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Counting patterns in permutations and words

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The dissertation of Jeffrey Edward Liese is approved, and it is acceptable in quality and form for publication on microfilm:

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I dedicate this thesis to my parents for their continued love and support, even through year after year of education!

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Chapter 3 appears as a publication in the Annals of Combinatorics 11 (2007) 481-506 as *Counting Descents and Ascents Relative to Equivalence Classes mod* k by Jeffrey Liese. Permission to reproduce this publication as a chapter in this thesis has been sought and obtained. I would like to thank both the journal and the publisher Birkäuser Verlag, Basel, 2007 for allowing me to include these results here.

Research in mathematics is definitely a collaborative effort. I would also like to acknowledge a few other individuals who I have had the pleasure to work with as a graduate student. A large portion of Chapter 4 consists of joint work which appears in preprint: J. Hall, J. Liese, and J. Remmel. *q-Counting descent pairs with prescribed tops and bottoms*. In addition, Chapter 5 consists of joint work which appears in preprint: S. Kitaev, J. Liese, J. Remmel, B. Sagan. *Rationality, irrationality, and Wilf equivalence in generalized factor order*. I sincerely appreciate the coauthor's permission and unconditional willingness to allow me to reproduce these results here.

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ABSTRACT OF THE DISSERTATION

Counting patterns in permutations and words

by

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The study of permutations and permutation statistics dates back hundreds of years to the time of Euler and before. In this thesis, we examine several generalizations of classical permutation statistics, most often generalizing the descent statistic, $des(\sigma)$. Chapter 1 is dedicated to providing some history and background to the work presented in later chapters. Chapter 2 reviews permutations, notations and the study of several classic permutation statistics. It is interesting to note that many surprising identities and connections to other areas of combinatorics arise as we refine the descent statistic. In Chapter 3, we consider a more refined pattern matching condition where we take into account conditions involving the equivalence classes of the elements of a descent mod k for some integer $k \geq 2$. In general, when one includes parity conditions or conditions involving equivalence mod k, then the problem of counting the number of pattern matchings becomes more complicated. We then proceed to provide q-analogues to these findings and present them in Chapter 4. In Chapter 5, we prove some results on patterns in words. In particular we show that the generating functions for words embedding specific patterns are rational functions. In fact we also develop a method to obtain these generating functions using a finite state automaton. Thus, we can compare generating functions for words embedding different patterns. Sometimes these generating functions are the same, so many bijective questions arise from this study.

We will then review some work of Jeff Remmel and Anthony Mendes. In particular, they were able to find generating functions which count occurrences of consecutive sequences in a permutation or a word which matches a given pattern by exploiting the combinatorics associated with symmetric functions. They were able to take the generating function for the number of permutations which do not contain a certain pattern and give generating functions refining permutations by both the total number of pattern matches and the number of non-overlapping pattern matches. However, as a corollary, the generating function that they produced involved a term counting the number of permutations that have consecutive overlapping patterns at certain positions. We begin to enumerate these for permutations in S_4 and S_5 in Chapter 6. Lastly, we look at yet another generalization of the descent statistic where we require the descent to be equal to a fixed value, k. Our results in this area are presented in Chapter 7.

1

Historical introduction

We will now provide a historical introduction to the remaining chapters of the thesis. In particular, we will review previous results in these subjects and highlight some of the main new results that will be presented. Most of the text found in this historical introduction can also be found in the introduction to each chapter.

First we will review some work of Kitaev and Remmel. Given any sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, we let $red(\sigma)$ be the permutation that results by replacing the ith largest integer that appears in the sequence σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $red(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, we define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a τ**-match** at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let τ-$mch(\sigma)$ be the number of τ-matches in the permutation σ. To prevent confusion, we note that a permutation not having a τ-match is different than a permutation being τ-avoiding. A permutation is called τ**-avoiding** if there are no indices $i_1 < \cdots < i_j$ such that $red[\sigma_{i_1} \cdots \sigma_{i_j}] = \tau$. For example, if $\tau = 2\ 1\ 4\ 3$, then the permutation $3\ 2\ 1\ 4\ 6\ 5$ does not have a τ-match but it does not avoid τ since $red[2\ 1\ 6\ 5] = \tau$.

In the case where $|\tau| = 2$, then τ-$mch(\sigma)$ reduces to familiar permutation statistics. That is, if $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. Then it is easy to see that $(2\ 1)$-$mch(\sigma) = des(\sigma) = |Des(\sigma)|$ and $(1\ 2)$-$mch(\sigma) = rise(\sigma) = |Rise(\sigma)|$.

A number of recent publications have analyzed the distribution of τ-matches in permutations. See, for example, [1, 2, 3]. A number of interesting results have been proved. For example, let τ-nlap(σ) be the maximum number of non-overlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers. Then Kitaev [2, 3] proved the following.

Theorem 1.1.

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x)+x(1-t)A(t)} \tag{1.1}$$

where $A(t) = \sum_{n\geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$.

In other words, if the exponential generating function for the number of permutations in S_n without any τ-matches is known, then so is the exponential generating function for the entire distribution of the statistic $\tau-$nlap.

In Chapter 3, we generalize the work of Kitaev and Remmel and consider a more refined pattern matching condition where we take into account conditions involving equivalence mod k for some integer $k \geq 2$. That is, suppose we fix $k \geq 2$ and we are given some sequence of distinct integers $\tau = \tau_1 \cdots \tau_j$. Then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a τ-k-**equivalence match** at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \ldots, j-1\}$, $\sigma_{i+s} = \tau_{1+s}$ mod k. For example, if $\tau = 1\ 2$ and $\sigma = 5\ 1\ 7\ 4\ 3\ 6\ 8\ 2$, then σ has τ-matches starting at positions 2, 5, and 6. However, if $k = 2$, then only the τ-match starting at position 5 is a τ-2-equivalence match. Later, it will be explained that the τ-match starting a position 2 is a (1 3)-2-equivalence match and the τ-match starting a position 6 is a (2 4)-2-equivalence match. Let τ-k-**emch**(σ) be the number of τ-k-equivalence matches in the permutation σ. Let τ-k-**enlap**(σ) be the maximum number of non-overlapping τ-k-equivalence matches in σ where two τ-matches are said to overlap if they contain any of the same integers.

More generally, if Υ is a set of sequences of distinct integers of length j, then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a Υ-k-**equivalence match** at place i provided there is a $\tau \in \Upsilon$ such that $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \dots, j-1\}$, $\sigma_{i+s} = \tau_{1+s} \mod k$. Let Υ-k-**emch**(σ) be the number of Υ-k-equivalence matches in the permutation σ and Υ-k-**enlap**(σ) be the maximum number of non-overlapping Υ-k-equivalence matches in σ.

Chapter 3, presents the study of the polynomials

$$T_{\tau,k,n}(x) = \sum_{\sigma \in S_n} x^{\tau\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} T^s_{\tau,k,n} x^s \text{ and} \tag{1.2}$$

$$U_{\Upsilon,k,n}(x) = \sum_{\sigma \in S_n} x^{\Upsilon\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} U^s_{\Upsilon,k,n} x^s. \tag{1.3}$$

In particular, we shall focus on certain special cases of these polynomials where we consider only patterns of length 2. That is, fix $k \geq 2$ and let A_k equal the set of all sequences $(a\ b)$ such that $1 \leq a < b \leq 2k$ where there is no lexicographically smaller sequence $x\ y$ having the property that $x \equiv a \mod k$ and $y \equiv b \mod k$. For example,

$$A_4 = \{1\ 2, 1\ 3, 1\ 4, 1\ 5, 2\ 3, 2\ 4, 2\ 5, 2\ 6, 3\ 4, 3\ 5, 3\ 6, 3\ 7, 4\ 5, 4\ 6, 4\ 7, 4\ 8\}.$$

Let $D_k = \{b\ a : a\ b \in A_k\}$ and $E_k = A_k \cup D_k$. Thus E_k consists of all k-equivalence patterns of length 2 that we could possibly consider. Note that if $\Upsilon = A_k$, then Υ-k-$emch(\sigma) = rise(\sigma)$ and if $\Upsilon = D_k$, then Υ-k-$emch(\sigma) = des(\sigma)$.

Our goal is to give explicit formulas for the coefficients of $T^s_{\tau,k,n}$ and $U^s_{\Upsilon,k,n}$. First we will show that we can use inclusion-exclusion to find a formula for $U^s_{\Upsilon,k,n}$ for any $\Upsilon \subset E_k$ in terms of certain rook numbers of a sequence of boards associated with Υ. While this approach is straightforward, it is unsatisfactory since it reduces the computation of $U^s_{\Upsilon,k,n}$ to another difficult problem: namely, computing rook numbers for general boards. However, we can give two other more direct formulas for the coefficients $U^s_{\Upsilon,k,n}$ where $\Upsilon \subseteq A_k$ is a subset of the form $\{(x_1, y_1), (x_2, y_2), \dots, (x_t, y_t)\}$ where for all i, j $y_i \equiv y_j \mod k$. That is, if we define $y = min(\{y_1, \dots, y_t\})$ and $\alpha = |\{x_i : x_i < y\}|$, then we shall show that for

$y-k \le j \le y-1$ and all $s \le n$ such that $kn+j>0$, we have

$$\begin{aligned} &\frac{U^s_{\Upsilon,k,kn+j}}{((k-1)n+j)!} \\ = &\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{s-r}\Gamma(r,j,n) \quad (1.4)\\ = &\sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{n-s-r}\Omega(r,n) \quad (1.5)\end{aligned}$$

$\Gamma(r,j,n) = \prod_{i=0}^{n-1}((k-1)n+r+j+1-\alpha-i(|\Upsilon|-1))$ and
$\Omega(r,n) = \prod_{i=0}^{n-1}(r+\alpha+i(|\Upsilon|-1))$.

Furthermore, in the special case where $\Upsilon = \{(1\ k)\}$, our results will imply that for all $0 \le s \le n$ and for all $0 \le j \le k-1$,

$$\begin{aligned} &\frac{T^s_{(1\ k),k,kn+j}}{((k-1)n+j)!} \quad (1.6)\\ = &\frac{1}{((k-1)n+j)!}\sum_{r=s}^{n}(-1)^{r-s}(kn+j-r)!\binom{r}{s}S_{n+1,n+1-r}\\ = &\sum_{r=0}^{s}(-1)^{s-r}((k-1)n+j+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}\\ = &\sum_{r=0}^{n-s}(-1)^{n-s-r}(1+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}\end{aligned}$$

where $S_{n,k}$ is the Stirling number of the second kind, i.e., $S_{n,k}$ is the number of partitions of an n-set into k parts. These formulas lead to interesting identities in their own right. For example, from above, we see that for all $k \ge 2$, $0 \le s \le n$ and $0 \le j \le k-1$,

$$\sum_{r=0}^{s}(-1)^{s-r}((k-1)n+j+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r} = \sum_{r=0}^{n-s}(-1)^{n-s-r}(1+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}.$$

The general problem of finding explicit expressions for the coefficients $U^s_{\Upsilon,k,n}$ for arbitrary Υ is open. However, Kitaev and Remmel [4, 5] have developed formulas for $U^s_{\Upsilon,k,n}$ in certain other special cases. In particular, Kitaev and Remmel studied permutation statistics which classified the descents of a permutation according to whether either the first element or the second element of a descent pair is equivalent to $0 \mod k$. In our language, they computed explicit formulas for $U^s_{\Upsilon,k,n}$ where either $\Upsilon = \{b\ a : (b\ a) \in D_k \ \&\ b \equiv 0 \mod k\}$ or $\Upsilon = \{b\ a : (b\ a) \in D_k \ \&\ a \equiv 0 \mod k\}$. In this chapter, we shall generalize some of their results by deriving explicit formulas for $U^s_{\Upsilon,k,n}$ in the special cases where Υ is a subset of the form $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ where for all i, j $y_i \equiv y_j \mod k$ and either $\Upsilon \subseteq A_k$ or $\Upsilon \subseteq D_k$.

Much of the work in Chapter 3 was later generalized by Hall and Remmel. Following this, I collaborated with Hall and Remmel to generalize their results by deriving some q-analogues of their formulas. All of these results are presented in Chapter 4. We first begin with a review of their work.

As before, let S_n denote the set of permutations of the set $[n] = \{1, 2, \ldots, n\}$. Given subsets $X, Y \subseteq \mathbb{N}$ and a permutation $\sigma \in S_n$, let

$$\begin{aligned} \mathrm{Des}_{X,Y}(\sigma) &= \{i : \sigma_i > \sigma_{i+1} \ \&\ \sigma_i \in X \ \&\ \sigma_{i+1} \in Y\}, \text{ and} \\ \mathrm{des}_{X,Y}(\sigma) &= |\mathrm{Des}_{X,Y}(\sigma)|. \end{aligned}$$

If $i \in \mathrm{Des}_{X,Y}(\sigma)$, then we call the pair (σ_i, σ_{i+1}) an (X, Y)-*descent*. For example, if $X = \{2, 3, 5\}, Y = \{1, 3, 4\}$, and $\sigma = 54213$, then $\mathrm{Des}_{X,Y}(\sigma) = \{1, 3\}$ and $\mathrm{des}_{X,Y}(\sigma) = 2$.

For fixed n we define the polynomial

$$P_n^{X,Y}(x) = \sum_{s \geq 0} P_{n,s}^{X,Y} x^s := \sum_{\sigma \in S_n} x^{\mathrm{des}_{X,Y}(\sigma)}. \tag{1.7}$$

Thus the coefficient $P_{n,s}^{X,Y}$ is the number of $\sigma \in S_n$ with exactly s (X, Y)-descents. Again, note that with the correct choices of sets for X and Y, this coefficient generalizes $U^s_{\Upsilon,k,n}$.

Hall and Remmel [6] gave direct combinatorial proofs of a pair of formulas for

$P_{n,s}^{X,Y}$. First of all, for any set $A \subseteq \mathbb{N}$, let

$$\begin{aligned} A_n &= A \cap [n], \text{ and} \\ A_n^c &= (A^c)_n = [n] - A. \end{aligned}$$

Then Hall and Remmel [6] proved the following theorem.

Theorem 1.2.

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{s} (-1)^{s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1+r+\alpha_{X,n,x}+\beta_{Y,n,x}), \quad (1.8)$$

and

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{|X_n|-s-r} \prod_{x \in X_n} (r+\beta_{X,n,x}-\beta_{Y,n,x}), \quad (1.9)$$

where for any set A and any $j, 1 \leq j \leq n$, we define

$$\begin{aligned} \alpha_{A,n,j} &= |A^c \cap \{j+1, j+2, \ldots, n\}| = |\{x : j < x \leq n \ \& \ x \notin A\}|, \text{ and} \\ \beta_{A,n,j} &= |A^c \cap \{1, 2, \ldots, j-1\}| = |\{x : 1 \leq x < j \ \& \ x \notin A\}|. \end{aligned}$$

Now we will present our generalizations of their work. There are two natural approaches to finding q-analogues of (4.2) and (4.3). The first approach is to use q-analogues of the simple recursions that are satisfied by the coefficients $P_{n,s}^{X,Y}$. This approach naturally leads us to recursively define of a pair of statistics $\mathrm{stat}_{X,Y}(\sigma)$ and $\overline{\mathrm{stat}}_{X,Y}(\sigma)$ on permutations σ so that if we define

$$P_{n,s}^{X,Y}(q) = \sum_{\sigma \in S_n, \mathrm{des}_{X,Y}(\sigma)=s} q^{\mathrm{stat}_{X,Y}(\sigma)} \quad (1.10)$$

and

$$\bar{P}_{n,s}^{X,Y}(q) = \sum_{\sigma \in S_n, \mathrm{des}_{X,Y}(\sigma)=s} q^{\overline{\mathrm{stat}}_{X,Y}(\sigma)}, \quad (1.11)$$

then we can prove the following formulas:

$$\frac{P_{n,s}^{X,Y}(q)}{[|X_n^c|]_q!} = \quad (1.12)$$

$$\sum_{r=0}^{s} (-1)^{s-r} q^{\binom{s-r}{2}} \begin{bmatrix} |X_n^c|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{x \in X_n} [1+r+\alpha_{X,n,x}+\beta_{Y,n,x}]_q$$

and

$$\frac{\bar{P}^{X,Y}_{n,s}(q)}{[|X^c_n|]_q!} = \tag{1.13}$$
$$\sum_{r=0}^{|X_n|-s} \frac{q^{\binom{|X_n|-s-r}{2}}}{(-1)^{|X_n|-s-r}} \begin{bmatrix} |X^c_n|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{x\in X_n} [r+\beta_{X,n,x}-\beta_{Y,n,x}]_q \,.$$

The second approach is to q-analogue the combinatorial proofs of (4.2) and (4.3). We will see that this approach also works and leads to a more direct definition of $\mathrm{stat}_{X,Y}(\sigma)$ and $\overline{\mathrm{stat}}_{X,Y}(\sigma)$, involving generalizations of classical permutation statistics such as inv and maj.

In Chapter 5, we move from the study of patterns in permutations to patterns in words.

Let P be a set and consider the corresponding *free monoid* or *Kleene closure* of all words over P:

$$P^* = \{w = w_1 w_2 \dots w_\ell \ : \ n \geq 0 \text{ and } w_i \in P \text{ for all } i\}.$$

Let $\exp(t(x-y))$ be the empty word and for any $w \in P^*$ we denote its cardinality or *length* by $|w|$. Given $w, w' \in P^*$ we say that w' is a *factor* of w if there are words u, v with $w = uw'v$, where adjacency denotes concatenation. For example, $u = 322$ is a factor of $w = 12213221$ starting with the fifth element of w. *Factor order* on P^* is the partial order obtained by letting $u \leq_{\mathrm{fo}} w$ if and only if there is a factor w' of w with $u = w'$.

Now suppose that we have a poset $\mathcal{P} = (P, \leq)$. We define *generalized factor order* (relative to $\mathcal{P}$) on P^* by letting $u \leq_{\mathrm{gfo}} w$ if there is a factor w' of w such that

(a) $|u| = |w'|$, and

(b) $u_i \leq w'_i$ for $1 \leq i \leq |u|$.

We call w' an *embedding* of u into w, and if the first element of w' is the jth element of w, we call j an *embedding index* of u into w. We also say that in this embedding u_i is in *position* $j+i-1$. To illustrate, suppose $P=\mathbb{P}$, the positive integers with the usual order relation. If $u=322$ and $w=12213431$ then $u \leq_{\text{gfo}} w$ because of the embedding factor $w'=343$ which has embedding index 5, and the two 2's of u are in positions 6 and 7. Note that we obtain ordinary factor order by taking P to be an antichain. Also, we will henceforth drop the subscript gfo since context will make it clear what order relation is meant. Generalized factor order is the focus of this chapter.

Returning to the case where P is an arbitrary set, let $\mathbb{Z}\langle\langle P\rangle\rangle$ be the algebra of formal power series with integer coefficients and having the elements of P as noncommuting variables. In other words,

$$\mathbb{Z}\langle\langle P\rangle\rangle = \left\{ f = \sum_{w\in P^*} c(w)w \ : \ c(w)\in\mathbb{Z} \text{ for all } w \right\}.$$

If $f\in\mathbb{Z}\langle\langle P\rangle\rangle$ has no constant term, i.e., $c(\exp(t(x-y)))=0$, then define

$$f^* = \exp(t(x-y)) + f + f^2 + f^3 + \cdots = (\exp(t(x-y)) - f)^{-1}.$$

(We need the restriction on f to make sure that the sums are well defined as formal power series.) We say that f is *rational* if it can be constructed from the elements of P using only a finite number of applications of the algebra operations and the star operation.

A *language* is any $\mathcal{L}\subseteq P^*$. It has an associated generating function

$$f_{\mathcal{L}} = \sum_{w\in\mathcal{L}} w.$$

The language $\mathcal{L}$ is *regular* if $f_{\mathcal{L}}$ is rational.

Consider generalized factor order on P^* and fix a word $u\in P^*$. There is a corresponding language and generating function

$$\mathcal{F}(u) = \{w \ : \ w\geq u\} \quad \text{and} \quad F(u) = \sum_{w\geq u} w.$$

One of our main results is as follows.

Theorem 1.3. *If $\mathcal{P} = (P, \leq)$ is a finite poset and $u \in P^*$, then $F(u)$ is rational.*

This is an analogue of a result of Björner and Sagan [7] for generalized subword order on P^*. *Generalized subword order* is defined exactly like generalized factor order except that w' is only required to be a subword of w, i.e., the elements of w' need not be consecutive in w.

We will demonstrate Theorem 5.1 by constructing a NFA accepting the language for $F(u)$. It is a well-known theorem that, for $|P|$ finite, a language $\mathcal{L} \subseteq P^*$ is regular if and only if there is a NFA accepting $\mathcal{L}$. (See, for example, the text of Hopcroft and Ullman [8, Chapter 2].) In fact, the NFA still exists even if P is infinite, suggesting that more can be said about the generating function in this case.

We are particularly interested in the case of $P = \mathbb{P}$ with the usual order relation. So $\mathbb{P}^*$ is just the set of *compositions* (ordered integer partitions). Given $w = w_1 w_2 \ldots w_\ell \in \mathbb{P}^*$, we define its *norm* to be

$$\Sigma(w) = w_1 + w_2 + \cdots + w_\ell.$$

Let t, x be commuting variables. Replacing each $n \in w$ by tx^n we get an associated monomial called the *weight* of w

$$\mathrm{wt}(w) = t^{|w|} x^{\Sigma(w)}.$$

For example, if $w = 213221$ then

$$\mathrm{wt}(w) = tx^2 \cdot tx \cdot tx^3 \cdot tx^2 \cdot tx^2 \cdot tx = t^6 x^{11}.$$

We also have the associated *weight generating function*

$$F(u; t, x) = \sum_{w \geq u} \mathrm{wt}(w).$$

Our NFA will demonstrate, via the transfer-matrix method, that this is also a rational function of t and x. The details will be given in Section 5.3. Upon computing $F(u; t, x)$ we can find words u and v with $u \neq v$ but $F(u; t, x) =$

$F(w;t,x)$ which we can view as a type of equivalence relation. In fact, we model our notion of equivalence after Wilf equivalence, which is used in the theory of pattern avoidance. We first present the notion of Wilf equivalence in pattern avoidance.

Given a sequence of distinct numbers $s = s_1 \cdots s_n$, we let $red(s)$ denote the permutation of S_n whose elements have the same relative order at $s_1 \cdots s_n$. For example, red(5276) = 2143. Then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_m$ occurs in permutation $\tau = \tau_1 \cdots \tau_n$ if there is a subsequence $1 \leq i_1 < \cdots < i_m \leq n$ such that $red(\tau_{i_1} \cdots \tau_{i_m}) = \sigma$. For example, if $\tau = 7\ 1\ 4\ 2\ 3\ 5\ 6$, then $2\ 1\ 3$ occurs in τ. Notice that $red(\tau_3\tau_4\tau_6) = 213$.

We let $S_n(\sigma) = |\{\tau \in S_n : \tau \text{ avoids } \sigma\}|$. Then, we say that $\sigma, \tau \in S_m$ are **Wilf equivalent** if $S_n(\sigma) = S_n(\tau)$ for all n. See the survey article of Wilf [9] for more information about this subject.

In our studies, we introduce a concept called Wilf equivalence as well. Call $u, w \in \mathbb{P}^*$ *Wilf equivalent* if $F(u;t,x) = F(v;t,x)$. When we mention Wilf equivalence after this point, we will not be referring to the definition from the theory of pattern avoidance. Section 5.4 is devoted to proving various Wilf equivalences. Although these results were discovered by having a computer construct the corresponding generating functions, the proofs we give are purely combinatorial. In the next two sections, we investigate a stronger notion of equivalence and compute generating functions for two families of compositions.

Chapter 5 provides a nice transition back to the study of permutations in the following way.

Suppose we define $\text{nlap}(u,w)$ to be the maximum number of nonoverlapping embeddings of u into w. Notice that the results of Chapter 5 give us a way to compute a generating function involving the statistic $\text{nlap}(u,w)$. Any word w with $\text{nlap}(u,w) = k$ for some $k \geq 1$ can easily be factored into k words embedding u as a suffix and a word avoiding u. Thus we obtain the following.

$$\sum_{w \in \mathbb{P}^*} z^{\text{nlap}(u,w)} x^{\Sigma w} t^{|w|} = \sum_{n \geq 0} z^n S(u)^n A(u) \tag{1.14}$$

So, the study of words in this fashion gives us a relatively easy way to compute the generating function for the statistic nlap. We now turn to a way to try and get a handle on the statistic mentioned before, τ-nlap(σ).

Remmel and Mendes proved Kitaev's Theorem, 3.1, with an alternate approach. We will not present the proof here, but it can be found in [10]. We will first introduce some notation that was adopted by Remmel and Mendes and then will state a corollary that was the impetus to the material in Chapter 6.

We now define a few auxiliary sets associated with a given permutation $\tau \in S_j$. For a permutation $\sigma \in S_j$, let

$$\begin{aligned} \mathrm{Mch}_\tau(\sigma) &= \{i : \mathrm{red}(\sigma_{i+1} \cdots \sigma_{i+j}) = \tau\} \qquad \text{and} \\ I_\tau &= \{1 \le i < j : \text{there exist } \sigma \in S_{j+i} \text{ such that } \mathrm{Mch}_\tau(\sigma) = \{0, i\}\}. \end{aligned}$$

One (or one's computer) can find every element in the set I_τ for any $\tau \in S_j$ by finding the set $\mathrm{Mch}_\tau(\sigma)$ for all $\sigma \in S_{j+i}$ for $i = 1, \ldots, j-1$.

Let I_τ^* be the set of all words with letters in the set I_τ. We let ϵ denote the empty word. If $w = w_1 \cdots w_n \in I_\tau^*$ is word with n-letters, we define

$$\ell(w) = n, \qquad \sum w = \sum_{i=1}^{n} w_i, \qquad \text{and} \qquad ||w|| = j + \sum w.$$

In the special case where $w = \epsilon$, we let $\ell(w) = 0$ and $\sum w = 0$. Let

$$\begin{aligned} A_\tau &= \{w \in I_\tau^* : \ell(w) \ge 2 \text{ and } \sum w < j\} \qquad \text{and} \\ B_{u,\tau} &= \{w_1 \cdots w_n \in I_\tau^* : \sum w_2 \cdots w_n + \sum u < j \le \sum w_1 \cdots w_n + \sum u\} \end{aligned}$$

for each word $u \in I_\tau^*$ with $\sum u < j$.

To help understand these sets, see the examples presented in Chapter 6.

Form a new alphabet

$$K_\tau = \{\overline{u} : u \in I_\tau\} \cup \{\overline{w} : w \in A_\tau\}.$$

We let $\Psi : K_\tau^* \to I_\tau^*$ be the function such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$. For example, if $\tau = \gamma = 2\,1\,4\,3\,5\,6$ as above, then $\Psi(\overline{5}\,\overline{22}\,\overline{2}\,\overline{2}\,\overline{5}) = 522225$.

Define $\overline{J}_\tau$ in the following manner.

1. $\epsilon \in \overline{J}_\tau$.

2. $\overline{v} \in \overline{J}_\tau$ for all $v \in I_\tau$.

3. If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$, then $\overline{u}\, \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$ for all $u \in B_{w_1,\tau}$.

4. The only words in $\overline{J}_\tau$ are the result of applying one of the above rules.

Take $J_\tau = \Psi(\overline{J}_\tau)$.

The final set we would like to define is as follows. Let

$$\mathcal{P}_w^\tau = \{\sigma \in S_{||w||} : \mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}\}$$

and $\mathcal{P}_\epsilon^\tau = \{\tau\}$.

We will now make a key observation. Suppose that $\overline{u} = \overline{u}_k \cdots \overline{u}_1 \in \overline{J}_\tau$ and $\Psi(\overline{u}) = w_1 \cdots w_n$. Thus $w = w_1 \cdots w_n \in J_\tau$. Now suppose that $\sigma \in \mathcal{P}_w^\tau$. Then

$$\mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}.$$

We can scan σ to discover that there are τ-matches at positions $1, 1+w_1, 1+w_1+w_2, \ldots, 1+w_1+w_2+\cdots+w_n$ so that we can recover $w_1, \ldots, w_n$ from σ. In addition, we claim that $\overline{u}$ can also be recovered. Let us now describe an algorithm which does this.

Step 1. Set $\overline{u}_1 = \overline{w}_n$. (This must be the case since the only words of length 1 in $\overline{J}_\tau$ are of the form $\overline{v}$ for $v \in I_\tau$.)

Step s+1. Suppose that we have recovered $\overline{u}_1, \ldots, \overline{u}_s$, $\Psi(\overline{u}_s \cdots \overline{u}_1) = w_b \cdots w_n$, and $\Psi(\overline{u}_s) = w_b \cdots w_c$ where $b \leq c$. It must be the case that $w_b + \cdots + w_c = r$ for some $1 \leq r \leq j-1$. If $w_{b-1} + r \geq j$, then set $\overline{u}_{s+1} = \overline{w}_{b-1}$. Otherwise, let a be the unique integer such that $1 \leq a < b-1$ such that $w_a + \cdots + w_c \geq j$, but $w_{a+1} + \cdots + w_c < j-1$ and set $\overline{u}_{s+1} = \overline{w_a \cdots w_{b-1}}$. (In this latter case, it follows from our definitions that $w_a \cdots w_{b-1}$ is the unique cofinal sequence of $w_1 \cdots w_{b-1}$ such that $w_a \cdots w_{b-1} \in B_{w_b \cdots w_c, \tau}$ so that it must be the case that $\overline{u}_{s+1} = \overline{w_a \cdots w_{b-1}}$.)

Remmel and Mendes proved the following corollary when they proved Kitaev's formula using their methods. For any permutation τ,

$$1+\sum_{n=1}^{\infty}\frac{t^n}{n!}|\{\sigma\in S_n : \tau\text{-mch}(\sigma)=0\}| = \frac{1}{1-t+\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{w\in J_\tau,||w||=n}(-1)^{\overline{\ell}(w)}|\mathcal{P}^{\tau}_{w}|}. \tag{1.15}$$

They noted that if the generating function for the number of permutations without any τ-matches is known, then Theorem 3.1 is able to refine all permutations of n by the maximum number of nonoverlapping τ-matches. However, Theorem 3.1 does not give any direct way to find the number of permutations without τ-matches and, in general, it is difficult to count the number of permutations σ in S_n with τ-mch$(\sigma)=0$. However, (6.2) provides an alternative approach to finding the number of permutations σ in S_n with τ-mch$(\sigma)=0$. That is, instead of counting the number of permutations in S_n with τ-mch$(\sigma)=0$, we may try to understand the sum on the right hand side of the statement of the corollary.

As an example of this phenomenon, suppose we wanted to find out more about the maximum number of nonoverlapping τ-matches when $\tau = 1\ 3\ 2$. The set I_τ contains only the integer 2. This greatly simplifies matters since we have that $J_\tau = I_\tau^* = \{2\}^*$. Suppose that $\sigma_1\ldots\sigma_{2n+3}\in P^{\tau}_{2^n}$. Since there is τ-match starting at position $2i+1$ for $i=0,\ldots,n$, it must be the case that $\sigma_{2i+1}<\sigma_{2i+2},\sigma_{2i+3}$ for $i=0,\ldots,n$. It follows that $\sigma_1=1$ and $\sigma_3=2$. It is not difficult to see that σ_2 can be any element of $\{3,\ldots,2n+3\}$ and that red$(\sigma_3\ldots\sigma_{2n+3})\in P^{\tau}_{2^{n-1}}$ if $n\geq 1$. It follows that $|P^{\tau}_{2^n}|=(2n+1)|P^{\tau}_{2^{n-1}}|$ if $n\geq 1$. Since $P^{\tau}_{2^0}=1$, it follows by induction that $|P^{\tau}_{2^n}|=(2n+1)!!=(2n+1)(2n-1)\cdots 3\cdot 1$ for $n\geq 0$. Thus, from 6.2, we have that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}|\{\sigma\in S_n:\text{Mch}_{132}(\sigma)=\emptyset\}| &= \frac{1}{1-t+\sum_{n=0}^{\infty}(-1)^n|P^{132}_{2^n}|\frac{t^{2n+3}}{(2n+3)!}}\\ &= \frac{1}{1-t+\sum_{n=0}^{\infty}(-1)^n(\prod_{i=0}^{n}(2i+1))\frac{t^{2n+3}}{(2n+3)!}}\\ &= \frac{1}{1-\int exp(-t^2/2)dt}.\end{aligned}$$

From Theorem 3.1, the generating function refining the permutations of n by the maximum number of nonoverlapping τ-matches is equal to

$$\left(1 - tx + (x-1)\int \exp\left(-t^2/2\right)\, dt\right)^{-1}$$

in the case $\tau = 1\ 3\ 2$.

We begin to investigate $\sum_{w \in J_\tau, ||w||=n} (-1)^{\bar{\ell}(w)} |\mathcal{P}_w^\tau|$ in an attempt to apply these techniques for $\tau \in S_4$.

In Chapter 7, we will alter the descent statistic in another way. For example, suppose we only wanted to count descents that drop by a fixed value, k. The motivation for this research actually came from a comment made at my advancement talk by one of my committee members, Ramamohan Paturi. He noted that it was similar to the problem posed in Chapter 3.

If we let

$$\sigma = 3\ 8\ 6\ 2\ 4\ 7\ 5\ 1,$$

we can see that $Des(\sigma) = \{2, 3, 4\}$ and thus $des(\sigma) = 3$. We can see that descent in the second position is from 8 to 6, and thus we would consider this a drop by 2, or a 2-drop. In general, if $\sigma_i - \sigma_{i+1} = k > 0$, then we call the descent at position i a k-drop.

We can now try to enumerate permutations by the number of k-drops. To be more rigorous, suppose we define, for $k > 0$,

$$\begin{aligned} Des_k(\sigma) &= \{i | \sigma_i - \sigma_{i+1} = k\} \\ des_k(\sigma) &= |Des_k(\sigma)| \\ P_{n,k}(x) &= \sum_{\sigma \in S_n} x^{des_k(\sigma)} = \sum_{s=0}^{n-k} P_{n,k,s} x^s. \end{aligned}$$

Chapter 7 begins the study of the coefficients $P_{n,k,s}$ and also presents two q-analogues of these coefficients as well.

2

Permutation basics

2.1 What is a permutation?

A permutation is defined as an ordered list of distinguishable objects. Naturally, you can assume your objects are the integers from 1 to n. For example,

$$\sigma = 2\ 4\ 5\ 7\ 3\ 1\ 6$$

is a permutation of the integers from 1 to 7.

However, we could also think of a permutation as a bijective mapping from a set X onto itself. In our previous example where $\sigma = 2\ 4\ 5\ 7\ 3\ 1\ 6$, we can associate a mapping f which maps the element i to the i^{th} element of σ, σ_i. The associated mapping in this example is

$$\begin{array}{cccc} f(1) = 2 & f(2) = 4 & f(3) = 5 & f(4) = 7 \\ f(5) = 3 & f(6) = 1 & f(7) = 6 & \end{array}.$$

In this setting, we can construct a group, called a permutation group whose elements are permutations of a given set, where the group operation is composition of these bijective mappings. We will define S_n to be the symmetric group on n elements. In other words, for our previous example, $\sigma \in S_7$.

There are several notations used when presenting permutations. We will describe two of them now. The first is the method that we used previously and will adopt from here on. We will not use other notations unless we explicitly make mention of it.

2.1.1 One line notation

As we mentioned earlier, we can think of a permutation in S_n as a bijective mapping from the set $[n] := \{1, \ldots, n\}$ onto itself. We will present a permutation in one-line notation, $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$ if and only i is mapped to σ_i. So for example, if $\sigma = 1\ 5\ 4\ 2\ 3$, then σ is the permutation that maps 1 to 1, 2 to 5, 3 to 4, 4 to 2, and 5 to 3.

2.1.2 Cycle notation

Another notation which is commonly used with permutations is cycle notation. The idea with this notation is to repeatedly apply the bijective mapping to a particular element to generate a cycle. For example, if we again let $\sigma = 1\ 5\ 4\ 2\ 3$ then we can observe that 2 maps to 5. If we now apply the mapping to 5, we see that 5 maps to 3, which maps to 4, which maps to 2. So the progression of these elements, namely $(2, 5, 3, 4)$ is called a cycle. Also, note that 1 maps to 1, so we consider (1) a cycle. Typically in cycle notation, cycles are ordered by increasing minimal element. Thus, in cycle notation $\sigma = (1)(2, 5, 3, 4)$.

