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Pseudo-rotations and symplectic topology

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PSEUDO-ROTATIONS AND SYMPLECTIC TOPOLOGY

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Abstract

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by

Erman Çineli

In the context of symplectic dynamics, pseudo-rotations are Hamiltonian diffeomorphisms with finite and minimal possible number of periodic points. These maps are of interest in both dynamics and symplectic topology. In this thesis, principally in relation with the Conley conjecture, we study pseudo-rotations from two different perspectives. In the first part, we prove a variant of the Chance–McDuff conjecture. We show that a closed monotone symplectic manifold, which admits a non-degenerate pseudo-rotation, must have a deformed quantum Steenrod square of the top degree element and hence non-trivial holomorphic spheres. In the second part, we give a simple proof of a slightly weaker version of a recent theorem by Shelukhin which extends Franks' "two-or-infinitely-many" theorem to Hamiltonian diffeomorphisms in higher dimensions. More precisely, we show that for a certain class of closed monotone symplectic manifolds (e.g. $\mathbb{CP}^n$) pseudo-rotations are the only strongly non-degenerate counterexamples to the Conley conjecture. In addition, we show that every non-degenerate pseudo-rotation of $\mathbb{CP}^2$ is balanced by using equivariant pair-of-pants product and quantum Steenrod squares.

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Chapter 1

Introduction

In this thesis, principally in connection with the Conley conjecture, we study pseudo-rotations from two different perspectives. The Conley conjecture asserts that for a broad class of closed symplectic manifolds every Hamiltonian diffeomorphism, in other words, every time-one map in a possibly time-dependent Hamiltonian system, has infinitely many simple (un-iterated) periodic points. It is easy to see that the conjecture does not hold unconditionally: an irrational rotation of $S^2 \subset \mathbb{R}^3$ about the z-axis has only two periodic points – the North and the South Poles. More generally, the conjecture fails for all manifolds admitting a Hamiltonian circle (or torus) action with isolated fixed points – a generic element of the circle (or the torus) gives rise to a Hamiltonian diffeomorphism with finitely many periodic points.

The state of the art result proved in [GG17] and then by a different method in [Çi] is that when a closed symplectic manifold (M, ω) admits a Hamiltonian diffeomorphism with finitely many periodic points, there exists a class $A \in \pi_2(M)$ such that the integrals $\langle [\omega], A\rangle > 0$ and $\langle c_1(TM), A\rangle > 0$. For example, the Conley conjecture holds when $\omega|_{\pi_2(M)} = 0$ or when M is negative monotone. For many manifolds the conjecture is also known to hold C^∞-generically (see [GG09b, Su21a]); we refer the reader to [GG15] for a detailed survey and further references.

The failure of the Conley conjecture for a manifold is expected to have strong symplectic topological consequences (in addition to the topological consequences mentioned above). For instance, in [McD], it is shown that a symplectic manifold admitting a Hamiltonian circle action is uniruled (has a non-zero Gromov–Witten invariant with one of the homology classes being the point class). The outstanding problem in this direction, inspired by the results in [McD] and referred to as the Chance–McDuff conjecture, is that whenever the Conley conjecture fails some Gromov-Witten invariants of the manifold are non-zero.

In every known counterexample to the Conley conjecture, all periodic points are strongly non-degenerate fixed points and the number of them is equal to the sum of the Betti numbers of the manifold. This is the minimal number allowed by the Arnold conjecture. Such maps are examples of non-degenerate pseudo-rotations. In the literature, there are a few slightly different definitions of pseudo-rotations, all reflecting the same condition that the map must have the least possible number of periodic orbits; [ÇGG19, Sh19a, Sh20]. In this thesis we define a non-degenerate pseudo-rotation as a Hamiltonian diffeomorphism φ such that all iterates φ^k, $k \in \mathbb{N}$, are non-degenerate and the Floer differential (over $\mathbb{F}_2 = \mathbb{Z}_2$) vanishes for all φ^k. (Here and throughout the thesis, unless stated otherwise, all cohomology groups are with $\mathbb{F}_2$-coefficients.) These are rare but very interesting maps. For instance, some manifolds (e.g., $\mathbb{CP}^n$) admit pseudo-rotations with finite number of ergodic measures; see [AK, FK, LeRS]. These pseudo-rotations are constructed from true rotations using the conjugation method and hence they are balanced; see [GG18b]. It is expected that, [GK], all pseudo-rotations have this property. For instance, in [GG18b] it was proved that most pseudo-rotations of $\mathbb{CP}^2$ are balanced with some possible exceptions called ghost pseudo-rotations. In this thesis, using quantum Steenrod squares from [Wi18, Wi20], we show that all non-degenerate pseudo-rotations of $\mathbb{CP}^2$ are balanced.

Recently, symplectic topological methods have been employed to study the dynamics of pseudo-rotations and its connections with symplectic topological properties of the underlying manifold in all dimensions; see [AS, Ban, Br15a, Br15b, ÇGG19, ÇGG20a, GG18a, GG18b, LeRS, Sh19b, Sh20]. In particular, several variants of the Chance–McDuff conjecture have been established for pseudo-rotations. In [ÇGG19] it was proved that a weakly monotone symplectic manifold with minimal Chern number $N > 1$, admitting a pseudo-rotation φ, must have deformed quantum product and, in particular, non-vanishing Gromov–Witten invariants, under certain additional index assumptions on φ. These extra assumptions appear to be satisfied for most, but certainly not all, pseudo-rotations. Simultaneously and independently, in [Sh20], along with some other results the quantum Steenrod square of the top degree class was shown to be deformed for monotone symplectic manifolds M^{2n} with Poincaré duality property (e.g., when $N \geq n + 1$), admitting pseudo-rotations. (These pseudo-rotations need not be non-degenerate.)

In this thesis we further investigate connections between pseudo-rotations and symplectic topology of the underlying manifold; and present a result, [ÇGG20a], that partially generalizes [Sh20]. We show that a closed monotone symplectic manifold, which admits a non-degenerate Hamiltonian pseudo-rotation, must have a deformed quantum Steenrod square of the top degree element and, as a consequence, non-trivial holomorphic spheres. This result complements the results from [ÇGG19]; and in [Sh19b] it is generalized, using a different method, for a broader class of pseudo-rotations allowing for some degenerations.

The quantum Steenrod square is a symplectic topological invariant introduced in [Se] and then studied in [Wi20, Wi18]; see also [Be, BC, Fu, He, SS, ShZa] for some relevant work. It is a cohomology operation $\mathcal{QS}$ on the quantum cohomology $\mathrm{HQ}^*(M)$, which is a deformation of the standard Steenrod square. In other words, $\mathcal{QS}(\alpha) = \mathrm{Sq}(\alpha) + O(\mathrm{q})$, where q is the generator of the Novikov ring and $\alpha \in \mathrm{H}^*(M)$. Roughly speaking, a deformed quantum Steenrod square, just as a deformed quantum product $*$, detects certain holomorphic spheres in M, but in general these spheres need not be related to Gromov–Witten invariants. Furthermore, $\mathcal{QS}$ can also be viewed as a deformation of the standard quantum square $\alpha \mapsto \alpha * \alpha$, with respect to a different parameter h, in the same sense as Sq can be thought of as a deformation of the cup square $\alpha \mapsto \alpha \cup \alpha$. On the Floer cohomology side, $\mathcal{QS}$ is closely related to another quantum cohomology operation also introduced in [Se], the equivariant pair-of-pants product $\wp$, which plays a crucial role in our proof. We will briefly discuss both of these operations in Chapter 3.

Let ϖ be the generator of the top degree cohomology group $\mathrm{H}^{2n}(M^{2n})$. Our result, Theorem 2.1.1, asserts that $\mathcal{QS}(\varpi)$ is different from $\mathrm{Sq}(\varpi) = \mathrm{h}^{2n}\varpi$ whenever M is monotone and admits a non-degenerate pseudo-rotation. Here we treat the Steenrod square Sq as a degree doubling map

$$\mathrm{Sq}\colon\ \mathrm{H}^*(M) \to \mathrm{H}^*(M)[[\mathrm{h}]], \quad \mathrm{Sq}(\alpha) = \sum_{i=0}^{|\alpha|} \mathrm{h}^{|\alpha|-i}\, \mathrm{Sq}^i(\alpha),$$

where $|\mathrm{h}| = 1$ and $\mathrm{Sq}^i(\alpha)$ is the i-th standard Steenrod square of $\alpha \in \mathrm{H}^*(M)$, and $\mathrm{H}^*(M)[[\mathrm{h}]]$ is the space of formal power series with coefficients in $\mathrm{H}^*(M)$, [Se, Wi20, Wi18]. As a consequence, there is a non-trivial holomorphic sphere through every point of M.

It is difficult to compare Theorem 2.1.1 and the results from [ÇGG19] detecting a deformed quantum product; for these theorems hold under different conditions and provide different symplectic topological information. Note however that the statement that $\mathcal{QS}$ is deformed is obviously much weaker than that its 0-th order term in h, the quantum square, is deformed, i.e., $\alpha * \alpha \neq \alpha \cup \alpha$ for some $\alpha \in \mathrm{H}^*(M)$.

From a different prespective, it is expected that a Hamiltonian diffeomorphism of a closed symplectic manifold has infinitely many periodic points whenever it has "more than absolutely necessary" fixed points. This statement is referred to as the Hofer–Zehnder conjecture, [HZ, p. 263]. It is a generalization of the celebrated theorem of Franks which asserts that every area preserving diffeomorphism of S^2 has either exactly two or infinitely many periodic points, [Fr92, Fr96]. (Moreover, in the setting of Franks' theorem, there are also strong growth rate results; see, e.g, [FH, LeC, Ke].)

The lower bound "more than absolutely necessary" is usually interpreted as a lower bound arising from some version of the Arnold conjecture, e.g., as the sum of the Betti numbers. For $\mathbb{CP}^n$, the expected threshold is $n+1$ regardless of the non-degeneracy assumption. In particular, it is 2 for $S^2 = \mathbb{CP}^1$ as in Franks' theorem. A slightly different interpretation of the conjecture, not directly involving the count of fixed points, is that the presence of a fixed or periodic point that is unnecessary from a homological or geometrical perspective is already sufficient to force the existence of infinitely many periodic points. We refer the reader to [GG14, Gü13, Gü14, Su21b] for some results in this direction.

The non-degenerate case of the Hofer–Zehnder conjecture can be rephrased as that all strongly non-degenerate counterexamples to the Conley conjecture are non-degenerate pseudo-rotations. Recently, for a certain class of manifolds (e.g. $\mathbb{CP}^n$), this was established in [Sh19a]; see also [Al]. (In fact, the results from [Sh19a] and [Al] allow some degeneracy.) In this thesis we give a simple proof, [ÇGG20b], of a slightly weaker version of Shelukhin's theorem, [Sh19a, Thm. A].

The original proof of Franks' theorem utilized methods from low-dimensional dynamics, and the first purely symplectic topological proof was given in [CKRTZ]. However, that proof and also a different approach from [BH] were still strictly low-dimensional, and Shelukhin's theorem, [Sh19a, Thm. A], is the first sufficiently general higher-dimensional variant of Franks' theorem. (Strictly speaking, [Sh19a, Thm. A] and our Theorem 2.2.1 and Corollary 2.2.2, which are overall slightly weaker, still fall short of completely reproving Franks' theorem in dimension two; we will discuss and compare these results in Section 2.2.1.) Similarly to [Sh19a], the key ingredient of our proof is Seidel's $\mathbb{Z}_2$-equivariant pair-of-pants product, [Se]. (While we use the original version of the product, [Sh19a] relies on its $\mathbb{Z}_p$-equivariant version from [ShZa].) Our proof also uses several simple ingredients from persistent homology theory in the form developed in [UZ] (see also [PS]), although to a much lesser degree than [Sh19a].

Chapter 2

Main results

In this chapter we state and discuss the main results. The conventions and basic definitions are reviewed in Chapter 3.

2.1 From pseudo-rotations to holomorphic curves

The Chance–McDuff conjecture asserts that whenever the Conley conjecture fails some Gromov-Witten invariants of the manifold are non-zero. In this thesis we prove a variant of this conjecture. We show that a closed monotone symplectic manifold, which admits a non-degenerate Hamiltonian pseudo-rotation, must have a deformed quantum Steenrod square $\mathcal{QS}$ of the top degree element and, as a consequence, non-trivial holomorphic spheres, [ÇGG20a]. Note that in general these spheres need not be related to Gromov–Witten invariants. As of this writing, all known Hamiltonian diffeomorphisms with finitely many periodic orbits are non-degenerate pseudo-rotations and these two classes might well coincide; see Section 2.2.

Theorem 2.1.1. *Assume that a closed monotone symplectic manifold* (M^{2n}, ω) *admits a non-degenerate pseudo-rotation. Then the quantum Steenrod square* $\mathcal{QS}$ *of the top degree cohomology class* $\varpi \in \mathrm{H}^{2n}(M; \mathbb{F}_2)$ *is deformed:* $\mathcal{QS}(\varpi) \neq \mathrm{h}^{2n}\varpi$.

The proof of Theorem 2.1.1 hinges on the same idea as the argument in [ÇGG19], although the latter proof is considerably more involved. In both cases, a non-trivial deformation comes roughly speaking from constant (to be more precise, zero energy) pair-of-pants solutions of the Floer equation: equivariant in the present case and standard for the quantum product.

For the standard pair-of-pants product, a zero energy curve is easily seen to be automatically regular provided that the Conley–Zehnder indices allow this. Namely, consider the cohomology pair-of-pants product of iterated capped periodic orbits $\bar{x}^{k_1} * \ldots * \bar{x}^{k_r}$. Then the least action term in this product is $\bar{x}^k$ with $k = k_1 + \ldots + k_r$, i.e.,

$$\bar{x}^{k_1} * \ldots * \bar{x}^{k_r} = \bar{x}^k + \ldots,$$

where the dots stand for higher action terms, if and only if $\bar{x}^k$ has the "right" Conley–Zehnder index. Explicitly, with our conventions, the latter index condition is that

$$\mu(\bar{x}^k) = \mu(\bar{x}^{k_1}) + \ldots + \mu(\bar{x}^{k_r}) + (r-1)n,$$

and the main difficulty in the proof in [ÇGG19] is to guarantee that this requirement is satisfied for some orbit $\bar{x}$ and that the resulting product is different from the cup product.

On the other hand, for the equivariant pair-of-pants product, $\bar{x}^2$ (or, to be more precise, its product with a suitable power of h) is always, without any index requirement, the least action term in $\wp(\bar{x} \otimes \bar{x})$, although now this is a non-trivial fact proved in [Se]; see also [ShZa]. This is sufficient to show that the quantum Steenrod square of ϖ is deformed whenever M admits a pseudo-rotation, by using simultaneously the action and h-adic filtrations of the equivariant Floer cohomology.

Remark 2.1.2. *We expect Theorem 2.1.1 to have several generalizations accessible by the same method with relatively minor modifications. Namely, one might be able to replace the assumption that M is monotone by the condition that it is weakly monotone; for one can expect the constructions of the equivariant pair-of-pants product $\wp$ from [Se] and of the quantum Steenrod square $\mathcal{QS}$ to extend to this setting with some modifications; see [SW]. One might also be able to extend Theorem 2.1.1 to the quantum Steenrod $\mathbb{Z}_p$ cohomology operations (see [ShZa]). On the other hand, in [Sh19b], using a different method, Theorem 2.1.1 is generalized for a broader class of pseudo-rotations allowing for some degenerations.*

2.2 Another look at the Hofer–Zehnder conjecture

Let φ be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold M. We view φ as the time-one map in a time-dependent Hamiltonian flow and denote by $\mathcal{P}_k(\varphi)$ the set of its k-periodic points, arising from contractible k-periodic orbits. The Hamiltonian diffeomorphism φ is said to be *k-perfect* if $\mathcal{P}_k(\varphi) = \mathcal{P}_1(\varphi)$ and *perfect* if φ is k-perfect for all $k \in \mathbb{N}$. We call φ a non-degenerate *pseudo-rotation* over a field $\mathbb{F}$ if it is strongly non-degenerate, perfect and the differential in the Floer complex of φ over $\mathbb{F}$ vanishes. This condition is independent of the choice of an almost complex structure and, by Arnold's conjecture, equivalent to that the number of 1-periodic orbits $|\mathcal{P}_1(\varphi)|$ is equal to the sum of Betti numbers of M over $\mathbb{F}$. Denote by $\beta(\varphi)$ the *boundary depth* of φ over $\mathbb{F}$, i.e., the length of the maximal finite bar in the *barcode* of φ; see [Us, UZ] and also Section 4.2.3.

One of the goals of this thesis is to give a simple proof, [ÇGG20b], of the following theorem proved in a slightly different form in [Sh19a].

