and not the (rather complicated) Lorentz properties of the Feynman rules that are responsible for the graph by graph finiteness in these theories.

Since N=2 theories in general have infinities at one loop a light cone superspace formulation cannot exhibit finiteness through power counting. However N=2 light cone superspace is worth investigating to see which, if any, cancellations of the N=4 theory remain. Tollsten [6] has already investigated this problem for the propagator at one loop in a superspace formulation different from the one given here.

The inclusion of N=2 matter in the right representations is known to

produce 1 loop finite theories [13]. Using some very simple considerations, we can exploit the graph by graph finiteness of the N=4 theory to derive the one loop finiteness conditions for the N=2 theory with matter. We further sketch a strategy for a general proof of finiteness at higher loops. Although incomplete, it immediately leads, without detailed computation, to the two loop finiteness of 1 loop finite theories.

An important issue for supersymmetric field theories is the choice of regulator. In the particular scheme of regularization by dimensional reduction (RDR) a breakdown is known to occur [11] for N=2 Yang-Mills at the three loop level. While the authors in [11] exhibit this breakdown through different renormalizations at the vector-ghost-ghost and scalar-fermion-fermion vertices, their calculation disagrees with the three loop calculation of reference [24]. Besides this (possible) three loop breakdown of RDR, various authors [11,25,26] have suggested that RDR is inadequate. While this (possible) breakdown does not stop us from discussing one and two loop finite theories it does interfere with finiteness proofs to all orders. While we do not offer a suitable regulator we can use some of the results developed here to prove the graph by graph finiteness of N=4 Yang-Mills. This has already been done in the conventional formalism by Lindgren [12].

For N=2 Yang-Mills, why do we need another computation of the two loop β function when it is already known to vanish [13,14] within RDR? Due to the difficulty of doing lightcone integrals and until recently the lack of a suitable $1/p_+$ prescription [2] no one has done any complete 2 loop calculation in the lightcone gauge [15]. Hopefully, this work will encourage more such efforts. Working in the lightcone gauge we encounter no offshell infrared infinities in any graph. As opposed to a previous calculation [13] there are no infrared divergent, ultraviolet finite integrals involved. Finally, the infinite part of the 2 loop counterterm is independent of the one loop subtraction prescription.

II N=2 Lagrangian and Lightcone Superspace

The starting point is the N=2 Yang-Mills lagrangian of Fayet [7] and Brink et.al. [8] . It contains a gluon field v_μ, two Weyl spinors λ and ψ, and a complex scalar A. All the fields are massless and in the adjoint representation of the gauge group. The lagrangian is

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - i\,\overline{\lambda}^{\dot{a}}\,\sigma^\mu_{\dot{a}a}D_\mu\lambda^a - i\,\overline{\psi}^{\dot{a}}\,\sigma^\mu_{\dot{a}a}D_\mu\psi^a - D_\mu A^* D^\mu A \\ + ig\sqrt{2}\lambda^a\left(\psi_a \times A\right) + ig\sqrt{2}\,\overline{\psi}^{\dot{a}}\left(\overline{\lambda}_{\dot{a}} \times A^*\right) + \frac{g^2}{2}\left(A^* \times A\right)^2\end{aligned} \tag{2.1}$$

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - g\left(v_\mu \times v_\nu\right) \tag{2.2}$$

$$D_\mu\phi = \partial_\mu\phi - g\left(v_\mu \times \phi\right) \tag{2.3}$$

$$\left(A \times B\right)^\alpha = f^{\alpha\beta\gamma}A^\beta B^\gamma \tag{2.4}$$

$f^{\alpha\beta\gamma}$ are the group structure constants while the spinor indices $a, \dot{a}$ run over 1,2. This lagrangian is just an N=1 Yang-Mills theory interacting with N=1 matter in the adjoint representation. $\mathcal{L}$ has a global $SU(2)$ invariance with λ, ψ transforming as a doublet. This is the origin of the two supercurrents in the model.

Next we impose the $v_+ = 0$ gauge condition through a delta function in the functional integral. With $v_+ = 0$ the $v_- = v_0 - v_3$ integral is gaussian and the fermion determinants for $\overline{\lambda}^2$, λ^2, $\overline{\psi}^2$, ψ^2 are field independent. So all these fields can be integrated out and we are left with an expression in which we only integrate over the fields $v = v_1 + iv_2$, , $v^* = v_1 - iv_2$, $\overline{\lambda}^1$, λ^1, $\overline{\psi}^1$, ψ^1, A, and A^*. These are the light cone fields and through them we can linearly realize supersymmetry. We will not display the rather lengthy expression for the light cone lagrangian since it's superspace form is so compact. By integrating out some of the fields we have not lost supersymmetry because it is really the constraint equations which allow this, and they are themselves supersymmetric. Half of the supersymmetry algebra closes on the light cone fields [9] (e.g. the remaining fields in the functional integral). If we look at the N=2 supercharge algebra,

$$\{Q^L_a, Q^M_b\} = \{\bar{Q}_{L\dot{a}}, \bar{Q}_{M\dot{b}}\} = 0 \tag{2.5}$$

$$\{Q^L_a, \bar{Q}_{M\dot{b}}\} = -2^{3/2}\delta^L_M p_{a\dot{b}} \tag{2.6}$$

$$p_{a\dot{a}} = \begin{pmatrix} -p_- & p \\ p^* & -p_+ \end{pmatrix} \tag{2.7}$$

it is the $a = \dot{a} = 2$ transformations we can realize with light cone fields alone. The supersymmetry transformations are implemented by $\sum_{a\dot{a}} \alpha^a Q^1_a + \beta^a Q^2_a + \overline{\alpha}^{\dot{a}}\, \bar{Q}_{1\dot{a}} + \overline{\beta}^{\dot{a}}\, \bar{Q}_{2\dot{a}}$ where the α's and β's are Grassmann parameters.

Here L,M are internal indices running over 1,2. In particular, the known N=2 transformations specialize for $\alpha^1 = \overline{\alpha}^1 = \beta^1 = \overline{\beta}^1 = 0$ to

$$\delta A = -2^{3/4}\alpha^2\psi^1 - 2^{3/4}\beta^2\lambda^1 \tag{2.8}$$

$$\delta\psi^1 = 2^{3/4}\overline{\alpha}^2\, p_+\, A - i2^{1/4}\beta^2\, p_+\, v \tag{2.9}$$

$$\delta\lambda^1 = 2^{3/4}\overline{\beta}^2\, p_+\, A - i2^{1/4}\alpha^2\, p_+\, v \tag{2.10}$$

$$\delta v = 2^{5/4} i\overline{\alpha}^2\lambda^1 + 2^{5/4} i\overline{\beta}^2\psi^1 \tag{2.11}$$

It can be checked that the commutator of two supersymmetry transformations (2.3) is consistent with the algebra (2.1) at $a = \dot{a} = 2$. In fact only the light cone fields close part of the supersymmetry algebra without auxillary fields.

At this point we would like to represent the charge algebra

$$\{Q^L_2, Q^M_2\} = \{\overline{Q}_{L2}, \overline{Q}_{M2}\} = 0 \tag{2.12}$$

$$\{Q^L_2, \overline{Q}_{M2}\} = 2^{3/2}\, p_+\, \delta^L_M \tag{2.13}$$

in terms of linear differential operators. We do this by formulating the lagrangian in a superspace whose coordinates are space-time coordinates and two Grassmann parameters θ_1 and θ_2. This is Mandelstam's form [2] of superspace. The absence of θ's makes it analogous to the description of N=1 superspace in terms of the variables $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$, θ^j. So with the correspondences

$$Q_2^L \rightarrow D_L = \frac{i\partial}{\partial\theta_L} - i2^{1/2}\, p_+\, \theta_L \tag{2.14}$$

$$-\overline{Q}_2^L \rightarrow \overline{D}_L = \frac{\partial}{\partial\theta_L} + 2^{1/2}\, p_+\, \theta_L \tag{2.15}$$

, $(L = 1, 2)$, we can construct superfields Φ, Φ^* which respond to supersymmetry transformations as

$$\delta\Phi = \sum_{L=1}^{2} \left(\alpha_L D_L + \beta_L \overline{D}_L\right) \Phi \tag{2.16}$$

$$\delta\Phi^* = \sum_{L=1}^{2} \left(-\alpha_L D_L + \beta_L \overline{D}_L\right) \Phi^* \tag{2.17}$$

The explicit superfields can be constructed as

$$\Phi = \frac{1}{2}v + \frac{\theta_1}{2^{1/4}}\lambda^1 + \frac{\theta_2}{2^{1/4}}\psi^1 + \theta_1\theta_2\, p_+\, A^* \tag{2.18}$$

$$\Phi^* = \frac{1}{2}v^* + \frac{\theta_1}{2^{1/4}}\,\overline{\lambda}^1 + \frac{\theta_2}{2^{1/4}}\,\overline{\psi}^1 + \theta_1\theta_2\, p_+\, A \tag{2.19}$$

and with a little guesswork the lagrangian written in terms of the superfields is

$$\mathcal{L} = \Phi^* \bullet \frac{p_\mu^2}{p_+} D\Phi + 2ig\frac{D}{p_+}\Phi \bullet \left(\frac{p}{p_+}\Phi^* \times \Phi^*\right)$$
$$+ 2ig\frac{D}{p_+}\Phi^* \bullet \left(\frac{p^*}{p_+}\Phi \times \Phi\right) + 2g^2\left(\frac{D}{p_+}\Phi \times \Phi^*\right) \bullet \frac{D}{{p_+}^2}\left(\frac{D}{p_+}\Phi^* \times \Phi\right) \tag{2.20}$$

. D is defined by $D = D_1 D_2$.

The action is $S = \int d\theta_2 d\theta_1 \mathcal{L}$. A better form for Feynman rules is got by replacing the present fields with

$$\phi^* = \frac{1}{p_+}\Phi^*, \qquad \phi = \frac{D}{{p_+}^2}\Phi \qquad or \qquad \Phi = \frac{-D}{2}\phi.$$

Then we have

$$i\mathcal{L} = -i\phi^* \bullet p_\mu^2\, p_+\, \phi - 2g\, p_+\, \phi \bullet (p\phi^* \times p_+\, \phi^*)$$
$$- \frac{g}{2}D\phi^* \bullet \left(\frac{p^*}{p_+}D\phi \times D\phi\right) - ig^2\,(p_+\,\phi \times p_+\,\phi^*) \bullet \frac{D}{{p_+}^2}\,(D\phi^* \times D\phi) \tag{2.21}$$

In this form $\mathcal{L}$ depends on θ only through ϕ and ϕ^*. This is seen by writing out (2.21) using the definition of the D's. There results

$$
\begin{aligned}
i\mathcal{L} = {} & \phi_a^* \left(-ip_\mu^2 p_+ \delta^{ab}\right) \phi_b - 2g f^{abc} p_+ \phi_a p \phi_b^* p_+ \phi_c \\
& + f^{abc}\phi_a^* [- \left(p_+ \, p^* \phi_b\right) \left(\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\phi_c\right) - \left(p^* \frac{\partial}{\partial\theta_2}\phi_b\right)\left(p_+ \frac{\partial}{\partial\theta_1}\phi_c\right) \\
& + \left(p^* \frac{\partial}{\partial\theta_1}\phi_b\right)\left(p_+ \frac{\partial}{\partial\theta_2}\phi_c\right) - \left(\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\frac{p^*}{p_+}\phi_b\right)\left(p_+^2 \phi_c\right)] \\
& - 2ig^2 f^{ac\alpha} f^{db\alpha}\left[\frac{1}{p_+^2}\left(p_+\phi_a p_+ \phi_c^*\right)\right] [\left(p_+^2 \phi_d^*\right)\left(\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\phi_b\right) \\
& + \left(p_+^2 \phi_b\right)\left(\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\phi_d^*\right) - \left(p_+ \frac{\partial}{\partial\theta_1}\phi_d^*\right)\left(p_+ \frac{\partial}{\partial\theta_2}\phi_b\right) + \\
& \left(p_+ \frac{\partial}{\partial\theta_2}\phi_d^*\right)\left(p_+ \frac{\partial}{\partial\theta_1}\phi_b\right)]
\end{aligned}
\tag{2.22}
$$