2.1.3 Permutation statistics

We will now define some classical permutation statistics on a permutation σ including: the descent statistic, $des(\sigma)$, the major index statistic, $maj(\sigma)$, the inversion statistic, $inv(\sigma)$, the rise statistic, $ris(\sigma)$, the comajor index statistic, $comaj(\sigma)$ and the coinversion statistic, $coinv(\sigma)$. We also define the descent set of σ, $Des(\sigma)$ and the rise set of σ, $Ris(\sigma)$.

1. $Des(\sigma) = \{i | \sigma_i > \sigma_{i+1}\}$

2. $des(\sigma) = |Des(\sigma)|$

3. $maj(\sigma) = \sum_{i \in Des(\sigma)} i$

4. $inv(\sigma) = \sum_{1 \leq i < j \leq n} \chi(\sigma_i > \sigma_j)$, where $\chi(A) = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$

5. $Ris(\sigma) = \{i | \sigma_i < \sigma_{i+1}\}$

6. $ris(\sigma) = |Ris(\sigma)|$

7. $comaj(\sigma) = \sum_{i \in Ris(\sigma)} i$

8. $coinv(\sigma) = \sum_{1 \leq i < j \leq n} \chi(\sigma_i < \sigma_j)$,

Here are some basic things to note about these statistics:

1. $Ris(\sigma) \cup Des(\sigma) = \{1, 2, \ldots, n-1\}$

2. $ris(\sigma) + des(\sigma) = n - 1$

3. $maj(\sigma) + comaj(\sigma) = 1 + 2 + \ldots + n - 1 = \binom{n}{2}$

4. $inv(\sigma) + coinv(\sigma) = \binom{n}{2}$

For example, if $\sigma = 6\ 3\ 2\ 1\ 5\ 4$, then $Des(\sigma) = \{1, 2, 3, 5\}$, $des(\sigma) = 4$, $maj(\sigma) = 1 + 2 + 3 + 5 = 11$, and $inv(\sigma) = 9$.

We will go on to talk more about some generating functions and what is known about some of these permutation statistics mentioned above with a focus on the statistic $des(\sigma)$. However, first we include a review of symmetric functions as they will play a role in proving the generating function for $des(\sigma)$.

2.2 The combinatorics of symmetric functions

We will be presenting a review of the symmetric functions from a purely combinatorial standpoint. The proofs for a variety of the statements made in this section will be omitted.

We first define an operation on a polynomial in N variables, $P(x_1, \ldots, x_N)$ in the following way,

$$\sigma P(x_1, \ldots, x_N) = P(x_{\sigma_1}, \ldots, x_{\sigma_N}).$$

We are now able to define $\Lambda(\overline{x})$ as the space of symmetric functions in variables $x_1, \ldots, x_N$. If for all $\sigma \in S_n$, $\sigma P(x_1, \ldots, x_N) = P(x_1, \ldots, x_N)$, then P is symmetric.

We also define $\Lambda_n(\overline{x})$ as the space of homogeneous symmetric polynomials of degree n, $n < N$. For example, if $N = 3$, the polynomial

$$x_1 + x_2 + x_3 + x_1^2 + x_2^2 + x_3^2$$

is clearly symmetric but is not homogenous, because not all of the monomials are of the same degree. It is clear though that we can take a polynomial and group the monomials that occur by their degree. This means that

$$\Lambda(\overline{x}) = \bigoplus_n \Lambda_n(\overline{x}).$$

We will now begin to examine several bases for $\Lambda_n(\overline{x})$, starting with the most basic. Suppose we have $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $0 < \lambda_1 \leq \ldots \leq \lambda_k$, is a partition of n, i.e., $|\lambda| = \lambda_1 + \ldots + \lambda_k = n$, written, $\lambda \vdash n$.

2.2.1 The monomial symmetric functions

We define a monomial symmetric function,

$$m_\lambda(\overline{x}) = \sum_{\substack{p_1, \ldots, p_N \\ \tau(p_1, \ldots, p_N) = \lambda}} x_1^{p_1} \ldots x_n^{p_n},$$

where $\tau(p_1, \ldots, p_N)$ is the weakly increasing rearrangement of $p_1, \ldots, p_N$. For example,

$$m_{2,1}(\overline{x}) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2.$$

Theorem 2.1. *$\{m_\lambda(\overline{x}) | \lambda \vdash n\}$ is a basis for $\Lambda_n(\overline{x})$.*

2.2.2 The elementary, homogeneous, power, and Schur symmetric functions

We will now define four other types of symmetric functions which we will eventually verify that they form a basis of $\Lambda_n(\overline{x})$ as well. First we define,

$$\begin{aligned} H(t) &= \prod_{i=1}^{N} \frac{1}{1-tx_i} = \sum_{n\geq 0} h_n(\overline{x})t^n \\ E(t) &= \prod_{i=1}^{N}(1+tx_i) = \sum_{n\geq 0} e_n(\overline{x})t^n. \end{aligned}$$

This means that we have

$$\begin{aligned} h_n(\overline{x}) &= \sum_{1\leq i_1\leq\ldots\leq i_n\leq N} x_{i_1}\cdots x_{i_n} \\ e_n(\overline{x}) &= \sum_{1\leq i_1<\ldots<i_n\leq N} x_{i_1}\cdots x_{i_n}. \end{aligned}$$

If we also define, $p_n(\overline{x}) = x_1^n + \ldots + x_N^n$, we can now define the homogeneous, elementary and power symmetric functions in the following way.

1. $h_\lambda(\overline{x}) = h_{\lambda_1}(\overline{x})\cdots h_{\lambda_k}(\overline{x})$

2. $e_\lambda(\overline{x}) = e_{\lambda_1}(\overline{x})\cdots e_{\lambda_k}(\overline{x})$

3. $p_\lambda(\overline{x}) = p_{\lambda_1}(\overline{x})\cdots p_{\lambda_k}(\overline{x})$

We will also now define the Schur functions. Given a partition λ, we can imagine filling the cells of the Ferrers diagram corresponding to λ with the rules:

1. the entries are weakly increasing in rows from left to right, and

2. the entries are strictly increasing in columns from bottom to top.

A Ferrers diagram with the prescribed filling is called a column strict tableaux of shape λ. For each filling, T, we define the weight of T to be

$$w(T) = \prod_i x_i^{\text{\# of occurences of i}}.$$

For example, suppose T is the column strict tableaux shown in Figure 2.1, then $w(T) = x_1^4 x_2^3 x_3^3 x_4^3 x_5^2 x_6^2$.

T=

6					
4	6				
3	4	4	5		
2	2	3	3		
1	1	1	1	2	5

Figure 2.1: The tableaux, T.

We define the Schur function,

$$S_\lambda(\overline{x}) = \sum_{T \in CS(\lambda)} w(T), \tag{2.1}$$

where $CS(\lambda)$ is the set of all column strict tableaux of shape λ.

We will now note several observations. First, note that $S_{(n)}(\overline{x}) = h_n(\overline{x})$ and $S_{1^n}(\overline{x}) = e_n(\overline{x})$. Second, note that as defined in (2.1), $S_\lambda(\overline{x})$ is not obviously symmetric, but it can be shown that indeed it is.

Theorem 2.2. $\{h_\lambda(\overline{x})|\lambda \vdash n\}$, $\{e_\lambda(\overline{x})|\lambda \vdash n\}$, $\{p_\lambda(\overline{x})|\lambda \vdash n\}$, $\{S_\lambda(\overline{x})|\lambda \vdash n\}$ *are all bases of* $\Lambda_n(\overline{x})$.

We will start by proving that the elementary symmetric functions are indeed a basis. First we will introduce some notation. Suppose we have $\{a_\lambda(\overline{x})|\lambda \vdash n\}$, and $\{b_\lambda(\overline{x})|\lambda \vdash n\}$ are bases of $\Lambda_n(\overline{x})$. Then if we think of a row vector, $< b_\lambda(\overline{x}) >_{\lambda \vdash n}$, we should be able to express this in terms of the a basis using a matrix. In other words,

$$< b_\lambda(\overline{x}) >_{\lambda \vdash n} = < a_\lambda(\overline{x}) >_{\lambda \vdash n} M(a, b)$$

Thus,

$$b_\lambda(\overline{x}) = \sum_{\mu \vdash n} a_\mu(\overline{x}) M(a, b)_{\mu,\lambda}.$$

So, we will try to write the elementary symmetric functions as linear combinations of the monomial symmetric functions, i.e.,

$$e_\lambda(\overline{x}) = \sum_{\mu \vdash n} m_\mu(\overline{x}) M(m, e)_{\mu,\lambda}.$$

Table 2.1: How many times does a certain monomial appear in e_λ?

x_1x_2	1	1	0
x_1	1	0	0
$x_1^2x_2$	2	1	0

x_1x_2	1	1	0
x_2	0	1	0
$x_1x_2^2$	1	2	0

x_1x_2	1	1	0
x_3	0	0	1
$x_1x_2x_3$	1	1	1

x_1x_3	1	0	1
x_1	1	0	0
$x_1^2x_3$	2	0	1

x_1x_3	1	0	1
x_2	0	1	0
$x_1x_2x_3$	1	1	1

x_1x_3	1	0	1
x_3	0	0	1
$x_1x_3^2$	1	0	2

x_2x_3	0	1	1
x_1	1	0	0
$x_1x_2x_3$	1	1	1

x_2x_3	0	1	1
x_2	0	1	0
$x_2^2x_3$	0	2	1

x_2x_3	0	1	1
x_3	0	0	1
$x_2x_3^2$	0	1	2

To show this, we will look at an example. Suppose $\lambda = \{1, 2\}$, since $e_\lambda = e_1 \cdot e_2$, we have that

$$\begin{aligned} e_2 \cdot e_1 &= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 3x_1x_2x_3, \end{aligned}$$

which we can see is equal to $m_{2,1}+3m_{1,1,1}$. However, we can construct the following table, Table 2.1, to keep track of how many times a certain monomial appears in e_λ. As there are nine terms after we take the product of e_1 and e_2, we will look at nine matrices which have a one in the i^{th} row and j^{th} column if x_j shows up in e_{λ_i}. If we multiply two of the elements involved in one of the nine products, we see that this corresponds to summing the columns. In other words, the sum of the entries in the i^{th} column is the exponent corresponding to x_i in the product. So we can actually give a combinatorial interpretation to these coefficients. In particular,

$$e_\lambda(\overline{x}) \mid_{m_\mu(\overline{x})} = e_\lambda(\overline{x}) \mid_{x_1^{\mu_1} \cdots x_k^{\mu_k}},$$

where $\mu = (\mu_1 \geq \ldots \geq \mu_k > 0)$, which is the number of (0,1) valued $k \times N$ matrices M such that the row sums of M are λ and the column sums are μ where k is the number of parts of λ. Notice that in our example,

$$e_{2,1}(\overline{x}) \mid_{m_{1,1,1}(\overline{x})} = 3,$$

as there are three (0,1) valued 2×3 matrices whose row sums are 2,1 and whose column sums are 1,1,1.

We will now define a dominance order on partitions of n. Suppose we have two partitions, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$, and $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0)$. We say that $\lambda \geq_D \mu$, or λ dominates μ if and only if for all j,

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i.$$

Notice that sometimes we will have two shapes that are incomparable by this definition. Figure 2.2 shows the dominance order on partitions of 6.

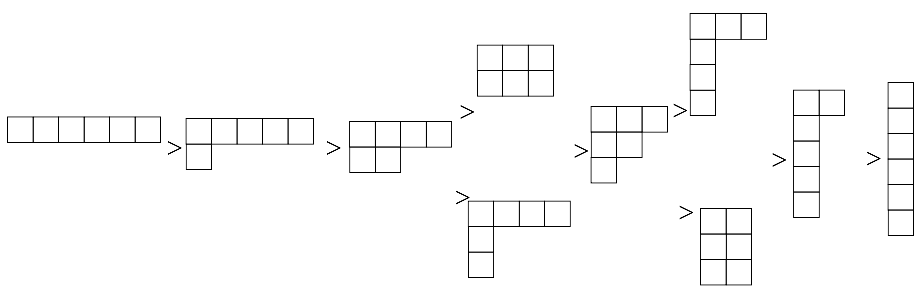

Figure 2.2: The dominance order on partitions of 6.

A natural question to ask is "what is the largest μ, in dominance order, such that $e_\lambda \mid_{m_\mu} = e_\lambda \mid_{x_1^{\mu_1} \cdots x_k^{\mu_k}} \neq 0$?" Thinking in terms of (0,1)-valued matrices, you would want to put the ones as far to the left as possible. Once this is done, the partition obtained is λ', called lambda conjugate. Thus, we have that

$$e_\lambda \mid_{m_\mu} \neq 0 \Rightarrow \mu \leq_D \lambda'.$$

It can also be shown that

$$\lambda \leq_D \mu \Leftrightarrow \mu' \leq_D \lambda'.$$

We create some linear order of partitions of n, $\lambda^1, \lambda^2, \dots \lambda^{P(n)}$ compatible with $\leq_D$, meaning that if $\lambda^i \leq_D \lambda^j$, then $i < j$. We can construct the change of basis matrix, $\|M(m,e)_{\mu,\lambda}\|$. It will look like the following:

$$\begin{array}{ccccc} & \lambda^1 & \lambda^2 & \dots & \lambda^{P(n)} \\ \lambda^1 & 1 & & & \\ \lambda^2 & 0 & 1 & & \\ \vdots & 0 & 0 & 1 & \\ \lambda^{P(n)} & 0 & 0 & 0 & 1 \end{array}$$

Most importantly, note that it is unit upper triangular, which means that it is invertible. Thus, we could also write the monomial symmetric functions in terms of the elementary symmetric functions. This shows that the elementary symmetric functions are also a basis of $\Lambda_n(\overline{x})$.

Recall that we had defined the following functions,

$$H(t) = \prod_{i=1}^{N} \frac{1}{1 - x_i t} = \sum_{n \geq 0} h_n(\overline{x}) t^n$$
$$E(t) = \prod_{i=1}^{N} (1 + x_i t) = \sum_{n \geq 0} e_n(\overline{x}) t^n.$$

Just by this definition, we have the obvious identity that

$$H(t)E(-t) = 1.$$

By definition, this implies that

$$\sum_{n \geq 0} t^n \left(\sum_{k=0}^{n} h_k(\overline{x}) (-1)^{n-k} e_{n-k}(\overline{x}) \right) = 1.$$

In particular, for $n > 0$,

$$\sum_{k=0}^{n} h_k(\overline{x}) (-1)^{n-k} e_{n-k}(\overline{x}) = 0. \tag{2.2}$$

In fact, this can be shown combinatorially as well. If we think of the homogenous symmetric functions $h_k = \sum_{T \in CS((k))} w(T)$ and $e_k = \sum_{S \in CS(1^k)} w(S)$, the above equation becomes

$$\sum_{k=0}^{n} (-1)^{n-k} \left(\sum_{T \in CS((k))} w(T) \right) \left(\sum_{S \in CS(1^{n-k})} w(S) \right) = 0.$$

We can now think of summing over all pairs $T \in CS((k))$ and $S \in CS(1^{n-k})$, yielding

$$\sum_{(T,S)} (-1)^{|S|} w(T) w(S) = 0.$$

We will now introduce a sign reversing involution on pairs of tableaux as follows. If the largest element of T is larger than the top element of S, move this element from T to the top of S to obtain T' and S'. If the largest element of T is less than or equal to the top element of S, move the top element from S to the end of T to obtain T' and S'. See Figure 2.3 for how this involution acts on an example. It is also clearly sign reversing and an involution.

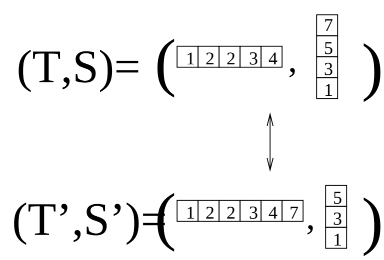

Figure 2.3: An involution on pairs of tableaux.

Notice also that we can take (2.2) and solve the equation for either $h_n(\overline{x})$ or $e_n(\overline{x})$. Doing this we obtain

$$\begin{aligned} h_n(x) &= \sum_{k=0}^{n-1} h_k(\overline{x})(-1)^{n-k-1} e_{n-k}(\overline{x}), \quad \text{and similarly,} \\ (-1)^n e_n(x) &= \sum_{k=1}^{n} h_k(\overline{x})(-1)^{n-k-1} e_{n-k}(\overline{x}), \quad \text{which implies} \\ e_n(x) &= \sum_{k=0}^{n-1} e_k(\overline{x})(-1)^{n-k-1} h_{n-k}(\overline{x}). \end{aligned}$$

We notice that we have a type of symmetry between the es and hs. Moreover, staring at the above equations, we see that we should be able to write e_n in terms of the hs by induction. In other words, we know $e_1 = h_1$. So, by induction, for $i < n$ we can write e_i in terms of the hs and thus using the above equation, we can write e_n in terms of the hs; therefore e_λ could also be written in terms of the hs. We could have switched the roles of h and e in the above argument and would have reached the conclusion that we could also write the hs in terms of the es.

We will now try to find a change of basis matrix such that

$$h_\lambda(\overline{x}) = \sum_{\mu \vdash n} e_\mu M(e,h)_{\mu,\lambda}.$$

However, before we do this, we will introduce brick tabloids.

2.2.3 Brick tabloids

First, we shall define a brick tabloid of type μ and shape λ. We think of filling the Ferrers diagram of λ with bricks whose sizes are the components of μ in such a way that

1. each brick lies entirely in a row, and
2. the bricks do not overlap.

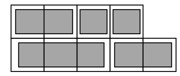

Figure 2.4: A brick tabloid of type μ and shape λ.

For example, suppose $\mu = (1,1,2,2,3)$ and $\lambda = (4,5)$. Figure 2.4 illustrates a brick tabloid of type μ and shape λ.

We also define $B_{\mu,\lambda}$ as the number of brick tabloids of type μ and shape λ. For any partition λ, $l(\lambda) =$ the number of parts of λ.

Theorem 2.3.

$$M(e,h)_{\mu,\lambda} = M(h,e)_{\mu,\lambda} = (-1)^{n-l(\lambda)} B_{\mu,\lambda}$$

Proof. We want to write

$$\begin{aligned} h_n(\overline{x}) = \sum_{\mu \vdash n} e_\mu(\overline{x}) M(e,h)_{\mu,(n)} &= \sum_{k=0}^{n-1} h_k(\overline{x})(-1)^{n-k-1} e_{n-k}(\overline{x}) \\ &= \sum_{k=0}^{n-1} (\sum_{\alpha \vdash k} e_\alpha(\overline{x}) M(e,h)_{\alpha,(k)})(-1)^{n-k-1} e_{n-k}(\overline{x}). \end{aligned}$$

We will now introduce some notation. If j is a part of μ, then μ/j is the partition that results by removing the part j from μ. If j is not a part of μ, then $M(e,h)_{\mu/j,(n-j)} = 0$. We have that

$$M(e,h)_{\mu,(n)} = \sum_{j=1}^{n} (-1)^{j-1} M(e,h)_{\mu/j,(n-j)}. \tag{2.3}$$

This is simply because, looking at the above equations, the only way you can get something involving e_μ, is if e_α and e_{n-k} multiply to produce e_μ. This happens exactly when $n-k$ is a part of μ. So we introduce j as the values of $n-k$ and sum over all possible j.

Also, if we suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ where $l \geq 2$, and $\lambda^- = (\lambda_2, \ldots, \lambda_l)$, on one hand, we have that

$$h_\lambda(\overline{x}) = \sum_{\mu \vdash n} e_\mu M(e,h)_{\mu,\lambda}.$$

However, we also have that

$$h_\lambda(\overline{x}) = h_{\lambda_1}(\overline{x}) \cdot h_{\lambda^-}(\overline{x}) = (\sum_{\alpha \vdash \lambda_1} e_\alpha(\overline{x}) M(e,h)_{\alpha,(\lambda_1)})(\sum_{\beta \vdash \lambda^-} e_\beta(\overline{x}) M(e,h)_{\beta,(\lambda^-)}).$$

This gives us that

$$M(e,h)_{\mu,\lambda} = \sum_{\alpha \vdash \lambda_1, \beta \vdash \lambda^-, \alpha+\beta=\mu} M(e,h)_{\alpha,(\lambda_1)} M(e,h)_{\beta,\lambda^-} \tag{2.4}$$

by the same reasoning as before.

You can see that we've developed two recursions for the coefficients, namely (2.3) and (2.4).

We now shift our attention back to brick tabloids. We define the weight of a brick in a brick tabloid to be $w(b) = (-1)^{|b|-1}$. We extend this to define the weight of a brick tabloid T of type μ to be

$$w(T) = \prod_{b \in T} w(b) = \prod_{b \in T} (-1)^{|b|-1} = (-1)^{n-l(\mu)}.$$

So, $(-1)^{n-l(\mu)} B_{\mu,\lambda} = \sum_{T \in \mathcal{B}_{\mu,\lambda}} w(T)$, where $\mathcal{B}_{\mu,\lambda}$ is the set of brick tabloids of type μ and shape λ.

We can, however classify the sum,

$$\sum_{T \in \mathcal{B}_{\mu,(n)}} w(T)$$

in several ways. First, we will classify the bricks tabloids by the size of their first brick, j. Then the sum becomes

$$\sum_{j=1}^{n} (-1)^{j-1} \chi(j \text{ is part of } \mu) \sum_{T' \in \mathcal{B}_{\mu/j,(n-j)}} w(T').$$

In other words, we have the recursion

$$(-1)^{n-l(\mu)} B_{\mu,(n)} = \sum_{j=1}^{n} (-1)^{j-1} (-1)^{n-j-l(\mu/j)} B_{\mu/j,(n-j)}. \tag{2.5}$$

We could also classify the sum

$$\sum_{T\in\mathcal{B}_{\mu,\lambda}} w(T)$$

by the filling in the first part of λ. Then the sum becomes

$$\sum_{T\in\mathcal{B}_{\alpha,\lambda_1},T'\in\mathcal{B}_{\beta,\lambda^-},\alpha+\beta=\mu} w(T)w(T')$$

$$= \sum_{T\in\mathcal{B}_{\alpha,\lambda_1},T'\in\mathcal{B}_{\beta,\lambda^-},\alpha+\beta=\mu} (-1)^{|\alpha|-l(\lambda_1)}B_{\alpha,\lambda_1}(-1)^{|\beta|-l(\lambda^-)}B_{\beta,\lambda^-}.$$

So we also have the following recursion:

$$(-1)^{n-l(\lambda)}B_{\mu,\lambda} = \sum_{T\in\mathcal{B}_{\alpha,\lambda_1},T'\in\mathcal{B}_{\beta,\lambda^-},\alpha+\beta=\mu} (-1)^{|\alpha|-l(\lambda_1)}B_{\alpha,\lambda_1}(-1)^{|\beta|-l(\lambda^-)}B_{\beta,\lambda^-}. \quad (2.6)$$

We notice that $M(e,h)_{\mu,\lambda}$ and $(-1)^{n-l(\lambda)}B_{\mu,\lambda}$ satisfy the same recursions, as (2.3) and (2.5) are identical and (2.4) and (2.6) are as well. They are also completely determined by these recursions.

Since we know $h_1(\overline{x}) = e_1(\overline{x})$, we have that

$$M(e,h)_{(1),(1)} = 1 \text{ and also } (-1)^{1-l((1))}B_{(1),(1)} = 1.$$

Thus we have that $M(e,h)_{\mu,\lambda} = (-1)^{n-l(\lambda)}B_{\mu,\lambda}$. It is the same argument to verify that $M(h,e)_{\mu,\lambda} = (-1)^{n-l(\lambda)}B_{\mu,\lambda}$. □

We have previously shown that $\{e_\lambda(\overline{x})|\lambda \vdash n\}$ is a basis for $\Lambda_n(\overline{x})$. This implies that $\{1, e_1(\overline{x}), e_2(\overline{x}), \ldots\}$ generate the ring of symmetric polynomials. Here we do not need to have a finite number of variables: $\overline{x}$ could be $\{x_1, x_2, \ldots\}$.

2.2.4 Using ring homomorphisms in permutation enumeration

We will define a ring homomorphism from $\bigoplus_{n=0}^{\infty}\Lambda_n(\overline{x}) = \Lambda(\overline{x}) \to Q[x]$. It will be defined on the elementary symmetric functions and we will define

$$\xi(e_n(\overline{x})) = \frac{(1-x)^{n-1}}{n!}.$$

Theorem 2.4.

$$n!\xi(h_n(\overline{x})) = \sum_{\sigma \in S_n} x^{des(\sigma)}$$

We will delay the proof of this theorem for a moment while we show what we can prove with this theorem. If we assume the theorem,

$$\begin{aligned}\xi(H(t)) &= \xi\left(\sum_{n\geq 0} h_n(\overline{x})t^n\right) \\ &= \sum_{n\geq 0} \xi(h_n)(\overline{x})t^n \\ &= \sum_{n\geq 0} \frac{t^n}{n!}\left(\sum_{\sigma\in S_n} x^{des(\sigma)}\right)\end{aligned}$$

On the other hand, remember that we have

$$H(t)E(-t) = 1 \text{ or } H(t) = \frac{1}{E(-t)}.$$

So,

$$\begin{aligned}\xi(H(t)) = \xi\left(\frac{1}{E(-t)}\right) &= \frac{1}{1+\xi(e_n)(-t)^n} \\ &= \frac{1}{1+\sum_{n\geq 1}\frac{(1-x)^{n-1}}{n!}(-t)^n} \\ &= \frac{1-x}{1-x+\sum_{n\geq 1}\frac{(1-x)^n}{n!}(-t)^n} \\ &= \frac{1-x}{-x+e^{(x-1)t}}\end{aligned}$$

Thus using this ring homomorphism we can get a generating function for the permutation statistic, $des(\sigma)$. Now we will prove the theorem.

Proof.

$$
\begin{aligned}
n!\xi(h_n(\overline{x})) &= n!\xi\left(\sum_{\mu\vdash n} e_\mu(\overline{x})(-1)^{n-l(\mu)}B_{\mu,(n)}\right)\\
&= n!\sum_{\mu\vdash n}(-1)^{n-l(\mu)}B_{\mu,(n)}\xi(e_\mu)(\overline{x})\\
&= n!\sum_{\mu\vdash n}(-1)^{n-l(\mu)}B_{\mu,(n)}\prod_{i=1}^{l(\mu)}\xi(e_{\mu_i})(\overline{x})\\
&= n!\sum_{\mu\vdash n}(-1)^{n-l(\mu)}B_{\mu,(n)}\prod_{i=1}^{l(\mu)}\frac{(1-x)^{\mu_i-1}}{\mu_i!}(\overline{x})\\
&= \sum_{\mu\vdash n}B_{\mu,(n)}(x-1)^{n-l(\mu)}\binom{n}{\mu_1,\ldots,\mu_{l(\mu)}}(\overline{x})
\end{aligned}
$$

We will now take this sum and interpret it combinatorially. We will construct a set of objects $\mathcal{O}$ and we will interpret the sum as

$$\sum_{O\in\mathcal{O}} w(O).$$

To describe these objects, we first will interpret the $B_{\mu,(n)}$ as a brick tabloid of type μ and shape (n). We interpret the multinomial coefficient as filling the bricks with the numbers $\{1,\ldots,n\}$ where inside a given brick, the numbers are decreasing. We interpret the $(x-1)^{n-l(\mu)}$ as weighting each cell of (n) except those cells which are the last cell in a brick with either an x or a -1. We will assume that the cells that are the last cell in a brick are weighted with a 1. The weight of an object of this type is just the product of the weights of all the cells. For example, if $n=9$ and O is the object pictured in Figure 2.5, $w(O)=(-1)^3x^3$.

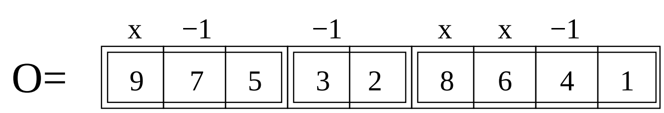

Figure 2.5: An example of the object, $O\in\mathcal{O}$.

We will now define a sign-reversing involution, I, on the objects in $\mathcal{O}$. The involution is the following: scan the brick tabloid from left to right until either

1. you find a cell with a weight of (-1), in which case you split the brick in two after this cell, which means we must change the weight of this cell to a 1 as it is now the last cell of a brick, or

2. you see a decrease between bricks, in which case you merge them and weight the last cell of the first brick with a (-1).

The involution is fairly simple, and it is clear that I^2 is the identity and I is sign reversing. Figure 2.6 shows how I acts on our previous example.

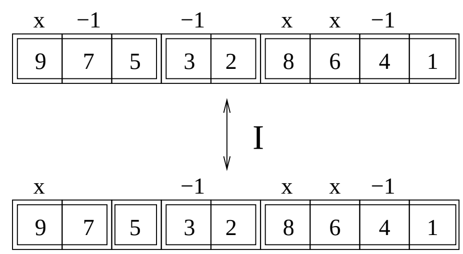

Figure 2.6: Illustrating the involution, I.

What are the fixed points of the involution I? They must have the two following properties.

1. There must be no cells weighted with a (-1).

2. There must be no decreases between bricks.

However, we can just think of the numbers inside the tabloid as a permutation, and since each cell except the last cell of each brick is the position of a descent, the exponent of x counts the number of descents. This proves the theorem.

□

2.3 Generating function for descents and Eulerian polynomials

As shown above, we have the following generating function for descents.

Table 2.2: The Eulerian numbers, $E_{n,k}$

1									
1	1								
1	4	1							
1	11	11	1						
1	26	66	26	1					
1	57	302	302	57	1				
1	120	1191	2416	1191	120	1			
1	247	4293	15619	15619	4293	247	1		
1	502	14608	88234	156190	88234	14608	502	1	
1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

Theorem 2.5.

$$\sum_{n\geq 0}\frac{t^n}{n!}\left(\sum_{\sigma\in S_n} x^{des(\sigma)}\right) = \frac{1-x}{-x+e^{(x-1)t}}.$$

The Eulerian polynomials $E_n(x)$, are defined by the following

$$E_n(x) = \sum_{\sigma\in S_n} x^{des(\sigma)}.$$

Sometimes an alternate definition is given as $E_n(x) = \sum_{\sigma\in S_n} x^{1+des(\sigma)}$.

Suppose $E_n(x) = \sum_{k=0,n-1} E_{n,k}x^k$, we call the coefficients of these polynomials, $E_{n,k}$, the Eulerian numbers. Table 2.2 is a table of the Eulerian numbers, $E_{n,k}$ where n is the row index and $k+1$ is the column index.

Most of the analysis we will present in this thesis involves trying to produce explicit formulas for coefficients similar to $E_{n,k}$, but counting a slightly modified statistic rather than producing generating functions. To illustrate some of the techniques that will be used, we will make mention of how we might be able to prove the following known theorem that gives an explicit formula for the Eulerian numbers.

Theorem 2.6.

$$E_{n,k} = \sum_{i=0}^{k+1} (-1)^i \binom{n+1}{i} (k-i+1)^n.$$

Proof. In order to show that this formula for the Eulerian numbers is correct, we will first derive some recursions for which the Eulerian numbers must satisfy. This is fairly simple to do. Recall that $E_{n,k}$ counts the number of permutations in S_n having k descents. Suppose we were to build up a permutation in S_n with k descents from a permutation in S_{n-1}. We could begin with a permutation in S_{n-1} that has k descents and then insert n into this permutation somewhere that will not create a new descent. There are exactly $k+1$ of these positions, namely the positions between existing descents and the position at the end of the permutation, thus there is a contribution of $(k+1)E_{n-1,k}$ to $E_{n,k}$. However, we could have also begun with a permutation in S_{n-1} that has $k-1$ descents and then insert n into this permutation somewhere that will create a new descent. There are exactly $(n-k)$ of these positions, namely those positions which are not between existing descents or at the end as there are $n-(k-1)-1 = n-k$ such positions, thus there is also a contribution of $(n-k)E_{n-1,k-1}$ to $E_{n,k}$. Hence we have that the Eulerian numbers satisfy the following recursion.

$$E_{n,k} = (k+1)E_{n-1,k} + (n-k)E_{n-1,k-1}.$$

If we were able to show that the proposed formula satisfies this recursion, we could construct an inductive argument that would verify the theorem. It is not a simple exercise in this particular example, and is thus omitted. However, we will be using such arguments to prove explicit formulas in the latter chapters, so it is good discussion nonetheless. $\square$

3

Enumerating descents with specified equivalence classes

3.1 Introduction

Given any sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, we let $red(\sigma)$ be the permutation that results by replacing the ith largest integer that appears in the sequence σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $red(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, we define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a **τ-match** at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let τ-$mch(\sigma)$ be the number of τ-matches in the permutation σ. To prevent confusion, we note that a permutation not having a τ-match is different than a permutation being τ-avoiding. A permutation is called **τ-avoiding** if there are no indices $i_1 < \cdots < i_j$ such that $red[\sigma_{i_1} \cdots \sigma_{i_j}] = \tau$. For example, if $\tau = 2\ 1\ 4\ 3$, then the permutation $3\ 2\ 1\ 4\ 6\ 5$ does not have a τ-match but it does not avoid τ since $red[2\ 1\ 6\ 5] = \tau$.

In the case where $|\tau| = 2$, then τ-$mch(\sigma)$ reduces to familiar permutation statistics. That is, if $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. Then it is easy to see that $(2\ 1)$-$mch(\sigma) = des(\sigma) = |Des(\sigma)|$ and $(1\ 2)$-$mch(\sigma) = rise(\sigma) = |Rise(\sigma)|$.

A number of recent publications have analyzed the distribution of τ-matches in permutations. See, for example, [1, 2, 3]. A number of interesting results have been proved. For example, let τ-nlap(σ) be the maximum number of non-overlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers. Then Kitaev [2, 3] proved the following.

Theorem 3.1.

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x)+x(1-t)A(t)} \tag{3.1}$$

where $A(t) = \sum_{n\geq 0}\frac{t^n}{n!}|\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$.

In other words, if the exponential generating function for the number of permutations in S_n without any τ-matches is known, then so is the exponential generating function for the entire distribution of the statistic $\tau-$nlap.

In this chapter, we consider a more refined pattern matching condition where we take into account conditions involving equivalence mod k for some integer $k \geq 2$. That is, suppose we fix $k \geq 2$ and we are given some sequence of distinct integers $\tau = \tau_1 \cdots \tau_j$. Then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a τ-k-**equivalence match** at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0,\ldots,j-1\}$, $\sigma_{i+s} = \tau_{1+s} \mod k$. For example, if $\tau = 1\ 2$ and $\sigma = 5\ 1\ 7\ 4\ 3\ 6\ 8\ 2$, then σ has τ-matches starting at positions 2, 5, and 6. However, if $k = 2$, then only the τ-match starting at position 5 is a τ-2-equivalence match. Later, it will be explained that the τ-match starting a position 2 is a (1 3)-2-equivalence match and the τ-match starting a position 6 is a (2 4)-2-equivalence match. Let τ-k-**emch**(σ) be the number of τ-k-equivalence matches in the permutation σ. Let τ-k-**enlap**(σ) be the maximum number of non-overlapping τ-k-equivalence matches in σ where two τ-matches are said to overlap if they contain any of the same integers.

More generally, if Υ is a set of sequences of distinct integers of length j, then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a Υ-k-**equivalence match** at place i provided there is a $\tau \in \Upsilon$ such that $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0,\ldots,j-1\}$, $\sigma_{i+s} = \tau_{1+s} \mod k$. Let Υ-k-**emch**(σ) be the number of Υ-

k-equivalence matches in the permutation σ and Υ-k-**enlap**(σ) be the maximum number of non-overlapping Υ-k-equivalence matches in σ.