Theorem 2.2.1 (Shelukhin's Theorem, [Sh19a]; see also [ÇGG20b])**.** *Assume that φ is strongly non-degenerate and perfect and that $\beta(\psi)$ over $\mathbb{F}_2 := \mathbb{Z}_2$ is bounded from above for all Hamiltonian diffeomorphisms ψ of M or at least for all iterates $\psi = \varphi^{2^k}$ (e.g., $M = \mathbb{CP}^n$). Then φ is a non-degenerate pseudo-rotation.*

Applying this to the iterates φ^{2^k} we obtain

Corollary 2.2.2 ([Sh19a]; see also [ÇGG20b])**.** *Assume that φ is strongly non-degenerate, $\beta\big(\varphi^{2^k}\big)$ over $\mathbb{F}_2$ is bounded from above (e.g., $M = \mathbb{CP}^n$), and $|\mathcal{P}_1(\varphi)|$ is strictly greater than the sum of Betti numbers of M over $\mathbb{F}_2$. Then $\big|\mathcal{P}_{2^k}(\varphi)\big| \to \infty$ as $k \to \infty$.*

This theorem is proved in Section 4.2.2 as an easy consequence of Theorem 4.2.1. On the conceptual level, our proof of Theorem 4.2.1 is also a subset of Shelukhin's argument, although the inclusion is rather implicit. Namely, our proof focuses on the behavior of the shortest bar $\beta_{\min}$ in the barcode of φ (rather than the longest finite bar, a.k.a. the boundary depth, $\beta \geq \beta_{\min}$, [UZ]) or, to be more precise, of the shortest Floer arrow under the iteration from φ to φ^2; see Section 4.2.1. In particular, we show in Theorem 4.2.1 that when φ is 2-perfect the shortest arrow persists under such an iteration, although it may migrate into the equivariant domain for φ^2, and the length of the arrow doubles. The shortest non-equivariant arrow for φ^2 is at least as long as the equivariant one. Hence $\beta_{\min}(\varphi^2) \geq 2\beta_{\min}(\varphi)$, and Theorem 2.2.1 readily follows from Theorem 4.2.1 applied to a sequence of period doubling iterations; see Section 4.2.2. The key ingredient in the proof of Theorem 4.2.1 is the equivariant pair-of-pants product, introduced in [Se], having a very strong non-vanishing property also proved therein (see Proposition 3.2.6).

2.2.1 Comparison with Shelukhin's theorem

In this section we discuss some of the differences between Theorem 2.2.1 and the original Shelukhin's theorem, [Sh19a, Thm. A]. From our perspective the key new feature of the proof presented here is its simplicity. Note however that there are two major differences (discussed in detail below) between Theorem 2.2.1 and [Sh19a, Thm. A] which make the former a weaker statement. Our argument extends with minor modifications to cover these differences, which we briefly discuss, but we prefer to omit the proofs which would divert us from our goal.

First of all, in the most recent version of [Sh19a, Thm. A] there are no restrictions on the ground coefficient field $\mathbb{F}$ while here $\mathbb{F} = \mathbb{F}_2$. When $\mathbb{F}$ is $\mathbb{Q}$, the assertion is that $\mathcal{P}_p(\varphi)$ contains a simple periodic orbit for every large prime p. As a consequence, one obtains the growth of order at least $O(k/\log k)$ for the number of simple periodic orbits of period up to k. This difference stems from the fact that the main tool used in

[Sh19a] is the $\mathbb{Z}_p$-equivariant pair-of-pants product introduced in [ShZa] while we rely on a somewhat simpler $\mathbb{Z}_2$-equivariant pair-of-pants product defined in [Se]. We touch upon the p-iterated analogues of Theorem 2.2.1 and Corollary 2.2.2 in Remark 4.2.12.

Secondly, [Sh19a, Thm. A] allows for some degeneracy of φ. Namely, in the setting of Corollary 2.2.2, the number of 1-periodic orbits $|\mathcal{P}_1(\varphi)|$ in the condition that $|\mathcal{P}_1(\varphi)|$ is strictly greater than the sum of Betti numbers is replaced by

$$\sum_{x\in\mathcal{P}_1(\varphi)} \dim_{\mathbb{F}} \mathrm{HF}(x;\mathbb{F}), \tag{2.1}$$

where $\mathrm{HF}(x;\mathbb{F})$ is the local Floer (co)homology of x with coefficients in a field $\mathbb{F}$ (see, e.g., [GG10]). Note that, as a consequence, Corollary 2.2.2 still holds without the non-degeneracy assumption, provided that the number of 1-periodic orbits with $\mathrm{HF}(x;\mathbb{F})\neq 0$ is greater than the sum of Betti numbers. In the setting of this thesis, one should take $\mathbb{F}=\mathbb{F}_2$ and we will further discuss the degenerate case of Theorem 2.2.1 and Corollary 2.2.2 in Section 4.2.5. Overall, the role of the condition that $\mathrm{HF}(x;\mathbb{F})\neq 0$ is unclear to us beyond the case of S^2. Franks' theorem has an analogue for a certain class of symplectomorphisms of surfaces and then, interestingly, this condition becomes essential; see [Bat, GG09b].

2.2.2 Upper bound on β

In this section we discuss the requirement in Theorem 2.2.1 and Corollary 2.2.2 that $\beta(\psi)$ is bounded from above. First of all, note that while it would be sufficient to only have an upper bound on $\beta_{\min}(\psi)$ where $\psi=\varphi^{2^k}$ or, as in [Sh19a, Thm. A], on $\beta(\psi)$ where $\psi = \varphi^p$, all relevant results proved to date are more robust and give an upper bound on $\beta(\psi)$ for all ψ. (This is the curse (and the blessing) of symplectic topological methods in dynamics: they are very robust and general, but not particularly discriminating; they often tell the same thing about all maps. There are, however, exceptions.)

The simplest manifold for which such an *a priori* bound is established is $\mathbb{CP}^n$ for any coefficient field (suppressed in the notation), and the result essentially goes back to [EP]. The argument is roughly as follows. (We use here the notation and conventions from Section 3.1.) First recall that

$$\beta(\psi) \leq \gamma(\psi). \tag{2.2}$$

Here $\gamma(\psi)$ is the γ-norm of ψ defined, using cohomology, as

$$\gamma(\psi) = -\big(\mathrm{c}_{\mathbb{1}}(\psi) + \mathrm{c}_{\mathbb{1}}(\psi^{-1})\big),$$

where $\mathrm{c}_\alpha(\psi)$ is the spectral invariant associated with a quantum cohomology class $\alpha \in \mathrm{HQ}(M)$ and $\mathbb{1}$ is the unit in the ordinary cohomology $\mathrm{H}(M)$ of M. (We suppress the grading in the cohomology notation when it is irrelevant.) The upper bound (2.2) holds for any closed monotone symplectic manifold and its proof is similar to the proof in [Us] of the upper bound for β by the Hofer norm, but with continuation maps replaced by the multiplications by the image of $\mathbb{1}$ in $\mathrm{HF}(\psi)$ and $\mathrm{HF}\left(\psi^{-1}\right)$. (We refer the reader to [KS] for some further results along these lines.) Applying the Poincaré duality in Floer cohomology (see [EP]), it is not hard to show that $\mathrm{c}_{\mathbb{1}}(\psi^{-1}) = -\mathrm{c}_{\varpi}(\psi)$ when $N \geq n+1$, where ϖ is the generator of $\mathrm{H}^{2n}(M)$ and N is the minimal Chern number of M^{2n}. In particular, this is true for $M = \mathbb{CP}^n$ since then $N = n+1$. By construction, for any two classes α and ζ in $\mathrm{HQ}(M)$ the spectral invariants satisfy the Lusternik–Schnirelmann inequality $\mathrm{c}_{\alpha*\zeta}(\psi) \geq \mathrm{c}_\alpha(\psi)$. Thus, from the identity $\varpi * \zeta = \mathrm{q}\mathbb{1}$ where ζ is the generator of $\mathrm{HQ}^2(\mathbb{CP}^n)$, we conclude that $\mathrm{c}_{\mathbb{1}}(\psi) \leq \mathrm{c}_\varpi(\psi) \leq \mathrm{c}_{\mathbb{1}}(\psi) + \pi$. These inequalities, combined with (2.2), show that

$$\beta(\psi) \leq \gamma(\psi) \leq \pi$$

for any Hamiltonian diffeomorphism ψ of $\mathbb{CP}^n$.

A similar upper bound on β holds for all closed monotone manifolds M such that $\mathrm{HQ}^{even}(M; \mathbb{F})$ for some field $\mathbb{F}$ is semi-simple, i.e., splits as an algebra into a direct sum of fields. This is [Sh19a, Thm. B] and, interestingly, this result bypasses the upper bound (2.2) in its original form. In fact, $\mathrm{HQ}(S^2 \times S^2; \mathbb{Q})$ is semi-simple, but γ is not bounded from above for $S^2 \times S^2$; see [Sh19a, Rmk. 7] and also [PR, Thm. 6.2.6]. We are not aware of any algebraic criteria for an *a priori* bound on the γ-norm. Nor do we know how large the class of monotone symplectic manifolds with semi-simple $\mathrm{HQ}^{even}(M; \mathbb{F})$ is. In addition to $\mathbb{CP}^n$ (with any $\mathbb{F}$), the complex Grassmannians, $S^2 \times S^2$, and the one point blow-up of $\mathbb{CP}^2$ with standard monotone symplectic structures are in this class when *char* $\mathbb{F} = 0$ (see [EP] and references therein); but $S^2 \times S^2$ is not for $\mathbb{F} = \mathbb{F}_2$.

2.3 Pseudo-rotations of $\mathbb{CP}^2$

Let φ be a non-degenerate pseudo-rotation of $\mathbb{CP}^n$ and let $\bar{x}_i$ be the capped 1-periodic orbits of φ with Conley–Zehnder index $\mu(\bar{x}_i) \in [-n, n]$. The pseudo-rotation

φ is called *balanced* if the sum $\sum \hat{\mu}(\bar{x}_i)$ of the mean indices $\hat{\mu}(\bar{x}_i)$ of $\bar{x}_i$ is equal to zero. Such pseudo-rotations are of interest since, for instance, one can show that, [GG18b], every balanced pseudo-rotation of $\mathbb{CP}^2$ has a matching true rotation with the same fixed point data; which is consistent with the conjugation method (see [AK, FK, LeRS]). On the other hand, a pseudo-rotation of $\mathbb{CP}^n$ which is not balanced, if exists, cannot come from the conjugation method since all true rotations are balanced; see [GG18b].

It was proved in [GG18b] that most pseudo-rotations of $\mathbb{CP}^2$ are balanced with some possible exceptions called ghost pseudo-rotations. In this thesis, following [GG18b, Rmk. 5.12] and using the quantum Steenrod square from [Wi20, Wi18], we prove the following index theorem which as a corollary implies that all non-degenerate pseudo-rotations of $\mathbb{CP}^2$ are balanced.

Theorem 2.3.1. *Let φ be a pseudo-rotation of $\mathbb{CP}^2$. Assume that φ, φ^2 are non-degenerate and let $\bar{x}_i$ be the capped 1-periodic orbits of φ with indices $\mu(\bar{x}_1) = -2$, $\mu(\bar{x}_2) = 0$ and $\mu(\bar{x}_3) = 2$. Then we have $\mu(\bar{x}_1^2) + \mu(\bar{x}_2^2) + \mu(\bar{x}_3^2) = 0$.*

For a non-degenerate pseudo-rotation φ of $\mathbb{CP}^2$, by applying Theorem 2.3.1 to the iterates of the form φ^{2^k} one can conclude that φ is balanced, i.e., $\hat{\mu}(\bar{x}_1) + \hat{\mu}(\bar{x}_2) + \hat{\mu}(\bar{x}_3) = 0$. Note that, in the setting of Theorem 2.3.1, although the sum of indices does not change

$$0 = \sum \mu(\bar{x}_i) = \sum \mu(\bar{x}_i^2),$$

individual terms in the sum might change and can go out of the support $\{0, \pm 2\}$. For instance, this is the case (for $\mathbb{CP}^1$) for the second pseudo-rotation considered in Example

3.2.15. As a consequence, when applying Theorem 2.3.1 to the pair φ^2, φ^4 one might need to recap the capped orbits $\bar{x}_i^2$ so that their indices are supported in $\{0, \pm 2\}$. However in that case, since $\sum \mu(\bar{x}_i^2) = 0$, the total index change caused by the recapping would be zero. One still concludes that

$$0 = \sum \mu(\bar{x}_i) = \sum \mu(\bar{x}_i^2) = \sum \mu(\bar{x}_i^4) = \cdots ,$$

and as a result

$$0 = \lim_{k \to \infty} \sum \frac{\mu(\bar{x}_i^{2^k})}{2^k} = \sum \hat{\mu}(\bar{x}_i).$$

Hence we have

Corollary 2.3.2. *Every non-degenerate pseudo-rotation of $\mathbb{CP}^2$ is balanced.*

A generalization of this result to $\mathbb{CP}^n$ for all $n > 2$ is work in progress.

Chapter 3

Preliminaries

3.1 Conventions and notation

Throughout this thesis, the underlying symplectic manifold (M, ω) is assumed to be closed and strictly monotone, i.e., $[\omega]|_{\pi_2(M)} = \lambda c_1(TM)|_{\pi_2(M)} \neq 0$ for some $\lambda > 0$. The *minimal Chern number* of M is the positive generator N of the subgroup $\langle c_1(TM), \pi_2(M)\rangle \subset \mathbb{Z}$ and the *rationality constant* is the positive generator $\lambda_0 = 2N\lambda$ of the group $\langle \omega, \pi_2(M)\rangle \subset \mathbb{R}$.

A *Hamiltonian diffeomorphism* $\varphi = \varphi_H = \varphi_H^1$ is the time-one map of the time-dependent flow $\varphi^t = \varphi_H^t$ of a 1-periodic in time Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. Such time-one maps form the group $Ham(M, \omega)$ of Hamiltonian diffeomorphisms of M. In what follows, it will be convenient to view Hamiltonian diffeomorphisms together with the path φ_H^t, $t \in [0, 1]$, up to homotopy with fixed end points, i.e., as elements of the universal covering $\widetilde{Ham}(M,\omega)$ of the group of Hamiltonian diffeomorphisms.

Let $x\colon S^1 \to M$ be a contractible loop. A *capping* of x is an equivalence class of maps $A\colon D^2 \to M$ such that $A|_{S^1} = x$. Two cappings A and A' of x are equivalent if the integral of ω (or of $c_1(TM)$ since M is strictly monotone) over the sphere obtained by attaching A to A' is equal to zero. A capped closed curve $\bar{x}$ is, by definition, a closed curve x equipped with an equivalence class of cappings, and the presence of capping is indicated by a bar.

The action of a Hamiltonian H on a capped closed curve $\bar{x} = (x, A)$ is

$$\mathcal{A}_H(\bar{x}) = -\int_A \omega + \int_{S^1} H_t(x(t))\,dt.$$

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of $\mathcal{A}_H$ on this space are exactly the capped 1-periodic orbits of X_H.

The k-periodic *points* of φ are in one-to-one correspondence with the k-periodic *orbits* of H, i.e., of the time-dependent flow φ^t. Recall also that a k-periodic orbit of H is called *simple* if it is not iterated. A k-periodic orbit x of H is said to be *non-degenerate* if the linearized return map $D\varphi^k\colon T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. A Hamiltonian H is non-degenerate if all its 1-periodic orbits are non-degenerate and H is strongly non-degenerate if all periodic orbits of H (of all periods) are non-degenerate. We denote the collection of capped k-periodic orbits of H by $\bar{\mathcal{P}}_k(\varphi)$.

Let $\bar{x}$ be a non-degenerate capped periodic orbit. The *Conley–Zehnder index* $\mu(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, as in [Sa, SZ]. In this thesis, we normalize μ so that

$\mu(\bar{x}) = n$ when x is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian.

The *mean index* $\hat{\mu}(\bar{x}) \in \mathbb{R}$ measures, roughly speaking, the total angle swept by certain (Krein–positive) unit eigenvalues of the linearized flow $d\varphi^t|_{\bar{x}}$ with respect to the trivialization associated with the capping; see [Lo, SZ]. The mean index is defined even when x is degenerate, and we always have the inequality $\big|\, \hat{\mu}(\bar{x}) - \mu(\bar{x}) \big| \leq n$. Moreover, if x is non-degenerate, the inequality is strict:

$$\big|\, \hat{\mu}(\bar{x}) - \mu(\bar{x}) \big| < n. \tag{3.1}$$

The mean index is homogeneous with respect to iteration: $\hat{\mu}\left(\bar{x}^k\right) = k\,\hat{\mu}(\bar{x})$. (The capping of $\bar{x}^k$ is obtained from the capping of $\bar{x}$ by taking its k-fold cover branched at the origin.)

