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DOUBLE DISPERSION RELATIONS AND UNITARITY AS THE BASIS FOR A DYNAMICAL THEORY OF STRONG INTERACTIONS*

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I. INTRODUCTION

The discovery by Mandelstam in 1958 of a prescription for extending collision amplitudes into the complex plane, as functions simultaneously of energy and of momentum transfer,[1] has brought Heisenberg's 1943 idea of a dynamical S-matrix theory[2] close to realization. Heisenberg recognized the importance of the unitarity and Lorentz invariance of the S matrix, and understood the close relation between bound states and poles of the scattering amplitude. However, the role of the interaction (or force) in the S-matrix approach remained obscure until field theoretical considerations were applied.

In 1955, Chew and Low showed for the static model of the pion-nucleon interaction that one was dealing with an analytic function, that the "forces" could be associated with singularities of the scattering amplitude in unphysical regions, and that a knowledge of the location and strength of these singularities was probably sufficient to determine the S matrix.[3] Furthermore, these authors showed how crossing relations could be used to calculate the unphysical singularities. Of course the static

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* A series of lectures to be delivered in part at the Summer School of Theoretical Physics, Les Houches, July 1960; and in part at the Scottish Universities Summer School in Physics, Edinburgh, August 1960.

model is not Lorentz-invariant, and it fails to include many interactions which must be important. That Lorentz invariance was not a difficulty was suggested by the form of relativistic fixed-momentum transfer-dispersion relations, proposed in 1955 by Goldberger[4] and by Karplus and Ruderman,[5] to which the static Low equations were shown to bear a striking resemblance.[6] These "one-dimensional" relations, however, do not describe all the unphysical singularities and are insufficient to determine the S matrix. To include all the forces requires a knowledge of singularities in momentum transfer as well as energy; this information is provided by the double-dispersion relations, proposed two years ago by Mandelstam.

Actually, a generalization of Mandelstam's ideas to elements of the S matrix involving more than two particles is required before the theory can be regarded as complete. Such a generalization has not yet been achieved, but there is no reason to think that more than mathematical ingenuity is involved in treating the rapidly increasing number of degrees of freedom. From a practical standpoint the lack of generalization is not yet of major consequence, since the one- and two-particle S-matrix elements continue to saturate the theorists' capacity for calculation. However, three-particle states may soon become a center of attention.

Stated vaguely, the general principle emerging from the work of the past 5 years based on field theory is that the S matrix is the boundary value of an analytic function of momentum variables, satisfying the substitution law (to be discussed below) and with only those singularities required by unitarity. It is to be hoped that this notion of "maximal analyticity" will be made precise by future developments; at the moment, one can say that, given the requirements of Lorentz invariance

and unitarity, the only prescription yet discovered for extending two-particle S-matrix elements into the complex plane, consistent with the substitution law, is that given by Mandelstam. This simple fact is in striking contrast to the tortuous methods used in current attempts to derive the Mandelstam representation from principles of field theory. When one realizes that some of these "principles" (such as microcausality) can be expressed only through concepts (such as the local field) about which grave doubts have been raised, one wonders whether a field-theoretical starting point is worth the trouble. If the notion of "maximal analyticity" can be generally and precisely formulated, it would seem a promising candidate to replace the framework of field theory which has never been satisfactory.

Such a point of view has been expressed forcefully by Landau,[7] who feels that any further work with field theory is a waste of time. I concur in the belief that field theory is inconsistent and will eventually die, but am impressed by the many apparently valid general principles (such as the substitution law) that have been discovered by studying the dubious concept of the local field. In any case, in this series of lectures field theory is not used; we accept the Mandelstam representation as a starting point and investigate the consequences, hoping to make plausible the conclusion that analyticity and the substitution law, together with Lorentz invariance and unitarity, are sufficient to determine the S matrix.

Such a conclusion cannot be firm until (as stated above) we know how to handle general S-matrix elements, involving more than two particles. These elements are important even in discussing elastic scattering because, through unitarity, they determine the "strength" of important singularities in the elastic amplitude. How, then, can we expect to deduce any meaningful

consequences from an incomplete theory? The answer rests on two general features of the Mandelstam representation:

(a) The location of singularities is determined by the total "masses" of actual physical systems; the higher the mass the farther from the origin is the associated singularity. Now, among the strongly interacting particles there are none of zero mass;* thus, the total "mass" of strongly interacting physical systems systematically tends to increase with the number of particles, and the singularities near the origin tend to be determined by one- and two-particle configurations. If there are aspects of the physical problem that are controlled mainly by "near-by" singularities, then one can make a meaningful comparison of theory with experiment without a complete understanding of "faraway" singularities in which multiparticle configurations play a role.

(b) The "strength" of singularities is related to physical cross sections and restricted by unitarity, so that in a limited region of the complex plane the behavior of an S-matrix element tends to be controlled by the closest singularities. More precisely, an analytic function is determined through the Cauchy relations by a kind of Coulomb's law for a potential due to point charges (poles) and line charges (branch cuts).

* Note that problems involving large numbers of low-frequency virtual photons, such as Coulomb bound states or low-velocity Coulomb scattering, cannot be handled by the approach described in these lectures. Because of the zero mass of the photon, there is no separation of single-photon and multiphoton singularities. For high particle velocities, of course, the small magnitude of the fine-structure constant often makes it possible to neglect multiple photon contributions.

The line-charge "density" is the discontinuity across the cut, which we shall see is proportional to physical cross sections and therefore limited in magnitude. There is assurance therefore that the "Coulomb's law" reciprocal dependence on distance, which favors near-by singularities, will not be overwhelmed by an increasing strength of singularity with distance. From a practical standpoint, this feature of the S-matrix approach is of tremendous importance to a theory of strong interactions, permitting an orderly and systematic series of approximations whose validity is subject to realistic appraisal without any assumption as to the magnitudes of coupling constants.

We shall see in what follows that the range of a force in the conventional point of view corresponds to the reciprocal distance from the origin in the complex (momentum) plane of the associated singularity. Thus the "near-by" singularities, associated with one- and two-particle configurations, are the "long-range forces." The forces we cannot calculate reliably (but only put limits on) are those of short range. This way of assessing the situation suggests the two kinds of predictions we can expect to make with the incomplete theory:

(a) Scattering in states of large orbital angular momentum should be more or less completely predictable, since the centrifugal "barrier" shields these states from the unknown short-range forces. In other words, high-angular-momentum collisions are controlled by near-by singularities that our theory is able to handle.

(b) In states of low angular momentum, experience with potential scattering suggests that the short-range interaction, even

though complicated and exerting a significant influence, can be represented by a small number of parameters so far as low-energy experiments are concerned. Boundary-condition treatments of the hard core in the nuclear force are based on this circumstance, as are effective range formulas in general. The Coulomb potential analogy to our S-matrix problem in the complex plane suggests a general explanation. Any collection of source charges (singularities), if sufficiently distant, can be replaced by a single point charge (pole) at infinity, so far as the potential (scattering amplitude) in a local region is concerned. If one wishes to represent the first derivative of the potential, that is nonzero because of the finite distance of the actual charges, an equivalent point charge at a finite distance can be found. For higher derivatives, more poles or perhaps multipoles may be added, but it is clear that faraway singularities generally produce only smooth variations and can be represented by a small number of parameters. The near-by singularities, in contrast, may be expected to produce strong and characteristic variations in the amplitude that can be identified in experimental results. These strong variations are predictable in the incomplete theory.

The inverse relation between range of interaction and distance in momentum space is of course traceable to the uncertainty principle. The unphysical singularities of an elastic-scattering amplitude correspond to the systems that can be "exchanged" between the particles undergoing scattering. Only by such exchanges can a force be transmitted, and it is well known that according to the uncertainty principle the range of the

force is $\sim E^{-1}$, if E is the total energy necessary to create the exchanged system. The incomplete theory allows us to calculate forces due to one- and two-particle exchange, while three-particle and higher-multiplicity exchange must at this stage be treated phenomenologically. Let us consider some specific situations, remembering that the possible system to be exchanged must obey all the conservation laws of strong interactions.

1. Nucleon-nucleon scattering. Here the longest-range force (or the nearest unphysical singularity) comes from single pion exchange, while the next longest is due to two pions. Both of these are calculable, but forces of range shorter than one-third of a pion Compton wave length must await a generalization of the Mandelstam representation. Note that although we could attempt to calculate the force due to K-$\overline{K}$ exchange there is not much point in doing so because the mass of two kaons is as great as that of seven pions. If a hitherto undiscovered particle exists, of zero strangeness and mass less than three pions, then its contribution to the nuclear force should be experimentally identifiable.

2. Pion-pion scattering. Here all odd-pion exchanges are forbidden, so the longest-range force is due to pion pairs, and the incomplete theory carries us down to one-quarter of a pion Compton wave length in the force range.

3. Pion-nucleon scattering. Here there are two kinds of long-range forces. The "ordinary" forces arise from exchange of systems of zero baryon number, of which pion pairs are the least massive and pion quartets the first configuration that must be treated phenomenologically. (Odd-pion exchange is again forbidden.) However, a very important "baryon exchange" force also must be considered because of the large difference in

mass between pion and nucleon. That is to say, the original nucleon can "emit" a virtual nucleon, becoming a pion, with a violation of energy conservation that is determined not by the nucleon mass alone but rather (it turns out) by the geometric mean of nucleon and pion masses. This virtual nucleon moves across to the initial pion and is absorbed, transforming it into a nucleon and transmitting a force whose range is (by the uncertainty principle) approximately $(m_\pi\ m_N)^{-1/2}$, comparable to the range of 2π exchange. The incomplete theory can handle also the corresponding force when a single pion accompanies the exchanged nucleon, but more than one pion "fellow traveler" is beyond our powers at present. We shall see that the complicated nature of the pion-nucleon force is reflected in a complicated arrangement of singularities in the complex plane. By contrast, the singularities of the π-π and N-N amplitudes have an exceedingly simple structure.

It is clear that because of their small mass pions play a central role in this kind of approach to a theory of strong interactions. It will be impossible to go any distance without understanding the two-pion system, which occurs prominently not only in the long-range parts of the above-listed interactions but in many other processes as well. This circumstance alone would justify devoting much of our attention in these lectures to the two-pion configuration; another reason, however, is that, of all strongly interacting systems, the π-π is the easiest to handle, while at the same time it contains all the essential features of the general S-matrix approach. If one understands clearly for the π-π problem what can be done and what cannot be done in the Mandelstam framework, a satisfactory foundation for discussing all strong-interaction problems will have been laid.

One other general observation may be worth while before we plunge into the complex plane. This concerns what is meant by a "complete" dynamical theory. Ultimately, of course, we hope to have a theory that explains the masses and coupling constants of all particles as well as the symmetries that their interactions are observed to possess. However, the S-matrix approach to be described here makes no attempt to explain symmetries (charge independence, parity, etc.); these are accepted and imposed from the beginning as conditions to be satisfied by all matrix elements. In this respect we are on the same footing as conventional field theory. On the other hand, the question as to how many particles are to be regarded as "elementary" is not so clear. The deuteron is certainly not elementary--its existence is to be explained by showing that a pole of definite position and residue must occur in that N-N amplitude which has the quantum numbers of the deuteron, if we start with a knowledge of certain other singularities. Within the framework of approach outlined above, based on an ordering of masses, the pion, the nucleon, and the kaon have to be accepted as elementary, since by a substantial margin they are the least massive systems with strangeness and baryon quantum numbers (0, 0), (0, 1), and (1, 0) respectively. It is possible, however, that in the future an increase in our ability to handle multiparticle systems of high mass may lead to the conclusion that the poles corresponding to one or more of the above particles have residues (coupling constants) or positions (masses)--or both--that cannot be arbitrarily assigned but are in fact controlled by distant singularities. Even within our present approach it is possible that some of the hyperons may emerge as "bound states;" we shall discuss below possible criteria for distinguishing bound states from elementary particles within the S-matrix framework.

Though a certain number of "elementary" particle masses may be accepted as necessary input information, there is still the question how many additional arbitrary parameters are allowed by the combined requirements of Lorentz invariance, unitarity, and analyticity. Mandelstam has conjectured that this number is the same as the number of renormalizable interactions in conventional field theory.[7] We shall be able to verify in several special cases that the number is not larger; it seems, therefore, that the S-matrix approach is at least as "complete" as conventional field theory is supposed to be. Of course, with the current limitation to one- and two-body configurations, the phenomenological representation of short-range forces requires extra parameters.

II. THE LORENTZ-INVARIANT AMPLITUDE AND THE SUBSTITUTION LAW

For reasons of convenience, Möller in his 1945 paper[8] introduced a matrix S-1 and then in addition factored out an energy-momentum delta function, leaving a Lorentz-invariant function of $n-1$ momentum variables, where n is the total number of particles involved (ingoing plus outgoing). This reduced matrix has been called by various names; we shall be mainly concerned with the case $n = 4$, where the reduced matrix element will simply be referred to as the "invariant amplitude." The normalization of the invariant amplitude has not been standardized; we shall choose it as close as possible to the "physical amplitude" $f(\theta)$, which is defined, except for a phase, by the barycentric-system differential cross-section formula

$$\frac{d\sigma}{d\Omega} = \frac{q_f}{q_i} \left| f(\theta) \right|^2 . \qquad \text{(II.1)}$$

Here q_f and q_i are the final and initial magnitudes of three momenta and θ the angle between; q_i and q_f are of course equal for elastic scattering. The phase of $f(\theta)$ will be conventional, i.e., $f(\theta)$ becomes real as the interaction becomes weak, positive for attraction and negative for repulsion in the elastic case. More precisely, for elastic scattering of particles with zero spin,

$$f(\theta) = \frac{1}{q} \sum_{\ell=0}^{\infty} (2\ell + 1)\, e^{i\delta_\ell} \sin \delta_\ell \, P_\ell(\cos \theta) , \qquad \text{(II.2)}$$

where δ_ℓ is the phase shift in the state of orbital angular momentum ℓ. Möller showed that either for inelastic or elastic scattering the factor connecting f to the invariant amplitude for zero spin is simply W, the

total energy in the barycentric system.[8] Thus we normalize our invariant amplitude A by the formula

$$A = \frac{W}{2} f \, . \tag{II.3}$$

The invariant-cross-section formula (for an arbitrary coordinate system) in terms of $|A|^2$ may be found in Moller's article, as well as in many textbooks. All we need here is the knowledge that A is Lorentz-invariant.

With zero spin, A can depend only on the invariants that may be formed from the three independent four-momenta remaining after energy-momentum conservation is applied. To maintain a maximum symmetry let us assign four-momenta p_1, p_2, p_3, p_4, all of which correspond formally to ingoing particles. Two of these momenta will always be positive timelike, representing the actual ingoing particles, while the other two are negative timelike and represent the actual outgoing antiparticles. Energy-momentum conservation is stated through the condition

$$p_1 + p_2 + p_3 + p_4 = 0 \, , \tag{II.4}$$

while the particle masses are introduced through the four constraints

$$p_i^2 = m_i^2 \, . \tag{II.5}$$

It is convenient to define three invariants

$$\begin{aligned} s_1 &= (p_1 + p_4)^2 = (p_2 + p_3)^2 \, , \\ s_2 &= (p_2 + p_4)^2 = (p_1 + p_3)^2 \, , \\ s_3 &= (p_3 + p_4)^2 = (p_1 + p_2)^2 \, , \end{aligned} \tag{II.6}$$

each of which is the square of the total energy in the barycentric system for a particular pairing of incoming and outgoing particles. With the constraints (II.4) and (II.5), s_1, s_2, and s_3 are not independent of one another but satisfy the relation

$$s_1 + s_2 + s_3 = m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \qquad \text{(II.7)}$$

Thus, any two of the s variables are to be considered as independent, with the third determined by (II.7). We now assert that our invariant amplitude A is a function only of the two independent s variables.

It is trivial to verify that no further independent scalars can be formed from $p_1 \ldots p_4$. That there are two and only two could have been anticipated by realizing that in the barycentric system the scattering depends on energy and angle and nothing more. What is not trivial, however, is to say that A cannot depend on which of the four-vectors p_i is positive timelike and which negative, i.e., on which particles are incoming and which outgoing. A Lorentz transformation cannot interchange positive and negative timelike vectors, so we are going beyond Lorentz invariance; we are invoking the notion of TCP invariance.

Consider the six reactions represented by Fig. 1. We can classify these by pairing the particles--two incoming and two outgoing---to define three "channels." Channel I is that for which s_1 is the square of the total energy in the barycentric system, pairing p_1 with p_4 and p_2 with p_3. It describes the reaction

$$\pi^- + \bar{\Lambda} \rightarrow K_0 + \bar{p} \ ,$$

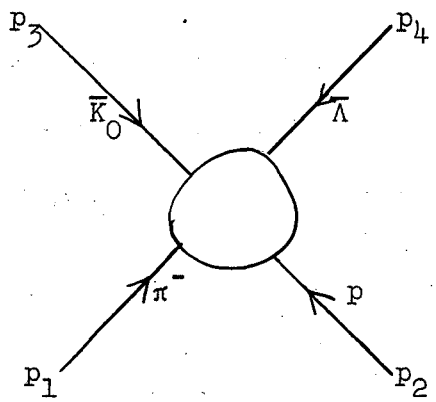

Fig. 1. Diagram describing the reactions

$$\text{I.}\quad \bar{K}_0 \rightarrow p \rightarrow \pi^+ + \Lambda\ ,$$

$$\text{II.}\quad \pi^- + \bar{K}_0 \rightarrow \Lambda + \bar{p}\ ,$$

$$\text{III.}\quad \pi^- + p \rightarrow K_0 + \Lambda\ ,$$

as well as the corresponding antiparticle reactions.

as well as the TCP equivalent antiparticle reaction*

$$\bar{K}_0 + p \rightarrow \pi^+ + \Lambda .$$

Reactions with π^- ingoing have p_{10} positive, while those with π^+ outgoing have p_{10} negative. The signs of the energy components of the other four momenta obey a similar rule.

Thus Channel I is characterized simply by the fact that s_1 is positive and greater than some "threshold" value. Channel II is that for which s_2 is greater than some positive threshold, and includes the two reactions corresponding to the pairing (2, 4) and (1, 3), while Channel III makes the final pairing (3, 4) and (1, 2) and of course has s_3 above threshold. It is possible to verify that the physical regions of the s variables for the different channels are nonoverlapping. Consider the simple case in which all four masses are equal, e.g., the diagram of Fig. 2, which includes the three basic reactions, $n + p \rightarrow n + p$, $n + \bar{p} \rightarrow n + \bar{p}$, and $p + \bar{p} \rightarrow n + \bar{n}$. If q and θ are the barycentric system three-momenta and angle for neutron-proton scattering, then

$$\begin{aligned} s_1 &= -2q^2(1 + \cos\theta) , \\ s_2 &= -2q^2(1 - \cos\theta) , \\ s_3 &= 4(q^2 + M^2) , \end{aligned} \qquad \text{(II.8)}$$

* Because strong interactions have special symmetries, time reversal, charge conjugation, and charge independence, a single invariant amplitude actually can describe many more than the two TCP equivalent reactions. It is confusing, however, to invoke these additional symmetries before understanding the general features of the S-matrix approach.