In this chapter, we shall begin the study of the polynomials

$$T_{\tau,k,n}(x) = \sum_{\sigma \in S_n} x^{\tau\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} T^s_{\tau,k,n} x^s \text{ and} \tag{3.2}$$

$$U_{\Upsilon,k,n}(x) = \sum_{\sigma \in S_n} x^{\Upsilon\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} U^s_{\Upsilon,k,n} x^s. \tag{3.3}$$

In particular, we shall focus on certain special cases of these polynomials where we consider only patterns of length 2. That is, fix $k \geq 2$ and let A_k equal the set of all sequences $(a\ b)$ such that $1 \leq a < b \leq 2k$ where there is no lexicographically smaller sequence $x\ y$ having the property that $x \equiv a \mod k$ and $y \equiv b \mod k$. For example,

$$A_4 = \{1\ 2, 1\ 3, 1\ 4, 1\ 5, 2\ 3, 2\ 4, 2\ 5, 2\ 6, 3\ 4, 3\ 5, 3\ 6, 3\ 7, 4\ 5, 4\ 6, 4\ 7, 4\ 8\}.$$

Let $D_k = \{b\ a : a\ b \in A_k\}$ and $E_k = A_k \cup D_k$. Thus E_k consists of all k-equivalence patterns of length 2 that we could possibly consider. Note that if $\Upsilon = A_k$, then Υ-k-$emch(\sigma) = rise(\sigma)$ and if $\Upsilon = D_k$, then Υ-k-$emch(\sigma) = des(\sigma)$.

Our goal is to give explicit formulas for the coefficients of $T^s_{\tau,k,n}$ and $U^s_{\Upsilon,k,n}$. First we shall show that we can use inclusion-exclusion to find a formula for $U^s_{\Upsilon,k,n}$ for any $\Upsilon \subset E_k$ in terms of certain rook numbers of a sequence of boards associated with Υ. While this approach is straightforward, it is unsatisfactory since it reduces the computation of $U^s_{\Upsilon,k,n}$ to another difficult problem, namely, computing rook numbers for general boards. However, we can give two other more direct formulas for the coefficients $U^s_{\Upsilon,k,n}$ where $\Upsilon \subseteq A_k$ is a subset of the form $\{(x_1, y_1), (x_2, y_2), \ldots, (x_t, y_t)\}$ where for all i, j $y_i \equiv y_j \mod k$. That is, if we define $y = min(\{y_1, \ldots, y_t\})$ and $\alpha = |\{x_i : x_i < y\}|$, then we shall show that for

$y-k \leq j \leq y-1$ and all $s \leq n$ such that $kn+j > 0$, we have

$$\frac{U^s_{\Upsilon,k,kn+j}}{((k-1)n+j)!}$$
$$= \sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{s-r}\Gamma(r,j,n) \quad (3.4)$$
$$= \sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{n-s-r}\Omega(r,n) \quad (3.5)$$

$\Gamma(r,j,n) = \prod_{i=0}^{n-1}((k-1)n+r+j+1-\alpha-i(|\Upsilon|-1))$ and
$\Omega(r,n) = \prod_{i=0}^{n-1}(r+\alpha+i(|\Upsilon|-1))$.

Furthermore, in the special case where $\Upsilon = \{(1\ k)\}$, our results will imply that for all $0 \leq s \leq n$ and for all $0 \leq j \leq k-1$,

$$\frac{T^s_{(1\ k),k,kn+j}}{((k-1)n+j)!} \quad (3.6)$$
$$= \frac{1}{((k-1)n+j)!}\sum_{r=s}^{n}(-1)^{r-s}(kn+j-r)!\binom{r}{s}S_{n+1,n+1-r}$$
$$= \sum_{r=0}^{s}(-1)^{s-r}((k-1)n+j+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}$$
$$= \sum_{r=0}^{n-s}(-1)^{n-s-r}(1+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}$$

where $S_{n,k}$ is the Stirling number of the second kind, i.e., $S_{n,k}$ is the number of partitions of an n-set into k parts. These formulas lead to interesting identities in their own right. For example, from above, we see that for all $k \geq 2$, $0 \leq s \leq n$ and $0 \leq j \leq k-1$,

$$\sum_{r=0}^{s}(-1)^{s-r}((k-1)n+j+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r} =$$
$$\sum_{r=0}^{n-s}(-1)^{n-s-r}(1+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}.$$

The general problem of finding explicit expressions for the coefficients $U^s_{\Upsilon,k,n}$ for arbitrary Υ is open. However, Kitaev and Remmel [4, 5] have developed formulas for $U^s_{\Upsilon,k,n}$ in certain other special cases. In particular, Kitaev and Remmel studied permutation statistics which classified the descents of a permutation according to whether either the first element or the second element of a descent pair is equivalent to $0 \mod k$. In our language, they computed explicit formulas for $U^s_{\Upsilon,k,n}$ where either $\Upsilon = \{b\ a : (b\ a) \in D_k \ \&\ b \equiv 0 \mod k\}$ or $\Upsilon = \{b\ a : (b\ a) \in D_k \ \&\ a \equiv 0 \mod k\}$. In this chapter, we shall generalize some of their results by deriving explicit formulas for $U^s_{\Upsilon,k,n}$ in the special cases where Υ is a subset of the form $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ where for all i, j $y_i \equiv y_j \mod k$ and either $\Upsilon \subseteq A_k$ or $\Upsilon \subseteq D_k$.

The outline of this chapter is as follows. In Section 3.2, we shall discuss some of the previous results of Kitaev and Remmel [4, 5] and give some examples of the polynomials $T_{\tau,k,n}(x)$. In Section 3.3, we will show how to one can use inclusion-exclusion to derive an $U_{\Upsilon,k,n}(x)$ in terms of certain rook numbers. In Section 3.4, we shall prove formulas in the case where Υ consists of a sequences of pairs $\{(x_1, y_1), \ldots, (x_t, y_t)\} \subseteq A_k$ such that for all i and j, $y_i = y_j \mod k$ using simple recursions. Finally in Section 3.5, we will present some simple combinatorial and bijective proofs of some of our results. We will also make a few comments about the problem of finding $U_{\Upsilon,k,n}(x)$ for arbitrary Υ.

Note that using the bijection which sends each permutation $\sigma = \sigma_1 \cdots \sigma_n$ to its reverse, $\sigma^r = \sigma_n \cdots, \sigma_1$, one can show that the same formulas hold for $\Upsilon^r = \{(y_1, x_1), \ldots, (y_t, x_t)\} \subseteq D_k$.

3.2 Previous results and examples

In this section, we shall state some previous results and give some examples of the polynomials $T_{\tau,k,n}(x)$ and $U_{\Upsilon,k,n}(x)$. As mentioned in the introduction, Kitaev and Remmel [4, 5] found explicit formulas for the coefficients $U^s_{\Upsilon,k,n}$ in certain special cases. In particular, they studied descents according to the equivalence class $\mod k$ of either the first or second element in a descent pair. That is, for any set $X \subseteq \{0, 1, 2, \ldots\}$, define

- $\overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_i \in X\}$ and $\overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|$
- $\overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_{i+1} \in X\}$ and $\overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|$

In [4], Kitaev and Remmel studied the following polynomials.

1. $R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_E(\sigma)} = \sum_{k=0}^n R_{k,n} x^k$,
2. $P_n(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_E(\sigma)} z^{\chi(\sigma_1 \in E)} = \sum_{k=0}^n \sum_{j=0}^1 P_{j,k,n} z^j x^k$
3. $M_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_O(\sigma)} = \sum_{k=0}^n M_{k,n} x^k$, and
4. $Q_n(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_O(\sigma)} z^{\chi(\sigma_1 \in O)} = \sum_{k=0}^n \sum_{j=0}^1 Q_{j,k,n} z^j x^k$.

where $E = \{0, 2, 4, \ldots,\}$ is the set of even numbers, $O = \{1, 3, 5, \ldots\}$ is the set of odd numbers, and for any statement A, we let $\chi(A) = 1$ is A is true and $\chi(A) = 0$ if A is false. Thus, for example, in our language, $R_n(x) = U_{\Upsilon,2,n}(x)$ where $\Upsilon = \{2\ 1, 4\ 2\}$ and $P_n(x,1) = U_{\Upsilon,2,n}(x)$ where $\Upsilon = \{3\ 2, 4\ 2\}$. In this case, there are some surprisingly simple formulas for the coefficients of this polynomials. For example, Kitaev and Remmel [4] proved the following.

Theorem 3.2.

$$\begin{aligned}
R_{k,2n} &= \binom{n}{k}^2 (n!)^2, \\
R_{k,2n+1} &= (k+1)\binom{n}{k+1}^2 (n!)^2 + (2n+1-k)\binom{n}{k}^2 (n!)^2 \\
&= \frac{1}{k+1}\binom{n}{k}^2 ((n+1)!)^2, \\
P_{1,k,2n} &= \binom{n-1}{k}\binom{n}{k+1}(n!)^2,
\end{aligned}$$

$$\begin{aligned}
P_{0,k,2n} &= \binom{n-1}{k}\binom{n}{k}(n!)^2, \\
P_{0,k,2n+1} &= (k+1)\binom{n}{k}\binom{n+1}{k+1}(n!)^2 = (n+1)\binom{n}{k}^2 (n!)^2, \ \textit{and} \\
P_{0,k,2n+1} &= \binom{n}{k}(n!)^2 \left(n\binom{n-1}{k} + (k+1)\binom{n}{k} \right).
\end{aligned}$$

In [5], Kitaev and Remmel studied the polynomials

1. $A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$ and

2. $B_n^{(k)}(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j$.

where $kN = \{0, k, 2k, \ldots\}$. Again both $A_n^{(k)}(x)$ and $B_n^{(k)}(x,z)$ are special cases of $U_{\Upsilon,k,n}(x)$. When $k \geq 2$, the formulas for $A_n^{(k)}(x)$ and $B_n^{(k)}(x,z)$ become more complicated. Nevertheless, certain nice formulas arise. For example, Kitaev and Remmel [5] proved the following.

Theorem 3.3. *For all* $0 \leq j \leq k-1$ *and all* $n \geq 0$, *we have*

$$\begin{aligned} & \frac{A_{s,kn+j}^{(k)}}{((k-1)n+j)!} \\ = & \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=0}^{n-1} (r+1+j+(k-1)i) \\ = & \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r} \prod_{i=1}^{n} (r+(k-1)i) \end{aligned}$$

In general, when one includes parity conditions or conditions involving equivalence mod k, then the problem of counting the number of pattern matchings become more complicated. For example, if $\tau = 2\ 1$, then the number of permutations of S_n with no τ-matches is 1 since the only permutation of S_n with no (2 1)-matches is the identity permutation $1\ 2\ \cdots\ n-1\ n$. However, according to Theorem 3.2, the number of permutations of S_m with no $\{(2\ 1), (4\ 2)\}$-2-equivalences matches is $(n!)^2$ if $m = 2n$ and is $((n+1)!)^2$ if $m = 2n+1$. Similarly, the analogue of the Kitaev's result (3.1) fails to hold in general. For example, in the case where $k = 2$ and $\tau = 1\ 2$, then (3.6) implies that for $n \geq 1$, $T^0_{(1\ 2),2,2n} = n^n(n!)$ and $T^0_{(1\ 2),2,2n+1} = (n+1)^n((n+1)!)$, then

$$\begin{aligned} A(t) &= \sum_{n \geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : (1\ 2)\text{-}2\text{-}emch(\sigma) = 0\}| \\ &= 1 + \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} n^n(n!) + \sum_{n \geq 0} \frac{t^{2n+1}}{(2n+1)!} (n+1)^n(n+1)!. \end{aligned}$$

Moreover for any $\sigma \in S_n$, $(1\ 2)\text{-}2\text{-}emch(\sigma) = (1\ 2)\text{-}2\text{-}enlap(\sigma)$. But is easy to check that

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{(1\ 2)\text{-}2\text{-}emch(\sigma)} \neq \frac{A(t)}{(1-x)+x(1-t)A(t)}.$$

Next we give some examples of our polynomials. Table 3.1 lists $T_{(ab),k,n}(x)$ for all possible values of a and b where $k = 3$ and $2 \leq n \leq 7$. Glancing at these values, certain things become apparent. First, observe that for each of these polynomials all the coefficients are divisible by the coefficient of the highest power of x appearing in the polynomial. Second, one can observe that polynomials $T_{(ab),3,n}(x)$ depend only on b. Finally, one can also observe that for any given n, the function $T_{(ab),k,n}(x)$ takes at most three distinct values. For example, when $n = 5$, one can see that all the polynomials $T_{(ab),3,5}(x)$ are equal to one of $T_{(12),3,5}(x)$, $T_{(13),3,5}(x)$, or $T_{(36),3,5}(x)$ and that these three polynomials are distinct. All of these facts are true in general for any k and n since they follow from our closed forms for $T_{(ab),k,n}(x)$.

Table 3.1: Some select values of $T_{(ab),k,n}(x)$

$T_{(12),3,2}(x) = 1 + x$	$T_{(13),3,2}(x) = 2$	$T_{(14),3,2}(x) = 2$
$T_{(12),3,3}(x) = 4 + 2x$	$T_{(13),3,3}(x) = 4 + 2x$	$T_{(14),3,3}(x) = 6$
$T_{(12),3,4}(x) = 18 + 6x$	$T_{(13),3,4}(x) = 18 + 6x$	$T_{(14),3,4}(x) = 18 + 6x$
$T_{(12),3,5}(x) = 54 + 60x + 6x^2$	$T_{(13),3,5}(x) = 96 + 24x$	$T_{(14),3,5}(x) = 96 + 24x$
$T_{(12),3,6}(x) = 384 + 312x + 24x^2$	$T_{(13),3,6}(x) = 384 + 312x + 24x^2$	$T_{(14),3,6}(x) = 600 + 120x$
$T_{(12),3,7}(x) = 3000 + 1920x + 120x^2$	$T_{(13),3,7}(x) = 3000 + 1920x + 120x^2$	$T_{(14),3,7}(x) = 3000 + 1920x + 120x^2$
$T_{(23),3,2}(x) = 2$	$T_{(24),3,2}(x) = 2$	$T_{(25),3,2}(x) = 2$
$T_{(23),3,3}(x) = 4 + 2x$	$T_{(24),3,3}(x) = 6$	$T_{(25),3,3}(x) = 6$
$T_{(23),3,4}(x) = 18 + 6x$	$T_{(24),3,4}(x) = 18 + 6x$	$T_{(25),3,4}(x) = 24$
$T_{(23),3,5}(x) = 96 + 24x$	$T_{(24),3,5}(x) = 96 + 24x$	$T_{(25),3,5}(x) = 96 + 24x$
$T_{(23),3,6}(x) = 384 + 312x + 24x^2$	$T_{(24),3,6}(x) = 600 + 120x$	$T_{(25),3,6}(x) = 600 + 120x$
$T_{(23),3,7}(x) = 3000 + 1920x + 120x^2$	$T_{(24),3,7}(x) = 3000 + 1920x + 120x^2$	$T_{(25),3,7}(x) = 4320 + 720x$
$T_{(34),3,2}(x) = 2$	$T_{(35),3,2}(x) = 2$	$T_{(36),3,2}(x) = 2$
$T_{(34),3,3}(x) = 6$	$T_{(35),3,3}(x) = 6$	$T_{(36),3,3}(x) = 6$
$T_{(34),3,4}(x) = 18 + 6x$	$T_{(35),3,4}(x) = 24$	$T_{(36),3,4}(x) = 24$
$T_{(34),3,5}(x) = 96 + 24x$	$T_{(35),3,5}(x) = 96 + 24x$	$T_{(36),3,5}(x) = 120$
$T_{(34),3,6}(x) = 600 + 120x$	$T_{(35),3,6}(x) = 600 + 120x$	$T_{(36),3,6}(x) = 600 + 120x$
$T_{(34),3,7}(x) = 3000 + 1920x + 120x^2$	$T_{(35),3,7}(x) = 4320 + 720x$	$T_{(36),3,7}(x) = 4320 + 720x$

3.3 Finding the coefficients for $U_{\Upsilon,k,n}(x)$ by inclusion-exclusion

In this section, we shall show how we can use inclusion-exclusion to obtain an expression for $U_{\Upsilon,k,n}(x)$ for any $\Upsilon \subset E_k$. The idea is as follows. Suppose that we fix k and $\Upsilon \subseteq E_k$. Given any two element sequence $ab \in E_k$, we shall write $ab \approxeq xy \mod k$ if (i) $x \equiv a \mod k$, (ii) $y \equiv b \mod k$, (iii) $a < b$ implies $x < y$, and (iv) $a > b$ implies $x > y$. Then for each $n \geq 1$, we let $\Upsilon_n = \{xy : 1 \leq x, y \leq n \ \&\ xy \approxeq ab \mod k \text{ where } (ab) \in \Upsilon\}$. For each $xy \in \Upsilon_n$, we let $C_{xy,n}$ equal the set of all $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ such that there exist an $1 \leq i < n$ such that $\sigma_i = x$ and $\sigma_{i+1} = y$. Given $\sigma \in S_n$, we define $Pr_{\Upsilon,n}(\sigma)$, the property set of σ relative to Υ, to be the set of all $xy \in \Upsilon_n$ such that $\sigma \in C_{xy,n}$. Then we define the following.

1. For each $T \subseteq \Upsilon_n$, let $E_{=T,\Upsilon,n} = \{\sigma \in S_n : Pr_{\Upsilon,n}(\sigma) = T\}$ and $\beta_{T,\Upsilon,n} = |E_{=T,\Upsilon,n}|$.
2. For each $T \subseteq \Upsilon_n$, let $E_{\supseteq T,\Upsilon,n} = \{\sigma \in S_n : Pr_{\Upsilon,n}(\sigma) \supseteq T\}$ and $\alpha_{T,\Upsilon,n} = |E_{\supseteq T,\Upsilon,n}|$.
3. For each $r \geq 0$, let $\beta_{r,\Upsilon,n} = \sum_{S\subseteq\Upsilon_n,|S|=r} \beta_{S,\Upsilon,n}$ and $\alpha_{r,\Upsilon,n} = \sum_{S\subseteq\Upsilon_n,|S|=r} \alpha_{S,\Upsilon,n}$.

It is an easy consequence of the inclusion-exclusion principle that

$$\sum_{t\geq 0} \beta_{t,\Upsilon,n} x^t = \sum_{t\geq 0} \alpha_{t,\Upsilon,n}(x-1)^t. \tag{3.7}$$

It is also easy to see from our definitions that

$$\sum_{t\geq 0} \beta_{t,\Upsilon,n} x^t = U_{\Upsilon,k,n}(x). \tag{3.8}$$

Thus we get an expression for $U_{\Upsilon,k,n}(x)$ by calculating the RHS of (3.7).

Next we observe that it is easy to compute $\alpha_{T,\Upsilon,n}$. We say that $T \subseteq \Upsilon_n$ is consistent if there do not exist distinct ab and cd in T such that either $a = c$ or $b = d$. For example, if $k = 4$ and $\Upsilon = \{12, 34, 32, 46\}$, then $\Upsilon_7 = \{12, 16, 56, 34, 32, 72, 76, 46\}$. Then $T_1 = \{12, 16, 34\}$ and $T_2 = \{12, 32, 76\}$ are not consistent while $T_3 = \{12, 34, 46\}$ is consistent. First we claim that if T is consistent, then $\alpha_{T,\Upsilon,n} = (n - |T|)!$. That is, we need to construct $E_{\supseteq T,\Upsilon,n}$ which consists of all permutations $\sigma \in S_n$ such that each pattern in T occurs consecutively in σ. We do this by first constructing the maximal blocks of elements of $\{1, \ldots, n\}$ where xy occurs consecutively in a block if and only if $xy \in T$. For example, if $n = 7$ and $T = T_3$ as given above, then the maximal blocks constructed from T are 12, 346, 5 and 7. Then it is easy to see that any permutation of the maximal blocks constructed from T corresponds to a permutation $\sigma \in E_{\supseteq T,\Upsilon,n}$. For example, the permutation of the maximal blocks 346 5 12 7 corresponds to the permutation 3 4 6 5 1 2 7 $\in E_{\supseteq T_3,\Upsilon,7}$. Now it is easy to see that the number of maximal blocks of $\{1, \ldots, n\}$ constructed from T is $n - |T|$. Thus $\alpha_{T,\Upsilon,n} = |E_{\supseteq T,\Upsilon,n}| = (n - |T|)!$. Of course, if T is inconsistent, there is no permutation $\sigma \in S_n$ such that all the sequences in T occur consecutively in σ. In this situation, $\alpha_{T,\Upsilon,n} = 0$.

Thus to compute $\alpha_{t,\Upsilon,n}$, we need only count the number of consistent subsets of size t in Υ_n. We can think of this problems as counting the number of rook placements of size t in a certain board associated with Υ_n. That is, given Υ_n, let $B_{\Upsilon,n}$ be the set of all (x, y) such that $xy \in \Upsilon_n$. For example, if $k = 4$ and $\Upsilon = \{12, 34, 32, 46\}$ so that $\Upsilon_7 = \{12, 16, 56, 34, 32, 72, 76, 46\}$, then $B_{\Upsilon,7}$ consists of the shaded cells on the board pictured in Figure 3.1.

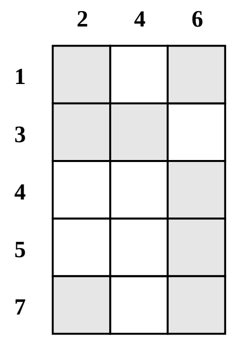

Figure 3.1: The board $B_{\Upsilon,7}$.

Given any board $B \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$, we let $r_k(B)$ denote the number of placements of k rooks in B such that no two rooks lie in the same row or the same column. It is then easy to see that number of consistent subsets of size t in Υ_n equals $r_t(B_{\Upsilon,n})$ and thus, $\alpha_{t,\Upsilon,n} = (n-t)!r_t(B_{\Upsilon,n})$. It follows that

$$\begin{aligned} U_{\Upsilon,k,n}(x) &= \sum_{t\geq 0} \beta_{t,\Upsilon,n} x^t = \sum_{t\geq 0} \alpha_{t,\Upsilon,n}(x-1)^t \\ &= \sum_{t\geq 0} (n-t)! r_t(B_{\Upsilon,n}) \sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} x^s \\ &= \sum_{s\geq 0} x^s \sum_{t=s}^{n} (n-t)!(-1)^{t-s} \binom{t}{s} r_t(B_{\Upsilon,n}). \end{aligned} \tag{3.9}$$

The problem with the formula in (3.9) is that we obtain an expression for the coefficients of $U_{\Upsilon,k,n}(x)$ in terms of the numbers $r_t(B_{\Upsilon,n})$ which are not easy to compute in general. There are however some special cases of (3.9) where the numbers $r_t(B_{\Upsilon,n})$ are familiar. That is, suppose $\Upsilon = \{(1k)\}$. Then it is easy to see that $B_{\Upsilon,kn+j}$ consists of the set of cells $\{(1+ik, jk) : 0 \le i < j \le n\}$. For example, if $k = 3$ and $\Upsilon = \{(13)\}$, then $B_{\Upsilon,12}$ consists of the shaded cells on the board pictured in Figure 3.2.

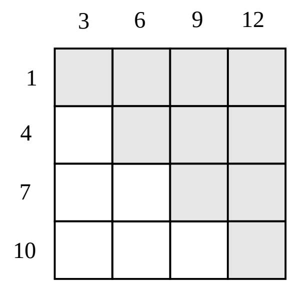

Figure 3.2: The board $B_{\{13\},12}$.

It is well known that the Stirling number of the second kind, $S_{n+1,k}$, is the number of placements of $n+1-k$ rooks on the staircase board, consisting of columns of heights $0, 1, \ldots, n$ reading from right to left, so that no two rooks lie in the same row or column. It then easily follows that

$$T^s_{(1k),k,kn+j} = U^s_{\{(1k)\},k,kn+j} = \sum_{r=s}^{n} (-1)^{r-s} \binom{r}{s} (kn+j-r)! S_{n+1,n+1-r}. \tag{3.10}$$

Another case that involves the Stirling numbers is when $\Upsilon = D_k$. As pointed out in the introduction, in that case, Υ-k-$emch(\sigma) = des(\sigma)$. In this case the board the $B_{\Upsilon,n}$ equals $\{(j,i) : 0 \le i < j \le n\}$ which is equivalent to a staircase board with column heights $0, 1, \ldots, n-1$.

It is also well known that the Eulerian number $E_{m,n}$ counts the number of permutations in S_m that have exactly n descents. Thus we can derive the following formula for the Eulerian numbers in terms of the Stirling numbers.

$$E_{n,s} = U^s_{A_k,k,n}(x) = \sum_{r=s}^{n} (-1)^{r-s} \binom{r}{s} (n-r)! S_{n,n-r}. \tag{3.11}$$

In some other cases, we have been able to derive formulas that involve sums over products of Stirling numbers. For example, if $k = 4$ and $\Upsilon = \{13, 24\}$ so that

$\Upsilon_{12} = \{13, 17, 1\ 11, 57, 5\ 11, 9\ 11, 24, 28, 2\ 12, 68, 6\ 12, 10\ 12\}$, then $B_{\Upsilon,12}$ consists of the shaded cells on the board pictured in Figure 3.3.

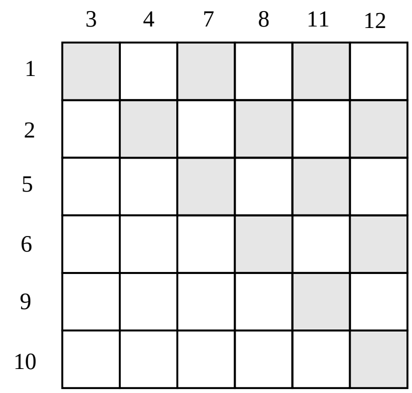

Figure 3.3: The board $B_{\{13,24\},12}$.

In such cases, it is easy to see that the board $B_{\Upsilon,n}$ naturally breaks up as a disjoint union of staircase boards. It follows that

$$U^s_{\{13,24\},4,4n} = \sum_{r=s}^{2n} (-1)^{r-s} \binom{r}{s} (kn + j - r)! \sum_{i=0}^{r} S_{n+1,n+1-i} S_{n+1,n+1-(r-i)}.$$

3.4 Finding the coefficients of $U_{\Upsilon,k,n}$ and $T_{\tau,k,n}$ by iterating recursions

In this section, we shall give an alternative approach to finding $U^s_{\Upsilon,k,n}$ and $T^s_{\tau,k,n}$ that exploits the fact that we can find simple recursions for the polynomials $U_{\Upsilon,k,n}$. Note that $T_{\tau,k,n}$ is a special case of $U_{\Upsilon,k,n}$ so will derive general formulas for $U^s_{\Upsilon,k,n}$.

Given any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we label with the integers from 0 to n (from left to right) the possible positions of where we can insert $n+1$ to get a permutation in S_{n+1}. In other words, inserting $n+1$ in position 0 means that we insert $n+1$ at the beginning of σ and for $i \geq 1$, inserting $n+1$ in position i means we insert $n+1$ immediately after σ_i. In such a situation, we let $\sigma^{(i)}$ denote the permutation of S_{n+1} that results by inserting $n+1$ in position i.

Throughout the rest of this section, we shall assume that $k \geq 2$ and $\Upsilon \subseteq A_k$ is a subset of the form $\{(x_1, y_1), (x_2, y_2), \ldots, (x_t, y_t)\}$ where for all i, j $y_i \equiv y_j \mod k$. Now, define $y = min(\{y_1, \ldots, y_t\})$ and $\alpha = |\{x_i : x_i < y\}|$. We then let $Asc_{\Upsilon,k}(\sigma) = \{i : \sigma_i < \sigma_{i+1} \;\&\; \sigma_i \equiv x_j \mod k \;\&\; \sigma_{i+1} \equiv y_j \mod k \text{ for some } (x_j, y_j) \in \Upsilon\}$. We shall call the elements of $Asc_{\Upsilon,k}(\sigma)$ the Υ-ascents of σ.

For $j = y - k + 1, \ldots, y - 1$, let Δ_{kn+j} be the operator which sends x^s to $sx^{s-1} + (kn + j - s)x^s$ and Γ_{kn+y} be the operator that sends x^s to $((k - |\Upsilon|)n + y + s - \alpha)x^s + (|\Upsilon|n + \alpha - s)x^{s+1}$. Then we have the following.

Theorem 3.4. *Given Υ, y, and α as described above, the polynomials $\{U_{\Upsilon,k,n}(x)\}_{n\geq 1}$ satisfy the following recursions.*

1. $U_{\Upsilon,k,1}(x) = 1$,
2. *For* $j = y - k + 1, \ldots, y - 1$, $U_{\Upsilon,k,kn+j}(x) = \Delta_{kn+j}(U_{\Upsilon,k,kn+j-1}(x))$, *and*
3. $U_{\Upsilon,k,kn+y}(x) = \Gamma_{kn+y}(U_{\Upsilon,k,kn+y-1}(x))$.

Proof. Part (1) is trivial.

For part (2), fix j such that $y - k + 1 \leq j \leq y - 1$. Now suppose $\sigma = \sigma_1 \cdots \sigma_{kn+j-1} \in S_{kn+j-1}$ and $asc_{\Upsilon,k}(\sigma) = s$. It is then easy to see that if we insert $kn + j$ in position i where $i \in Asc_{\Upsilon,k}(\sigma)$, then $asc_{\Upsilon,k}(\sigma^{(i)}) = s - 1$. However, if we insert $kn + j$ in position i where $i \notin Asc_{\Upsilon,k}(\sigma)$, then $asc_{\Upsilon,k}(\sigma^{(i)}) = s$. Thus $\{\sigma^{(i)} : i = 0, \ldots, kn + j - 1\}$ gives a contribution of $sx^{s-1} + (kn + j - s)x^s$ to $U_{\Upsilon,k,kn+j}$.

For part (3), suppose $\sigma = \sigma_1 \cdots \sigma_{kn+y-1} \in S_{kn+y-1}$ and $asc_{\Upsilon,k}(\sigma) = s$. In this situation we can create a Υ-ascent, but we can't lose one. That is, if we place $kn + y$ after any element equivalent to $x_i \mod k$ for some $(x_i, y_i) \in \Upsilon$ which isn't already part of a Υ-ascent, we would create an additional Υ-ascent. There are $|\Upsilon|n + \alpha - s$ such locations. This means that the number of locations that keep the number of ascents the same must be $(k - |\Upsilon|)n + y + s - \alpha$ as the two must sum to $kn + y$. Thus $\{\sigma^{(i)} : i = 0, \ldots, kn + y - 1\}$ gives a contribution of $((k - |\Upsilon|)n + y + s - \alpha)x^s + (|\Upsilon|n + \alpha - s)x^{s+1}$ to $U_{\Upsilon,k,kn+y}$. □

We can give combinatorial proofs of two simple formulas for the extreme coefficients of $U_{\Upsilon,k,n}(x)$.

Theorem 3.5. *Let Υ, y, and α be as described above. Then for all $k \geq 2$, for all $j = y-k, \ldots, y-1$ and n such that $kn+j > 0$,*

$$U^0_{\Upsilon,k,kn+j} = ((k-1)n+j)! \prod_{i=0}^{n-1}((k-1)n+j+1-\alpha-i(|\Upsilon|-1)) \quad (3.12)$$

$$U^n_{\Upsilon,k,kn+j} = ((k-1)n+j)! \prod_{i=0}^{n-1}(\alpha+i(|\Upsilon|-1)) \quad (3.13)$$

Proof. Clearly when $n = 0$, the only $j \in \{y-k, \ldots, y-1\}$ such that $kn+j > 0$ are $j = 1, \ldots y-1$. In these cases, no permutation σ of S_j can have an Υ-k-equivalence match so that $U_{\Upsilon,k,j}(x) = j!$. By convention, we assume the empty product is equal to 1 so that our formulas holds when $n = 0$.

Next assume that $n \geq 1$ and $\Upsilon = \{(x_i, y_i) : i = 1, \ldots t\}$ where $x_1, \ldots x_\alpha$ consist of those x_i's such that $x_i < y$. Suppose that $j \in \{y-k, \ldots, y-1\}$.

First we consider those permutations $\sigma \in S_{kn+j}$ such that Υ-k-$emch(\sigma) = 0$. We claim that we can construct all such σ as follows. By our definition, there are $(k-1)n+j$ elements in $\{1, \ldots, kn+j\}$ which are not equivalent to $y \mod k$. We can arrange these elements in $((k-1)n+j)!$ ways. Given an arrangement τ of the elements in $\{1, \ldots, kn+j\}$ which are not equivalent to $y \mod k$, we can extend τ to a permutation $\sigma \in S_{kn+j}$ such that Υ-k-$emch(\sigma) = 0$ as follows. First we can insert y into τ so that we do not create any Υ-k-equivalence matches. Clearly this can be done in $(k-1)n+j+1-\alpha$ ways since all we have to do is to ensure that we do not insert y immediately after any of $x_1, \ldots x_\alpha$. Now suppose τ_1 is a sequence that results from inserting y into τ so that we do not create any Υ-k-equivalence matches. Then, the number of ways to insert $y+k$ into τ_1 so that we do not create any Υ-k-equivalence matches is $(k-1)n+j+1-\alpha-(|\Upsilon|-1)$. That is there are $(k-1)n+j+2$ possible ways to insert $y+k$ into τ_1 but that are α+$|\Upsilon|$ elements z such that if we insert $y+k$ after z, then we would form an Υ-k-equivalence match. Now suppose τ_2 is a sequence that results from inserting $y+k$ into τ_1 so that we do not create any Υ-k-equivalence matches. Then, the number of ways to insert $y+2k$ into τ_2 so that we do not create any Υ-k-equivalence matches is $(k-1)n+j+1-\alpha-2(|\Upsilon|-1)$. That is there are $(k-1)n+j+3$ possible ways to insert $y+2k$ into τ_2 but that are $\alpha+2|\Upsilon|$ elements z such that if we insert $y+2k$ after z, then we would form an Υ-k-equivalence match. Continuing in this way, we see that $U^0_{\Upsilon,k,kn+j} = ((k-1)n+j)!\prod_{i=0}^{n-1}((k-1)n+j+1-\alpha-i(|\Upsilon|-1))$.

Next we consider those permutations $\sigma \in S_{kn+j}$ such that Υ-k-$emch(\sigma) = n$. We claim that we can construct all such σ as follows. By our definition, there are $(k-1)n+j$ elements in $\{1,\ldots,kn+j\}$ which are not equivalent to $y \mod k$. We can arrange these elements in $((k-1)n+j)!$ ways. Given an arrangement τ of the elements in $\{1,\ldots,kn+j\}$ which are not equivalent to $y \mod k$, we can extend τ to a permutation $\sigma \in S_{kn+j}$ such that Υ-k-$emch(\sigma) = n$ as follows. Clearly, we must insert $y, y+k, \ldots, y+(n-1)k$ in such a way that each of these elements create an Υ-k-equivalence match. Thus we must insert y into τ so that it immediately follows one of $x_1,\ldots,x_\alpha$. Hence we have α ways to insert y. Now suppose τ_1 is a sequence that results from inserting y into τ so that we did create a Υ-k-equivalence match. Then the number of ways to insert $y+k$ into τ_1 so that we create another Υ-k-equivalence match is $\alpha+(|\Upsilon|-1)$ since there $\alpha+|\Upsilon|$ elements $x < y+k$ such that $(x\ (y+k))$ would be an Υ-k-equivalence match and we can not insert $y+k$ immediately before y. Now suppose τ_2 is a sequence that results from inserting $y+k$ into τ_1 so that we have created a second Υ-k-equivalence match. Then the number of ways to insert $y+2k$ into τ_2 so that we create an additional Υ-k-equivalence matches is $\alpha+2(|\Upsilon|-1)$ since there $\alpha+2|\Upsilon|$ elements $x < y+k$ such that $(x\ (y+2k))$ would be an Υ-k-equivalence match and we can not insert $y+2k$ immediately before y or $y+2k$. Continuing in this way, we see that $U^n_{\Upsilon,k,kn+j} = ((k-1)n+j)!\prod_{i=0}^{n-1}(\alpha+i(|\Upsilon|-1))$. □

Next we shall derive our general formulas (3.4) and (3.5) for $U^s_{\Upsilon,k,n}$ by using the recursions implicit in Theorem 3.4. That is, it is easy to see from Theorem 3.4 that we have two following recursions for the coefficients $U^s_{\Upsilon,k,n}$.

For $y-k+1 \leq j \leq y-1$,

$$U^s_{\Upsilon,k,kn+j} = (kn+j-s)U^s_{\Upsilon,k,kn+j-1} + (s+1)U^{s+1}_{\Upsilon,k,kn+j-1} \tag{3.14}$$

Similarly, we have

$$U^s_{\Upsilon,k,kn+y} = ((k-|\Upsilon|)n+y-\alpha+s)U^s_{\Upsilon,k,kn+y-1} + (|\Upsilon|n+\alpha-s+1)U^{s-1}_{\Upsilon,k,kn+y-1} \tag{3.15}$$

In order to give an inductive proof of (3.4) via these recursions, we first need to establish the following lemma.

