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ABSTRACT

The technique used by Faddeev to obtain connected equations for the nonrelativistic three-body T matrix is generalized for four particles. It is shown that the four-body equations are completely determined by the solutions of all the possible two-body subsystems, as is the case in the three-body problem. This approach can be extended to more complicated multiparticle systems.

I. INTRODUCTION

The study of nonrelativistic scattering processes that involve more than two particles has recently received considerable attention.[1-4] When the particles interact only by pairs, and there are no multiparticle forces, the problem cannot be approached by means of the Lippmann-Schwinger equation.[5] The main reason for this is that the Lippmann-Schwinger kernel $\sum_{i<j} V_{ij}(E - H_o)^{-1}$ is the sum of disconnected parts in each of which (N-2) particles are not interacting. In momentum space, this yields (N-2) delta functions in addition to the overall delta function representing conservation of momentum. Consequently, the kernel is unbounded and the equation is strongly singular. This difficulty cannot be removed by iterating the equations; any iterated kernel will still contain disconnected parts. The only possibility of obtaining equations that may be solvable by one of the standard methods is to apply one of the usual tricks for handling singular integral equations. It consists of solving in some way the singular part of the kernel in a closed form, in such a way that the remaining equation is nonsingular. In the case we are considering, it amounts to recasting the Lippmann-Schwinger equation into a connected form, by previously solving some pieces of the kernel in an explicit way.

This problem was solved for the general N-body problem by Weinberg.[3] We refer to his paper for a very lucid discussion of the difficulties associated with the multiparticle scattering problem. Huntziker[6] has given a general proof of the compactness of the

Weinberg kernel, providing certain assumptions are made about the potentials. In the four-body problem, for example, the Weinberg equations require a knowledge of the solutions of all the possible two- and three-body problems involved, as well as of the potentials V_{ij} .

In the three-body problem, another possible solution was proposed previously by Faddeev.[2] In place of having only one equation for the three-body T matrix, he proposed a set of three coupled integral equations. But the counterpart of this slight complication is that the Faddeev equations do not depend upon the original potentials. The inhomogeneous term and the kernel of the Faddeev equations are completely determined by the off-the-energy-shell two-body amplitudes. This property of the Faddeev equations has been used by Lovelace[2] to propose a practical theory for three-particle processes, in which experimental information about the two-particle subsystems is used to determine partially the off-shell two-body amplitudes.

The purpose of this paper is to generalize the Faddeev approach to the four-body problem; that is, to get connected equations in which the two-body potentials do not appear explicitly. It is possible to go on and get similar equations for more than four particles, but we will not do so explicitly in this paper because the four-body problem is sufficiently complicated to illustrate the general technique. In Section II we review briefly the derivation of the three-body Fadeev equations. In Section III the four-body problem is formulated and some preliminary results are derived. In Section III, the four-body equations are derived; and finally their properties and possible importance are discussed in Section V.

II. THE THREE-BODY PROBLEM

Consider the Hamiltonian

$$H = H_o + V , \tag{2.1}$$

where

$$H_o = \sum_{i=1}^{3} \frac{\vec{p}_i^{\,2}}{2m_i} \quad \text{and} \quad V = V_{12} + V_{13} + V_{23} . \tag{2.2}$$

When the resolvent operators of H_o and H ,

$$G_o(z) = (z - H_o)^{-1} ; \qquad G(z) = (z - H)^{-1} , \tag{2.3}$$

are introduced, the three-body T matrix is defined by

$$G(z) = G_o(z) + G_o(z)\, \hat{T}(z)\, G_o(z) . \tag{2.4}$$

Using the resolvent identity

$$G(z) = G_o(z) + G_o(z)\, V\, G(z) , \tag{2.5}$$

one obtains the Lippmann-Schwinger equation,[6]

$$\hat{T}(z) = V + V\, G(z) V = V + V\, G_o(z)\, \hat{T}(z) . \tag{2.6}$$

Faddeev[1] defines the following operators

$$\hat{T}_{ij}(z) = V_{ij} + V_{ij}\, G(z)\, V . \tag{2.7}$$

Clearly, the three-body T matrix is given by the sum

$$\hat{T}(z) = \hat{T}_{12}(z) + \hat{T}_{13}(z) + \hat{T}_{23}(z) . \tag{2.8}$$

The Faddeev equations are coupled integral equations for the $\hat{T}_{ij}(z)$. In order to obtain them, let us consider the resolvent of the Hamiltonian $H_{ij} = H_o + V_{ij}$,

$$G_{ij}(z) = \left[z - H_{ij} \right]^{-1} . \tag{2.9}$$

The two-body T matrix for particles i and j in the three-body Hilbert space--i.e., with particle k as a spectator particle--is defined by