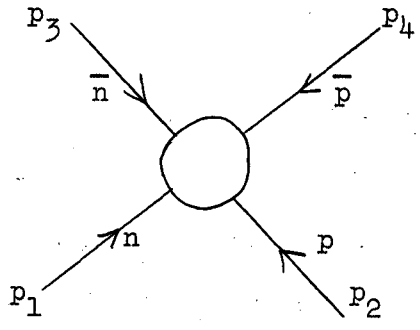

Fig. 2. Diagram for nucleon-nucleon and nucleon-antinucleon scattering.

$$\text{I.} \quad n + \bar{p} \rightarrow n + \bar{p} \,,$$

$$\text{II.} \quad p + \bar{p} \rightarrow n + \bar{n} \,,$$

$$\text{III.} \quad n + p \rightarrow n + p \,.$$

and we see that in the physical region of Channel III ($n + p \to n + p$), s_3 is positive and greater than $4M^2$, while s_1 and s_2 are both negative. Obviously, in the physical region for Channel II ($p + \bar{p} \to n + \bar{n}$), s_2 is greater than $4M^2$ while s_1 and s_3 are negative; for Channel I ($n + \bar{p} \to n + \bar{p}$) the positive variable is s_1. In general those two s variables that for a particular channel are not the square of the total energy may be interpreted as the negative squares of momentum transfer and have physical ranges that extend to minus infinity.

We may now state the substitution law, which in the present framework takes a very simple form. We postulate that a single analytic function of two variables, $A(s_1, s_2)$, describes all three channels corresponding to a given diagram, the channel being selected merely by assigning the values of the variables. The key word here is "analytic." Since the ranges for the three channels are nonoverlapping, one must have a procedure of continuation to give such a postulate any physical content. In the Feynman-diagram approach to field theory, the substitution law is a direct consequence of the Feynman rules and has a clear meaning for any diagram whose singularities have been analyzed. All diagrams analyzed to date satisfy the Mandelstam representation, which postulates a singularity structure permitting analytic continuation between the three physical regions. The substitution law has such a simple and plausible appearance that its power for many years was not appreciated, but it is now recognized as playing a key role in the dynamical S-matrix approach.

A principle related to the substitution law, which applies when there are two or more identical particles among the four involved in a particular diagram, is the so-called "crossing symmetry." Exchanging two

identical particles at most changes the sign of the amplitude, and such an interchange means switching two of the s variables, leaving the third alone. For example, suppose particles 1 and 3 are identical. Then, depending on whether these are bosons or fermions, the amplitude is either symmetric or antisymmetric under exchange of p_1 and p_3, which means interchanging s_1 and s_3, leaving s_2 alone. (Note that such an exchange is consistent with the constraint (II.7).) If 1 and 3 are both incoming or both outgoing--i.e., $\sqrt{s_2}$ is the energy--the symmetry in question is just the Pauli principle. If one is incoming and the other outgoing, however, the symmetry cannot be so identified. In this case, if one starts with physical values of the s variables, the exchange in question necessarily leads to nonphysical values because of the above-mentioned nonoverlapping nature of the energy and momentum-transfer ranges. Thus crossing symmetry has a general meaning only when continuation of the amplitude into unphysical regions is possible. Such a continuation is of course allowed by the Mandelstam representation.

III. THE MANDELSTAM REPRESENTATION

It is clear that the three s variables are on an equivalent footing, even though only two are independent, and the Mandelstam prescription for analytic continuation is most simply expressed by a representation exhibiting all three. Mandelstam postulates that, except for possible subtractions needed if the integrals do not converge, the invariant amplitude A can be written in the form

$$A(s_1, s_2, s_3) = \frac{1}{\pi} \int ds_1' \frac{\rho_1(s_1')}{s_1' - s_1} + \frac{1}{\pi} \int ds_2' \frac{\rho_2(s_2')}{s_2' - s_2}$$

$$+ \frac{1}{\pi} \int ds_3' \frac{\rho_3(s_3')}{s_3' - s_3} + \frac{1}{\pi^2} \iint ds_1' ds_2' \frac{\rho_{12}(s_1', s_2')}{(s_1' - s_1)(s_2' - s_2)}$$

$$+ \frac{1}{\pi^2} \iint ds_1' ds_3' \frac{\rho_{13}(s_1', s_3')}{(s_1' - s_1)(s_3' - s_3)}$$

$$+ \frac{1}{\pi^2} \iint ds_2' ds_3' \frac{\rho_{23}(s_2', s_3')}{(s_2' - s_2)(s_3' - s_3)} , \qquad \text{(III.1)}$$

where the spectral functions ρ_i and ρ_{ij} are real and the integrations in each s' variable go over a region of the positive real axis extending to infinity. More precisely, the spectral functions fail to vanish only when an argument is equal to the square of the mass of an actual physical system that has the quantum numbers of the corresponding channel.

For an illustration, let us refer to Fig. 3, which describes reactions with two nucleons and two pions. Channel I has charge +1, baryon number +1, and zero strangeness. The lightest system with these quantum numbers is the proton with mass M; the next lightest are (π^o, p) and (π^+, n), with a range of masses starting from $M + 1$ * and extending to infinity. More massive systems, containing 3, 4, ... etc. particles, fall in this range, so we conclude that the spectrum $\rho_1(s_1')$ has a "line" at $s_1' = M^2$ and a "continuum" for $(M + 1)^2 < s_1' < \infty$. The "line" obviously leads to a pole in (III.1) of the form $R_1(M^2 - s_1)^{-1}$, and once this is removed the lower limit on the ds_1' integration is equal to $(M + 1)^2$.

For Channel II there is no pole because we know of no single particle that has the same quantum numbers as two pions, i.e., zero baryon number, zero strangeness, and parity $(-1)^J$. The continuum starts here with the two-pion system at $s_2' = 4$. Channel III is similar to Channel I; in fact when the notion of charge independence is introduced we shall be able to relate these two channels by crossing symmetry.

The one-dimensional integrals in (III.1) correspond to Feynman diagrams in which at some point a single-particle line joins the initial and final configurations. That is, the integral containing ρ_1 represents

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* We shall use the pion mass as a unit, neglecting the difference between the masses of charged and neutral pions. We also take $\hbar = c = 1$.

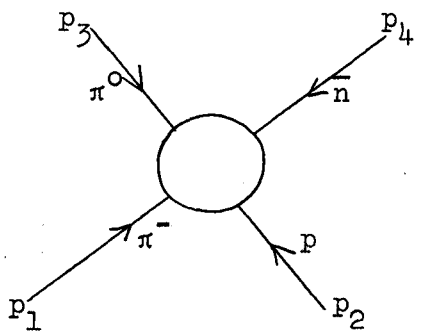

Fig. 3. Diagram for the three channels:

$$\text{I.} \quad \pi^o + p \rightarrow \pi^+ + n \,,$$

$$\text{II.} \quad \pi^- + \pi^o \rightarrow n + \bar{p} \,,$$

$$\text{III.} \quad \pi^- + p \rightarrow \pi^o + n \,.$$

all diagrams of the type of Fig. 4,

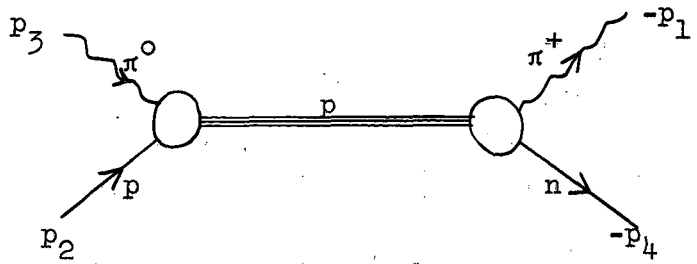

Fig. 4.

which, if spins are ignored, sum up to

$$F^p_{\pi^o p}(s_1)\ S^p(s_1)\ F^p_{\pi^+ n}(s_1)\ , \qquad \text{(III.2)}$$

where the F's are vertex functions on the mass shell for the two particles indicated in the subscripts but considered as a function of the square of the mass of the intermediate proton, and S^p is the renormalized proton propagator. It is well known that $S^p(s_1)$ has a pole of unit residue at $s_1 = M^2$, while the vertex functions are analytic in the neighborhood of this point. Thus the residue of the corresponding pole in our invariant scattering amplitude is

$$R_1 = F^p_{\pi^o p}(M^2)\ F^p_{\pi^+ n}(M^2) = g_{0p}\, g_c\ , \qquad \text{(III.3)}$$

since the renormalized coupling constants g are defined as the value of

the appropriate pion-nucleon vertex function with all three particles on the mass shell. It is evident that these considerations apply to any pole that may occur, always leading to the residue as a product of two vertex functions each on the mass shell.

It is possible to deduce the above recipe for relating poles in scattering amplitudes to coupling constants without reference to Feynman diagrams, but the machinery is cumbersome. Particularly when spin is present it is a great convenience to know that the Feynman rules, applied to diagrams of lowest order, lead to the correct connection between residues and conventional coupling constants. This fact of course explains why perturbation theory, blindly applied, occasionally gives sensible answers even in strong-interaction problems. Poles dominate the behavior of the scattering amplitude in their immediate neighborhood, so if one happens to be discussing experiments close to a pole, the lowest-order perturbation formula may be reliable.

In addition to motivating the pole, Formula (III.2) also shows the origin of the continuum part of $\rho_1(s_1')$ in the well-known branch cuts extending from $(M + 1)^2$ to infinity in the vertex and propagator functions. The double integrals in (III.1) arise, however, from Feynman diagrams in which two or more particles are always present in intermediate configurations for <u>all three</u> channels. The fourth-order box diagram of Fig. 5 is a simple example. Mandelstam has analyzed the fourth-order diagrams[1] to show that they satisfy the double-integral representation, and many higher-order diagrams have by now been similarly analyzed by others.[9] It turns out that for certain mass ratios one encounters so-called "anomalous thresholds," that is, lower limits on the double integrals not determined simply by the masses of possible intermediate states. However,

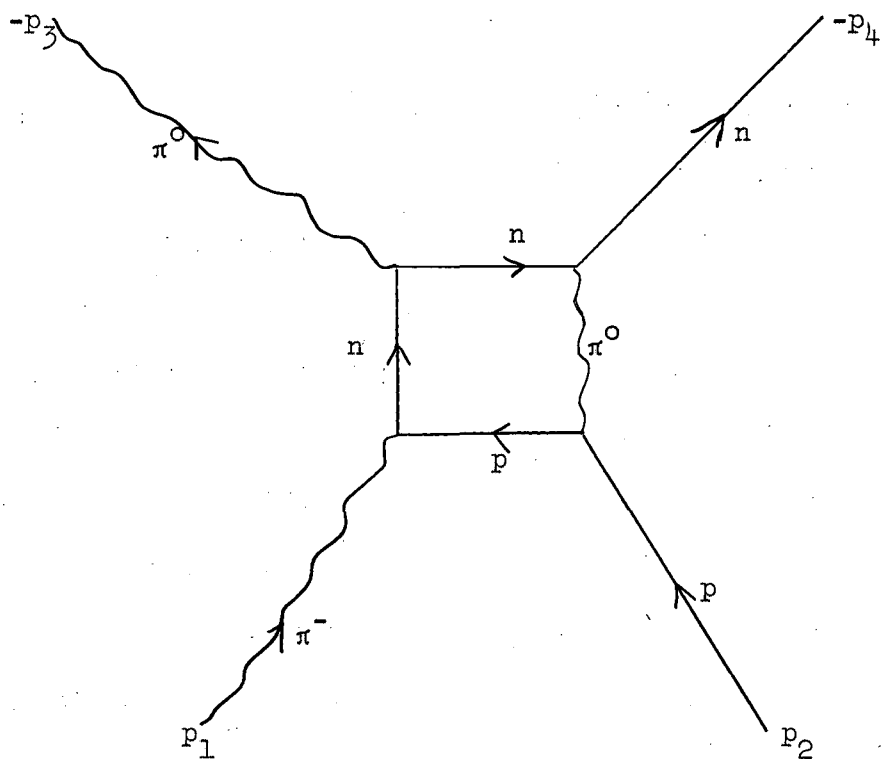

Fig. 5. A fourth-order "box" diagram.

all the problems we shall discuss in these lectures have normal thresholds, and the anomalous cases in any event do not seem to present an essential difficulty. Techniques for handling them are already being developed.[10]

It turns out that constraints exist on the possible masses that can occur in intermediate configurations for one channel, given the mass of the intermediate configuration for another channel. In other words, the region in which a double spectral function, say $\rho_{12}(s_1', s_2')$, fails to vanish is not rectangular but bounded by curves asymptotic to the square of the lowest mass of a multiparticle system with the appropriate quantum numbers. For our pion-nucleon example of Fig. 3, these asymptotes have been shown above to be $s_1' = (M + 1)^2$ and $s_2' = 4$. The detailed forms of the boundary curves have been calculated by Mandelstam both on the basis of Feynman diagrams and by consistency considerations involving unitarity.[1] In general these curves can be correctly obtained from an analysis of the lowest order diagrams contributing to the double spectral functions.

For a variety of reasons it is useful to exhibit the connection between the Mandelstam representation (III.1) and one-dimensional dispersion relations. First observe that in the physical region for Channel I, say, the only denominators that vanish in Formula (III.1) are those containing the factor $s_1' - s_1$. Remembering

$$\frac{1}{s_1' - s_1} = P \frac{1}{s_1' - s_1} + i\pi\,\delta(s_1' - s_1)\,, \qquad \text{(III.4)}$$

we can then easily calculate the imaginary part of A in this region to be

$$A_1(s_1, s_2, s_3) = \rho_1(s_1) + \frac{1}{\pi}\int ds_2' \frac{\rho_{12}(s_1, s_2')}{s_2' - s_2}$$

$$+ \frac{1}{\pi}\int ds_3' \frac{\rho_{13}(s_1, s_3')}{s_3' - s_3} \quad . \tag{III.5}$$

Outside this region we shall <u>define</u> $A_1(s_1, s_2, s_3)$ by Formula (III.5), and always refer to it as the absorptive part for Channel I even though it will itself become complex when one of the remaining denominators in (III.5) vanishes. Note that for fixed s_1, A_1 is an analytic function of s_2 (or s_3) with s_3 (or s_2) determined by (II.7).

In a similar way we can define A_2 and A_3 to be the absorptive parts for Channels II and III, respectively. It is then simply a matter of algebra to verify that (III.1) can be written in three possible ways in terms of A_1, A_2, and A_3:

$$A(s_1, s_2, s_3) = \frac{1}{\pi}\int ds_1' \frac{\rho_1(s_1')}{s_1' - s_1} + \frac{1}{\pi}\int ds_2' \frac{A_2(s_1, s_2', \Sigma m^2 - s_1 - s_2')}{s_2' - s_2}$$

$$+ \frac{1}{\pi}\int ds_3' \frac{A_3(s_1, \Sigma m^2 - s_1 - s_3', s_3')}{s_3' - s_3} , \tag{III.6a}$$

(Equation continued)

$$A(s_1, s_2, s_3) = \frac{1}{\pi} \int ds_2' \frac{\rho_2(s_2')}{s_2' - s_2} + \frac{1}{\pi} \int ds_1' \frac{A_1(s_1', s_2, \Sigma m^2 - s_1' - s_2)}{s_1' - s_1}$$

$$+ \frac{1}{\pi} \int ds_3' \frac{A_3(\Sigma m^2 - s_2 - s_3', s_2, s_3')}{s_3' - s_3} , \qquad \text{(III.6b)}$$

$$= \frac{1}{\pi} \int ds_3' \frac{\rho_3(s_3')}{s_3' - s_3} + \frac{1}{\pi} \int ds_1' \frac{A_1(s_1', \Sigma m^2 - s_1' - s_3, s_3)}{s_1' - s_1}$$

$$+ \frac{1}{\pi} \int ds_2' \frac{A_2(\Sigma m^2 - s_2' - s_3, s_2', s_3)}{s_2' - s_2} . \qquad \text{(III.6c)}$$

The first form is usually called the one-dimensional dispersion relation for fixed s_1, the second for fixed s_2, and the third for fixed s_3. In our pion-nucleon example the form that has received most attention is that for s_2 fixed at a negative value in its momentum-transfer range, where the two absorptive parts that occur both correspond to pion-nucleon scattering in a more or less physical region. In what follows, however, we shall often find it necessary to put the fixed variable into its positive (energy) range; the absorptive parts then are nonphysical and have a meaning only through (III.5).

IV. GENERALIZATION TO INCLUDE CHARGE AND SPIN

The possibility of degrees of freedom of charge and spin has so far been ignored. However, internal degrees of freedom may always be absorbed into invariant matrices, whose coefficients are invariant functions of the s variables only. The number of such functions depends on the complexity of the internal degrees of freedom, and generally the vector addition rule can be used in counting. For example, the pion has zero spin and isotopic spin 1, while the nucleon has spin $1/2$ and isotopic spin $1/2$. The different possible total I-spin values for a 2π system are 0, 1, and 2, so that we expect three independent invariant π-π amplitudes. The possible total I-spin values for a π-N system are $1/2$ and $3/2$, but in addition the nucleon spin can combine in two ways with any given orbital angular momentum ℓ to form $J = \ell + 1/2$ or $J = \ell - 1/2$. Thus there are four independent invariant π-N amplitudes. The N-N system is even more complicated; if one counts carefully here, the result is ten.