Alternatively, using the method of Brink et.al. [19] we can immediately get the N=2 Yang-Mills lagrangian from the N=4 lagrangian of Mandelstam [2]. The basic observation is that since Grassmann integration = Grassmann differentiation, writing the N=4 action as;

$$
S_{N=4} = \int d\theta_1 d\theta_2 d\theta_3 d\theta_4 \mathcal{L}\left(\phi\right) = \int d\theta_1 d\theta_2 \left[\frac{\partial}{\partial\theta_3}\frac{\partial}{\partial\theta_4}\mathcal{L}\left(\phi\right)\right]|_{\theta_3=\theta_4=0} \tag{2.23}
$$

and the quantity in square brackets is the N=2 decomposition of the N=4 lagrangian. That is;

$$
\frac{\partial}{\partial\theta_3}\frac{\partial}{\partial\theta_4}\mathcal{L}\left(\phi\right)|_{\theta_3=\theta_4=0} = N = 2 \quad Lagrangian \tag{2.24}
$$

To carry out this decomposition we need to seperate the N=2 matter and N=2 Yang-Mills parts of ϕ. The field decomposition of ϕ (eq. 3.4 in ref. [2]) is;

$$
\phi = \frac{i}{2\,p^+}v + \frac{1}{2\,p^+}\theta^\alpha\psi_\alpha + \frac{i}{4}\theta^\alpha\theta^\beta\rho^a_{\alpha\beta}A_a + \frac{1}{6}\theta^\alpha\theta^\beta\theta^\gamma\epsilon_{\alpha\beta\gamma\delta}\overline{\psi}^\delta + 2i\,p^+\,\theta_1\theta_2\theta_3\theta_4\,v^* \tag{2.25}
$$

.Here ρ and ϵ are numerical tensors but more importantly v and v^* are the gluon fields. By noting which derivatives of ϕ contain gluons we can make the following identifications;

N=2 Yang-Mills superfields (bosonic)

$$
\Phi^* = \frac{\partial}{\partial\theta_3}\frac{\partial}{\partial\theta_4}\phi \qquad \Phi = \phi|_{\theta_3=\theta_4=0} \tag{2.26}
$$

N=2 matter superfields (fermionic)

$$
\chi^* = \frac{\partial}{\partial\theta_3}\phi|_{\theta_4=0} \qquad \chi = \frac{\partial}{\partial\theta_4}\phi|_{\theta_3=0} \tag{2.27}
$$

. Since we are not interested in the matter part of the lagrangian we can immediately set $\chi = \chi^* = 0$. The result of this calculation is just eq. 2.20 after a suitable field redefinition.

From the form (2.22) , which depends only on derivatives in superspace, we can easily write the Feynman rules in momentum space for the x's and the θ's. Functional derivatives of superfields are just delta functions so deriving the Feynman rules is simple. These Feynman rules are given in figure 1, with notation summarized in the appendix. The four point vertex is conveniently expressed as a sum of asymmetric four point vertices. This proves useful in the classification of divergent graphs. With the Feynman rules and some light cone integrals (appendix) we can compute the one and two loop counterterms of the theory.

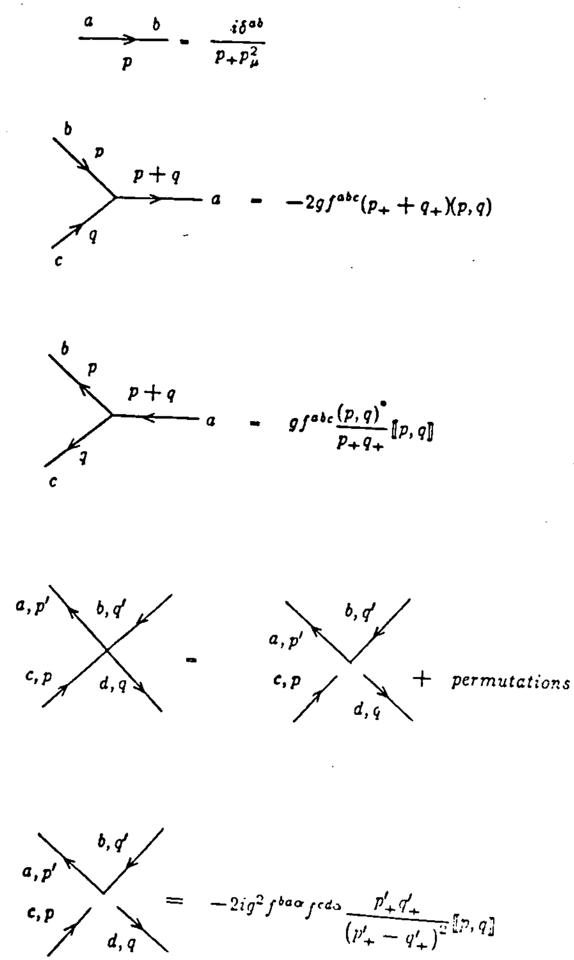


Figure 1: Feynman rules for N=2 Yang-Mills. Momentum flows in the arrow's direction and the four point vertex is represented as a sum of asymmetric four point vertices.

Since we are using the $p_+ \to p_+ + i\epsilon\, p_-$ or Mandelstam [2] prescription to treat the $1/p_+$ singularities, conventional power counting is valid and we can eliminate many graphs from consideration on this basis. However some refinement of the usual power counting procedure is needed for light cone integrals. In theories with manifest Lorentz covariance (euclidean covariance in the Wick rotated integrals) all 4 components of momentum in a given loop integration are on the same footing, e.g. the integrand transforms as some tensor representation of $O(4)$. For light cone integrals the transverse p_1, p_2 and longitudinal p_0, p_3 components appear on different footings and euclidean light cone integrals transform under $O(2) \times O(2)$ not $O(4)$. In the plus-minus direction the $O(2)$ transformation properties arise from the combination $p_+ \to ip_0 + p_3$ which changes by a phase under rotations in the p_0, p_3 plane. To check the convergence of any multiple integral Weinberg's [10] theorem states that every integration and subintegration must converge, and to ascertain the convergence/divergence of any integration we can use ordinary power counting. Because of the $O(4)$ symmetry in covariant integrals we can use $\int d^4p$ as the lowest subintegration in the nested hierarchy of integrations making up a multiloop integral. As is well known this gives rise to the usual prescription for determining which subtractions are required in a given graph. Due to the $O(2) \times O(2)$ symmetry, in light cone integrals we can no longer use $\int d^4p$ as the lowest subintegration, in it's place we use $\int d^2p_T$ and $\int d^2p_L$. So instead of one 4 dimensional momentum flow in each loop,

we must consider two 2 dimensional flows in the application of Weinberg's theorem.

The effect of this modification of power counting properties is mild. For a graph with $n_{3/4}$ three/four point vertices power counting in the transverse dimensions yields a degree of divergence $D_T = 2 - 2n_4 - n_3$. So the only graphs in the theory which diverge in the transverse dimensions are in figure 2. Of these only A and B are actually divergent. Had we used the power counting customary to covariant integrals we would have found $D = -2$ for graph A and concluded (wrongly) it's convergence.

Application of this revised form of power counting to N=4 Yang-Mills [2] does not change the conclusion that the theory is finite. This is so since all of the potentially divergent graphs in $\int d^2p_T$ are topologically the same as A,C,D,E in figure 2. But since the finiteness of the 2 point function follows from the finiteness of 3 point graphs via the Ward identity these divergences in the propagator must cancel.

Returning to the lagrangian with the matter fields; we could derive it using the procedure outlined above, however it is much simpler to make use of Brink et.al.'s [19] decomposition of the N=4 lagrangian into N=2 matter and Yang-Mills superfields. However, this lagrangian is not in the form we desire. It is written in terms of chiral and antichiral superfields in a superspace containing 2 θ's and 2 $\bar{\theta}$'s. We need the lagrangian in Mandelstam's

[2,28] form of superspace to facilitate the classification of infinite graphs. Since we will not enumerate the infinite graphs on the basis of their external vertex configurations we will not actually need the explicit form of the lagrangian. However we will outline it's derivation from Brink's [19]. From this N=2 decomposition of the N=4 lagrangian we easily get the N=2 lagrangian with the matter fields acting in an arbitrary representation of the gauge group by replacing (in the appropriate places) the structure constants by the generators of the matter representation of interest.

The N=2 decomposition of N=4 Yang-Mills as stated in [19] involves 2 N=2 Yang-Mills fields (bosonic) $\overline{\phi}^a$, ϕ^a and 2 N=2 matter superfields (fermionic) $\overline{\psi}^a$, ψ^a. $\overline{\phi}^a$ and $\overline{\psi}^a$ are antichiral, $\overline{d}_m\overline{\phi}= 0$ m=1,2 while ψ^a, ϕ^a are chiral, $d_m\phi = 0$ m=1,2. The superspace derivatives are;

$$d_m = -\frac{\partial}{\partial\,\overline{\theta}_m} - \frac{i}{\sqrt{2}}\theta_m\,\partial^+ \tag{2.28}$$

$$\overline{d}_m = \frac{\partial}{\partial\theta_m} + \frac{i}{\sqrt{2}}\,\overline{\theta}_m\partial^+ \tag{2.29}$$

and satisfy the representation independent anticommutators

$$\{d_m, \overline{d}_n\} = -i\,\sqrt{2}\partial^+\,\delta_{mn}$$

$$\{d_m, d_n\} = \{\overline{d}_m, \overline{d}_n\} = 0 \tag{2.29}$$

∂^+ and the rest of the notation for the lagrangian is in [19]. We reproduce here the lagrangian of ref. [19] (eq. 3.4):

$$\begin{aligned} S = \int d^4x d\theta_1 d\theta_2 d\,\overline{\theta}_1\,d\,\overline{\theta}_2 & \left[-\frac{1}{4}\,\overline{\phi}^a\,\partial_\mu^2\phi^a - \frac{i}{2\,\sqrt{2}}\,\overline{\psi}^a\,\frac{\partial_\mu^2}{\partial^+}\psi^a\right] \\ & + g f^{abc}\left[\partial^+\,\phi^a\,\overline{\phi}^b\,\overline{\partial}\phi^c + \frac{i}{\sqrt{2}}\,\partial^+\,\psi^a\frac{1}{\partial^+}\,\overline{\psi}^b\,\overline{\partial}\phi^c - \frac{i}{\sqrt{2}}\,\partial^+\,\phi^a\frac{1}{\partial^+}\,\overline{\psi}^b\,\overline{\partial}\psi^c + c.c.\right] \\ & + g f^{abc} f^{ade}\left[-\frac{1}{2}\,\frac{d_1d_2}{\partial^+}\left(\partial^+\,\phi^b\,\overline{\phi}^c\right)\frac{\overline{d}_1\overline{d}_2}{\partial^+}\left(\partial^+\overline{\phi}^d\,\phi^e\right)\right. \\ & - \frac{i}{2\,\sqrt{2}}\,\frac{d_1d_2}{\partial^+}\left(\overline{\psi}^b\,\psi^c\right)\frac{\overline{d}_1\overline{d}_2}{\partial^+}\left(\phi^d\,\partial^+\overline{\phi}^e\right) - \frac{1}{2\,\sqrt{2}}\,\frac{d_1d_2}{\partial^+}\left(\overline{\phi}^b\partial^+\,\phi^c\right)\frac{\overline{d}_1\overline{d}_2}{\partial^+}\left(\psi^d\,\overline{\psi}^e\right) \\ & - \frac{i}{2\,\sqrt{2}}\,d_1d_2\left(\overline{\phi}^b\,\psi^c\right)\frac{\overline{d}_1\overline{d}_2}{\partial^+}\left(\phi^d\,\overline{\psi}^e\right) - \frac{i}{\sqrt{2}}\,\overline{\psi}^b\partial^+\overline{\phi}^c\,\frac{1}{\partial^+}\left(\partial^+\,\phi^d\psi^e\right) \\ & \left. + \,\frac{1}{4}\,\frac{d_1d_2}{\partial^+}\left(\overline{\psi}^b\,\psi^c\right)\frac{\overline{d}_1\overline{d}_2}{\partial^+}\left(\psi^d\,\overline{\psi}^e\right) + \frac{1}{4}\,\overline{\psi}^b\,\psi^c\psi^d\,\overline{\psi}^e\right] \end{aligned} \tag{2.30}$$