Lemma 3.6. *For all positive integers $k, n, j, z_1, \ldots, z_n$, $0 \leq z_i \leq (k-1)n+j$,*

$$\sum_{r=0}^{n+1} (-1)^{n+1-r} \binom{kn+j+1}{(k-1)n+j, r, n+1-r} \prod_{i=1}^{n} (z_i + r) = 0. \tag{3.16}$$

Proof. To begin, we will give a combinatorial interpretation to the LHS of (3.16). It is well known that

$$\binom{kn+j+1}{(k-1)n+j, r, n+1-r} = |R(0^{(k-1)n+j}, 1^r, 2^{n+1-r})|.$$

Now given $\beta \in R(0^{(k-1)n+j}, 1^r, 2^{n+1-r})$, we first let $a_1 < \ldots a_r$ be the positions of the 1's in β reading from left to right and $a_{r+1} < \cdots < a_{r+(k-1)n+j}$ be the positions of 0's in β reading from left to right. Then we will consider the set of functions $\mathcal{F}_{z_1,\ldots,z_n,\beta}$ which map $\{1, \ldots, n\}$ into the set of the positions of the of 0's and 1's in β such that for all $1 \leq i \leq n$, $f(i)$ is mapped to one of the positions $a_1, a_2, \ldots, a_{z_i+r}$. Clearly,

$$\prod_{i=1}^{n} (z_i + r) = |\mathcal{F}_{z_1,\ldots,z_n,\beta}|.$$

We define the set $C_{n,j,z_1,\ldots,z_n}$ to consist of the set of all pairs $c = (\beta, f)$ such that $\beta \in R(0^{(k-1)n+j}, 1^r, 2^{n+1-r})$ for some $0 \leq r \leq s$ and $f \in \mathcal{F}_{z_1,\ldots,z_n,\beta}$. If $c = (\beta, f) \in C_{n,j,z_1,\ldots,z_n}$ where $\beta \in R(0^{(k-1)n+j}, 1^r, 2^{n+1-r})$, then we define $sgn(c) = (-1)^{n+1-r}$.

That is, the sign of any $c = (\beta, f) \in C_{n,j,z_1,\ldots,z_n}$ is just $(-1)^{two(\beta)}$ where $two(\beta)$ is the number of 2's in β. It is then easy to see that

$$\sum_{c \in C_{n,s,z_1,\ldots,z_n}} sgn(c) = \sum_{r=0}^{n+1} (-1)^{n+1-r} \binom{kn+j+1}{(k-1)n+j, r, n+1-r} \prod_{i=1}^{n} (z_i + r). \quad (3.17)$$

Thus to prove (3.16), we need only define a sign reversing involution $I : C_{n,s,z_1,\ldots,z_n} \to C_{n,s,z_1,\ldots,z_n}$ that has no fixed points. Suppose that we are given $c = (\beta, f) \in C_{n,j,z_1,\ldots,z_n}$ where $\beta \in R(0^{(k-1)n+j}, 1^r, 2^{n+1-r})$. Let $\beta = \beta_1 \cdots \beta_{kn+j+1}$. Then we will say that β_t is free for (β, f) if either $\beta_t = 2$ or $\beta_t = 1$ and t is not in the range of f. Then our involution is very simple: namely, we let $I(c)$ be the configuration (β^*, f^*) where β^* results from β by finding the left most free element β_t in β and changing β_t into a 1 if β_t is equal to 2 and changing β_t into a 2 if β_t is equal to 1. Then we simply let $f = f^*$. That is, for $i = 1, \ldots, n$, then f^* maps i to position t_i if f maps i to position t_i. For example, suppose that $k = 3$, $n = 3$, $j = 1$, $r = 2$ and $(z_1, z_2, z_3) = (2, 4, 4)$ so that $kn + j + 1 = 11$. Now if $\beta = 0\ 2\ 1\ 0\ 0\ 2\ 0\ 1\ 0\ 0\ 0$, then the positions of the 1s, reading from right to left, are $a_1 = 3$ and $a_2 = 8$. The positions of the 0s, reading from right to left, are $a_3 = 1$, $a_4 = 4$, $a_5 = 5$, $a_6 = 7$, $a_7 = 9$, $a_8 = 10$, and $a_9 = 11$. Thus $(a_1, \ldots, a_9) = (3, 8, 1, 4, 5, 7, 9, 10, 11)$. Thus for a pair (β, f) to be in $C_{3,1,z_1,z_2,z_3}$, we must have that f maps 1 to one of the positions $a_1, \ldots, a_4$, f maps 2 to one of the positions $a_1, \ldots, a_6$, and f maps 3 to one of the positions $a_1, \ldots, a_6$. Now suppose that $f(1) = a_4 = 4$, $f(2) = a_1 = 3$, and $f(3) = a_6 = 7$. Then the left most free element is in β is β_2 so that $\beta^* = 0\ 1\ 1\ 0\ 0\ 2\ 0\ 1\ 0\ 0\ 0$. Note that in this case, the positions of the 1's, reading from right to left, are $a_1^* = 2$, $a_2^* = 3$ and $a_3^* = 8$. The positions of the 0's, reading from right to left, are $a_4^* = 1$, $a_5^* = 4$, $a_6^* = 5$, $a_7^* = 7$, $a_8^* = 9$, $a_9^* = 10$, and $a_{10}^* = 11$. Thus $(a_1^*, \ldots, a_{10}^*) = (2, 3, 8, 1, 4, 5, 7, 9, 10, 11)$. Thus, in this case, $f^*(1) = 4 = a_5^*$, $f^*(2) = 3 = a_2^*$, and $f^*(3) = 7 = a_7^*$. Note that in this case, $f^*(i)$ mapped to the same position but relative to the new order of the positions $(a_1^*, \ldots, a_{10}^*)$, $f^*(i)$ can stay the same or increase by 1. That is, in general, if $f(i) = a_{t_i}$, then $f^*(i) = a_{t_i}^*$ or $a_{t_{i+1}}^*$ depending on whether the new 1 is inserted before or after the element a_{t_i}. Now we must check that (β^*, f^*) is still in $C_{n,j,z_1,\ldots,z_n}$. First, it is easy to see that changing a 2 to 1 will increase r by 1 so that $f^*(i)$ must still be among $a^*_1, \ldots, a^*_{z_i+r+1}$. Thus if we change a 2 to a 1 to form β^*, then $(\beta^*, f^*) \in C_{n,j,z_1,\ldots,z_n}$.

Similarly, if the leftmost free element is a 1 and we change it to a 2 and $f(i) = a_{t_i}$, then either $f^*(i) = a^*_{t_i-1}$ or $f^*(i) = a^*_{t_i}$ depending on whether the 1 that was removed lies to left or right of the a_{t_i} in the sequence $(a_1, \ldots, a_{(k-1)n+j+r})$. In this case, it is easy to see that $f^*(i) = a^*_{t_i}$ only if the element in position a_{t_i} in β was a 1 because we listed the positions of the 1s first and then we listed the positions of the 0s to form $(a_1, \ldots, a_{(k-1)n+j+r})$. Thus if $f^*(i) = a^*_{t_i}$, then $t_i < r$ so that $f^*(i)$ is among $a^*_1, \ldots, a^*_{z_i+r-1}$. If $f^*(i) = a^*_{t_i-1}$, then we automatically have $f^*(i)$ is among $a^*_1, \ldots, a^*_{z_i+r-1}$. Thus if we change a 1 to 2 to form β^*, then $(\beta^*, f^*) \in C_{n,j,z_1,\ldots,z_n}$.

Note that there are total of $n+1$ 2s and 1s in any β such that $(\beta, f) \in C_{n,j,z_1,\ldots,z_n}$ so that there is at least one free element in β. Thus, $I(c)$ is always defined. Since we change the number of 2s by one, it follows that $sgn(I(c)) = -sgn(c)$ for all c. Finally, since our map does not change the positions of the free elements in the sequence, it follows that $I(I(c)) = c$ for all c. Hence I shows that elements on the LHS side of (3.17) cancel off in pairs so that the sum is equal to 0 as claimed. □

We are now in a position prove the first of our closed formulas for $U^s_{\Upsilon,k,kn+j}$. This formula was obtained by using (3.14) and iterating these recursions from the bottom up.

Theorem 3.7. *For all $y - k \leq j \leq y - 1$ and all $s \leq n$ such that $kn + j > 0$, we have*

$$\frac{U^s_{\Upsilon,k,kn+j}}{((k-1)n+j)!} = \left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+r+j}{r} \binom{kn+j+1}{s-r} \Gamma(r,j,n)\right]$$

where $\Gamma(r,j,n) = \prod_{i=0}^{n-1}((k-1)n+r+j+1-\alpha-i(|\Upsilon|-1))$.

Proof. We shall prove by induction on $kn+j$ that our formulas hold. The formulas trivially hold when $kn + j = 1$ since the formula just reduces to $U^0_{\Upsilon,k,1} = 1$.

Next we will show that our formulas satisfy the recursions (3.14) and (3.15). In order to simplify the algebra, we will convert the form used in the theorem to the following.

$$U^s_{\Upsilon,k,kn+j} = \sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+j)!(kn+j+1)!\Gamma(r,j,n)}{(kn+j-s+r+1)!r!(s-r)!} \tag{3.18}$$

So, for $y-k+1 \leq j \leq y-1$ plugging in the above form into the RHS of (3.14) gives

$$(kn+j-s)\left[\sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+j-1)!(kn+j)!\Gamma(r,j-1,n)}{(kn+j-s+r)!r!(s-r)!}\right]$$
$$+(s+1)\left[\sum_{r=0}^{s+1} \frac{(-1)^{s+1-r}((k-1)n+r+j-1)!(kn+j)!\Gamma(r,j-1,n)}{(kn+j-s+r-1)!r!(s+1-r)!}\right]$$

Removing the $s+1$ term from the second summand, recognizing that $\Gamma(r,j-1,n) = \Gamma(r-1,j,n)$ and combining the rest of the terms yields

$$\sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+j-1)!(kn+j)!\Gamma(r-1,j,n)\left[-r(kn+j+1)\right]}{(kn+j-s+r)!r!(s+1-r)!}$$
$$+\frac{((k-1)n+s+j)!\Gamma(s+1,j-1,n)}{s!}$$

Since there is a factor of r in the numerator, we may omit the $r=0$ term from the summand, shift indices and recognize that $\Gamma(s+1,j-1,n) = \Gamma(s,j,n)$ to get

$$\sum_{r=0}^{s-1} \frac{(-1)^{s-r}((k-1)n+r+j)!(kn+j+1)!\Gamma(r,j,n)}{(kn+j-s+r+1)!r!(s-r)!}$$
$$+\frac{((k-1)n+s+j)!\Gamma(s,j,n)}{s!}$$
$$= \sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+j)!(kn+j+1)!\Gamma(r,j,n)}{(kn+j-s+r+1)!r!(s-r)!} = U^s_{\Upsilon,k,kn+j}$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (3.14) for $y-k+1 \leq j \leq y-1$. We will now show that our formula satisfies (3.15). The RHS of (3.15) becomes

$$((k-|\Upsilon|)n+s+y-\alpha)\left[\sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y)!\Gamma(r,y-1,n)}{(kn+y-s+r)!r!(s-r)!}\right]$$
$$+(|\Upsilon|n+\alpha-s+1)\left[\sum_{r=0}^{s-1} \frac{(-1)^{s-r-1}((k-1)n+r+y-1)!(kn+y)\Gamma(r,y-1,n)!}{(kn+y-s+r+1)!r!(s-r-1)!}\right]$$

Removing the s term from the first summand, and combining the rest of the terms yields

$$
\begin{aligned}
& \sum_{r=0}^{s-1} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y)!\Gamma(r,y-1,n)\left[(kn+y+1)(kn-n|\Upsilon|+r+y-\alpha)\right]}{(kn+y-s+r+1)!r!(s-r)!} \\
& + \frac{((k-|\Upsilon|)n+s+y-\alpha)((k-1)n+y+s-1)!\Gamma(s,y-1,n)}{s!} \\
= & \sum_{r=0}^{s-1} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y+1)!\Gamma(r,y-1,n)((k-|\Upsilon|)n+r+y-\alpha)}{(kn+y-s+r+1)!r!(s-r)!} \\
& + \frac{((k-|\Upsilon|)n+s+y-\alpha)((k-1)n+y+s-1)!\Gamma(s,y-1,n)}{s!} \\
= & \sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y+1)!\Gamma(r,y-1,n)((k-|\Upsilon|)n+r+y-\alpha)}{(kn+y-s+r+1)!r!(s-r)!} \\
= & \sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y+1)!\Gamma(r,y-k,n+1)}{(kn+y-s+r+1)!r!(s-r)!} = U^s_{\Upsilon,k,kn+y}
\end{aligned}
$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (3.15) as desired.

We can complete our induction as follows. Assume that our formulas for $U^s_{\Upsilon,k,kn+j-1}$ hold for all $s \leq n$ where $y-k+1 \leq j \leq y$ such that $kn+j-1>0$. Our induction hypothesis and the fact that our formulas satisfy the recursions (3.14) immediately implies that our formula for $U^s_{\Upsilon,k,kn+j}$ must be true for all $s<n$ where $y-k+1 \leq j \leq y-1$. However, the case when $s=n$ needs to be examined more closely. The problem is that our formula for $U^{n+1}_{\Upsilon,k,kn+j-1}$ make sense even though by our definitions, it must be the case that $U^{n+1}_{\Upsilon,k,kn+j-1} = 0$ since there are no permutations in S_{kn+j-1} which have $n+1$ Υ-k-matches. Thus when we consider the recursion

$$
U^n_{\Upsilon,k,kn+j} = (kn+j-n)U^n_{\Upsilon,k,kn+j-1} + (n+1)U^{n+1}_{\Upsilon,k,kn+j-1}, \tag{3.19}
$$

we must interpret $U^{n+1}_{\Upsilon,k,kn+j-1}$ as 0. Hence, to show that this recursion is satisfied by our formulas, we need an independent proof that

$$
\sum_{r=0}^{n+1}(-1)^{n+1-r}\binom{(k-1)n+r+j-1}{r}\binom{kn+j}{n+1-r}\Gamma(r,j-1,n) = 0. \tag{3.20}
$$

However, it is easy to see that (3.20) is just a special case of Lemma 3.6.

It is also easy to see that our induction assumptions and the fact that our formulas satisfy the recursions (3.15) will allow us to prove our formulas hold for

$U^s_{\Upsilon,k,k(n+1)+y-k}$, for all $1 < s < n$. However, the $s = 0$ and $s = n+1$ cases need to be examined separately. That is, first consider the recursion

$$U^0_{\Upsilon,k,kn+y} = ((k - |\Upsilon|)n + y - \alpha)U^0_{\Upsilon,k,kn+y-1} + (|\Upsilon|n + \alpha + 1)U^{-1}_{\Upsilon,k,kn+y-1}. \quad (3.21)$$

Since there is no permutation $\sigma \in S_{kn+y-1}$ which has -1 Υ-k-matches, we must also interpret $U^{-1}_{\Upsilon,k,kn+y-1}$ as 0. However our formula for $U^{-1}_{\Upsilon,k,kn+y-1}$ is also equal to 0 since we have to interpret the sum in the formula as the empty sum which equals 0 by definition. Thus our formulas also satisfy (3.21) which allows us to establish and our formula for $U^0_{\Upsilon,k,kn+y}$ is true. Finally, consider the recursion

$$\begin{aligned} U^{n+1}_{\Upsilon,k,k(n+1)+y} &= ((k - |\Upsilon|)n + y - \alpha + n + 1)U^{n+1}_{\Upsilon,k,k(n+1)+y-1} \\ &\quad + (|\Upsilon|n + \alpha - s + 1)U^{n}_{\Upsilon,k,k(n+1)n+y-1}. \end{aligned}$$

Again, we have to interpret $U^{n+1}_{\Upsilon,k,k(n+1)+y-1}$ as 0 since there are no permutation $\sigma \in S_{k(n+1)+y-1}$ which has $n+1$ Υ-k-equivalence matches. Thus we need an independent proof that

$$\sum_{r=0}^{n+1} (-1)^{n+1-r} \binom{(k-1)n + r + y - 1}{r} \binom{kn + j}{n + 1 - r} \Gamma(r, y-1, n) = 0. \quad (3.22)$$

Again, (3.22) is just a special case of Lemma 3.6 so that our formula for $U^{n+1}_{\Upsilon,k,k(n+1)+y-k}$ must also be true. □

Here is another formula for $U^s_{\Upsilon,k,kn+j}$. This one was obtained by iterating the recursions (3.14) and (3.15) from the top down.

Theorem 3.8. *For all $y - k \leq j \leq y - 1$ and all $s \leq n$ such that $kn + j > 0$, we have*

$$\frac{U^s_{\Upsilon,k,kn+j}}{((k-1)n + j)!} = \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n + r + j}{r} \binom{kn + j + 1}{n - s - r} \Omega(r, n)$$

where $\Omega(r, n) = \prod_{i=0}^{n-1} (r + \alpha + i(|\Upsilon| - 1))$.

Proof. We shall prove by induction on $kn+j$ that our formulas hold. The formulas trivially hold when $kn+j=1$ since the formula just reduces to $U^0_{\Upsilon,k,1}=1$.

First we shall show that our formulas satisfy the recursions (3.14) and (3.15). In order to simplify the algebra, we will again convert the form used in the theorem to the following.

$$U^s_{\Upsilon,k,kn+j} = \sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+j)!(kn+j+1)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r)!}$$

So, for $y-k+1 \le j \le y-1$ plugging in the above form into the RHS of (3.14) gives

$$(kn+j-s)\left[\sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+j-1)!(kn+j)!\Omega(r,n)}{((k-1)n+j+s+r)!r!(n-s-r)!}\right]$$
$$+(s+1)\left[\sum_{r=0}^{n-s-1} \frac{(-1)^{n-s-r-1}((k-1)n+r+j-1)!(kn+j)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r-1)!}\right]$$

Removing the $n-s$ term from the first summand, and combining the rest of the terms yields

$$\sum_{r=0}^{n-s-1} \frac{(-1)^{n-s-r}((k-1)n+r+j-1)!(kn+j)!\Omega(r,n)\left[(kn+j+1)((k-1)n+j+r)\right]}{((k-1)n+j+s+r+1)!r!(n-s-r)!}$$
$$+\frac{(kn+j-s)!\Omega(n-s,n)}{(n-s)!}$$
$$= \sum_{r=0}^{n-s-1} \frac{(-1)^{n-s-r}((k-1)n+r+j)!(kn+j+1)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r)!} + \frac{(kn+j-s)!\Omega(n-s,n)}{(n-s)!}$$
$$= \sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+j)!(kn+j+1)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r)!} = U^s_{\Upsilon,k,kn+j}$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (3.14) for $y-k+1 \le j \le y-1$. We will now show that our formula satisfies (3.15). The RHS of (3.15) becomes

$$((k-|\Upsilon|)n+s+y-\alpha)\left[\sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+y-1)!(kn+y)!\Omega(r,n)}{((k-1)n+y+s+r)!r!(n-s-r)!}\right]$$
$$+(|\Upsilon|n+\alpha-s+1)\left[\sum_{r=0}^{n-s+1} \frac{(-1)^{n-s-r+1}((k-1)n+r+y-1)!(kn+y)!\Omega(r,n)}{((k-1)n+y+s+r-1)!r!(n-s-r+1)!}\right]$$

Removing the $n-s+1$ term from the second summand, and combining the rest of the terms yields

$$\sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+y-1)!(kn+y)!\Omega(r,n)\left[(-1)(\alpha+r+n(|\Upsilon|-1))(kn+y+1)\right]}{((k-1)n+y+s+r)!r!(n-s-r+1)!}$$
$$+\frac{(|\Upsilon|n+\alpha-s+1)(kn+y-s)!\Omega(n-s+1,n)}{(n-s+1)!}$$
$$\sum_{r=0}^{n-s+1} \frac{(-1)^{n-s-r+1}((k-1)n+r+y-1)!(kn+y+1)!\Omega(r,n)(\alpha+r+n(|\Upsilon|-1))}{((k-1)n+y+s+r)!r!(n-s-r+1)!}$$
$$\sum_{r=0}^{n-s+1} \frac{(-1)^{n-s-r+1}((k-1)n+r+y-1)!(kn+y+1)!\Omega(r,n+1)}{((k-1)n+y+s+r)!r!(n-s-r+1)!} = U^s_{\Upsilon,k,kn+y}$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (3.15) as desired.

Now we can complete our induction as follows. Assume that our formulas for $U^s_{\Upsilon,k,kn+j-1}$ hold for all $s \leq n$ where $y-k+1 \leq j \leq y$ such that $kn+j-1 > 0$.

It is easy to see that recursion (3.14) will allow us to prove our formulas hold for $U^s_{\Upsilon,k,kn+j}$ for all $s < n$ where $y-k+1 \leq j \leq y-1$. Once again, the case when $s = n$ needs to be examined separately. That is, in the recursion

$$U^n_{\Upsilon,k,kn+j} = (kn+j-n)U^n_{\Upsilon,k,kn+j-1} + (n+1)U^{n+1}_{\Upsilon,k,kn+j-1}, \qquad (3.23)$$

we must interpret $U^{n+1}_{\Upsilon,k,kn+j-1}$ as 0 since there are no permutations in S_{kn+j-1} with $n+1$ Υ-k-equivalences matches. In this case, it is easy to see that we must interpret the sum in formula for $U^{n+1}_{\Upsilon,k,kn+j-1}$ as the empty sum which equals 0 by definition so that our formulas do satisfy this recursion. Thus our formula for $U^n_{\Upsilon,k,kn+j}$ holds.

It is also easy to see that recursion (3.15) will allow us to prove our formulas hold for $U^s_{\Upsilon,k,k(n+1)+y-k}$, for all $1 < s < n$. Again, the cases $s = 0$ and $s = n+1$ need to be examined separately. Again, the problem is that our formulas make sense in the case

$$U^0_{\Upsilon,k,kn+y} = ((k-|\Upsilon|)n+y-\alpha)U^0_{\Upsilon,k,kn+y-1} + (|\Upsilon|n+\alpha+1)U^{-1}_{\Upsilon,k,kn+y-1} \qquad (3.24)$$

and when

$$U^{n+1}_{\Upsilon,k,k(n+1)+y-k} = ((k-|\Upsilon|)n+y-\alpha+n+1)U^{n+1}_{\Upsilon,k,kn+y-1} + (|\Upsilon|n+\alpha-s+1)U^n_{\Upsilon,k,kn+y-1}. \qquad (3.25)$$

However, by our definitions, we must interpret $U^{-1}_{\Upsilon,k,kn+y-1}$ and $U^{n+1}_{\Upsilon,k,kn+y-1}$ to be 0. Again, it is easy to verify $U^{n+1}_{\Upsilon,k,kn+y-1} = 0$ since we would again interpret the sum in the formula as the empty sum. However, in order to establish that $U^{-1}_{\Upsilon,k,kn+y-1} = 0$, we need an independent proof that

$$((k-1)n+j)!\left[\sum_{r=0}^{n+1}(-1)^{n+1-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{n+1-r}\Omega(r,n)\right] = 0. \tag{3.26}$$

for any $y-k \le j \le y-1$. As in the proof of our previous formulas, (3.26) is a special case of Lemma 3.6. Thus our formula for $U^{0}_{\Upsilon,k,k(n+1)+y-k}$ also holds. □

Note if we compare the formulas for $U^{s}_{\Upsilon,k,kn+j}$ from Theorem 3.7 and from Theorem 3.8, we obtain this very interesting result.

Corollary 3.9. *For all $y-k \le j \le y-1$ and all $s \le n$ such that $kn+j > 0$, we have*

$$\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{s-r}\Gamma(r,j,n) \tag{3.27}$$

$$= \sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{n-s-r}\Omega(r,n) \tag{3.28}$$

$$\textit{where } \Gamma(r,j,n) = \prod_{i=0}^{n-1}((k-1)n+r+j+1-\alpha-i(|\Upsilon|-1))$$

$$\textit{and } \Omega(r,n) = \prod_{i=0}^{n-1}(r+\alpha+i(|\Upsilon|-1)).$$

3.5 Combinatorial proofs and bijective questions

Looking at Theorem 3.8, letting $s = n-1$ and $\Upsilon = \{(ab)\}$, we arrive at the following corollary. We will present a direct counting argument which verifies the obvious consequence of the theorem.

Corollary 3.10. *For all $b-k \le j \le b-1$ and for all n such that $kn+j > 0$,*

$$T^{n-1}_{(ab),k,kn+j} = ((k-1)n+j)!\,[2^n((k-1)n+j+1)-(kn+j+1)]$$

Proof. We will use a particular construction to verify this corollary. Suppose that we would like to construct $\sigma = \sigma_1 \cdots \sigma_{kn+j}$ such that $asc_{ab,k}(\sigma) = n-1$. Consider the following construction.

We again will first place the elements which can not be the larger element in an a, b ascent. Recall that there are $(k-1)n + j$ elements of this type giving $((k-1)n+j)!$ placements of these numbers.

Now we will count the number of ways of inserting the remaining elements so that we create $n-1$ a, b ascents. We will organize the argument by first deciding which element will not be creating an a, b ascent. After this decision, it is clear that the each of the remaining elements to be placed must create an a, b ascent.

Assume that b, the first element, will not cause an a, b ascent. Now consider placing $2b, 3b, \ldots, nb$ from smallest to largest. There are two locations that you can place $2b$ to create an a, b ascent, namely after a or after $a+k$. Once the $2b$ has been placed, no other element could be placed between it and the previous element as this would mean that $2b$ would not be causing an a, b ascent. So you can see that when you next place $3b$, you've lost a location to place, but you also gain a location, namely the position following $a+2k$. So in the case where b does not cause an a, b ascent there are 2^{n-1} ways of placing the elements $2b, 3b, \ldots, nb$. Now when we turn to place b, the number of positions available that will not create an a, b ascent depends on if there was an element placed directly after a. If there were an element placed directly after a, there would be $(k-1)n+j+1$ positions to insert b. If there were not an element placed directly after a, there would be $(k-1)n+j$ positions to insert b. The difference being that in the second case, placing b directly following a would cause an a, b ascent. The only way to be in the second case is if you placed $b+ik$ directly after $a+ik$. So, given this argument, the number of ways to place these remaining elements when b does not create an a, b ascent is given by the following.

$$
\begin{aligned}
& (2^{n-1}-1)((k-1)n+j+1)+(k-1)n+j \\
= \quad & 2^{n-1}((k-1)n+j+1)-1
\end{aligned}
$$

Now assume that $2b$, the second element, will not cause an a, b ascent. Now consider placing $b, 3b, \ldots, nb$ from smallest to largest. Here there is only one location that you can place b to create an a, b ascent, namely after a. Now there are two locations that you can place $3b$ to create an a, b ascent, namely after $a+k$ or after $a+2k$. As per the above argument each element needing to be placed after $3b$ has been placed will have precisely 2 locations that could cause an a, b ascent. So in the case where $2b$ does not cause an a, b ascent there are 2^{n-2} ways of placing the elements $b, 3b, \ldots, nb$. Now when we turn to place $2b$, the number of positions available that will not create an a, b ascent depends on if there was an element placed directly after $a+k$. Similar to before, If there were an element placed directly after $a+k$, there would be $(k-1)n+j+1$ positions to insert b. If there were not an element placed directly after $a+k$, there would be $(k-1)n+j$ positions to insert b. So, given this argument, the number of ways to place these remaining elements when b does not create an a, b ascent is given by the following.

$$
\begin{aligned}
& (2^{n-2}-1)((k-1)n+j+1)+(k-1)n+j \\
= \quad & 2^{n-2}((k-1)n+j+1)-1
\end{aligned}
$$

This argument easily generalizes to the following. The number of ways to place these remaining elements when ib, the i^{th} element, does not create an a, b ascent is given by the following.

$$
\begin{aligned}
& (2^{n-i}-1)((k-1)n+j+1)+(k-1)n+j \\
= \quad & 2^{n-i}((k-1)n+j+1)-1
\end{aligned}
$$

This means that

$$\begin{aligned}
T^{n-1}_{(ab),k,kn+j} &= ((k-1)n+j)!(\textstyle\sum_{i=1}^{n} 2^{n-i}((k-1)n+j+1)-1) \\
&= ((k-1)n+j)!(\textstyle\sum_{i=0}^{n-1} 2^{i}((k-1)n+j+1)-1) \\
&= ((k-1)n+j)!((k-1)n+j+1)(\textstyle\sum_{i=0}^{n-1} 2^{i})-n) \\
&= ((k-1)n+j)!((k-1)n+j+1)(2^n-1)-n) \\
&= ((k-1)n+j)!\,[2^n((k-1)n+j+1)-(kn+j+1)]
\end{aligned}$$

□

Strangely enough, we were able to give a combinatorial proof of $T^{n-1}_{(ab),k,kn+j}$ in Corollary 3.10, but unable to use the same type of direct counting argument to give a combinatorial proof to the formula for $T^{1}_{(ab),k,kn+j}$ in Theorem 3.7. However, if we apply a similar argument we do arrive at yet another formula and it is the following.

Theorem 3.11. *We have for* $j = b-k, \ldots, b-1$,

$$T^{1}_{(ab),k,kn+j}(x) = ((k-1)n+j)!\sum_{i=1}^{n}(n-i+1)((k-1)n+j)^{n-i}((k-1)n+j+1)^{i-1} \tag{3.29}$$

Proof. Like usual, we will present a construction that will verify the theorem. As before, we will first place those elements which can not be the larger element in an a, b ascent, there are $(k-1)n+j$ elements of this type giving rise to the factor $((k-1)n+j)!$ in (3.29). Now we will count the number of ways to create exactly 1 a, b ascent by first choosing which element will cause the ascent.

Suppose the minimal element, b causes the ascent. We will first place b, then $b+k \ldots b+(n-1)k$. This forces b to be placed directly after a, keeping the number of positions to insert $b+k$ at $(k-1)n+j+1$. However we are unable to place $b+k$ after $a+k$ so we lose a position. Now when we place $b+2k$, we have gained a position to place following $b+k$, but have lost a position after $a+2k$. You can see this continues in this fashion and you end up with $(k-1)n+j$ choices to place each element $b+k\dots b+(n-1)k$. Giving a total of $((k-1)n+j)^{n-1}$ ways to place $b+k\dots b+(n-1)$ so that b causes the ascent.

Now, suppose $b+k$ causes the ascent. We will first place $b+k$, then $b, b+2k\dots b+(n-1)k$. This forces $b+k$ to be placed directly after a, or directly after $a+k$. If $b+k$ was placed directly after $a+k$, you notice that when we turn to place b, we no longer are losing a spot, as b could follow $a+k$. So there would be $(k-1)n+j+1$ choices to place b. When you look to place $b+2k$, you do lose a position to place since $b+2k$ can't follow $a+2k$ and this continues giving $(k-1)n+j$ choices to place each element $b+2k\dots b+(n-1)k$. If $b+k$ was placed directly after $a+k$, you notice that when we turn to place b, we are losing a spot, as b can't follow a. So there would be $(k-1)n+j$ choices to place b. When you look to place $b+2k$, you gain and lose a position to place and this continues giving $(k-1)n+j$ choices to place each element $b, b+2k\dots b+(n-1)k$. Giving a total of $((k-1)n+j+1)((k-1)n+j)^{n-2}+((k-1)n+j+1)^{n-1}$ ways to place $b, b+2k\dots b+(n-1)k$ so that $b+k$ causes the ascent.

This generalizes to the following: suppose $b+rk$ causes the ascent and you place $b+rk$ after $a+sk$. Then when you place the elements $b+ik$ where $s \leq i < r$, there will be $(k-1)n+j+1$ choices and when you place the remaining elements $b+ik$ where $i < s$ or $i > r$ there will be$(k-1)n+j+1$ choices. This gives the total number of placements that have one a, b ascent to be

$$\begin{aligned} & ((k-1)n+j)! \textstyle\sum_{i=1}^{n} (\sum_{t=1}^{i} ((k-1)n+j)^{n-t}((k-1)n+j+1)^{t-1}) \\ = \; & ((k-1)n+j)! \textstyle\sum_{i=1}^{n} (n-i+1)((k-1)n+j)^{n-i}((k-1)n+j+1)^{i-1} \end{aligned}$$

which is the desired result. □

We end this section with a bijective proof of one more of the consequences of our formulas. Examining Table 3.1 from Section 3.2, we noted that $T_{(ab),k,m}$ is identical for certain values of a, b and m. By Theorem 3.7 and Theorem 3.8, it is clear why this is the case. However, we will now provide a bijection between the two as another way to show this fact.

First observe that for a fixed k and any given m, there are unique N and j such that $m = kn + j$ where $b - k \leq j \leq b - 1$. Note that n and j are independent of a. In such a situation, we shall define

$$\overline{Quotient}_{k,b}(m) := n.$$

Note also, that for any given m, there are exactly $\overline{Quotient}_{k,b}(m)$ elements that could be the larger element in an a, b ascent.

Theorem 3.12. *Given m, $(a, b), (c, d) \in A_k$, then $T_{(ab),k,m}(x) = T_{(cd),k,m}(x)$ whenever $\overline{Quotient}_{k,b}(m) = \overline{Quotient}_{k,d}(m)$.*

Proof. Let $n = \overline{Quotient}_{k,b}(m)$ and C_n be the set of sequences $c_1 \cdots c_n$ of length n such that $c_i \in \{1, \ldots, m - n + i\}$ for $i = 1, \ldots, n$. Observe that $|C_n| = (m - n + 1)(m - n + 2) \ldots m = (m - n + 1)_n$.

To define our bijection, we will first will first define a bijection $\Theta_{a,b,m} : S_m \mapsto (S_{m-n} \times C_n)$. It is easy to see from our definitions that S_m and $(S_{m-n} \times C_n)$ have the same cardinality. That is,

$$|(S_{m-n} \times C_n)| = |S_{m-n}||C_n| = (m - n)!(m - n + 1)_n = m! = |S_m|.$$

The map $\Theta_{a,b,m}$ is simple. Given $\sigma \in S + m$, we first remove every element of the form $b + k(i - 1)$, where i is a positive integer from σ and define the resulting sequence to be α_1. We then define $\tau = red(\alpha_1)$. For example, suppose $k = 3$, $a = 2$, $b = 3$, $m = 10$ and $\sigma = 2\ 6\ 1\ 8\ 9\ 4\ 5\ 3\ 7\ 10$. In this case, $n = \overline{Quotient}_{3,3}(10) = 3$, since we would write $10 = 3 * 3 + 1$. So, to obtain α we remove all elements of the form $3 + 3(i - 1)$ where i is a positive integer from σ, in other words we remove 3, 6 and 9. In this example, $\alpha_1 = 2\ 1\ 8\ 4\ 5\ 7\ 10$ and thus $\tau = red(\alpha_1) = red(2\ 1\ 8\ 4\ 5\ 7\ 10) = 2\ 1\ 6\ 3\ 4\ 5\ 7$. Next we shall show how we can obtain a $c \in C_n$ from α_1 via the insertion process that creates σ from α_1. That is, we shall insert the elements of the form $b + k(i - 1)$ in order starting with α_1 and use the sequence $c = c_1 \cdots c_n$ to record the spaces into which the elements

$b, b+k, \ldots, b+kn$ need to be inserted to recreate σ. Let α_i for $i = 1, ldots, n-1$ be the sequence that results by removing $b+k(i-1), \ldots, b+kn$ from σ. At the i^{th} step of this process, label the spaces into which we can insert $b+k(i-1)$ into α_i with the integers from 1 to $m-n+i$ where we label the spaces following the elements $a, a+k, \ldots, a+nk$ with the numbers $1, \ldots, n$ and then label the remaining positions from left to right.