A proper choice of invariant spin matrices leads to invariant amplitudes with exactly the same singularity structure as expressed by Eq. (III.1) for the zero-spin case. The choice of charge matrices is obviously irrelevant to analyticity properties, although certain choices may be more convenient than others for calculation. To obtain correct spin matrices, the current procedure is to guess the answer and then check. This has worked so far in all cases involving particles of spin zero and spin $1/2$, as well as photons. The procedure has been described in some detail for the nontrivial N-N case by Goldberger, Grisaru, McDowell, and Wong.[11] We shall not delve into the spin problem here but simply state for illustration the well-known charge-spin result for the π-N system.[6]

For a diagram of the type of Fig. 3, rather than specifying the charge let us label the pion lines each by the conventional isotopic vector index that takes on the values 1, 2, 3. With the pion of momentum p_1 associate the index α, and with p_3 the index β. Nucleon-charge degrees of freedom may be suppressed into initial and final isospinors, and the problem is then to form two charge-independent combinations of nucleon isotopic-spin operators, τ_k. A possible choice is $\frac{1}{2}[\tau_\beta \tau_\alpha + \tau_\alpha \tau_\beta] = \delta_{\beta\alpha}$ and $\frac{1}{2}[\tau_\beta, \tau_\alpha]$, the one symmetric and the other antisymmetric under pion exchange. Nucleon spin may be similarly suppressed into initial and final (4-component) spinors, and a choice made of two independent Lorentz invariants constructed from the Dirac matrices γ_μ and the four-momenta. Here the correct choice is essentially unique and turns out to be 1 and $\frac{1}{2}\, i\, \gamma\cdot(p_1 - p_3)$. A linear combination of these matrices with <u>constant</u> coefficients is of course satisfactory, but s polynomials must be avoided in the coefficients or, as explained below, extra poles may be produced in the invariant amplitudes.

The complete amplitude for a π-N diagram of the type of Fig. 3 may be written

$$\bar{u}_{-p_4}\Big\{\delta_{\beta\alpha}[-A^0(s_1, s_2, s_3) + \tfrac{1}{2}\, i\, \gamma\cdot(p_1 - p_3)\, B^0(s_1, s_2, s_3)] + \tfrac{1}{2}[\tau_\beta, \tau_\alpha][-A^1(s_1, s_2, s_3) + \tfrac{1}{2}\, i\gamma\cdot(p_1 - p_3)\, B^1(s_1, s_2, s_3)]\Big\} u_{p_2}, \tag{IV.1}$$

and the connection with amplitudes for well-defined total I spin is easily obtained. For example, for Channel III,

$$\pi(p_1, \alpha) + N(p_2) \rightarrow \pi(-p_3, \beta) + N(-p_4) ,$$

we find for the two values $I = 1/2, \ 3/2$

$$A_{III}^{1/2} = A^0 + 2A^1 , \qquad B_{III}^{1/2} = B^0 + 2B^1 ,$$

$$A_{III}^{3/2} = A^0 - A^1 , \qquad B_{III}^{3/2} = B^0 - B^1 .$$

On the other hand, for Channel II,

$$\pi(p_1, \alpha) + \pi(p_3, \beta) \rightarrow N(-p_2) + N(-p_4) ,$$

we find that except for normalization the amplitudes for the two total isotopic spin values 0, 1 are just the quantities already labeled with these superscripts.[12] Our particular choice of matrices above is motivated by crossing symmetry. Under interchange of the two pions, $\alpha \leftrightarrow \beta$ and $p_1 \leftrightarrow p_3$, so that A^0 and B^1 are symmetric under interchange of s_1 and s_3 while A^1 and B^0 are antisymmetric.

It can be verified that the four invariant functions $A^{0,1}$, $B^{0,1}$, satisfy the Mandelstam representation--that is, have only "dynamical" singularities arising from intermediate states in the various channels--if one accepts the corresponding conjecture for zero spin. However, had we used an essentially different choice of matrices, e.g., 1 and $\gamma \cdot p_1 \ \gamma \cdot p_2$, we should have found additional (kinematical) poles in the corresponding invariant functions to compensate for zeros in the spin matrices. Care must therefore be used in setting up problems with spin, but once a correct choice of amplitudes has been made the use of unitarity and analyticity to determine the S matrix follows the same line as for zero spin.

V. PHYSICAL INTERPRETATION OF SINGULARITIES

We are now in a position to see in detail the connection between forces and singularities that was emphasized in the introduction. The singularities in the Mandelstam representation occur only when one or more of the denominators in (III.1) vanish, and this in turn happens only when an s variable is equal to the square of the mass of a strongly interacting physical system having the quantum numbers of the associated channel. If we are focusing attention on one particular channel, e.g., Channel III of Fig. 2 ($n + p \to n + p$), then we shall refer to the singularities "belonging" to this channel--that is, due to the vanishing of $s_3' - s_3$ denominators--as "physical" singularities. Most of these occur in the actual physical region of s_3 for Channel III, although there may be some extending for a distance below the true physical threshold. These "physical" singularities may be thought of as consequences of unitarity for Channel III, in contrast to the "unphysical" singularities associated with Channels I and II which may be thought of as the "forces" giving rise to the Channel III reaction.

To bring out these ideas in a familiar situation, let us study Formula (III.6c) as applied to the amplitude for Fig. 2, where we use (II.8) to replace s_1, s_2, s_3 by q^2 and $\cos\theta$, the barycentric-system variables for n-p scattering. Still ignoring nucleon spin and not worrying about a possible n-p bound state, we have

$$A(q^2, \cos\theta) = \frac{1}{\pi}\int_0^\infty dq'^2\ \frac{\rho_3(q'^2)}{q'^2 - q^2} + \frac{1}{\pi}\int ds_1'\ \frac{A_1(s_1', 4M^2 - s_1' - s_3, s_3)}{s_1' + 2q^2(1 + \cos\theta)}$$

$$+ \frac{1}{\pi}\int ds_2'\ \frac{A_2(4M^2 - s_2' - s_3, s_2', s_3)}{s_2' + 2q^2(1 - \cos\theta)} \quad . \tag{V.1}$$

One of the important features of this expression is that the $\cos\theta$ dependence of the n-p amplitude comes entirely from the denominators of the last two terms. Let us compare the form (V.1) to the Born approximation for scattering by a Yukawa potential of range r_0. This is well known to have the form

$$\frac{\text{constant}}{\frac{1}{r_0^2} + 2q^2(1 - \cos\theta)} \quad . \tag{V.2}$$

Thus we see that the last term of (V.1) looks like the Born scattering due to a superposition of Yukawa potentials, where the range is $1/\sqrt{s_2'}$. The second term in (V.1) can similarly be identified with an exchange potential, but the first term is of a different type. This contributes only to S-wave n-p scattering, and is to be associated with the failure of the Born approximation when the S-phase shift is large. If some higher phase shifts also are large, further terms are needed in (V.1), corresponding to subtractions in the original Mandelstam representation.

For sufficiently high angular momentum, however, it is reasonable to expect the phase shift to be small when q^2 is small, so that the interpretation of the second two terms as the Fourier transforms of "potentials" is plausible. Now let us consider the strength and range of the "potentials," as controlled by the functions A_1 and A_2. The longest-range forces will come from the lowest values of s_1' and s_2', and these in turn are determined by the lowest masses of physical systems with the quantum numbers of Channel I and Channel II, respectively. Consider Channel I, $(n + \bar{p} \to n + \bar{p})$, which gives us our exchange forces, and refer to Formula (III.5) to see the structure of A_1. The term $\rho_1(s_1)$

contains a delta function corresponding to a discrete π^- state, and the coefficient of the delta function is g_c^2. Thus the longest-range exchange force has a range of one pion Compton wave length and a strength determined by the pion-nucleon coupling constant.

The next longest range force comes from the two-pion parts of the ρ_1, ρ_{12}, and ρ_{13} spectra. To calculate this we recall the form of the unitarity condition for a matrix $T = \frac{1}{2i}(S - 1)$:

$$\text{Im } T_{ba} = \sum_m T^*_{mb} T_{ma} \quad . \tag{V.3}$$

Our invariant amplitude A is proportional to T and has the same phase, so in the physical region for Channel I we can derive an expression of the form

$$A_1(s_1, s_2, s_3) = \text{constant} \times \text{Im} \langle n(-p_3), \bar{p}(-p_2) \,|T|\, n(p_1), \bar{p}(p_4) \rangle \tag{V.4}$$

$$= \text{constant} \times \sum_m \langle m|T|\, n(-p_3), \bar{p}(-p_2)\rangle^* \langle m|T|n(p_1), \bar{p}(p_4)\rangle \quad .$$

It is possible to extend this formula to unphysical regions, and since all the elements of T conserve energy and momentum, we see that it is the unitarity condition which makes A_1 vanish except for values of $s_1 = (p_1 + p_4)^2$ equal to the squares of masses of systems that can be reached both from the initial and the final states of Channel I. For the range $4 < s_1 < 9$, only 2π states contribute to the sum in (V.4), so if we have some means of calculating the matrix element connecting nucleon-antinucleon states to two-pion states, we can calculate the "strength" of the exchange force for ranges between one-half and one-third

of a pion Compton wave length. Since this matrix element corresponds to a four-line diagram (it is in fact Channel II of Fig. 3), there is hope that the calculation can be performed.

Formula (V.4) tells us how to calculate shorter-range forces due to multiparticle exchange if the matrix elements connecting these states to the n-$\bar{p}$ system are known. We do not yet have a method for obtaining multiparticle matrix elements, but we observe that they are bounded in magnitude because of unitarity and therefore there is a limit to the possible strength of the forces that they generate.

Formula (V.1) is actually suitable as it stands for calculating the high-angular-momentum parts of the n-p scattering amplitude, which are determined by the lower range of s_1' and s_2'. The modifications to take account of spin and charge have been worked out by several authors and expressions given for the one and two-pion parts of A_1 and A_2.[11,13] The very high ℓ-phase shifts are of course controlled by the one-pion parts alone, and thus by the pion mass and the pion-nucleon coupling constant, quantities already known with good accuracy. This circumstance has been exploited in recent phase-shift analyses of nucleon-nucleon scattering.

It should be obvious that the considerations of this section are general. The forces producing a certain reaction are due to the intermediate states that occur in the two "crossed" reactions belonging to the same diagram. The range of a given part of the force is determined by the mass of the intermediate state producing it, and the strength of the force by the matrix elements connecting that state to the initial and final states of the crossed reaction. By considering all three channels on this basis we have a self-determining situation. One channel provides forces for the

other two--which in turn generate the first. Our task now is to understand how many arbitrary parameters there are in such a situation, as well as how to handle low angular momentum--or, in other words, how to calculate the first term on the right-hand side of Eq. (V.1). These two questions are closely related.

VI. PARTIAL-WAVE AMPLITUDES

There is no need to treat high-angular-momentum scattering one partial wave at a time, and in fact good reason to avoid doing so. Not only is simplicity lost, but the partial-wave expansion converges only in the neighborhood of the physical region and not throughout the complex plane. However, there seems no way to avoid a special discussion of certain low-angular-momentum partial waves. It is here that arbitrary parameters may enter, corresponding to an asymptotic behavior that requires subtractions, and it is here that the difficult multiparticle singularities are most important. In our S-matrix approach, the lowest partial waves generate the higher waves, as we shall see in detail below. In a very direct sense, then, the low-ℓ waves constitute the heart of the problem.

Let us consider the equal-mass spin-zero case and define a partial-wave amplitude for Channel III by the formula

$$A_\ell^{III}(q^2) = \frac{1}{2} \int_{-1}^{+1} d\cos\theta \, P_\ell(\cos\theta) \, A(q^2, \cos\theta) , \qquad (VI.1)$$

where q^2 and $\cos\theta$ are related to the s variables by (II.8). In a certain region--including the physical interval, $q^2 > 0$, $-1 < \cos\theta < +1$--the full amplitude can be represented by the series

$$A(q^2, \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) \, A_\ell^{III}(q^2) \, P_\ell(\cos\theta) , \qquad (VI.2)$$

but even outside this region we may define $A_\ell(q^2)$ by (IV.1). We shall in fact be interested in extending $A_\ell(q^2)$ to the entire q^2 complex plane, where we shall find three sets of singularities, corresponding to the three channels of the problem.

Formula (V.1) is well suited to carrying out the projection (VI.2). The first term of (V.1) contributes only to $\ell = 0$, and there it survives the projection unchanged. The second and third terms lead to integrals of the form

$$\frac{1}{2} \int_{-1}^{+1} d \cos\theta \; P_\ell(\cos\theta) \; \frac{1}{s' + 2q^2(1 \pm \cos\theta)} \; , \qquad \text{(VI.3)}$$

which are simply related to associated Legendre functions of the second kind. Certain important properties of these integrals may be seen by inspection:

(a) For small q^2 they behave like $(q^2)^\ell$.

(b) For large s' they behave like $(s')^{-\ell-1}$.

(c) They are analytic functions of q^2 in the cut plane, where the cut should be chosen to run from $-s'/4$ to $-\infty$ if the function is to be real for $q^2 > 0$.

The behavior near $q^2 = 0$ is maintained after the integrations over ds_1' and ds_2' and is a well-known and general property of partial-wave amplitudes, related to the centrifugal barrier effect. The large s' behavior simply confirms the remarks of the preceding section about the range of the interaction and the magnitude of s', i.e., as ℓ increases the large values of s_1' and s_2' become less and less important, since they correspond to short-range interactions. This is again a centrifugal barrier effect.

The property we are most concerned with is that of analyticity, and to get the full story here we have to look also at the q^2 (or s_3) dependence of the functions A_1 and A_2, which appear in the numerators of the integrands in Formula (V.1). Referring to Formula (III.5)

for $A_1(s_1', 4M^2 - s_1' - s_3, s_3)$, we see that the $s_3' - s_3$ denominator leads to a cut running from 0 to ∞ in q^2, while there seems to be a second cut associated with the denominator $s_2' - (4M^2 - s_3 - s_1')$. It may be verified, however, that this second apparent singularity is canceled by a corresponding part of the expression for $A_2(4M^2 - s_2' - s_3, s_2', s_3)$. After multiplying (VI.3) by A_1 or A_2 and integrating over ds_1' or ds_2', we thus produce an analytic function of q^2 with two cuts, one running from 0 to ∞ along the positive real axis, to be called the "right-hand" or "physical" cut, and one running along the negative real axis from $-\infty$ to $-\frac{1}{4} s_{min}'$, where s_{min}' is the lowest square of a mass appearing in the spectra associated with Channels I and II. This latter will be called the "left-hand" or "unphysical" cut. In addition, if there exists a stable single-particle state with the quantum numbers of a particular partial wave of Channel III, there is a corresponding pole on the negative q^2 real axis.

In general, if the four particle masses are not all equal,* there are two distinct unphysical cuts, one for Channel I and one for Channel II, and these do not lie entirely on the real axis, but wander into the complex plane. The location of the cuts is always completely determined by kinematical considerations, however, and there are no new essential complications beyond the equal-mass case. We continue, therefore, to concentrate on that case.

Notice that our partial-wave amplitude is a real analytic function. That is, if we define $\nu = q^2$, we have

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* See, for example, McDowell.[14]

$$A_\ell^{III^*}(\nu) = A_\ell^{III}(\nu^*) \quad . \tag{VI.4}$$

Thus the discontinuity across a cut along the real axis is twice the imaginary part of the function at that point, while the function is real on the real axis in the gap between $-\frac{1}{4} s'_{min}$ and the origin. In the next section it will be shown that $A_\ell(\nu)$ is determined by the discontinuities across its cuts, so it is important to be able to calculate the imaginary part along the real axis. On the right-hand cut we are in the physical region and the required imaginary part is given by the unitarity condition. Comparing (VI.2) with (II.2) and (II.3) we see, for $\nu > 0$,

$$A_\ell^{III}(\nu) = \sqrt{\frac{\nu + m^2}{\nu}}\, e^{i\delta_\ell} \sin \delta_\ell \quad , \tag{VI.5}$$

so that in the elastic region, where the phase shifts are real, we have

$$\text{Im}\, A_\ell^{III}(\nu) = \sqrt{\frac{\nu}{\nu + m^2}}\, \left| A_\ell^{III}(\nu) \right|^2 \quad , \text{ for } \nu > 0. \tag{VI.6}$$

In general a factor R_ℓ should be added to the right-hand side of (VI.6), where R_ℓ is the ratio of total to elastic partial-wave cross sections. On the left-hand cut we may most easily calculate the imaginary part from Formula (V.1) before integrating over $\cos\theta$. Remembering that A_1 and A_2 may be considered real in this region if we anticipate the above-mentioned cancellation between the two, we find*

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* We define the partial-wave amplitude along both cuts as the limit as the cut is approached from above.

$$\text{Im } A_\ell^{III} = -\frac{1}{2}\int_{-1}^{+1} d\cos\theta\, P_\ell(\cos\theta) \times \left\{ A_1\left(-2\nu(1+\cos\theta), -2\nu(1-\cos\theta), 4(\nu+m^2)\right) + A_2\left(-2\nu(1+\cos\theta), -2\nu(1-\cos\theta), 4(\nu+m^2)\right)\right\},$$

$$\text{for } \nu < 0 . \qquad \text{(VI.7)}$$

Note that the possible presence of the first term in (V.1) for the case $\ell = 0$ does not affect either Formula (VI.6) or (VI.7).

If we think of A_1 (or A_2) as made up of a sum of contributions from different kinds of intermediate states in Channel I (or II), as expressed by Formula (V.4), then, according to (VI.7), $\text{Im } A_\ell^{III}$ is similarly composed of additive parts and these are nonzero along different portions of the negative real axis. It is evident, in fact, that a Channel I intermediate state of mass m_1 gives a nonzero contribution to $\text{Im } A_\ell^{III}$, according to (VI.7), in the interval between $-\infty$ and $-\frac{1}{4}m_1^2$. Thus the least massive intermediate states control $\text{Im } A_\ell^{III}$ on the "near-by" portion of the left-hand cut. As we go farther to the left, more and more massive intermediate states come into the picture. Taking the nucleon-nucleon problem again as an example, we find the left-hand cut begins at $-\frac{1}{4}$, and up to -1 is completely determined by one-pion exchange. Between -1 and $-9/4$, the two-pion contribution must be added, between $-9/4$ and -4 the three-pion contribution, and so on. Each new threshold can be shown to be a branch point, with the associated cut running to the left.

The right-hand cut of the N-N amplitude of course begins at $\nu = 0$, where there is a branch point, and the next branch point does not occur until $\nu = \frac{1}{4}(2M + 1)^2 - M^2 \approx 7$, the threshold for single-pion production in nucleon-nucleon collisions. In the interval $0 < \nu < 7$, the simple Formula (VI.6) is exact; for $\nu > 7$ the factor R_ℓ should be added. In the $J = 1$ even-parity amplitude there is a pole corresponding to the deuteron and falling in the gap between cuts at $\nu = \frac{1}{4} M_D^2 - M^2 \approx -\frac{1}{10}$. We shall see that this pole need not be postulated in advance, but is a necessary consequence of the left and right cuts.

VII. DETERMINATION OF A PARTIAL-WAVE AMPLITUDE FROM ITS UNPHYSICAL SINGULARITIES

We now come to the fundamental dynamical problem, the determination of a partial-wave amplitude--given the discontinuity across its unphysical ("left-hand") cuts. This discontinuity plays the role of the interaction potential energy in a nonrelativistic scattering problem, and what we require now is the equivalent of the Schrödinger equation to allow us to calculate the phase shift from a knowledge of the interaction. The basic technique was discovered by Chew and Low,[3] working with the static model of the pion-nucleon interaction, and was subsequently modified by Chew and Mandelstam[15] for use in a more general class of problems.