$$t_{ij}(z) = V_{ij} + V_{ij} \, G_{ij}(z) \, V_{ij} , \tag{2.10}$$

and satisfies the Lippmann-Schwinger equation,

$$t_{ij}(z) = V_{ij} + V_{ij} \, G_o(z) \, t_{ij}(z) . \tag{2.11}$$

It is trivially related to the solutions of the two-body problem, $\hat{t}_{ij}(z)$, by

$$\langle \vec{p}_i \vec{p}_j \vec{p}_k \,|\, t_{ij}(z) \,|\, \vec{p}_i{}' \vec{p}_j{}' \vec{p}_k{}' \rangle = \delta(\vec{p}_k - \vec{p}_k{}') \langle \vec{p}_i \vec{p}_j \,|\, \hat{t}_{ij}(z - \frac{\vec{p}_k^{\,2}}{2m_k}) \,|\, \vec{p}_i{}' \vec{p}_j{}' \rangle . \tag{2.12}$$

The identity

$$G(z) = G_{ij}(z) + G_{ij}(z) \left[V_{ik} + V_{jk} \right] G(z) ; \quad i \neq j \neq k \tag{2.13}$$

can easily be shown, and by inserting (2.13) into (2.7) we get

$$\hat{T}_{ij}(z) = V_{ij} + V_{ij}\, G_{ij}(z)V + V_{ij}\, G_{ij}(z)\left[V_{ik} + V_{jk}\right]G(z)V$$

$$= V_{ij} + V_{ij}\, G_{ij}(z)V_{ij} + V_{ij}G_{ij}(z)\left[V_{ij} + V_{jk}\right] + V_{ij}G_{ij}(z)\left[V_{ij}+V_{jk}\right]G(z)V.$$

By using (2.11), and also the Lippmann-Schwinger equation for $t_{ij}(z)$ in the form $G_{ij}(z)V_{ij} = G_o(z)t_{ij}(z)$, one obtains

$$\hat{T}_{ij}(z) = t_{ij}(z) + t_{ij}(z)\, G_o(z)\left[V_{ik} + V_{ik}\, G(z)V + V_{jk} + V_{jk}\, G(z)V\right].$$

Finally, using the definitions (2.7), this equation becomes

$$\hat{T}_{ij}(z) = t_{ij}(z) + t_{ij}(z)\, G_o(z)\left[\hat{T}_{ik}(z) + \hat{T}_{jk}(z)\right] \qquad (2.14)$$

$$\text{for} \quad i,j,k = 1,2,3$$
$$\text{and} \quad i \neq j \neq k \;.$$

These are the Faddeev equations. Because of the fact that $T_{ij}(z)$ is not coupled to itself, the first iterated kernel is connected. Assuming that the potential satisfies

$$|v_{ij}(\vec{q} - \vec{q}')| = C\left[1 + (q - q')\right]^{-1-\epsilon_o} ; \qquad \epsilon_o > 0 , \qquad (2.15)$$

Faddeev proved that the first iterated kernel is compact, except when z is on the real positive axis.[1] It is also possible to prove[7] that the fifth iternated kernel is compact for any value of z.

III. THE FOUR-BODY PROBLEM

In this section, we consider a Hamiltonian of the form $H = H_o + V$, where

$$H_o = \sum_{i=1}^{4} \frac{\vec{p}_i^{\,2}}{2m_i} , \tag{3.1}$$

$$V = \sum_{i<j} V_{ij}, \text{ for } i, j = 1,2,3,4. \tag{3.2}$$

Here again we define

$$G_o(z) = (z-H_o)^{-1} , \quad G(z) = (z-H)^{-1} . \tag{3.3}$$

The four-body amplitude $\mathcal{T}(z)$ is defined by the relations

$$G(z) = G_o(z) + G_o(z)\,\mathcal{T}(z)\,G_o(z) \tag{3.4}$$

or

$$\mathcal{T}(z) = V + V\,G(z)\,V . \tag{3.5}$$

We introduce next six operators, in analogy with (2.7),

$$\mathcal{T}_{ij}(z) = V_{ij} + V_{ij}\,G(z)\,V, \text{ for } i < j \text{ ; } i,j = 1,2,3,4 . \tag{3.6}$$

The four-body $\mathcal{T}(z)$ operator is then given by the sum:

$$\mathcal{T}(z) = \sum_{i<j} \mathcal{T}_{ij}(z). \tag{3.7}$$

Our aim is to get a set of coupled integral equations for the $\mathcal{T}_{ij}(z)$; such that they are connected, and do not contain the potentials. This will be done in the next section. Here, for the sake of clarity, we want to make a few comments about the notation we will use in the rest of the paper. If we use the indices i, j, k, ℓ it will be understood that their range of values is from 1 to 4. When we use the subindices ij, ijk, or $ijk\ell$ in an operator, it will also be understood that $i<j$; $i<j<k$, and $i<j<k<\ell$ respectively. The two-body amplitudes of particles i and j in the four-body Hilbert space will be denoted by $t_{ij}(z)$; the three-body amplitude of particles ijk in the four-body Hilbert space will be denoted by $T^{(\ell)}(z)$, where the upper index indicates the spectator particle. We will use $\mathcal{T}(z)$ for the four-body amplitudes.