. Followng the methodology of [28] to bring 2.30 into the required form we make use of the trick [27] of changing variables to $x^+ \rightarrow x^+ + \frac{i}{\sqrt{2}}\left(\overline{\theta}_1\,\theta_1 + \overline{\theta}_2\,\theta_2\right)$, all other variables remaining the same. The effect on the superspace derivatives is;

$$d_m \to -\frac{\partial}{\partial\,\overline{\theta}_m} \tag{2.31}$$

$$\overline{d}_m \to \frac{\partial}{\partial\theta_m} + i\,\sqrt{2}\overline{\theta}_m\partial^+ \tag{2.32}$$

with the anticommutators 2.28, 2.29 remaining the same. In these new variables the chirality condition on ϕ^a, ψ^a is just the statement that these fields are independent of $\overline{\theta}_1$, $\overline{\theta}_2$. By virture of the identity,

$$\overline{\phi} = \overline{d}_1\overline{d}_2\,\frac{1}{2\,\partial^{+2}}\,d_1d_2\overline{\phi} \tag{2.33}$$

which is valid for antichiral fields $\overline{\phi}$, we can express an antichiral field in terms of a chiral one ϕ^* as:

$$\overline{\phi} = \overline{d}_1\overline{d}_2\,\frac{1}{\partial^+}\phi^* \tag{2.34}$$

$$\phi^* = d_1d_2\,\frac{1}{2\,\partial^+}\,\overline{\phi} \tag{2.35}$$

Similarly for the fermionic field;

$$\overline{\psi} = \overline{d}_1\overline{d}_2\,\frac{1}{\partial^+}\,\psi^* \tag{2.36}$$

where ψ^* is a chiral superfield. Rewriting the action 2.30 in terms of chiral fields as defined by 2.34 and 2.36 and using the fact that

$$\int d\,\overline{\theta}_1\,d\,\overline{\theta}_2\,\mathcal{L} = d_1d_2\,\mathcal{L}\,|_{\overline{\theta}_1=\overline{\theta}_2=0} \tag{2.37}$$

we get:

$$S = \int d^4x d\theta_1 d\theta_2 - \frac{1}{4}\,d_1d_2\left(\frac{\overline{d}_1\overline{d}_2}{\partial^+}\,\phi^{*a}\partial_\mu^2\phi^a\right)_{\overline{\theta}_1=\overline{\theta}_2=0}$$
$$-\frac{i}{2\,\sqrt{2}}\,d_1d_2\left(\frac{\overline{d}_1\overline{d}_2}{\partial^+}\psi^{*a}\,\frac{\partial_\mu^2}{\partial^+}\psi^a\right)_{\overline{\theta}_1=\overline{\theta}_2=0}$$
$$+\,g f^{abc}\left[d_1d_2\left(\partial^+\,\phi^a\,\frac{\overline{d}_1\overline{d}_2}{\partial^+}\,\phi^{*b}\overline{\partial}\phi^c\right) + \frac{i}{\sqrt{2}}\,d_1d_2\left(\partial^+\,\psi^a\,\frac{\overline{d}_1\overline{d}_2}{\partial^+}\psi^{*b}\,\overline{\partial}\phi^c\right)\right.$$
$$\left. + \cdots \quad \right]_{\overline{\theta}_1=\overline{\theta}_2=0}$$
$$+\,g^2 f^{abc} f^{ade}\left[-\frac{1}{2}\,d_1d_2\left(\frac{d_1d_2}{\partial^+}\left(\partial^+\,\phi^b\,\frac{\overline{d}_1\overline{d}_2}{\partial^+}\,\phi^{*c}\right)\frac{\overline{d}_1\overline{d}_2}{\partial^+}\left(\overline{d}_1\overline{d}_2\,\phi^{*d}\phi^e\right)\right)\right.$$
$$\left. +\,\frac{1}{4}\,d_1d_2\left(\frac{\overline{d}_1\overline{d}_2}{\partial^+}\psi^{*b}\,\psi^c\psi^d\,\frac{\overline{d}_1\overline{d}_2}{\partial^+}\psi^{*e}\right) + \cdots\,\right]_{\overline{\theta}_1=\overline{\theta}_2=0} \tag{2.38}$$

where we have written down just some of the terms in the action. The ϕ, ϕ^* part of 2.38 gives us 2.22 after a suitable rescaling of the fields. Notice, 2.38 contains no explicit θ's , only derivatives in θ so that like 2.22 we can formulate the theory in θ momentum space. As an example of the reduction , consider the quartic term in ϕ of 2.38, it is:

$$-\frac{g^2}{2} f^{abc} f^{ade}\, d_1 d_2 \left[\frac{d_1 d_2}{\partial^+}\left(\partial^+\, \phi^b\, \frac{\bar{d}_1\bar{d}_2}{\partial^+}\, \phi^{*c}\right) \frac{\bar{d}_1\bar{d}_2}{\partial^+}\left(\bar{d}_1\bar{d}_2\, \phi^{*d}\phi^e\right)\right]_{\bar{\theta}_1=\bar{\theta}_2=0} \quad (2.39)$$

Since $d_m = -\frac{\partial}{\partial\theta_m}$ and only $\bar{d}_m$ depends on $\bar{\theta}$ 2.39 becomes:

$$-\frac{g^2}{2} f^{abc} f^{ade} \left[\frac{1}{\partial^+}\left(\partial^+\, \phi^b 2\, \partial^+\, \phi^{*c}\right) \frac{d_1 d_2}{\partial^+}\left(\bar{d}_1\bar{d}_2\, \phi^{*d}\, \bar{d}_1\bar{d}_2\, \phi^e\right)\right]_{\bar{\theta}_1=\bar{\theta}_2=0}$$

$$= -2g^2 f^{abc} f^{ade} \left[\frac{1}{\partial^+}\left(\partial^+\, \phi^b\, \partial^+\, \phi^{*c}\right) \frac{1}{\partial^+}\left(\partial^{+2}\, \phi^{*d}\, \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\, \phi^e\right.\right.$$

$$\left.\left. -\, \partial^+ \frac{\partial}{\partial\theta_1}\, \phi^{*d}\, \partial^+ \frac{\partial}{\partial\theta_2}\, \phi^e + \partial^+ \frac{\partial}{\partial\theta_2}\, \phi^{*d}\, \partial^+ \frac{\partial}{\partial\theta_1}\, \phi^e + \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\, \phi^{*d}\, \partial^{+2}\, \phi^e\right)\right] \quad (2.40)$$

which is the same as the quartic term in 2.22 apart from normalization.

Matter in an arbitrary representation with (anti-hermitian) generators $(T^a)_B^A$ satisfying $\left[T^a, T^b\right] = f^{abc}T^c$ is incorporated by simple replacements. Thus for a Yukawa term;

$$f^{abc}\psi^a\, \psi^{*b}\, \phi^c \rightarrow T_A^{cB}\psi^A\, \psi^*{}_B\, \phi^c$$

and for a quartic term in ψ;

$$f^{abc} f^{ade}\, \psi^{*b}\, \psi^c \psi^d\, \psi^{*e} \rightarrow T_C^{aB} T_D^{aE}\, \psi^*{}_B\, \psi^C \psi^D\, \psi^*{}_E$$

If, for example, the matter is in 2 irreducible multiplets with generators T_1 and T_2 the lagrangian follows the above prescription but with T in block form;

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

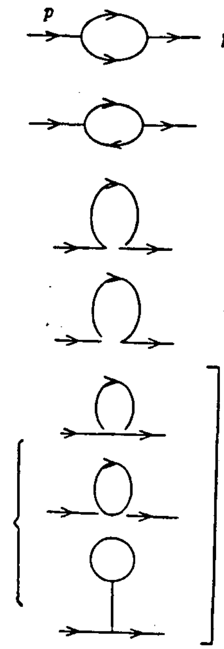
. The generalization to more than 2 multiplets is clear.

After reducing 2.38 to a form depending only on derivatives of θ we can formulate the Feynman rules for N=2 matter in an arbitrary representation interacting with N=2 Yang-Mills and get those of fig. 1 as a subset. However since we will not be classifying the divergent diagrams as in section 3 we do not need the lagrangian or the corresponding Feynman rules in full detail. But knowing how to go from the N=4 lagrangian written with N=2 superfields to the N=2 lagrangian with arbitrary matter allows us to determine without detailed calculation which N=2 theories are one loop finite. Since the N=4 theory is graph by graph finite (with the previous exceptions) [12] it's N=2 decomposistion is finite when we sum topologically similar graphs. That is when the internal lines contain N=2 adjoint rep. matter and N=2 Yang-Mills lines in all possible ways. For example in fig. 13 graph a is a contribution to the N=4 three point function and it is finite. If we restrict the external legs of graph a to be N=2 Yang-Mills fields it is still finite and has the decomposition given by graphs b and c. In addition to their propagator and vertex factors, graphs b and c contain color weight factors. Since b+c =finite when the matter is in the adjoint representation, if we consider graph c when the matter is in some arbitrary representation then if the color weight of c is the same as when the matter is in the adjoint representation, we get b+c=finite. So b+c= finite when

$$f^{a\alpha\beta}f^{b\alpha\beta} = Tr\left(T^aT^b\right) = \sum_i n_i Tr\left(T_i^aT_i^b\right) \tag{2.41}$$

when there are n_i matter fields in the representation i. This is the same result for one loop finiteness as found in [13].

For the other 3 point one loop graphs, the argument and result are word for word the same as above. Since all the three point couplings are 1 loop finite, Lorentz symmetry forces the other one loop couplings to be finite. At 2 loops the above argument breaks down in the sense that 2.41 alone is not sufficent to produce 2 loop fintieness although it is known to be sufficent [13]. However the 2 loop analysis does not exploit already finite graphs since this requires a classification of divergent graphs in the N=2 theory with matter. So that although this case might not be trivial like the one loop case it's computation would be simpler than that given in [13].



A: $p_+^3\int\frac{d^4q\,(p,q)^*(p,q)}{(p_+-q_+)^2q_+^2(p_\mu-q_\mu)^2q_\mu^2} = infinite$

B: $p_+\int\frac{d^4q\,(p,q)^*(p,q)}{q_+^2q_\mu^2(q_\mu+p_\mu)^2} = infinite$

C: $p_+^3\int\frac{d^4q}{(p_+-q_+)^2q_\mu^2} = finite$

D: $p_+^3\int\frac{d^4q}{(p_+-q_+)^2q_\mu^2} = finite$

E: $= 0$

from θ integration or structure constants

Figure 2: One loop graphs contributing to the two point function and their corresponding momentum integrals

III Classification of Divergent Diagrams

What greatly simplifies the calculation of the two loop counterterms is the small number of divergent graphs compared to the total number of graphs at a given order. Analysis of the loop integrals over the θ momenta is central. Circulating in each loop along with ordinary four-momentum p_μ are 2 θ momenta $\overline{p}_1$, $\overline{p}_2$ which are integrated over $\int d\,\overline{p}_2\, d\,\overline{p}_1$. For purposes of power counting $\overline{p}_1 \sim m^{1/2}$. From the Feynman rules (fig. 1) we see that the θ momenta appear in four point vertices and one type of three point vertex and then only in the combination (called brackets) $[\![p,q]\!] \equiv (p,q)_1\,(p,q)_2$ where $(p,q)_i \equiv p_+\,\overline{q}_i - q_+\overline{p}_i$. One fact (the cancellation theorem) we need about brackets is: if $I = [\![q_1,q_2]\!] \cdots [\![q_{2n-1},q_{2n}]\!]$ (n brackets in all) and the q_i's are linear combinations of $\leq$ n independent momenta, then $I = 0$ identically. This theorem immediately implies the vanishing of all vacuum diagrams since an l loop diagram must have $\geq$l brackets to 'saturate' the θ momenta integrals but the brackets contain only l independent momenta ('saturate' means for each $d^2\overline{p}$ there is at least one pair $\overline{p}_1\overline{p}_2$ in the integrand). So in this case I=0.

If $J_i \equiv (q_1,q_2)_i \cdots (q_{2n-1},q_{2n})_i = 0$, then it suffices to prove $J_1 = 0$ when the q's are linear combinations of $\leq$n momenta since $I = (-1)^{n+1}\,J_1 J_2$. Write the 2n q's in terms of the n momenta p_i, $q_j = \sum_k A_{jk}p_k$ where some of the p's may vanish if there are less than n independent momenta. Abbreviating

$(j_1,j_2) \equiv (p_{j_1},p_{j_2})_1$ J_1 becomes

$$J_1 = \sum_{j_1\cdots j_{2n}=1}^{n} A_{1j_1}\cdots A_{2nj_n}\,(j_1,j_2)\cdots(j_{2n-1},j_{2n}) \qquad (3.1)$$

Each term in the sum 3.1 vanishes since every term contains a closed chain and closed chains vanish. For example, (1,2)(1,3)(3,4)(5,1)(1,4) contains the closed chain (1,3)(3,4)(4,1) (from their definition (a,b)=-(b,a)). Proving closed chains vanish is just a calculation utilizing the identity;

$$\left(p^{(1)},p^{(2)}\right)\left(p^{(2)}p^{(3)}\right)\cdots\left(p^{(n-1)},p^{(n)}\right) =$$

$$p_+^{(2)}\cdots p_+^{(n-1)}\left(p_+^{(1)}\,\overline{p}_1^{(2)}\cdots\overline{p}_1^{(n)} - \overline{p}_1^{(1)}\,p_+^{(2)}\,\overline{p}_1^{(3)}\cdots\overline{p}_1^{(n)} + \cdots\right. \qquad (3.2)$$

$$\left. + (-1)^{n+1}\,\overline{p}_1^{(1)}\cdots\overline{p}_1^{(n-1)}\,p_+^{(n)}\right)$$

with $p^{(1)} = p^{(n)}$. This identity is also used for computing the θ momenta integrals occuring in loop graphs.