Let us, momentarily at least, regard the imaginary part of A_ℓ^{III} along the left-hand cut as a given quantity, to be called $f_\ell(\nu)$. Along the right-hand cut the imaginary part of A_ℓ^{III} is proportional to $|A_\ell^{III}|^2$, according to (VI.6), so that the imaginary part of the reciprocal amplitude is a given function, at least in the elastic region where $R_\ell = 1$:

$$\operatorname{Im}\left(A_\ell^{III}\right)^{-1}_{\nu > 0} = -\frac{\operatorname{Im} A_\ell^{III}}{|A_\ell^{III}|^2} = -\sqrt{\frac{\nu}{\nu + m^2}}\, R_\ell \ . \qquad \text{(VII.1)}$$

That the imaginary part of the direct function is known along the left-hand cut, while the imaginary part of the reciprocal is known along the right-hand cut, suggests that we write

$$A_\ell^{III}(\nu) = N_\ell(\nu)/D_\ell(\nu) \ , \qquad \text{(VII.2)}$$

where the numerator function $N_\ell(\nu)$ has only the left-hand cut and the denominator function $D_\ell(\nu)$ has only the right-hand cut. Such a separation is certainly possible if inelastic scattering is neglected,* because the explicit form of $D(\nu)$ in terms of the phase shift has been given by Omnes[16] as

$$D_\ell(\nu) = \exp\left(-\frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu' \frac{\delta_\ell(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}\right), \qquad \text{(VII.3)}$$

where we normalize to unity at the arbitrary point on the real axis $\nu = \nu_0$. The Omnes function clearly has only the right-hand cut, and in addition it has the phase $e^{-i\delta_\ell}$ along this cut. The numerator function, defined by (VII.2), is therefore real along the positive real axis, since A_ℓ^{III} here has the phase $e^{i\delta_\ell}$; therefore the right-hand cut is absent for N_ℓ.

Formula (VII.3) is useful for many purposes but does not solve our problem. We go back instead to the defining properties of N_ℓ and D_ℓ, which allow us to write

$$\begin{aligned} \text{Im } N_\ell(\nu) &= f_\ell(\nu)\, D_\ell(\nu) && \text{for } \nu < \nu_L \\ &= 0 && \text{for } \nu > \nu_L\ , \end{aligned} \qquad \text{(VII.4)}$$

if ν_L is the end of the left-hand cut. At the same time, according to (VII.1), we have

$$\begin{aligned} \text{Im } D_\ell &= -N_\ell \sqrt{\frac{\nu}{\nu+m^2}}\, R_\ell && \text{for } \nu > 0\ , \\ &= 0 && \text{for } \nu < 0\ . \end{aligned} \qquad \text{(VII.5)}$$

* The argument here is due to Mandelstam (unpublished).

Let us assume tentatively that N_ℓ vanishes at infinity so that, using the Cauchy integral theorem, we get

$$N_\ell(\nu) = \frac{1}{\pi} \int_{-\infty}^{\nu_L} d\nu' \, \frac{\operatorname{Im} N_\ell(\nu')}{\nu' - \nu} \,, \tag{VII.6}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\nu_L} d\nu' \, \frac{f_\ell(\nu') \, D_\ell(\nu')}{\nu' - \nu} \,.$$

We are allowed to normalize D_ℓ arbitrarily to unity at $\nu = \nu_0$, so that it is necessary only that D_ℓ/ν vanish at infinity for us to write

$$D_\ell(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^\infty d\nu' \, R_\ell(\nu') \sqrt{\frac{\nu'}{\nu' + m^2}} \, \frac{N_\ell(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} \,. \tag{VII.7}$$

We now have two coupled linear integral equations for N_ℓ and D_ℓ that can be converted into a single equation for one unknown function. For example, if we substitute (VII.6) into (VII.7) we find

$$D_\ell(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{\nu_L} d\nu' \, H_\ell(\nu, \nu') \, f_\ell(\nu') \, D_\ell(\nu') \,, \tag{VII.8}$$

where

$$H_\ell(\nu, \nu') = \frac{1}{\pi} \int_0^\infty d\nu'' \, \frac{R_\ell(\nu'') \sqrt{\dfrac{\nu''}{\nu'' + m^2}}}{(\nu'' - \nu_0)(\nu'' - \nu)(\nu'' - \nu')} \,. \tag{VII.9}$$

Assuming that R_ℓ is known and that $R_\ell(\nu'')/\nu''$ vanishes as $\nu'' \to \infty$, and provided that $f_\ell(\nu')$ vanishes as $\nu' \to \infty$ (which is implied by our original assumption that N_ℓ vanishes at ∞), the linear integral

equation (VII.8) is of the nonsingular Fredholm type and can be solved by any number of standard methods. This is the "Schrödinger equation" for the S-matrix theory.

Now let us consider the possibility of zeros in the denominator function, which correspond to poles in A_ℓ^{III}. If these occur off the real axis they are not consistent with the original Mandelstam representation. However, a pole on the real axis for $-m^2 < \nu < 0$ may be interpreted as a bound state.* This is the sense in which we said earlier that the deuteron pole is a consequence of other singularities. If f_ℓ is given correctly, both the position and the residue of a bound-state pole follow from the solution of Eq. (VII.8).

To illustrate these considerations let us consider a very crude approximation for S-wave scattering in which we take $f_\ell(\nu)$ to be a delta function. In other words, we approximate the left-hand cut (a line charge) by a pole (a point charge). This approximation is reasonable for a region along the positive real axis whose extent is short compared with the "average" distance to the important left-hand singularities. Figure 6 shows some of the distances for the case of n-p scattering. We see that if the 1π exchange force is not too strong, the replacement of the left-hand cut by a pole may be reasonable for kinetic energies (lab) of 0 to 10 Mev.

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* Of course, in a calculation with approximate input functions, f_ℓ and R_ℓ, one may find zeros in D_ℓ outside the range where they can correspond to bound states. Such a situation should not be interpreted as a contradiction of Mandelstam's hypothesis, but only as an indication that the particular approximations used are inadequate.

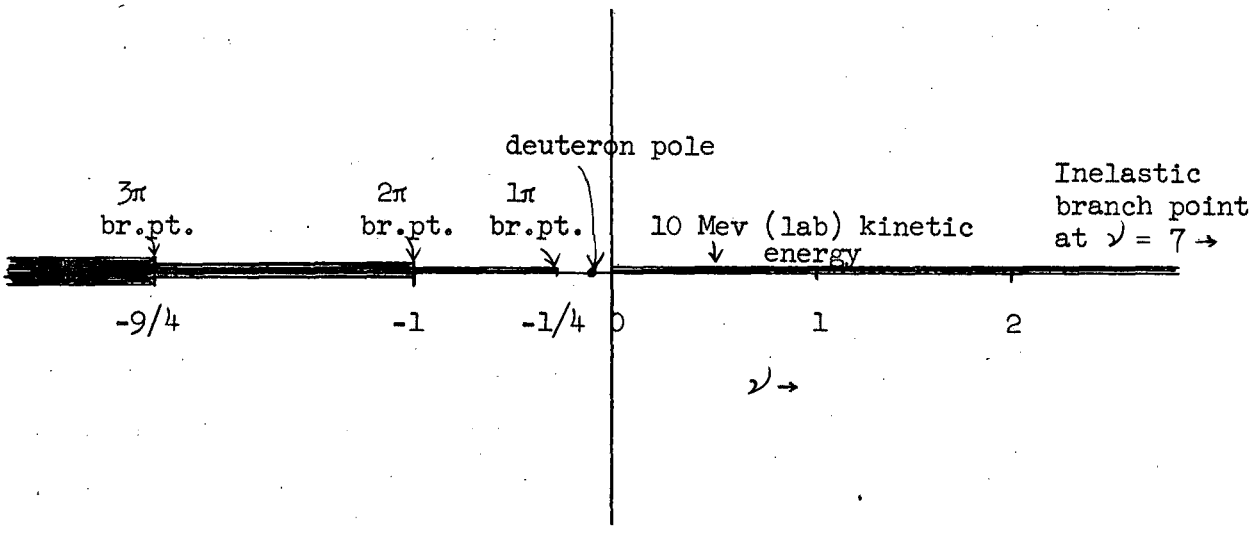

Fig. 6. "Near-by" singularities of a partial-wave n-p amplitude.

Suppose we locate the interaction pole at $\nu = -\nu_i$ and normalize D_0 to unity at this point (i.e., choose $\nu_0 = -\nu_i$). Then if we introduce a parameter Γ, to characterize the strength of the interaction, by writing

$$f_0(\nu) = -\pi \Gamma \delta(\nu + \nu_i) , \qquad \text{(VII.10)}$$

we have, from (VII.6),

$$N_0(\nu) = \frac{\Gamma}{\nu_i + \nu} , \qquad \text{(VII.11)}$$

and from (VII.7),

$$D_0(\nu) = 1 - \frac{\Gamma}{\pi} (\nu + \nu_i) \int_0^\infty d\nu' \sqrt{\frac{\nu'}{\nu' + m^2}} \frac{R_0(\nu')}{(\nu' + \nu_i)^2(\nu' - \nu)} , \qquad \text{(VII.12)}$$

so in this simple case we do not even have to solve an integral equation. If both ν and ν_i are small compared with the inelastic threshold, we may set $R_0(\nu') = 1$. The integral in (VII.12) can then easily be performed and the nonrelativistic result (i.e., for ν and ν_i both small compared with m^2 --an excellent approximation for n-p scattering where $m^2 = 44$) is

$$D_0(\nu) = 1 - \frac{\Gamma}{m} \frac{\nu + \nu_i}{2\sqrt{\nu_i}\left(\sqrt{\nu_i} + \sqrt{-\nu}\right)^2} \quad . \tag{VII.13}$$

In the physical region, $\nu > 0$, we then find

$$\mathrm{Re}\ \frac{D_0(\nu)}{N_0(\nu)} = \frac{\sqrt{\nu}}{m} \cot \delta_0 = \left(\frac{\nu_i}{\Gamma} - \frac{\sqrt{\nu_i}}{2m} \right) + \nu \left(\frac{1}{\Gamma} + \frac{1}{2m\sqrt{\nu_i}} \right) \quad . \tag{VII.14}$$

Comparing this with the standard nonrelativistic effective range formula,[17]

$$q \cot \delta_0 = \frac{1}{a} + \frac{1}{2} r q^2 \quad ,$$

we see

$$\frac{1}{a} = \frac{m}{\Gamma} \nu_i - \frac{1}{2} \sqrt{\nu_i} \quad , \tag{VII.15a}$$

$$\frac{1}{2} r = \frac{m}{\Gamma} + \frac{1}{2\sqrt{\nu_i}} = \frac{1}{\sqrt{\nu_i}} + \frac{1}{a \nu_i} \quad . \tag{VII.15b}$$

Let us study the dependence of our result on the input parameters Γ and ν_i. First, if ν_i is held fixed and Γ is small, we see that the scattering length a is proportional to Γ and has the same sign, exactly what we expect if Γ determines the magnitude and sign of the

interaction. Evidently positive Γ means an attractive force. If Γ is negative (repulsive), increasing its magnitude does not make a indefinitely large; the scattering length never exceeds $2/\sqrt{\nu_i}$ in absolute value. This is exactly the behavior of a repulsive potential of range $\sim 2/\sqrt{\nu_i}$.

If Γ is positive (attractive), increasing its magnitude makes the scattering length increase and in fact become infinite for

$$\Gamma = 2n\sqrt{\nu_i} \quad . \tag{VII.16}$$

This is the condition for a bound state to appear; for larger values of Γ one may easily calculate from (VII.13) that $D_0(\nu)$ has a zero at

$$-\nu = \alpha^2 = \nu_i \left(\frac{\Gamma - 2m\sqrt{\nu_i}}{\Gamma + 2m\sqrt{\nu_i}} \right)^2 \quad . \tag{VII.17}$$

Thus we can calculate the binding energy--if there is a bound state--from a knowledge of f_0.

Problem 1:

(a) Show that, for our single-interaction pole example, when there is a bound state one may write

$$\sqrt{\nu} \cot \delta_0 = -\alpha + (\nu + \alpha^2)\left(\frac{m}{\Gamma_B} + \frac{1}{2\alpha} \right) ,$$

where Γ_B is the residue of the bound-state pole and $-\alpha^2$ its position.

(b) By comparison with the nonrelativistic formula for the effective range in terms of an integral over the square of the bound-state configuration-space wave function,[18] identify the residue Γ_B with the square of the asymptotic normalization coefficient for the bound-state function.

Finally we remark that if we are near the condition for a bound state, so that the scattering length is large, then according to (VII.15) the effective range is approximately $2/\sqrt{\nu_i}$. This is a second confirmation that the inverse distance to an unphysical singularity corresponds to the interaction range.

In the actual case of n-p scattering the effective-range formula is extremely accurate in the interval 0 to 10 Mev, and the empirically observed values of the scattering length and effective range for both singlet and triplet states have been shown by Noyes and Wong[19] to imply a value of ν_i in the above formula approximately equal to unity. In other words the "average" position of the left-hand discontinuity in the n-p amplitude occurs near the beginning of the 2π contribution. This is an understandable circumstance if both 1π and 2π forces are important.

A two-pole approximation to the left-hand cut is of course better than a one pole approximation. The problem can again be solved exactly, as it can be for any finite number of poles, and one finds a result corresponding to the superposition of potentials of different ranges. In short, the discontinuity along the unphysical cuts seems to play a role closely analogous to the potential function in nonrelativistic scattering theory; any intuition developed from experience with the latter is likely to be useful in the S-matrix approach. Take, for example, the question of $\ell = 0$ resonances at energies well above the physical threshold. In a potential model it is clear that such resonances cannot occur unless there is a strong inner attraction surrounded by a strong outer repulsion, so that the wave can be "trapped." We would find, correspondingly, by studying the properties of Eq. (VII.8), that a single sign for f_0 cannot produce a resonance, i.e., a zero in the real part of $D_0(\nu)$ for $\nu > 0$;

rather, f_0 must change sign along the left-hand cut, being negative (attractive) in the far region and positive (repulsive) in the near region. In contrast we expect that resonances can occur quite naturally in states with angular momentum whenever there is a sufficiently strong attraction, because the centrifugal barrier does the "trapping." Let us look now for the centrifugal-barrier mechanism in the S-matrix approach; this question is of importance for many reasons, not just in connection with resonances.

The barrier mechanism arises from the property, derived from Eq. (VI.3), that $A_\ell^{III}(\nu)$ behaves like ν^ℓ near $\nu = 0$. We may assign a corresponding behavior to our numerator function N_ℓ and make an appropriate number of subtractions in Formula (VII.6):

$$N_\ell(\nu) = \frac{\nu^\ell}{\pi} \int_{-\infty}^{\nu_L} d\nu' \frac{f_\ell(\nu') D_\ell(\nu')}{\nu'^\ell(\nu' - \nu)} \quad . \tag{VII.18}$$

It now appears that N_ℓ behaves at infinity like $\nu^{\ell-1}$, but we cannot have changed the asymptotic behavior by making subtractions, so for $\ell > 0$ the function f_ℓ must oscillate in a special way that guarantees consistency. (This property is of course precisely that required to make the original integral (VII.6) vanish at the origin like ν^ℓ.) The subtracted form, (VII.18), shows that large negative values of ν' become less and less important as ℓ increases; this is the centrifugal barrier suppressing the influence of short-range interactions. Obviously the S wave is most sensitive to the far discontinuity on the left-hand cut, the P wave next most sensitive--and so on. The ℓ value which first becomes sensitive to multiparticle singularities varies from one problem to another and depends on the precise asymptotic behavior of the partial-

wave amplitudes. This asymptotic behavior also controls the number of truly independent parameters allowed in each problem.

VIII. ASYMPTOTIC BEHAVIOR OF PARTIAL-WAVE AMPLITUDES AND FUNDAMENTAL INTERACTION CONSTANTS

We have assumed above that $N_\ell(\nu)$ vanishes at infinity and that $D_\ell(\nu)$ behaves like a constant. This is an oversimplification, since it corresponds to the vanishing of the partial-wave amplitude itself at infinity, which is not necessarily the case. A glance at Formula (VI.5) for the spin-zero partial-wave amplitude, for example, shows that according to the unitarity condition $A_\ell^{III}(\nu)$ may approach a constant that is less than or equal to unity in absolute value. If it does approach a constant then a subtraction must be made in the Cauchy formula, (VII.6), for the numerator function. The necessary subtraction can be made at $\nu = 0$, as in (VII.18), for all ℓ greater than zero, without introducing a new parameter; however, one arbitrary constant seems allowable in the S-wave amplitude even if the entire left-hand discontinuity f_0 is calculable. It is natural to make the S-wave subtraction at the point ν_0, where the denominator function is normalized to unity, and to put ν_0 in the singularity-free gap between ν_L and 0, so that the subtraction constant is real. We then have

$$N_0(\nu) = a_0 + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{\nu_L} d\nu' \, \frac{f_0(\nu')\, D_0(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} \,. \qquad \text{(VIII.1)}$$

The substitution law implies a relation, between the three different channels for a given diagram, that permits a single real parameter to determine the subtraction constants for all three. Also various symmetry principles can be invoked to relate the arbitrary constant for one diagram to that for another. We shall see below how all this works for π-π scattering.

Formula (VI.5), on which the above analysis rests, holds only when all four particles in the diagram have zero spin. Several other cases have been studied and it is found that the unitarity limitation on asymptotic behavior of partial-wave amplitudes depends on spin. For example, Frazer and Fulco[20] have shown that for π-N scattering an analytic function of $W = \sqrt{s_3}$, which in the physical region is related to the phase shift by the formula

$$\frac{2W^2}{(W+M)^2 - 1} \frac{e^{i\delta_{\ell,J}} \sin \delta_{\ell,J}}{q} , \qquad \text{(VIII.2)}$$

plays a role analogous to that of $A_\ell^{III}(\nu)$ in our zero-spin example. Such a function, however, is required by unitarity to vanish at infinity and permits no arbitrary constants in its determination. The corresponding analysis for N-N scattering has been carried out by Goldberger, Grisaru, McDowell, and Wong,[11] and again allows no arbitrary constants. Diagrams containing one or two photons also have been analyzed with the same result. What is the underlying principle here? When are arbitrary constants allowed and when are they not? We do not yet have a clear and complete answer, but Mandelstam[1] has emphasized a striking parallelism with the theory of quadrilinear renormalizable interactions.