Fixing an almost complex structure, which will be suppressed in the notation, we denote by $(\mathrm{CF}(\varphi), d_{Fl})$ and $\mathrm{HF}(\varphi)$ the Floer complex and cohomology of φ over $\mathbb{F}_2 = \mathbb{Z}_2$; see, e.g., [MS, Sa]. (Throughout this thesis, all complexes and cohomology groups are over $\mathbb{F}_2$.) The complex $\mathrm{CF}(\varphi)$ is generated by the capped 1-periodic orbits $\bar{x}$ of H, graded by the Conley–Zehnder index, and filtered by the action. The filtration level (or the action) of a chain $\xi \in \mathrm{CF}(\varphi)$ is defined by

$$\mathcal{A}(\xi) = \min\{\mathcal{A}(\bar{x}_i)\}, \text{ where } \xi = \sum \bar{x}_i. \tag{3.2}$$

(Note that the filtration depends on H, not just on φ, making of the notation $\mathrm{CF}(\varphi)$ somewhat misleading.) The differential d_{Fl} is the upward Floer differential: it increases the action and also the index by one. The Floer complex $\mathrm{CF}(\varphi)$ is also a finite-dimensional free module over the Novikov ring Λ. There are several choices of Λ; see, e.g., [MS]. For our purposes, it is convenient to take the field of Laurent series $\mathbb{F}_2((\mathrm{q}))$ with $|\mathrm{q}| = 2N$ as Λ. With this choice, Λ naturally acts on $\mathrm{CF}(\varphi)$ by recapping, and multiplication by q corresponds to the recapping by $A \in \pi_2(M)$ with $\langle c_1(TM), A\rangle = N$. Furthermore, $\mathrm{CF}(\varphi)$ is a finite-dimensional vector space over Λ with a preferred basis formed by 1-periodic orbits with arbitrarily fixed capping.

Notationally, it is convenient to equip $\mathrm{CF}(\varphi)$ with a non-degenerate $\mathbb{F}_2$-valued pairing $\langle\, ,\rangle$ for which $\bar{\mathcal{P}}_1(\varphi)$ is an orthogonal basis: $\langle \bar{x},\, \bar{y}\rangle = \delta_{\bar{x}\bar{y}}$. Then, essentially by definition,

$$d_{Fl}\bar{x} = \sum \langle d_{Fl}\bar{x},\, \bar{y}\rangle\, \bar{y}.$$

There is a canonical, grading-preserving isomorphism

$$\Phi\colon\ \mathrm{HQ}^*(M) \xrightarrow{\cong} \mathrm{HF}^*(\varphi)[n], \tag{3.3}$$

where $\mathrm{HQ}(M)$ is the quantum cohomology of M; see, e.g., [Sa, MS] and references therein. (Depending on the context, this is the PSS-isomorphism or the continuation map or a combination of the two.) The cohomology groups $\mathrm{HQ}(M)$ and $\mathrm{HF}(\varphi)$ are also modules over a Novikov ring Λ, and $\mathrm{HQ}(M) \cong \mathrm{H}(M) \otimes \Lambda \cong \mathrm{HF}(\varphi)$ (as a module).

For instance, assume that H is C^2-small and autonomous (i.e., independent of t), and has a unique maximum and a unique minimum. Then the top degree cohomology class $\varpi \in \mathrm{H}^{2n}(M) \subset \mathrm{HQ}^{2n}(M)$ corresponds to the maximum of H, which has degree n in $\mathrm{HF}^*(\varphi)$; the unit $\mathbb{1} \in \mathrm{HQ}^0(M)$ corresponds to the minimum of H which has degree $-n$ in $\mathrm{HF}^*(\varphi)$. We denote by $|\alpha|$ the degree of $\alpha \in \mathrm{HQ}^*(M)$ or $\alpha \in \mathrm{HF}^*(\varphi)$.

When φ is a pseudo-rotation (or, more generally, if $d_{Fl} = 0$), the isomorphism (3.3) turns into the natural identification

$$\mathrm{HQ}^*(M)[-n] \cong \mathrm{HF}^*(\varphi) \cong \mathrm{CF}^*(\varphi).$$

Since any iterate φ^k is also a pseudo-rotation, we have

$$\mathrm{HQ}^*(M)[-n] \cong \mathrm{HF}^*\left(\varphi^k\right) \cong \mathrm{CF}^*\left(\varphi^k\right).$$

It is worth emphasizing that the resulting isomorphism between $\mathrm{CF}^*(\varphi) = \mathrm{HF}^*(\varphi)$ and $\mathrm{CF}^*\left(\varphi^k\right) = \mathrm{HF}^*\left(\varphi^k\right)$, which is given by the continuation map, is usually different from the iteration map $\bar{x} \mapsto \bar{x}^k$. For instance, unless M is aspherical the iteration map is not onto and, in general, $\mu(\bar{x}) \neq \mu\left(\bar{x}^k\right)$ even in the aspherical case.

The quantum homology $\mathrm{HQ}^*(M)$ carries the *quantum product*, denoted here by $*$, which makes it into a graded-commutative algebra over Λ with unit $\mathbb{1}$. This product is a deformation (in q) of the cup product: $\alpha * \beta = \alpha \cup \beta + O(\mathrm{q})$. For instance, $\alpha_1 * \alpha_n = \mathrm{q}\mathbb{1}$ in $\mathrm{HQ}^*(\mathbb{CP}^n)$, where α_l stands for the generator (i.e., the only non-zero element) of $\mathrm{H}^{2l}(\mathbb{CP}^n)$. In Floer cohomology, the quantum product corresponds to the so-called *pair-of-pants product*

$$\mathrm{HF}^*(\varphi) \otimes \mathrm{HF}^*(\varphi) \to \mathrm{HF}^*\left(\varphi^2\right)[n],$$

which we also denote by $*$. We emphasize that with our conventions $|\alpha * \beta| = |\alpha| + |\beta| + n$ in Floer cohomology. When (3.3) is applied to φ and φ^2, the pair-of-pants product turns into the quantum product:

$$\mathrm{HQ}^*(M) \otimes \mathrm{HQ}^*(M) \cong \mathrm{HF}^*(\varphi) \otimes \mathrm{HF}^*(\varphi) \to \mathrm{HF}^*\left(\varphi^2\right) \cong \mathrm{HQ}^*(M).$$

Here, for the sake of simplicity, we suppressed the shifts of degree as we often will in what follows.

3.2 Equivariant Floer cohomology and the pair-of-pants product

Drawing heavily from [Se] and also [ShZa, Wi20, Wi18], we recall the construction of the equivariant Floer cohomology and equivariant pair-of-pants product together with its relation to the quantum Steenrod square in the case where the ambient manifold M is closed and monotone. Then we will take a closer look at the effect of the additional condition that the Hamiltonian diffeomorphism φ is a pseudo-rotation.

3.2.1 Equivariant Floer cohomology

The *equivariant Floer cohomology* $\mathrm{HF}_{eq}\left(\varphi^2\right)$, introduced in [Se], is the homology of a certain complex $\left(\mathrm{CF}_{eq}\left(\varphi^2\right), d_{eq}\right)$ called the *equivariant Floer complex.* As a graded $\mathbb{F}_2$-vector space or as a Λ-module,

$$\mathrm{CF}^*_{eq}\left(\varphi^2\right) := \mathrm{CF}^*\left(\varphi^2\right)[[\mathrm{h}]] = \mathrm{CF}^*\left(\varphi^2\right) \otimes_\Lambda \Lambda[[\mathrm{h}]] \tag{3.4}$$

where $|\mathrm{h}| = 1$, and the differential d_{eq} has the form

$$d_{eq} = d_{Fl} + \mathrm{h}\, d_1 + \mathrm{h}^2\, d_2 + \ldots = d_{Fl} + O(\mathrm{h}).$$

Remark 3.2.1. *For monotone symplectic manifolds there are other choices in the definition of* $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$*, e.g.,* $\mathrm{CF}^*\left(\varphi^2\right)[\mathrm{h}]$*; see Remark 3.2.12. We utilize this flexibility and use the polynomial version* $\mathrm{CF}^*\left(\varphi^2\right)[\mathrm{h}]$ *in the proof of Theorem 4.2.1.*

The differential d_{eq} is $\Lambda[[\mathrm{h}]]$-linear and non-strictly action-increasing. It is roughly speaking defined as follows, mimicking Borel's construction of the $\mathbb{Z}_2$-equivariant Morse cohomology.

Fix a family $\tilde{J}$ of 2-periodic in t almost complex structures on M parametrized by the unit infinite-dimensional sphere $S^\infty \subset \mathbb{R}^\infty$. Here $\mathbb{R}^\infty$ is the direct sum of infinitely many copies of $\mathbb{R}$, i.e., its elements $\xi = (\xi_0, \xi_1, \ldots)$ have only finitely many non-zero components, and $S^\infty = \{\|\xi\| = 1\}$ with $\|\xi\|^2 = \sum_k |\xi_k|^2$. The almost complex structure $\tilde{J}$ is required to satisfy the symmetry condition $\tilde{J}_{-\xi} = \tilde{J}'_\xi$, where $\tilde{J}'_\xi$ is obtained from $\tilde{J}_\xi$ by the time-shift $t \mapsto t+1$. Consider the self-indexing quadratic form $f(\xi) = \sum_k k|\xi_k|^2$ on S^∞ and an antipodally symmetric metric such that the natural equatorial embedding $S^\infty \to S^\infty$ given by $(\xi_0, \xi_1, \ldots) \mapsto (0, \xi_0, \ldots)$ is an isometry. (Note also that the pull back of f by this embedding is $f+1$.) The almost complex structure $\tilde{J}$ must furthermore be constant in ξ near the critical points of f, invariant under the equatorial embedding, and satisfy a certain regularity requirement. Denote by $w_k^\pm$ the critical points of f of index k.

Next, consider the hybrid Morse-Floer complex of $\mathcal{A} + f$ with respect to $\tilde{J}$ and the metric on S^∞. This complex has pairs $(\bar{x}, w_k^\pm)$ with $\bar{x} \in \bar{\mathcal{P}}_2(\varphi)$ as generators and carries a natural $\mathbb{Z}_2$-action, free on the generators, sending $(\bar{x}, w_k^\pm)$ to $(\bar{x}', w_k^{-\pm})$, where $\bar{x}'$ is the time-shift of $\bar{x}$. It is easy to see that the homology of this hybrid complex is equal to $\mathrm{HF}(\varphi^2)$. By definition, $\mathrm{CF}_{eq}\left(\varphi^2\right)$ is the $\mathbb{Z}_2$-invariant part of this hybrid complex, where we write $\bar{x}\,\mathrm{h}^k$ for $(\bar{x}, w_k^+) + (\bar{x}', w_k^-)$. The fact that the differential is h-linear follows from the requirement that f (up to a constant) and the auxiliary data are invariant under the equatorial embedding. Thus, in self-explanatory notation,

$$d_k\bar{x} = \sum \left\langle d_k\bar{x},\, \mathrm{h}^k\bar{y} \right\rangle \bar{y}, \text{ where } \mu(\bar{y}) = \mu(\bar{x}) + 1 - k$$

and $\langle d_k\bar{x},\, \mathrm{h}^k\bar{y}\rangle$ counts mod 2 the total number of continuation Floer trajectories from $\bar{x}$ to $\bar{y}$ along gradient lines of f connecting w_0^+ to w_k^+ and from $\bar{x}$ to $\bar{y}'$ along gradient lines of f connecting w_0^+ to w_k^-. Clearly, the complex (and hence its cohomology) is filtered by the action $\mathcal{A}$ in addition to the filtration by $\mathcal{A}+f$. On the level of (co)chains the filtration is defined similarly to (3.2), but with the powers of h ignored:

$$\mathcal{A}(\xi) = \min\{\mathcal{A}(\bar{x}_i)\}, \text{ where } \xi = \sum \mathrm{h}^{m_i}\bar{x}_i.$$

The equivariant complex and the cohomology has natural continuation properties; see [Se].

Example 3.2.2. *Assume that φ is 2-perfect and φ^2 admits a regular 1-periodic almost complex structure J, i.e., for every pair $\bar{x}$ and $\bar{y}$ of 2-periodic orbits the space of Floer trajectories connecting $\bar{x}$ to $\bar{y}$ has dimension $\mu(\bar{y}) - \mu(\bar{x})$. In particular, this space is empty when $\mu(\bar{y}) \leq \mu(\bar{x})$, except when $\bar{y} = \bar{x}$ and the space comprises one constant trajectory. Set $\tilde{J} = J$ to be a constant (i.e., independent of ξ) almost complex structure. Then $\tilde{J}$ is also regular and $d_j = 0$ for $j \geq 1$ since continuation trajectories for a constant homotopy are just Floer trajectories. Thus, in this case, $\mathrm{HF}_{eq}\left(\varphi^2\right) = \mathrm{HF}(\varphi)[[\mathrm{h}]]$ for any interval of action. These conditions are met, for instance, when $\varphi = \varphi_H$ is generated by a C^2-small autonomous Hamiltonian H. As a consequence, for any φ the global cohomology $\mathrm{HF}_{eq}\left(\varphi^2\right)$ is not a particularly interesting object: it is simply isomorphic to*

$\mathrm{HQ}(M)[[\mathrm{h}]]$ *via the equivariant continuation (or the PSS map); see [Wi18, Wi20] for further details.*

Remark 3.2.3. *In this connection we point out that there are two slightly different constructions of the equivariant Floer cohomology for Hamiltonians with symmetry, e.g., the* S^1*-symmetry for autonomous Hamiltonians and the* $\mathbb{Z}_k$*-symmetry for* k*-iterated Hamiltonian diffeomorphisms. The first construction uses a parametrized perturbation of the original Hamiltonian and the action functional; see [BO, Vi99] and also [GG19]. This is a Floer theoretic analogue of taking a Morse perturbation of the pull-back (which is Morse–Bott) of the original Morse function to the Borel quotient. In the second construction one keeps the Hamiltonian and the action functional unchanged, but uses a parametrized almost complex structure and continuation maps along the gradient lines of an auxiliary Morse function on the classifying space to define the differential; see [Hu, Se, SS]. This approach results in a complex and cohomology* a priori *better behaving with respect to the action filtration. This is the complex considered here. (The difference becomes apparent in the context of the filtered Leray spectral sequence converging to the equivariant cohomology and associated with the* h*-adic filtration: it is not even clear how to define the* h*-adic filtration in the framework of the first construction without additional assumptions on the perturbation; see [BO].)*

3.2.2 Equivariant pair-of-pants product

The target space of the *equivariant pair-of-pants product* is the $\mathbb{Z}_2$-equivariant cohomology $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$ and the domain is the ordinary group cohomology of $\mathbb{Z}_2$ with coefficients in the complex $\mathrm{CF}^*(\varphi)\otimes_{\mathbb{F}_2}\mathrm{CF}^*(\varphi)$ with the $\mathbb{Z}_2$-action given by the involution ι interchanging the two factors. In other words, this is the cohomology of the complex

$$\mathrm{C}^*\left(\mathbb{Z}_2;\mathrm{CF}^*(\varphi)\otimes_{\mathbb{F}_2}\mathrm{CF}^*(\varphi)\right):=\left(\mathrm{CF}^*(\varphi)\otimes_{\mathbb{F}_2}\mathrm{CF}^*(\varphi)\right)[[\mathrm{h}]]$$

equipped with the differential $d_{\mathbb{Z}_2}=d_{Fl}+\mathrm{h}(id+\iota)$, where the first term stands for the differential induced by d_{Fl} on the tensor product. This complex also carries two increasing filtrations: the action filtration and the h-adic filtration. We note that the complex is "unaware" of simple 2-periodic orbits of φ.

The *equivariant pair-of-pants product* is the $\mathbb{F}_2[[\mathrm{h}]]$-linear map

$$\wp\colon\ \mathrm{H}^*\left(\mathbb{Z}_2;\mathrm{CF}^*(\varphi)\otimes_{\mathbb{F}_2}\mathrm{CF}^*(\varphi)\right)\to\mathrm{HF}^*_{eq}\left(\varphi^2\right)$$

induced by the chain map

$$\mathrm{C}^*\left(\mathbb{Z}_2;\mathrm{CF}^*(\varphi)\otimes_{\mathbb{F}_2}\mathrm{CF}^*(\varphi)\right)\to\mathrm{CF}^*_{eq}\left(\varphi^2\right)$$

constructed in [Se]. This product is a deformation in h of the pair-of-pants product, i.e., on the chain level the equivariant pair-of-pants product $\wp(c_1\otimes c_2)$ of c_1 and c_2 in $\mathrm{CF}^*_{eq}(\varphi)$ has the form $c_1*c_2+O(\mathrm{h})$. Furthermore, on the chain level, $\wp$ is bi-linear, i.e., $\wp(\mathrm{q}c_1\otimes c_2)=\wp(c_1\otimes \mathrm{q}c_2)=\mathrm{q}\wp(c_1\otimes c_2)$. Note, however, that $\mathrm{q}c_1\otimes c_2\neq c_1\otimes\mathrm{q}c_2$, since the tensor product is taken over $\mathbb{F}_2$.