Then c_i is defined to be the label of the position into which $b+k(i-1)$ was inserted into α_i to create α_{i+1}. In our example there will be 3 steps. At the first step, since the position following 2 is the only position that could cause a $(2,3)$ ascent it is labeled with a one and then the rest are labeled from left to right as follows.

$$_{\overline{4}}2_{\overline{1}}1_{\overline{5}}8_{\overline{3}}4_{\overline{6}}5_{\overline{2}}7_{\overline{7}}10_{\overline{8}}$$

We see that 3 follows 5 in σ so 3 must be inserted in position 2 and thus $c_1 = 2$. We then label α_2 as follows.

$$_{\overline{4}}2_{\overline{1}}1_{\overline{5}}8_{\overline{3}}4_{\overline{6}}5_{\overline{2}}3_{\overline{7}}7_{\overline{8}}10_{\overline{9}}$$

We see that 6 follows 2 in σ so 6 must be inserted in position 1 and thus $c_2 = 1$. We then label α_3 as follows.

$$_{\overline{4}}2_{\overline{1}}6_{\overline{5}}1_{\overline{6}}8_{\overline{3}}4_{\overline{7}}5_{\overline{2}}3_{\overline{8}}7_{\overline{9}}10_{\overline{10}}$$

We see that 9 follows 8 in σ so 9 must be inserted in position 1 and thus $c_3 = 3$. Hence $C = 6\ 1\ 1$ and $\Theta_{2,3,10}(2\ 6\ 1\ 8\ 9\ 4\ 5\ 3\ 7\ 10) = (2\ 1\ 6\ 3\ 4\ 5\ 7, 2\ 1\ 3)$.

It is easy to see that we can recreate σ from $(\tau, c) = \Theta_{a,b,m}(\sigma)$ so that $\Theta_{a,b,m}$ is a bijection. Moreover, if $\Theta_{a,b,m}(\sigma) = (\tau, c)$, then we claim that we can determine the number of (a,b) ascents in σ directly from c. The reasoning is as follows. At the first step of the insertion process, when we are inserting b into α_1, there is only one position to insert b that will cause an (a,b) ascent, namely after a. This position is labeled with a 1. Note that at later stages we may insert elements of the form $b + k(i-1)$ immediately after a but not matter what happens a will be the start of an (a, b) ascent in σ. Thus we know that a is the smaller element of an (a, b) ascent if c_1 is 1. Similarly, if $c_i = r \in \{1, \dots, i\}$, then we have inserted $b + k(i-1)$ immediately after $a + k(r-1)$ so that no matter how we insert $b + ki, \dots, b + ni$, $a + k(r-1)$ will start an (a, b) ascent in σ. It easily follows that the set of i such that $a + k(i-1)$ start an (a, b)-ascent in σ is equal to the set of all i such that there exists a $j \geq i$ such that $c_j = i$. For any $c \in C_n$, we set $s_c = |\{i : 1 \leq i \leq n \;\&\; (\exists j \geq i)(c_j = i)\}|$. then if follows that for any σ such that $\Theta_{a,b,m}(\sigma) = (\tau, c)$, σ has s_c (a, b) ascents.

Now let $S_m^{(a,b),s}$ denote the set of permutations $\sigma \in S_m$ such that σ has s (a, b) ascents and C_n^s equal the set of $c \in C_n$ such that $s_c = s$. Then we have proved that $\Theta_{a,b,m}$ induces a bijection between $S_m^{(a,b),s}$ and $S_{m-n} \times C_n^s$ for all s. We can use the same argument to show $\Theta_{c,d,m}$ induces a bijection between $S_m^{(c,d),s}$ and $S_{m-n} \times C_n^s$ for all s. Thus the map Δ_s which is equal to $\Theta_{c,d,m}^{-1} \circ \Theta_{a,b,m}$ restricted to $S_m^{(a,b),s}$ is a bijection from $S_m^{(a,b),s}$ to $S_m^{(c,d),s}$ for all s. $\square$

3.6 Acknowledgement

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4

q-Counting descent pairs with prescribed tops and bottoms

4.1 Introduction to q-analogues

We will now define the following q-analogues.

1. $[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$
2. $[n]_q! = [n]_q [n-1]_q \ldots [1]_q$
3. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$
4. $\begin{bmatrix} n \\ \lambda_1, \lambda_2, \ldots, \lambda_k \end{bmatrix}_q = \frac{[n]_q!}{[\lambda_1]_q![\lambda_2]_q! \ldots [\lambda_k]_q!}$, where $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$

Now we will present some combinatorial interpretations of these analogues.

Theorem 4.1.

$$[n]_q! = \sum_{\sigma \in S_n} q^{inv(\sigma)} = \sum_{\sigma \in S_n} q^{maj(\sigma)}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(1^k, 0^{n-k})} q^{inv(r)} = \sum_{r \in R(1^k, 0^{n-k})} q^{maj(r)}$$

$$\begin{bmatrix} n \\ \lambda_1, \lambda_2, \ldots, \lambda_k \end{bmatrix}_q = \sum_{r \in R(1^{\lambda_1}, 2^{\lambda_2}, \ldots, k^{\lambda_k})} q^{inv(r)} = \sum_{r \in R(1^{\lambda_1}, 2^{\lambda_2}, \ldots, k^{\lambda_k})} q^{maj(r)}$$

where $R(1^{\lambda_1}, 2^{\lambda_2}, \ldots, k^{\lambda_k})$ is the set of all rearrangements of λ_1 1's, λ_2 2's, ... and λ_k k's.

Proof. We will begin by proving the first statement of the theorem. It is a simple proof by induction. Suppose we have a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_{n-1} \in S_{n-1}$. If we were to extend σ to a permutation in S_n, we would need to insert n somewhere in σ. Now, we will label the positions where we could insert n in the following manner, namely with the labels 0 through $n-1$ from right to left.

$$\overline{_{n-1}}\sigma_1\overline{_{n-2}}\sigma_2\overline{_{n-3}}\sigma_3 \ldots \overline{_2}\,\sigma_{n-2}\overline{_1}\sigma_{n-1}\overline{_0}$$

It is easy to see that inserting n into the position labeled with i will create i new inversions. Thus,

$$\sum_{\sigma \in S_n} q^{inv(\sigma)} = \sum_{\sigma \in S_{n-1}} q^{inv(\sigma)}(1 + q + \ldots + q^{n-1}) = [n-1]_q!\,[n]_q = [n]_q!.$$

To prove the second statement of the theorem we will first note that we want to show

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(1^k, 0^{n-k})} q^{inv(r)}.$$

Multiplying both sides of this expression by $[k]_q!\,[n-k]_q!$ yields

$$[n]_q! = [k]_q!\,[n-k]_q! \sum_{r \in R(1^k, 0^{n-k})} q^{inv(r)}.$$

So, using the first statement in the theorem this is equivalent to showing that

$$\begin{aligned} \sum_{\sigma \in S_n} q^{inv(\sigma)} &= \Big(\sum_{\alpha \in S_k} q^{inv(\alpha)}\Big)\Big(\sum_{\beta \in S_{n-k}} q^{inv(\beta)}\Big)\Big(\sum_{r \in R(1^k, 0^{n-k})} q^{inv(r)}\Big) \\ &= \sum_{(\alpha,\beta,r) \in S_k \times S_{n-k} \times R(1^k, 0^{n-k})} q^{inv(\alpha)+inv(\beta)+inv(r)}. \end{aligned}$$

To show this, we will produce a bijection, $\Gamma : S_k \times S_{n-k} \times R(1^k, 0^{n-k}) \mapsto S_n$ such that $\Gamma(\alpha, \beta, r) = \sigma \Rightarrow inv(\alpha) + inv(\beta) + inv(r) = inv(\alpha)$. We will now define the bijection by showing how it acts on a simple example. Suppose $k = 3$, $n = 7$, and that $\alpha = 1\ 3\ 2$, $\beta = 1\ 4\ 2\ 3$, and $r = 0100110$. First we will add $n - k$ to the elements of α, yielding $\alpha' = 5\ 7\ 6$. Note that doing this ensures all the elements of α' are larger than every element of β. We then think of associating the elements of α' with the 1's of r and the elements of β with the 0's of r. We obtain σ by scanning the elements of r from left to right and choosing the leftmost unused element of α if $r_i = 1$ or of β if $r_i = 0$. In our example, $\sigma = 1\ 5\ 4\ 2\ 7\ 6\ 3$.

Furthermore, Γ is a bijection, because starting with σ, we can associate 1 with 765 and 0 with 4321. This will give us back α, β and r. It is also weight preserving because $inv(\sigma) =$ the number of inversions involving 5,6,7 + the number of inversions involving 1,2,3,4 + the number of inversions involving 1,2,3,4 with 5,6,7 which is precisely $inv(\alpha) + inv(\beta) + inv(r)$.

This bijection can be generalized to show the third statement of the theorem. Similarly it is equivalent to showing that

$$\begin{aligned} & \sum_{\sigma \in S_n} q^{inv(\sigma)} \\ = \ & \Big(\sum_{\alpha_1 \in S_{\lambda_1}} q^{inv(\alpha_1)}\Big)\Big(\sum_{\alpha_2 \in S_{\lambda_2}} q^{inv(\alpha_2)}\Big) \cdots \Big(\sum_{\alpha_k \in S_{\lambda_k}} q^{inv(\alpha_k)}\Big)\Big(\sum_{r \in R(1^{\lambda_1}, 2^{\lambda_2}, \ldots, k^{\lambda_k})} q^{inv(r)}\Big) \\ = \ & \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_k, r) \in S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_k} \times R(1^{\lambda_1}, 2^{\lambda_2}, \ldots, k^{\lambda_k})} q^{inv(\alpha_1) + inv(\alpha_2) + \ldots + inv(\alpha_k) + inv(r)}. \end{aligned}$$

Like before we will describe how Γ acts on $(\alpha_1, \alpha_2, \ldots, \alpha_k, r)$. First, for each j, we will add $\sum_{i=1}^{j-1} \lambda_i$ to α_j to get α'_j. This ensures that each element of α'_j is larger than every element of α'_t when $t < j$. Like before we associate the elements of α'_j with j. We obtain σ by scanning the elements of r from left to right and choosing the leftmost unused element of α'_j if $r_i = j$. By similar reasoning, Γ is a weight preserving bijection.

We will now show that

$$[n]_q! = \sum_{\sigma \in S_n} q^{maj(\sigma)}.$$

Again, we will show this by induction by labeling the possible places to insert $n+1$ into a permutation in S_n and seeing what happens to the maj statistic. Here is the labeling. We first label the rightmost space with a 0, then label the spaces following descents from right to left with (1,2,..., $des(\sigma)$), and lastly we label the remaining spaces from left to right with ($des(\sigma)$+1,...,n).

Lemma 4.2. *Using the above described labeling, if we insert* $n+1$ *into the space labeled* i *in* σ *to obtain* $\sigma^{(i)}$*, then* $maj(\sigma^{(i)}) = i + maj(\sigma)$*.*

Proof. **Case 1:** $n+1$ **is inserted into a position labeled with 0,1,...,**$des(\sigma)$
If $n+1$ is inserted in the position labeled 0, it is clear that $maj(\sigma^{(i)}) = maj(\sigma)$. In the case where $n+1$ is inserted between a descent pair, inserting $n+1$ into this place will eliminate the descent at this position, but create a new descent at the following position. This gives a +1 contribution to the major index statistic. However, the index of every descent pair following the newly created descent pair will also get shifted by 1. By our labeling, there are precisely $i-1$ such pairs and thus the net contribution to the major index statistic is $1 + (i-1) = i$.

Case 2: $n+1$ **is inserted into a position labeled with** $des(\sigma)+1, \ldots, n$
If $n+1$ is inserted in the position labeled $des(\sigma)+1$, it is clear that $maj(\sigma^{(i)}) = maj(\sigma) + des(\sigma) + 1$. This is because this position is the leftmost position to insert and will create a new descent in the first position, giving the +1, and also shift the index of every other descent pair, giving the $+des(\sigma)$. Now we will look at the difference in contribution between placing $n+1$ in a position labeled $p+1$ compared with placing $n+1$ in a position labeled p, where $p+1 > des(\sigma)+1$. Suppose that there are i elements of σ before the position labeled p and k elements of σ before the position labeled $p+1$. Inserting at position p will cause a new descent at index $i+1$, but will also shift the index of every following descent. Inserting at position $p+1$ will cause a new descent at index $k+1$ and will also shift the index of every following descent. So, the difference between the two is that inserting at position p shifted all the descents among $\sigma_{i+1} \ldots \sigma_k$, and added a descent at index $i+1$, giving a contribution of $(i+1)+(i+2)+\ldots+k$. Inserting at position $p+1$ did not shift the descents among $\sigma_{i+1} \ldots \sigma_k$, so they remained with a contribution of $(i+1)+(i+2)+\ldots+(k-1)$, but it created a descent at $k+1$. Thus, the

difference in contribution is 1. This verifies the lemma. □

Thus, using the lemma,

$$\sum_{\sigma \in S_{n+1}} q^{maj(\sigma)} = \sum_{\sigma \in S_n} q^{maj(\sigma)}(1 + q + \ldots + q^n) = [n]_q! \, [n+1]_q = [n+1]_q!.$$

In addition, these labeling schemes will give us a bijection, $\Theta : S_n \mapsto S_n$ where $\Theta(\sigma) = \tau \Rightarrow inv(\sigma) = maj(\tau)$. It is simple to describe, just reverse the labeling process of σ using the inversion labeling and record the position numbers of inserting $n+1$. Now, construct τ by using the same position numbers, but using the major index labeling.

□

4.2 q-Analogues of the formulas from Chapter 3

Much of the work in Chapter 3 was later generalized by Hall and Remmel. Following this, we all worked together to generalize their results by deriving some q-analogues of their formulas.

As before, let S_n denote the set of permutations of the set $[n] = \{1, 2, \ldots, n\}$. Given subsets $X, Y \subseteq \mathbb{N}$ and a permutation $\sigma \in S_n$, let

$$\begin{aligned} \mathrm{Des}_{X,Y}(\sigma) &= \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_i \in X \ \& \ \sigma_{i+1} \in Y\}, \text{ and} \\ \mathrm{des}_{X,Y}(\sigma) &= |\mathrm{Des}_{X,Y}(\sigma)|. \end{aligned}$$

If $i \in \mathrm{Des}_{X,Y}(\sigma)$, then we call the pair (σ_i, σ_{i+1}) an (X,Y)-*descent*. For example, if $X = \{2, 3, 5\}, Y = \{1, 3, 4\}$, and $\sigma = 54213$, then $\mathrm{Des}_{X,Y}(\sigma) = \{1, 3\}$ and $\mathrm{des}_{X,Y}(\sigma) = 2$.

For fixed n we define the polynomial

$$P_n^{X,Y}(x) = \sum_{s \geq 0} P_{n,s}^{X,Y} x^s := \sum_{\sigma \in S_n} x^{\mathrm{des}_{X,Y}(\sigma)}. \tag{4.1}$$

Thus the coefficient $P_{n,s}^{X,Y}$ is the number of $\sigma \in S_n$ with exactly s (X,Y)-descents. Again, note that with the correct choices of sets for X and Y, this coefficient generalizes $U_{\Upsilon,k,n}^{s}$.

Hall and Remmel [6] gave direct combinatorial proofs of a pair of formulas for $P_{n,s}^{X,Y}$. First of all, for any set $A \subseteq \mathbb{N}$, let

$$\begin{aligned} A_n &= A \cap [n], \text{ and} \\ A_n^c &= (A^c)_n = [n] - A. \end{aligned}$$

Then Hall and Remmel [6] proved the following theorem.

Theorem 4.3.

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{s} (-1)^{s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}), \quad (4.2)$$

and

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{|X_n|-s-r} \prod_{x \in X_n} (r+\beta_{X,n,x}-\beta_{Y,n,x}), \quad (4.3)$$

where for any set A and any $j, 1 \le j \le n$, we define

$$\begin{aligned} \alpha_{A,n,j} &= |A^c \cap \{j+1, j+2, \ldots, n\}| = |\{x : j < x \le n \ \& \ x \notin A\}|, \ \textit{and} \\ \beta_{A,n,j} &= |A^c \cap \{1, 2, \ldots, j-1\}| = |\{x : 1 \le x < j \ \& \ x \notin A\}|. \end{aligned}$$

Example 4.4. Suppose $X = \{2,3,4,6,7,9\}, Y = \{1,4,8\}$, and $n = 6$. Thus $X_6 = \{2,3,4,6\}, X_6^c = \{1,5\}, Y_6 = \{1,4\}, Y_6^c = \{2,3,5,6\}$, and we have the following table of values of $\alpha_{X,6,x}, \beta_{Y,6,x}$, and $\beta_{X,6,x}$.

x	2	3	4	6
$\alpha_{X,6,x}$	1	1	1	0
$\beta_{Y,6,x}$	0	1	2	3
$\beta_{X,6,x}$	1	1	1	2

Equation (4.2) gives

$$\begin{aligned} P_{6,2}^{X,Y} &= 2! \sum_{r=0}^{2} (-1)^{2-r} \binom{2+r}{r} \binom{7}{2-r} (2+r)(3+r)(4+r)(4+r) \\ &= 2\,(1 \cdot 21 \cdot 2 \cdot 3 \cdot 4 \cdot 4 - 3 \cdot 7 \cdot 3 \cdot 4 \cdot 5 \cdot 5 + 6 \cdot 1 \cdot 4 \cdot 5 \cdot 6 \cdot 6) \\ &= 2(2016 - 6300 + 4320) \\ &= 72. \end{aligned}$$

while (4.3) gives

$$\begin{aligned} P_{6,2}^{X,Y} &= 2! \sum_{r=0}^{2} (-1)^{2-r} \binom{2+r}{r} \binom{7}{2-r} (1+r)(0+r)(-1+r)(-1+r) \\ &= 2\,(1 \cdot 21 \cdot 1 \cdot 0 \cdot (-1) \cdot (-1) - 3 \cdot 7 \cdot 2 \cdot 1 \cdot 0 \cdot 0 + 6 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1) \\ &= 2(0 - 0 + 36) \\ &= 72. \end{aligned}$$

Now we shall present the methods used to arrive at q-analogues of their results. There are two natural approaches to finding q-analogues of (4.2) and (4.3). The first approach is to use q-analogues of the simple recursions that are satisfied by the coefficients $P_{n,s}^{X,Y}$. This approach naturally leads us to recursively define of a pair of statistics $\text{stat}_{X,Y}(\sigma)$ and $\overline{\text{stat}}_{X,Y}(\sigma)$ on permutations σ so that if we define

$$P_{n,s}^{X,Y}(q) = \sum_{\sigma \in S_n, \text{des}_{X,Y}(\sigma)=s} q^{\text{stat}_{X,Y}(\sigma)} \tag{4.4}$$

and

$$\bar{P}_{n,s}^{X,Y}(q) = \sum_{\sigma \in S_n, \text{des}_{X,Y}(\sigma)=s} q^{\overline{\text{stat}}_{X,Y}(\sigma)}, \tag{4.5}$$

then we can prove the following formulas:

$$\frac{P_{n,s}^{X,Y}(q)}{[|X_n^c|]_q!} = \tag{4.6}$$

$$\sum_{r=0}^{s} (-1)^{s-r} q^{\binom{s-r}{2}} \begin{bmatrix} |X_n^c| + r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{x \in X_n} [1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}]_q$$

and

$$\frac{\bar{P}^{X,Y}_{n,s}(q)}{[|X^c_n|]_q!} = \tag{4.7}$$
$$\sum_{r=0}^{|X_n|-s} \frac{q^{\binom{|X_n|-s-r}{2}}}{(-1)^{|X_n|-s-r}} \begin{bmatrix} |X^c_n|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{x\in X_n} [r+\beta_{X,n,x}-\beta_{Y,n,x}]_q .$$

The second approach is to q-analogue the combinatorial proofs of (4.2) and (4.3). We will see that this approach also works and leads to a more direct definition of $\text{stat}_{X,Y}(\sigma)$ and $\overline{\text{stat}}_{X,Y}(\sigma)$, involving generalizations of classical permutation statistics such as inv and maj.

4.2.1 Recursions for $P^{X,Y}_{n,s}(q)$

In this section, we shall give q-analogues of the recursions for the coefficients $P^{X,Y}_{n,s}$ developed by Hall and Remmel [6].

Given $X, Y \subseteq \mathbb{N}$, let $P^{X,Y}_0(x,y) = 1$, and for $n \geq 1$, define

$$P^{X,Y}_n(x,y) = \sum_{s,t\geq 0} P^{X,Y}_{n,s,t} x^s y^t := \sum_{\sigma\in S_n} x^{\text{des}_{X,Y}(\sigma)} y^{|Y^c_n|}.$$

Let Φ_{n+1} and Ψ_{n+1} be the operators defined as

$$\Phi_{n+1} : x^s y^t \longrightarrow s x^{s-1} y^t + (n+1-s) x^s y^t$$
$$\Psi_{n+1} : x^s y^t \longrightarrow (s+t+1) x^s y^t + (n-s-t) x^{s+1} y^t.$$

Then Hall and Remmel proved the following.

Proposition 4.5. *For any sets $X, Y \subseteq \mathbb{N}$, the polynomials $P^{X,Y}_n(x,y)$ satisfy*

$$P^{X,Y}_{n+1}(x,y) = \begin{cases} y\cdot \ \ \Phi_{n+1}(P^{X,Y}_n(x,y)) & \text{if } n+1 \notin X \text{ and } n+1 \notin Y, \\ \quad\ \Phi_{n+1}(P^{X,Y}_n(x,y)) & \text{if } n+1 \notin X \text{ and } n+1 \in Y, \\ y\cdot \ \ \Psi_{n+1}(P^{X,Y}_n(x,y)) & \text{if } n+1 \in X \text{ and } n+1 \notin Y, \\ \quad\ \Psi_{n+1}(P^{X,Y}_n(x,y)) & \text{if } n+1 \in X \text{ and } n+1 \in Y. \end{cases}$$

It is easy to see that Proposition 4.5 implies that the following recursion holds for the coefficients $P_{n,s,t}^{X,Y}$ for all $X, Y \subseteq \mathbb{N}$ and $n \geq 1$.

$$P_{n+1,s,t}^{X,Y} = \tag{4.8}$$
$$\begin{cases} (s+1)P_{n,s+1,t-1}^{X,Y} + (n+1-s)P_{n,s,t-1}^{X,Y} & \text{if } n+1 \notin X \text{ and } n+1 \notin Y, \\ (s+1)P_{n,s+1,t}^{X,Y} + (n+1-s)P_{n,s,t}^{X,Y} & \text{if } n+1 \notin X \text{ and } n+1 \in Y, \\ (s+t)P_{n,s,t-1}^{X,Y} + (n+2-s-t)P_{n,s-1,t-1}^{X,Y} & \text{if } n+1 \in X \text{ and } n+1 \notin Y, \\ (s+t+1)P_{n,s,t}^{X,Y} + (n+1-s-t)P_{n,s-1,t}^{X,Y} & \text{if } n+1 \in X \text{ and } n+1 \in Y. \end{cases}$$

We define two q-analogues of the operators Φ_{n+1} and Ψ_{n+1} as follows. Let Φ_{n+1}^q and Ψ_{n+1}^q be the operators defined as

$$\Phi_{n+1}^q \; : \; x^s y^t \longrightarrow [s]_q x^{s-1} y^t + q^s [n+1-s]_q x^s y^t$$
$$\Psi_{n+1}^q \; : \; x^s y^t \longrightarrow [s+t+1]_q x^s y^t + q^{s+t+1} [n-s-t]_q x^{s+1} y^t,$$

and let $\bar{\Phi}_{n+1}^q$ and $\bar{\Psi}_{n+1}^q$ be the operators defined as

$$\bar{\Phi}_{n+1}^q \; : \; x^s y^t \longrightarrow q^{n+1-s} [s]_q x^{s-1} y^t + [n+1-s]_q x^s y^t$$
$$\bar{\Psi}_{n+1}^q \; : \; x^s y^t \longrightarrow q^{n-s-t} [s+t+1]_q x^s y^t + [n-s-t]_q x^{s+1} y^t.$$

Given subsets $X, Y \subseteq \mathbb{N}$, we define the polynomials $P_n^{X,Y}(q,x,y)$ by $P_0^{X,Y}(q,x,y) = 1$ and

$$P_{n+1}^{X,Y}(q,x,y) = \tag{4.9}$$
$$\begin{cases} y \cdot \; \Phi_{n+1}^q(P_n^{X,Y}(q,x,y)), & \text{if } n+1 \notin X, n+1 \notin Y, \\ \Phi_{n+1}^q(P_n^{X,Y}(q,x,y)), & \text{if } n+1 \notin X, n+1 \in Y, \\ y \cdot \; \Psi_{n+1}^q(P_n^{X,Y}(q,x,y)), & \text{if } n+1 \in X, n+1 \notin Y, \\ \Psi_{n+1}^q(P_n^{X,Y}(q,x,y)), & \text{if } n+1 \in X, n+1 \in Y. \end{cases}$$

Similarly, we define the polynomials $\bar{P}_n^{X,Y}(q,x,y)$ by $\bar{P}_0^{X,Y}(q,x,y) = 1$ and

$$\bar{P}_{n+1}^{X,Y}(q,x,y) = \begin{cases} y \cdot \; \bar{\Phi}_{n+1}^q(\bar{P}_n^{X,Y}(q,x,y)), & \text{if } n+1 \notin X, n+1 \notin Y, \\ \bar{\Phi}_{n+1}^q(\bar{P}_n^{X,Y}(q,x,y)), & \text{if } n+1 \notin X, n+1 \in Y, \\ y \cdot \; \bar{\Psi}_{n+1}^q(\bar{P}_n^{X,Y}(q,x,y)), & \text{if } n+1 \in X, n+1 \notin Y, \text{ and} \\ \bar{\Psi}_{n+1}^q(\bar{P}_n^{X,Y}(q,x,y)), & \text{if } n+1 \in X, n+1 \in Y. \end{cases} \tag{4.10}$$

It is easy to see that (4.9) implies that

$$P^{X,Y}_{n+1,s,t}(q) = \tag{4.11}$$
$$\begin{cases} [s+1]_q P^{X,Y}_{n,s+1,t-1}(q) + q^s[n+1-s]_q P^{X,Y}_{n,s,t-1}(q) & \text{if } n+1 \notin X \text{ and } n+1 \notin Y, \\ [s+1]_q P^{X,Y}_{n,s+1,t}(q) + q^s[n+1-s]_q P^{X,Y}_{n,s,t}(q) & \text{if } n+1 \notin X \text{ and } n+1 \in Y, \\ [s+t]_q P^{X,Y}_{n,s,t-1}(q) + q^{s+t-1}[n+2-s-t]_q P^{X,Y}_{n,s-1,t-1}(q) & \text{if } n+1 \in X \text{ and } n+1 \notin Y, \text{ and} \\ [s+t+1]_q P^{X,Y}_{n,s,t}(q) + q^{s+t}[n+1-s-t]_q P^{X,Y}_{n,s-1,t}(q) & \text{if } n+1 \in X \text{ and } n+1 \in Y. \end{cases}$$

Next we describe an insertion statistic $\mathrm{stat}_{X,Y}(\sigma)$ so that

$$P^{X,Y}_n(q,x,y) = \sum_{\sigma \in S_n} q^{\mathrm{stat}_{X,Y}(\sigma)} x^{\mathrm{des}_{X,Y}(\sigma)} y^{|Y^c_n|}. \tag{4.12}$$

We define $\mathrm{stat}_{X,Y}(\sigma)$ by recursion. For any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, there are $n+1$ positions where we can insert $n+1$ to obtain a permutation in S_{n+1}. That is, we either insert $n+1$ at the end or immediately before σ_i for $i = 1, \ldots, n$. We next describe a labeling procedure for these possible positions. That is, if $n+1 \notin X$, then we first label positions which are between an X,Y-descent from left to right with the integers from 0 to $\mathrm{des}_{X,Y}(\sigma) - 1$ and then label the remaining positions from right to left with the integers from $\mathrm{des}_{X,Y}(\sigma)$ to n. If $n+1 \in X$, then we label the positions which lie between an X,Y-descent or are immediately in front of an element of Y^c_n plus the position at the end from left to right with the integers $0, \ldots, \mathrm{des}_{X,Y}(\sigma) + |Y^c_n|$ and then label the remaining positions from right to left with the integers $\mathrm{des}_{X,Y}(\sigma) + |Y^c_n| + 1$ to n.

Example 4.6. Suppose that $X_7 = \{2,3,6,7\}$ and $Y_7 = \{1,2,3,4\}$ and $\sigma = 6\ 3\ 1\ 4\ 5\ 7\ 2$. Then if $8 \notin X$, then we would label the positions of σ as

$$\sigma = \underset{7}{-}\ 6\ \underset{0}{-}\ 3\ \underset{1}{-}\ 1\ \underset{6}{-}\ 4\ \underset{5}{-}\ 5\ \underset{4}{-}\ 7\ \underset{2}{-}\ 2\ \underset{3}{-}.$$

If $8 \in X$, then we would label the positions of σ as

$$\sigma = \underset{0}{-}\ 6\ \underset{1}{-}\ 3\ \underset{2}{-}\ 1\ \underset{7}{-}\ 4\ \underset{3}{-}\ 5\ \underset{4}{-}\ 7\ \underset{5}{-}\ 2\ \underset{6}{-}.$$

We then define $\sigma^{(k)}$ to be the permutation in S_{n+1} that is obtained by inserting $n+1$ into the position labeled with a k using the above labeling scheme and recursively define $\mathrm{stat}_{X,Y}(\sigma)$ by declaring that

1. $\mathrm{stat}_{X,Y}(\sigma) = 0$ if $\sigma \in S_1$ and

2. $\mathrm{stat}_{X,Y}(\sigma) = \mathrm{stat}_{X,Y}(\tau) + k$ if $\sigma = \tau^{(k)}$ for some $\tau \in S_n$ if $\sigma \in S_{n+1}$.

Example 4.7. Suppose that $X_7 = \{2, 3, 6, 7\}$ and $Y_7 = \{1, 2, 3, 4\}$ and $\sigma = 6\ 3\ 1\ 4\ 5\ 7\ 2$. Then we can compute $\mathrm{stat}_{X,Y}(\sigma)$ by recursion using the labeling scheme as follows.

σ restricted to $\{1, \ldots, k\}$	Contribution to $\mathrm{stat}_{X,Y}(\sigma)$
$\sigma \restriction_{\{1\}} = {}_{1}\ 1\ {}_{0}$	0
$\sigma \restriction_{\{1,2\}} = {}_{2}\ 1\ {}_{1}\ 2\ {}_{0}$	0
$\sigma \restriction_{\{1,2,3\}} = {}_{3}\ 3\ {}_{0}\ 1\ {}_{2}\ 2\ {}_{1}$	2
$\sigma \restriction_{\{1,2,3,4\}} = {}_{4}\ 3\ {}_{0}\ 1\ {}_{3}\ 4\ {}_{2}\ 2\ {}_{1}$	2
$\sigma \restriction_{\{1,2,3,4,5\}} = {}_{5}\ 3\ {}_{0}\ 1\ {}_{4}\ 4\ {}_{1}\ 5\ {}_{3}\ 2\ {}_{2}$	2
$\sigma \restriction_{\{1,2,3,4,5,6\}} = {}_{0}\ 6\ {}_{1}\ 3\ {}_{2}\ 1\ {}_{6}\ 4\ {}_{3}\ 5\ {}_{5}\ 2\ {}_{4}$	5
$\sigma = 6\ 3\ 1\ 4\ 5\ 7\ 2$	5

Thus $\mathrm{stat}_{X,Y}(\sigma) = 16$ in this case.

Note that for any $\sigma \in S_n$,

$$\sum_{k=0}^{n} q^{\mathrm{stat}_{X,Y}(\sigma^{(k)})} = (1 + q + \cdots + q^n) q^{\mathrm{stat}_{X,Y}(\sigma)} = [n+1]_q q^{\mathrm{stat}_{X,Y}(\sigma)}$$

from which it easily follows by induction that

$$\sum_{\sigma \in S_n} q^{\mathrm{stat}_{X,Y}(\sigma)} = [n]_q!.$$

Thus our statistic is Mahonian. Moreover, it is easy to check that if we define

$$P_{n,s,t}^{X,Y}(q) = \sum_{\sigma \in S_n, \mathrm{des}_{X,Y}(\sigma)=s, |Y_n^c|=t} q^{\mathrm{stat}_{X,Y}(\sigma)}, \tag{4.13}$$

then the $P_{n,s,t}^{X,Y}(q)$'s satisfy the recursions (4.11). For example, suppose that $n+1 \notin X$ and $n+1 \notin Y$. Then to obtain a permutation $\sigma \in S_{n+1}$ which contributes to $P_{n+1,s,t}^{X,Y}(q)$, we can either (i) start with an element $\alpha \in S_n$ such that $\mathrm{des}_{X,Y}(\alpha) = s+1$ and insert $n+1$ in any position that lies between an X,Y-descent in α because that will destroy that X,Y-descent or (ii) start with an element $\beta \in S_n$ such that $\mathrm{des}_{X,Y}(\beta) = s$ and insert $n+1$ in any position other than those that lie between an X,Y-descent in β since such an insertion will preserve the number of X,Y-descents. In case (i), our labeling ensures that such an α will contribute $(1+q+\cdots+q^s)q^{\mathrm{stat}_{X,Y}(\alpha)} = [s+1]_q q^{\mathrm{stat}_{X,Y}(\alpha)}$ to $P_{n+1,s,t}^{X,Y}(q)$ so that we get a total contribution of $[s+1]_q P_{n,s+1,t}^{X,Y}(q)$ to $P_{n+1,s,t}^{X,Y}(q)$ from the permutations in case (i). Similarly, our labeling ensures that each such β contributes $(q^s+\cdots+q^n)q^{\mathrm{stat}_{X,Y}(\beta)} = q^s[n+1-s]_q$ to $P_{n+1,s,t}^{X,Y}(q)$ so that we get a total contribution of $q^s[n+1-s]_q P_{n,s+1,t}^{X,Y}(q)$ to $P_{n+1,s,t}^{X,Y}(q)$ from the permutations in case (ii). The other cases are proved in a similar manner.

It is easy to see that (4.10) implies that

$$\bar{P}_{n+1,s,t}^{X,Y}(q) = \tag{4.14}$$
$$\begin{cases} q^{n-s}[s+1]_q \bar{P}_{n,s+1,t-1}^{X,Y}(q) + [n+1-s]_q \bar{P}_{n,s,t-1}^{X,Y}(q) & \text{if } n+1 \notin X \text{ and } n+1 \notin Y, \\ q^{n-s}[s+1]_q \bar{P}_{n,s+1,t}^{X,Y}(q) + [n+1-s]_q \bar{P}_{n,s,t}^{X,Y}(q) & \text{if } n+1 \notin X \text{ and } n+1 \in Y, \\ q^{n+1-s-t}[s+t]_q \bar{P}_{n,s,t-1}^{X,Y}(q) + [n+2-s-t]_q \bar{P}_{n,s-1,t-1}^{X,Y}(q) & \text{if } n+1 \in X \text{ and } n+1 \notin Y, \text{ and} \\ q^{n-s-t}[s+t+1]_q \bar{P}_{n,s,t}^{X,Y}(q) + [n+1-s-t]_q \bar{P}_{n,s-1,t}^{X,Y}(q) & \text{if } n+1 \in X \text{ and } n+1 \in Y. \end{cases}$$

Again, we can recursively define an insertion statistic $\overline{\mathrm{stat}}_{X,Y}(\sigma)$ so that

$$\bar{P}_x^{X,Y}(q,x,y) = \sum_{\sigma \in S_n} q^{\overline{\mathrm{stat}}_{X,Y}(\sigma)} x^{\mathrm{des}_{X,Y}(\sigma)} y^{|Y_n^c|}. \tag{4.15}$$

The only difference in this case is that if a possible insertion position p was labeled with i relative to $\mathrm{stat}_{X,Y}$, then position p should be labeled with $n-i$ relative to $\overline{\mathrm{stat}}_{X,Y}$. It is easy to see that this labeling can be described as follows. If

$n+1 \notin X$, then we first label positions which are not between an X, Y-descent from left to right with the integers from 0 to $n - \text{des}_{X,Y}(\sigma)$ and then label the remaining positions from right to left with the integers from $n - \text{des}_{X,Y}(\sigma)+1$ to n. If $n+1 \in X$, then we label the positions which do not lie between an X, Y-descent or are not immediately in front of an element of Y_n^c or are not at the end from left to right with the integers $0, \ldots, n-(\text{des}_{X,Y}(\sigma)+|Y_n^c|)-1$ and then label the remaining positions from right to left with the integers $n-(\text{des}_{X,Y}(\sigma)+|Y_n^c|)$ to n.