In conventional perturbation field theory it is not possible to introduce interactions formed by quadrilinear products of arbitrary fields, even when all symmetry principles are obeyed. Most of these combinations, such as the Fermi beta-decay interaction product of four spinor fields, are nonrenormalizable and give divergent results in higher orders of perturbation theory. The one interaction that <u>is</u> renormalizable, however, is the product of four scalar (or pseudoscalar) fields. In all cases

studied so far the impossibility of an arbitrary constant in the S-matrix approach coincides with the nonrenormalizability of the corresponding quadrilinear interaction.

There exist, of course, a number of renormalizable trilinear interactions, so that it is natural to ask where the corresponding constants appear in the S-matrix theory. We have anticipated the answer in Section III above: The residues of our poles are products of coupling constants associated with trilinear interactions. If particles, other than photons, with spin greater than 1/2 are involved, the interaction is probably not renormalizable in perturbation theory; but where such a particle appears we have so far been able to regard it as a bound state, so that the residue of the pole is determined by other singularities. Such is the case with the deuteron pole in the diagram of Fig. 2.

When we consider strange-particle processes, such as (K^-p) scattering, there are poles corresponding to the Λ and Σ particles, and a question arises whether the Λ and Σ are "elementary" or bound states. The trilinear interaction here is renormalizable, so from that point of view they may be elementary. The S-matrix approach outlined above allows another test, however, at least in principle. If we solve our N/D problem, ignoring the existence of the hyperons, and find that the denominator functions develop zeros at the correct point, we should have demonstrated that they are bound states. If on the other hand the poles must be inserted at the beginning of the calculation, then we should regard these particles as elementary. Of course, in practice we don't know enough yet about the unphysical singularities of the amplitude to perform such a calculation. However, the (K^-p) amplitude in the <u>physical</u> region has a characteristically different behavior in the two

cases, so it may be possible to decide the question by a sufficiently complete experimental study of K^--p scattering.

This line of discussion brings us to one of the most tricky aspects of the S-matrix approach. How do we know that all "elementary" particles are stable with respect to strong interactions? Unstable particles apparently have no place in our scheme, yet in conventional field theory there is no difficulty in introducing a field corresponding to a particle that becomes unstable after strong interactions are "turned on." Actually, in the procedure outlined in the preceding section for calculating the denominator function there is a loophole that was first emphasized by Castillejo, Dalitz, and Dyson.[21] This loophole seems the natural place to introduce either stable or unstable "elementary" particles having the quantum numbers of Channel III; it is the following: We can always add poles on the real axis to our expression (VII.7) for $D_\ell(\nu)$. Each such CDD pole produces a zero in $D_\ell(\nu)$, close to the pole if the residue of the latter is small; and if the position of the CDD pole occurs on the negative real axis where D_ℓ is real, the associated zero is also likely to be on the negative real axis. In such a case we would have a pole in the partial-wave amplitude $A_\ell^{III}(\nu)$ that could be interpreted as a stable particle. Two new parameters would have been introduced through the position and residue of the CDD pole in the denominator function, which then determine the position and residue of the pole in A_ℓ^{III} itself. The latter two quantities correspond to the mass and coupling constant of the new particles, so we again have complete agreement with perturbation theory in the number of arbitrary parameters.

Actually, for stable-particle poles such as the nucleon pole in Channel III of Fig. 3, it is inconvenient in practice to go through the

above procedure because one would like to introduce the nucleon mass and the pion-nucleon coupling constant at the beginning of the calculation. It is more convenient and completely equivalent to put such a pole into the numerator function. However, a CDD denominator pole on the positive real axis is a horse of a different color because here, owing to the complexity of D_ℓ, the associated zero usually occurs away from the real axis. If the sign of the residue of the CDD pole is correctly chosen, the zero of D_ℓ can be made to appear on the so-called "unphysical sheet" of the Riemann surface, not the sheet on which we are working, and therefore it does not violate our postulated properties of analyticity. The corresponding pole on the unphysical sheet for A_ℓ^{III} represents the unstable particle, which again is characterized by two parameters.

When a CDD denominator pole is present with not too large a residue, the real part of the denominator vanishes somewhere near the pole, so that the phase shift goes through 90 deg and we have a resonance. The question often comes up whether such a resonance, due to an unstable elementary particle, can be distinguished from a "dynamical" resonance due to attractive forces arising in the discontinuity across the left-hand cut. In general the answer is certainly affirmative, because the phase shift must change sign at the position of the CDD pole, which is presumably near the resonance. It is possible, of course, for the numerator function to have a zero with no CDD pole present, but there is no reason for this to occur in close association with a resonance. More generally, it is obvious that the behavior of a phase shift is more complicated when an unstable elementary particle exists, because of the two new parameters in its functional form. It seems quite clear, for example, that the low-energy

$I = 3/2$, $J = 3/2$ resonance in the pion-nucleon system is not due to some "hidden" particle. The 33 phase shift does not vanish anywhere near the resonance, and its energy dependence, as we shall see below, is entirely compatible with the complete absence of CDD poles.

To summarize our point of view about unstable elementary particles: Some may exist although to date there are no indications of any. If some do exist we can insert them into the S-matrix framework with exactly the same number of new parameters as would be involved in conventional field theory.

IX. PION-PION SCATTERING: GENERAL FORMULATION

Let us turn now to a specific consideration of π-π scattering, where we can see in detail how the principles described above work out in practice. Mandelstam and I have been intensively studying this simplest of strong-interaction problems for about two years,[15,22] but more work still needs to be done, as will be seen.

The diagram in question is that of Fig. 7, where we are again using

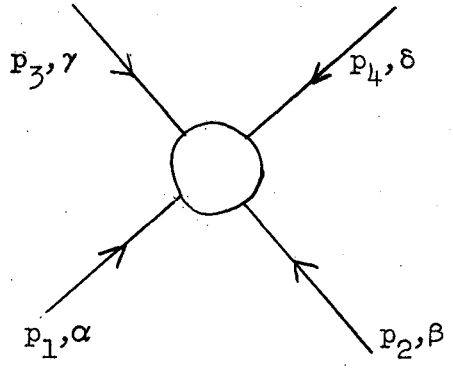

Fig. 7. Diagram for pion-pion scattering.

the isotopic vector index to label pion charge states. Assuming charge independence, we write the complete amplitude as

$$A(s_1, s_2, s_3)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + B(s_1, s_2, s_3)\delta_{\alpha\gamma}\,\delta_{\beta\delta} + C(s_1, s_2, s_3)\delta_{\alpha\delta}\,\delta_{\beta\gamma}\,, \tag{IX.1}$$

and observe that crossing symmetry leads to the relations

$$\left.\begin{matrix} A \leftrightarrow A \\ B \leftrightarrow B \end{matrix}\right\} \qquad s_1 \leftrightarrow s_2, \quad s_3 \leftrightarrow s_3,$$

$$\left.\begin{matrix} A \leftrightarrow B \\ C \leftrightarrow C \end{matrix}\right\} \qquad s_2 \leftrightarrow s_3, \quad s_1 \leftrightarrow s_1,$$

$$\left.\begin{matrix} A \leftrightarrow C \\ B \leftrightarrow B \end{matrix}\right\} \qquad s_1 \leftrightarrow s_3, \quad s_2 \leftrightarrow s_2 . \tag{IX.2}$$

The connection between A, B, C and the amplitudes for well-defined total I spin in Channel III turns out to be

$$\begin{aligned} A^{III,0} &= 3A + B + C , \\ A^{III,1} &= B - C , \\ A^{III,2} &= B + C . \end{aligned} \tag{IX.3}$$

Problem 2: Derive formulas (IX.3).

There are no poles and the continuous spectrum in the Mandelstam representation begins at 4 in each of the s variables. The region in which any double spectral function $\rho(x, y)$ fails to vanish turns out to be bounded by the curves

$$\begin{aligned} x &= \frac{16y}{y - 4} \qquad \text{for} \quad x > y , \\ y &= \frac{16x}{x - 4} \qquad \text{for} \quad y > x , \end{aligned} \tag{IX.4}$$

as shown in Fig. 8. The large distance to the boundary from the corner

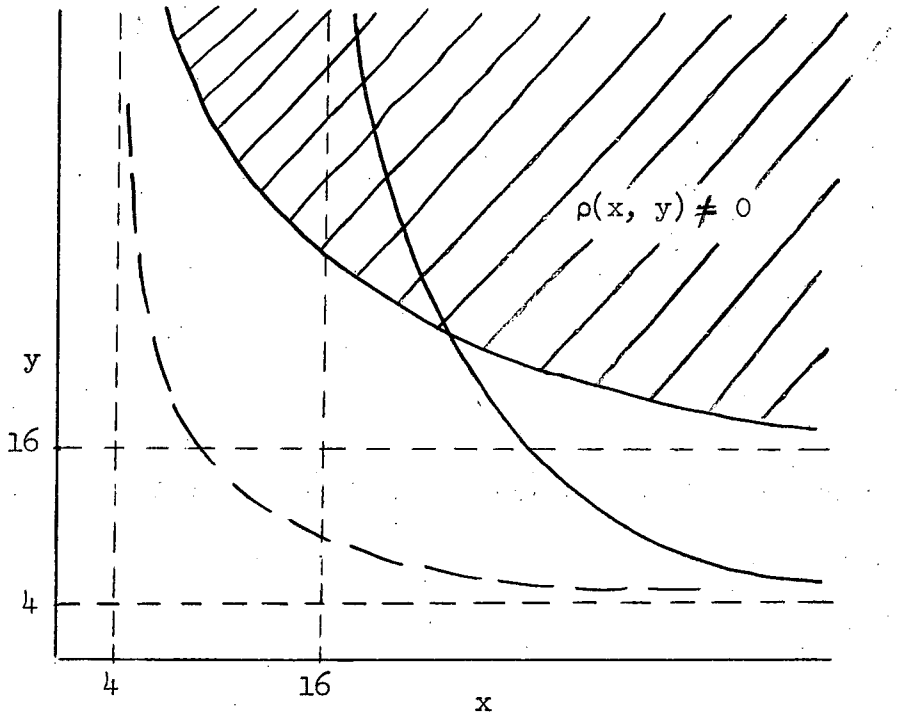

Fig. 8. The region in which the two-dimensional π-π spectral functions are nonvanishing. The dashed curve indicates the boundary if there were a 3π vertex.

$y = x = 4$ is associated with the absence of a three-pion vertex, and a study of Fig. 8 reveals a circumstance emphasized in the first paper by Mandelstam:[1] If multiparticle singularities--in this case due to 4π and higher configurations--are consistently to be ignored, then in the π-π problem one need not worry at all about the double spectral functions, but can represent the entire amplitude by one-dimensional dispersion integrals. Such would not be the case if a three-pion vertex existed, since the boundary then would be the dashed curve. (The double spectral function in the near region, however, can always be explicitly calculated in closed form in terms of the coupling constant associated with the vertex. This is an important consideration in the N-N problem.[11])

In the π-π problem, the double spectral functions do not begin at all until the region where they involve 4π states, and therefore they elude our present calculational power. We must not ignore double spectral functions at the beginning of our considerations, however, or we shall find ourselves faced with a contradiction if any phase shifts for $\ell > 0$ turn out to be large. Keeping only the single-dispersion integrals in (III.1) obviously corresponds to assuming that only the S wave has a nonzero imaginary part in the physical region. This is a consistent assumption if all higher phase shifts are small because we have

$$\frac{\text{Im } A_\ell^{III,I}}{\text{Re } A_\ell^{III,I}} = \tan \delta_\ell^I \quad . \tag{IX.5}$$

We shall find, however, some reasons for believing that the P wave π-π phase shift is large at low energies.

When we introduce the barycentric-system variables, ν and $\cos\theta$ for Channel III, the crossing conditions (IX.2) become

$$\begin{aligned} A(\nu, \cos\theta) &= A(\nu, -\cos\theta) \quad , \\ B(\nu, \cos\theta) &= C(\nu, -\cos\theta) \end{aligned} \tag{IX.6}$$

and

$$A(\nu, \cos\theta) = B(\nu', \cos\theta') \quad , \tag{IX.7}$$

where

$$\nu' = \frac{\nu}{2}(1 + \cos\theta) - (\nu + 1) \tag{IX.8}$$

and

$$\cos\theta' = \frac{\frac{\nu}{2}(1+\cos\theta) + (\nu+1)}{\frac{\nu}{2}(1+\cos\theta) - (\nu+1)} . \qquad \text{(IX.9)}$$

There is a point of maximum symmetry at $s_1 = s_2 = s_3$ or

$$\nu = \nu' = \nu_0 = -2/3 ,$$
$$\cos\theta = \cos\theta' = 0 . \qquad \text{(IX.10)}$$

At this point the three functions A, B, C are all real and equal to each other. It is therefore natural to introduce the arbitrary parameter, which in Section VIII we saw would be needed, by the definition

$$A(-2/3,\ 0) = B(-2/3,\ 0) = C(-2/3,\ 0) = -\lambda . \qquad \text{(IX.11)}$$

Using (IX.3) we find, accordingly (dropping the superscript III henceforth),

$$A^0(-2/3,\ 0) = -5\lambda ,$$
$$A^1(-2/3,\ 0) = 0 , \qquad \text{(IX.12)}$$
$$A^2(-2/3,\ 0) = -2\lambda .$$

It is also possible to derive an infinite number of derivative crossing conditions at the symmetry point. Remembering that A^0 and A^2 are even functions of $\cos\theta$, while A^1 is an odd function, we find the first-derivative conditions,

$$\left\{ \frac{\partial}{\partial \cos\theta} \left(\frac{A^1}{\nu}\right) = -\frac{\partial}{\partial \nu} A^2 = \frac{1}{2} \frac{\partial}{\partial \nu} A^0 = \lambda_1 \right\}_{\substack{\nu = -2/3 \\ \cos\theta = 0}} , \qquad \text{(IX.13)}$$

where λ_1 can in principle be determined as a function of λ by following the procedure of Section VII. We shall see, however, that if the P-phase shift is large, implying that 4π and higher singularities are important, we shall not actually be able to calculate λ_1. Higher-derivative conditions at the symmetry point will not be written down here. A

Problem 3: Derive formula (IX.13), as well as formulas (IX.8) and (IX.9) on which it is based.

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consistent procedure of calculation based on the 2π singularities alone should satisfy all conditions involving higher derivatives, since these are insensitive to distant singularities.

To carry out the dynamical program of Section VII we need the discontinuity across the left-hand cut of the partial-wave amplitudes as given by Formula (VI.7). Making use of crossing symmetry to refer all absorptive parts to Channel III, and changing variables from $\cos\theta$ to ν' as given by (IX.8), we then find

$$f_\ell^I(\nu) = \operatorname{Im} A_\ell^I(\nu)$$
$$\nu < -1$$

$$= \frac{1}{\nu} \int_0^{-\nu-1} d\nu' \; P_\ell\left(1 + 2\frac{\nu' + 1}{\nu}\right) \sum_{I'=0,1,2} \alpha_{II'} A_3^{I'}\left(\nu', 1 + 2\frac{\nu+1}{\nu'}\right), \qquad \text{(IX.14)}$$

where

$$\alpha_{II'} = \begin{pmatrix} 2/3 & 2 & 10/3 \\ 2/3 & 1 & -5/3 \\ 2/3 & -1 & 1/3 \end{pmatrix} . \qquad \text{(IX.15)}$$

The basic machinery is now complete; to proceed further we must be able to evaluate in (IX.14) the absorptive parts $A_3^I(\nu, \cos\theta)$ in the physical region for ν but in the $\cos\theta$ unphysical region $-\infty < \cos\theta < -1$. We can attempt the necessary extension by a polynomial expansion

$$A_3^I(\nu', 1 + 2\frac{\nu+1}{\nu'}) = \sum_{\ell=0}^{\infty} (2\ell+1)\, \mathrm{Im}\, A_\ell^I(\nu')\, P_\ell(1 + 2\frac{\nu+1}{\nu'}) , \qquad \text{(IX.16)}$$

but the singularities of A_3^I as expressed by the equivalent of Formula (III.5) for Channel III, together with Fig. 8, restrict the region of convergence of the series (IX.16), for the range of ν' needed in (IX.14), to $\nu > -9$. If we terminate the series (IX.16) after the first few terms, we may hope to have an approximate formula that works somewhat further to the left on the unphysical cut, but eventually the polynomial method fails. One must go back to the equivalent of Formula (III.5) to find a better technique of continuation, and there we encounter the elusive double-spectral functions which involve four-pion singularities. This is the frontier at present; whether we can improve the polynomial method without at the same time understanding diagrams with 6 pion-lines is an open question.

Backing away from the frontier, it is still interesting to ask how much one can predict about π-π scattering if the discontinuity across the left-hand cut can be handled for $\nu > -9$. After all, 1 Bev pion

kinetic energy (lab) corresponds to only $\nu = 3.5$ on the right-hand cut, so that on the basis of our Coulomb-law analogy we should be able to make useful predictions up to lab energies of 2 to 3 Bev. Another limitation with basically the same origin--is inelastic scattering on the right-hand cut. The actual threshold occurs at $\nu = 3$, but experience with π-N inelastic scattering suggests that, until each of the produced pions can have a momentum in the barycentric system of order unity, phase-space factors will hold the inelastic cross section to a small fraction of the elastic. Thus we expect to be able to set R_ℓ equal to unity for $\nu \lesssim 10$.