To get a better understanding of how the map $\wp$ works, note first that the graded space $\mathrm{CF}^*\left(\varphi^2\right)$ has a canonical involution ι' given by the shift of time $x(t)\mapsto x(t+1)$; for the generators of this space are the 2-periodic orbits of φ. In general, this linear map, extended to $\mathrm{CF}^*\left(\varphi^2\right)[[\mathrm{h}]]$, does not commute with d_{Fl} unless there is a regular 1-periodic (rather than 2-periodic) almost complex structure. However, when this is the case, one can replace the complex $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$ by the former complex with the differential $d_{Fl} + \mathrm{h}(id + \iota')$. Then $\wp$ is a deformation of the map induced by the pair-of-pants product map $\mathrm{CF}(\varphi) \otimes_{\mathbb{F}_2} \mathrm{CF}(\varphi) \to \mathrm{CF}\left(\varphi^2\right)$ of the "coefficient" complexes.

Remark 3.2.4. *Recall that even when ι' does not commute with d_{Fl}, it becomes an isomorphism of complexes when the target is equipped with the Floer differential associated with the time-shifted almost complex structure $J'_t = J_{t+1}$. Then, once composed with the continuation map, ι' induces an involution of $\mathrm{HF}^*\left(\varphi^2\right)$. In our setting, the global Floer cohomology $\mathrm{HF}^*\left(\varphi^2\right)$ and the involution are independent of φ, and hence this involution is the identity map.*

The key property, [Se, Thm. 1.3], of the equivariant pair-of-pants product map $\wp$ is that when, for example, (M, ω) is symplectically aspherical it becomes an isomorphism once h^{-1} is attached to the ground ring, i.e., after taking tensor product with the ring of Laurent series $\mathbb{F}_2((\mathrm{h}))$. This yields the Floer theoretic analogue of Borel's localization relating the filtered cohomology $\mathrm{HF}^*(\varphi)$ and $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$ and, as a consequence, a variant of Smith's inequality, cf. [ÇG, He, Sh19a, ShZa]. Although in this thesis we do not directly use any of these results, we will briefly revisit them in Remark 3.2.8.

Next, consider the map $\mathcal{S}\colon \mathrm{CF}^*(\varphi) \to \mathrm{CF}^*(\varphi) \otimes_{\mathbb{F}_2} \mathrm{CF}^*(\varphi)$ given by $c \mapsto c \otimes c$ for all $c \in \mathrm{CF}^*(\varphi)$. When needed, we extend this map to $\mathrm{CF}^*(\varphi)[[\mathrm{h}]]$ by setting $\mathcal{S}(\mathrm{h}c) = \mathrm{h}\mathcal{S}(c)$. Note that, since we tensor over $\mathbb{F}_2$, the map $\mathcal{S}$ is not homogeneous in q. In general, $\mathcal{S}$ is neither linear (even over $\mathbb{F}_2$) nor, when linear, is it a chain map. However, $\mathcal{S}$ is well-defined on the level of cohomology and becomes $\mathbb{F}_2[[\mathrm{h}]]$-linear when multiplied by h, i.e., as a map

$$\mathrm{h}\,\mathcal{S}\colon\ \mathrm{HF}^*(\varphi)[[\mathrm{h}]] \to \mathrm{H}^*\left(\mathbb{Z}_2; \mathrm{CF}^*(\varphi) \otimes_{\mathbb{F}_2} \mathrm{CF}^*(\varphi)\right). \tag{3.5}$$

For instance, to see the linearity it suffices to observe that

$$\mathrm{h}\big((c_1 + c_2)^2 - c_1^2 - c_2^2\big) = d_{\mathbb{Z}_2}(c_1 \otimes c_2),$$

when $d_{Fl}(c_1) = 0 = d_{Fl}(c_2)$. As a consequence, $\mathcal{S}$ itself is defined and $\mathbb{F}_2[[\mathrm{h}]]$-linear on the level of cohomology when the target has no h-torsion. Then, composing the cohomology map $\mathcal{S}$ with the equivariant pair-of-pants map $\wp$, we obtain a map

$$\mathcal{PS}\colon\ \mathrm{HF}^*(\varphi)[[\mathrm{h}]] \to \mathrm{HF}^*_{eq}\left(\varphi^2\right);$$

the notation is borrowed from [Wi20, Wi18]. This map is $\mathbb{F}_2[[\mathrm{h}]]$-linear whenever the target has no h-torsion and, as we will soon see, this condition is automatically satisfied in the case we are interested in.

Remark 3.2.5. *Following [Se, Sect. 2.1] note that for purely algebraic reasons there is a canonical isomorphism*

$$\mathrm{H}^*\left(\mathbb{Z}_2; \mathrm{CF}^*(\varphi) \otimes_{\mathbb{F}_2} \mathrm{CF}^*(\varphi)\right) \cong \mathrm{H}^*\left(\mathbb{Z}_2; \mathrm{HF}^*(\varphi) \otimes_{\mathbb{F}_2} \mathrm{HF}^*(\varphi)\right)$$

Hence the cohomology group on the right can also be thought of as the domain of the equivariant pair-of-pants product $\wp$ and the target of the map $\mathcal{S}$. Furthermore, the map

$$\mathcal{S}\colon\ \mathrm{HF}^*(\varphi)((\mathrm{h})) \to \mathrm{H}^*\left(\mathbb{Z}_2; \mathrm{CF}^*(\varphi) \otimes_{\mathbb{F}_2} \mathrm{CF}^*(\varphi)\right) \otimes_{\mathbb{F}_2[[\mathrm{h}]]} \mathbb{F}_2((\mathrm{h})),$$

which is $\mathbb{F}_2((\mathrm{h}))$*-linear regardless of whether or not the target of* $\mathrm{h}\mathcal{S}$ *in* (3.5) *has* h-*torsion, is also an isomorphism, again for purely algebraic reasons.*

In general, we still have $\mathcal{PS}$ defined on the chain level as a map

$$\mathrm{CF}^*(\varphi)[[\mathrm{h}]] \to \mathrm{CF}^*_{eq}\left(\varphi^2\right)$$

such that $\mathcal{PS}(c) = c * c + O(\mathrm{h})$ for $c \in \mathrm{CF}^*(\varphi)$, but it is neither $\mathbb{F}_2$-linear nor a chain homomorphism. Recall that the map $\mathcal{S}$ itself is not homogeneous in q. However the composition $\mathcal{PS}$ is homogeneous, since $\wp$ satisfies $\wp(\mathrm{q}c_1 \otimes c_2) = \wp(c_1 \otimes \mathrm{q}c_2)$. (Here we have once again ignored the shift of degree: by construction, $|\mathcal{PS}(c)| = 2|c| + n$.)

By construction the equivariant pair-of-pants map $\wp$ and the map $\mathcal{PS}$ preserve the action filtration; cf. Remark 3.2.9.

One of the key ingredients in the proof of [Se, Thm. 1.3] is the following result, which also plays a central role in our argument and which, slightly deviating from [Se], we state for the map $\mathcal{PS}$ rather than for $\wp$.

Proposition 3.2.6 ([Se], Prop. 6.7)**.** *Consider a collection of orbits* $\bar{x}_i \in \bar{\mathcal{P}}_1(\varphi)$, $i = 1, \dots, \ell$, *such that* $\mathcal{A}_H(\bar{x}_i) = a$ *for* $i = 1, \dots, \ell_0 \leq \ell$ *and* $\mathcal{A}_H(\bar{x}_i) > a$ *for the remaining orbits. Then*

$$\mathcal{PS}\colon \sum_{i=1}^{\ell} \bar{x}_i \mapsto \sum_{i=1}^{\ell_0} \mathrm{h}^{m_i} \bar{x}_i^2 + \dots,$$

where $\bar{x}_i^2 \in \bar{\mathcal{P}}_2(\varphi)$ *is the second iterate of* $\bar{x}_i$ *and*

$$m_i = 2\mu(\bar{x}_i) - \mu\left(\bar{x}_i^2\right) + n \tag{3.6}$$

and the dots stand for a sum of capped orbits with action strictly greater than $2a$.

For instance, $\mathcal{PS}(\bar{x}) = \mathrm{h}^m \bar{x}^2 + \ldots$, where $m = 2\mu(\bar{x}) - \mu(\bar{x}^2) + n$ and the remaining terms have action higher than that of $\bar{x}^2$. In particular, $\bar{x}^2$ with some power of h is necessarily present in $\mathcal{PS}(\bar{x})$.

Proposition 3.2.6 is an equivariant analogue of the standard fact that a constant solution of the Floer or Cauchy–Riemann equation is automatically regular whenever the relative index of the solution is zero, which in turn is a consequence of that the kernel of the linearized operator at the constant solution with suitable boundary conditions is trivial; see, e.g., [ÇGG19, Lemma 3.1], [MS, Lemma 6.7.6], [Se, p. 971] and [Sa, Sect. 2.7]. However, the step from a non-equivariant to equivariant setting is non-trivial. We refer the reader to [Se] for the proof; see also [ShZa] for an alternative approach and generalizations.

Remark 3.2.7. *A generalization of the equivariant pair-of-pants product to the p-th iterates φ^p, where p is a prime, replacing $\mathbb{Z}_2$ by $\mathbb{Z}_p$ and $\mathbb{F}_2$ by $\mathbb{F}_p$ is constructed in [ShZa]. This construction and the analogue of Seidel's non-vanishing theorem for the p-th iterate plays a crucial role in the original proof of Shelukhin's theorem in [Sh19a]; cf. Remark 4.2.12.*

Remark 3.2.8 (Borel's localization theorem according to [Se])**.** *As has been mentioned above, one consequence of Proposition 3.2.6 is [Se, Thm. 1.3] asserting, in particular, that for a symplectically aspherical manifold the equivariant pair-of-pants product $\wp$ in the filtered Floer cohomology (i.e., the homology of the Floer complex restricted to an action interval) becomes an isomorphism after tensoring with $\mathbb{F}_2((\mathrm{h}))$. (The theorem* *follows from the proposition via applying the spectral sequence comparison theorem to the action filtration spectral sequence.) Since $\mathcal{S}$ is an isomorphism modulo* h*-torsion for purely algebraic reasons (see Remark 3.2.5), this yields that the map $\mathcal{PS}$ is also an isomorphism, as well as the variants of Borel's localization and Smith's inequality in the filtered Floer cohomology.*

On the other hand, while for a closed symplectic manifold M the analogues of Borel's localization and Smith's inequality hold trivially in the global Floer cohomology, the filtered version of Borel's localization (in the most naive form) fails without the assumption that (M, ω) is symplectically aspherical; see, however, [Sh19a]. Moreover, $\mathcal{PS}$ need not be an isomorphism even globally without this assumption. In fact, as Example 3.2.10 shows, $\mathcal{PS}$ is not an epimorphism already for $M = S^2$.

Remark 3.2.9 (Regularity)**.** *To ensure that the regularity condition is satisfied for the equivariant pair-of-pants product, a certain arbitrarily small "inhomogeneous perturbation", i.e., an s-dependent perturbation of the Hamiltonian, is introduced to the Floer equation in [Se]. This perturbation is compactly supported in s and thus does not affect the initial and terminal Hamiltonians and the actions. However, it does affect the relation between the energy of a pair-of-pants curve and the action difference. As a consequence, the equivariant pair-of-pants product $\wp$ is now action increasing only up to an ϵ-error, which goes to zero with the size of the perturbation. Therefore, $\mathcal{PS}$ also preserves the action filtration only up to an ϵ-error. In particular, if all fixed points of φ have distinct actions and $\epsilon > 0$ is small, $\mathcal{PS}$ literally preserves the action filtration. This would already be sufficient for our purposes; see Remark 4.1.1. However, since the* *pair-of-pants curves connecting different orbits must have energy* a priori *bounded away from zero (cf. [GG17, Prop. 2.2]), the map $\mathcal{PS}$ always preserves the action filtration when the inhomogeneous perturbation is small enough, and Proposition 3.2.6 holds as stated.*

3.2.3 Quantum Steenrod square

The counterpart of the map $\mathcal{PS}$ on the side of the quantum cohomology is the *quantum Steenrod square* $\mathcal{QS}$. This quantum cohomology operation is studied in detail in [Wi20, Wi18], but the first Morse/Floer theoretic descriptions of the Steenrod squares go back to [Be, BC, Fu]. Throughout this section it is essential that the manifold M is closed.

Following [Se, Wi20, Wi18] and slightly changing the usual notation, let us define the *Steenrod square* as the degree doubling linear map

$$\mathrm{Sq}\colon \mathrm{H}^*(M) \to \mathrm{H}^*(M)[[\mathrm{h}]], \quad \mathrm{Sq}(\alpha) = \sum_{i=0}^{|\alpha|} \mathrm{h}^{|\alpha|-i}\,\mathrm{Sq}^i(\alpha), \tag{3.7}$$

where Sq^i are the standard Steenrod squares. In particular, $|\,\mathrm{Sq}^i(\alpha)| = |\alpha| + i$, and $\mathrm{Sq}^0(\alpha) = \alpha$ and $\mathrm{Sq}^{|\alpha|}(\alpha) = \alpha \cup \alpha$. For instance, for the generator ϖ of $\mathrm{H}^{2n}(M^{2n})$,

$$\mathrm{Sq}(\varpi) = \mathrm{h}^{2n}\varpi. \tag{3.8}$$

The *quantum Steenrod square* is a degree doubling map

$$\mathcal{QS}\colon \mathrm{H}^*(M) \to \mathrm{HQ}^*(M)[[\mathrm{h}]],$$

which is a certain deformation of Sq in q:

$$\mathcal{QS}(\alpha) = \mathrm{Sq}(\alpha) + O(\mathrm{q}) \tag{3.9}$$

for $\alpha \in \mathrm{H}^*(M)$. For instance,

$$\mathcal{QS}(\varpi) = \mathrm{h}^{2n}\varpi + O(\mathrm{q}),$$

and $\mathcal{QS}$ is undeformed at $\varpi \in \mathrm{H}^{2n}(M)$ if and only if the higher order terms in q vanish.

It is convenient to formally extend Sq and $\mathcal{QS}$ to the maps

$$\mathrm{Sq}\colon\ \mathrm{HQ}^*(M)[[\mathrm{h}]] \to \mathrm{HQ}^*(M)[[\mathrm{h}]]$$

and

$$\mathcal{QS}\colon\ \mathrm{HQ}^*(M)[[\mathrm{h}]] \to \mathrm{HQ}^*(M)[[\mathrm{h}]],$$

which are linear over $\mathbb{F}_2[[h]]$ and homogeneous of degree two in q. More precisely, for instance, we set $\mathcal{QS}(\mathrm{h}\alpha) = \mathrm{h}\mathcal{QS}(\alpha)$ and $\mathcal{QS}(\mathrm{q}\alpha) = \mathrm{q}^2\mathcal{QS}(\alpha)$. Since the extension is linear over $\mathbb{F}_2[[h]]$, the maps are no longer degree doubling. In what follows, unless stated otherwise, Sq and $\mathcal{QS}$ we will refer to the extended maps above. Note that $\mathcal{QS}$ is still a deformation of Sq in q in the sense of (3.9).

The next ingredient we need is the equivariant continuation/PSS map introduced in [Wi18]. This is the map Φ_{eq} from the $\mathbb{Z}_2$-equivariant cohomology of $\mathrm{HQ}^*(M)$ with trivial action to the equivariant Floer cohomology of φ^2:

$$\Phi_{eq}\colon\ \mathrm{HQ}^*_{eq}(M) \to \mathrm{HF}^*_{eq}\left(\varphi^2\right)[n].$$

Since the $\mathbb{Z}_2$-action in the cohomology is trivial, we have

$$\mathrm{HQ}^*_{eq}(M) = \mathrm{HQ}^*(M)[[\mathrm{h}]]$$

and one can also think of the cohomology on the left hand side as the target space of $\mathcal{QS}$.

Just as an ordinary continuation/PSS map Φ, its equivariant counterpart Φ_{eq} is an $\mathbb{F}_2[[\mathrm{h}]]$-linear isomorphism. As a consequence, $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$ has no h-torsion and the map $\mathcal{PS}$ is linear.