Example 4.8. Suppose that $X_7 = \{2,3,6,7\}$ and $Y_7 = \{1,2,3,4\}$ and $\sigma = 6\ 3\ 1\ 4\ 5\ 7\ 2$. Then if $8 \notin X$, then we would label the positions of σ as

$$\sigma = \underset{0}{\_}\ 6\ \underset{7}{\_}\ 3\ \underset{6}{\_}\ 1\ \underset{1}{\_}\ 4\ \underset{2}{\_}\ 5\ \underset{3}{\_}\ 7\ \underset{5}{\_}\ 2\ \underset{4}{\_}.$$

If $8 \in X$, then we would label the positions of σ as

$$\sigma = \underset{7}{\_}\ 6\ \underset{6}{\_}\ 3\ \underset{5}{\_}\ 1\ \underset{0}{\_}\ 4\ \underset{4}{\_}\ 5\ \underset{3}{\_}\ 7\ \underset{2}{\_}\ 2\ \underset{1}{\_}.$$

We then define $\sigma^{(\bar{k})}$ to be the permutation in S_{n+1} that is obtained by inserting $n+1$ into the position labeled with a k using the above labeling scheme and recursively define $\overline{\text{stat}}_{X,Y}(\sigma)$ by declaring that

1. $\overline{\text{stat}}_{X,Y}(\sigma) = 0$ if $\sigma \in S_1$ and
2. $\overline{\text{stat}}_{X,Y}(\sigma) = \overline{\text{stat}}_{X,Y}(\tau) + k$ if $\sigma = \tau^{(\bar{k})}$ for some $\tau \in S_n$ if $\sigma \in S_{n+1}$.

Example 4.9. Suppose that $X_7 = \{2,3,6,7\}$ and $Y_7 = \{1,2,3,4\}$ and $\sigma = 6\ 3\ 1\ 4\ 5\ 7\ 2$. Then we can compute $\overline{\text{stat}}_{X,Y}(\sigma)$ by recursion using the labeling scheme as follows.

σ restricted to $\{1,\ldots,k\}$	Contribution to $\overline{\text{stat}}_{X,Y}(\sigma)$
$\sigma \restriction_{\{1\}} = \underset{0}{-}\ 1\ \underset{1}{-}$	0
$\sigma \restriction_{\{1,2\}} = \underset{0}{-}\ 1\ \underset{1}{-}\ 2\ \underset{2}{-}$	1
$\sigma \restriction_{\{1,2,3\}} = \underset{0}{-}\ 3\ \underset{3}{-}\ 1\ \underset{1}{-}\ 2\ \underset{2}{-}$	0
$\sigma \restriction_{\{1,2,3,4\}} = \underset{0}{-}\ 3\ \underset{4}{-}\ 1\ \underset{1}{-}\ 4\ \underset{2}{-}\ 2\ \underset{3}{-}$	1
$\sigma \restriction_{\{1,2,3,4,5\}} = \underset{0}{-}\ 3\ \underset{5}{-}\ 1\ \underset{1}{-}\ 4\ \underset{4}{-}\ 5\ \underset{2}{-}\ 2\ \underset{3}{-}$	2
$\sigma \restriction_{\{1,2,3,4,5,6\}} = \underset{6}{-}\ 6\ \underset{5}{-}\ 3\ \underset{4}{-}\ 1\ \underset{0}{-}\ 4\ \underset{3}{-}\ 5\ \underset{1}{-}\ 2\ \underset{2}{-}$	0
$\sigma = 6\ 3\ 1\ 4\ 5\ 7\ 2$	1

Thus $\overline{\text{stat}}_{X,Y}(\sigma) = 5$ in this case.

Again it is easy to check that

$$\sum_{\sigma \in S_n} q^{\overline{\text{stat}}_{X,Y}(\sigma)} = [n]_q!$$

so that $\overline{\text{stat}}_{X,Y}$ is also a Mahonian statistic. We then define

$$\bar{P}^{X,Y}_{n,s,t}(q) = \sum_{\sigma \in S_n, \text{des}_{X,Y}(\sigma)=s, |Y_n^c|=t} q^{\overline{\text{stat}}_{X,Y}(\sigma)}. \tag{4.16}$$

Again is straightforward to see that the $\bar{P}^{X,Y}_{n,s,t}(q)$ satisfy the recursion (4.14). Finally, it is easy to prove by induction that if $\sigma \in S_n$, then

$$\text{stat}_{X,Y}(\sigma) = \binom{n}{2} - \overline{\text{stat}}_{X,Y}(\sigma). \tag{4.17}$$

It follows that

$$q^{\binom{n}{2}} P^{X,Y}_{n,s,t}(1/q) = \bar{P}^{X,Y}_{n,s,t}(q). \tag{4.18}$$

It is possible to show that one can prove (4.6) and (4.7) from the recursions (4.11) and (4.14). We shall not give such proofs here, but instead give direct combinatorial proofs of (4.6) and (4.7) which will give us non-recursive descriptions of the statistics $\text{stat}_{X,Y}$ and $\overline{\text{stat}}_{X,Y}$.

4.2.2 Combinatorial proofs

In this section, we shall show how to modify the combinatorial proofs of (4.2) and (4.3) found in Hall and Remmel [6] to give combinatorial proofs of (4.6) and (4.7). We start with the proof of (4.6).

Theorem 4.10. *Let*

$$P_n^{X,Y}(q,x) = \sum_{s\geq 0} P_{n,s}^{X,Y}(q)x^s := \sum_{\sigma\in S_n} q^{\mathrm{inv}_{X^c}(\sigma)+\mathrm{rlmaj}_{X,Y}(\sigma)+y^c x\mathrm{coinv}_{X,Y}(\sigma)} x^{\mathrm{des}_{X,Y}(\sigma)},$$

where

$$\begin{aligned} \mathrm{inv}_{X^c}(\sigma) &= \sum_{i=1}^{n} (\#j \in X^c \text{ s.t. } j \text{ appears to the left of } i \text{ and } j > i) \\ \mathrm{rlmaj}_{X,Y}(\sigma) &= \sum_{i\in Des_{X,Y}(\sigma)} (n-i)\,, \text{ and} \\ y^c x\mathrm{coinv}_{X,Y}(\sigma) &= \sum_{x\in X_n} (\#z \in Y^c \text{ s.t. } z \text{ appears to the left of } x \text{ and } z < x)\,. \end{aligned}$$

Then

$$\frac{P_{n,s}^{X,Y}(q)}{[|X_n^c|]_q!} = \sum_{r=0}^{s}(-1)^{s-r}q^{\binom{s-r}{2}} \begin{bmatrix} |X_n^c|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{x\in X_n} [1+r+\alpha_{X,n,x}+\beta_{Y,n,x}]_q\,, \tag{4.19}$$

where

$$\begin{aligned} X_n &= X\cap[n], \\ X_n^c &= (X^c)_n = [n]-X, \end{aligned}$$

and

$$\begin{aligned} \alpha_{X,n,j} &= |X^c\cap\{j+1,j+2,\ldots,n\}| = |\{z : j < z \leq n \;\&\; z\notin X\}|, \\ \beta_{X,n,j} &= |X^c\cap\{1,2,\ldots,j-1\}| = |\{z : 1\leq z < j \;\&\; z\notin X\}|, \text{ and} \\ \beta_{Y,n,j} &= |Y^c\cap\{1,2,\ldots,j-1\}| = |\{z : 1\leq z < j \;\&\; z\notin Y\}|. \end{aligned}$$

Proof. The proofs are analogous to those presented in [6], with the addition of q-weights on the objects of the sign-reversing involutions.

Let X, Y, n, and s be given. For r satisfying $0 \leq r \leq s$, we define the set of what we call $(n,s,r)^{X,Y}$-*configurations*. An $(n,s,r)^{X,Y}$-configuration c consists of an array of the numbers $1, 2, \ldots, n$, r $+$'s, and $(s-r)$ $-$'s, satisfying the following two conditions:

(i) each $-$ is either at the very beginning of the array or immediately follows a number, and

(ii) if $x \in X$ and $y \in Y$ are consecutive numbers in the array, and $x > y$, i.e., if (x,y) forms an (X,Y)-descent pair in the underlying permutation, then there must be at least one $+$ between x and y.

Note that in an $(n,s,r)^{X,Y}$-configuration, the number of $+$'s plus the number of $-$'s equals s.

For example, if $X = \{2,3,5,6\}$ and $Y = \{1,3\}$, the following is a $(6,5,3)^{X,Y}$-configuration:

$$c = 5 + 2 - +46 + 13 - .$$

In this example, the underlying permutation is 5 2 4 6 1 3.

In general, we will let $c_1 c_2 \cdots c_n$ denote the underlying permutation of the $(n,s,r)^{X,Y}$-configuration c.

Let $C_{n,s,r}^{X,Y}$ be the set of all $(n,s,r)^{X,Y}$-configurations. We claim that

$$\left|C_{n,s,r}^{X,Y}\right| = |X_n^c|! \binom{|X_n^c| + r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}).$$

That is, we can construct the $(n,s,r)^{X,Y}$-configurations as follows. First, we pick an order for the elements in X_n^c. This can be done in $|X_n^c|!$ ways. Next, we insert the r $+$'s. This can be done in $\binom{|X_n^c|+r}{r}$ ways. Next, we insert the elements of $X_n = \{x_1 < x_2 < \cdots < x_{|X_n|}\}$ in increasing order. After placing $x_1, x_2, \ldots, x_{i-1}$, the next element x_i can be placed

- immediately before any of the β_{Y,n,x_i} elements of $\{1, 2, \ldots, x_{i-1}\}$ that is not in Y, or
- immediately before any of the α_{X,n,x_i} elements of $\{x_i+1, x_i+2, \ldots, n\}$ that is not in X, or
- immediately before any of the r +'s, or
- at the very end of the array.

Thus we can place the elements of X_n in $\prod_{i=1}^{|X_n|}(1+r+\alpha_{X,n,x}+\beta_{Y,n,x})$ ways. Note that although x_i might also be in Y, and might be placed immediately *after* some other element of X_n, condition (ii) is not violated because the elements of X_n are placed in increasing order. Finally, since each $-$ must occur either at the very start of the configuration or immediately following a number, we can place the $-$'s in $\binom{n+1}{s-r}$ ways.

Let the q-weight $w_q(c)$ of an $(n, s, r)^{X,Y}$-configuration c be

$$(-1)^{s-r} q^{\mathrm{inv}_{X^c}(c) + \mathrm{rlmaj}_{X,Y}(c) + y^c x\mathrm{coinv}_{X,Y}(c)},$$

where

$$\begin{aligned}
\mathrm{inv}_{X^c}(c) &= \sum_{i=1}^{n} (\#j \in X^c \text{ s.t. } j \text{ appears to the left of } i \text{ and } j > i), \\
\mathrm{rlmaj}_{X,Y}(c) &= \sum_{i=1}^{n} (\#\text{signs to the left of } i), \text{ and} \\
y^c x\mathrm{coinv}_{X,Y}(c) &= \sum_{x \in X_n} (\#z \in Y^c \text{ s.t. } z \text{ appears to the left of } x \text{ and } z < x)
\end{aligned}$$

In our example, with $X = \{2, 3, 5, 6\}, Y = \{1, 3\}$, and c the $(6, 5, 3)^{X,Y}$-configuration

$$5 + 2 - +46 + 13-,$$

we have

$$\begin{aligned}\mathrm{inv}_{X^c}(c) &= 0+0+0+0+1+1=2,\\ \mathrm{rlmaj}_{X,Y}(c) &= 0+1+3+3+4+4=15, \text{ and}\\ y^c x\mathrm{coinv}_{X,Y}(c) &= 0+0+3+1=4.\end{aligned}$$

The q-weight of this configuration is thus $w_q(c) = (-1)^{5-3}q^{2+15+4} = q^{21}$.

Next we must show that

$$S_{n,s,r} = \sum_{c \in C^{X,Y}_{n,s,r}} w_q(c) = \tag{4.20}$$
$$\frac{q^{\binom{s-r}{2}}}{(-1)^{s-r}} [|X^c_n|]_q! \begin{bmatrix} |X^c_n|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{x \in X_n} [1+r+\alpha_{X,n,x_i}+\beta_{Y,n,x_i}]_q .$$

We shall use two well-known results to help us prove (4.20). That is, for any sequence $s = s_1 \cdots s_n$ of natural numbers, we let $\mathrm{inv}(s) = \sum_{1\le i<j\le n} \chi(s_i > s_j)$ and $\mathrm{coinv(s)} = \sum_{1\le i<j\le n} \chi(s_i < s_j)$ where for any statement A, $\chi(A) = 1$ is A is true and $\chi(A) = 0$ if A is false. Then for any $n \geq 1$,

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{coinv}(\sigma)} = [n]_q!. \tag{4.21}$$

Similarly we let $\mathcal{R}(1^k 0^{n-k})$ denote the set of rearrangement of k 1's and $n-k$ 0's. Then

$$\sum_{r \in \mathcal{R}(1^k 0^{n-k})} q^{\mathrm{inv}\ (r)} = \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{4.22}$$

If we count the inversions caused by the 1's reading from right to left in any $r \in \mathcal{R}(1^k 0^{n-k})$, it follows that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{0 \le i_1 \le \cdots \le i_k \le n-k} q^{i_1+\cdots+i_k}, \tag{4.23}$$

and if we replace each i_s in (4.23) by $j_s = i_s + s - 1$, then it is easy to see that

$$q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{0 \le j_1 < \cdots < j_k \le n-1} q^{j_1+\cdots+j_k}. \tag{4.24}$$

Now consider how we constructed the elements of $C_{n,s,r}^{X,Y}$ above. We first put down a permutation of the elements of X_n^c. Since each inversion among these elements contributes 1 to $\mathrm{inv}_{X^c}(c)$, these placements contribute a factor of $[|X_n^c|]_q!$ to $S_{n,s,r}$ by (4.21). Next, we insert the r +'s. Since each + contributes 1 for each element of X_n^c to its right to $\mathrm{rlmaj}_{X,Y}(c)$, the q-count over all possible ways of inserting the r +'s into our permutation of X_n^c is the same as the number of inversions between r 1's and $|X_n^c|$ 0's. Thus the insertion of the r +'s contributes a factor of $\begin{bmatrix} |X_n^c|+r \\ r \end{bmatrix}_q$ to $S_{n,s,r}$ by (4.22). Next, we insert the elements of $X_n = \{x_1 < x_2 < \cdots < x_{|X_n|}\}$ in increasing order. After placing $x_1, x_2, \ldots, x_{i-1}$, the next element x_i can go

- immediately before any of the β_{Y,n,x_i} elements of $\{1, 2, \ldots, x_{i-1}\}$ that is not in Y, or
- immediately before any of the α_{X,n,x_i} elements of $\{x_i+1, x_i+2, \ldots, n\}$ that is not in X, or
- immediately before any of the r +'s, or
- at the very end of the array.

Note that each of the elements counted by β_{Y,n,x_i} to left of x_i contributes 1 to $y^c x\mathrm{coinv}_{X,Y}(c)$, each of the elements counted by α_{X,n,x_i} to the left of x_i contributes 1 to $\mathrm{inv}_{X^c}(c)$, and each of the +'s to left of x_i contributes 1 to $\mathrm{rlmaj}_{X,Y}(c)$. Thus it follows that the placement of x_i contributes a factor of $1+q+\cdots+q^{r+\alpha_{X,n,x_i}+\beta_{Y,n,x_i}} = [1+r+\alpha_{X,n,x_i}+\beta_{Y,n,x_i}]_q$ to $S_{n,s,r}$. Finally we must insert the $(s-r)$ $-$'s. Since each $-$ must occur either at the very start of the configuration or immediately following a number and each $-$ contributes the number of elements of $\{1, \ldots n\}$ that lie to its right to $\mathrm{rlmaj}_{X,Y}(c)$, it follows that the contribution over all such placements to $S_{n,s,r}$ is

$$\sum_{0 \le j_1 < \cdots < j_{s-r} \le n} q^{j_1 + \cdots j_{s-r}} = q^{\binom{s-r}{2}} \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \tag{4.25}$$

by (4.24). Thus we have established that the right-hand side of (4.19) is the sum of the $w_q(c)$ over all possible configurations.

We now prove (4.19) by exhibiting a weight-preserving sign-reversing involution I on the set $C^{X,Y}_{n,s} = \bigsqcup_{r=0}^{s} C^{X,Y}_{n,s,r}$, whose fixed points correspond to permutations $\sigma \in S_n$ such that $\mathrm{des}_{X,Y}(\sigma) = s$. We say that a sign can be "reversed" if it can be changed from $+$ to $-$ or from $-$ to $+$ without violating conditions (i) and (ii). To apply I to a configuration c, we scan from left to right until we find the first sign that can be reversed. We then reverse that sign, and we let $I(c)$ be the resulting configuration. If no signs can be reversed, we set $I(c) = c$.

In our example, with $X = \{2,3,5,6\}, Y = \{1,3\}$, and c the $(6,5,3)^{X,Y}$-configuration

$$5 + 2 - +46 + 13-,$$

the first sign we encounter is the $+$ following 5. This $+$ can be reversed, since 52 is not an (X,Y)-descent. Thus $I(c)$ is the configuration shown below:

$$I(c) = 5 - 2 - +46 + 13 - .$$

It is easy to see that $I(I(c)) = c$ in this case, since applying I again we change the $-$ following 5 back to a $+$.

Conditions (i) and (ii) are clearly preserved by the very definition of I. It is also clear that I is sign-reversing, since if $I(c) \neq c$, then $I(c)$ either has one more $-$ than c, or one fewer $-$ than c. I also preserves the q-weight, since $\mathrm{inv}_{X^c}(c), \mathrm{rlcomaj}_{X,Y}(c)$, and $y^c x\mathrm{coinv}_{X,Y}(c)$ depend only on the underlying permutation and the distribution of signs (without regard to $+$ or $-$), neither of which is changed by I. To see that I is in fact an involution, we note that the only signs that are *not* reversible are single $+$'s occurring in the middle of an (X,Y)-descent pair, and $+$'s that immediately follow another sign. In either case, it is clear that a sign is reversible in a configuration c if and only if the corresponding sign is reversible in $I(c)$. Thus, if a sign is the first reversible sign in c, the corresponding sign in $I(c)$ must also be the first reversible sign in $I(c)$. It follows that $I(I(c)) = c$

for all $c \in C_{n,s}^{X,Y}$. We therefore have

$$\sum_{r=0}^{s} \sum_{c \in C_{n,s,r}^{X,Y}} w_q(c) = \sum_{r=0}^{s} \sum_{\substack{c \in C_{n,s,r}^{X,Y} \\ I(c) = c}} w_q(c).$$

Now, consider the fixed points of I. Suppose that $I(c) = c$. Then c clearly can have no $-$'s, and thus $r = s$. This implies that the sign associated with the configuration c is positive. It must also be the case that no $+$'s can be reversed. Thus each of the s $+$'s must occur singly in the middle of an (X,Y)-descent pair. It follows that the underlying permutation has exactly s (X,Y)-descents.

Finally, we should observe that if $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a permutation with exactly s (X,Y)-descents, then we can create a fixed point of I simply by placing a $+$ in the middle of each (X,Y)-descent pair.

For example, if $X = \{2,4,6,9\}$, $Y = \{1,4,7\}$, $n = 9$, $s = 2$, and $\sigma = 528941637$, then the corresponding fixed point is

$$c = 5289 + 4 + 1637.$$

Note that in such a case, if $\sigma_i > \sigma_{i+1}$ is an X,Y-descent in σ, then in the corresponding fixed point c of I, we will have a $+$ between σ_i and σ_{i+1}. This means that each of $\sigma_{i+1}, \ldots, \sigma_n$ will contribute 1 to $\text{rlmaj}_{X,Y}(c)$ for the $+$ between σ_i and σ_{i+1}. Hence it follows that

$$\text{rlmaj}_{X,Y}(c) = \sum_{i \in \text{Des}_{X,Y}(\sigma)} (n-i).$$

Thus for any permutation $\sigma \in S_n$ with s X,Y-descents, the weight $w_q(c)$ of the corresponding configuration c which is a fixed point of I is

$$q^{\text{inv}_{X^c}(\sigma)+\text{rlmaj}_{X,Y}(\sigma)+y^c x\text{coinv}_{X,Y}(\sigma)}$$

where

$$\begin{aligned} \text{inv}_{X^c}(\sigma) &= \sum_{i=1}^{n} (\#j \in X^c \text{ s.t. } j \text{ appears to the left of } i \text{ and } j > i) \\ \text{rlmaj}_{X,Y}(\sigma) &= \sum_{i \in \text{Des}_{X,Y}(\sigma)} (n-i)\,, \text{ and} \\ y^c x\text{coinv}_{X,Y}(\sigma) &= \sum_{x \in X_n} (\#z \in Y^c \text{ s.t. } z \text{ appears to the left of } x \text{ and } z < x) \end{aligned}$$

as desired. □

Our next result shows that the statistics defined in 4.2.1 and 4.2.2 are in fact the same.

Theorem 4.11. *For all $\sigma \in S_n$,*

$$\text{stat}_{X,Y}(\sigma) = \text{inv}_{X^c}(\sigma) + \text{rlmaj}_{X,Y}(\sigma) + y^c x\text{coinv}_{X,Y}(\sigma)$$

Proof. By definition, it is the case that

$$\text{stat}_{X,Y}(\sigma^{(k)}) = \text{stat}_{X,Y}(\sigma) + k.$$

We will now show that

$$\begin{aligned} & \text{inv}_{X^c}(\sigma^{(k)}) + \text{rlmaj}_{X,Y}(\sigma^{(k)}) + y^c x\text{coinv}_{X,Y}(\sigma^{(k)}) \\ = \; & \text{inv}_{X^c}(\sigma) + \text{rlmaj}_{X,Y}(\sigma) + y^c x\text{coinv}_{X,Y}(\sigma) + k, \end{aligned}$$

which when combined with the fact that all of the statistics are 0 on $\sigma \in S_1$, verifies the theorem. Suppose $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ where $\text{des}_{X,Y}(\sigma) = s$ and $|Y_n^c| = t$.

Case 1: $n+1 \notin X$ **and** $\textbf{des}_{X,Y}(\sigma^{(k)}) = s-1$

Assume further that $\sigma^{(k)}_{j+1} = n+1$. Inserting $n+1$ at position $j+1$ means that it will contribute $n-j$, the number of elements following $n+1$, to the inv_{X^c} statistic. So we have that

$$\text{inv}_{X^c}(\sigma^{(k)}) = \text{inv}_{X^c}(\sigma) + n - j.$$

Since we destroyed an X, Y-descent at position j we lose a contribution of $n-j$ to the $\text{rlmaj}_{X,Y}$ statistic. On the other hand, each of the k X, Y-descents preceding $n+1$ will contribute 1 to this statistic. Thus,

$$\text{rlmaj}_{X,Y}(\sigma^{(k)}) = \text{rlmaj}_{X,Y}(\sigma) - (n-j) + k.$$

Since $n+1 \notin X$, we have that

$$y^c x\text{coinv}_{X,Y}(\sigma^{(k)}) = y^c x\text{coinv}_{X,Y}(\sigma).$$

Case 2: $n+1 \notin X$ **and** $\textbf{des}_{X,Y}(\sigma^{(k)}) = s$

Assume further that there are d X, Y-descents to the left of $n+1$ in $\sigma^{(k)}$. Inserting $n+1$ at a position labeled k means that it will contribute $k-s+s-d = k-d$, the number of elements following $n+1$, to the inv_{X^c} statistic. So we have that

$$\text{inv}_{X^c}(\sigma^{(k)}) = \text{inv}_{X^c}(\sigma) + k - d.$$

Since we are dealing with a permutation of length $n+1$, each of the d X, Y-descents preceding $n+1$ will contribute 1 to the $\text{rlmaj}_{X,Y}$ statistic. Thus,

$$\text{rlmaj}_{X,Y}(\sigma^{(k)}) = \text{rlmaj}_{X,Y}(\sigma) + d.$$

Again, since $n+1 \notin X$, we have that

$$y^c x\text{coinv}_{X,Y}(\sigma^{(k)}) = y^c x\text{coinv}_{X,Y}(\sigma).$$

Case 3: $n+1 \in X$ **and** $\textbf{des}_{X,Y}(\sigma^{(k)}) = s$

Assume further that there are d X, Y-descents to the left of $n+1$ in $\sigma^{(k)}$. Since $n+1 \in X$, we have that

$$\text{inv}_{X^c}(\sigma^{(k)}) = \text{inv}_{X^c}(\sigma).$$

Each of the d X, Y-descents preceding $n+1$ will contribute 1 to the $\text{rlmaj}_{X,Y}$ statistic. Thus,

$$\text{rlmaj}_{X,Y}(\sigma^{(k)}) = \text{rlmaj}_{X,Y}(\sigma) + d.$$

Inserting $n+1$ at a position labeled k means that it will contribute $k-d$, the number of elements preceding $n+1$ that are in Y_n^c, to the statistic $y^c x\text{coinv}_{X,Y}$. So we have that

$$y^c x\text{coinv}_{X,Y}(\sigma^{(k)}) = y^c x\text{coinv}_{X,Y}(\sigma) + k - d.$$

Case 4: $n+1 \in X$ **and** $\mathbf{des}_{X,Y}(\sigma^{(k)}) = s+1$

Assume further that there are d X,Y-descents to the left of $n+1$ in $\sigma^{(k)}$ and that $\sigma_{j+1}^{(k)} = n+1$. Again, since $n+1 \in X$, we have that

$$\text{inv}_{X^c}(\sigma^{(k)}) = \text{inv}_{X^c}(\sigma).$$

However, since we have created an X,Y-descent at position $j+1$ we gain a contribution of $n+1-(j+1) = n-j$ to the $\text{rlmaj}_{X,Y}$ statistic. On the other hand, each of the d X,Y-descents preceding $n+1$ will also contribute 1 to this statistic. Thus,

$$\text{rlmaj}_{X,Y}(\sigma^{(k)}) = \text{rlmaj}_{X,Y}(\sigma) + d + (n-j).$$

Inserting $n+1$ at a position labeled k means that it will contribute $j-(n-k)-d$, the number of elements preceding $n+1$ that are in Y_n^c, to the statistic $y^c x\text{coinv}_{X,Y}$. So we have that

$$y^c x\text{coinv}_{X,Y}(\sigma^{(k)}) = y^c x\text{coinv}_{X,Y}(\sigma) + j - (n-k) - d.$$

In each case we see that inserting $n+1$ into the position labeled with a k contributes k to each statistic and thus they are equivalent. □

Next we consider the proof of (4.7).

Theorem 4.12. *Let*

$$\bar{P}_n^{X,Y}(q,x) = \sum_{s\geq 0} \bar{P}_{n,s}^{X,Y}(q)x^s := \sum_{\sigma\in S_n} q^{\text{coinv}_{X^c}(\sigma)+\text{rlcomaj}_{X,Y}(\sigma)-y^c x\text{coinv}(\sigma)} x^{\text{des}_{X,Y}(\sigma)},$$

where

$$\begin{aligned}\text{coinv}_{X^c}(\sigma) &= \sum_{i=1}^{n} (\#j \in X^c \text{ s.t. } j \text{ appears to the left of } i \text{ and } j < i) \\ \text{rlcomaj}_{X,Y}(\sigma) &= \sum_{i \notin Des_{X,Y}(\sigma), \sigma_i \in X_n} (n-i)\,, \text{ and} \\ y^c x\text{coinv}_{X,Y}(\sigma) &= \sum_{x \in X_n} (\#z \in Y^c \text{ s.t. } z \text{ appears to the left of } x \text{ and } z < x)\,.\end{aligned}$$

Then

$$\frac{\bar{P}^{X,Y}_{n,s}(q)}{[|X^c_n|]_q!} = \tag{4.26}$$
$$\sum_{r=0}^{|X_n|-s} \frac{q^{\binom{|X_n|-s-r}{2}}}{(-1)^{|X_n|-s-r}} \begin{bmatrix} |X^c_n|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ |X_n|-s-r \end{bmatrix}_q \prod_{x \in X_n} [r + \beta_{X,n,x} - \beta_{Y,n,x}]_q\,,$$

where

$$\begin{aligned} X_n &= X \cap [n], \\ X^c_n &= (X^c)_n = [n] - X, \end{aligned}$$

and

$$\begin{aligned} \alpha_{X,n,j} &= |X^c \cap \{j+1, j+2, \ldots, n\}| = |\{z : j < z \leq n \ \&\ z \notin X\}|, \\ \beta_{X,n,j} &= |X^c \cap \{1, 2, \ldots, j-1\}| = |\{z : 1 \leq z < j \ \&\ z \notin X\}|, \text{ and} \\ \beta_{Y,n,j} &= |Y^c \cap \{1, 2, \ldots, j-1\}| = |\{z : 1 \leq z < j \ \&\ z \notin Y\}|. \end{aligned}$$

Proof. Let X, Y, n, and s be given. For r satisfying $0 \leq r \leq |X_n| - s$, an $\overline{(n,s,r)}^{X,Y}$-*configuration* consists of an array of the numbers $1, 2, \ldots, n$, r $+$'s, and $(|X_n|-s-r)$ $-$'s, satisfying the following three conditions:

(i) each $-$ is either at the very beginning of the array or immediately follows a number,

(ii) if $c_i \in X, 1 \leq i < n$, and (c_i, c_{i+1}) is not an (X,Y)-descent pair of the underlying permutation, then there must be at least one $+$ between c_i and c_{i+1}, and

(iii) if $c_n \in X$, then at least one $+$ must occur to the right of c_n.

Note that in an $\overline{(n,s,r)}^{X,Y}$-configuration, the number of $+$'s plus the number of $-$'s equals $|X_n| - s$.

For example, if $X = \{2,3,6\}$ and $Y = \{1,2,5\}$, then the following is a $\overline{(6,1,1)}^{X,Y}$-configuration:

$$c = 213+6-54.$$

Let $\overline{C}^{X,Y}_{n,s,r}$ be the set of all $\overline{(n,s,r)}^{X,Y}$-configurations. Then we claim that

$$\left|\overline{C}^{X,Y}_{n,s,r}\right| = |X_n^c|!\binom{|X_n^c|+r}{r}\binom{n+1}{|X_n|-s-r}\prod_{x\in X_n}(r+\beta_{X,n,x}-\beta_{Y,n,x}).$$

That is, we can construct the $\overline{(n,s,r)}^{X,Y}$-configurations as follows. First, we pick an order for the elements in X_n^c. This can be done in $|X_n^c|!$ ways. Next, we insert the r $+$'s. This can be done in $\binom{|X_n^c|+r}{r}$ ways. Next, we insert the elements of $X_n = \{x_1 < x_2 < \cdots < x_{|X_n|}\}$ in increasing order. We can place x_1 in $r + \beta_{X,n,x_1} - \beta_{Y,n,x_1}$ ways, since x_1 can either go immediately before any of the r $+$'s or immediately before any of the $x_1 - 1 - \beta_{Y,n,x_1} = \beta_{X,n,x_1} - \beta_{Y,n,x_1}$ elements of Y which are less than x_1. We note here that $\beta_{X,n,x_i} = x_i - i$ for all $i, 1 \le i \le |X_n|$. There are now two cases to consider for the placement of x_2.

Case 1. x_1 was placed immediately in front of some element of $y \in Y$. In this case, x_2 cannot be placed immediately in front of y, since this would violate condition (ii). x_2 can be placed before any $+$ or immediately in front of any element of Y which is less than x_2, except y. Hence, x_2 can be placed in

$$\begin{aligned} r + x_2 - 1 - \beta_{Y,n,x_2} - 1 &= r + x_2 - 2 - \beta_{Y,n,x_2} \\ &= r + \beta_{X,n,x_2} - \beta_{Y,n,x_2} \end{aligned}$$

ways.

Case 2. x_1 was placed immediately before a $+$. In this case, x_2 cannot be placed immediately before the same $+$, since again we would violate condition (ii). x_2 can be placed immediately before any of the other $+$'s or immediately before any element of Y which is less than x_2. Hence x_2 can be placed in

$$\begin{aligned} r-1+x_2-1-\beta_{Y,n,x_2} &= r+x_2-2-\beta_{Y,n,x_2} \\ &= r+\beta_{X,n,x_2}-\beta_{Y,n,x_2} \end{aligned}$$

ways.

In general, having placed $x_1, x_2, \ldots, x_{i-1}$, we cannot place x_i immediately before some $y \in Y, y < x_i$, which earlier had an element of $\{x_1, x_2, \ldots, x_{i-1}\}$ placed before it. Similarly, we cannot place x_i immediately before any $+$ which earlier had an element of $\{x_1, x_2, \ldots, x_{i-1}\}$ placed before it. It then follows that there are

$$\begin{aligned} r+x_i-1-\beta_{Y,n,x_i}-(i-1) &= r+x_i-i-\beta_{Y,n,x_i} \\ &= r+\beta_{X,n,x_i}-\beta_{Y,n,x_i} \end{aligned}$$

ways to place x_i. Thus, there are total of $\prod_{i=1}^{|X_n|}(r+\beta_{X,n,x_i}-\beta_{Y,n,x_i})$ ways to place $x_1, x_2, \ldots, x_{|X_n|}$, given our placement of the elements of X_n^c. Finally, we can place the $-$'s in $\binom{n+1}{|X_n|-s-r}$ ways.

Now suppose that $c = c_1 \cdots c_n$ is a configuration in $\overline{C}_{n,s,r}^{X,Y}$. We define the q-weight $\bar{w}_q(c)$ of an $\overline{(n,s,r)}^{X,Y}$-configuration c to be

$$(-1)^{|X_n|-s-r} q^{\mathrm{coinv}_{X^c}(c)+\mathrm{rlcomaj}_{X,Y}(c)-y^c x\mathrm{coinv}_{X,Y}(c)},$$

where

$$\begin{aligned} \mathrm{coinv}_{X^c}(c) &= \sum_{i=1}^{n} (\# j \in X^c \text{ s.t. } j \text{ appears to the left of } i \text{ and } j < i) \\ \mathrm{rlcomaj}_{X,Y}(c) &= \sum_{i=1}^{n} (\#\text{signs to the left of } i)\,, \text{ and} \\ y^c x\mathrm{coinv}_{X,Y}(c) &= \sum_{x \in X_n} (\# z \in Y^c \text{ s.t. } z \text{ appears to the left of } x \text{ and } z < x)\,. \end{aligned}$$

In our example, with $X = \{2,3,4\}, Y = \{1,3,5\}$, and c the $\overline{(6,0,3)}^{X,Y}$-configuration

$$2 + 5413 + +6,$$

we have,

$$\begin{aligned}\text{coinv}_{X^c}(c) &= 3, \\ \text{rlcomaj}_{X,Y}(c) &= 7, \text{ and} \\ y^c x\text{coinv}_{X,Y}(c) &= 2.\end{aligned}$$

The q-weight of this configuration is thus $\bar{w}_q(c) = (-1)^{3-0-3} q^{3+7-2} = q^8$.

Note that the definition of $\text{rlcomaj}_{X,Y}$ is identical to that of $\text{rlmaj}_{X,Y}$ of Theorem 4.10. However, because the signs play a different role here, it will not be the case that $\text{rlmaj}_{X,Y}$ and $\text{rlcomaj}_{X,Y}$ reduce to the same statistics on S_n.