In this low-energy physical region $0 < \nu \lesssim 10$, we expect phase shifts for sufficiently high ℓ to be small and the corresponding partial-wave amplitudes to be almost real. It is then appropriate not to discuss separate ℓ values but to evaluate the entire amplitude for ℓ greater than some critical value from Formula (V.1). For $I = 0, 2$, only even ℓ values occur, so it is natural to separate out only the S wave for special attention. Using the same notation as in (IX.14), we find

$$A^I(\nu, \cos\theta) = \underset{I=0,2}{} A_0^I(\nu) + \frac{1}{\pi} \int_0^\infty d\nu' \sum_{I'=0,1,2} \alpha_{II'} A_3^{I'}(\nu', 1 + 2\frac{\nu+1}{\nu'}) \times \left\{ \frac{1}{2} \left[\frac{1}{\nu' + 1 + \frac{\nu}{2}(1 - \cos\theta)} + \frac{1}{\nu' + 1 + \frac{\nu}{2}(1 + \cos\theta)} \right] - \frac{1}{\nu} \ln\left(1 + \frac{\nu}{\nu' + 1}\right) \right\} . \quad \text{(IX.16)}$$

Correspondingly, for $I = 1$ we separate out the P wave:

$$A^1(\nu, \cos\theta) = 3\cos\theta\, A_1^{\,1}(\nu) + \frac{1}{\pi}\int_0^\infty d\nu' \sum_{I'=0,1,2} \alpha_{1I'}\, A_3^{\,I'}(\nu', 1 + 2\frac{\nu+1}{\nu'})$$

$$\times \left\{ \frac{1}{2}\left[\frac{1}{\nu' + 1 + \frac{\nu}{2}(1 - \cos\theta)} - \frac{1}{\nu' + 1 + \frac{\nu}{2}(1 + \cos\theta)} \right] \right.$$

$$\left. - \frac{3\cos\theta}{\nu}\left[(1 + 2\frac{\nu+1}{\nu'}) \ln(1 + \frac{\nu}{\nu'+1}) - 2 \right] \right\} . \qquad \text{(IX.17)}$$

With ν held fixed in the low-energy region, the brackets in the above integrals fall off rapidly as ν' increases, so that the region of large ν' where the polynomial expansion fails should not be important. We may then expect to consistently approximate $A_3^{\,I'}$ by keeping only the $\ell = 0, 1$ terms of the series (IX.16). That is, for $0 < \nu' \lesssim 10$,

$$A_3^{\,0,2}(\nu', \cos\theta') \approx \operatorname{Im} A_0^{\,0,2}(\nu') , \qquad \text{(IX.18a)}$$

$$A_3^{\,1}(\nu', \cos\theta') \approx 3\cos\theta' \operatorname{Im} A_1^{\,1}(\nu') . \qquad \text{(IX.18b)}$$

Problem 4: For $\nu = 3$, find the maximum value of ν' for which the polynomial expansion for $A_3(\nu', 1 + 2\frac{\nu+1}{\nu'})$ converges, on the basis of the double-spectral-function boundary formulas (IX.4). Remember that a Legendre polynomial expansion converges within a singularity-free ellipse whose foci are at ± 1.

Note that while (IX.16) and (IX.17) are exact expressions, the imaginary part of the high-ℓ part of the amplitude is automatically real if we use a polynomial expansion for A_3^I. The imaginary part comes from the high-ν' range, where the expansion fails.

We have assumed here that all π-π phase shifts for $\ell > 1$ are small in the low-energy region and shown how, if this is the case, the higher-angular-momentum part of the amplitude can be calculated in terms of S- and P-phase shifts. The assumption is evidently subject to check a posteriori, and if D-phase shifts turn out to be large, for example, we can start over with the $\ell = 2$ amplitude separated for special treatment along with $\ell = 0$ and $\ell = 1$. Conversely, it may turn out that the P-phase shift is so small that there was no need to separate $\ell = 1$ in the first place. We must separate $\ell = 0$, of course, since the chain has to begin somewhere.

The physical basis for the scheme that arises here may be understood in terms of the energy of the pion pairs that are being exchanged and are responsible for the long-range pion-pion forces. The longest-range force comes from pion pairs with a minimum of excess kinetic energy, and these obviously tend to be in a relative S state. Slightly shorter-range forces involve pairs of slightly higher relative kinetic energies where P waves can contribute, and so on. Now, as we have emphasized many times, high values of the external angular momentum are sensitive only to the long-range forces; thus they are determined by what happens at low angular momentum in the other two channels. It is clear then that the core of our theoretical problem lies in the S and possibly the P waves. Fortunately, these also should be the most prominent parts of the scattering from an experimental point of view.

X. S-DOMINANT SOLUTIONS OF THE π-π PROBLEM

It turns out that for certain choices of the arbitrary parameter λ, which we shall refer to as the π-π coupling constant, one can find consistent solutions of the above equations (neglecting inelastic scattering) in which only the S-phase shifts are large at low energies. The solutions have been obtained first by a numerical iteration procedure using a 704 computer,[23] and then reproduced in an approximate form by analytical means.[22]

The two S-wave amplitudes were each represented by N/D, where the subtraction point was chosen as $\nu_0 = -2/3$ and the subtraction constants a_0^0 and a_0^2 related to λ by (IX.16) and (IX.12). If D and higher waves are small, then

$$a_0^0 \approx 5\lambda \ , \qquad a_0^2 \approx -2\lambda \ . \tag{X.1}$$

The iteration procedure essentially was to choose a value of λ and start with $A_0^{0,2} = a_0^{0,2}$ and $A_\ell^I = 0$ for $\ell > 0$. Formula (IX.18a) then allows us to calculate

$$A_3^{0,2}(\nu', \cos\theta') = \sqrt{\frac{\nu'+1}{\nu'}}\ \left| A_0^{0,2}(\nu') \right|^2 \ , \qquad A_3^1(\nu', \cos\theta') = 0 \ , \tag{X.2}$$

which may be substituted into (IX.14) to give $f_0^I(\nu)$, the S-wave discontinuity on the left-hand cut. The integral equation for the denominator function is nonsingular if $f_\ell^I(\nu)$ tends to zero at $-\infty$,

which turns out to be the case in the approximation (X.2). Thus we can solve the integral equation numerically and come up with new functions $A_0^I(\nu)$ with which to start a second cycle. Actually a slightly more complicated routine was used and the P wave was included after the first cycle. However, the $\ell = 1$ phase shift never became large and the result would have been the same if it has been omitted. Convergence was generally achieved in no more than 5 or 6 cycles even when λ was large enough to produce a bound state.

Formula (X.1) tells us that negative values of λ correspond to a net attractive force, and positive values to a net repulsion. In the approximation (X.2) our nonzero range forces are all attractive, as can be seen from the appropriate elements of the matrix (IX.15), together with Formula (IX.14). Because we have made a subtraction, we are in effect allowing a zero-range force, but in potential scattering theory it is well known that a repulsion of zero range has no effect. Thus we have no possibility of repulsion anywhere, and positive values of λ should be impossible. If we go ahead and choose a positive λ anyhow and solve the equations, we get a result but find that the denominator function has a zero on the left-hand cut that moves in from $-\infty$ as λ increases. The sign of the residue of the corresponding pole in $A_0^I(\nu)$ is such as to correspond to a repulsive delta function in $f_0^I(\nu)$. Thus we don't get a solution to the original problem, but instead to a problem with a repulsive force in addition to the attractive forces we fed in. It turns out that the repulsion has the minimum strength and range necessary to produce the required values of a_0^I.

So long as the position of the zero in D_0^I is far enough to the left, it may correspond to an actual force that we have not considered,

but if it moves closer than $\nu = -9$, say, we must reject the solution as inconsistent. A more quantitative criterion is to see how badly the derivative crossing condition (IX.13) is violated by the presence of the extra pole in A_0^I, since crossing conditions must be preserved by any sensible assignment of phenomenological singularities. On this basis it seems that values of λ greater than about 0.1 are not permitted for S-dominant solutions.

Negative values of λ give no trouble, except that--as one would expect--bound states eventually develop. This happens first for $I = 0$ at $\lambda = -0.46$. Since experimentally there seems to be no bound state of the π-π system we conclude that the possible range of λ for S-dominant solutions is limited to $-0.46 < \lambda \lesssim 0.1$. Within this range there can be no resonances, for the reason discussed above in Section VII: There is no repulsive barrier to trap the wave.

A rough but adequate analytic approximation to the S-dominant π-π problem may be obtained by replacing the continuous discontinuity across the left-hand cut by a delta function, just as in the n-p example above. The position and strength of the delta function for each I value can be determined from an extended use of crossing symmetry.[22]

Unfortunately the most definite indication about the actual π-π interaction, as we shall discuss below, is that there is a P resonance at fairly low energies. It will now be shown that such a circumstance requires an important role for the discontinuity on the left-hand cut beyond the region of convergence of our Legendre polynomial expansion. A second parameter must be introduced to represent this contribution.

XI. P-DOMINANT SOLUTIONS OF THE π-π PROBLEM

The structure of our equations is such that if the P- (or any higher ℓ-) phase shift becomes large at low energies, then the double spectral functions play an essential role and we cannot ignore the region along the left-hand cut where the polynomial expansion fails. To understand this point in a concrete way, let us consider the dispersion relation for a partial-wave amplitude, for example the P wave:

$$A_1^1(\nu) = \frac{\nu}{\pi} \int_{-\infty}^{-1} d\nu' \frac{f_1^1(\nu')}{\nu'(\nu' - \nu)} + \frac{\nu}{\pi} \int_0^{\infty} d\nu' \frac{\mathrm{Im}\, A_1^1(\nu')}{\nu'(\nu' - \nu)} , \tag{XI.1}$$

$$= A_{1L}^1(\nu) + A_{1R}^1(\nu) ,$$

remembering that because of (VI.5) the absolute value of $A_1^1(\nu)$ is bounded by $\sqrt{\frac{\nu+1}{\nu}}$ in the physical region $\nu > 0$. We have not used this relation directly in the N/D procedure of Section VII, but the solution of our equations must satisfy (XI.1). The existence of $A_{1R}^1(\nu)$, the integral over the right-hand cut in (XI.1), is guaranteed by unitarity, but unless there are oscillations, the existence of $A_{1L}^1(\nu)$, the integral over the left-hand cut, requires that $f_1^1(\nu)/\nu$ approach zero as ν approaches minus infinity. A glance then at Formulas (IX.14) and (IX.16) reveals that if in the polynomial expansion for A_3^I we keep any terms of order $\ell > 1$, the asymptotic behavior of $f_1^1(\nu)$ is inconsistent with (XI.1). The contribution from $\ell = 1$ is marginal; it may lead to no worse than constant asymptotic behavior for $f_1^1(\nu)$, but even this requires additional discussion.

If $f_1^1(\nu)$ approaches a constant c_1 without oscillation, then for large ν the function $A_{1L}^1(\nu)$ behaves like $\frac{1}{\pi} c_1 \ell n \nu$, which increases without bound. Since the sum of A_{1L}^1 and A_{1R}^1 is bounded by unitarity the only possible salvation is to have a similar logarithmic term in A_{1R}^1, with the opposite coefficient. However, $\text{Im } A_1^1$ is positive and bounded by unity, so the necessary cancellation can occur only if c_1 is positive and less than one.

Returning to Formula (IX.14), we may calculate c_1 from the $\ell = 1$ part of A_3^1 to be

$$c_1 = 6 \int_0^\infty d\nu' \frac{\text{Im } A_1^1(\nu')}{\nu'} , \qquad \text{(XI.2)}$$

which happens to be positive but which certainly exceeds unity if there is a P resonance of any reasonable width. The same difficulty occurs for $f_0^{0,2}(\nu)$, so that there is no possibility of carrying through any part of the simple program for a large P-wave situation without introducing at least one new parameter to represent the portion of the left-hand cut where the polynomial expansion for A_3^I fails. Notice that the difficulty exists whether or not we attribute the P resonance to an unstable elementary particle. In that case we would introduce two new (really fundamental) parameters with the CDD pole, but we would still need another phenomenological parameter to handle the trouble on the left-hand cut.

One may ask if the difficulty with a P resonance suggests that such a resonance is unlikely in the π-π system. We think not, for a variety of reasons. For one thing, one has exactly the same kind of trouble with the $(3/2,\ 3/2)$ resonance in the π-N system, which is firmly

established experimentally. But more important, there is simply no reason to think that we can avoid the double spectral functions. There must be deep physical content there that has not yet been appreciated. We ought not be surprised if such a fundamental question as the existence of a resonance depends at least in part on simultaneous analyticity properties in two dimensions of our amplitudes.

There are useful things to be done, nevertheless, before attacking the double spectral functions in a serious way. Unitarity, for example, guarantees that on the left-hand cut the partial-wave discontinuities $f_\ell^I(\nu)$ approach constants less than unity (or at least oscillate), so that some kind of cutoff procedure seems appropriate. One possible procedure, which Mandelstam and I are studying, is the following: We arbitrarily set $f_0^{0,2}(\nu)$ and $f_1^1(\nu)$ equal to zero beyond some point $\nu_c \lesssim -9$, and add delta functions at this point to represent the remainder of the cut. The strengths of the three delta functions are adjusted to satisfy the derivative crossing relations (IX.13) so that they are all determined by the single new parameter λ_1. Of course there is some arbitrariness in the choice of cutoff, but we believe that for a definite choice of λ and λ_1 the solutions do not depend sensitively on the cutoff position unless it is unreasonably close or unreasonably far away. This belief stems from preliminary calculations in which the near and known part of the left-hand cut for each amplitude was replaced by poles.

These preliminary calculations have yielded some interesting qualitative results:

(a) The main attractive force to produce a P resonance can come from the exchange of a resonating P-wave pion pair. In other words, the

contribution to f_1^1 from A_3^1, as given by (IX.18b), is predominantly attractive and sufficiently large--if there is a P resonance--to support a P resonance. We thus have a "bootstrap" mechanism. Furthermore, this P-wave force has a long-range component that is repulsive, so that the resonance can be quite sharp.

(b) The exchange of a resonating P-wave pair produces a strong repulsive force in the $I = 0$ S state and a strong attraction in the $I = 2$ S state, so that these amplitudes have a behavior quite different from the S-dominant situation even for the same value of λ. A resonance in the $I = 0$ state is possible (although not probable) now that long-range repulsion is present, and the absolute value of (negative) λ at which a bound $I = 0$ state appears is substantially reduced.

(c) An upper limit still exists on positive λ because there continues to be no long-range repulsion in the $I = 2$ state. This limit may be slightly larger than in the S-dominant situation because we now have no problem with the $I = 0$ state in this respect.

No calculations have yet been done with a CDD pole, because so far there seems no need to complicate the situation. It is of course hoped that in the future we can understand enough about the double spectral functions to dispense with λ_1 as an independent parameter. Even if that is possible, however, we may still find more than one solution for a given value of λ. Our so-called "coupling constant," after all, is essentially the value of the S-wave amplitudes at a point near zero kinetic energy, and it is well known that quite different sets of forces can lead to the same scattering at one particular energy.

XII. THE VERTEX FUNCTION OF ONE VARIABLE: PION ELECTROMAGNETIC STRUCTURE AS AN EXAMPLE

If we were completely to exclude weak and electromagnetic interactions from our considerations there would be no need to study the so-called vertex functions--which are associated with three-particle diagrams. These have entered our strictly strong-interaction approach only when all three particles are on the mass shell--where one is dealing with a single number, not a function. If we want to take advantage, however, of the established validity of lowest-order perturbation theory with respect to the fine-structure constant and the Fermi weak interaction constant, in discussing certain experiments involving photons or leptons as well as strong interactions, then we find it necessary to analyze three-particle vertices where the mass of one of the particles is considered a complex variable.

Suppose, for example, that we want to describe the deviations from Coulomb scattering of an electron by a charged pion, deviations due to the "structure" of the pion associated with its strong interactions. That is to say, a real pion can emit virtual strongly interacting systems of appropriate quantum numbers that produce a charge distribution in its neighborhood. By electron elastic scattering of sufficiently large momentum transfer we should be able to probe the "structure" of this charge distribution and learn something about the strong interactions that produce it, if we assume that the electromagnetic aspects of the situation are completely understood. The latter statement can be given a concrete meaning in terms of Fig. 9, which represents

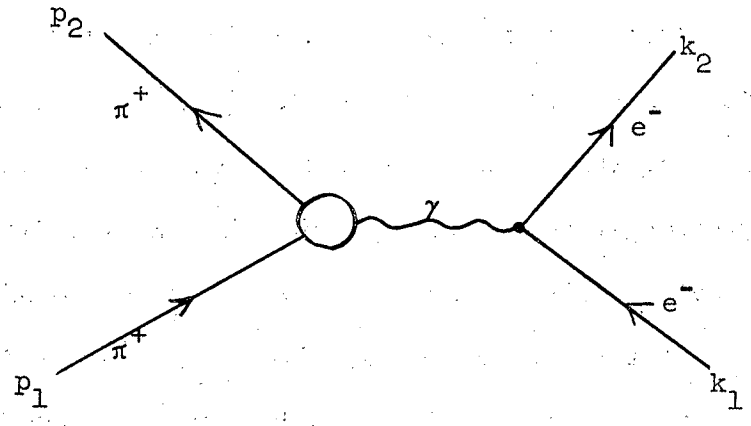

Fig. 9. Diagram for electron-pion elastic scattering via a single virtual photon.

all diagrams in which a single photon is exchanged between the electron and the pion; multiple photon exchanges should be weaker in order of magnitude by a factor $e^2 = 1/137$. The single-photon part of the amplitude, ignoring electron and photon spin,* can be written

$$F^{\gamma}_{\pi^+\pi^-}(t)\, S^{\gamma}(t)\, F^{\gamma}_{e^+e^-}(t)\,, \qquad \text{(XII.1)}$$

just as for Fig. 4, where S^{γ} is the photon propagator and $F^{\gamma}_{\pi^+\pi^-}$ and $F^{\gamma}_{e^+e^-}$ are vertex functions, all three depending on

$$t = (p_1 - p_2)^2 = (k_1 - k_2)^2 = -2q^2(1 - \cos\theta), \qquad \text{(XII.2)}$$

if as usual q and $\cos\theta$ are the barycentric-system variables. What is meant when we say we "understand" the purely electromagnetic aspects

* Since the pion has zero spin, Fig. 9 involves only one invariant function of t even when the electron and photon are correctly described.

of the problem is the assumption that, to an accuracy of order e^2, we have

$$S^{\gamma}(t) = \frac{1}{t} , \qquad \text{(XII.3)}$$

and

$$F^{\gamma}_{e^+e^-}(t) = e . \qquad \text{(XII.4)}$$

There are of course known modifications of the photon propagator and the electron-photon vertex that play an important role in low-t electrodynamical experiments of high accuracy; however, these modifications, being of order e^2, are much smaller than expected effects from the pion-photon vertex.

It is not certain that (XII.3) and (XII.4) are correct for the large values of t in which we shall be interested, since these formulas have been tested only for relatively small t. A failure for large t is what is commonly referred to as "a breakdown of quantum electrodynamics at short distances," and a search for such a breakdown is the main object of the extensive electron accelerator program at Stanford University. No evidence has yet been developed, however, that we cannot trust (XII.3) and (XII.4) to order e^2 for all t; therefore we shall assume here that a measurement of electron-pion elastic scattering is essentially a measurement of the pion-photon vertex function.

The physical range of t in elastic scattering, according to (XII.2), is $t < 0$. However, one can in addition measure electron-positron annihilation to produce a $\pi^+\pi^-$ pair, a process that according to the substitution law is also determined by (XII.1). In this case the physical range is $t > 4$. We can in principle, then, measure $F_{\pi^+\pi^-}(t)$ over the whole real axis except for the gap, $0 < t < 4$; so let us now see what predictions can be made about this vertex function.