The spaces and maps we have introduced fit together into the following commutative diagram, where we again suppressed the shifts of degree by the continuation/PSS maps:

$$\begin{array}{ccc} \mathrm{HQ}^*(M)[[\mathrm{h}]] & \xrightarrow[\cong]{\Phi} & \mathrm{HF}^*(\varphi)[[\mathrm{h}]] \\ \downarrow{\scriptstyle \mathcal{QS}} & & \downarrow{\scriptstyle \mathcal{PS}} \\ \mathrm{HQ}^*_{eq}(M) & \xrightarrow[\cong]{\Phi_{eq}} & \mathrm{HF}^*_{eq}\left(\varphi^2\right) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle F} \\ \mathrm{HQ}^*(M)[[\mathrm{h}]] & \xrightarrow[\cong]{\Phi} & \mathrm{HF}^*\left(\varphi^2\right)[[\mathrm{h}]] \end{array} \tag{3.10}$$

We emphasize that here the continuation/PSS maps Φ for φ and φ^2, and Φ_{eq} (i.e., the horizontal arrows) are $\mathbb{F}_2[[\mathrm{h}]]$-linear isomorphisms. The requirement that the top square is commutative can be viewed as the definition of $\mathcal{QS}$, [Wi18]. Likewise, the condition that the bottom square is commutative is the definition of F, i.e.,

$$F = \Phi\Phi_{eq}^{-1}.$$

It is worth pointing out that in general the map F need not preserve the action filtration.

Example 3.2.10. *Let $M = S^2$. Then $\mathcal{QS}(\mathbb{1}) = \mathbb{1}$ and $\mathcal{QS}(\varpi) = \mathrm{h}^2\varpi + \mathrm{q}\mathbb{1}$, where $\mathbb{1}$ is the generator of $\mathrm{H}^0(S^2)$; see [Wi20]. Here $\mathrm{q}\mathbb{1}$ is the deformation term, which is equal to the quantum square of ϖ. Continuing the discussion from Remark 3.2.8 and making* h *invertible, it is now easy to see that neither ϖ nor $\mathrm{q}\mathbb{1}$ nor $\mathrm{q}\varpi$ is in the image of*

$$\mathcal{QS}\colon\ \mathrm{HQ}^*(S^2)((\mathrm{h})) \to \mathrm{HQ}^*_{eq}(S^2) \otimes_{\mathbb{F}_2[[\mathrm{h}]]} \mathbb{F}_2((\mathrm{h})) \cong \mathrm{HQ}^*(S^2) \otimes_\Lambda \Lambda((\mathrm{h})).$$

As a consequence of the diagram (3.10)*,*

$$\mathcal{PS}\colon\ \mathrm{HF}^*(\varphi)((\mathrm{h})) \to \mathrm{HF}^*_{eq}(\varphi^2) \otimes_{\mathbb{F}_2[[\mathrm{h}]]} \mathbb{F}_2((\mathrm{h}))$$

is not onto for any Hamiltonian diffeomorphism $\varphi\colon S^2 \to S^2$*. Note that at the same time linearly extending the classical Steenrod square map* (3.7) *over* $\mathbb{F}_2((\mathrm{h}))$ *gives a linear isomorphism* $\mathrm{H}^*(M)((\mathrm{h})) \to \mathrm{H}^*(M)((\mathrm{h}))$ *for any closed manifold* M*.*

Finally, as has been mentioned above, the complex $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$ carries the h-adic filtration:

$$\mathrm{CF}^*_{eq}\left(\varphi^2\right) \supset \mathrm{h}\,\mathrm{CF}^*_{eq}\left(\varphi^2\right) \supset \mathrm{h}^2\,\mathrm{CF}^*_{eq}\left(\varphi^2\right) \supset \dots .$$

The associated spectral sequence (the Leray spectral sequence in the equivariant cohomology) converges to the graded space E_∞ associated with the h-adic filtration of $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$. It readily follows from the fact that Φ_{eq} is an isomorphism that this spectral sequence collapses on the E_1-page: $E_1 = \mathrm{HF}^*\left(\varphi^2\right)[[\mathrm{h}]] = E_\infty$. Indeed, the fact that E_1 in every bi-degree has the same dimension over $\mathbb{F}_2$ as E_∞ forces all higher order differentials to vanish. Alternatively, we can view $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$ as an ungraded complex over Λ with h-adic filtration. Then E_1 is finite-dimensional over Λ in every degree with $\dim_\Lambda E_1^* = \dim_\Lambda E_\infty^*$, which again implies that the spectral sequence collapses in the E_1-term. With this in mind we can view $\mathrm{HF}^*\left(\varphi^2\right)[[\mathrm{h}]]$ as the graded space associated with the h-adic filtration on $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$.

Remark 3.2.11. *It is essential to acknowledge an abuse of terminology in the paragraph above and in what follows. Strictly speaking, since the* h*-adic filtration is infinite, to keep the isomorphism between* E_∞ *and* $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$ *we would need to define the* E_∞*-page of the* *spectral sequence as the direct product of individual degree terms rather than the direct sum. Then E_∞ becomes a filtered (rather than graded) module which is isomorphic to* $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$ *as a filtered module, but not to the graded module resulting from the* h*-adic filtration of* $\mathrm{HF}^*_{eq}\left(\varphi^2\right)$*. The problem here stems from the difference between the direct product and the direct sum; e.g.,* $\mathbb{F}_2[[h]]$ *is the product of its fixed degree terms and not a graded ring, whereas* $\mathbb{F}_2[h]$ *is the direct sum and graded. Below we will ignore this issue, which is not uncommon in symplectic topology literature, for it is essentially of terminological nature. Furthermore, the problem can be completely avoided as explained in the next remark.*

Remark 3.2.12. *In* (3.4)*, we defined* $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$ *as the space of formal power series with coefficients in* $\mathrm{CF}^*\left(\varphi^2\right)$*. There are, however, other choices of* $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$ *although the difference is purely technical, cf. [Se, Wi20, Wi18]. For instance, by examining the action/index change one can readily see that for a closed monotone symplectic manifold, the expansion of* d_{eq} *involves only a finite number of non-zero terms. Thus the equivariant Floer homology can be defined over* $\mathbb{F}_2[\mathrm{h}]$ *as* $\mathrm{CF}^*\left(\varphi^2\right)[\mathrm{h}]$*. In a similar vein, the product* $\mathcal{PS}$*, the quantum Steenrod square* $\mathcal{QS}$*, and the continuation/PSS maps* Φ_{eq} *and* Φ^{-1}_{eq} *are also polynomial in* h*. (Note that as a consequence, F is also polynomial in* h*, which is* a priori *non-obvious.) Therefore, in all constructions mentioned above one can replace formal power series in* h *by polynomials, avoiding the terminological issue pointed out in Remark 3.2.11. Furthermore, we utilize the polynomial version* $\mathrm{CF}^*\left(\varphi^2\right)[\mathrm{h}]$ *in the proof of Theorem 4.2.1. This difference is essential for our proof as at some point in the argument we evaluate the elements of* $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$ *at* $\mathrm{h}=1$*. Finally, we also note* *that one can choose a middle way and replace the space of formal power series by the tensor product with* $\mathbb{F}_2[[\mathrm{h}]]$ *over* $\mathbb{F}_2$*, e.g., setting* $\mathrm{CF}^*_{eq}\left(\varphi^2\right) = \mathrm{CF}^*\left(\varphi^2\right) \otimes_{\mathbb{F}_2} \mathbb{F}_2[[\mathrm{h}]]$.

We will need the following simple observation:

Lemma 3.2.13. *For any Hamiltonian diffeomorphism* φ*, the equivariant continuation map* Φ_{eq} *induces the ordinary continuation map* Φ *on the graded vector space* $E_\infty = \mathrm{HF}^*\left(\varphi^2\right)[[\mathrm{h}]]$*, i.e.,*

$$\Phi_{eq} = \Phi + O(h). \tag{3.11}$$

Proof. Let f be a C^2-small Morse function on (M, ω), unrelated to the pseudo-rotation φ. Set $\mathrm{QC}^*(M) = \mathrm{CM}^*(f) \otimes \Lambda$ where $\mathrm{CM}^*(f)$ is the Morse complex of f. The complex $\mathrm{QC}^*(M)[[\mathrm{h}]]$ has a natural h-adic filtration and the resulting spectral sequence collapses on the E_1-page; for h is not involved in the differential. Furthermore, recall that $\mathrm{HF}^*\left(\varphi\right)[[\mathrm{h}]]$ is the E_1-page associated with the h-adic filtration of $\mathrm{CF}^*_{eq}\left(\varphi^2\right)$. We claim that, in the obvious notation,

$$E_1(\Phi_{eq}) = \Phi. \tag{3.12}$$

Then, since both spectral sequences collapse on the E_1-page, (3.11) readily follows from (3.12).

It remains to establish (3.12). Let ψ_f be the Hamiltonian diffeomorphism generated by f. Following [Wi18], we write the chain level definitions of Φ_{eq} and Φ as the compositions

$$\mathrm{QC}^*(M)[[\mathrm{h}]] \xrightarrow{\Psi_{eq}} \mathrm{CF}^*\left(\psi_f^2\right)[[\mathrm{h}]] \xrightarrow{C_{eq}} \mathrm{CF}^*\left(\varphi^2\right)[[\mathrm{h}]] = \mathrm{CF}^*_{eq}\left(\varphi^2\right)$$

and

$$\mathrm{QC}^*(M)[[\mathrm{h}]] \xrightarrow{\Psi} \mathrm{CF}^*\left(\psi_f^2\right)[[\mathrm{h}]] \xrightarrow{C} \mathrm{CF}^*\left(\varphi^2\right)[[\mathrm{h}]].$$

Here the map Ψ is the PSS map for f or, to be more precise, for $2f$. Furthermore, C is the continuation from the Floer complex of φ^2 to the Floer complex of ψ_f^2 for a fixed (e.g., linear) homotopy between f and the Hamiltonian H generating φ. The maps Ψ_{eq} and C_{eq} are the equivariant counterparts of Ψ and, respectively, C. (Note that the differential in the Morse complex of f, and hence in the Floer complex of ψ_f^2, might be non-trivial; for M need not admit a perfect Morse function.) By definition (see [Wi18]), $\Psi_{eq} = \Psi + O(\mathrm{h})$ and $C_{eq} = C + O(\mathrm{h})$, and (3.12) follows. □

3.2.4 Enter pseudo-rotations

Assume now that φ is a pseudo-rotation or more generally that every 2-periodic point is a fixed point and that $d_{Fl} = 0$ for φ and hence for φ^2. Then $\mathrm{HF}^*(\varphi) = \mathrm{CF}^*(\varphi)$ and $\mathrm{HF}^*\left(\varphi^2\right) = \mathrm{CF}^*\left(\varphi^2\right)$.

Furthermore, from the collapse of the Leray spectral sequence it then follows inductively that $d_{eq} = 0$ and we have the identifications

$$F_0\colon\ \mathrm{HF}^*_{eq}\left(\varphi^2\right) = \mathrm{CF}^*_{eq}\left(\varphi^2\right) = \mathrm{CF}^*\left(\varphi^2\right)[[\mathrm{h}]], \tag{3.13}$$

which, in contrast with the natural map F, are specific to the case of pseudo-rotations, but might exist under somewhat less restrictive conditions. Of course, $F_0 = id$, but we prefer to use a different notation at this point to emphasize the fact that F_0 is defined only under some additional assumptions on φ and φ^2.

The next important (but simple) ingredient of our proof, which we use to establish (4.4) below, is the following lemma.

Lemma 3.2.14. *We have* $F = F_0 + O(\mathrm{h})$.

This lemma is an immediate consequence of Lemma 3.2.13 and the identifications (3.13).

Example 3.2.15. *Returning to Example 3.2.10 consider the rotation* φ *of* S^2 *about the z-axis in an angle* θ. *Let* $\bar{x}$ *and* $\bar{y}$ *be the South and North poles respectively, equipped with trivial cappings. Thus* $\mu(\bar{x}) = -1$ *and* $\mu(\bar{y}) = 1$. *Passing to the second iterate* φ^2, *we still have* $\mu(\bar{x}^2) = -1$ *and* $\mu(\bar{x}^2) = 1$ *when* $\theta \in (0, \pi)$. *Then* $\Phi_{eq}(\mathbb{1}) = \bar{x}^2 = \Phi(\mathbb{1})$ *and* $\Phi_{eq}(\varpi) = \bar{y}^2 = \Phi(\varpi)$ *and* $F = id$. *In general,* $F = id$ *whenever* φ *is sufficiently close to* id. *On the other hand, if* $\theta \in (\pi, 2\pi)$ *we have* $\mu(\bar{x}^2) = -3$ *and* $\mu(\bar{y}^2) = 3$. *Then* $\Phi_{eq}(\mathbb{1}) = \mathrm{q}^{-1}\bar{y}^2 + \mathrm{h}^2\bar{x}^2$, *while* $\Phi(\mathbb{1}) = \mathrm{q}^{-1}\bar{y}^2$, *and* $\Phi_{eq}(\varpi) = \mathrm{q}\bar{x}^2 = \Phi(\varpi)$. *(This can be proved by using the information about* $\mathcal{QS}$ *from Example 3.2.10 along with index/action analysis and Lemma 3.2.14.) As a consequence,* $F(\bar{x}^2) = \bar{x}^2$ *and* $F(\bar{y}^2) = \bar{y}^2 + \mathrm{h}^2\mathrm{q}\bar{x}^2$. *We are not aware of any general method of explicitly calculating the map* F.

Chapter 4

Proofs

4.1 Proof of Theorem 2.1.1

For the sake of simplicity, we will assume that all capped periodic orbits have distinct action: the argument extends to the general case in a straightforward way and the difference is purely expository. (See also Remark 4.1.1.)

We will argue by contradiction: throughout the proof we assume that $\mathcal{QS}(\varpi)$ is undeformed, i.e., $\mathcal{QS}(\varpi) = \mathrm{h}^{2n}\varpi$; see (3.8) and (3.9). The proof comprises two steps, and this assumption, which we aim to disprove, is used in both steps.

Write

$$\Phi(\varpi) = \bar{x} + \ldots, \tag{4.1}$$

where the dots stand for the terms with action strictly greater than the action of $\bar{x}$. Thus $\mu(\bar{x}) = n$. In the first step we prove the theorem under the additional condition that the index of $\bar{x}$ jumps from $\bar{x}$ to $\bar{x}^2$, i.e.,

$$\mu\big(\bar{x}^2\big) > \mu(\bar{x}) = n. \tag{4.2}$$

This condition might or might not hold for φ and in the second step we show that (4.2) is necessarily satisfied for a sufficiently high iterate of φ. This will complete the proof of the theorem.

Step 1. Thus let us prove the theorem under the additional condition (4.2), where $\bar{x}$ is given by (4.1). From the top square in the diagram (3.10) and Proposition 3.2.6, we obtain the following commutative square:

$$\begin{array}{ccc} \varpi & \xrightarrow{\Phi} & \bar{x} + \dots \\ {\scriptstyle \mathcal{QS}}\downarrow & & \downarrow{\scriptstyle \mathcal{PS}} \\ \mathrm{h}^{2n}\varpi & \xrightarrow{\Phi_{eq}} & \mathrm{h}^m \bar{x}^2 + \dots \end{array}$$

In the bottom right corner the dots again stand for higher action terms and, by (3.6) and (4.2),

$$m = 3n - \mu\big(\bar{x}^2\big) < 2n.$$

This contradicts the fact that Φ_{eq} is $\mathbb{F}_2[[\mathrm{h}]]$-linear.

Step 2. To finish the proof it remains to make sure that (4.2) is satisfied after, if necessary, replacing φ by its iterate.

The condition that $\mu(\bar{x}) = n$ guarantees that, by (3.1), $\hat{\mu}(\bar{x}) > 0$ and hence $\mu\big(\bar{x}^k\big) \to \infty$, and furthermore that the sequence $\mu\big(\bar{x}^k\big)$ is increasing (but not necessarily strictly increasing):

$$\mu\big(\bar{x}^k\big) \nearrow \infty. \tag{4.3}$$

There are several ways to show this. For instance, let us adopt the argument from [GG18b, Sect. 4]; see also [GG18b, Formula (6.1)]. Namely, let $P \in \widetilde{\mathrm{Sp}}(2n)$ be the linearized flow along $\bar{x}$. Since the index sequence $\mu\big(P^k\big)$ is invariant under iso-spectral deformations, we can assume without loss of generality that $P(1)$ is semi-simple. Then P can be expressed as the product of a loop ϕ and $P \in \widetilde{\mathrm{Sp}}(2n)$ which decomposes as a direct sum of elements of $\widetilde{\mathrm{Sp}}(2)$ or $\widetilde{\mathrm{Sp}}(4)$ of the following three types: short rotations of $\mathbb{R}^2$ (by an angle $\theta \in (-\pi, \pi)$), positive and complex hyperbolic transformations of $\mathbb{R}^2$ or $\mathbb{R}^4$ with zero index, and negative hyperbolic transformations of $\mathbb{R}^2$. (A negative hyperbolic transformation is the counterclockwise rotation in π composed with a positive hyperbolic transformation with zero index.) Then, using the condition that $\mu(P) = n$, we can redistribute the loop part ϕ among individual terms and write P as $\bigoplus P_i$, where P_i is either a counterclockwise rotation by $\theta \in (0, 2\pi)$ or a counterclockwise negative hyperbolic transformation. Clearly, each of the sequences $\mu\big(P_i^k\big)$ is increasing, and hence so is $\mu\big(P^k\big)$.