So it suffices to show that if $j_1,\cdots j_{2n}$ are chosen in any way from the set 1,$\cdots$n then $(j_1,j_2)\,(j_3,j_4)\cdots(j_{2n-1},j_{2n})$ contains a closed chain. This is obviously true for n=1 or 2. Assume it's true for $n \leq N-1$. There are two cases to consider. First, among the N brackets each number from the set $1\cdots N$ does not occur exactly twice. If one number does not occur at all, throw away any one bracket and the inductive hypothesis insures the existence of a closed chain. But if some number $j \geq 1$ of the brackets contain the only instance of j numbers, (in the previous example, j=2 and 2 and 5 occur once each) forget about the j aforementioned brackets and we are left with N-j brackets containing N-j different numbers which by the inductive hypothesis contains a closed chain.

In the second case, among the N brackets each of the numbers $1, \cdots N$ occurs exactly twice. So remove from the N brackets the bracket $(1, a_1)$ and continue by removing the bracket(s) containing 1 and/or a_1. Keep removing brackets until our collection of brackets contains each number in the collection exactly twice. If our collection contains $\leq N-1$ brackets the inductive hypothesis insures they form a closed chain while if our collection contains N brackets we can make the closed chain

$$(1, a_2)\,(a_2, a_3)\,(a_3, a_4) \cdots (a_{n-1}, a_n)\,(a_n, 1) \tag{3.3}$$

Now, onto the classification of graphs. The degree of divergence (=D) of a diagram is strongly dependent on the configuration of it's external legs. Thus in figure [3] diagram a is divergent while diagram b is convergent. (overall, both contain a divergent subdiagram). This difference is due to different θ integrals and external vertex factors, the factors from internal lines and vertices being identical in the two cases. Thus for diagram 3.a the external vertex factors and θ integrations make diagram a~

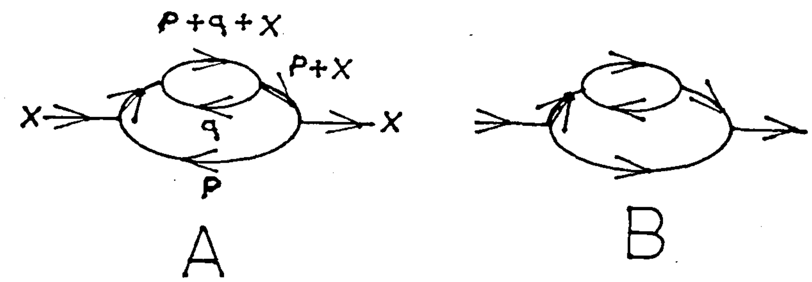


Figure 3: An example of how different external leg configurations lead to different degrees of divergence. Diagram a has D=0 so is divergent while diagram b has D=-2 so is convergent (overall) . Diagram b has it's momenta labeled the same way.

$$(p, X)\,(p, X)^{*}\,\frac{(p_+ + X_+)}{p_+\, X_+} \int d^2\, \overline{p} d^2\, \overline{q} [\![\, p + X, q \,]\!] [\![\, p, X \,]\!] \bullet (internal) \tag{3.4}$$

while diagram b~

$$(p + X, -p)\,(p + X, -p)^{*}\,\frac{X_+}{(p_+ + X_+)\, p_+} \int d^2\, \overline{p} d^2\, \overline{q}\, [\![\, p + X, q \,]\!] [\![\, p + X, -p \,]\!] \bullet (internal) \tag{3.5}$$

where *internal*=internal propagators and vertices. For the θ integrals make use of $[\![p,q]\!]=[\![p,q+\lambda p]\!]$ and symmetry of the bracket to get the θ integral for diagram a into the form:

$$\theta \int = \int d^2\,\bar{p} d^2\,\bar{q} [\![q, p+X]\!][\![p+X, X]\!]$$

and then make use of 3.2 to get:

$$\theta \int =$$

$$- \int d^2\,\bar{p} d^2\,\bar{q}\,(p_+ + X_+)^2 \left(q_+ \left(\bar{p}_1 + \overline{X}_1\right) \overline{X}_1 - \bar{q}_1\,(p_+ + X_+)\,\overline{X}_1\right.$$

$$+\,\bar{q}_1 \left(\bar{p}_1 + \overline{X}_1\right) X_+ \tag{3.6}$$

$$\bullet \left(q_+ \left(\bar{p}_2 + \overline{X}_2\right) \overline{X}_2 - \bar{q}_2\,(p_+ + X_+)\,\overline{X}_2 + \bar{q}_2 \left(\bar{p}_2 + \overline{X}_2\right) X_+\right)$$

$$= -\,(p_+ + X_+)^2\, X_+{}^2 \int d^2\,\bar{p} d^2\,\overline{q} \bar{q}_1 \bar{p}_1 \bar{q}_2 \bar{p}_2 = (p_+ + X_+)^2\, X_+{}^2\,.$$

.For diagram b the θ integral is done the same way with the same result. The internal propagators and vertices are identical for graphs a and b and contribute D=-4. So the respective degrees of divergence of diagrams a and b are D_a=4-4=0 and D_b=2-4=0. Thus the convergence or divergence of similar diagrams is dependent on their external vertex configurations.

Now for some general powercounting considerations. Each loop has a

$d^4 p d\,\bar{p}_1\; d\,\bar{p}_2$ integration; this is 4-1=3 powers of p. Each vertex has 3 powers of p and each propagator -3 powers. So the total degree of divergence D of a diagram is D=3·loops -3·internal lines +3·vertices -external powers =3·(loops -internal lines + vertices) -external powers = 3 -external powers. Examining the Feynman rules and considering all possible ways a line or pair of lines can be external we see that every external vertex (a vertex with at least one external line) carries at least one power of external momentum. So it suffices to consider graphs with three or fewer external vertices. since only they are infinite. Another easily proven topological relation is that the number of brackets in a graph is, l−1 + (number of outgoing external lines), where l= number of loops. Outgoing external line = line with an outgoing arrow.

First we will classify the divergent four point functions with two incoming and two outgoing lines (fig 4). Such l loop graphs have l+1 brackets. Those graphs with 3 external vertices are listed in fig.5. They always contain a four point vertex with 2 external lines. The only configurations for external lines from the 4 point vertex which contribute only 1 external power of momentun are in fig.6. So among the graphs of fig.5 the divergent candidates have θ momenta integrals that are one of the two following forms:

$$\int d^2\,\bar{q}_1 \cdots d^2\,\bar{q}_l [\![X, q_1]\!][\![q_2, q_3]\!] \cdots [\![q_{2l}, q_{2l+1}]\!] \tag{3.7}$$

$$\int d^2\,\overline{q}_1 \cdots d^2\,\overline{q}_l [\![\, X, Y \,]\!] [\![\, q_1, q_2 \,]\!] \cdots [\![\, q_{2l-1}, q_{2l} \,]\!] \qquad (3.8)$$

where the total number of brackets in each case is l+1. X and Y are external momenta while the q_i are linear combinations of internal and external momenta. The most pessimistic analysis ,which we always use, would say expression 3.7 $=\overline{X}_1\overline{X}_2 q_+{}^{2l+2} + O\left(X_+{}^2\overline{X}_1\overline{X}_2\right)$ corresponding to D=0 for the diagram. In fact expression 3.7 equals $\overline{X}_1\overline{X}_2 X_+{}^2 q_+{}^{2l} + O\left(X_+{}^4\overline{X}_1\overline{X}_2\right)$ which implies D=-2. This follows from the cancellation theorem. Say the only external vertex factor we got from 3.7was $\overline{X}_1\overline{X}_2$. This factor comes from the first bracket. The remaining terms are polynomial in $X_+{}^2$ (X_+ is a generic external momentum) and setting the external momentum equal to zero in the remaining l brackets gives the coefficent of the leading ($X_+{}^0$) term of this polynomial. But this coefficent consists of the product of l brackets depending on l independent momenta and by the cancellation theorem this product vanishes identically. So the form of 3.7 is as stated.

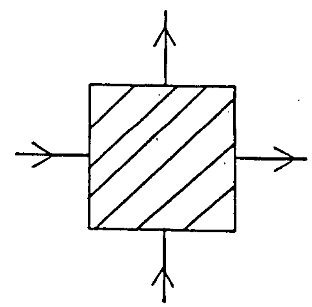


Figure 4: A class of four point functions which are analyzed for divergences.

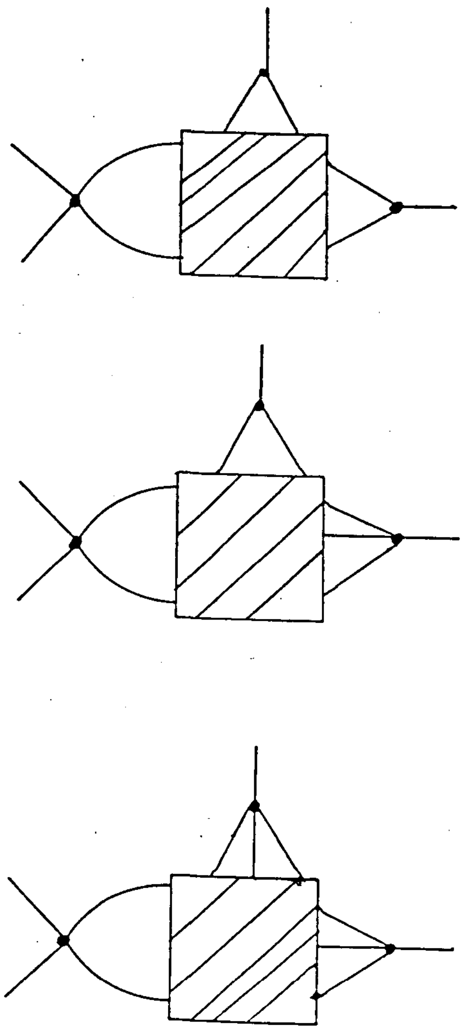


Figure 5: Four point function with three external vertices. Arrows on the external lines and the adjoining lines have been omitted as they can be labeled in many different ways.

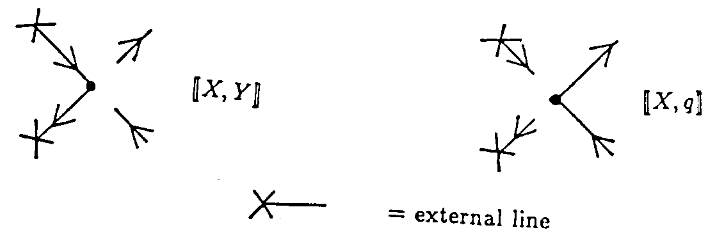


Figure 6: The only configurations of four point vertices with 2 external lines that contribute 1 external power. At right is the form of the corresponding bracket, X,Y=external momenta q=internal momenta.

Applying the same sort of argument as above to expression 3.8 shows it to have the form $[\![X, Y]\!] X_+^{\,2}$ +more optimistic terms. Therefore diagrams with θ momenta integrations as in 3.8 are also convergent, i.e. the graphs in fig. 5.

Having shown the diagrams with 3 external vertices to be finite we examine those with 2 external vertices. Such diagrams have 2 external four point vertices, each vertex with 2 external legs. For such diagrams the θ momentum integral either vanishes or has the form

$$\int d^2\,\bar{q}_1 \cdots d^2\,\bar{q}_l [\![Y, Z]\!][\![X, q_1]\!][\![q_2, q_3]\!] \cdots [\![q_{2l-2}, q_{2l-1}]\!] \qquad (3.9)$$

where there are l+1 brackets in all and X,Y,Z are external momenta. Using arguments similar to those before, 3.9 has the form $[\![Y, Z]\!] X_+^{\,2} q_+^{\,2l+2}$ providing 1 more external power than the most pessimistic estimate. So the only diagrams with two external vertices that diverge have external vertex configurations as in fig.6. Examining them we see that only diagrams of the type in fig.7 are divergent.

So far we have only discussed four point functions with two incoming and two outgoing lines. Other variations of incoming and outgoing lines are possible, however their divergence would correspond to counterterms in the lagrangian of a form not present from the begining (e.g. $\phi^2\phi^{*2}$). Such terms can be ruled out on the basis of renormalizability. Furthermore an analysis like the one above shows them to be finite.