Let

$$\bar{S}_{n,s,r} = \sum_{c \in \overline{C}^{X,Y}_{n,s,r}} \bar{w}_q(c).$$

We must show that

$$\bar{S}_{n,s,r} = \tag{4.27}$$
$$\frac{q^{\binom{|X_n|-s-r}{2}} [|X_n^c|]_q!}{(-1)^{|X_n|-s-r}} \begin{bmatrix} |X_n^c| + r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ |X_n| - s - r \end{bmatrix}_q \prod_{x \in X_n} [r + \beta_{X,n,x_i} - \beta_{Y,n,x}]_q .$$

Now consider how we constructed the elements of $\overline{C}^{X,Y}_{n,s,r}$ above. We first put down a permutation of the elements of X_n^c. Since each coinversion among these elements contributes 1 to $\text{coinv}_{X^c}(c)$, these elements contribute a factor of $[|X_n^c|]_q!$ to $\bar{S}_{n,s,r}$. Next, we put down the r +'s. Since each + contributes 1 for each element of X_n^c to its right to $\text{rlcomaj}_{X,Y}(c)$, the q-count over all possible ways of inserting the r +'s into our permutation of X_n^c is the same as the number of inversions between r 1's and $|X_n^c|$ 0's. Thus the insertion of the r +'s contributes a factor of $\begin{bmatrix} |X_n^c| + r \\ r \end{bmatrix}_q$ to $\bar{S}_{n,s,r}$. Next, we insert the elements of $X_n = \{x_1 < x_2 < \cdots < x_{|X_n|}\}$ in increasing order. Each x_i must go either before a $+$ or before an element $y \in Y$ such that $y < x_i$. However, having placed $x_1, x_2, \ldots, x_{i-1}$, we cannot place x_i immediately before some $y \in Y, y < x_i$, which earlier had an element of $\{x_1, x_2, \ldots, x_{i-1}\}$ placed before it. Similarly, we cannot place x_i immediately before any $+$ which earlier had an element of $\{x_1, x_2, \ldots, x_{i-1}\}$ placed before it. So the number of ways to place x_i is the difference between the sum of the number of $+$'s and the number of elements smaller than x_i which are in Y, and the number of $x \in X$ already placed, which is $i-1$. This difference is $r+x_i-1-\beta_{Y,n,x_i}-(i-1) = r+\beta_{X,n,x_i}-\beta_{Y,n,x_i}$. Now when we place x_i, each $+$ to the left of x_i contributes 1 to $\mathrm{rlcomaj}_{X,Y}(c)$, each $z \in X^c, z < x_i$, to the left of x_i contributes 1 to $\mathrm{coinv}_{X^c}(c)$, and each $z \in Y^c, z < x_i$, contributes 1 to $y^c x\mathrm{coinv}_{X,Y}(\mathrm{c})$. The key here is to realize that the difference between the number of $y \in Y, y < x_i$ to the left of x_i and the number of $x \in X, x < x_i$ to the left of x_i is equal to the difference between the number of $z \in X_n^c, z < x_i$ to the left of x_i and the number of $z \in Y_n^c, y < x_i$ to the left of x_i. In any rate, when we place x_i, we effectively get a factor of q for each $+$ and each element $y \in Y$ satisfying $y < x_i$ which lies to the left of x_i, and a factor of q^{-1} for each element $x \in X$ satisfying $x < x_i$ which lies to the left of x_i. Thus the net effect is that we have a factor of q for each place before the position of x_i which was a possible position where we could have placed x_i. By our argument above, there are exactly $r+\beta_{X,n,x_i}-\beta_{Y,n,x_i}$ positions so that the contribution over all possible placements of x_i at this stage is $1+q+\cdots q^{r+\beta_{X,n,x_i}-\beta_{Y,n,x_i}-1} = [r+\beta_{X,n,x_i}-\beta_{Y,n,x_i}]_q$. Thus the total contribution from the placements of elements in X_n is $\prod_{x\in X_n}[r+\beta_{X,n,x}-\beta_{Y,n,x}]_q$. Finally we must insert the $(|X_n|-s-r)$ $-$'s. In this case, we can argue exactly in the proof of Theorem 4.10, that all possible insertions contribute a factor of

$$\sum_{0\le j_1<\cdots<j_{|X_n|-s-r}\le n} q^{j_1+\cdots j_{|X_n|-s-r}} = q^{\binom{|X_n|-s-r}{2}} \begin{bmatrix} n+1 \\ |X_n|-s-r \end{bmatrix}_q \tag{4.28}$$

to $\bar{S}_{n,s,r}$. Thus we have established that the right-hand side of (4.26) is the sum of the $\bar{w}_q(c)$ over all possible configurations.

We now prove the theorem by exhibiting a sign-reversing involution I on the set $\overline{C}_{n,s}^{X,Y} = \bigsqcup_{r=0}^{|X_n|-s} \overline{C}_{n,s,r}^{X,Y}$ whose fixed points correspond to permutations $\sigma \in S_n$ such that $\mathrm{des}_{X,Y}(\sigma) = s$. We define I exactly as in the proof of Theorem 4.10. That is, we scan from left to right and reverse the first sign that we can reverse without violating conditions (i)-(iii).

For example, suppose $X = \{2, 3, 4\}$, $Y = \{1, 3, 5\}$, and we have the $\overline{(6,0,2)}^{X,Y}$-configuration

$$c = 213 + 6 - 54 + .$$

We cannot reverse the $+$ following 3 without violating condition (ii), since $3 \in X$ and 36 is not an (X,Y)-descent. Thus, we reverse the $-$ following 6 to get

$$I(c) = 213 + 6 + 54 + .$$

It is clear that $\bar{w}_q(c) = \pm \bar{w}_q(I(c))$, since $\mathrm{coinv}_{X^c}(c) = \mathrm{coinv}_{X^c}(I(c))$, $\mathrm{rlcomaj}_{X,Y}(c) = \mathrm{rlcomaj}_{X,Y}(I(c))$, and $y^c x\mathrm{coinv}_{X,Y}(c) = y^c x\mathrm{coinv}_{X,Y}(I(c))$. If $I(c) \neq c$, then $I(c)$ either has one more $-$ than c, or one fewer $-$ than c, and so I is sign-reversing weight preserving involution.

It follows that

$$\sum_{r=0}^{|X_n|-s} \sum_{c \in \overline{C}_{n,s,r}^{X,Y}} \bar{w}_q(c) = \sum_{r=0}^{|X_n|-s} \sum_{\substack{c \in \overline{C}_{n,s,r}^{X,Y} \\ I(c) = c}} \bar{w}_q(c).$$

Now, consider a fixed point c of I. As in the proof of Theorem 4.10, c can have no $-$'s, and thus $r = |X_n| - s$ and thus $w(c)$ is positive. No string of multiple $+$'s can occur, since the first $+$ in such a string could be reversed. Thus, each of the $(|X_n| - s)$ $+$'s appears singly, and must either

- immediately follow some $c_i \in X, 1 \leq i < n$, such that (c_i, c_{i+1}) is not an (X,Y)-descent pair of the underlying permutation, or
- immediately follow $c_n \in X$.

Thus $|X_n|-s$ elements of X_n immediately precede a $+$ that cannot be reversed, and are thus not the tops of (X,Y)-descent pairs. It follows that each of the remaining s elements of X_n do not immediately precede a $+$, and as such each must be the top of an (X,Y)-descent pair. Thus the underlying permutation $c_1c_2\cdots c_n$ has exactly s (X,Y)-descents.

Again, we observe that if $\sigma_1\sigma_2\cdots\sigma_n$ is a permutation with exactly s (X,Y)-descents, then we can create a fixed point of I by inserting a $+$ after every element of X_n that is not the top of an (X,Y)-descent pair. For example, if $X=\{2,3,4,6,8,9\}$, $Y=\{1,2,3,5\}$, $n=9$, $s=4$, and $\sigma = 9\ 5\ 8\ 6\ 2\ 1\ 4\ 3\ 7$, then the corresponding configuration would be

$$958+62143+7.$$

Finally, observe that if such a σ corresponds to a fixed point of I, then for every pair (σ_i,σ_{i+1}) such that $\sigma_i\in X_n$ and (σ_i,σ_{i+1}) is not a (X,Y)-descent pair, the $+$ between σ_i and σ_{i+1} in the corresponding configuration contributes 1 for each of $\sigma_{i+1},\ldots,\sigma_n$. It then follows that

$$\mathrm{rlcomaj}_{X,Y}(c) = \sum_{\sigma_i\in X_n, i\notin DesX,Y(\sigma)} (n-i) = \mathrm{rlcomaj}_{X,Y}(\sigma).$$

□

For any permutation σ, we now have

$$\begin{aligned}
&\mathrm{stat}(\sigma)+\mathrm{coinv}_{X^c}(\sigma)+\mathrm{rlcomaj}_{X,Y}(\sigma)-y^cx\mathrm{coinv}(\sigma) = \\
&\mathrm{inv}_{X^c}(\sigma)+\mathrm{rlmaj}_{X,Y}(\sigma)+y^cx\mathrm{coinv}(\sigma)+ \\
&\mathrm{coinv}_{X^c}(\sigma)+\mathrm{rlcomaj}_{X,Y}(\sigma)-y^cx\mathrm{coinv}(\sigma) = \\
&\mathrm{inv}_{X^c}(\sigma)+\mathrm{coinv}_{X^c}(\sigma)+\mathrm{rlmaj}_{X,Y}(\sigma)+\mathrm{rlcomaj}_{X,Y}(\sigma).
\end{aligned}$$

But for any $\sigma=\sigma_1\cdots\sigma_n$, $i\in\{1,\ldots,n\}$, and $X,Y\subseteq\mathbb{N}$, it will be that case that if $\sigma_i\in X_n^c$, then we get a contribution of 1 to $\mathrm{inv}_{X^c}(\sigma)+\mathrm{coinv}_{X^c}(\sigma)$ for each σ_j with $j>i$ so that

$$\mathrm{inv}_{X^c}(\sigma)+\mathrm{coinv}_{X^c}(\sigma) = \sum_{\sigma_i\in X_n^c}(n-i). \tag{4.29}$$

On the other hand, it easy to see from our definition that

$$\text{rlmaj}_{X,Y}(\sigma) + \text{rlcomaj}_{X,Y}(\sigma) = \sum_{\sigma_i \in X_n} (n-i). \tag{4.30}$$

Thus

$$\text{inv}_{X^c}(\sigma) + \text{coinv}_{X^c}(\sigma) + \text{rlmaj}_{X,Y}(\sigma) + \text{rlcomaj}_{X,Y}(\sigma) = \sum_{i=1}^{n}(n-i) = \binom{n}{2}. \tag{4.31}$$

It thus follows that

$$\text{coinv}_{X^c}(\sigma) + \text{rlcomaj}_{X,Y}(\sigma) - y^c x \text{coinv}(\sigma) = \binom{n}{2} - \text{stat}(\sigma) = \overline{\text{stat}}(\sigma).$$

Thus we have proved the following theorem.

Theorem 4.13. *For all permutations σ,*

$$\overline{stat}(\sigma) = coinv_{X^c}(\sigma) + rlcomaj_{X,Y}(\sigma) - y^c x\, coinv(\sigma). \tag{4.32}$$

Remark 4.14. Note that we have shown that $stat_{X,Y} = inv_{X^c} + rlmaj_{X,Y} + y^c x\, coinv_{X,Y}$ is a Mahonian statistic for all X and Y, and interpolates between inv, $rlmaj$, and $coinv$, in the sense that $stat_{\emptyset,\emptyset}(\sigma) = inv(\sigma)$, $stat_{\mathbb{N},\mathbb{N}}(\sigma) = rlmaj(\sigma)$, and $stat_{\mathbb{N},\emptyset}(\sigma) = coinv(\sigma)$. Similarly, $\overline{stat}_{X,Y} = coinv_{X^c} + rlcomaj_{X,Y} - y^c x\, coinv_{X,Y}$ is a Mahonian statistic for all X and Y, and interpolates between $coinv$ and $rlcomaj$, in the sense that $\overline{stat}_{\emptyset,\emptyset}(\sigma) = coinv(\sigma)$ and $\overline{stat}_{\mathbb{N},\mathbb{N}}(\sigma) = rlcomaj(\sigma)$.

4.2.3 Applications

Note that the results of 4.2.1 and 4.2.2 show that for all $X, Y \subseteq \mathbb{N}$ and $n \geq 1$,

$$\bar{P}_{n,s}^{X,Y}(q) = q^{\binom{n}{2}} P_{n,s}^{X,Y}(1/q).$$

But then by Theorem 4.10, we have that

$$\frac{\bar{P}_{n,s}^{X,Y}(q)}{[|X_n^c|]_{1/q}!} = \tag{4.33}$$

$$q^{\binom{n}{2}} \sum_{r=0}^{s} \frac{q^{-\binom{s-r}{2}}}{(-1)^{s-r}} \begin{bmatrix} |X_n^c| + r \\ r \end{bmatrix}_{1/q} \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_{1/q} \prod_{x \in X_n} [1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}]_{1/q}$$

But then we can use the identities

$$\begin{aligned} [n]_{1/q} &= q^{-(n-1)} [n]_q , \\ [n]_{1/q}! &= q^{-\binom{n}{2}}([n]_q!), \text{ and} \\ \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} &= q^{\binom{k}{2}+\binom{n-k}{2}-\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \end{aligned}$$

to rewrite (4.33) as

$$\frac{\bar{P}^{X,Y}_{n,s}(q)}{[|X^c_n|]_q!} = \sum_{r=0}^{s} \frac{q^A}{(-1)^{s-r}} \begin{bmatrix} |X^c_n|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{x\in X_n} \frac{[1+r+\alpha_{X,n,x}+\beta_{Y,n,x}]_q}{q^{(r+\alpha_{X,n,x}+\beta_{Y,n,x})}} \tag{4.34}$$

where

$$\begin{aligned} A &= \binom{n}{2} - \binom{|X^c_n|}{2} - \binom{s-r}{2} + \binom{r}{2} + \binom{|X^c_n|}{2} - \binom{|X^c_n|+r}{2} + \\ &\quad \binom{s-r}{2} + \binom{n+1-(s-r)}{2} - \binom{n+1}{2} \\ &= \binom{r}{2} - \binom{|X^c_n|+r}{2} + \binom{n+1-(s-r)}{2} - n \\ &= \binom{r}{2} - \left(\binom{|X^c_n|}{2} + \binom{r}{2} + r|X^c_n|\right) + \binom{n+1-(s-r)}{2} - n \\ &= \binom{n+1-(s-r)}{2} - \binom{|X^c_n|}{2} - n - r|X^c_n|. \end{aligned}$$

We can then factor out a q^r for each term in the product to get a factor of $q^{-r|X_n|}$ which can be combined with the factor $q^{-r|X^c_n|}$ to obtain a factor $q^{-r(|X_n|+|X^c_n|)} = q^{-rn}$. It then follows that

$$\bar{P}^{X,Y}_{n,s}(q) = \tag{4.35}$$
$$[|X^c_n|]_q! \sum_{r=0}^{s} (-1)^{s-r} q^{\binom{n+1-(s-r)}{2} - \binom{|X^c_n|}{2} - (r+1)n} \begin{bmatrix} |X^c_n| + r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \times$$
$$\prod_{x \in X_n} q^{-(\alpha_{X,n,x} + \beta_{Y,n,x})} \left[1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}\right]_q$$

Comparing (4.35) and (4.26), we have proved the following identity.

Theorem 4.15. *For all $X, Y \subseteq \mathbb{N}$ and $1 \le s \le n$,*

$$[|X^c_n|]_q! \sum_{r=0}^{s} (-1)^{s-r} q^{\binom{n+1-(s-r)}{2} - \binom{|X^c_n|}{2} - (r+1)n} \begin{bmatrix} |X^c_n| + r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \times$$
$$\prod_{x \in X_n} q^{-(\alpha_{X,n,x} + \beta_{Y,n,x})} \left[1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}\right]_q =$$
$$[|X^c_n|]_q! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} q^{\binom{|X_n|-s-r}{2}} \begin{bmatrix} |X^c_n| + r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ |X_n| - s - r \end{bmatrix}_q \times$$
$$\prod_{x \in X_n} \left[r + \beta_{X,n,x} - \beta_{Y,n,x}\right]_q$$

For some sets X and Y this identity can be rewritten in terms of hypergeometric series; hence we obtain combinatorial proofs of identities such as the following, which is a special case of an integral form of a transformation of Karlsson-Minton type basic hypergeometric series due to Gasper [11]. Here we employ the notation of basic hypergeometric series, in particular

$${}_{m+1}\phi_m \left[\begin{matrix} a_0, & a_1, & a_2, & \ldots, & a_m \\ & b_1, & b_2, & \ldots, & b_m \end{matrix} ; q, x \right] := \sum_{r=0}^{\infty} \frac{(a_0;q)_r (a_1;q)_r (a_2;q)_r \cdots (a_m;q)_r}{(q;q)_r (b_1;q)_r (b_2;q)_r \cdots (b_m;q)_r} x^r.$$

Corollary 4.16. *Let $u = (u_1, \ldots, u_k)$ be a weakly increasing array of non-negative integers, and $v = (v_1, \ldots, v_k)$ an array of positive integers. Then for $n \ge \sum_{i=1}^{k} v_i +$*

max$\{u_i + v_i : 1 \leq i \leq k\} - u_1$, *we have*

$$q^{\binom{n+1}{2}}(q^{-(a+s)};q)_a(q^{-(u_1+v_1+s)};q)_{v_1}\cdots(q^{-(u_k+v_k+s)};q)_{v_k} \times {}_{k+2}\phi_{k+1}\left[\begin{matrix} q^{-(n+1)}, & q^{-s}, & q^{-(s+u_1)}, & \ldots, & q^{-(s+u_k)} \\ & q^{-(s+a)}, & q^{-(s+u_1+v_1)}, & \ldots, & q^{-(s+u_k+v_k)}, \end{matrix} ;q,q\right] =$$

$$(-1)^n(q^{n+1-a-s};q)_a(q^{n+1-u_1-v_1-s};q)_{v_1}\cdots(q^{n+1-u_k-v_k-s};q)_{v_k} \times {}_{k+2}\phi_{k+1}\left[\begin{matrix} q^{-(n+1)}, & q^{-(n-a-s)}, & q^{-(n-u_1-v_1-s}, & \ldots, & q^{-(n-u_k-v_k-s)} \\ & q^{-(n-s)}, & q^{-(n-u_1-s)}, & \ldots, & q^{-(n-u_k-s)} \end{matrix} ;q,q\right], \tag{4.36}$$

where $a = n - \sum_{i=1}^{k} v_i$.

Proof. For each $i, 1 \leq i \leq k$, let

$$f(i) = |\{j : u_j \leq i \leq u_j + v_j - 1\}|.$$

Define $M = \max\{m : f(m) > 0\}$ and set $b = a + 1 - M - u_1$. Let X be the subset of $\mathbb{N}$ defined by the binary sequence

$$\tau = \underbrace{0\ldots0}_{b}\underbrace{1\ldots1}_{f(M)}0\underbrace{1\ldots1}_{f(M-1)}0\underbrace{1\ldots1}_{f(M-2)}0\ldots0\underbrace{1\ldots1}_{f(u_1+1)}0\underbrace{1\ldots1}_{f(u_1)}\underbrace{0\ldots0}_{u_1}.$$

That is, let $i \in X$ if and only if $\tau_i = 1$. The identity follows directly from Theorem 4.15 with X as above and $Y = \mathbb{N}$. □

Finally , we end this section by giving some examples showing that in some special cases, we can considerable simplify the formulas for the $P_{n,s}^{X,Y}(q)$.

Corollary 4.17. *Let* $X = 2\mathbb{N}$ *and* $Y = \mathbb{N}$. *Then*

$$P_{2n,s}^{X,Y}(q) = q^{s^2}\left([n]_q!\right)^2 \begin{bmatrix} n \\ s \end{bmatrix}_q^2,$$

which was originally derived by Liese and Remmel [12] using the recursion (9.1).

Proof. By the main theorem, we have

$$\begin{aligned}
P^{X,Y}_{2n,s}(q) &= [n]_q! \sum_{r=0}^{s} (-1)^{s-r} q^{\binom{s-r}{2}} \begin{bmatrix} n+r \\ r \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ s-r \end{bmatrix}_q \prod_{i=1}^{n} [r+i]_q \\
&= \frac{(q^{s+1})_n^2}{(1-q)^{2n}} \, {}_3\phi_2 \left[\begin{matrix} q^{-s}, & q^{-s}, & q^{-(2n+1)} \\ & q^{-(n+s)}, & q^{-(n+s)} \end{matrix} ; q, q \right] \\
&= \frac{(q^{s+1})_n^2}{(1-q)^{2n}} \frac{(q^{n+1-s})_s (q^{n+1-s})_s}{(q^{n+1})_s (q^{n+1})_s} \\
&= q^{s^2} \left([n]_q!\right)^2 \begin{bmatrix} n \\ s \end{bmatrix}_q^2,
\end{aligned}$$

where we employ Jackson's q-analogue of the Pfaff-Saalschütz ${}_3F_2$ summation formula (see [13])

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \\ & c, & abc^{-1}q^{-n+1} \end{matrix} ; q, q \right] = \frac{(c/a)_n (c/b)_n}{(c)_n (c/(ab))_n}.$$

□

Remark 4.18. More generally, if $X = \{u+2, u+4, \cdots, u+2m\}$ and $Y = \mathbb{N}$, a similar computation gives

$$P^{X,Y}_{2m+u+v,s}(q) = q^{s^2+vs} [m+u]_q! [m+v]_q! \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} m+u+v \\ v+s \end{bmatrix}_q.$$

4.3 Acknowledgement

Chapter 4, in full, is a reprint of the material as it now appears in preprint as "q-Counting descent pairs with prescribed tops and bottoms" by John Hall, Jeffrey Liese and Jeffrey Remmel. The paper is in preparation. The dissertation author was a co-author of this paper.

5

Rationality, irrationality, and Wilf equivalence in generalized factor order

5.1 Introduction and definitions

Let P be a set and consider the corresponding *free monoid* or *Kleene closure* of all words over P:

$$P^* = \{w = w_1 w_2 \ldots w_\ell \ : \ n \geq 0 \text{ and } w_i \in P \text{ for all } i\}.$$

Let $\exp(t(x-y))$ be the empty word and for any $w \in P^*$ we denote its cardinality or *length* by $|w|$. Given $w, w' \in P^*$ we say that w' is a *factor* of w if there are words u, v with $w = uw'v$, where adjacency denotes concatenation. For example, $u = 322$ is a factor of $w = 12213221$ starting with the fifth element of w. *Factor order* on P^* is the partial order obtained by letting $u \leq_{\text{fo}} w$ if and only if there is a factor w' of w with $u = w'$.

Now suppose that we have a poset $\mathcal{P} = (P, \leq)$. We define *generalized factor order* (relative to $\mathcal{P}$) on P^* by letting $u \leq_{\text{gfo}} w$ if there is a factor w' of w such that

(a) $|u| = |w'|$, and

(b) $u_i \leq w'_i$ for $1 \leq i \leq |u|$.

We call w' an *embedding* of u into w, and if the first element of w' is the jth element of w, we call j an *embedding index* of u into w. We also say that in this embedding u_i is in *position* $j + i - 1$. To illustrate, suppose $P = \mathbb{P}$, the positive integers with the usual order relation. If $u = 322$ and $w = 12213431$ then $u \leq_{\text{gfo}} w$ because of the embedding factor $w' = 343$ which has embedding index 5, and the two 2's of u are in positions 6 and 7. Note that we obtain ordinary factor order by taking P to be an antichain. Also, we will henceforth drop the subscript gfo since context will make it clear what order relation is meant. Generalized factor order is the focus of this chapter.

Returning to the case where P is an arbitrary set, let $\mathbb{Z}\langle\langle P\rangle\rangle$ be the algebra of formal power series with integer coefficients and having the elements of P as noncommuting variables. In other words,

$$\mathbb{Z}\langle\langle P\rangle\rangle = \left\{ f = \sum_{w \in P^*} c(w)w \ : \ c(w) \in \mathbb{Z} \text{ for all } w \right\}.$$

If $f \in \mathbb{Z}\langle\langle P\rangle\rangle$ has no constant term, i.e., $c(\exp(t(x-y))) = 0$, then define

$$f^* = \exp(t(x-y)) + f + f^2 + f^3 + \cdots = (\exp(t(x-y)) - f)^{-1}.$$

(We need the restriction on f to make sure that the sums are well defined as formal power series.) We say that f is *rational* if it can be constructed from the elements of P using only a finite number of applications of the algebra operations and the star operation.

A *language* is any $\mathcal{L} \subseteq P^*$. It has an associated generating function

$$f_{\mathcal{L}} = \sum_{w \in \mathcal{L}} w.$$

The language $\mathcal{L}$ is *regular* if $f_{\mathcal{L}}$ is rational.

Consider generalized factor order on P^* and fix a word $u \in P^*$. There is a corresponding language and generating function

$$\mathcal{F}(u) = \{w \ : \ w \geq u\} \quad \text{and} \quad F(u) = \sum_{w \geq u} w.$$

One of our main results is as follows.

Theorem 5.1. *If $\mathcal{P} = (P, \leq)$ is a finite poset and $u \in P^*$, then $F(u)$ is rational.*

This is an analogue of a result of Björner and Sagan [7] for generalized subword order on P^*. *Generalized subword order* is defined exactly like generalized factor order except that w' is only required to be a subword of w, i.e., the elements of w' need not be consecutive in w.

To prove the previous theorem, we will use finite automata. Given any set, P, a *nondeterministic finite automaton* or *NFA* over P is a digraph (directed graph) Δ with vertices V and arcs $\vec{E}$ having the following properties.

1. The elements of V are called *states* and $|V|$ is finite.

2. There is a designated *initial state* α and a set Ω of *final states*.

3. Each arc of $\vec{E}$ is labeled with an element of P.

Given a (directed) path in Δ starting at α, we construct a word in P^* by concatenating the elements on the arcs along the path in the order in which they are encountered. The *language accepted by* Δ is the set of all such words which are associated with a path ending in a final state. It is a well-known theorem that, for $|P|$ finite, a language $\mathcal{L} \subseteq P^*$ is regular if and only if there is a NFA accepting $\mathcal{L}$. (See, for example, the text of Hopcroft and Ullman [8, Chapter 2].)

We will demonstrate Theorem 5.1 by constructing a NFA accepting the language for $F(u)$. This will be done in the next section. In fact, the NFA still exists even if P is infinite, suggesting that more can be said about the generating function in this case.

We are particularly interested in the case of $P = \mathbb{P}$ with the usual order relation. So $\mathbb{P}^*$ is just the set of *compositions* (ordered integer partitions). Given $w = w_1 w_2 \ldots w_\ell \in \mathbb{P}^*$, we define its *norm* to be

$$\Sigma(w) = w_1 + w_2 + \cdots + w_\ell.$$

Let t, x be commuting variables. Replacing each $n \in w$ by tx^n we get an associated monomial called the *weight* of w

$$\mathrm{wt}(w) = t^{|w|} x^{\Sigma(w)}.$$

For example, if $w = 213221$ then

$$\mathrm{wt}(w) = tx^2 \cdot tx \cdot tx^3 \cdot tx^2 \cdot tx^2 \cdot tx = t^6 x^{11}.$$

We also have the associated *weight generating function*

$$F(u; t, x) = \sum_{w \geq u} \mathrm{wt}(w).$$

Our NFA will demonstrate, via the transfer-matrix method, that this is also a rational function of t and x. The details will be given in Section 5.3.

Call $u, w \in \mathbb{P}^*$ *Wilf equivalent* if $F(u; t, x) = F(v; t, x)$. This definition is modeled on the one used in the theory of pattern avoidance. See the survey article of Wilf [9] for more information about this subject. Section 5.4 is devoted to proving various Wilf equivalences. Although these results were discovered by having a computer construct the corresponding generating functions, the proofs we give are purely combinatorial. In the next two sections, we investigate a stronger notion of equivalence and compute generating functions for two families of compositions.

5.2 Construction of automata

We will now introduce two other languages which are related to $\mathcal{F}(u)$ and which will be useful in proving Theorem 5.1 and in demonstrating Wilf equivalence. We say that u is a *suffix* (respectively, *prefix*) of w if $w = vu$ (respectively, $w = uv$) for some word v. Let $\mathcal{S}(u)$ be all the $w \in \mathcal{F}(u)$ such that, in the definition of generalized factor order, the only possible choice for w' is a suffix of w. Let $S(u)$ be the corresponding generating function.

We say that $w \in P^*$ *avoids* u if $w \not\geq u$ in generalized factor order. Let $\mathcal{A}(u)$ be the associated language with generating function $A(u)$. The next result follows easily from the definitions and so we omit the proof. In it, we will use the notation Q to stand both for a subset of P and for the generating function $Q = \sum_{a \in Q} a$. Context will make it clear which is meant.

Lemma 5.2. *Let $\mathcal{P} = (P, \leq)$ be any poset and let $u \in P^*$. Then we have the following relationships:*

1. $\mathcal{F}(u) = \mathcal{S}(u)P^*$ *and* $F(u) = S(u)(\exp(t(x-y)) - P)^{-1}$,
2. $\mathcal{A}(u) = P^* - \mathcal{F}(u)$ *and* $A(u) = (\exp(t(x-y)) - P)^{-1} - F(u)$. □

We will now prove that all three of the languages we have defined are accepted by NFAs. An example follows the proof so the reader may want to read it in parallel.

Theorem 5.3. *Let $\mathcal{P} = (P, \leq)$ be any poset and let $u \in P^*$. Then there are NFAs accepting $\mathcal{F}(u)$, $\mathcal{S}(u)$, and $\mathcal{A}(u)$.*

Proof. We first construct an NFA, Δ, for $\mathcal{S}(u)$. Let $\ell = |u|$. The states of Δ will be all subsets T of $\{1, \ldots, \ell\}$. The initial state is $\emptyset$. Let $w = w_1 \ldots w_m$ be the word corresponding to a path from $\emptyset$ to T. The NFA will be constructed so that if the path is continued, the only possible embedding indices are those in the set $\{m - t + 1 \;:\; t \in T\}$. In other words, for each $t \in T$ we have

$$u_1 u_2 \ldots u_t \leq w_{m-t+1} w_{m-t+2} \ldots w_m, \tag{5.1}$$

for each $t \in \{1, \ldots, \ell\} - T$ this inequality does not hold, and $u \not\leq w'$ for any factor w' of w starting at an index smaller than $m - \ell + 1$. From this description it is clear that the final states should be those containing ℓ.

The definition of the arcs of Δ is forced by the interpretation of the states. There will be no arcs out of a final state. If T is a nonfinal state and $a \in P$ then there will be an arc from T to

$$T' = \{t+1 \ : \ t \in T \cup \{0\} \text{ and } u_{t+1} \le a\}.$$

It is easy to see that (5.1) continues to hold for all $t' \in T'$ once we append a to w. This finishes the construction of the NFA for $\mathcal{S}(u)$.

To obtain an automaton for $\mathcal{F}(u)$, just add loops to the final states of Δ, one for each $a \in P$. An automaton for $\mathcal{A}(u)$ is obtained by just interchanging the final and nonfinal states in the automaton for $\mathcal{F}(u)$. □

As an example, consider $P = \mathbb{P}$ and $u = 132$. We will do several things to simplify writing down the automaton. First of all, certain states may not be reachable by a path starting at the initial state. So we will not display such states. For example, we cannot reach the state $\{2,3\}$ since we can always start an embedding at w_n. Also, given states T and U there may be many arcs from T to U, each having a different label. So we will replace them by one arc bearing the set of labels of all such arcs. Finally, set braces will be dropped for readability. The resulting digraph is displayed in Figure 5.1.

Consider what happens as we build a word w starting from the initial state $\emptyset$. Since $u_1 = 1$, any element of $\mathbb{P}$ could be the first element of an embedding of u into w. That is why every element of the interval $[1, \infty) = \mathbb{P}$ produces an arrow from the initial state to the state $\{1\}$. Now if $w_2 \le 2$, then an embedding of u could no longer start at w_1 and so these elements give loops at the state $\{1\}$. But if $w_2 \ge 3$ then an embedding could start at either w_1 or at w_2 and so the corresponding arcs all go to the state $\{1,2\}$. The rest of the automaton is explained similarly.

As an immediate consequence of the previous theorem we get the following result which includes Theorem 5.1.

Theorem 5.4. *Let $\mathcal{P} = (P, \le)$ be a finite poset and let $u \in P^*$. Then the generating functions $F(u)$, $S(u)$, and $A(u)$ are all rational.* □

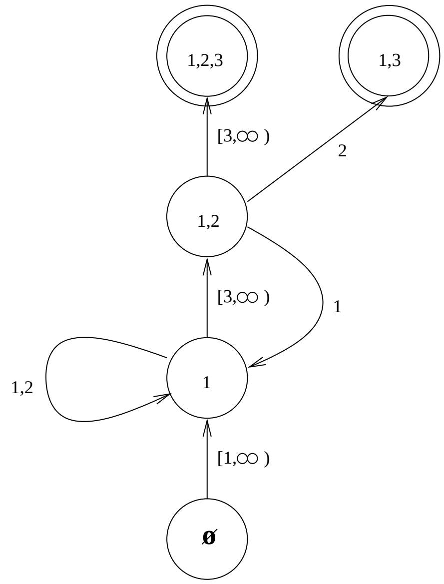


Figure 5.1: A NFA accepting $\mathcal{S}(132)$

5.3 The positive integers

If $P = \mathbb{P}$ then Theorem 5.4 no longer applies to the generating functions $F(u)$, $S(u)$, and $A(u)$. However, we can still show rationality of the weight generating function $F(u;t,x)$ as defined in the introduction. Similarly, we will see that the series

$$S(u;t,x) = \sum_{w\in\mathcal{S}(u)} \mathrm{wt}(w) \quad \text{and} \quad A(u;t,x) = \sum_{w\in\mathcal{A}(u)} \mathrm{wt}(w)$$

are rational.

Note first that Lemma 5.2 still holds for $\mathbb{P}$ and can be made more explicit in this case. Extend the function wt to all of $\mathbb{Z}\langle\langle\mathbb{P}\rangle\rangle$ by letting it act linearly. Then

$$\begin{aligned} \mathrm{wt}(\exp(t(x-y)) - \mathbb{P})^{-1} &= \frac{1}{1-\sum_{n\geq 1} tx^n} \\ &= \frac{1}{1-tx/(1-x)} \\ &= \frac{1-x}{1-x-tx}. \end{aligned}$$

We now plug this into the lemma.

Corollary 5.5. *We have*

1. $F(u;t,x) = \dfrac{(1-x)S(u;t,x)}{1-x-tx}$ *and*

2. $A(u;t,x) = \dfrac{1-x}{1-x-tx} - F(u;t,x).$ □

It follows that if any one of these three series is rational then the other two are as well.

We will now use the NFA, Δ, constructed in Theorem 5.3 to show that $S(u;t,x)$ is rational. This is essentially an application of the transfer-matrix method. See the text of Stanley [14, Section 4.7] for more information about this technique. The *transfer matrix* M for Δ has rows and columns indexed by the states with

$$M_{T,U} = \sum_n \mathrm{wt}(n)$$

where the sum is over all n which appear as labels on the arcs from T to U. For example, consider the case where $w = 132$ as done at the end of the previous section. If we list the states in the order

$$\emptyset,\ \{1\},\ \{1,2\},\ \{1,3\},\ \{1,2,3\}$$

then the transfer matrix is

$$M = \begin{bmatrix} 0 & \dfrac{tx}{1-x} & 0 & 0 & 0 \\ 0 & t(x+x^2) & \dfrac{tx^3}{1-x} & 0 & 0 \\ 0 & tx & 0 & tx^2 & \dfrac{tx^3}{1-x} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now M^k has entries $M^k_{T,U} = \sum_w \mathrm{wt}(w)$ where the sum is over all words w corresponding to a directed walk of length k from T to U. So to get the weight generating function for walks of all lengths one considers $\sum_{k\geq 0} M^k$. Note that this sum converges in the algebra of matrices over the formal power series algebra $\mathbb{Z}[[t,x]]$ because none of the entries of M has a constant term. It follows that

$$L := \sum_{k\geq 0} M^k = (I-M)^{-1} = \frac{\operatorname{adj}(I-M)}{\det(I-M)} \tag{5.2}$$

where adj denotes the adjoint.

Now

$$S(u;t,x) = \sum_T L_{\emptyset,T}$$

where the sum is over all final states of Δ. So it suffices to show that each entry of L is rational. From equation (5.2), this reduces to showing that each entry of M is rational. So consider two given states T, U. If T is final then we are done since the Tth row of M is all zeros. If T is not final, then consider

$$T' = \{t+1 \ : \ t \in T \cup \{0\}\}. \tag{5.3}$$

If $U = T'$ then there will be an $N \in \mathbb{P}$ such that all the arcs out of T with labels $n \geq N$ go to T'. So $M_{T,T'}$ will contain $\sum_{n\geq N} tx^n = tx^N/(1-x)$ plus a finite number of other terms of the form tx^m. Thus this entry is rational. If $U \neq T'$, then there will only be a finite number of arcs from T to U and so $M_{T,U}$ will actually be a polynomial. This shows that every entry of M is rational and we have proved, with the aid of the remark following Corollary 5.5, the following result.