As a consequence of (4.3), there exists $r = 2^{\ell_0} \geq 1$ such that $\mu\big(\bar{x}^k\big) = n$ for $k \leq r$ but $\mu\big(\bar{x}^{2r}\big) > n$.

We claim that

$$\Phi(\varpi) = \bar{x}^{2^\ell} + \dots \tag{4.4}$$

as long as $\ell \leq \ell_0$, where the dots stand again for the terms of higher action. In particular, we can replace φ by φ^r to guarantee that (4.2) is satisfied.

To prove (4.4), arguing by induction, it is enough to show that (4.1) still holds for φ^2, i.e.,

$$\Phi(\varpi) = \bar{x}^2 + \ldots, \tag{4.5}$$

provided, of course, that it holds for φ and $\mu\big(\bar{x}^2\big) = n = \mu(\bar{x})$.

To establish (4.5), let us trace the image of ϖ through the diagram (3.10). We have

$$\begin{array}{ccc}
\varpi & \xrightarrow{\Phi} & \bar{x} + \ldots \\
\downarrow{\scriptstyle \mathcal{QS}} & & \downarrow{\scriptstyle \mathcal{PS}} \\
\mathrm{h}^{2n}\varpi & \xrightarrow{\Phi_{eq}} & \mathrm{h}^{2n}\big(\bar{x}^2 + R\big) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle F} \\
\mathrm{h}^{2n}\varpi & \xrightarrow{\Phi} & \mathrm{h}^{2n}\big(\bar{x}^2 + R'\big).
\end{array}$$

We emphasize that in the left column of the diagram, we have used, as in Step 1, the background assumption that $\mathcal{QS}(\varpi)$ is undeformed. Next, let us take a closer look at what R and R' are.

The remainder R in $\mathcal{PS}(\bar{x} + \ldots)$ is a sum of capped 2-periodic orbits of φ with action strictly greater than the action of $\bar{x}^2$ and possibly h-dependent coefficients. The condition that

$$\mathcal{PS}(\bar{x} + \ldots) = \Phi_{eq}\big(\mathrm{h}^{2n}\varpi\big) = \mathrm{h}^{2n}\Phi_{eq}(\varpi)$$

guarantees that $\mathcal{PS}(\bar{x} + \ldots)$ is divisible by h^{2n}. Let us write

$$R = \sum \bar{z}_j + O(\mathrm{h}),$$

where $\bar{z}_j$ are some capped 2-periodic orbits of φ with action strictly greater than the action of $\bar{x}^2$.

By Lemma 3.2.14, $F\big(\bar{x}^2\big) = \bar{x}^2 + O(\mathrm{h})$ and

$$R' = \sum \bar{z}_j + O(\mathrm{h}).$$

However, Φ is a non-equivariant continuation/PSS map and thus $\Phi(\varpi) \in \mathrm{CF}^*\left(\varphi^2\right)$. Therefore, $R' \in \mathrm{CF}^*\left(\varphi^2\right)$ since

$$\mathrm{h}^{2n}\big(\bar{x}^2 + R'\big) = \mathrm{h}^{2n}\Phi(\varpi).$$

This proves (4.5) and completes the proof of the theorem. □

Remark 4.1.1. *Returning to the regularity question (see Remark 3.2.9), note that, the general case of the theorem can also be reduced to the case considered above where all periodic orbits of φ have distinct action. Indeed, it is clear from the proof that it suffices to assume that φ is a pseudo-rotation up to a certain iteration order r, which is completely determined by the indices and the mean indices of 1-periodic orbits. Then the actions can be made distinct by an arbitrarily small perturbation of φ keeping it a pseudo-rotation up to arbitrarily large iteration order. (A somewhat similar type of perturbation is used, for instance, in [GG17, Sect. 3.2] in the proof of a Conley conjecture type result.)*

4.2 Proof of theorem 2.2.1

Throughout this section, we use the polynomial version $\mathrm{CF}(\varphi^2)[\mathrm{h}]$ of the equivariant complex $\mathrm{CF}_{eq}(\varphi^2)$; see Remark 3.2.12. This is essential for our proof of Theorem 4.2.1 below, since at some point in the argument we evaluate the elements of $\mathrm{CF}_{eq}(\varphi^2)$ at $\mathrm{h} = 1$.

4.2.1 Floer graphs

Let φ be a non-degenerate Hamiltonian diffeomorphism of a closed monotone symplectic manifold M. Consider the directed graph $\Gamma(\varphi)$ whose vertices are capped fixed points of φ, and two vertices $\bar{x}$ and $\bar{y}$ are connected by an arrow (from $\bar{x}$ to $\bar{y}$) if and only if $\mu(\bar{y}) = \mu(\bar{x}) + 1$ and there is an odd number of Floer trajectories from $\bar{x}$ to $\bar{y}$, i.e., $\langle d_{Fl}\bar{x}, \bar{y}\rangle = 1$. The length of an arrow is the difference of actions of $\bar{y}$ and $\bar{x}$. We call $\Gamma(\varphi)$ the *Floer graph* of φ.

When M is strictly monotone as is always assumed in this thesis, the group $\mathbb{Z}$ acts freely on $\Gamma(\varphi)$ by simultaneous recapping, preserving the arrow length. Sometimes it is convenient to consider the *reduced Floer graph* $\tilde{\Gamma}(\varphi) := \Gamma(\varphi)/\mathbb{Z}$. The length of an arrow in $\tilde{\Gamma}(\varphi)$ is still well-defined. Note that, unless M is symplectically aspherical, both $\Gamma(\varphi)$ and $\tilde{\Gamma}(\varphi)$ are infinite, but the latter has finitely many arrows. In particular, if $d_{Fl} \neq 0$, there exists a shortest arrow. Such an arrow might not be unique, although it is unique for a generic φ, but obviously all shortest arrows have the same length.

The *equivariant Floer graph* $\Gamma_{eq}(\varphi^2)$ of φ^2 is defined in a similar fashion. (We are assuming that φ^2 is non-degenerate, and hence φ is also non-degenerate.) Its vertices are capped two-periodic orbits of φ. The vertices $\bar{x}$ and $\bar{y}$ are connected by an arrow if and only if $\bar{y}$ enters $d_{eq}(\bar{x})$ with non-zero coefficient. In other words, now we do not require the index difference to be 1, and $\bar{x}$ and $\bar{y}$ are connected by an arrow if and only if $\bar{x}$ and $\mathrm{h}^m\bar{y}$, where $m = \mu(\bar{x}) - \mu(\bar{y}) + 1$, are connected by an odd number of equivariant Floer trajectories. The length of an arrow is again the difference of actions. As in the non-equivariant case, the *reduced equivariant Floer graph* $\tilde{\Gamma}_{eq}(\varphi^2) := \tilde{\Gamma}_{eq}(\varphi^2)/\mathbb{Z}$ has only finitely many arrows, and hence the shortest arrows exist.

We note that $\Gamma(\varphi^2)$ and $\Gamma_{eq}(\varphi^2)$ (and their reduced counterparts) have the same vertices. Furthermore, since $d_{eq} = d_{Fl} + O(\mathrm{h})$, every arrow in $\Gamma(\varphi^2)$ is also an arrow in $\Gamma_{eq}(\varphi^2)$, i.e., the equivariant Floer graph is obtained from its non-equivariant counterpart by adding arrows. Note that in the process the shortest arrow length can only get shorter or remain the same. Also, observe that there is a natural one-to-one map from the vertices of $\tilde{\Gamma}(\varphi)$ to the vertices of $\tilde{\Gamma}(\varphi^2)$ sending $\bar{x}$ to $\bar{x}^2$; likewise for un-reduced graphs. However, even when φ is 2-perfect, this map is not onto unless M is symplectically aspherical.

The following theorem relates the Floer graphs for φ and its second iterate φ^2.

Theorem 4.2.1. *Assume that φ is 2-perfect and φ^2 is non-degenerate. Then $\bar{x}$ and $\bar{y}$ are connected by one of the shortest arrows in $\Gamma(\varphi)$ if and only if $\bar{x}^2$ and $\bar{y}^2$ are connected by one of the shortest arrows in $\Gamma_{eq}(\varphi^2)$.*

This theorem is proved in Section 4.2.4 after we recall in Section 4.2.3 a few relevant facts about barcodes.

Remark 4.2.2 (The role of an almost complex structure)**.** *The Floer graph of φ depends on the choice of an almost complex structure J, and hence should rather be denoted by $\Gamma(\varphi, J)$. Likewise, the equivariant Floer graph depends on the parametrized almost complex structure. However, in both cases, the collection of shortest arrows is independent of this choice. This fact implicitly follows from Theorem 4.2.1 or can be proved directly* *by a continuation argument.*

Note also that Floer graphs are stable under small perturbations of φ and J. To be more precise, $\Gamma(\varphi, J) = \Gamma(\tilde{\varphi}, \tilde{J})$ whenever $\tilde{\varphi}$ is sufficiently close to φ and $\tilde{J}$ is close to J. The same is true in the equivariant setting.

4.2.2 Implications and the proof of Theorem 2.2.1

Theorem 4.2.1 shows that when φ is perfect, the shortest arrow (or, to be more precise, every shortest arrow) persists from φ to φ^2, although in the process it might move to the equivariant domain. This happens exactly when the difference of indices changes: $\mu(\bar{y}) - \mu(\bar{x}) = 1$ but $\mu\big(\bar{y}^2\big) - \mu\big(\bar{x}^2\big) \neq 1$. Moreover, in this case, we necessarily have $\mu\big(\bar{y}^2\big) - \mu\big(\bar{x}^2\big) < 1$. On the other hand, if the difference of indices remains equal to one, the orbits continue to be connected by one of the shortest non-equivariant arrows.

Denote by $\beta_{\min}(\varphi) = \mathcal{A}(\bar{y}) - \mathcal{A}(\bar{x})$ the length of a shortest arrow. As follows from Proposition 4.2.6, $\beta_{\min}(\varphi)$ is exactly equal to the shortest bar in the barcode of φ. Since every non-equivariant arrow for φ^2 is also an equivariant arrow, the shortest equivariant arrow length $\beta^{eq}_{\min}\big(\varphi^2\big)$ for φ^2 does not exceed $\beta_{\min}\big(\varphi^2\big)$, i.e.,

$$\beta^{eq}_{\min}\big(\varphi^2\big) \leq \beta_{\min}\big(\varphi^2\big).$$

In the setting of Theorem 4.2.1,

$$\beta^{eq}_{\min}\big(\varphi^2\big) = \mathcal{A}\big(\bar{y}^2\big) - \mathcal{A}\big(\bar{x}^2\big) = 2\beta_{\min}(\varphi).$$

We conclude that

$$2\beta_{\min}\big(\varphi^{2^k}\big) \leq \beta_{\min}\big(\varphi^{2^{k+1}}\big)$$

as long as the iterates of φ remain perfect and non-degenerate, and hence

$$2^k \beta_{\min}(\varphi) \leq \beta_{\min}\big(\varphi^{2^k}\big).$$

In particular, when φ is perfect, the longest finite bar $\beta(\varphi)$ (and even the shortest bar) in the barcode cannot be bounded from above for the iterates of φ. This proves Theorem 2.2.1.

Remark 4.2.3. *An interesting question that arises from Theorem 4.2.1 is if a shortest arrow could persist in the non-equivariant domain for all iterates* φ^{2^k}*, assuming that* φ *is perfect. As discussed above, this would be the case if and only if* $\mu\big(\bar{y}^{2^k}\big) - \mu\big(\bar{x}^{2^k}\big) = 1$ *for all* $k \in \mathbb{N}$*. Using a slightly simplified version of the index divisibility theorem from [GG18b] one can show that this is impossible when* φ *is replaced by a suitable iterate* φ^m*. (This is non-obvious.) Passing to an iterate is apparently essential because there exist pairs of strongly non-degenerate elements* A *and* B *in* $\widetilde{\mathrm{Sp}}(2n)$ *such that* $\mu\big(A^{2^k}\big) - \mu\big(B^{2^k}\big) = 1$ *for all* $k = 0, 1, 2, \ldots$*.*

4.2.3 A few words about the shortest bar

In this section we recall a few facts about persistent homology in the context of Hamiltonian Floer theory. All results discussed here are contained in, e.g., [UZ], although in some instances implicitly and usually in a much more general setting. A reader sufficiently familiar with the material can easily skip this section. There are, however, two points the reader might want to keep in mind. Namely, our emphasis here is on the shortest bar rather than the longest finite bar (aka the boundary depth) which is more frequently used in applications to dynamics. Secondly, our sign conventions are different from those in [UZ] due to the fact that we are working with Floer cohomology.

Consider the Floer complex $\mathcal{C} := \mathrm{CF}(\varphi)$ of a non-degenerate Hamiltonian diffeomorphism φ of a strictly monotone symplectic manifold, equipped with the standard action filtration. Clearly, $\mathcal{C}$ is a finite-dimensional vector space over Λ and the collection of 1-periodic orbits of φ with fixed capping forms a basis of $\mathcal{C}$.

A finite set of vectors $\xi_i \in \mathcal{C}$ is said to be *orthogonal* if for any collection of coefficients $\lambda_i \in \Lambda$ we have

$$\mathcal{A}\big(\sum \lambda_i \xi_i\big) = \min \mathcal{A}(\lambda_i \xi_i).$$

(Recall that with our conventions,

$$\mathcal{A}(\xi) := \min \mathcal{A}(\bar{x}_i) \text{ when } \xi = \sum \bar{x}_i;$$

see (3.2).) It is not hard to show that an orthogonal set is necessarily linearly independent over Λ.

Example 4.2.4. *Assume that all capped 1-periodic orbits of φ have distinct actions. Write $\xi_i = \bar{x}_i + \ldots$, where the dots stand for the orbits with action strictly greater than $\bar{x}_i$. Then it is easy to see that the set ξ_i is orthogonal if and only if the capped orbits $\bar{x}_i$ are distinct.*

Definition 4.2.5. *A basis $\mathcal{B} = \{\alpha_i, \eta_j, \gamma_j\}$ of $\mathcal{C}$ over Λ is said to be a* singular decomposition *if*

- $d_{Fl}\alpha_i = 0$,

- $d_{Fl}\eta_j = \gamma_j$,
- *$\mathcal{B}$ is orthogonal.*

It is shown in [UZ, Sections 2 and 3] that $\mathcal{C}$ admits a singular decomposition. For the sake of brevity we omit the proof of this fact. In what follows we will order the pairs (η_j, γ_j) so that

$$\mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1) \leq \mathcal{A}(\gamma_2) - \mathcal{A}(\eta_2) \leq \dots . \tag{4.6}$$

This increasing sequence is usually referred to as the *barcode* of φ (or to be more precise the collection of finite bars). The maximal entry in the sequence is called the *boundary depth* $\beta(\varphi)$, [Us]. The barcode is independent of the choice of a singular decomposition (see, e.g., [UZ]), but here we do not use this fact. Instead, we need the following characterization of the shortest bar $\beta_{\min} = \beta_{\min}(\varphi)$:

Proposition 4.2.6 ([UZ]). *Set*

$$\beta_{\min} := \mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1).$$

Then

$$\beta_{\min} = \inf \{\mathcal{A}(\bar{y}) - \mathcal{A}(\bar{x}) \mid \langle d_{Fl}\bar{x}, \bar{y}\rangle = 1\} \tag{4.7}$$

$$= \inf \{\mathcal{A}(d_{Fl}\xi) - \mathcal{A}(\xi) \mid \xi \in \mathcal{C}, \xi \neq 0\} . \tag{4.8}$$

Here, in the first equality, the infimum is taken over all capped 1-periodic orbits $\bar{x}$ and $\bar{y}$ such that $\bar{y}$ enters $d_{Fl}\bar{x}$ with non-zero coefficient and, in the second, over all non-zero $\xi \in \mathcal{C}$. In particular, $\beta_{\min}(\varphi)$ is the shortest arrow length in $\Gamma(\varphi)$.

Note that the infimums in (4.7) and (4.8) are actually attained and thus can be replaced by minima, and that the proposition can be thought of as an analogue for $\mathcal{C}$ of the Courant-Fischer minimax theorem giving a variational interpretation of the eigenvalues of a quadratic form. For the sake of completeness we include a proof of Proposition 4.2.6.