The matrix elements of $t_{ij}(z)$ and $T^{(\ell)}(z)$ can be written in terms of the matrix elements of the operators defined in the previous section, in the following way[2]:

$$\langle \vec{p}_i\vec{p}_j\vec{p}_k\vec{p}_\ell | t_{ij}(z) | \vec{p}_i{}'\vec{p}_j{}'\vec{p}_k{}'\vec{p}_\ell{}' \rangle = \delta(\vec{p}_k-\vec{p}_k{}')\delta(\vec{p}_\ell-\vec{p}_\ell{}')\langle \vec{p}_i\vec{p}_j | \hat{t}_{ij}(z-\omega_k-\omega_\ell) | \vec{p}_i{}'\vec{p}_j{}' \rangle , \quad (3.8)$$

$$\langle \vec{p}_i\vec{p}_j\vec{p}_k\vec{p}_\ell | T^{(\ell)}(z) | \vec{p}_i{}'\vec{p}_j{}'\vec{p}_k{}'\vec{p}_\ell{}' \rangle = \delta(\vec{p}_\ell-\vec{p}_\ell{}')\langle \vec{p}_i\vec{p}_j\vec{p}_k | \hat{T}(z-\omega_\ell) | \vec{p}_i{}'\vec{p}_j{}'\vec{p}_k{}' \rangle , \quad (3.9)$$

where $\omega_i = \dfrac{\vec{p}_i^{\,2}}{2m_i}$.

Let us define the operators

$$H_{ij} = H_o + V_{ij} \quad , \tag{3.10a}$$

$$H_{ijk} = H_o + V_{ij} + V_{ik} + V_{jk} \, , \tag{3.11a}$$

$$H_{ij,k\ell} = H_o + V_{ij} + V_{k\ell} \quad , \tag{3.12a}$$

and their resolvents;

$$G_{ij}(z) = (z-H_{ij})^{-1} \quad , \tag{3.10b}$$

$$G_{ijk}(z) = (z-H_{ijk})^{-1} \quad , \tag{3.11b}$$

$$G_{ij,k\ell}(z) = (z-H_{ij,k\ell})^{-1} \quad . \tag{3.12b}$$

We will need to use several properties of the two- and three-body amplitudes. The two-body amplitudes are given by

$$t_{ij}(z) = V_{ij} + V_{ij} \, G_{ij}(z) \, V_{ij} \quad , \tag{3.13}$$

and the Lippmann-Schwinger equations read:

$$V_{ij} \, G_{ij}(z) = t_{ij}(z) \, G_o(z) . \tag{3.14}$$

The three-body amplitudes $T^{(\ell)}(z)$ are defined by the equation

$$T^{(\ell)}(z) = (V_{ij}+V_{ik}+V_{jk}) + (V_{ij}+V_{ik}+V_{jk})\, G_{ijk}(z)\, (V_{ij}+V_{ik}+V_{jk}), \quad (3.15)$$

and the Faddeev operators (2.7) in the four-body Hilbert space read

$$T_{ij}^{(\ell)}(z) = V_{ij} + V_{ij}\, G_{ijk}(z)\, (V_{ij}+V_{ik}+V_{jk}) . \quad (3.16)$$

Their matrix elements are trivially related to the matrix elements of the operators $\hat{T}_{ij}(z)$ studied in the preceding section; the relation is given by Eq.(3.9) by writing $T_{ij}^{(\ell)}(z)$ and $\hat{T}_{ij}(z-\omega_\ell)$ in place of $T^{\ell}(z)$ and $\hat{T}(z-\omega_\ell)$, respectively.

The Faddeev equations for $T_{ij}^{(\ell)}(z)$ are:

$$T_{ij}^{(\ell)}(z) = t_{ij}(z) + t_{ij}(z)\, G_o(z) \left[T_{ik}^{(\ell)}(z) + T_{jk}^{(\ell)}(z)\right] . \quad (3.17)$$

Before going on to derive the four-body equations, it is convenient to consider in some detail the Green's function, $G_{ij,k\ell}(z)$. To calculate it is to solve a four-body problem in which the only nonvanishing potentials are V_{ij} and $V_{k\ell}$. We shall show that this problem can be solved in a closed form, in terms of the two-body amplitudes $\hat{t}_{ij}(z)$ and $\hat{t}_{k\ell}(z)$ only.

