Title
On embedding 4-manifolds in R7

Journal
Mathematical Proceedings of the Cambridge Philosophical Society, 61(3)

ISSN
0305-0041

Author
Hirsch, Morris W

Publication Date
1965-07-01

DOI
10.1017/s0305004100038998

Peer reviewed

Proc. Camb. Phil. Soc. (1965), **61**, 657
Printed in Great Britain

On embedding 4-manifolds in R^7

BY MORRIS W. HIRSCH
University of California

(*Received* 4 *January* 1965)

Let M denote a connected, compact, unbounded, piecewise linear ($= PL$) manifold of dimension 4. Denote by $\overline{W}_3(M)$ the normal Stiefel–Whitney class in dimension 3, with integer coefficients (twisted if M is not orientable).

It is well known that $\overline{W}_3(M) = 0$ if M is orientable; see (5), for example.

Let R^7 denote Euclidean 7-space.

THEOREM. *If M is orientable, or, more generally, if $\overline{W}_3(M) = 0$, then there is a PL embedding $M \to R^7$.*

(*Note.* This theorem, with the same proof, is due independently to C. T. C. Wall.)

Proof. Let $\Delta \subset M$ be a 4-simplex, and put $M_0 = M - \operatorname{int}\Delta$. Let S be the 3-sphere $\partial M_0 = \partial\Delta$.

The embedding $M \to R^7$ is constructed in two stages, following the method of Haefliger–Hirsch ((1)). First a PL embedding $g: M_0 \to R^7$ is found such that $g|S \simeq 0$ in $R^7 - g(\operatorname{int} M_0)$; call such an embedding *untwisted*. Then a theorem of Irwin is applied to extend g to a PL embedding of Δ in $R^7 - g(\operatorname{int} M_0)$.

To find an untwisted PL embedding $g: M_0 \to R^7$, the argument of § 4 of (1) is applied to a smoothing $\tilde{M}_0$ of M_0 (which exists because M_0 is 4-dimensional ((7))) to produce a *smooth* untwisted embedding $f: \tilde{M}_0 \to R^7$; then Whitehead's approximation theorem ((6)) is invoked to obtain g.

(For completeness, the construction of the smooth embedding $f: \tilde{M}_0 \to R^7$ is outlined. In (2) it is proved that there is a smooth embedding $h: \tilde{M}_0 \to R^7$, which might be twisted. If $\overline{W}_3(M) = 0$, then h admits a normal vector field, thought of as a map $u: M_0 \to R^7 - h(M_0)$. If M is orientable, $u(S)$ bounds the chain $u(M_0)$ in $R^7 - h(M_0)$. Alexander duality and Hurewicz isomorphism then imply $u(S) \simeq 0$ in $R^7 - h(M_0)$, and it follows that h is untwisted. If M is not orientable, then $u(S)$ bounds mod 2; in this case g is obtained by twisting $h(M_0)$ around the 3-handles of a handle decomposition.)

Assume then that $M_0 \subset R^7$ as a subcomplex, and $S \simeq 0$ in $R^7 - M_0$. Let $N \subset R^7$ be a 'regular neighbourhood' of M_0 with $S \subset \partial N$ ((3)). Put $W = R^7 - \operatorname{int} N$. The PL 7-manifold W is easily seen to be 2-connected, since it has the same homotopy type as $R^7 - M_0$.

Clearly $S \simeq 0$ in W, so there exists a map $v: \Delta \to W$ with $v|S =$ identity. Theorem 1 of Irwin ((4)) implies that v is homotopic rel S to a PL embedding $\phi: \Delta \to W$. Since $W \cap M_0 = S$, the embedding ϕ completes the embedding of M in R^7.

REFERENCES

(1) Haefliger, A. and Hirsch, M. Existence and classification of differentiable embeddings. *Topology*, **2** (1963), 129–135.

(2) Hirsch, M. On imbedding differentiable manifolds in euclidean space. *Ann. of Math.* **73** (1961), 566–671.

(3) Hudson, J. F. P. and Zeeman, E. C. On regular neighbourhoods. *Proc. London Math. Soc.* (3) **14** (1964), 719–745.

(4) Irwin, M. Combinatorial embeddings of manifolds. *Bull. American Math. Soc.* **68** (1962), 25–27.

(5) Massey, W. and Peterson, F. On the dual Stiefel–Whitney classes of a manifold. *Bol. Soc. Mat. Mexicana* (1963), pp. 1–13.

(6) Whitehead, J. H. C. On C^1 complexes. *Ann. of Math.* **41** (1940), 809–824.

(7) Whitehead, J. H. C. Manifolds with transverse fields in euclidean space. *Ann. of Math.* **73** (1961), 154–212.

Proc. Camb. Phil. Soc. (1965), 6
Printed in Great Britain

Unkn
and

We shall present this pa
though all our arguments
knottedness hypotheses.
p, S^{p-1} for the standard u
Kosinski proved in (5),
I *Let T be a submanif*
of its complementary comp
II *Let T_1, T_2 be as in I.*
III *Let S_1, S_2 be subm*
and $S^{p+q+1} - S_2$ are diffeon
The present paper is m
improved. To fix notatio
Lemma 1. *Let T^{p+q} be a*
that each component has A
joint union of two open s
homology S^p, and if $p \neq 1$
We shall call a manifol
of a sphere with a disc or
sphere throwing the man
diffeomorphic to the prod
Lemma 1, our first main r
Theorem 2. *Assume p*
(A) *Suppose C_p is a ho*
$p = 1$ and $q = 3$, assume th
(B) *Suppose $p, q \neq 1$ an*
torus.
(C) *Suppose $q = 1$, and*
Then T is unknotted if and
Conjecture. Any h-cobor
It should perhaps be
geometric terms, we have
to a purely algebraic (com
between our conjecture a

Proceedings of the

CAMBRIDGE PHILOSOPHICAL SOCIETY

(MATHEMATICAL AND PHYSICAL SCIENCES)

VOLUME 61

CAMBRIDGE
AT THE UNIVERSITY PRESS
1965