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Computation of the One-Dimensional Free-Space Periodic Green's Function for Leaky Waves using the Ewald Method

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Abstract – **This paper examines an extension of the Ewald method for evaluating the periodic free-space Green's function due to an array of point sources, when the wavenumber of the phased sources is allowed to be complex. This makes the Ewald method useful for treating leaky modes on periodic structures.**

1 INTRODUCTION

The Ewald method that is used in evaluating the free-space periodic Green's function for an array of point sources is extended here to leaky modes by allowing for complex wavenumbers. It is shown that care must be taken when choosing the path of integration in the complex plane that is used to define the exponential integral function that appears in the Ewald method. An analytic continuation of the exponential integral function that appears when the wavenumber is real is used, giving rise to a "generalized exponential integral function." By doing so, one can obtain a simple rule for how to modify the exponential integral calculation to obtain solutions that correspond to physical leaky-wave solutions.

This extension of the Ewald method to complex wavenumbers allows for the treatment of periodic leaky-wave antennas as well as metamaterial structures such as one-dimensional chains of particles, including plasmonic nanoparticles.

2 PERIODIC GREEN'S FUNCTION

We consider here an infinite one-dimensional array of point sources located along the z axis at $z = nd$, where d is the period of the array. The usual space-domain form of the free-space (or homogeneous medium, to be more general) Green's function is

$$G(\mathbf{r},\mathbf{r}') = \sum_{n=-\infty}^{\infty} e^{-j k_{z0} n d} \, \frac{e^{-jkR_n}}{4\pi R_n}, \qquad (1)$$

where k_{z0} is the phasing wavenumber and R_n is the distance from the nth source point to the observation point at (ρ, z) in cylindrical coordinates, given by

$$R_n = \sqrt{\rho^2 + (z - nd)^2} \, . \qquad (2)$$

(A time-harmonic convenient of $\exp(j\omega t)$ is assumed and suppressed.) This summation is slowly converging for real wavenumbers k_{z0}, and it fails to converge for complex wavenumbers, as would be encountered for a leaky mode on a periodic structure.

An alternative representation is the spectral form of the periodic Green's function, given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{d} \sum_{q=-\infty}^{\infty} e^{-j k_{zq} z} \, \frac{H_0^{(2)}\left(k_{\rho q}\rho\right)}{4j}, \qquad (3)$$

where

$$k_{\rho q} = \sqrt{k^2 - k_{zq}^2}$$

and

$$k_{zq} = \beta_q = k_{z0} + 2\pi q/d \, .$$

This form converges for complex wavenumbers, but converges more slowly as the radial distance ρ from the z axis decreases. Unfortunately, small values of radial distance are encountered in a numerical moment-method solution of periodic structures, e.g., when treating the self-terms of the impedance matrix. Hence, it is highly desirable to have an efficient method for calculating the free-space periodic Green's function for complex wavenumbers.

The Ewald method is a very efficient method for calculating the periodic free-space Green's function, which casts the result as the sum of a modified spatial series and a modified spectral series, called here the "Ewald spatial" and the "Ewald spectral" series. Each of these two series has Gaussian convergence, and hence is very rapidly converging for all observation points. The Ewald method for the one-dimensional array of point sources in a homogenous medium has been discussed in [1] for the case of a real wavenumber k_{z0}. The result is given as

$$G(\mathbf{r},\mathbf{r}') = G^{E}_{spectral}(\mathbf{r},\mathbf{r}') + G^{E}_{spatial}(\mathbf{r},\mathbf{r}') \qquad (4)$$

where

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$$G^{E}_{spatial}(\mathbf{r},\mathbf{r}') = \frac{1}{8\pi}\sum_{n=-\infty}^{\infty}\frac{e^{-jk_{z0}nd}}{R_n} \cdot\left[e^{jkR_n}\,\mathrm{erfc}\left(R_nE + j\frac{k}{2E}\right) + e^{-jkR_n}\,\mathrm{erfc}\left(R_nE - j\frac{k}{2E}\right)\right] \tag{5}$$

is the Ewald spatial series and

$$G^{E}_{spectral}(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi d}\sum_{q=-\infty}^{\infty} e^{-jk_{zq}z} \sum_{p=0}^{\infty}(-1)^p\frac{(\rho E)^{2p}}{p!}E_{p+1}\left(-\frac{k_{\rho q}^2}{4E^2}\right) \tag{6}$$

is the Ewald spectral series. In these expressions erfc(z) is the complementary error function and $E_p(z)$ is the exponential integral function of order p, which is related to the fundamental exponential integral function of order 1 through the recurrence relation

$$E_{p+1}(z) = \frac{1}{p}\left(e^{-z} - z\,E_p(z)\right),\ p = 1,2,3,\ldots. \tag{7}$$

The fundamental exponential integral function is defined as

$$E_1(z) = \int_z^{\infty}\frac{e^{-t}}{t}dt\,,\quad -\pi \le \arg z \le \pi\ , \tag{8}$$

where the path C in the complex t plane in Eq. (8) starts at the point z and ends at infinity on the positive real axis, staying above or below the simple pole at t = 0, depending on whether z is in the upper half plane or the lower half plane.

3 EXTENSION OF THE EWALD METHOD TO COMPLEX WAVENUMBERS

Equation (5) is already applicable to complex wavenumbers and requires no modification. Equation (6) requires modification, however, when k_{z0} is complex. We consider any particular term q in the summation, corresponding to a particular space harmonic (Floquet wave). By carefully examining the location of the point

$$z = -k_{\rho q}^2 / 4E^2 \tag{9}$$

in Eq. (8), we can observe how the path C in the t plane changes as we move from the situation where k_{z0} is real to the situation where $k_{z0} = \beta - j\alpha$ is complex. In some cases the path changes so that it detours around the pole at t = 0; e.g., by starting at a value of z that is in the third quadrant of the complex t plane, and then detouring above the pole, and then heading to infinity on the positive real axis. For such a path, the function $E_1(z)$ gets modified by adding a residue contribution of $-2\pi j$ due to the pole at the origin that is detoured around. We call the resulting function a "generalized" exponential integral and denote it as $E_1^G(z)$.

The final result is as follows:

$$E_1^G(z) = E_1(z) + 2\pi jm\ , \tag{10}$$

where m is an integer that is chosen according to whether or not a residue has been captured. The value of m is given by

$$\begin{aligned} |\beta_q| > k, \quad & m = 0 \\ -k < \beta_q < 0, \quad & m = 0 \\ 0 < \beta_q < k, \quad & m = -1. \end{aligned} \tag{11}$$

Physically, the above rule means that the generalized exponential integral is required in Eq. (6) (m is different from zero) whenever the phase constant of a particular space harmonic lies within the region $0 < \beta_q < k$. This means that this particular space harmonic is in the radiating fast-wave region, and is a forward propagating wave (having a positive phase constant). Such a wave is physically chosen as improper, meaning that the radial wavenumber $k_{\rho q}$ has a positive imaginary part.

The above rule applies for the calculation of the field from the space harmonics of a guided mode that is assumed to be "physical," where each space harmonic is chosen as proper or improper according to the value of β_q [2]. For nonphysical modes, other values of m may be needed.

Results (omitted here) verify that the Ewald method as extended above gives the correct results (it agrees with the pure spectral method when $k_{\rho q}$ is chosen according to the physical proper/improper rule in the pure spectral method) and is faster than the pure spectral method when the radial distance ρ is less than about a quarter of a wavelength.

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