Let us call $\mathcal{Q}_{ij,k\ell}(z)$ the four-body amplitude associated with the Hamiltonian $H_{ij,k\ell}$. Obviously, we have:

$$\mathcal{Q}_{ij,k\ell}(z) = (V_{ij}+V_{k\ell}) + (V_{ij}+V_{k\ell})\, G_{ij,k\ell}(z)\, (V_{ij}+V_{k\ell}). \qquad (3.18)$$

Again following Faddeev's idea, we introduce the operators :

$$\mathcal{Q}_{ij}(z) = V_{ij} + V_{ij}\, G_{ij,k\ell}(z)\, (V_{ij} + V_{kl})\,, \qquad (3.19)$$

$$\mathcal{Q}_{k\ell}(z) = V_{k\ell}+ V_{k\ell}\, G_{ij,k\ell}(z)\, (V_{ij}+ V_{k\ell})\,, \qquad (3.20)$$

$$\mathcal{Q}_{ij,k\ell}(z) = \mathcal{Q}_{ij}(z) + \mathcal{Q}_{k\ell}(z)\,, \qquad (3.21)$$

Using the identity:

$$G_{ij,k\ell}(z) = G_{ij}(z) + G_{ij}(z)\, V_{k\ell}\, G_{ij,k\ell}(z)\,. \qquad (3.22)$$

One can very simply obtain for $\mathcal{Q}_{ij}(z)$ and $\mathcal{Q}_{k\ell}(z)$ the equations :

$$\begin{cases} \mathcal{Q}_{ij}(z) = t_{ij}(z) + t_{ij}(z)\, G_o(z)\mathcal{Q}_{k\ell}(z)\,, \\ \mathcal{Q}_{k\ell}(z) = t_{k\ell}(z) + t_{k\ell}(z)\, G_o(z)\mathcal{Q}_{ij}(z)\,. \end{cases} \qquad (3.23)$$

Although these equations will help us in simplifying the algebra in the next section it is not necessary to solve them to calculate $\mathcal{Q}_{ij}(z)$, for example. Remember that the Hamiltonian $H_{ij,k\ell}$ is

$$H_{ij,k\ell} = H_o + V_{ij} + V_{k\ell} = h_o^{(ij)} + V_{ij} + h_o^{(k\ell)} + V_{k\ell} = h_{ij} + h_{k\ell} , \qquad (3.24)$$

where

$$h_o^{(ij)} = \frac{\vec{p}_i^{\,2}}{2m_i} + \frac{\vec{p}_j^{\,2}}{2m_j} = \omega_i + \omega_j, \text{ for } \omega = \frac{\vec{p}^{\,2}}{2m}$$

$$h_o^{(k\ell)} = \omega_k + \omega_\ell \quad . \qquad (3.25)$$

Therefore, $G_{ij,k\ell}(z)$ is the resolvent of the sum of the two-body Hamiltonians h_{ij} and $h_{k\ell}$. These two operators commute, because they act upon different spaces. Therefore, we know that if $g_{ij}(z) = (z-h_{ij})^{-1}$, and $g_{k\ell}(z) = (z-h_{k\ell})^{-1}$, the resolvent of $h_{ij} + h_{k\ell}$ is given by[3,8]

$$G_{ij,k\ell}(z) = \frac{1}{2\pi i}\int_c g_{ij}(z')\, g_{k\ell}(z-z')\, dz' \quad , \qquad (3.26)$$

where the contour of integration encircles the spectrum of $g_{ij}(z')$ in a counterclockwise way (or the spectrum of $g_k(z-z')$ in a clockwise way). The reader should bear in mind that $g_{ij}(z)$ and $g_{k\ell}(z)$ are the two-body Green s functions in different two-body Hilbert spaces. Therefore, the matrix element of the right-hand side is trivial,

$$\langle \vec{p}_i\vec{p}_j\vec{p}_k\vec{p}_\ell | G_{ij,k\ell}(z) | \vec{p}_i^{\,\prime}\ \vec{p}_j^{\,\prime}\ \vec{p}_k^{\,\prime}\ \vec{p}_\ell^{\,\prime} \rangle =$$

$$= \frac{1}{2\pi i}\int_c \langle \vec{p}_i\vec{p}_j | g_{ij}(z') | \vec{p}_i^{\,\prime}\vec{p}_j^{\,\prime} \rangle \ \langle \vec{p}_k\vec{p}_\ell | g_{k\ell}(z-z') | \vec{p}_k^{\,\prime}\vec{p}_\ell^{\,\prime} \rangle\, dz' \ . \qquad (3.27)$$