Examining the other Green functions in the theory we can compute their degree of divergence from their external line configurations. As expected, the only other divergent graphs are 2 and 3 point functions and their external line configurations are listed in figs.8 and 9. Every divergent graph has only 2 external vertices, therefore the corresponding Feynman integrals depend on only one external momentum. There are a large number of types of divergent two point functions.

We have discussed the divergence of diagrams as a whole implicitly assuming a negative D implies convergence of the corresponding diagram. This is false but there is only one exception, the one loop propogator diagram in fig.2a previously discussed.

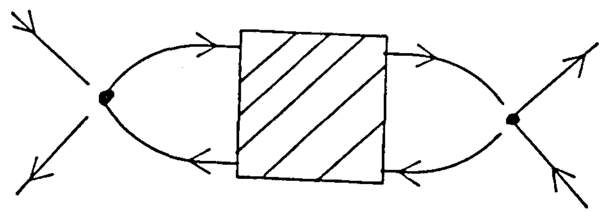


Figure 7: The only configuration of external lines producing a divergent four point function.

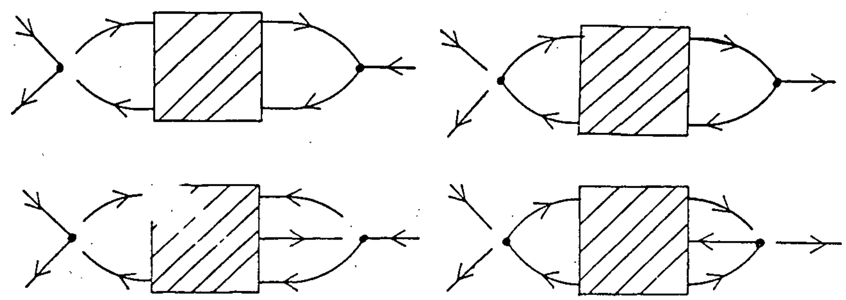


Figure 8: General forms for the divergent three point functions.

With the classification of the divergent graphs we can discuss their possible relevance to a general finiteness proof. From fig. 8 we see that the divergent part of the three point function $\phi^*\phi^*\ \phi$ is related in a simple way to the same complete 3 point function. This is illustrated in fig. 14 and has the following quantitative formulation. The $\phi^*\phi^*\ \phi$ term in the effective action can be shown to take the form;

$$\int d^6p d^6q\, \phi^{*a}(p)\, \phi^{*b}(q)\, \phi^c(-p-q)\, f^{abc} G(p_\mu, q_\mu) \qquad (3.10)$$

, G depends only on the 4-momenta p_μ and q_μ. Similarly, the 2 point function takes the form;

$$\int d^6p\, \phi^{*a}(p)\, \phi^a(-p)\, F(p_\mu) \qquad (3.11)$$

.Calling G_D the divergent part of the 3 point function, fig. 14 is the graphical statement of (neglecting numerical factors)

$$G_D(X,Y) = g^2 (X_+ + Y_+) Y_+ \int d^4p F^{-1}(p) F^{-1}(p+X) G(p,X) \quad -(X \leftrightarrow Y) \qquad (3.12)$$

.Writing the propagator and vertex as a power series in g, 3.12 relates the divergent part of the 3 point function at order n to the propagator and vertex at lower orders in g. The idea of a finiteness proof based on 3.12 is that if we can sufficently restrict the functional form of F and G through the Lorentz

Ward identities, the projection of G as defined by the RHS of 3.12 cannot have the form of a counterterm, that is $G_D \neq$constant•bare$\phi^*\phi^*\ \phi$ vertex. An (trivial) analogy with QED can be made. If $\Gamma_\mu^{(n)}$ is the electron-photon vertex through order α^n after we have made all subtractions through order α^{n-1}, then Tr$[\sigma^{\mu\nu}\Gamma_\nu^{(n)}]$ is finite since from Lorentz symmetry $\Gamma_\mu^{(n)}$ has the form

$$\Gamma_\mu^{(n)} = \gamma_\mu F^{(n)}(p,q) + \sigma_{\mu\nu} G^{\nu(n)}(p,q)$$

and only $\gamma_\mu F^{(n)}(p,q)$ has the form of a counterterm in QED. For N=2 Yang-Mills in the modified lightcone gauge the only linear Lorentz transformations are the boosts; $p_+ \to \lambda p_+$, $p_- \to \lambda^{-1}\ p_-$ and transverse rotations; $p \to e^{i\theta}p$, $p^* \to e^{-i\theta}\ p^*$. For G and F the differential equations summarizing their response under these 2 transformations are readily solved to give;

$$F(p_\mu) = p_+ f\left(p\, p^*, p_\mu^2\right)$$

$$G(p_\mu, q_\mu) = (p\, q_+ - q p_+)\left[(p\, q_+ - q p_+)(p^* q_+ - q^*\ p_+)\right]^{\frac{1}{2}} \bullet$$

$$g\left[(p\, q_- - q\, p_-)(p^* q_+ - q^*\ p_+), (p^* q_- - q^* p_-)(p\, q_+ - q p_+),\right.$$

$$\frac{(p\, q_+ - q p_+)(p^* q_+ - q^*\ p_+)}{(p_+ + q_+)^2}, p_\mu^2, q_\mu^2, p_\mu q_\mu$$

. Solving some of the nonlinear Lorentz Ward identities is a necessary but as yet undone part of this program.

One of the difficulties of this approach is distinguishing the one loop case (which is infinite) from the higher loop cases (which are presumably finite). This may not be a problem if we add N=2 matter which makes the theory one loop finite. Then if the divergent part of the three point function satisfies the generalization of eq. 3.12 and fig. 14 to include internal matter lines, the 2 loop finiteness of the theory is immediate. This is so because the color weight of a 2 loop graph of the form in fig. 14 are the same as the color weight of the same graph but with matter fields in the adjoint representation (remember, 1 loop finiteness ⇔ eq. 2.41). Two loop finiteness then follows from the arguments of section 2. The previous 2 loop finiteness proof makes use of plausible but unproven conjectures about the classification of divergent graphs in the N=2 theory with matter. For this reason alone, the classification of infinite graphs in the N=2 matter theory is interesting.

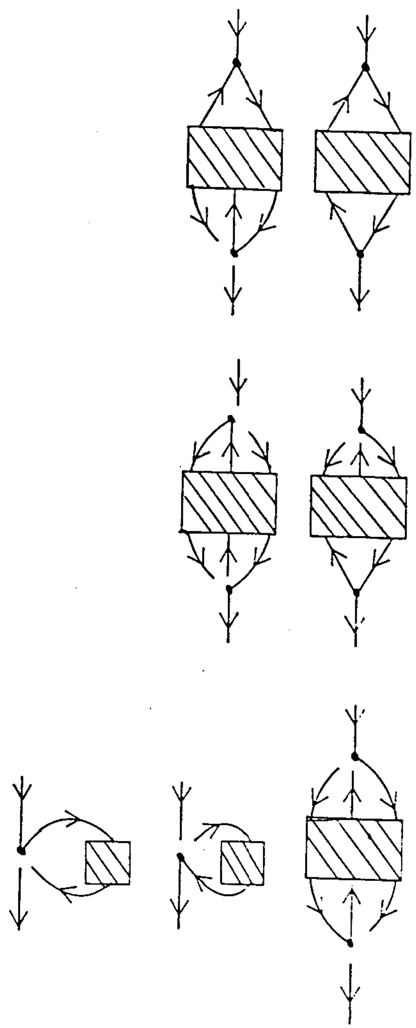


Figure 9: The external vertex configurations for the divergent propagator graphs.

IV One Loop Counterterms

First we compute the propagator counterterms. After doing the Grassmann momentum integrals as in sect. 3 we are left with ordinary four dimensional integrals. In figure 2 the one loop propagator graphs and their corresponding momentum integrals are given. The momentum integral for graph 2A is:

$$I_A = p\, p^*\, J_1 - pp_+ J_2 - p^*\, p_+ J_3 + p_+^2 J_4 \tag{4.1}$$

with

$$J_{1,2,3,4} = \int \frac{d^4q}{(p_\mu - q_\mu)^2\, q_\mu^2\, (p_+ - q_+)^2} \left(1, \frac{q^*}{q_+}, \frac{q}{q_+}, \frac{q\, q^*}{q_+{}^2}\right) \tag{4.2}$$

. By power counting only J_4 is divergent and only in the transverse dimensions. Calculation of J_4 is in the appendix and the infinite part of I_A is:

$$I_A = \frac{-4i\pi^2}{\epsilon} \frac{p_-}{p_+} \tag{4.3}.$$

For graph B the coresponding loop integral is:

$$p_+ pp^* I_1 - p_+^2 p I_2 - p_+^2 p^* I_3 + p_+^3 I_4 \tag{4.4}$$

with

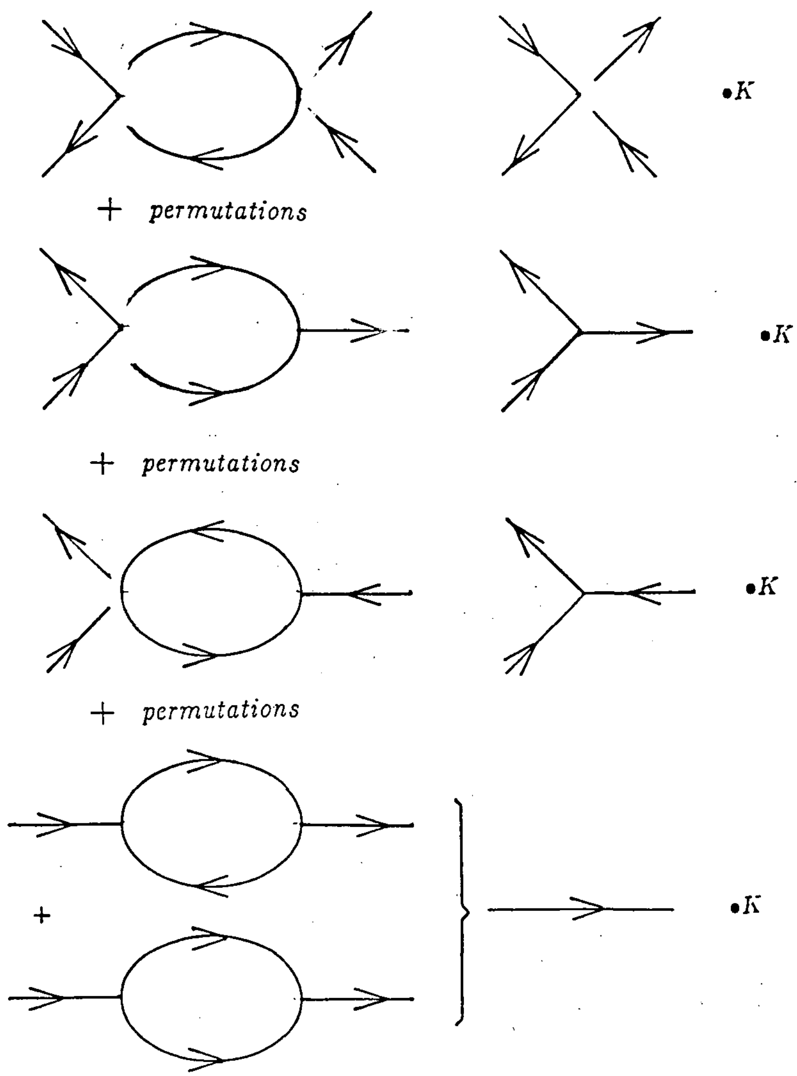
$$I_{1,2,3,4} = \int \frac{d^4q}{q_\mu^2\, (p_\mu + q_\mu)^2} \left(1, \frac{q^*}{q_+}, \frac{q}{q_+}, \frac{qq^*}{q_+{}^2}\right) \tag{4.5}.$$

Of these integrals only I_1 is actually infinite (see appendix) although they are all logarithmically divergent by power counting. So the infinite part of I_B is,

$$I_B = \frac{2i\pi^2}{\epsilon} p_+ p\, p^* \tag{4.6}$$

. When we take account of the symmetry factors of graphs A and B their combined contribution is proportional to $p_+ p_\mu^2$.

For the three and four point functions, only one graph contributes to the corresponding counterterms at one loop. The absence of infinite graphs at one loop with more than 2 propagators follows from the previous classification of infinite graphs. Infinite parts for these graphs are proportional to the invariant integral I_1 in (4.5). Figure 10 lists the infinite two, three and four point graphs and their associated counterterms. Notice that taken together the one loop counterterms are proportional to the Lagrangian.



$$K = \frac{-g^2}{\pi^2 \epsilon} C_2$$

Figure 10: The infinite one loop graphs and their corresponding counterterms.