The principles to be used can be applied to many other vertex problems, involving nucleons rather than pions and weak rather than electromagnetic "probing" interactions. Actually the nucleon-photon vertex is the one for which most data are available, but we shall see in the following section that to understand nucleon structure a prior understanding of pion structure is required. In addition there are the usual complications due to nucleon spin that tend to obscure the essential ideas involved.

It has been shown by a study of Feynman diagrams, as well as by more rigorous methods for certain special cases, that a vertex function of one variable, except for a possible subtraction, satisfies the representation

$$F(t) = \frac{1}{\pi} \int dt' \frac{g(t')}{t' - t} , \qquad \text{(XII.5)}$$

where the real spectral function $g(t)$ is nonvanishing* only for t equal to the square of the mass of a possible intermediate physical state having the quantum numbers of the single particle whose mass is the variable, as well as of the pair of particles on the other side of the vertex. The single external particle itself, however, is to be excluded from the spectrum. In our example we need to enumerate strongly interacting states having the quantum numbers of a photon as well as a $\pi^+\pi^-$ pair. Even though an external photon is involved, we need not worry

* As in the case of the Mandelstam representation, there may be anomalous thresholds for certain mass ratios. (See, for example, Karplus, Sommerfield, and Wichmann.[24]) We shall not consider such cases here.

about photons in intermediate states, because these give contributions to our spectral function $g(t)$ which are small, of order e^2.

Using charge independence and charge-conjugation symmetry, as well as other well-known conservation laws, we conclude that our intermediate states must have $I = 1$, $I_3 = 0$, $J = 1$, be odd under both space reflection and charge conjugation, and of course have zero strangeness and zero baryon number. The lightest such state is the P-wave pion pair, then we jump to 4, 6, 8...pions, $K\bar{K}$ pairs, etc. It follows that the lower limit of integration in (XII.3) is at $t' = 4$ and that $F_\pi(t)$ is real for $t < 4$. For $t > 4$, we have

$$\text{Im } F_\pi(t) = g_\pi(t) , \tag{XII.6}$$

so if we remember that $F_\pi(t)$ is proportional to and has the same phase as the matrix element

$$\langle \pi^+(p_2), \pi^-(-p_1) \mid T \mid \gamma(p_2 - p_1) \rangle ,$$

where the fictitious initial state consists of a photon with mass $\sqrt{t}$, and also remember the unitarity condition (V.3), then we see

$$g_\pi(t) = \text{real constant} \times \sum_m \langle m \mid T \mid \pi^+(p_2), \pi^-(-p_1) \rangle^* \times \langle m \mid T \mid \gamma(p_2 - p_1) \rangle . \tag{XII.7}$$

For $4 < t < 16$, only the 2π intermediate state contributes, and we have

$$g_\pi(t) = \text{real constant} \times A_1^{1*}(t)\, F_\pi(t) , \quad \text{for } 4 < t < 16 , \tag{XII.8}$$

where A_1^1 is the $I = 1$, $\ell = 1$ partial wave π-π amplitude introduced

earlier (but considered here as a function of $t = 4(\nu + 1)$, which is a more convenient variable). Now $g_\pi(t)$ is supposed to be real, so according to (XII.8) the vertex function $F_\pi(t)$ must have the same phase as the P-wave amplitude along the real axis for $4 < t < 16$; and to the extent that we do not expect inelastic π-π scattering to compete seriously for $t \lesssim 40$, we may use the simple phase condition over the wider interval.

In fact, it is consistent with our earlier neglect of π-π inelastic scattering (i.e. we set $R_\ell = 1$ over the entire positive real axis) to use the simple phase condition over the whole cut of the vertex function $F_\pi(t)$. If we do so, the solution of the pion-structure problem is amazingly simple:

$$F_\pi(t) = \frac{e\, D_1^1(0)}{D_1^1(t)} \quad . \tag{XII.9}$$

The point is that Formula (XII.5) simply requires $F_\pi(t)$ to be a real analytic function in the entire complex t plane, with the single cut running along the positive real axis from 4 to ∞. Otherwise, unitarity requires $F_\pi(t)$ to have the phase $\exp(i\delta_1^1)$ along the cut. These are exactly the defining properties of the reciprocal of our π-π P-wave denominator function, so we conclude that $F_\pi(t)$ must be proportional to the reciprocal of D_1^1. Gauge invariance guarantees

$$F_\pi(0) = e \quad ,$$

leading to the complete result (XII.9), first written in this form by Frazer and Fulco.[25] Earlier Federbush, Goldberger, and Treiman[26] had derived the result in the Omnes form (VII.3), which is suitable if one has no particular knowledge about analytic properties of the phase shift.

It is evident from (VII.3) and (XII.9) that if the P-wave π-π phase shift is small for $0 < \nu \lesssim 10$ (or $4 < t \lesssim 40$), then the pion charge structure factor differs very little from e over a corresponding range of t along both positive and negative axes. In other words, the pion behaves almost as a point charge for electron scattering, and the cross section for π^+-π^- production in e^--e^+ annihilation is "normal." Suppose, on the other hand, that there is a reasonably sharp dynamical resonance at $t = t_R$, that is to say, the real part of the denominator function vanishes at t_R, with a nearly linear dependence extending down to $t = 0$.* (See Fig. 10.) Such a behavior is typical of resonances due

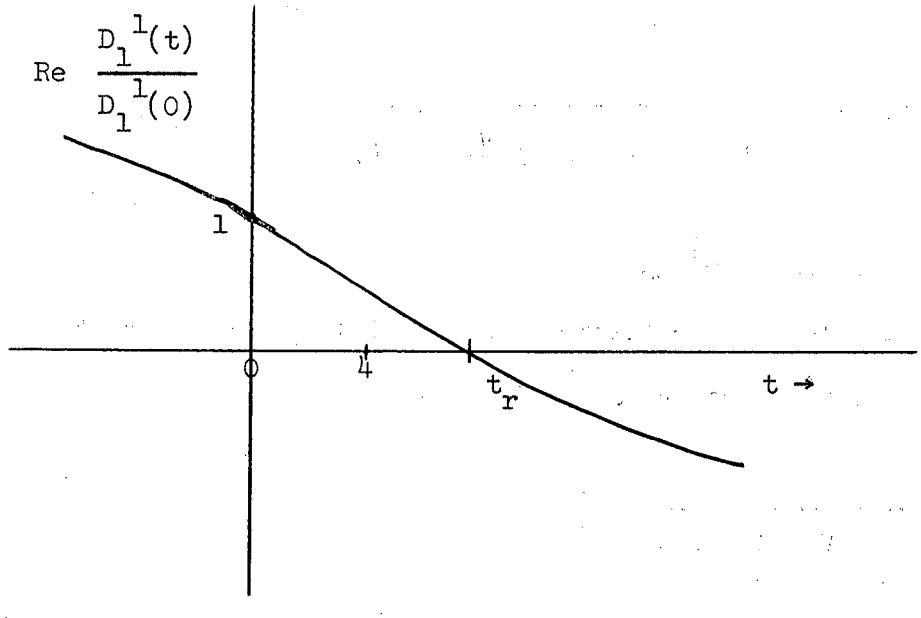

Fig. 10. Typical behavior of the real part of the denominator function for a dynamical resonance.

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* There is of course a branch point at $t = 4$, but it produces a discontinuity only in second and higher derivatives of $D_1^1(t)$.

to attractive "forces" inside a repulsive barrier. (In contrast, a resonance due to an unstable elementary particle has a pole in the denominator function, close to t_r. Such a resonance generally causes a completely different type of behavior for the structure function from that described here.) At the same time, of course, the imaginary part of D_1^1 does not vanish. According to (VII.5) and (VII.18), we may expect a behavior

$$\text{Im } D_1^1 = - \sqrt{\frac{\nu^3}{\nu + 1}} \; \Gamma(t) \quad ,$$

where $\Gamma(t)$ is a slowly varying function if the important contributions from the left-hand cut in (VII.18) are not too close. Thus, we have the rough formula

$$\frac{1}{e} F_\pi(t) \approx \frac{t_r}{t_r - t - i \Gamma \sqrt{\frac{\nu^3}{\nu + 1}} \; \theta(\nu)} \qquad \text{(XII.10)}$$

that can be used for $|t| \lesssim t_r$.

In the electron-scattering region, $t < 0$, we see that the cross section is simply reduced by a smooth factor

$$\left[\frac{t_r}{t_r + 2q^2(1 - \cos \theta)} \right]^2 . \qquad \text{(XII.11)}$$

Comparing to what we would get from a classical extended charge $\rho(\vec{r})$,

$$F_\pi = \int d\vec{r} \; e^{i(\vec{q}_1 - \vec{q}_2)\cdot\vec{r}} \; \rho(r) \qquad \text{(XII.12)}$$

$$\approx e\left(1 - \frac{\overline{r^2}}{6} (\vec{q}_1 - \vec{q}_2)^2 + \ldots \right) , \qquad \text{(XII.13)}$$

we see that the "mean-square radius" of the pion is related to the resonance energy by

$$\frac{\overline{r^2}}{6} \approx \frac{1}{t_r} \quad . \tag{XII.14}$$

A completely different kind of effect would be observed in the cross section for $e^+ + e^- \rightarrow \pi^+ + \pi^-$. There the "normal" cross section is multiplied by a resonance factor

$$\frac{t_r^{\,2}}{(t_r - t)^2 + \Gamma^2 \frac{\nu^3}{\nu + 1}} \tag{XII.15}$$

that greatly enhances the reaction for t near t_r. Notice that for a fixed position of the resonance, t_r, the net enhancement becomes greater as the width of the resonance decreases because the maximum value of (XII.15) varies inversely as the square of the width Γ. This is a characteristic feature of a dynamical resonance that distinguishes it from a resonance associated with an unstable particle.

XIII. THE PROCESSES $\pi + N \leftrightarrow \pi + N$ AND $\pi + \pi \leftrightarrow N + \bar{N}$

Before we can discuss the electromagnetic structure of the nucleon it is necessary to analyze the matrix element for the process $\pi + \pi \leftrightarrow N + \bar{N}$, since the 2π state is expected to play an important role in the spectral function for the nucleon-photon vertex. If we are to understand one channel of a diagram of the type of Fig. 3, however, it is necessary to discuss the other two channels at the same time; thus pion-nucleon scattering is drawn into the picture. Of course, there is every reason to study π-N scattering in its own right; probably more experimental information is available here than for any of the other reactions that have been considered.

The basic invariance considerations for the π-N problem have been already outlined in Section IV. We have four invariant amplitudes $A^{0,1}$ and $B^{0,1}$ satisfying the Mandelstam representation. It turns out that $A^{0,1}$ have no poles, while the residues of the Channel I and Channel II poles in $B^{0,1}$ are just plus or minus the rationalized pion-nucleon coupling constant,

$$g^2 = \frac{g_r^2}{4\pi} \approx 14 . \qquad \text{(XIII.1)}$$

(The value of g^2 is obtained most accurately from a comparison with experiment of π-N dispersion relations at $s_2 = 0$, i.e., in the forward direction.) We find, in other words,

$$B^{\binom{0}{1}}_{\text{poles}}(s_1, s_2, s_3) = \frac{g_r^2}{M^2 - s_3} \mp \frac{g_r^2}{M^2 - s_1} , \qquad \text{(XIII.2)}$$

Considering first Channel II, $\pi + \pi \leftrightarrow N + \bar{N}$, we follow Frazer and Fulco[12] and introduce the barycentric-system variables q_2, p_2, and $\cos \theta_2$,

where q_2 is the magnitude of a pion momentum and p_2 that of a nucleon, while θ_2 is the production angle. We find

$$\begin{aligned} s_1 &= -p_2^2 - q_2^2 - 2\, p_2\, q_2 \cos\theta_2 \ , \\ s_2 &= 4(q_2^2 + 1) = 4(p_2^2 + M^2) \ , \\ s_3 &= -p_2^2 - q_2^2 + 2\, p_2\, q_2 \cos\theta_2 \ . \end{aligned} \tag{XIII.3}$$

Next we define*

$$\left(A_\ell^{II,T}(s_2);\ B_\ell^{II,T}(s_2)\right) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta_2\, P_\ell(\cos\theta_2) \times \left(A^T(s_2,\cos\theta_2);\ B^T(s_2,\cos\theta_2)\right) ,$$

realizing that the index ℓ here is not the orbital angular momentum. We know from crossing symmetry that A^0 and B^1 are even functions of $\cos\theta_2$, while A^1 and B^0 are odd functions, so that $A_\ell^{II,0}$ and $B_\ell^{II,1}$ vanish for odd ℓ while $A_\ell^{II,1}$ and $B_\ell^{II,0}$ vanish for even ℓ.

Frazer and Fulco then introduce helicity amplitudes, $f_\pm^J(s_2)$, for well-defined angular momentum J. The subscript (+) indicates that both nucleon and antinucleon have the same helicity, while (-) indicates opposite helicities. Using the technique of Jacob and Wick[27] they find, for each T,

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* We use T to designate isotopic spin here in order to avoid confusion with the channel index.

$$f_+^J = \frac{1}{4\pi}\left\{-\frac{p_2^2}{(p_2q_2)^J}A_J^{II} + \frac{M}{(2J+1)(p_2q_2)^{J-1}}[(J+1)B_{J+1}^{II} + J\,B_{J-1}^{II}]\right\}, \tag{XIII.5}$$

$$f_-^J = \frac{1}{4\pi}\,\frac{[J(J+1)]^{1/2}}{2J+1}\,\frac{1}{(p_2q_2)^{J-1}}\,[B_{J-1}^{II} - B_{J+1}^{II}], \tag{XIII.6}$$

and show that these helicity amplitudes are analytic functions of s_2, with a cut associated with Channel II running from 4 to ∞ along the positive real axis, and coincident cuts due to Channels I and III running from $s_2^L = 4 - 1/M^2$ to $-\infty$.

Problem 5: Show that a fixed value of s_3' in the spectrum of Channel III gives rise to a cut in the helicity amplitude running from

$$\frac{4M^2 - (s_3' - M^2 - 1)^2}{s_3'} \quad \text{to} \quad -\infty .$$

The discontinuity across the left-hand cut has two parts, one from the poles starting at s_2^L and one from the continuous π-N spectra starting at $s_2 = 0$. Frazer and Fulco give explicit formulas for this discontinuity in terms of g_r^2 and A_3^T and B_3^T, the absorptive parts of the pion-nucleon elastic-scattering amplitude. We shall not write down these formulas because of their complexity; they have basically the same structure as (IX.14), except for the additional term proportional to g_r^2 that controls the left-hand cut for $0 < s_2 < s_2^L$.

Next Frazer and Fulco consider the right-hand cut and show that for $4 < s_2 < 16$ unitarity requires the helicity amplitudes to have the

same phase as the π-π amplitude in the corresponding state (same J and T). The heuristic derivation of this result parallels the discussion of the preceding section, but one must be careful because N-$\bar{N}$ states cannot really exist for $s_2 < 4M^2 = 180$. Mandelstam, however, has recently given a rigorous justification for the naive extension of the unitarity condition in the $\pi + \pi \leftrightarrow N + \bar{N}$ channel.[10] The next step is to argue, as before, that if we are concerned only with $| s_2 | \lesssim 40$, we may use the simple phase condition for the entire right-hand cut; the helicity amplitudes may then be explicitly written down in terms of the discontinuity over the left-hand cut and the appropriate π-π denominator function:

$$f_{\pm}^{J,T}(s_2) = \frac{1}{\pi D_J^T(s_2)} \int_{-\infty}^{s_2^L} ds_2' \frac{\mathrm{Im}\, f_{\pm}^{J,T}(s_2') D_J^T(s_2')}{(s_2' - s_2)} \quad \text{(XIII.7)}$$

We see by inspection that this formula makes the helicity amplitude an analytic function with the two desired cuts, and at the same time assigns the correct phase on the right-hand cut and the correct discontinuity across the left-hand cut.

Two different attitudes may be adopted at this stage. A great deal of experimental information exists about Channels I and III, $\pi + N \leftrightarrow \pi + N$, so that the required discontinuity over the left-hand cut for the $\pi + \pi \leftrightarrow N + \bar{N}$ partial-wave amplitudes is empirically known for a substantial distance. Also the point $s_2 = 0$ is physical for π-N scattering, so that the $\pi + \pi \leftrightarrow N + \bar{N}$ helicity amplitudes and some of their derivatives are known at the origin. With all this information, D. Wong has shown that one can quite reliably calculate the helicity amplitudes along the right-hand cut for $s_2 \lesssim 40$, provided the π-π

amplitudes are also known in this region.[11,28] This approach seems sensible if one wants the $\pi + \pi \leftrightarrow N + \bar{N}$ amplitudes in order to predict such things as nucleon-electromagnetic structure and nucleon-nucleon scattering. If one proposes to predict π-N scattering, however, the logic of the situation becomes confused. Strictly speaking, one should regard the problem of $\pi + \pi \leftrightarrow N + \bar{N}$ and $\pi + N \leftrightarrow \pi + N$ amplitudes as one of codetermination, using the kind of iteration procedure described above for the π-π problem. If one were to do this, exactly the same kind of difficulties as encountered in the π-π problem would arise and arbitrary parameters would have to be introduced. How many such parameters are needed is not yet known, because the problem is much more complex than in the π+π case. Let us now take at least a brief look at some of the complications arising in a study of the Channel I and Channel III partial-wave amplitudes.