Using Eq. (3.26) for $G_{ij,k\ell}(z)$ one can obtain for the operator $\mathcal{A}_{ij}(z)$ the formula

$$\mathcal{A}_{ij}(z) = t_{ij}(z) + \frac{1}{2\pi i}\int_c \hat{t}_{ij}(z')g_o^{(k\ell)}(z-z')\,\hat{t}_{k\ell}(z-z')\,g_o^{(k\ell)}(z-z')\,dz'$$

$$+ \frac{1}{2\pi i}\int_c \hat{t}_{ij}(z')\,g_o^{(ij)}(z')\,g_o^{(k\ell)}(z-z')\,\hat{t}_{k\ell}(z-z')\,dz'. \qquad (3.28)$$

A proof of this formula is given in the Appendix. The matrix elements of $\mathcal{A}_{ij}(z)$ are given by:

$$\langle \vec{p}_i\vec{p}_j\vec{p}_k\vec{p}_\ell \,|\mathcal{A}_{ij}(z)|\, \vec{p}_i{}'\vec{p}_j{}'\vec{p}_k{}'\vec{p}_\ell{}' \rangle \qquad (3.29)$$

$$= \langle \vec{p}_i\vec{p}_j \,|\, \hat{t}_{ij}(z-\omega_k-\omega_\ell) \,|\, \vec{p}_i{}'\vec{p}_j{}'\rangle\, \delta(\vec{p}_k-\vec{p}_k{}')\,\delta(\vec{p}_\ell-\vec{p}_\ell{}')$$

$$+ \frac{1}{2\pi i}\int_c dz' \langle \vec{p}_i\vec{p}_j \,|\, \hat{t}_{ij}(z') \,|\, \vec{p}_i{}'\vec{p}_j{}'\rangle \frac{1}{(z-z')-(\omega_k+\omega_\ell)}$$

$$\langle \vec{p}_k\vec{p}_\ell \,|\, \hat{t}_{k\ell}(z-z') \,|\, \vec{p}_k{}'\vec{p}_\ell{}'\rangle \frac{1}{(z-z')-(\omega_k'+\omega_\ell')}$$

$$+ \frac{1}{2\pi i}\int_c dz' \langle \vec{p}_i\vec{p}_j \,|\, \hat{t}_{ij}(z') \,|\, \vec{p}_i{}'\vec{p}_j{}'\rangle \frac{1}{z'-(\omega_i'+\omega_j')}\;\frac{1}{(z-z')-(\omega_k+\omega_\ell)}$$

$$\langle \vec{p}_k\vec{p}_\ell \,|\, \hat{t}_{k\ell}(z-z') \,|\, \vec{p}_k{}'\vec{p}_\ell{}' \rangle ,$$

and in the case in which sample pole approximations are used for the two-body amplitudes, the integrals can easily be evaluated. A similar formula can be written for $\mathcal{A}_{k\ell}(z)$.

IV. THE FOUR-BODY EQUATIONS

In the previous section we have introduced the operators $\mathcal{T}_{ij}(z)$; our intention here is to derive the system of coupled integral equations satisfied by them. Let us consider, for example, the operator $\mathcal{T}_{12}(z)$, defined by

$$\mathcal{T}_{12}(z) = V_{12} + V_{12}\, G(z) V . \qquad (4.1)$$

When we derived the Faddeev equations for $\hat{T}_{ij}$ we used in the definition (2.7) of this operator an identity between $G(z)$ and $G_{ij}(z)$. We could use in (4.1) a similar resolvent identity connecting $G(z)$ with $G_{12}(z)$, but the resulting equations would not be connected. If we are to obtain connected equations, we must use in (4.1) an identity connecting $G(z)$ with $G_{12}(z)$ and all the other Green's functions containing the subindices 12, namely: $G_{123}(z)$, $G_{124}(z)$ and $G_{12,34}(z)$.

The following identities can be easily shown:

$$G(z) = G_{12}(z) + G_{12}(z)\left[V_{13} + V_{14} + V_{23} + V_{34} + V_{34}\right] G(z) , \qquad (4.2)$$

$$G(z) = G_{123}(z) + G_{123}(z)\left[V_{14} + V_{24} + V_{34}\right] G(z) , \qquad (4.3)$$

$$G(z) = G_{124}(z) + G_{124}(z)\left[V_{13} + V_{23} + V_{34}\right] G(z) , \qquad (44)$$

$$G(z) = G_{12,34}(z) + G_{12,34}(z)\left[V_{13} + V_{14} + V_{23} + V_{34}\right] G(z) . \qquad (4.5)$$