Proof. Let us denote the right-hand sides in (4.7) and (4.7) by $\beta'_{\min}$ and, respectively, $\beta''_{\min}$. We claim that $\beta'_{\min} = \beta''_{\min}$. Indeed, setting $\xi = \bar{x}$, in (4.8), it is easy to see that $\beta''_{\min} \leq \beta'_{\min}$. On the other hand, writing $\xi = \bar{x}_1 + \bar{x}_2 + \dots$ in the order of increasing action and $d_{Fl}\xi = \sum d_{Fl}\bar{x}_i = \bar{y} + \dots$, we observe that $\langle \bar{y}, d_{Fl}\bar{x}_i \rangle = 1$ for some i. Then

$$\begin{aligned}\mathcal{A}(d_{Fl}\xi) - \mathcal{A}(\xi) &= \mathcal{A}(\bar{y}) - \mathcal{A}(\bar{x}_1) \\ &\geq \mathcal{A}(\bar{y}) - \mathcal{A}(\bar{x}_i) \\ &\geq \beta'_{\min},\end{aligned}$$

and thus $\beta''_{\min} \geq \beta'_{\min}$.

Next, clearly, $\beta_{\min} \geq \beta''_{\min}$. Therefore, it remains to show that $\beta_{\min} \leq \beta''_{\min}$. To this end, let us decompose ξ in the basis $\mathcal{B}$ over Λ:

$$\xi = \sum \lambda_j \eta_j + \sum \lambda'_j \gamma_j + \sum \lambda''_i \alpha_i.$$

Then

$$d_{Fl}\xi = \sum \lambda_j \gamma_j.$$

By orthogonality,

$$\mathcal{A}(d_{Fl}\xi) = \min \mathcal{A}(\lambda_j \gamma_j) = \mathcal{A}(\lambda_k \gamma_k)$$

for some k, and, again by orthogonality,

$$\mathcal{A}(\xi) \leq \min \mathcal{A}(\lambda_j \eta_j) \leq \mathcal{A}(\lambda_k \eta_k).$$

Therefore,

$$\begin{aligned} \mathcal{A}(d_{Fl}\xi) - \mathcal{A}(\xi) &\geq \mathcal{A}(\lambda_k \gamma_k) - \mathcal{A}(\lambda_k \eta_k) \\ &= \mathcal{A}(\gamma_k) - \mathcal{A}(\eta_k) \\ &\geq \mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1) = \beta_{\min}. \end{aligned}$$

As a consequence, $\beta_{\min} \leq \beta''_{\min}$, which finishes the proof of the proposition. □

Remark 4.2.7. *In conclusion, we point out that all results in this section are purely algebraic and extend in a straightforward way to any un-graded finite-dimensional complex over Λ with an "action filtration" having expected properties; see [UZ].*

4.2.4 Proof of theorem 4.2.1

We begin by proving the theorem under the additional background assumption that *all actions and action differences for φ and φ^2 are distinct modulo the rationality constant λ_0*. Then, in the last step of the proof, we will show how to remove this extra assumption. Note that in particular this assumption guarantees that the shortest arrow is unique for $\Gamma(\varphi)$ and $\Gamma_{eq}(\varphi^2)$.

Remark 4.2.8. *It is worth pointing out that while this background assumption is satisfied C^∞-generically, it is not quite innocuous in the context of pseudo-rotations or* *perfect Hamiltonian diffeomorphisms. Indeed, in this case one can expect certain "resonance relations" between actions or actions and mean indices to hold; see [GK, GG09b].*

The proof is carried out in three steps.

Step 1: The shortest arrow for φ. In this step we simply apply the machinery from Section 4.2.3 to $\mathrm{CF}(\varphi)$. Let $\mathcal{B} = \{\alpha_i, \eta_j, \gamma_j\}$ be a singular decomposition for $\mathrm{CF}(\varphi)$ over Λ; see Definition 4.2.5. Due to the background assumption, the inequalities in (4.6) are strict:

$$\mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1) < \mathcal{A}(\gamma_2) - \mathcal{A}(\eta_2) < \dots. \tag{4.9}$$

Let us write

$$\gamma_1 = \bar{y}_* + \dots \text{ and } \eta_1 = \bar{x}_* + \dots,$$

where dots stand for higher action terms, and $\bar{x}_*$ and $\bar{y}_*$ are unique by the background assumption. Then, by definition,

$$\mathcal{A}(\gamma_1) = \mathcal{A}(\bar{y}_*) \text{ and } \mathcal{A}(\eta_1) = \mathcal{A}(\bar{x}_*),$$

and hence

$$\beta_{\min} := \mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1) = \mathcal{A}(\bar{y}_*) - \mathcal{A}(\bar{x}_*).$$

We claim that

$$\langle d_{Fl}\bar{x}_*, \bar{y}_*\rangle = 1. \tag{4.10}$$

Indeed, $\langle d_{Fl}\bar{x}, \bar{y}_*\rangle = 1$ for some $\bar{x}$ entering η_1. Then

$$\beta_{\min} = \mathcal{A}(\bar{y}_*) - \mathcal{A}(\bar{x}_*) \geq \mathcal{A}(\bar{y}_*) - \mathcal{A}(\bar{x}) \geq \beta_{\min}.$$

It follows that the first inequality is in fact an equality and $\bar{x} = \bar{x}_*$ due to the background assumption.

Therefore, by Proposition 4.2.6 and (4.10), $\bar{x}_*$ and $\bar{y}_*$ are connected by the shortest arrow in $\Gamma(\varphi)$.

Step 2: The shortest arrow for φ^2. In the previous step we have shown that $\bar{x}_*$ and $\bar{y}_*$ are connected by the shortest arrow in $\mathrm{CF}(\varphi)$. Our goal now is to prove the following key fact.

Lemma 4.2.9. *The iterated orbits $\bar{x}_*^2$ and $\bar{y}_*^2$ are connected by the shortest arrow in $\Gamma_{eq}(\varphi^2)$.*

Since under the background assumption the shortest arrows in $\tilde{\Gamma}(\varphi)$ and $\Gamma_{eq}(\varphi^2)$ are unique, this will establish the theorem.

Proof of Lemma 4.2.9. In the notation from Section 3.2, set

$$\hat{\alpha}_i = \wp(\alpha_i \otimes \alpha_i),$$

$$\hat{\eta}_j = \mathrm{h}\wp(\eta_j \otimes \eta_j) + \wp(\eta_j \otimes \gamma_j),$$

$$\hat{\gamma}_j = \wp(\gamma_j \otimes \gamma_j).$$

Then, by Seidel's non-vanishing theorem (Proposition 3.2.6),

$$\hat{\eta}_1 = \mathrm{h}^m \bar{x}_*^2 + \dots \text{ and } \hat{\gamma}_1 = \mathrm{h}^{m'} \bar{y}_*^2 + \dots$$

for some $m \geq 0$ and $m' \geq 0$, where the dots again stand for higher action terms.

Since $\wp$ is a chain map, i.e., $\wp \circ d_{\mathbb{Z}_2} = d_{eq} \circ \wp$, we have

$$d_{eq} \hat{\alpha}_i = 0$$

and

$$\begin{aligned} d_{eq}\hat{\eta}_j &= \mathrm{h}\wp(\gamma_j \otimes \eta_j) + \mathrm{h}\wp(\eta_j \otimes \gamma_j) \\ &\quad + \wp(\mathrm{h}\eta_j \otimes \gamma_j + \mathrm{h}\gamma_j \otimes \eta_j) \\ &\quad + \wp(\gamma_j \otimes \gamma_j) \\ &= \hat{\gamma}_j. \end{aligned}$$

This indicates that the collection $\hat{\mathcal{B}} := \{\hat{\alpha}_i, \hat{\eta}_j, \hat{\gamma}_j\}$ can be thought of as a singular decomposition of $\mathrm{CF}_{eq}\left(\varphi^2\right)$ with the minimal bar given by

$$\mathcal{A}(\hat{\gamma}_1) - \mathcal{A}(\hat{\eta}_1) = \mathcal{A}\big(\bar{y}^2_*\big) - \mathcal{A}\big(\bar{x}^2_*\big),$$

and, arguing similarly to Step 1, we should be able to show that $\bar{x}^2_*$ and $\bar{y}^2_*$ are connected by the shortest arrow. A minor technical difficulty that arises at this stage is that $\mathrm{CF}_{eq}\left(\varphi^2\right)$ does not fit in with the algebraic framework of Section 4.2.3 or [UZ]. Namely, $\mathrm{CF}_{eq}\left(\varphi^2\right)$ is not finite-dimensional over Λ; it is finite-dimensional over $\Lambda[\mathrm{h}]$, but the latter is not a field. We circumvent this difficulty by a trick which essentially amounts to setting $\mathrm{h} = 1$. (This is the point where our choice of working with polynomials in h rather than formal power series as in [Se] is essential; cf. Remark 3.2.12.)

Consider the ungraded complex $\tilde{\mathcal{C}}$ defined as follows: $\tilde{\mathcal{C}} := \mathrm{CF}\left(\varphi^2\right) \subset \mathrm{CF}_{eq}\left(\varphi^2\right)$ as a vector space over Λ with the differential $\tilde{d}\alpha := d_{eq}\alpha|_{\mathrm{h}=1}$ for $\alpha \in \tilde{\mathcal{C}}$. Since d_{eq} is h-linear, we have $\tilde{d}^2 = 0$. More formally, $\tilde{\mathcal{C}}$ is the quotient complex in the short exact sequence of ungraded complexes

$$0 \longrightarrow \mathrm{CF}_{eq}\left(\varphi^2\right) \xrightarrow{1+\mathrm{h}} \mathrm{CF}_{eq}\left(\varphi^2\right) \xrightarrow{\pi} \tilde{\mathcal{C}} \longrightarrow 0$$

over Λ, where π is the $\mathrm{h} = 1$ evaluation map.

Remark 4.2.10. *This exact sequence, for any action interval, gives rise to the exact triangle in Floer cohomology relating* $\mathrm{H}(\tilde{\mathcal{C}})$ *and* $\mathrm{HF}_{eq}\left(\varphi^2\right)$ *via multiplication by* $1+\mathrm{h}$. *As any map of the form* $id + O(\mathrm{h})$, *this multiplication map in Floer cohomology is one-to-one, and thus*

$$\mathrm{H}(\tilde{\mathcal{C}}) \cong \mathrm{HF}_{eq}\left(\varphi^2\right)/(1+\mathrm{h})\,\mathrm{HF}_{eq}\left(\varphi^2\right),$$

and hence $\dim_{\mathbb{F}_2} \mathrm{H}(\tilde{\mathcal{C}}) = \mathrm{rk}_{\mathbb{F}_2[\mathrm{h}]}\, \mathrm{HF}_{eq}\left(\varphi^2\right)$, *for any action interval. For global cohomology,* $\mathrm{H}(\tilde{\mathcal{C}}) \cong \mathrm{HF}\left(\varphi^2\right)$ *as ungraded* Λ*-modules by the continuation argument and Example 3.2.2.*

Since, by construction, $\tilde{\mathcal{C}}$ is a finite-dimensional vector space over Λ, now the machinery from [UZ] applies literally; see Remark 4.2.7. In self-explanatory notation,

$$\left\langle d_{eq}\bar{z}, \mathrm{h}^m \bar{z}'\right\rangle \neq 0 \text{ where } m = \mu(\bar{z}) - \mu(\bar{z}') + 1 \Longleftrightarrow \left\langle \tilde{d}\bar{z}, \bar{z}'\right\rangle \neq 0$$

for $\bar{z}$ and $\bar{z}'$ in $\bar{\mathcal{P}}_2(\varphi)$. Furthermore, we can also form the Floer graph for $\tilde{\mathcal{C}}$ and this graph is identical to the equivariant Floer graph $\Gamma_{eq}\left(\varphi^2\right)$.

Claim 4.2.11. *The subset* $\tilde{\mathcal{B}} := \pi(\hat{\mathcal{B}})$ *in* $\tilde{\mathcal{C}}$ *formed by* $\tilde{\alpha}_i := \pi(\hat{\alpha}_i)$ *and* $\tilde{\eta}_j := \pi(\hat{\eta}_j)$ *and* $\tilde{\gamma}_j := \pi(\hat{\gamma}_j)$ *is a singular decomposition for* $\tilde{\mathcal{C}}$.

Putting aside the proof of the claim, let us first show how Lemma 4.2.9 follows from it. Observe that

$$\mathcal{A}(\tilde{\gamma}_j) - \mathcal{A}(\tilde{\eta}_j) = 2\big(\mathcal{A}(\gamma_j) - \mathcal{A}(\eta_j)\big). \tag{4.11}$$

Indeed, set

$$\eta_j = \bar{x}_j + \dots,$$

$$\gamma_j = \bar{y}_j + \dots,$$

where as usual the dots stand for strictly higher action terms. (Thus $\bar{x}_* = \bar{x}_1$ and $\bar{y}_* = \bar{y}_1$.) By Seidel's non-vanishing theorem (Proposition 3.2.6), we have

$$\hat{\eta}_j = \mathrm{h}^{m_j} \bar{x}_j^2 + \dots,$$

$$\hat{\gamma}_j = \mathrm{h}^{m'_j} \bar{y}_j^2 + \dots$$

for some $m_j \geq 0$ and $m'_j \geq 0$, and hence

$$\tilde{\eta}_j = \bar{x}_j^2 + \dots,$$

$$\tilde{\gamma}_j = \bar{y}_j^2 + \dots.$$

Therefore,

$$\mathcal{A}(\tilde{\gamma}_j) - \mathcal{A}(\tilde{\eta}_j) = \mathcal{A}\big(\bar{y}_j^2\big) - \mathcal{A}\big(\bar{x}_j^2\big) = 2\big(\mathcal{A}(\bar{y}_j) - \mathcal{A}(\bar{x}_j)\big) = 2\big(\mathcal{A}(\gamma_j) - \mathcal{A}(\eta_j)\big),$$

which proves (4.11).

In particular, similarly to (4.9), we have

$$\mathcal{A}(\tilde{\gamma}_1) - \mathcal{A}(\tilde{\eta}_1) < \mathcal{A}(\tilde{\gamma}_2) - \mathcal{A}(\tilde{\eta}_2) < \dots.$$

Therefore,

$$\beta_{\min}(\tilde{\mathcal{C}}) := \mathcal{A}(\tilde{\gamma}_1) - \mathcal{A}(\tilde{\eta}_1) = \mathcal{A}\big(\bar{y}_*^2\big) - \mathcal{A}\big(\bar{x}_*^2\big)$$

is the shortest bar for $\tilde{\mathcal{C}}$. As in Step 1, we infer that

$$\left\langle \tilde{d}\bar{x}_*^2, \bar{y}_*^2 \right\rangle = 1.$$

Hence there is an arrow connecting these two orbits in the Floer graph for $\tilde{\mathcal{C}}$ and this is the shortest arrow. The Floer graph for $\tilde{\mathcal{C}}$ is defined similarly and in fact identical to the equivariant Floer graph $\Gamma_{eq}(\varphi^2)$. Therefore, this arrow is also the shortest arrow in $\Gamma_{eq}(\varphi^2)$, completing the proof of Lemma 4.2.9 modulo Claim 4.2.11.

Proof of Claim 4.2.11. Since π is a homomorphism of complexes, we have $\tilde{d}\tilde{\alpha}_i = 0$ and $\tilde{d}\tilde{\eta}_j = \tilde{\gamma}_j$. Therefore, we only need to show that $\tilde{\mathcal{B}}$ is an orthogonal basis. For this we do not need to distinguish between different types of elements of $\mathcal{B}$. Write $\mathcal{B} = \{\xi_i\}$, where $\xi_i = \bar{z}_i + \ldots$ with the dots denoting the entries of strictly higher action. Then, by the definition of $\hat{\mathcal{B}}$ and Seidel's non-vanishing theorem, $\tilde{\mathcal{B}} = \{\tilde{\xi}_i\}$ comprises the elements

$$\tilde{\xi}_i := \pi(\hat{\xi}_i) = \bar{z}_i^2 + \ldots.$$

Now, as in Example 4.2.4, the orthogonality for $\mathcal{B}$ is equivalent to that the orbits $\bar{z}_i$ are distinct. Similarly, the orthogonality for $\tilde{\mathcal{B}}$ is equivalent to that the orbits $\bar{z}_i^2$ are again distinct. It follows that $\tilde{\mathcal{B}}$ is orthogonal if (in fact, iff) $\mathcal{B}$ is orthogonal which is a part of its definition. As a consequence, $\tilde{\mathcal{B}}$ is linearly independent over Λ.