V Two Loops

N=2 Yang-Mills realizes Lorentz symmetry nonlinearly and therefore the wavefunction, three point, and four point renormalization constants are equal to one another provided we can regulate the theory in a covariant fashion. RDR at the one loop level produces equal renormalization constants as previously shown.

Also the infinite part of the two loop counterterm is independent of the finite one loop subtraction constant. The one loop renormalized two and four point couplings are schematicaly $Z\phi\phi^*$ and $Z\phi^2\phi^{*2}$ where $Z=1+C_2/\pi^2 \cdot g^2/\epsilon$ $+g^2$B at one loop and we have included the effect of adding an arbitrary finite subtraction constant B. Our subtraction scheme must respect Lorentz symmetry so B is the same for all four renormalization constants. So the four point coupling gets an additional factor Z and the propagator recieves a 1/Z. As we know the only infinite four point functions have external line connections as in figure 7. From figure 12 we see the only effect of one loop counterterms on the four point function is to multiply the one loop infinite graph by $Z^2 \cdot 1/Z^2=1$. So the infinite part of the two loop, four point counterterm will be independent of the finite part of the one loop counterterm. Similar considerations demonstrate the two loop scheme independence of the infinite parts of the propagator and three point counterterms.

As noted previously, Lorentz symmetry forces the four renormalization constants to be equal. This is only true so long as we have a regulator which respects Lorentz symmetry. As demonstrated in the previous section at one loop, RDR produces a counterterm proportional to the Lagrangian. If we were certain our regulator would produce equal renormalization constants the best calculational method is to evaluate the divergent part of the four point function at the required number of loops. At more than one loop this involves the fewest number of graphs. So we will calculate $Z_{4-point}$ and assume the Lorentz Ward identities hold at two loops.

There are six graphs contributing at two loops to the infinite part of the four point function. They and their corresponding integrals are listed in fig.11 and below. Notice that each graph, especially those with a one loop propagator insertion, are offshell infrared finite in contrast to the calculation in the N=1 superfield formalism using the Fermi-Feynman gauge where the result is IR divergent [13]. Dimensional regularization for lightcone theories consists of continuing the integrals in the transverse dimensions (e.g. in the directions other than p_+ and p_-). This has been done in ordinary Yang-Mills [15,17] and gravity [18] and there the continuation is unambiguous since the theories themselves can be formulated in any number of dimensions. As a result we naturally get tensor integrals in the transverse dimensions; in the numerator of a Feynman integral we have terms like, $p_i\,(p-q)_j$ with i,j running over transverse indices. Since as a theory N=2 Yang-Mills is only defined in four dimensions the continuation of Feynman integrals in dimension does not come naturally. Mechanically, the difficulty at two loops for N=2 Yang-Mills is in continuing the integrals listed below to an arbitrary number of transverse dimensions. However this is possible since the infinite parts of these integrals involve in the numerators only the quantity $(p,q)^*\,(q,p) = -p_+^2 q_T^2 - q_+{}^2\, p_T^2 + 2p_+\, q_+\, p_T \cdot q_T$, with q_T=transverse components of q_μ. The above quantity can obviously be continued in it's transverse dimension.

$$I' = \int \frac{1}{q_+{}^2} \frac{(q,p)\,(p,q)^*}{p^2\,(p+q)^4\,(p+q-X)^2\,q^2}$$

$$I_1' = \int \frac{(p_+ + q_+)^2}{q_+{}^2\,p_+^2} \frac{(q,p)\,(p,q)^*}{p^2\,(p+q)^4\,(p+q-X)^2\,q^2}$$

$$I_2' = \int \frac{(p_+ + q_+)^2}{q_+{}^2} \frac{1}{p^2\,(p+q)^4\,(p+q-X)^2}$$

$$I_3' = \int \frac{(p_+ + q_+)}{q_+{}^2\,p_+} \frac{(p-X,q)^*\,(q,p)}{p^2\,(p-X)^2\,(p+q)^2\,(p+q-X)^2\,q^2}$$

$$I_5' = \int \frac{1}{p^2\,(p-X)^2\,q^2\,(q-X)^2}$$

$$I_6' = \int \frac{p_+\,q_+}{(p_+ - q_+)^2} \frac{1}{p^2\,(p-X)^2\,q^2\,(q-X)^2}$$

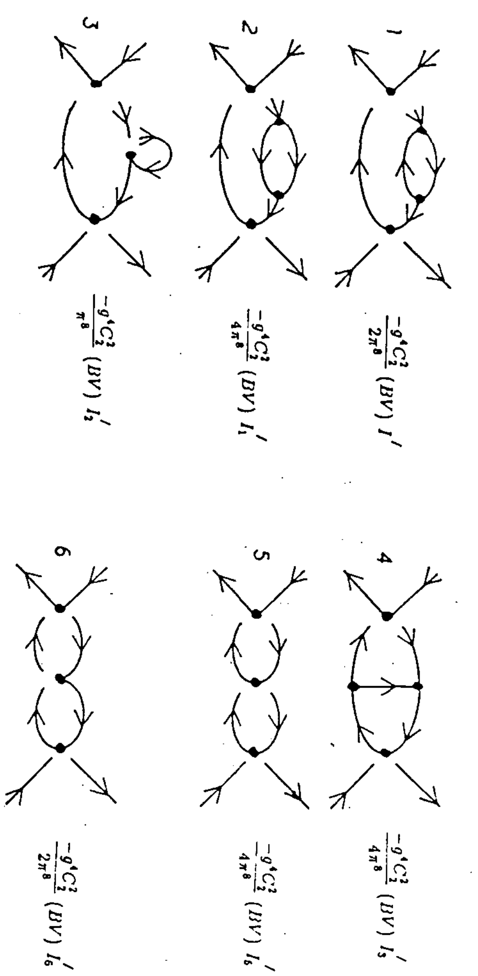


Figure 11: Divergent graphs and their corresponding expression. The I''s are listed in the text, BV=bare four point vertex (unsymmetrized).

Before calculating the two loop graphs remember that as illustrated in fig.12, the one loop counterterms inserted in one loop infinite graphs make no contribution to the two loop counterterm. Also, since the renormalized coupling g_R and the bare coupling are related by $g_R^2 = g^2/Z\,(g^2, \epsilon)$ there is no $1/\epsilon^2$ part in the sum of the six graphs, only (at most) a simple pole in ϵ.

Adding together the six graphs in fig.11 we get:

$$\frac{-g^4 C_2^2}{4\pi^8} \cdot (BV) \cdot [2I' + I_1' + 4I_2' + I_3' + I_5' + 2I_6'] \tag{5.1}$$

where BV is the bare four point vertex. Using the identity $(p,q)\,(q,p)^* = p_+\; q_+\; (p+q)^2 - p_+\,(p_+ + q_+)\,q^2 - q_+\;(p_+ + q_+)\,p^2$ which is also valid for the extension in transverse dimension we get:

$$\begin{aligned} 2I' + I_1' + 4I_2' + I_3' + I_5' + I_6' = \\ \int \frac{p_+}{q_+} \frac{1}{(p+q)^2\,(p+q-X)^2\,q^2} \left[\frac{1}{p^2} - \frac{1}{(p-X)^2}\right] \\ + \int \frac{(p_+ + q_+)^2}{p_+\; q_+} \frac{1}{(p+q)^2\,(p+q-X)^2\,q^2} \left[\frac{1}{p^2} - \frac{1}{(p-X)^2}\right] \\ -4 \int \frac{(p_+ + q_+)}{p_+} \frac{1}{p^2\,(p+q)^4\,(p+q-X)^2} \\ + \int \frac{p_+}{q_+} \frac{1}{p^2\,(p+q)^2\,(p+q-X)^2\,q^2} \\ + \int \frac{p_+^2}{{q_+}^2} \frac{1}{p^2\,(p-X)^2\,(p+q)^2\,(p+q-X)^2} \end{aligned} \tag{5.2}$$

In the above $\int \equiv \int d^{4-\epsilon}p\, d^{4-\epsilon}q$ and X= incoming momenta - outgoing momenta on one of the external vertices. By power counting the first two integrals in 5.2 are finite and after the shift of variable $q \to q - p$ the p integration in the third integral is $\int d^{4-\epsilon}p/p_+p^2 = 0$. The remaining two integrals we call L_1 and L_2. L_1 is calculated in the appendix and L_2 can be calculated using similar methods. The result for the RHS of 5.2 is $L_1 - L_2 =$

$$\frac{\pi^{4-\epsilon}}{X_\mu^{2\epsilon}}\left[\frac{2}{\epsilon^2}+\frac{4}{\epsilon}-\frac{2\gamma}{\epsilon}\right]-\frac{\pi^{4-\epsilon}}{X_\mu^{2\epsilon}}\left[\frac{2}{\epsilon^2}+\frac{4}{\epsilon}-\frac{2\gamma}{\epsilon}\right] = 0 + finite \quad (5.3)$$

. The integrals were calculated using the $1/p_+$ prescription of reference [2]. It is interesting that the graphs with double pole parts (first, second and fifth graph in fig 11) cancel against each other when only their double pole part is expected to cancel. This is most easily seen by adding together the graphs with only single pole parts (third,fourth and sixth graphs in fig.11). We need $4I_2' + I_3' + I_6'$. Using the formulas in the appendix we calculate

$$I_2' = -\frac{\pi^4}{2\epsilon} \qquad I_3' = \frac{2\pi^4}{\epsilon} \qquad I_6' = 0$$

which implies $4I_2' + I_3' + I_6' = 0$. Whether this generalizes to higher loops is unknown.

So in agreement with previous covariant calculations [13] utilizing RDR , the β function has zero contribution at two loops in the minimal subtraction scheme. In addition to the previously mentioned infrared finiteness the simplicity of the present calculation should encourage further work in multiloop

lightcone computations.

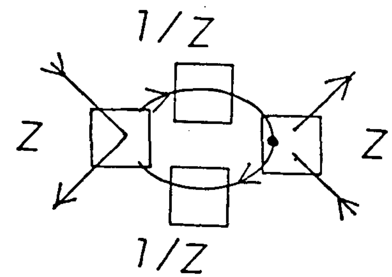


Figure 12: One loop counterterm contributions to the two loop β function. Only this configuration of external legs will produce an overall divergent four point function at any number of loops.

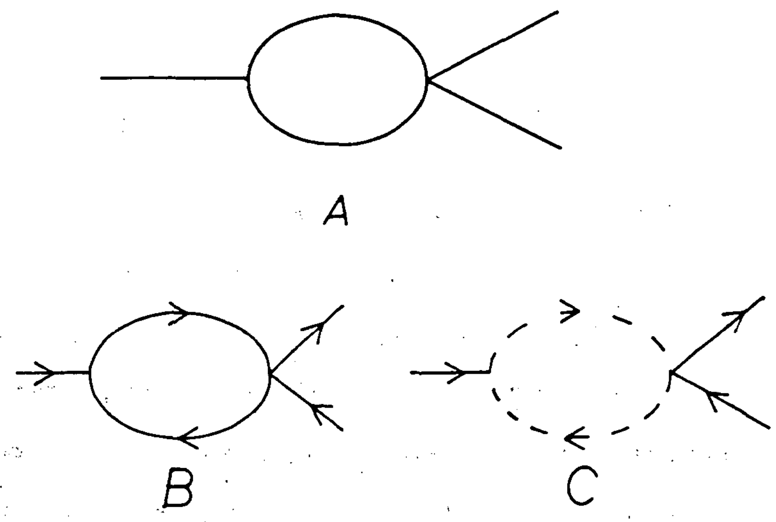


Figure 13: Exploiting the graph by graph finiteness of N=4 Yang-Mills to determine which 1 loop N=2 theories are finite. Solid lines = N=4 Yang-Mills fields, solid arrowed lines= N=2 Yang-Mills, dashed lines = N=2 matter. Graph A and B+C are finite.

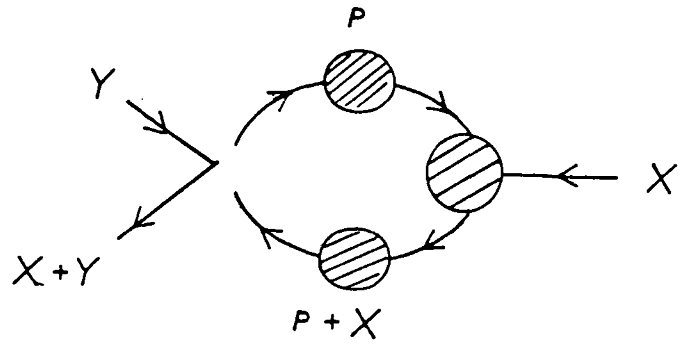


Figure 14: The divergent part of the 3 point function written in terms of the complete 2 and 3 point function.