Next we rewrite (4.2) as

$$G(z) = G_{12}(z) + G_{12}(z)\left[V_{13}+ V_{23}\right] G(z) + G_{12}(z)\left[V_{14}+ V_{24}\right]G(z)+ G_{12}(z)V_{34}G(z) , \tag{4.6}$$

and insert the identities (4.3), (4.4), and (4.5) in place of the $G(z)$ which are multipled (in the operator sense) by $\left[V_{13} + V_{23}\right]$, $\left[V_{14} + V_{24}\right]$, and V_{34}, respectively. In this way, we obtain

$$\begin{aligned} G(z) = {} & G_{12}(z) + G_{12}(z) \left[V_{13} + V_{23}\right] G_{123}(z) \\ & + G_{12}(z) \left[V_{14} + V_{24}\right] G_{124}(z) + G_{12}(z) V_{34} G_{12,34}(z) \\ & + G_{12}(z) \left[V_{13} + V_{23}\right] G_{123}(z) \left[V_{14} + V_{24} + V_{34}\right] G(z) \\ & + G_{12}(z) \left[V_{14} + V_{34}\right] G_{124}(z) \left[V_{13} + V_{23} + V_{34}\right] G(z) \\ & + G_{12}(z) V_{34} G_{12,34}(z) \left[V_{13} + V_{14} + V_{23} + V_{24}\right] G(z) . \end{aligned} \tag{4.7}$$

This is the resolvent identity which we next insert in (4.1) , to get an equation for $\mathcal{T}_{12}(z)$. Using (3.14), (3.16), (3.17), (3.18), and (3.6), we find

$$\mathcal{T}_{12}(z) = t_{12}(z) + \left[T_{12}^{(4)}(z) - t_{12}(z)\right] +$$

$$+ \left[T_{12}^{(3)}(z) - t_{12}(z)\right] + \left[\mathcal{Q}_{12}(z) - t_{12}(z)\right]$$

$$+ t_{12}(z)\, G_o(z) \left[V_{13} + V_{23}\right] G_{123}(z) \left[\mathcal{T}_{14}(z) + \mathcal{T}_{24}(z) + \mathcal{T}_{34}(z)\right] + \qquad (4.8)$$

$$+ t_{12}(z)\, G_o(z) \left[V_{14} + V_{24}\right] G_{124}(z) \left[\mathcal{T}_{13}(z) + \mathcal{T}_{23}(z) + \mathcal{T}_{34}(z)\right] +$$

$$+ t_{12}(z)\, G_o(z)\, V_{34}\, G_{12,34}(z) \left[\mathcal{T}_{13}(z) + \mathcal{T}_{14}(z) + \mathcal{T}_{23}(z) + \mathcal{T}_{24}(z)\right] .$$

The potentials can be completely eliminated from the equations by using the following relations, which may be obtained with the help of (3.16-19):

$$\left[V_{ik} + V_{jk}\right] G_{ijk}(z) = \left[T_{ik}^{(\ell)}(z) + T_{jk}^{(\ell)}(z)\right] G_o(z) , \qquad (4.9)$$

$$V_{k\ell}\, G_{ij,k\ell}(z) = \mathcal{Q}_{ij}(z)\, G_o(z) . \qquad (4.10)$$

Therefore, using again the Faddeev equations (3.17), as well as (3.23) one has

$$t_{12}(z)\, G_o(z) \left[V_{13} + V_{23}\right] G_{123}(z) = \left[T_{12}^{(4)}(z) - t_{12}(z)\right] G_o(z) = T_{12}^{(4)c}(z) G_o(z) , \qquad (4.11)$$

$$t_{12}(z)\, G_o(z)\left[V_{14}+V_{24}\right] G_{124}(z) = \left[T_{12}^{(3)}(z) - t_{12}(z)\right] G_o(z) = T_{12}^{(3)c}(z) G_o(z), \tag{4.12}$$

$$t_{12}\, G_o(z)\, V_{34}\, G_{12,34}(z) = \left[\mathcal{A}_{12}(z) - t_{12}(z)\right] G_o(z) = \mathcal{A}_{12}^{c}(z)\, G_o(z). \tag{4.13}$$

The final four-body equations are obtained by inserting (4.11-13) into (4.8). In general, they read:

$$\begin{aligned}\mathcal{T}_{ij}(z) &= t_{ij}(z) + T_{ij}^{(k)c}(z) + T_{ij}^{(\ell)c}(z) + \mathcal{A}_{ij}^{c}(z) + \\ &+ T_{ij}^{(k)}(z)\, G_o(z)\left[\mathcal{T}_{ik}(z) + \mathcal{T}_{jk}(z) + \mathcal{T}_{\ell k}(z)\right] + \\ &+ T_{ij}^{(\ell)(c)}(z)\, G_o(z)\left[\mathcal{T}_{i\ell}(z) + \mathcal{T}_{j\ell}(z) + \mathcal{T}_{\ell k}(z)\right] + \\ &+ \mathcal{A}_{ij}^{c}(z)\, G_o(z)\left[\mathcal{T}_{ik}(z) + \mathcal{T}_{jk}(z) + \mathcal{T}_{i\ell}(z) + \mathcal{T}_{j\ell}(z)\right] .\end{aligned} \tag{4.14}$$

The operator $T_{ij}^{(k)c}(z)$ is defined to be $\left[T_{ij}^{(k)}(z) - t_{ij}(z)\right]$. The operator $\mathcal{A}_{ij}^{c}(z)$ is also defined to be $\left[\mathcal{A}_{ij}(z) - t_{ij}(z)\right]$. Given the two-body scattering amplitudes $t_{ij}(z)$, one can calculate these operators by solving the three-body Faddeev equations and computing the integrals involved in our formulas (3.29).