Because of crossing symmetry it suffices to study Channel III, where we introduce the usual barycentric-system variables, q_3 and $\cos \theta_3$:

$$q_3^2 = \frac{[s_3 - (M+1)^2][s_3 - (M-1)^2]}{4 s_3},$$

$$s_2 = -2 q_3^2 (1 - \cos \theta), \qquad \text{(XIII.8)}$$

$$s_1 = \frac{1 - \cos \theta_3}{2} \frac{(M^2 - 1)^2}{s_3} - \frac{1 + \cos \theta_3}{2} (s_3 - 2M^2 - 2).$$

These formulas already suggest the woe in store for us as a result of the unequal masses of pion and nucleon. Next we define $A_\ell^{III}(s_3)$ and $B_\ell^{III}(s_3)$ in the usual way and relate these quantities to amplitudes for well-defined J and parity. The conventional notation here is to write $f_{\ell\pm}^{III}$, where ℓ is the orbital angular momentum and $(\pm)$ refers to $J = \ell \pm 1/2$. In the

physical region we have

$$f_{\ell\pm}^{III} = \frac{e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm}}{q_3} . \qquad \text{(XIII.9)}$$

Henceforth we shall drop the superscript III. The relation between A_ℓ, B_ℓ, and $f_{\ell\pm}$ turns out to be[14]

$$f_{\ell\pm}(W) = \frac{1}{8\pi W} \left\{ (E + M)[A_\ell + (W - M)B_\ell] + (E - M)[-A_{\ell\pm1} + (W + M)B_{\ell\pm1}] \right\} , \qquad \text{(XIII.10)}$$

where $W = \sqrt{s_3}$ and

$$E = \sqrt{q_3^2 + M^2} = \frac{W^2 + M^2 - 1}{2W} . \qquad \text{(XIII.11)}$$

Examination of (XIII.10) shows that $f_{\ell\pm}$, as a function of s_3, has a "kinematical" branch point at the origin but that as a function of W all the singularities are of the usual "dynamical" type--that is, they arise from the vanishing of denominators in the original Mandelstam representation. McDowell[14] emphasized that an interesting reflection property exists in the W plane--which encompasses two sheets of the usual s_3 plane. We see from (XIII.11) that we have

$$f_{\ell+}(-W) = - f_{(\ell+1)-}(W) , \qquad \text{(XIII.12)}$$

so if we work in the full W plane, the two amplitudes for the same J may be considered as a single analytic function in different halves of the plane. In view of this circumstance and for certain other reasons associated with the peculiarities of the kinematical relations (XIII.8)

and too complicated to discuss here, Frazer and Fulco[20] introduce

$$h_\ell(W) = \frac{W}{E + M} \frac{f_{\ell+}(W)}{q_3^{2\ell}} \qquad \text{(XIII.13)}$$

as the most convenient analytic function. For W real and greater than $M + 1$ this function is related to the phase shift for total angular momentum J and orbital angular momentum $\ell = J - 1/2$, while for W real and less than $-(M + 1)$ we are dealing with the phase shift for the same J but $\ell = J + 1/2$.

The singularities of $h_\ell(W)$ are very complicated. First there are the two physical cuts, running from $M + 1$ to $+\infty$ and from $-(M + 1)$ to $-\infty$, both on the real axis. For $J = 1/2$ and $I = 1/2$, there is also the nucleon pole in h_0 on the left half plane at $W = -M$. These are all the singularities arising from Channel III. The original pole in Channel I (also $\pi + N \leftrightarrow \pi + N$) leads to short branch cuts, on the positive real axis for $M - \frac{1}{M} \leqslant W \leqslant (M^2 + 2)^{1/2}$ and on a corresponding range of the negative real axis. These cuts are so short that they are well approximated by poles. A second branch cut arising from the original Channel I pole runs along the entire length of imaginary axis. Before proceeding further with our enumeration of singularities it is useful to discuss why the original nucleon pole leads in this way to both a long-range and a short-range force.

We have already in the introduction discussed the long-range force (due to the short cuts near the physical regions) a consequence of the "exchange" of a nucleon. (See Fig. 11(a).) The general structure of the theory, however, requires that the s_1 pole also represents the "exchange" of an antinucleon originating with the incident pion. (Fig. 11(b).) This

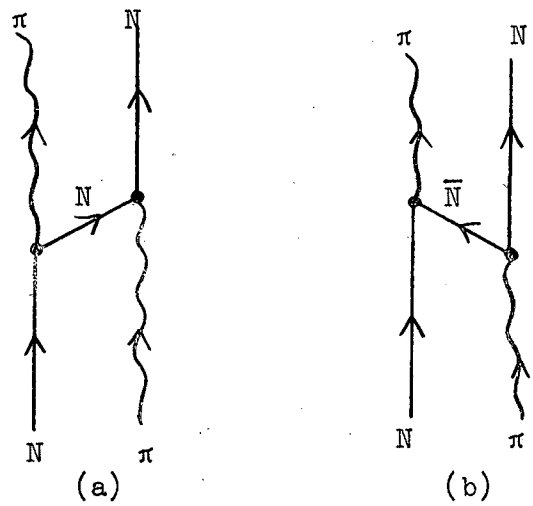

Fig. 11. The two exchange processes represented by the pole in the "crossed" πN channel.

force is of very short range because energy conservation is badly violated, and it corresponds to the cut along the imaginary axis--far from the physical region. (See Fig. 12)

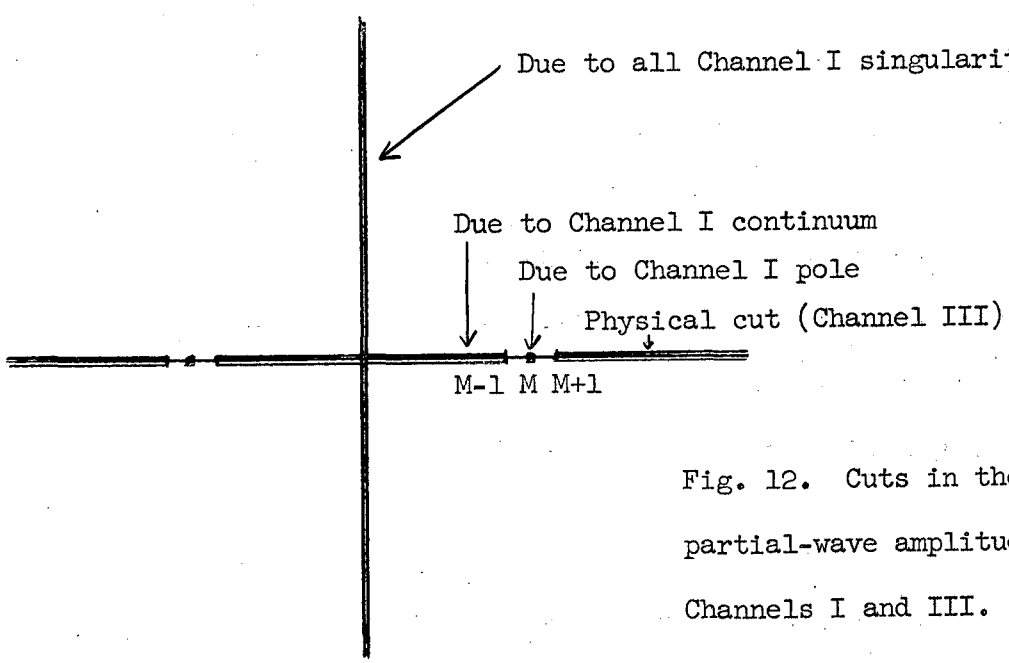

Fig. 12. Cuts in the Channel III partial-wave amplitudes due to Channels I and III.

The continuum singularities of Channel I, starting with the one-pion-one-nucleon contributions, similarly lead to two cuts, and for the same reason. The "near-by" cuts run from $(M - 1)$ and $-(M - 1)$ to the origin, while the far cuts again run along the imaginary axis.

The intermediate states of Channel II turn out to produce a complicated cut in the Channel III partial-wave amplitudes, partly on the imaginary W axis but also in part following a circle of radius M with center at the origin. The ends of this cut come close to the physical regions and can be interpreted as long-range π-N forces due to 2π exchange.

Formulas for the discontinuities across all the various cuts have been worked out in terms of absorptive parts for appropriate channels,[20,29] but one finds, just as for π-π, that the polynomial expansion of the absorptive parts suffices to calculate only the near-by discontinuities. An extensive investigation of just how much can be understood about the observed phase shifts in terms of near-by and calculable singularities has been undertaken by Frautschi and Walecka[29] as well as others. Final results are not yet available, but the qualitative success of the static model has been to a certain extent understood. It has been shown, for example, that the π-π cut has only a weak effect in the $I = 3/2$, $J = 3/2$ state, as does the near-by part of the crossed π-N continuum cut, so the dominant near-by singularity in this state is the short cut, near $W = M$, which can be approximated by a pole of residue $\sim g_r^2$. Replacing all other singularities by a distant pole then leads to the Chew-Low effective range behavior for δ_{33},[3]

$$\frac{4}{3} f^2 \frac{q_3^3}{W - M} \cot \delta_{33} \approx \frac{W_R - W}{W_R - M} , \qquad \text{(XIII.14)}$$

where $f^2 = \frac{1}{4M^2} g^2 \approx 0.08$ and W_R is the energy of the 3/2, 3/2 resonance. The value of W_R is related to the strength of the distant phenomenological pole and cannot be predicted until we have better calculational methods. However, the width of the resonance is correctly predicted in terms of f^2.

XIV. NUCLEON ELECTROMAGNETIC STRUCTURE

The basic methods of the S-matrix approach to strong interaction theory have now been outlined and several examples discussed. We conclude these lectures with an example of unusual importance from a practical standpoint. This is the problem of the nucleon-photon vertex, or in more familiar language, the electromagnetic structure of the nucleon. Recalling the considerations of Section XII, we realize that the nucleon-photon vertex function can be experimentally measured in the range $t < 0$ by electron-nucleon elastic scattering and for $t > 4M^2 \approx 180$ by $N + \bar{N} \leftrightarrow e^+ + e^-$. The latter range is very high from the point of view of our approach; however, the cut in the photon-nucleon vertex function does not begin at $4M^2$ but rather at 4, the two-pion threshold. By the usual arguments, we should expect that the discontinuity across the cut for $4 < t \lesssim 40$ should dominate the behavior of the function for $|t| \lesssim 40$; thus it is reasonable to try to understand electron-nucleon scattering up to a few Bev electron energy (lab) in terms of two- and three-pion contributions to the spectral function of the photon-nucleon vertex.

Actually four invariant functions are involved in the photon-nucleon vertex, because of the nucleon spin and charge degrees of freedom. Using the same kind of notation as in (IV.1), we would write[30]

$$\bar{u}_{p_2} \left\{ i\, \gamma\cdot\epsilon\, [G_1^S(t) + \tau_3 G_1^V(t)] \;+\; \gamma\cdot\epsilon\, \gamma\cdot(p_1 - p_2)[G_2^S(t) + \tau_3 G_2^V(t)] \right\} u_{p_1}, \tag{XIV.1}$$

where ϵ is the photon polarization vector and where the superscripts S and V refer to the isotopic scalar and vector parts of the electromagnetic

interaction.* The vertex functions $G(t)$ are related at $t = 0$ to the static nucleon charges and anomalous magnetic moments:

$$G_1^S(0) = G_1^V(0) = e/2 \ , \qquad \text{(XIV.2a)}$$

$$G_2^S(0) = \frac{\mu_P + \mu_N}{2} = -0.06 \frac{e}{2M} \ , \qquad \text{(XIV.2b)}$$

$$G_2^V(0) = \frac{\mu_P - \mu_N}{2} = 1.83 \frac{e}{2M} \ . \qquad \text{(XIV.2c)}$$

Each of the four functions $G(t)$ has a spectral representation of the type (XII.5) with the associated spectral functions given by a formula of the type (XII.7), except that the π^+-π^- state is replaced by an N-$\bar{N}$ state, with $I = 0$ for the isotopic scalar functions and $I = 1$ for the vector functions. The other quantum numbers are the same as for the pion-photon vertex.

It then turns out[26,31] that only even-pion intermediate states contribute to the isotopic vector part of nucleon electromagnetic structure and only odd-pion states to the scalar part. Considering first the vector part, let us calculate the spectral functions in the range $4 < t < 16$ where only 2π states contribute. Evidently the result is a product of the pion-photon vertex function $F_\pi(t)$ and the amplitudes for $\pi + \pi \leftrightarrow N + \bar{N}$ in the $I = 1$, $J = 1$ state. In the preceding section we saw that there are two such functions, which may be chosen to be the helicity amplitudes $f_\pm^{1,1}(t)$. Frazer and Fulco found the result[25]

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* Assuming "minimal" electromagnetic interactions, a photon can be thought of as carrying either isotopic spin 0 (scalar) or isotopic spin 1 (vector), but nothing more complicated. This follows from the fact that electric charge and the z component of I spin are linearly related.

$$g_1^V(t) = -F_\pi(t)\frac{\nu^{3/2}}{\sqrt{t}}\frac{M}{M^2-\frac{t}{4}}\left[\frac{S}{4\sqrt{2}\,M}f_-^{1,1}(t) - f_+^{1,1}(t)\right]^*,$$

(XIV.3)

$$g_2^V(t) = F_\pi(t)\frac{3/2}{\sqrt{t}}\frac{1}{M^2-\frac{t}{4}}\left[\frac{M}{\sqrt{2}}f_-^{1,1}(t) - f_+^{1,1}(t)\right]^*,$$

which may plausibly be used up to $t \sim 40$ if 4-pion states behave as we expect.

A priori we do not know how important in the spectral function is the region above that where the 2π state dominates. Experimentally, however, the measured slope of the functions $G(t)$ at $t = 0$ gives us the mean value of $1/t$ in weight function $g(t)/t$.[32] Thus if no subtraction is necessary, so that we write

$$G_i^V(t) = \frac{1}{\pi}\int_4^\infty dt' \frac{g_i^V(t')}{t'-t}, \qquad \text{(XIV.4)}$$

then we have

$$\left(\frac{1}{t}\right)_{Av,i}^V = \frac{\left[\frac{d}{dt}G_i^V(t)\right]_{t=0}}{G_i^V(0)}. \qquad \text{(XIV.5)}$$

There are substantial theoretical reasons for expecting no subtraction in $G_2^{S,V}(t)$ and weaker ones for $G_i^{S,V}(t)$. In any case, it is found experimentally that,[32] with (XIV.5) as a definition of $\left(\frac{1}{t}\right)_{Av.}$, we have

$$\left(\frac{1}{t}\right)_{Av,1}^V \approx \left(\frac{1}{t}\right)_{Av,2}^V \approx \frac{1}{12}. \qquad \text{(XIV.6)}$$

This circumstance strongly implies a dominant role for the 2π state in g_2^V and probably also in g_1^V.

Frazer and Fulco[25] proceeded to calculate $G_{1,2}^{V}(t)$ on the basis of (XIV.4) and (XIV.3), with various assumptions for the P-wave π-π phase shift, which controls $f_{\pm}^{1,1}(t)$ through Eq. (XIII.7) as well as $F_{\pi}(t)$ through Eq. (XII.9). They found, as had earlier been emphasized by Drell,[33] that with a small P-phase shift, the value of $G_{2}^{V}(0)$ (the vector anomalous moment) is too small by a factor of about five unless large contributions to (XIV.4) come from the unknown region $t' > 40$. In such a case, however, the experimental result (XIV.6) is incomprehensible. In order to achieve (XIV.6) Frazer and Fulco had to assume a resonance in the π-π P wave at $t_r \sim 12$ which greatly enhances this part of the spectrum. The mechanism of enhancement is quickly seen by reference to (XIV.3) and (XIII.7), from which follows $g_{2}^{V}(t) \sim |F_{\pi}(t)|^2$, so the spectral functions have the behavior (XII.15) in the neighborhood of the resonance. The average enhancement depends on the width parameter Γ; to achieve the experimental result (XIV.2c) it appears that $\Gamma \approx 0.4$ is required.

Thus, the large nucleon anomalous magnetic moment together with the large radius of this moment, $\left(\left(\frac{1}{t} \right)_{Av} = \frac{1}{6} \overline{r^2} \right)$, suggests a fairly narrow dynamical resonance in the P state of the π-π system, and in Section XI above we pointed out that such a resonance can come about through a "bootstrap" attractive force. Of course, if the resonance exists there must be other experimental manifestations; the cleanest suggested so far is the enhancement by the factor (XII.15) of pion-pair production in electron-positron annihilation. In general one would expect enhancement of this kind whenever pion pairs are produced, but usually there are other strongly interacting particles simultaneously present which confuse the situation. The other typical manifestation is the enhancement of the unphysical singularities in various amplitudes due to pion pairs. The

nucleon-photon vertex of course falls in this category, and we may remark now on the consequence of a P-wave π-π resonance for $G_1^V(t)$, the charge-vertex function, as well as on $G_2^V(t)$, the magnetic-moment vertex discussed above.

If we accept (XIV.4) for $G_1^V(t)$ without a subtraction, then the experimental near equality of the vector charge and magnetic-moment radii of the nucleon is immediately a consequence of a reasonably sharp π-π resonance. That is, both $g_1^V(t)$ and $g_2^V(t)$ are proportional to $|F_\pi(t)|^2$, so that the average value of $1/t$ should be $\sim 1/t_r$ for both. One may ask if (XIV.2a) is consistent with no subtractions in $G_1^V(t)$; a tentatively affirmative answer has been given by Frazer and Fulco,[25] considering the uncertainty in certain aspects of our knowledge of the $\pi + \pi \leftrightarrow N + \bar{N}$ helicity amplitudes $f_\pm^{1,1}(t)$. We have in the equality of charge and magnetic radii, therefore, some confirmation of the resonance hypothesis. Many other amplitudes are being similarly analyzed to discover the effect of a π-π P-wave resonance. These include N-N scattering,[11] π-N scattering,[29,34] and K-N scattering,[35] as well as $\gamma + N \leftrightarrow \pi + N$,[36] in each of which a strong 2π exchange interaction should have a marked effect. Calculations in all these cases are more complicated than for nucleon electromagnetic structure, and results are still inconclusive.

We conclude with a brief mention of $G_{1,2}^S(t)$, the isotopic scalar part of the nucleon-photon vertex. Experimentally $G_2^S(t)$ is very small over the entire range of t studied, while $G_1^S(t) \approx G_1^V(t)$. The latter circumstance means a low average value of t in $g_1^S(t)$ and, with no subtractions, suggests a low-energy resonance or perhaps even a bound state for the three-pion system with quantum numbers $J = 1$, $I = 0$

and odd parity. That such a state may feel an unusually strong total attraction follows from the fact that all three pairs are in the $I = 1$ configuration,[37] where we have conjectured a strong attractive force. Why the contribution to $g_2^S(t)$ from such a three-pion state should be small is not clear, but will not constitute a definite difficulty until we have methods for handling such multiparticle configurations. That the scalar charge radius is so close to the vector charge radius appears a coincidence from the present point of view. If it is not a coincidence, then the approach described in these lectures is deficient in some very basic respect.

XV. CONCLUSION

We have seen how the Mandelstam representation for scattering amplitudes can be used together with unitarity to make a theory of long-range strong interactions--more precisely, of those interactions due to exchange of one- and two-particle systems. The theory will not be complete until we understand how to handle general multiparticle exchanges, but we have seen that certain experimental questions can already be discussed in a meaningful way. That is to say, different kinds of experiments can be correlated even though the underlying short-range forces are not completely understood. We have also attempted here to make plausible that the S-matrix approach, when developed into a complete theory by extension to many-particle systems, will contain no more free parameters than conventional renormalizable field theory.

A more concrete aspect of these lectures has centered around the pion-pion interaction, which in a certain sense is fundamental for all strong-interaction questions. Pion-pion scattering is experimentally elusive, but we have seen that it must be understood before further substantial progress is possible in the theory of strong interactions.

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