VI Conclusion

The real consistency check for RDR in the lightcone formalism at two loops would be to calculate the propagator and both three point counterterms. We have not done this but it is a prerequisite for the three loop calculation. At three loops there does not appear to be any problem in continuing the integrals in transverse dimensions for the divergent part of the four point function. The vanishing of the infinite part of the two loop counterterm implies the infinite part of the three loop counterterm is independent of the finite one loop subtraction constant. Since the insertion of the one loop counterterm in the two loop graphs in all possible ways gives the contribution of the one loop counterterm to the three loop divergence, and this quantity is just $1/Z\cdot$(sum of two loop graphs) $= 1/Z \cdot$finite,the stated result follows.

In conclusion we have calculated the two loop β function in N=2 Yang-Mills by considering the divergent part of the four point function . The computation is greatly simplified by the small number of graphs that are infinite. There are no UV finite, IR divergent integrals encountered in the computation. It is hoped that the classification of divergent graphs will be an aid to a lightcone proof of ≥ 2 loop finiteness. It is also hoped that this two loop β function computation will encourage higher loop lightcone computations in other models where there is no problem continuing in transverse

dimensions, like Yang-Mills.

Probably the most important aspect of this work is not the actual results in N=2 Yang-Mills but instead the demonstration that multiloop calculations in lightcone field theories are feasible. In this regard, the appendix is the most important chapter since it establishes practical methods for calculating lightcone integrals.

VII Appendix

The notation used in the Feynman rules is

$$p_T = (p_1, p_2)$$

$$p = p_1 + ip_2 \qquad p_+ = p_0 + p_3$$

$$p_\mu^2 = -p_0^2 + p_1^2 + p_2^2 + p_3^2 = -p_+ \, p_- + p\, p^* = -p_+ \, p_- + p_T^2$$

$$p^* = p_1 - ip_2 \qquad p_- = p_0 - p_3$$

$$(p, q) = p\, q_+ - qp_+ \qquad (p, q)^* = p^* \, q_+ - q^* p_+$$

$$[\![p, q]\!] = (p_+\overline{q}_1 - q_+ \, \overline{p}_1)\, (p_+\overline{q}_2 - q_+ \, \overline{p}_2)$$

$\overline{p}_1, \overline{p}_2, \overline{q}_1, \overline{q}_2$ are the Grassmann momenta. Throughout the metric used is -+++.

Lightcone integrals are just momentum integrals with p_+'s in the denominator as opposed to covariant integrals which have only Lorentz invariants

in their denominators. The standard [22] method for calculating a covariant integral consists of combining the N factors in the denominator (propagators) by using N-1 Feynman parameters, then doing the resulting momentum integral (in $4 - \epsilon$ dimensions). Factors of momentum in the numerator do not change this in an essential way. The result is an integral over the N-1 Feynman parameters from which we must extract the $\epsilon \to 0$ divergences , overlapping and otherwise.

So lightcone integrals appear much more complicated to compute. Adopting the same strategy to compute a lightcone integral with N covariant and N' noncovariant (p_+) denominators we get, after doing the momentum integral, an integral over $(N + N' - 1)$ Feynman parameters. However the integrals over the N' parameters associated with the p_+ denominators can be explicitly done resulting in an $N - 1$ dimensional integral, the same as in the covariant case. Thus lightcone integrals are no more complicated than covariant integrals in the sense that in both cases the dimensionality of the parametric integral is equal to (number of propagators) -1 or less. Numerous [15,20,21] papers have been written containing technical details on the calculation of lightcone integrals. However none of them point out how similar are the evaluation of lightcone and covariant integrals and exploit the observation above.

There is one basic integral which establishes the claims made above. Denoting $D = aq_L^2 + bq_T^2 + 2q_Ll_L + 2q_Tl_T + m^2$, the integral of interest is:

$$I\left(l_\mu, a, b, m^2; \alpha, n\right) = \int \frac{d^{4-\epsilon}q}{[D]^\alpha} \frac{1}{{q_+}^n} \tag{7.1}$$

where n is an integer and α real. All other integrals of interest are got by differentiating I in one of it's parameters. If in an integral q_+ appears as

$$\frac{1}{(q_+ - a_+)^{n_1} (q_+ - b_+)^{n_2} \cdots}$$

we reduce it to a sum of partial fractions

$$\frac{A_1}{(q_+ - a_+)^{n'_1}} + \frac{A_2}{(q_+ - b_+)^{n'_2}} + \cdots$$

so it has the form of a sum of integrals of type I. With the abbreviations

$$H = abm^2 - al_T^2 - bl_T^2$$

$$K = abm^2 - bl_T^2$$

the formula for I is:

$$\begin{aligned} \int \frac{d^{4-\epsilon}q}{[D]^\alpha} \frac{1}{{q_+}^n} &= (-1)^n \, i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{b^{\alpha-3+\frac{\epsilon}{2}}}{a^{3-n-\alpha-\frac{\epsilon}{2}}} \frac{1}{l_+^n} \bullet \\ &\left(\frac{1}{[H]^{\alpha-2+\frac{\epsilon}{2}}} - \frac{1}{[K]^{\alpha-2+\frac{\epsilon}{2}}} \right) \\ &- (-1)^n \frac{i\pi^{2-\frac{\epsilon}{2}}}{\Gamma(\alpha)} \frac{b^{\alpha-3+\epsilon}}{a^{3-n-\alpha-\frac{\epsilon}{2}}} \sum_{j=1}^{n-1} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2} + j\right)}{\Gamma(j+1)} \bullet \\ &(-1)^j \frac{b^j \, {l_-}^j}{l_+^{n-j}} \frac{1}{K^{\alpha-2+\frac{\epsilon}{2}+j}} \end{aligned} \tag{7.2}$$

This formula is valid for integer n and real α.

From the form of I we see that any multiloop lightcone integral $\int d^4p_1 \cdots d^4p_l$ can be calculated by combining the N covariant denominators using N-1 Feynman parameters, using partial fractions on the p_{+1} denominators , calculating the p_1 integral using I or derivatives thereof then repeating this procedure for $p_{+2} \cdots p_{+l}$ successively. We never encounter momentum integrals we cannot do and the final result is an integral over N-1 Feynman parameters.

The derivation of 7.2 starts by considering another integral. With D as previously defined

$$\int \frac{d^{4-\epsilon}q}{[D]^\alpha} {q_-}^n = i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} a^{\alpha-n-3+\frac{\epsilon}{2}} b^{\alpha-3+\epsilon} \frac{{l_-}^n}{\left(abm^2 - al_T^2 - bl_L^2\right)^{\alpha-2+\frac{\epsilon}{2}}} \tag{7.3}$$

Equation 7.3 is easily derived by first Wick rotating, shifting the longitudinal variable $q_L \rightarrow q_L - l_L/a$,$q_L = (q_0, q_3)$ then doing the transverse, $d^{2-\epsilon}q_T$, and longitudinal, $d^2 q_L$, integrals. In all lightcone integrals the longitudinal integral is two dimensional while the transverse integral is done in $2-\epsilon$ dimensions.

Starting with the integral in 7.1 multiply numerator and denominator by $q_-{}^n$ and parameterize the denominator to get

$$I = \frac{\Gamma(n+\alpha)}{\Gamma(n)\,\Gamma(\alpha)} \int_0^1 dx x^{\alpha-1} (1-x)^{n-1} \bullet$$

$$\int \frac{d^{4-\epsilon} q q_-{}^n}{\left[(1-x+ax)\, q_L^2 + xbq_T^2 + 2xl_L q_L + 2xl_T q_T + xm^2\right]^{\alpha+n}} \tag{7.4}$$

Using 7.3 we do the q integration to get:

$$i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha+n-2+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)\,\Gamma(n)}\, l_-{}^n\, b^{\alpha+n-3+\epsilon} \bullet$$

$$\int_0^1 dx \frac{(1-x)^{n-1} \left[1+(a-1)\,x\right]^{\alpha-3+\frac{\epsilon}{2}}}{\left[(bm^2 - l_T^2) + (abm^2 - al_T^2 - bl_L^2 + l_T^2 - bm^2)\, x\right]^{\alpha+n-2+\frac{\epsilon}{2}}} \tag{7.5}$$

Now the x integral in 7.5 can be done by using:

$$\int_0^1 dx\, (1-x)^{n-1} \frac{(A+Bx)^{\beta-3}}{(C+Dx)^{\beta-2+n}} = \frac{(C+D)^{2-\beta}}{(A+B)^{3-n-\beta}}$$

$$\frac{1}{(BC-AD)^n} \int_1^{\frac{C}{A}\frac{A+B}{C+D}} ds\, (s-1)^{n-1}\, s^{2-n-\beta} \tag{7.6}$$

which is derived by making the substitution

$$s = \frac{x+\frac{C}{D}}{x+\frac{A}{B}}$$

followed by a trivial rescaling. Denoting the upper limit of the s integral in 7.6 by $1+y$ or

$$1+y = \frac{C}{A} \quad \frac{A+B}{C+D}$$

, sucessively integrate the RHS of 7.6 by parts to get:

$$\int_1^{1+y} ds\, (s-1)^{n-1}\, s^{\omega} =$$

$$(-1)^{n-1}\, (n-1)! \frac{\Gamma(\omega+1)}{\Gamma(\omega+n+1)} \left[(1+y)^{\omega+n} - 1\right] \tag{7.7}$$

$$+ \sum_{j=1}^{n-1} (-1)^{j+1} \frac{\Gamma(n)\,\Gamma(\omega+1)}{\Gamma(n-j+1)\,\Gamma(\omega+j+1)} y^{n-j}\, (1+y)^{\omega+j}$$

and there is no summation term for n=1.

Formula 7.2 is now derived putting these pieces together . Two useful special cases of 7.2 are $I\left(l_\mu, a, b, m^2; \alpha, n = 1, 2\right)$.

$$\int \frac{d^{4-\epsilon}q}{[D]^{\alpha}} \frac{1}{q_{+}} = - i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{a^{\alpha-2+\frac{\epsilon}{2}} b^{\alpha-3+\epsilon}}{l_{+}} \left(\frac{1}{[H]^{\alpha-2+\frac{\epsilon}{2}}} - \frac{1}{[K]^{\alpha-2+\frac{\epsilon}{2}}}\right) \tag{7.8}$$

$$\int \frac{d^{4-\epsilon}q}{[D]^{\alpha}} \frac{1}{{q_{+}}^{2}} = i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{a^{\alpha-1+\frac{\epsilon}{2}} b^{\alpha-3+\epsilon}}{l_{+}^{2}} \left(\frac{1}{[H]^{\alpha-2+\frac{\epsilon}{2}}} - \frac{1}{[K]^{\alpha-2+\frac{\epsilon}{2}}}\right) + i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 1 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} a^{\alpha-1+\frac{\epsilon}{2}} b^{\alpha-2+\epsilon} \frac{l_{-}}{l_{+}} \frac{1}{[K]^{\alpha-1+\frac{\epsilon}{2}}} \tag{7.9}$$

For any one loop compuation all we need is $I\left(l_{\mu}, a, b, m^{2}; \alpha, n\right)$ for $\omega = a = b$ or some derivative thereof. This is also true for the 2 loop computation considered in this work; the $\omega = a = b$ cases of 7.8, 7.9 along with standard covariant integrals (as in [23]) are needed.