Recalling that the four-body $\mathcal{T}(z)$ operator is the sum of all the $\mathcal{T}_{ij}(z)$ operators, one can check very easily that the sum of the inhomogeneous terms of the six equations yields correctly all the disconnected parts of the four-body amplitude. The first iterated kernel is connected because in (4.14) $\mathcal{T}_{ij}(z)$ is not coupled to itself.

V. CONCLUSIONS

This approach can in principle be generalized to the N-body problem. The basic idea is to introduce $\frac{N(N-1)}{2}$ amplitudes $\mathcal{T}_{ij}(z)$, in analogy with Eq.(3.6). In order to get an equation for $\mathcal{T}_{ij}(z)$ one has to insert in its definition the resolvent identity between the full N-body Green's function $G(z)$ and all the possible disconnected N-body Green's functions that contain the potential V_{ij}. These are known from the solutions for systems with a smaller number of particles, and from generalizations of Eq. (3.26). By following this approach, we are guaranteed that the potentials V_{ij} will not appear in the final equations.

We come then to the conclusion that, in the absence of multiparticle forces, the multiparticle $T(z)$ operators are completely determined by the two-body $t_{ij}(z)$ operators, with no reference to the original potentials whatsoever. However, one must bear in mind that in order to solve the Faddeev equations for the three-body problem, or the equations we proposed for the four-body problem, it is necessary to know the matrix elements of $t_{ij}(z)$ off the energy shell. The experimental data determine them only on the energy shell, so that all we can measure is $\langle\vec{p}|\hat{t}_{ij}(z)|\vec{p}\,'\rangle = t_{ij}(\vec{p},\vec{p}\,';z)$ when $\vec{p}^{\,2} = \vec{p}\,'^{2} = z$. The only way of obtaining the off-shell extension is through the Lippmann-Schwinger equation, which requires a knowledge of the potential. Nevertheless, the Faddeev approach still has its advantages in some cases. For example, if one is dealing with singular potentials, the mathematical difficulties associated with them need to be solved only at the two-body level, since they are not directly relevant to multiparticle calculations. If the two-body scattering amplitudes appear to be dominated by poles near the physical region--i.e., bound state or resonance poles--the problem of their off-shell extension can be overcome by using phenomenological form factors. If one considers an off-shell partial-wave amplitude $t_\ell(p,p';z)$ the poles will be poles in z. It is possible to prove that in the neighborhood of a pole z_p the off-shell amplitude is factorizable in its dependence upon the variables p and p'.[2] Therefore, one can write:

$$t_\ell(p,p';z) \simeq g_\ell(p)\, t_\ell(z)\, g_\ell(p') . \qquad (5.1)$$

A simple form for $t_\ell(z)$ is just a pole term, $\frac{1}{(z-zp)}$, in the case of a bound state. However, more complicated expressions for resonance poles can be used if one wants to satisfy two-body unitarity. The functions $g_\ell(p)$ are the so-called form factors; in the case of bound-state poles they are given in terms of the bound-state wave function by $(p^2 - E_p)\,\psi(p)$, E_p being the binding energy. In the case of a resonance, they are not so well defined, but in any case we know their behavior at the origin $(\sim \vec{p}^{\,\ell})$ and at infinity $(\sim \vec{p}^{\,\ell-2})$ for superpositions of Yukawa potentials.[9] They also contain the left-hand cuts of the partial-wave amplitudes,[9] and merely express the fact that the bound state and resonance poles by which one is approximating the two-body amplitudes are not elementary systems but composite ones with internal structure. All these requirements can be used to construct phenomonological expressions for the form factors. The Faddeev approach is very useful in performing semiphenomonological calculations to investigate the effect of two-body resonances and bound states in multiparticle systems.

After this paper was written, we received a paper by L. Rosenberg,[10] which includes most of the conclusions presented here.