Finally, since $\tilde{\mathcal{C}} = \mathrm{CF}\left(\varphi^2\right)$ as Λ-modules and φ is 2-perfect, we have

$$\dim_\Lambda \tilde{\mathcal{C}} = \dim_\Lambda \mathrm{CF}\left(\varphi^2\right) = \dim_\Lambda \mathrm{CF}(\varphi) = |\mathcal{B}| = |\tilde{\mathcal{B}}|,$$

and $\tilde{\mathcal{B}}$ is a basis. $\square$

This concludes the proof of Lemma 4.2.9. □

Step 3: Removing the background assumption. Recall that the Floer graphs $\Gamma(\varphi)$ and $\Gamma_{eq}\big(\varphi^2\big)$ are stable under small perturbations of φ. With this in mind, we can replace φ by a C^∞-small perturbation $\tilde{\varphi}$ meeting the background assumption, since the latter is a C^∞-generic condition. More precisely, one can change the action of a single orbit by a small amount (positive or negative) using a localized C^∞-small perturbation $\tilde{\varphi}$. Hence, given any arrow in the Floer graphs $\tilde{\Gamma}(\varphi)$ and $\tilde{\Gamma}_{eq}\big(\varphi^2\big)$, pick some small $\epsilon > 0$. Then one can apply local perturbations at the two ends to shorten its length by 2ϵ while not changing the lengths of the remaining arrows more than ϵ. It follows that every shortest arrow in the Floer graphs $\tilde{\Gamma}(\varphi)$ and $\tilde{\Gamma}_{eq}\big(\varphi^2\big)$ can be perturbed into the unique shortest arrow. Now, Theorem 4.2.1 for φ follows from that theorem for $\tilde{\varphi}$. □

Remark 4.2.12 (The $\mathbb{Z}_p$-equivariant analogue)**.** *This argument extends with only very minor changes to the pth iterates φ^p, where p is a prime, proving the analogue of Theorem 4.2.1 for $\mathbb{Z}_p$-equivariant cohomology of φ^p over $\mathbb{F}_p$ and relying on the results from [ShZa]; cf. Remark 3.2.7. As a consequence, as in the proof of Theorem 2.2.1, if φ is strongly non-degenerate, β is* a priori *bounded from above and $|\mathcal{P}(\varphi)|$ is greater than the sum of Betti numbers of M over $\mathbb{Q}$, then there exists a simple p-periodic orbit for every sufficiently large prime p as is shown in [Sh19a].*

4.2.5 Degenerate case

Perhaps, the simplest way to extend our arguments and, in particular, Theorem 2.2.1 and Corollary 2.2.2 to include some degenerate Hamiltonian diffeomorphisms as in [Sh19a] is by bypassing Theorem 4.2.1 and using a somewhat less precise argument. Below we outline the key steps of this generalization, some of which again overlap with [Sh19a]. The account is deliberately brief. The main new point here is the construction of the (equivariant) Floer graph in the degenerate case.

Assume that φ is 2-perfect and that the second iteration is admissible: -1 is not an eigenvalue of $D\varphi_x$ for any $x \in \mathcal{P}_1(\varphi)$. (The latter requirement is satisfied once φ is replaced by its sufficiently high iterate φ^{2^k}.) Then, as shown in [GG10], for every $\bar{x} \in \bar{\mathcal{P}}_1(\varphi)$ we have a canonical isomorphism in local Floer cohomology:

$$\mathrm{HF}(\bar{x}) \xrightarrow{\cong} \mathrm{HF}\left(\bar{x}^2\right) \tag{4.12}$$

up to a shift of grading. By the Smith inequality in local Floer cohomology, which can be proved by exactly the same argument as in [Se] (see also [ÇG, Sh19a]), we have $\mathrm{HF}_{eq}\left(\bar{x}^2\right) \cong \mathrm{HF}\left(\bar{x}^2\right)[\mathrm{h}]$, where, strictly speaking, on the left we have the graded module associated with the h-adic filtration of $\mathrm{HF}_{eq}\left(\bar{x}^2\right)$. (We expect that in this situation $d_{eq} = d_{Fl}$, and hence $\mathrm{HF}_{eq}\left(\bar{x}^2\right) \cong \mathrm{HF}\left(\bar{x}^2\right)[\mathrm{h}]$ literally, without passing to graded modules, but we have not been able to prove this.)

For every $\bar{x} \in \bar{\mathcal{P}}_1(\varphi)$, fix a basis $\xi_{i,\bar{x}}$ in $\mathrm{HF}(\bar{x})$ so that this system of bases is recapping-invariant. Applying (4.12) to this system, we obtain bases $\xi'_{i,\bar{x}}$ in $\mathrm{HF}\left(\bar{x}^2\right)$ with $\bar{x} \in \bar{\mathcal{P}}_1(\varphi)$, and this system extends to a recapping-invariant system over the entire

$\bar{\mathcal{P}}_2(\varphi)$.

We also have a recapping-invariant system of bases in $\mathrm{HF}_{eq}\left(\bar{x}^2\right)$ arising from $\wp(\xi_{i,\bar{x}} \otimes \xi_{i,\bar{x}}) \in \mathrm{HF}_{eq}\left(\bar{x}^2\right)$. To be more precise, it is convenient to replace the equivariant cohomology (local or global) by the homology of the ungraded complex $\tilde{\mathcal{C}}$ obtained by setting $\mathrm{h} = 1$ as in the proof of Theorem 4.2.1. For the sake of brevity, we keep the notation HF_{eq} for this cohomology suppressing the projection π in the notation. Set $\xi^{eq}_{\bar{x},i} := \wp(\xi_{i,\bar{x}} \otimes \xi_{i,\bar{x}})$. We claim that this is a basis in $\mathrm{HF}_{eq}\left(\bar{x}^2\right)$ which is now just a vector space over $\mathbb{F}_2$. Then, extending, we get a recapping invariant family of bases over $\bar{\mathcal{P}}_2(\varphi)$.

To show that $\{\xi^{eq}_{\bar{x},i}\}$ is indeed a basis, we first recall that, without changing $D\varphi_x$ and the local cohomology, φ can be deformed near x to the direct product of degenerate and totally non-degenerate maps; see [GG10, Sect. 4.5]. This essentially reduces the question to the case, which for the sake of brevity we will focus on, where x is totally degenerate, i.e., all eigenvalues of $D\varphi_x$ are equal to 1 and in particular φ can be made C^1-close to the identity. Furthermore, recall that $\mathrm{HF}(\bar{x}) \cong \mathrm{HF}(\varphi_f) \cong \mathrm{HM}(f)$ by [Gi, Sect. 3.3 and 6], where HM stands for the local Morse cohomology, f is the generating function of φ and φ_f is the germ of the Hamiltonian diffeomorphism generated by f. These isomorphisms come from continuation maps and there are similar isomorphisms (equivariant and non-equivariant) for $\bar{x}^2$ and $\varphi_{2f} = \varphi_f^2$, where we can replace the generating function for φ^2 by $2f$; see [GG10, Sect. 4.3]. Now, as in Example

3.2.2 and Remark 4.2.10, we arrive at the continuation map identifications

$$\mathrm{HF}_{eq}\left(\bar{x}^2\right) \cong \mathrm{HF}\left(\bar{x}^2\right) \cong \mathrm{HF}(\bar{x}) \cong \mathrm{H}(Y_f), \tag{4.13}$$

where Y_f is a certain topological space (the Conley index) associated with the critical point x of f. Furthermore, the map $\alpha \mapsto \wp(\alpha \otimes \alpha)$ turns into the Steenrod square Sq on $\mathrm{H}(Y_f)$; see [Wi20]. Thus, with these identifications in mind, $\xi_{\bar{x},i} = \xi'_{\bar{x},i}$ and

$$\xi^{eq}_{\bar{x},i} = \mathrm{Sq}(\xi_{\bar{x},i}) = \xi_{\bar{x},i} + \ldots, \tag{4.14}$$

where the dots stand for the terms of *higher degree* in $\mathrm{H}(Y_f)$. It follows that the vectors $\xi^{eq}_{\bar{x},i}$ are linearly independent and, since $\dim_{\mathbb{F}_2} \mathrm{HF}_{eq}\left(\bar{x}^2\right) = \dim_{\mathbb{F}_2} \mathrm{HF}(\bar{x})$ by (4.13), this system is a basis.

The action filtration spectral sequence in Floer cohomology has $E_1 = \bigoplus_{\bar{x}} \mathrm{HF}(\bar{x})$ and converges to $\mathrm{HF}(\varphi)$. With bases fixed, we can canonically collapse this spectral sequence into one complex with the same features as the ordinary Floer complex including the action filtration and cohomology equal to $\mathrm{HF}(\varphi)$; cf. [GG19, Sect. 2.1.3 and 2.5]. This data is sufficient to define the Floer graph $\Gamma(\varphi)$ of φ with vertices $\xi_{\bar{x},i}$. (Note that the orbits with $\mathrm{HF}(\bar{x}) = 0$ do not contribute to $\Gamma(\varphi)$ and the graph depends on the choice of the bases $\{\xi_{\bar{x},i}\}$.) It is also worth keeping in mind that even in the non-degenerate case this graph and the complex might differ from the Floer graph as defined in Section 4.2 and from the Floer complex. However, they have the same formal properties as $\mathrm{CF}(\varphi)$ and the original graph, and the resulting homology is isomorphic to the Floer cohomology $\mathrm{HF}(\varphi)$; cf. [GG19].

A similar construction applies to φ^2 in the ordinary and equivariant settings and $\xi'_{\bar{x},i} \leftrightarrow \xi^{eq}_{\bar{x},i}$ gives rise to an action-preserving one-to-one correspondence between the vertices of $\Gamma(\varphi^2)$ and $\Gamma_{eq}(\varphi^2)$. The condition that the sum (2.1) with $\mathbb{F} = \mathbb{F}_2$ is strictly greater than the sum of Betti numbers guarantees that the graph $\Gamma(\varphi)$, and hence $\Gamma(\varphi^2)$ and $\Gamma_{eq}(\varphi^2)$, have at least one arrow.

Denote by $\beta_{\min}$ the length of the shortest arrows in a Floer graph. Our goal is to show that φ cannot be 2^k-perfect, where k is sufficiently large, assuming an *a priori* upper bound on $\beta_{\min}(\varphi^{2^k})$ as in Theorem 2.2.1. (Note that in contrast with the non-degenerate case the Floer graphs are now sensitive to small perturbations of φ and we usually cannot make the shortest arrow unique without changing the graph unless $\dim_{\mathbb{F}_2} \mathrm{HF}(x) = 1$ for all $x \in \mathcal{P}_1(\varphi)$.)

The equivariant pair-of-pants product $\wp$ extends to the complexes we have constructed, and Seidel's non-vanishing theorem takes the form

$$\wp(\xi_{\bar{x},i} \otimes \xi_{\bar{x},i}) = \xi^{eq}_{\bar{x},i} + \dots, \tag{4.15}$$

where now the dots stand for terms with action greater than or equal to the action of $\xi^{eq}_{\bar{x},i}$, but with the provision that the first term enters the whole sum with non-zero coefficient. (This is a consequence of (4.14) and Seidel's non-vanishing theorem applied to the non-degenerate part in the splitting of φ at x.)

Pick one of the shortest arrows, say v, in $\Gamma_{eq}(\varphi^2)$. After recapping, we can ensure that the beginning of v has the form $\xi^{eq}_{\bar{x},i}$. Using (4.15) and the facts that $\wp$ is a chain map and v is a shortest arrow, it is not hard to see that $\xi_{\bar{x},i}$ is the beginning of an arrow in $\Gamma(\varphi)$ whose length is at most $\beta^{eq}_{\min}(\varphi^2)/2$. Hence,

$$2\beta_{\min}(\varphi) \leq \beta^{eq}_{\min}(\varphi^2). \tag{4.16}$$

(This proves a somewhat weaker version of Theorem 4.2.1: every shortest equivariant arrow comes from an arrow for φ.)

On the other hand,

$$\beta^{eq}_{\min}(\varphi^2) \leq \beta_{\min}(\varphi^2). \tag{4.17}$$

Indeed, $\dim_{\mathbb{F}_2} \mathrm{HF}^I(\varphi^2) \geq \mathrm{rk}_{\mathbb{F}_2[\mathrm{h}]} \mathrm{HF}^I_{eq}(\varphi^2)$ for any action interval I, as is easy to see from the h-adic filtration spectral sequence. Applying this to an interval tightly enclosing one of the shortest arrows in $\Gamma(\varphi^2)$ we obtain (4.17). In fact, we expect that, as in the non-degenerate case, $\Gamma_{eq}(\varphi^2)$ incorporates all arrows of $\Gamma(\varphi^2)$ (and, perhaps, more). This is a stronger statement than (4.17), but (4.17) is sufficient for our purposes.

Combining (4.16) and (4.17), we see that $2\beta_{\min}(\varphi) \leq \beta_{\min}(\varphi^2)$. As a consequence, $\beta_{\min}(\varphi^{2^k}) \geq 2^k\beta_{\min}(\varphi)$ as long as φ is 2^k-perfect. When $\beta_{\min}(\varphi^{2^k})$ is bounded from above, this is impossible for large k.

We note in conclusion that in the non-degenerate case this proof reduces to an argument which does not rely on persistence homology and is ultimately simpler and more direct, although arguably less structured, than our proof of Theorem 2.2.1 via Theorem 4.2.1.

4.3 Proof of Theorem 2.3.1

We carry out the proof in three steps. In the first step, we show that if the sum $\sum \mu(\bar{x}_i^2)$ is non-zero, then the difference $d_i := \mu(\bar{x}_i^2) - 2\mu(\bar{x}_i)$ is constant in x_i and equal to ± 2. In the second step, we compute $\mathcal{PS}(\bar{x}_i)$ under the assumption that $d_i = 2$. Finally in the last step, we rule out the case $d_i = 2$ by showing that $\mathcal{PS}(\bar{x}_i)$ forces the equivariant continuation/PSS map Φ_{eq} to be not polynomial in h; see Remark 3.2.12. This completes the proof since now the case $d_i = -2$ can be ruled out by replacing φ by φ^{-1}.

Step 1. Since φ is a pseudo-rotation, the total index jump $\sum \mu(\bar{x}_i^2) - \sum \mu(\bar{x}_i)$ is divisible by $2N = 6$. Hence, since $\sum \mu(\bar{x}_i) = 0$, the sum $\sum d_i$ is also divisible by 6. Then using the bound $|d_i| \leq n = 2$ (see (3.6) or for a more detailed account [ÇGG19, Sect. 5]), we conclude that d_i is constant and equal to ± 2 unless $\sum \mu(\bar{x}_i^2) = 0$.

Step 2. We claim that, under the assumption $d_i = 2$, $\mathcal{PS}(\bar{x}_i) = \bar{x}_i^2$. Write $\mathcal{PS}(\bar{x}_i) = \mathrm{h}^{m_i}\bar{x}_i^2 + R_i$. It follows from (3.6) that $m_i = 0$. Then R_i is divisible by h, since all capped periodic orbits of φ^2 have distinct index. In particular, any capped orbit that enters R_i should have index and hence action strictly less than $\bar{x}_i^2$; see [GG18a, Thm. 3.1] and also [GG09a]. On the other hand, by Proposition 3.2.6, the iterated orbit $\bar{x}_i^2$ is the least action term in $\mathcal{PS}(\bar{x}_i)$. Hence $R_i = 0$ and $\mathcal{PS}(\bar{x}_i) = \bar{x}_i^2$.

Step 3. We compute $\Phi_{eq}(\alpha^2)$ where α is the generator of $\mathrm{HQ}^2(\mathbb{CP}^2)$. Note that $\mathcal{QS}(\mathbb{1}) = \mathbb{1}$, $\mathcal{QS}(\alpha) = \alpha^2 + \mathrm{h}^2\alpha$ and $\mathcal{QS}(\alpha^2) = \mathrm{q}\alpha + \mathrm{h}^2\mathrm{q}\mathbb{1} + \mathrm{h}^4\alpha^2$; see [Wi20, Sect. 6].

We have

$$\Phi_{eq}^{-1}(\mathcal{PS}(\bar{x}_3)) = \Phi_{eq}^{-1}(\bar{x}_3^2) = \mathrm{q}\alpha + \mathrm{h}^2\mathrm{q}\mathbb{1} + \mathrm{h}^4\alpha^2,$$

$$\Phi_{eq}^{-1}(\mathcal{PS}(\bar{x}_2)) = \Phi_{eq}^{-1}(\bar{x}_2^2) = \alpha^2 + \mathrm{h}^2\alpha,$$

$$\Phi_{eq}^{-1}(\mathcal{PS}(\bar{x}_1)) = \Phi_{eq}^{-1}(\bar{x}_1^2) = \mathbb{1}$$

by the top square of the diagram (3.10). Then, by linearity,

$$\Phi_{eq}^{-1}(\mathrm{h}^2\mathrm{q}^{-1}\bar{x}_3^2 + \bar{x}_2^2 + \mathrm{h}^4\bar{x}_1^2) = (\mathbb{1} + \mathrm{h}^6\mathrm{q}^{-1})\alpha^2,$$

which implies that $\Phi_{eq}(\alpha^2)$ is not polynomial in h; see Remark 3.2.12. □

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