For reference we give the useful one loop formulas. Defining

$$D' = \omega q_{\mu}^{2} + 2 l_{\mu} q_{\mu} + m^{2}$$

$$H' = \omega m^{2} - l_{\mu}^{2}$$

$$K' = \omega m^{2} - l_{T}^{2}$$

$$\int \frac{d^{4-\epsilon}q}{[D']^{\alpha}} \frac{q_{j}}{q_{+}} = i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{l_{j}}{l_{+}} \omega^{\alpha-4+\epsilon} \left(\frac{1}{[H']^{\alpha-2+\frac{\epsilon}{2}}} - \frac{1}{[K']^{\alpha-2+\frac{\epsilon}{2}}}\right) \tag{7.10}$$

j runs over transverse indices.

$$\int \frac{d^{4-\epsilon}q}{[D']^{\alpha}} \frac{q_{T}^{2}}{q_{+}} = - i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 3 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{\omega^{\alpha-5+\epsilon}}{l_{+}} \left(\frac{1}{[H']^{\alpha-3+\frac{\epsilon}{2}}} - \frac{1}{[K']^{\alpha-3+\frac{\epsilon}{2}}}\right) - i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \omega^{\alpha-5+\epsilon} \frac{l_{T}^{2}}{l_{+}} \left(\frac{1}{[H']^{\alpha-2+\frac{\epsilon}{2}}} - \frac{1}{[K']^{\alpha-2+\frac{\epsilon}{2}}}\right) \tag{7.11}$$

$$\int \frac{d^{4-\epsilon}q}{[D']^{\alpha}} \frac{q_{j}}{{q_{+}}^{2}} = - i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 2 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \omega^{\alpha-3+\epsilon} \frac{l_{j}}{l_{+}^{2}} \left(\frac{1}{[H']^{\alpha-2+\frac{\epsilon}{2}}} - \frac{1}{[K']^{\alpha-2+\frac{\epsilon}{2}}}\right) - i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha - 1 + \frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \omega^{\alpha-3+\epsilon} \frac{l_{-} l_{j}}{l_{+}} \frac{1}{[K']^{\alpha-1+\frac{\epsilon}{2}}} \tag{7.12}$$

$$\int \frac{d^{4-\epsilon}q}{[D']^{\alpha}} \frac{q_T^2}{q_+{}^2} = i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha-3+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{\omega^{\alpha-4+\epsilon}}{l_+^2} \bullet \left(\frac{1}{[H']^{\alpha-3+\frac{\epsilon}{2}}} - \frac{1}{[K']^{\alpha-3+\frac{\epsilon}{2}}}\right)$$

$$+ i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha-2+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{l_-}{l_+} \frac{\omega^{\alpha-4+\epsilon}}{[K']^{\alpha-2+\frac{\epsilon}{2}}}$$

$$+ i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha-2+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{l_T^2}{l_+^2} \omega^{\alpha-4+\epsilon} \left(\frac{1}{[H']^{\alpha-2+\frac{\epsilon}{2}}} - \frac{1}{[H']^{\alpha-3+\frac{\epsilon}{2}}}\right)$$

$$+ i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha-1+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \frac{l_- l_T^2}{l_+} \frac{\omega^{\alpha-4+\epsilon}}{[K']^{\alpha-1+\frac{\epsilon}{2}}} \tag{7.13}$$

The usual covariant integrals are (see[23]);

$$\int \frac{d^{4-\epsilon}q}{[D']^{\alpha}} = i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha-2+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} \omega^{\alpha-4+\epsilon} \frac{1}{[H']^{\alpha-2+\frac{\epsilon}{2}}} \tag{7.14}$$

$$\int \frac{d^{4-\epsilon}q}{[D']^{\alpha}} q_\mu = -\, i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha-2+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} l_\mu\, \omega^{\alpha-5+\epsilon} \frac{1}{[H']^{\alpha-2+\frac{\epsilon}{2}}} \tag{7.15}$$

$$\int \frac{d^{4-\epsilon}q}{[D']^{\alpha}} q_\mu q_\nu = \frac{i\pi^{2-\frac{\epsilon}{2}}}{2} \frac{\Gamma\left(\alpha-3+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} g_{\mu\nu}\omega^{\alpha-6+\epsilon} \frac{1}{[H']^{\alpha-3+\frac{\epsilon}{2}}}$$

$$+ i\pi^{2-\frac{\epsilon}{2}} \frac{\Gamma\left(\alpha-2+\frac{\epsilon}{2}\right)}{\Gamma(\alpha)} l_\mu l_\nu \omega^{\alpha-6+\epsilon} \frac{1}{[H']^{\alpha-2+\frac{\epsilon}{2}}} \tag{7.16}$$

As an example of a one loop calculation, consider the 5 superficially divergent integrals eq. 4.2 and 4.5 encountered in computing the one loop

counterterms. Of these, I_1 is an ordinary covariant integral while I_2 and I_3 are essentially the same. The integral related to I_2 and I_3 is:

$$I = \int \frac{d^{4-\epsilon}q}{q^2 (q+p)^2} \frac{q_j}{q_+} \tag{7.17}$$

parameterizing the covariant denominators we get:

$$I = \int_0^1 d\alpha \int \frac{d^{4-\epsilon}q}{[q^2 + 2\alpha pq + \alpha p^2]^2} \frac{q_j}{q_+} \tag{7.18}$$

then using the formula 7.10 we get:

$$\int_0^1 d\alpha\, i\pi^{2-\frac{\epsilon}{2}}\, \Gamma\left(\frac{\epsilon}{2}\right) \frac{p_j}{p_+} \left(\frac{1}{[\alpha p^2 - \alpha^2 p^2]^{\frac{\epsilon}{2}}} - \frac{1}{[\alpha p^2 - \alpha^2 p_T^2]^{\frac{\epsilon}{2}}}\right) \tag{7.19}$$

The parametric integral has a finite $\epsilon \to 0$ limit and is:

$$I = i\pi^2 p_j \frac{p_-}{p_T^2} ln \left(\frac{-p_+\, p_-}{p_\mu^2}\right) + O(\epsilon) \tag{7.20}$$

So both I_2 and I_3 are finite. Next consider the integral I_4.

$$I_4 = \int \frac{d^{4-\epsilon}q}{q^2 (q+p)^2} \frac{q_T^2}{q_+{}^2} \tag{7.21}$$

Parameterizing the covariant denominators as before and using 7.13 to do the resulting q integral we get:

$$
\begin{aligned}
&i\pi^{2-\frac{\epsilon}{2}}\,\Gamma\left(-1+\frac{\epsilon}{2}\right)\frac{1}{p_+^2}\int_0^1\frac{d\alpha}{\alpha^2}\left(\left[\alpha p^2-\alpha^2p^2\right]^{1-\frac{\epsilon}{2}}-\left[\alpha p^2-\alpha^2p_T^2\right]^{1-\frac{\epsilon}{2}}\right)\\
&+i\pi^{2-\frac{\epsilon}{2}}\,\Gamma\left(\frac{\epsilon}{2}\right)\frac{p_-}{p_+}\int_0^1 d\alpha\left[\alpha p^2-\alpha^2p_T^2\right]^{-\frac{\epsilon}{2}}\\
&+i\pi^{2-\frac{\epsilon}{2}}\,\Gamma\left(\frac{\epsilon}{2}\right)\frac{p_T^2}{p_+^2}\int_0^1 d\alpha\left(\left[\alpha p^2-\alpha^2p^2\right]^{-\frac{\epsilon}{2}}-\left[\alpha p^2-\alpha^2p_T^2\right]^{-\frac{\epsilon}{2}}\right)\\
&+i\pi^{2-\frac{\epsilon}{2}}\,\Gamma\left(1+\frac{\epsilon}{2}\right)\frac{p_-\,p_T^2}{p_+^2}\int_0^1\alpha^2\left[\alpha p^2-\alpha^2p_T^2\right]^{-1-\frac{\epsilon}{2}}
\end{aligned}
\qquad (7.22)
$$

This result is finite as $\epsilon\to 0$ and equal to:

$$
i\pi^2\frac{p_-}{p_+}\left[ln\left(\frac{-p_+\,p_-}{p^2}\right)-\frac{p^2}{p_+\,p_-}\left(\frac{\pi^2}{6}+F\left(\frac{-p_T^2}{p^2}\right)\right)\right] \qquad (7.23)
$$

where

$$
F(x)=\int_0^x\frac{ds}{s}ln\,(1+s) \qquad (7.24)
$$

is the Spence function.

The remaining infinite integral is J_4.

$$
J_4=\int\frac{d^{4-\epsilon}q}{q^2\,(q-p)^2}\frac{q_T^2}{q_+{}^2\,(q_+-p_+)^2} \qquad (7.25)
$$

Repeatedly using

$$
\frac{1}{q_+\,(q_+-p_+)}=\frac{1}{p_+}\left(\frac{1}{q_+-p_+}-\frac{1}{q_+}\right) \qquad (7.26)
$$

and shifting integration variable when necessary we get:

$$
J_4=\int\frac{d^{4-\epsilon}q}{q^2\,(q-p)^2}\left(p_T^2-2p_Tq_T\right)\left(\frac{1}{p_+^2\,q_+{}^2}+\frac{2}{p_+^3\,q_+}\right) \qquad (7.27)
$$

then using the formulas 7.8,7.9,7.10 and 7.12 we find J_4 is infinite with infinite part equal to

$$
\frac{-4i\pi^2}{\epsilon}\frac{p_-}{p_+^3} \qquad (7.28)
$$

As an example of a two loop integral we compute L_1 as given in 5.2.

$$
L_1=\int\frac{d^{4-\epsilon}p\,d^{4-\epsilon}q}{p^2q^2\,(p+q)^2\,(p+q-X)^2}\frac{p_+}{q_+} \qquad (7.29)
$$

changing variables we have $L_1=-R_1-R_2$ where

$$
R_1=\int\frac{d^{4-\epsilon}q}{q_+\,q^2}\int\frac{d^{4-\epsilon}p\;p_+}{p^2\,(p-X)^2\,(p+q)^2} \qquad (7.30)
$$

$$
R_2=\int\frac{d^{4-\epsilon}p\,d^{4-\epsilon}q}{p^2\,(p-X)^2\,(p+q)^2\,q^2} \qquad (7.31)
$$

Evaluation of R_2 is routine and we get:

$$R_2 = -\frac{\pi^{4-\epsilon}}{\left(X_\mu^2\right)^\epsilon}\left[\frac{2}{\epsilon^2} + \frac{5}{\epsilon} - \frac{2\gamma}{\epsilon}\right] + \quad finite \tag{7.32}$$

For R_1, Feynman parameterizing the denominator in the p integral and then doing the p integral gives:

$$R_1 = i\pi^{2-\frac{\epsilon}{2}}\,\Gamma\left(1+\frac{\epsilon}{2}\right)\int d(\alpha\beta)\int \frac{d^{4-\epsilon}q\,(\beta\, q_+ - \alpha\, X_+)}{q_+\, q^2\left[\beta(1-\beta)\,q^2 + 2\alpha\beta X q + \alpha(1-\alpha)\,X^2\right]^{1+\frac{\epsilon}{2}}} \tag{7.33}$$

Parameterizing the q^2 denominator and doing the q integral gives:

$$R_1 = \frac{\pi^{4-\epsilon}}{\left(X_\mu^2\right)^\epsilon}\Gamma(\epsilon)\,(PI1 - PI2)$$

$$PI1 = \int \frac{d(\alpha\beta)\,\beta^{\frac{3}{2}\epsilon}}{(1-\beta)^{1-\frac{3}{2}\epsilon}}\int_0^1 \frac{dy\, y^{\frac{\epsilon}{2}}}{\left[\alpha\beta(1-\alpha)(1-\beta)\,y - \alpha^2\beta^2y^2\right]^\epsilon} \tag{7.34}$$

$$PI2 = \int \frac{d(\alpha\beta)\,(1-\beta)^{\frac{3}{2}\epsilon}}{\beta^{1-\frac{3}{2}\epsilon}}\int_0^1 \frac{dy}{y^{1-\frac{\epsilon}{2}}}\left[\frac{1}{\left[\alpha\beta(1-\alpha)(1-\beta)\,y - \alpha^2\beta^2\frac{X_T^2}{X^2}y^2\right]^\epsilon} - \frac{1}{\left[\alpha\beta(1-\alpha)(1-\beta)\,y - \alpha^2\beta^2y^2\right]^\epsilon}\right] \tag{7.35}$$

In the foregoing $\int d(\alpha\beta)$ denotes the integral over the region $\alpha, \quad \beta \geq 0 \quad \alpha+\beta \leq 1$. Setting $\epsilon = 0$ we get $PI1 = 1$ while letting $\epsilon \to 0$ in

$PI2$ gives after doing the y integral

$$PI2 = -\epsilon\int \frac{d(\alpha\beta)}{\beta}\left(F\left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)}\frac{X_T^2}{X^2}\right) - F\left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)}\right)\right)$$

where $F(a)$ is defined in 7.24. Changing variables $\beta \to (1-\alpha)\,t$ and $\alpha \to \alpha$ we get

$$PI2 = -\epsilon\int_0^1 d\alpha\int_0^1 \frac{dt}{t}\left[F\left(\alpha t\frac{X_T^2}{X^2}\right) - F(\alpha t)\right] = \epsilon \quad finite \tag{7.36}$$

Putting things together gives

$$R_1 = \frac{\pi^{4-\epsilon}}{(X^2)^\epsilon}\frac{1}{\epsilon} + \quad finite \tag{7.37}$$

so that

$$L_1 = \frac{\pi^{4-\epsilon}}{(X^2)^\epsilon}\left[\frac{2}{\epsilon^2} + \frac{4}{\epsilon} - \frac{2\gamma}{\epsilon}\right] + \quad finite \tag{7.38}$$

The other two loop integrals are done in an entirely similar manner. What greatly simplifies extracting pole parts of the parametric integrals is the fact that as $\epsilon \to 0$ the only divergences occur at the boundary of the parametric region of integration.

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