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APPENDIX

In order to prove Eq. (3.29), we have to use Eq. (3.26) for $G_{ij,k\ell}(z)$ in the definition (3.19) of the operator $\mathcal{Q}_{ij}(z)$. In so doing, we obtain:

$$\mathcal{Q}_{ij}(z) = V_{ij} + V_{ij}\,\frac{1}{2\pi i}\int_c g_{ij}(z')\,g_{k\ell}(z-z')dz'V_{ij} + V_{ij}\,\frac{1}{2\pi i}\int_c g_{ij}(z')g_{k\ell}(z-z')dz'V_{k\ell}\,. \qquad \text{(A.1)}$$

Using the Lippmann-Schwinger equations

$$V_{ij}\,g_{ij}(z') = \hat{t}_{ij}(z')\,g_o^{(ij)}(z')\,,$$

$$g_{k\ell}(z-z')\,V_{k\ell} = g_o^{(k\ell)}(z-z')\,\hat{t}_{k\ell}(z-z')\,,$$

and

$$V_{ij}\,g_{ij}(z')\,V_{ij} = \hat{t}_{ij}(z') - V_{ij}\,,$$

we obtain:

$$\begin{aligned}\mathcal{Q}_{ij}(z) = V_{ij} &+ \frac{1}{2\pi i}\int_c \left[\hat{t}_{ij}(z') - V_{ij}\right] g_{k\ell}(z-z')\,dz' \\ &+ \frac{1}{2\pi i}\int_c \hat{t}_{ij}(z')\,g_o^{(ij)}(z')\,g_o^{(k\ell)}(z-z')\,\hat{t}_{k\ell}(z-z')\,dz'\,.\end{aligned} \qquad \text{(A.2)}$$

Using next the definitions of $\hat{t}_{k\ell}(z)$;

$$g_{k\ell}(z-z') = g_o^{(k\ell)}(z-z') + g_o^{(k\ell)}(z-z')\, \hat{t}_{k\ell}(z-z')\, g_o^{(k\ell)}(z-z') ,$$

Equation (A.2) becomes:

$$\mathcal{a}_{ij}(z) = V_{ij}\left[1 - \frac{1}{2\pi i}\int_c g_{k\ell}(z-z')\,dz'\right] + \frac{1}{2\pi i}\int_c \hat{t}_{ij}(z')\, g_o^{(k\ell)}(z-z')\,dz' +$$

$$+ \frac{1}{2\pi i}\int_c \hat{t}_{ij}(z')\, g_o^{(k\ell)}(z-z')\, \hat{t}_{k\ell}(z-z')\, g_o^{(k\ell)}(z-z')\,dz'$$

$$+ \frac{1}{2\pi i}\int_c \hat{t}_{ij}(z')\, g_o^{(ij)}(z')\, g_o^{(k\ell)}(z-z')\, \hat{t}_{k\ell}(z-z')\,dz' . \tag{A.3}$$

By taking the contour c of integration as enclosing the singularities of $g_{k\ell}(z-z')$ and $g_o^{(k\ell)}(z-z')$ in a clockwise way, the first two terms of the right-hand side can be simplified. Recalling that

$$g_o^{(k\ell)}(z+i\epsilon) - g_o^{(k\ell)}(z-i\epsilon) = -2\pi i\, \delta\left[z-h_o^{(ij)}\right] ,$$

one gets

$$\frac{1}{2\pi i}\int_c \hat{t}_{ij}(z')\, g_o^{(k\ell)}(z-z')\,dz' = t_{ij}(z) . \tag{A.6}$$

The bracket multiplying the potential V_{ij} in Eq. (A.3) can be shown to vanish because of the completeness relation for the eigenstates of the Hamiltonian $h_{k\ell} = h_o^{(k\ell)} + V_{k\ell}$. We know that the Green's function $g_{k\ell}(z)$ can be represented as:

$$g_{k\ell}(z) = \sum_n \frac{|\psi_n\rangle \langle\psi_n|}{z + E_n} + \int_0^\infty dE \frac{|\psi(E)\rangle \langle\psi(E)|}{z - E} , \qquad (A.5)$$

where $|\psi_n\rangle$ are the discrete eigenstates of $h_{k\ell}$ with binding energy $(-E_n)$, and $|\psi(E)\rangle$ are these belonging to the continuum. Therefore

$$\frac{1}{2\pi i} \int_c g_{k\ell}(z-z')\, dz' = - \frac{1}{2\pi i} \int_{c'} g_{k\ell}(\omega)\, d\omega = \sum_n |\psi_n\rangle \langle\psi_n| + \int_0^\infty dE |\psi(E)\rangle \langle\psi(E)| ,$$

where the contour c' encloses the spectrum of $g_{k\ell}(\omega)$ in a clockwise way. Therefore, the completeness of the eigenstates of $h_{k\ell}(z)$ guarantees that

$$1 - \int_c g_{k\ell}(z-z')\, dz' = 0 . \qquad (A.6)$$

Using (A.4) and (A.6), one can reduce Eq. (A.3) to the Eq. (3.28) of the text.

FOOTNOTES AND REFERENCES

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