Theorem 5.6. *If $u \in \mathbb{P}^*$ then $F(u;t,x)$, $S(u;t,x)$, and $A(u;t,x)$ are all rational.* □

5.4 Wilf equivalence

Recall that $u, v \in \mathbb{P}^*$ are *Wilf equivalent*, written $u \sim v$, if $F(u;t,x) = F(v;t,x)$. By Corollary 5.5, this is equivalent to $S(u;t,x) = S(v;t,x)$ and to $A(u;t,x) = A(v;t,x)$. It follows that to prove Wilf equivalence, it suffices to find a weight-preserving bijection $f : \mathcal{L}(u) \to \mathcal{L}(v)$ where $\mathcal{L} = \mathcal{F}$, $\mathcal{S}$, or $\mathcal{A}$. Since $\sim$ is an equivalence relation, we can talk about the *Wilf equivalence class* of u which is $\{w \ : \ w \sim u\}$. It is worth noting that the automata for the words in a Wilf equivalence class need not bear a resemblance to each other.

Part of the motivation for this section is to try to explain as many Wilf equivalences as possible between permutations. For reference, in Section 5.8 the first table lists all such equivalences up through 5 elements.

First of all, we consider three operations on words in $\mathbb{P}^*$. The *reversal* of $u = u_1 \dots u_\ell$ is $u^r = u_\ell \dots u_1$. It will also be of interest to consider $1u$, the word gotten by prepending one to u. Finally, we will look at u^+ which is gotten by increasing each element of u by one, as well as u^- which performs the inverse operation whenever it is defined.

Lemma 5.7. *We have the following Wilf equivalences.*

(a) $u \sim u^r$*,*

(b) if $u \sim v$ *then* $1u \sim 1v$*,*

(c) if $u \sim v$ *then* $u^+ \sim v^+$*.*

Proof. (a) It is easy to see that the map $w \mapsto w^r$ is a weight-preserving bijection $\mathcal{F}(u) \to \mathcal{F}(u^r)$.

(b) We can assume we are given a weight-preserving bijection $f : \mathcal{S}(u) \to \mathcal{S}(v)$. Since 1 is the minimal element of $\mathbb{P}$,

$$\mathcal{S}(1u) = \{w \in \mathcal{S}(u) \ : \ |w| > |u|\}.$$

So f restricts to a weight-preserving bijection from $\mathcal{S}(1u)$ to $\mathcal{S}(1v)$.

(c) Now we consider a weight-preserving bijection $g : \mathcal{A}(u) \to \mathcal{A}(v)$. Any $w \in \mathbb{P}^*$ can be written in the form

$$w = y_1 \ z_1 \ y_2 \ z_2 \ \dots \ z_{k-1} \ y_k$$

where the y_i are factors containing all the ones in w (with y_1, y_k possibly empty), while the z_i contain only elements which are at least two. Since all elements of

u^+ are also at least two, $w \in \mathcal{A}(u^+)$ if and only if $z_i \in \mathcal{A}(u^+)$ for all i. This is equivalent to $z_i^- \in \mathcal{A}(u)$ for all i. Thus if we map w to

$$y_1 \; g(z_1^-)^+ \; y_2 \; g(z_2^-)^+ \; \ldots \; g(z_{k-1}^-)^+ \; y_k$$

then we will get the desired weight-preserving bijection $\mathcal{A}(u^+) \to \mathcal{A}(v^+)$. □

We can combine these three operations to prove more complicated Wilf equivalences. Since a word $w \in \mathbb{P}^*$ is just a sequence of positive integers, terms like "weakly increasing" and "maximum" have their usual meanings. Also, let w^{+m} be the result of applying the + operator m times. By using the previous lemma and induction, we obtain the following result. The proof is so straight forward that it is omitted.

Corollary 5.8. *Let y, y' be weakly increasing compositions and z, z' be weakly decreasing compositions such that yz is a rearrangement of $y'z'$. Then for any $u \sim v$ we have*

$$yu^{+m}z \sim y'v^{+m}z'$$

whenever $m \geq \max\{y, z\} - 1$. □

Applying the two previous results, we can obtain the Wilf equivalences in the symmetric group $\mathfrak{S}_3$ of all the permutation of $\{1, 2, 3\}$:

$$123 \sim 321 \sim 132 \sim 231 \quad \text{and} \quad 213 \sim 312.$$

These two groups are indeed in different equivalence classes as one can use equation (5.2) to compute that

$$S(123; t, x) = \frac{t^3x^6}{(1-x)^2(1-x-tx+tx^3-t^2x^4)}$$

while

$$S(213; t, x) = \frac{t^3x^6(1+tx^3)}{(1-x)(1-x+t^2x^4)(1-x-tx+tx^3-t^2x^4)}.$$

However, we will need a new result to explain some of the equivalences in $\mathfrak{S}_4$ such as $2134 \sim 2143$. Let u be a composition such that $\max u$ only occurs once.

Define a *pseudo-embedding* of u into w to be a factor w' of w satisfying the two conditions for an embedding except that the inequality may fail at the position of $\max u$. In particular, embeddings are pseudo-embeddings.

An example of the construction used in the next theorem follows the proof and can be read in parallel.

Theorem 5.9. *Let* $x, y, z \in \{1, \ldots, n-1\}^*$*. Then*

$$x(n-1)ynz \sim xny(n-1)z.$$

Proof. Let $u = x(n-1)ynz$ and $v = xny(n-1)z$. We will construct a weight-preserving bijection $\mathcal{A}(u) \to \mathcal{A}(v)$. To do this, it suffices to construct such a bijection between the set differences $\mathcal{A}(u) - \mathcal{A}(v) \to \mathcal{A}(v) - \mathcal{A}(u)$ since the identity map can be used on $\mathcal{A}(u) \cap \mathcal{A}(v)$.

Given $w \in \mathcal{A}(u) - \mathcal{A}(v)$, consider the set

$$\eta(w) = \{i \ : \ \text{there is an embedding of } v \text{ into } w \text{ with the } n \text{ in position } i\}.$$

For such i, $w_i \geq n$. It must also be that $w_{i+k} = n-1$ where $k = |y|+1$: Certainly $w_{i+k} \geq n-1$ because of the embedding. And if $w_{i+k} \geq n$ then there is also an embedding of u at the same position as the one for v, contradicting $w \in \mathcal{A}(u)$.

Now for each $i \in \eta(w)$ we consider the *string* at i

$$\sigma(i) = \{i, \ i+k, \ i+2k, \ \ldots, \ i+mk\}$$

where m is the first nonnegative integer such that there is no pseudo-embedding of v into w with the n in position $i+mk$. Note that m depends on i even though this is not reflected in our notation. Also, $m \geq 1$ since there is embedding of v into w with the n in position i. Finally, it is easy to see that $w_{i+k} = w_{i+2k} = \ldots = w_{i+mk} = n-1$ by an argument similar to that for w_{i+k}. This implies that any two strings are disjoint since $w_i \geq n$ for $i \in \eta(w)$.

Now map w to $\bar{w}$ which is constructed by switching the values of w_i and w_{i+mk} for every $i \in \eta(w)$. Since strings are disjoint, the switchings are well defined. We must show that $\bar{w} \in \mathcal{A}(v) - \mathcal{A}(u)$. We prove that $\bar{w} \in \mathcal{A}(v)$ by contradiction. The switching operation removes every embedding of v in w. If a new embedding was created then, because only elements of size at least $n-1$ move, the n in v must correspond to $\bar{w}_{i+mk}$ for some $i \in \eta(w)$. But now there is a pseudo-embedding of v into w with the n in position $i+mk$, contradicting the definition of m.

To show $\bar{w} \notin \mathcal{A}(u)$, we will actually prove the stronger statement that there is an embedding of u in $\bar{w}$ with the n in position $i+mk$ for each $i \in \eta(w)$ and these are the only embeddings. These embeddings exist because there is a pseudo-embedding of v into w with the n in position $i+(m-1)k$, $\bar{w}_{i+mk} \geq n$, and only elements of size at least $n-1$ move in passing from w to $\bar{w}$. They are the only ones because $w \in \mathcal{A}(u)$ and so any embedding of u in $\bar{w}$ would have to have the n in a position of the form $i+mk$.

Finally, we need to show that this map is bijective. But modifying the above construction by exchanging the roles of u and v and building the strings from right to left gives an inverse. This completes the proof. □

By way of illustration, suppose $n=6$, $u = 1\ 3\ 5\ 2\ 4\ 6\ 3$, and $v = 1\ 3\ 6\ 2\ 4\ 5\ 3$ so that $k=3$. We will write our example w in two line form with the upper line being the positions: $w =$

$$\begin{array}{cccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 1 & 1 & 2 & 4 & 8 & 3 & 9 & 5 & 4 & 5 & 5 & 4 & 5 & 5 & 3 & 3 & 3 & 6 & 6 & 5 & 5 & 3. \end{array}$$

Now there are three embeddings of v (and none of u) into w with the 6 in positions $\eta(w) = \{5, 7, 18\}$. For $i=5$ we have the string $\sigma(5) = \{5, 8, 11, 14\}$ since there are pseudo-embeddings of v with the n in positions $5, 8, 11$ but not in position 14. Similarly $\sigma(7) = \{7, 10, 13\}$ and $\sigma(18) = \{18, 21\}$. So $\bar{w}$ is obtained by switching w_5 with w_{14}, w_7 with w_{13}, and w_{18} with w_{21} to obtain $\bar{w} =$

$$\begin{array}{cccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 1 & 1 & 2 & 4 & 5 & 3 & 5 & 5 & 4 & 5 & 5 & 4 & 9 & 8 & 3 & 3 & 3 & 5 & 6 & 5 & 6 & 3. \end{array}$$

It is now easy to verify that our results so far suffice to explain all the Wilf equivalences in symmetric groups up through $\mathfrak{S}_4$. They also explain most, but not all, of the ones in $\mathfrak{S}_5$. We will return to the $n = 5$ case in the section on open questions.

One might wonder about the necessity of the requirement that the two equivalent words in Theorem 5.9 have a unique maximum. However, one can see from Table 5.2 in Section 5.8 that 122 and 212 are not Wilf equivalent. So if there is an analogue of this Theorem for more general words, another condition will have to be imposed.

One might also hope that it would be possible to do without the strings in the proof and merely switch w_i and w_{i+k} for all $i \in \eta(w)$ to get $\bar{w}$. This would only be invertible if the embedding indices for v in w would be the same as those for u in $\bar{w}$. Unfortunately, this does not always work as the following example shows. Consider $u = 231$, $v = 321$, and all w which are permutations of 1223. Then the members of $\mathcal{A}(u) - \mathcal{A}(v)$ are 1322, 3212, and 3221; while those of $\mathcal{A}(v) - \mathcal{A}(u)$ are 1232, 2313, and 2231. The embedding indices of v in the first three compositions are 2, 1, and 1 (respectively); while those of u in the second three are 2, 1, and 2. Thus preservation of the indices is not possible in this case. However, it is would be interesting to know when one can leave the indices invariant and this will be investigated in the next section.

5.5 Strong Wilf equivalence

Given $v, w \in \mathbb{P}^*$ we let

$$\mathrm{Em}(v, w) = \{j \ : \ j \text{ is an embedding index of } v \text{ into } w\}.$$

Call compositions u, v *strongly Wilf equivalent*, written $u \sim_s v$, if there is a weight-preserving map $f : \mathbb{P}^* \to \mathbb{P}^*$ such that

$$\mathrm{Em}(u, w) = \mathrm{Em}(v, f(w)) \tag{5.4}$$

for all $w \in \mathbb{P}^*$. In addition to being a natural notion, our interest in this concept is motivated by the fact that we were able to prove Theorem 5.12 below only under the assumption of strong Wilf equivalence, although we suspect it is true for ordinary Wilf equivalence. First, however, we will prove analogues of some of our results from the previous section in this setting.

Lemma 5.10. *If $u \sim_s v$ then*

(a) $1u \sim_s 1v$,

(b) $1u \sim_s v1$,

(c) $u^+ \sim_s v^+$.

Proof. Let $f : \mathbb{P}^* \to \mathbb{P}^*$ be a map satisfying (5.4). Define maps $g : \mathbb{P}^* \to \mathbb{P}^*$ and $h : \mathbb{P}^* \to \mathbb{P}^*$ by $g(\exp(t(x-y))) = h(\exp(t(x-y))) = \exp(t(x-y))$ and, for $w = by$ with $b \in \mathbb{P}$,

$$g(by) = bf(y) \quad \text{and} \quad h(by) = f(y)b.$$

It follows easily that these functions establish (a) and (b). Finally, the construction used in the proof of (c) in Lemma 5.7 can be carried over to prove the analogous case here. That is, if in the proof of part (c) of Lemma 4.1, we assume that $g : \mathbb{P}^* \to \mathbb{P}^*$ satisfies $\mathrm{Em}(u, w) = \mathrm{Em}(v, g(w))$ for all w, then map constructed in part (c) of Lemma 4.1 will witness that $u^+ \backsim_s v^+$.

□

As before, we can combine the previous result and induction to get a more general equivalence.

Corollary 5.11. *Let y, y' be weakly increasing compositions and z, z' be weakly decreasing compositions such that yz is a rearrangement of $y'z'$. Then for any $u \sim_s v$ we have*

$$yu^{+m}z \sim_s y'v^{+m}z'$$

whenever $m \geq \max\{y, z\} - 1$. □

Not every Wilf equivalence is a strong Wilf equivalence. From Lemma 5.7 (a) we know that $w \sim w^r$. But we can show that $2143 \not\sim_s 3412$ as follows. Consider how one could construct a word w of length 7 with $\Sigma(w)$ minimum and $\mathrm{Em}(2143, w) = \{1, 3, 4\}$. Construct a table with a copy of 2143 starting in the first, third, and fourth positions in rows 1, 2, and 3, respectively. Then take the maximum value in each column for the corresponding entries of w:

$$\begin{array}{ccccccccc} & & 2 & 1 & 4 & 3 & & & \\ & & & & 2 & 1 & 4 & 3 & \\ & & & & & 2 & 1 & 4 & 3 \\ \hline w & = & 2 & 1 & 4 & 3 & 4 & 4 & 3. \end{array}$$

By construction, w has the desired embedding indices and one sees immediately that it has no others. Note that this is the unique w satisfying the given restrictions and that $\mathrm{wt}(w) = t^7 x^{21}$. But applying the same process to 3412 gives $\bar{w} = 3434422$ with $\mathrm{wt}(\bar{w}) = t^7 x^{22}$. Since the weights do not agree, we can not have strong Wilf equivalence.

Finally, we come to the result alluded to at the beginning of this section. Given $b \in \mathbb{P}$ we let b^k denote the composition consisting of k copies of b.

Theorem 5.12. *Suppose $u = u_1 \dots u_n \sim_s v = v_1 \dots v_n$. Then for any $k \in \mathbb{P}$*

$$u_1^k \dots u_n^k \sim_s v_1^k \dots v_n^k.$$

Proof. Let $f : \mathbb{P}^* \to \mathbb{P}^*$ be a map satisfying (5.4). Given any $w \in \mathbb{P}^*$ and i with $1 \le i \le k$, consider the subword $w[i] = w_i w_{i+k} w_{i+2k} \dots$ of w. Then the embeddings of $u_1^k \dots u_n^k$ in w are completely determined by the embeddings of u in the $w[i]$ and vice-versa. So replacing each subword $w[i]$ by the subword $f(w[i])$ yields the desired map. $\square$

5.6 Computations

We will now explicitly calculate the generating functions $S(u;t,x)$ for two families of words u. Aside from providing an application of the ideas from the previous sections, these particular power series are of interest because they have numerators which are single monomials. This is not always the case. For example,

$$S(212;t,x) = \frac{t^3x^5(1+tx^2)}{(1-x)(1-x+t^2x^3)(1-x-tx+tx^2-t^2x^3)}.$$

In this case, one can use the theory of Groebner basis to show that $(1-x)(1-x+t^2x^3)(1-x-tx+tx^2-t^2x^3)$ is not in the ideal generated by $1+tx^2$ and $1+tx^2$ hence does not divide $(1-x)(1-x+t^2x^3)(1-x-tx+tx^2-t^2x^3)$. Thus we can not write $S(212;t,x)$ in the form $\frac{t^ax^b}{Q(x,t)}$ for some polynomial $Q(x,t)$.

We first determine the generating function for increasing permutations. It will be convenient to have the standard notation that, for a nonnegative integer k,

$$[k]_x = 1 + x + x^2 + \cdots + x^{k-1}.$$

Theorem 5.13. *For $n \geq 2$, define polynomials $B_n(t,x)$ by*

$$\begin{aligned} B_2(t,x) &= tx(1-x)^2, \\ B_{n+1}(t,x) &= tx^{n+1}B_n(t,x) + tx(1-x)^n(1-x^n). \end{aligned}$$

Then

$$S(12\ldots n;t,x) = \frac{t^n x^{\binom{n+1}{2}}}{(1-x)^n - B_n(t,x).}$$

Proof. Since $12\ldots n \sim n\ldots 21$ it suffices to compute the generating function for the latter. And in that case, one can simplify the automaton Δ constructed in Theorem 5.3.

Note that T is an accepting state for Δ if and only if $\max T = n$ (where we define $\max\emptyset = 0$). Furthermore, because of our choice of permutation, if there is an arc from T to U labeled a, then $\max U$ is completely determined by $\max T$ and a. So we can contract all the states with the same maximum into one. And when we do so, arcs of the same label will collapse together. The result for $n = 5$ is shown in Figure 5.2. For convenience in later indexing, the state labeled k is the one resulting from amalgamating those with maximum $n - k$.

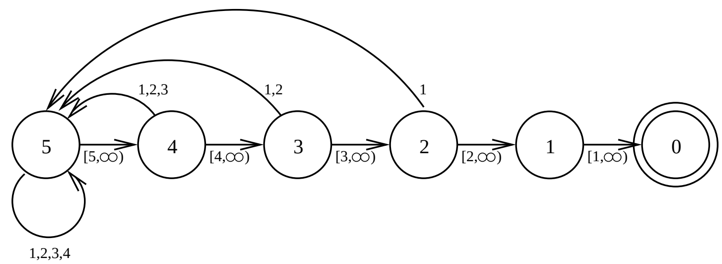


Figure 5.2: An automaton accepting $\mathcal{S}(54321)$.

Let $\mathcal{L}_k$ be the language of all words u such that the path for w starting at state k leads to the accepting state 0. Consider the corresponding generating function $L_k = \sum_{u \in \mathcal{L}_k} \mathrm{wt}(u)$. Directly from the automaton, we have $L_0 = 1$ and

$$L_k = \frac{tx^k}{(1-x)} L_{k-1} + tx[k-1]_x L_n$$

for $k \geq 1$. It is now easy to prove by induction that, for $k \geq 2$,

$$L_k = \frac{t^k x^{\binom{k+1}{2}} + B_k(t,x) L_n}{(1-x)^k}.$$

Plugging in $k = n$ and solving for $L_n = S(n \ldots 21; t, x)$ completes the proof. □

Theorem 5.14. *For any integers $k \geq 0$, $\ell \geq 1$, and $b \geq 2$ we have*

$$S(1^k b^\ell; t, x) = \frac{t^{k+\ell} x^{k+b\ell} (1-x)^{-(k+1)}}{\left((tx^b)^{\ell-1}(1 - tx[b-1]_x) + (1 - x - tx) \sum_{i=0}^{\ell-2} (1-x)^i (tx^b)^{\ell-2-i} \right)}.$$

Proof. Suppose $w = w_1 \ldots w_n \in \mathcal{S}(1^k b^\ell)$. Then to have $1^k b^\ell$ as a suffix, we must have $w_n, \ldots, w_{n-\ell+1} \geq b$.

There are now two cases depending on the length of w. If $|w| = k + \ell$ then $w_1, \ldots, w_k$ are arbitrary positive integers. If $|w| > k + \ell$ then write $w = yaz$

where $|z| = \ell$ and $a \in \mathbb{P}$. In order to make sure that $1^k b^\ell$ does not have another embedding intersecting z it is necessary and sufficient that $a < b$. And ruling out any embeddings inside y is equivalent to $y \in \mathcal{A}(1^k b^\ell)$. We must also make sure that $|y| \geq k$ in order to have $|w| > k + \ell$.

Let $S = S(1^k b^\ell; t, x)$ and $A = A(1^k b^\ell; t, x)$. Turning all the information about w into a generating function identity gives

$$S = \left(\frac{tx^b}{1-x}\right)^\ell \left[\left(\frac{tx}{1-x}\right)^k + tx[b-1]_x \left(A - [k]_{tx/(1-x)}\right)\right].$$

Also, combining the two parts of Corollary 5.5 gives

$$A = \frac{(1-x)(1-S)}{1-x-tx}.$$

Substituting this expression for A into our previous equation for S, one can easily solve for S to obtain that

$$S(1^k b^\ell; t, x)) = \frac{t^{k+\ell} x^{k+b\ell}(1-x-x^b t)}{(1-x)^{k+1}((1-x)^{\ell-1}(1-x-xt) + [b-1]_x x^{1+b\ell} t^{1+\ell})}.$$

Thus to finish the proof,one need only show that

$$\frac{((1-x)^{\ell-1}(1-x-xt) + [b-1]_x x^{1+b\ell} t^{1+\ell})}{(1-x-x^b t)} =$$

$$(x^b t)^{\ell-1}(1 - xt[b-1]_x) + \sum_{i=0}^{\ell-2} (1-x)^i (1-x-xt)(tx^b)^{l-2-i}.$$

which can be easily verified by long division.

5.7 Comments, conjectures, and open questions

5.7.1 Mixing factors and subwords

It is possible to create languages using combinations of factors and subwords. This is an idea that was first studied by Babson and Steingrímsson [15] in the context of pattern avoidance in permutations. Many of the results we have proved can be generalized in this way. We will indicate how this can be done for Theorem 5.3.

A *pattern* p over P is a word in P^* where certain pairs of adjacent elements have been overlined (barred). For example, in the pattern $p = 1\overline{133}24\overline{61}$ the pairs 13, 33, and 61 have been overlined. If $w \in P^*$ we will write $\overline{w}$ for the pattern where every pair of adjacent elements in w is overlined. So every pattern has a unique factorization of the form $p = \overline{y_1}\ \overline{y_2}\ \ldots\ \overline{y_k}$. In the preceding example, the factors are $y_1 = 1$, $y_2 = 133$, $y_3 = 2$, $y_4 = 4$, and $y_5 = 61$.

If $p = \overline{y_1}\ \overline{y_2}\ \ldots\ \overline{y_k}$ is a pattern and $w \in P^*$ then p *embeds* into w, written $p \to w$, if there is a subword $w' = z_1 z_2 \ldots z_k$ of w where, for all i,

1. z_i is a factor of w with $|z_i| = |y_i|$, and
2. $y_i \leq z_i$ in generalized factor order.

For example $\overline{324} \to 14235$ and there is only one embedding, namely 425. For any pattern p, define the language

$$\mathcal{F}(p) = \{w \in P^* \ :\ p \to w\}$$

and similarly for $\mathcal{S}(p)$ and $\mathcal{A}(p)$. The next result generalizes Theorem 5.3 to an arbitrary pattern.

Theorem 5.15. *Let $\mathcal{P} = (P, \leq)$ be any poset and let p be a pattern over P. Then there are NFAs accepting $\mathcal{F}(p)$, $\mathcal{S}(p)$, and $\mathcal{A}(p)$.*

Proof. As before, it suffices to build an NFA, Δ, for $\mathcal{S}(p)$. It will be simplest to construct an NFA with $\exp(t(x-y))$-moves, i.e., with certain arcs labeled $\exp(t(x-y))$ whose traversal does not append anything to the word being constructed. It is well known that the set of languages accepted by NFAs with $\exp(t(x-y))$-moves is still the set of regular languages.

Let $p = \overline{y_1}\ \overline{y_2}\ \ldots\ \overline{y_k}$ be the factorization of p and, for all i, let Δ_i be the automaton constructed in Theorem 5.3 for $\mathcal{S}(y_i)$. We can paste these automata together to get Δ as follows. For each i with $1 \leq i < k$, add an $\exp(t(x-y))$-arc from every final state of Δ_i to the initial state of Δ_{i+1}. Now let the initial state of

Δ be the initial state of Δ_1 and the final states of Δ be the final states of Δ_k. It is easy to see that the resulting NFA accepts the language $\mathcal{S}(p)$. ☐

5.7.2 Rationality for infinite posets

It would be nice to have a criterion that would imply rationality even for some infinite posets $\mathcal{P} = (P, \leq)$. To this end, assume that P is a subset of the positive integers and let $\mathbf{x} = \{x_1, \ldots, x_m\}$ be a set of commuting variables and consider the formal power series algebra $\mathbb{Z}[[t, x, \mathbf{x}]]$. Suppose we are given a function

$$\mathrm{wt} : P \to \mathbb{Z}[[\mathbf{x}]]$$

which then defines a weighting of words $w = w_1 \ldots w_\ell \in P^*$ by

$$\mathrm{wt}(w) = \prod_{i=1}^{m} \mathrm{wt}(w_i).$$

To avoid trivialities, we will assume that $\mathrm{wt}(a) \neq 0$ for at least one element a in P. For any language $L \subseteq P^*$, we say that $L(t, x, \mathbf{x}) = \sum_{w \in L} \mathrm{wt}(w) t^{|w|} x^{\Sigma(w)}$ is a *rational generating function* if there are polynomials $R(t, x, \mathbf{x})$ and $S(t, x, \mathbf{x})$ such that $L(\mathbf{x}) = \frac{R(t,x,\mathbf{x})}{S(t,x,\mathbf{x})}$. Note that we are guaranteed that $L(t, x, \mathbf{x})$ converges as a formal power series since for any (k, ℓ), there are only finitely words w such that $|w| = k$ and $\Sigma(w) = \ell$.

For $u \in P^*$, let

$$F(u; t, x, \mathbf{x}) = \sum_{w \geq u} \mathrm{wt}(w) t x^a$$

and similarly for $S(u; t, x, \mathbf{x})$ and $A(u; t, x, \mathbf{x})$. Suppose we want to make sure that $S(u; t, x, \mathbf{x})$ is a rational generating function. As done in Section 5.3, we can consider a transfer matrix with entries

$$M_{T,U} = \sum_{a} \mathrm{wt}(a) t x^a$$

where the sum is over all $a \in P$ occurring on arcs from T to U. Equation (5.2) remains the same, so it suffices to make sure that $M_{T,U}$ is always a rational generating function.

If there is an arc labeled a from T to U then we must have $U \subseteq T'$ where T' is given in equation (5.3). Recalling the definition of Δ from the proof of Theorem 5.3, we see that the a's appearing in the previous sum are exactly those satisfying

1. $a \geq u_{t+1}$ for $t+1 \in U$, and

2. $a \not\geq u_{t+1}$ for $t+1 \in T' - U$.

To state these criteria succinctly, for any subword y of u we write $a \geq y$ (respectively, $a \not\geq y$) if $a \geq b$ (respectively, $a \not\geq b$) for all $b \in y$. Finally, note that, from the proof of Theorem 5.3, similar transfer matrices can be constructed for $F(u; \mathbf{x})$ and $A(u; \mathbf{x})$. However in the automaton for those cases, there are additional loops on the final states for the automaton for $\mathcal{S}(u)$ which are labeled with every $a \in P$ so we must also assume that $\sum_{a \in P} \mathrm{wt}(a) t x^a$ is also a rational generating function. We have proved the following result which generalizes Theorem 5.6.

Theorem 5.16. *Let $\mathcal{P} = (P, \leq)$ be a poset with a weight function* $\mathrm{wt} : P^* \to \mathbb{Z}[[\mathbf{x}]]$, *and let $u \in P^*$. Suppose that for any two subwords y and z of u we have*

$$\sum_{\substack{a \geq y \\ a \not\geq z}} \mathrm{wt}(a) t x^a$$

is a rational generating function. Then so is $S(u; t, x, \mathbf{x})$. If in addition, $\sum_{a \in P} \mathrm{wt}(a) t x^a$ is a rational generating function, then so are $F(u; t, x, \mathbf{x})$ and $A(u; t, x, \mathbf{x})$. □

In fact, we can prove a slightly stronger theorem than Theorem 8.2 by using a slightly different automaton. That is, we have the following result.

Theorem 5.17. *Let $\mathcal{P} = (P, \leq)$ be a poset with a weight function $wt : P^* \to \mathbb{Z}[[\mathbf{x}]]$ and $u = u_1 \cdots u_n \in P^*$. Suppose that following generating functions are rational generating functions:*
(i) $\sum_{a \geq u_i} \mathrm{wt}(a) t x^a$ *for all i,*

(ii) $\sum_{a \not\geq u} \mathrm{wt}(a)tx^a$*, and*
(iii)$\sum_{a \in P} \mathrm{wt}(a)tx^a$.
Then $F(u;t,x,\mathbf{x})$, $S(u;t,x,\mathbf{x})$ *and* $A(u;t,x,\mathbf{x})$ *are rational generating functions.*

Proof. Our assumptions imply that $\sum_{a \in P} \mathrm{wt}(a)tx^a$ is of the form that $\frac{C(t,x,\mathbf{x})}{D(t,x,\mathbf{x})}$ and is not equal to 1. It follows that

$$\sum_{w \in P^*} \mathrm{wt}(w)t^{|w|}x^{\Sigma(w)} = \frac{1}{1 - \sum_{a \in P} \mathrm{wt}(a)tx^a} = \frac{R(t,x,\mathbf{x})}{S(t,x,\mathbf{x})}$$

for some non-zero polynomials $R(t,x,\mathbf{x})$ and $S(t,x,\mathbf{x})$. Thus since $\mathcal{F} = \mathcal{S}P^*$ and $\mathcal{A} = P^* - \mathcal{F}$, it follows that

$$\begin{aligned} F(u;t,x,\mathbf{x}) &= S(u;t,x,\mathbf{x})\frac{R(t,x,\mathbf{x})}{S(t,x,\mathbf{x})} \\ A(u;t,x,\mathbf{x}) &= \frac{R(t,x,\mathbf{x})}{S(t,x,\mathbf{x})} - F(u;t,x,\mathbf{x}). \end{aligned}$$

Thus we need only show that $F(u;t,x,\mathbf{x})$ is a rational generating function.

We construct an automaton Γ that accepts $\mathcal{F}(u)$. The automaton is more complicated that the automaton of Theorem 2.2, but it has the advantage that the corresponding functions $M_{U,T}$ are simpler.

Let U equal the set of letters in u. The set of states of Γ are the set of all $w \in U^*$ such that $|w| \leq n$ including the empty word ϵ. The start state is the empty word ϵ. The final states are the set of all $w \in U^n$ such that $u \leq w$. Now suppose that $w = w_1 \cdots w_k \in U^*$ and $|w| \leq n$. If w is a final state, then there is loop at w labeled a for every $a \in P$. If w is not a final state, then there is an edge from w to ϵ labeled with a for every a which is not comparable to any element of U. If w is not final state and $a \geq v$ where $v \in U$, then there an edge labeled with a from $w_1 \ldots w_k v$ if $k < n$ and an edge labeled with a from $w_2 \ldots w_n v$ if $k = n$.

It is easy Γ accepts $\mathcal{F}(u)$ and that all $M_{T,U}$ are either of the form $\sum_{a \geq v} \mathrm{wt}(a)tx^a$ for some $v \in U$, $\sum_{a \not\geq u} \mathrm{wt}(a)tx^a$, or $\sum_{a \in P} \mathrm{wt}(a)tx^a$. □

Remark. Note that in general for any language $L \subseteq P^*$, it is possible that $L(\mathbf{x}) = L(0,0,\mathbf{x}) = \sum_{a \in L} \mathrm{wt}(a)$ does not even converge as a formal power series since we did not make any assumptions about the weight function wt. However, it is the case that if $L(\mathbf{x})$ is well defined formal power series for all the languages L mentioned in Theorems 8.2 or 8.3, then we can conclude that $F(u; 0, 0, \mathbf{x})$, $S(u; 0, 0, \mathbf{x})$ and $A(u, 0, 0, \mathbf{x})$ are rational generating functions.

5.7.3 Irrationality for infinite posets

When P is infinite it is possible for the generating functions we have considered not to be rational. To illustrate this, suppose that $P = \mathbb{P}$ but with the following partial order. For any $A \subseteq \mathbb{P}$ with $1 \in A$, we define an order $\leq_A$ by insisting that the elements of $\mathbb{P} - \{1\}$ form an antichain, and that $1 \leq_A n$ if and only if $n \in A$. Consider the corresponding language $\mathcal{S}_A(1)$. Clearly $\mathcal{S}_A(1) = (\mathbb{P} - A)^* A$ and so no two of these languages are equal. It follows that the mapping $A \to S_A(1)$ is injective. So one of the $S_A(1)$ must be irrational since there are uncountably many possible A but only countably many rational functions in $\mathbb{Z}\langle\langle\mathbb{P}\rangle\rangle$.

5.7.4 Wilf equivalence and strong equivalence

There are a number of open problems and questions raised by our work on Wilf equivalence.

(1) **If $u \sim v$, then must v be a rearrangement of u?** This is the case for all the Wilf equivalences we have proved. Note that if the answer is "yes," then the Wilf equivalences given in Table 1 of Section 5.8 for the symmetric groups are actually Wilf equivalence classes.

(2) **Does $u \sim v$ imply that there is a bijection $\Theta : \mathbb{P}^* \to \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $w \in \mathcal{F}(u) \iff \Theta(w) \in \mathcal{F}(v)$ and $\Theta(w)$ is a rearrangement of w?** This is true for all the Wilf equivalences given in the tables in Section 5.8. That is, suppose that $[m]$ is the finite poset consisting of the integers $[m] = \{1, \ldots, m\}$ under the standard order. For any word $w \in [m]^*$ and $i \in [m]$, let $c_i(w)$ equal the number of occurrences of i in w. Then we can define the weight of w, $W_{[m]}(w) =$

$\prod_{i=1}^{m} x_i^{c_i(w)}$ and set

$$\begin{aligned} S(u;x_1,\ldots,x_m) &= \sum_{w\in\mathcal{S}(u)} W_{[m]}(w), \\ F(u;x_1,\ldots,x_m) &= \sum_{w\in\mathcal{F}(u)} W_{[m]}(w), \text{ and} \\ A(u;x_1,\ldots,x_m) &= \sum_{w\in\mathcal{A}(u)} W_{[m]}(w). \end{aligned}$$

Our results in Section 2 imply that $S(u;x_1,\ldots,x_m)$, $F(u;x_1,\ldots,x_m)$, and $A(u;x_1,\ldots,x_m)$ are always rational functions and that for $u,v \in [m]^*$, $S(u;x_1,\ldots,x_m) = S(v;x_1,\ldots,x_m)$ implies $F(u;x_1,\ldots,x_m) = F(v;x_1,\ldots,x_m)$. We have shown by computer that for all the Wilf equivalences $u \sim v$ given in the tables of Section 5.8 that $S(u;x_1,\ldots,x_5) = S(v;x_1,\ldots,x_5)$.

Clearly if $u,v \in [m]^*$ and $F(u;x_1,\ldots,x_m) = F(v;x_1,\ldots,x_m)$, then there is a weight preserving bijection $\Gamma : [m]^* \to [m]^*$ such that for all $w \in [m]^*$,

$$w \in \mathcal{F}(u) \iff \Gamma(w) \in \mathcal{F}(v).$$

Hence for any $w \in [m]^*$, $\Gamma(w)$ is a rearrangement of w. This bijection can then be lifted to our desired bijection Θ as follows. Suppose we are given a word $w = w_1 \ldots w_n \in \mathbb{P}^*$. Then let $1 \le i_1 < \cdots < i_k \le n$ be the sequence of all indices i such that $w_i \ge m$. Let $\bar{w}$ be the word in $[m]^*$ that results by replacing w_{i_s} by m for $s = 1,\ldots,k$. Clearly $u \le w$ if and only if $u \le \bar{w}$. Since $\Gamma(\bar{w}) = z_1 \ldots z_n$ is a rearrangement of $\bar{w}$, there are is a sequence $1 \le j_1 < \cdots < j_k \le n$ consisting of all the indices j such that $z_j = m$. Then we simply let $\Theta(w)$ be the result of replacing z_{j_s} by w_{i_s} for $s = 1,\ldots,k$.

Note also that if the answer to this question is yes, then it will be that case that $u,v \in [m]^*$ and $u \sim v$ relative to $\mathbb{P}$ implies $F(u;x_1,\ldots,x_m) = F(v;x_1,\ldots,x_m)$.

(3) **If $u^+ \sim v^+$, then is $u \sim v$?** In other words, does the converse of Lemma 5.7 (c) hold? We note that the converse of (b) is true. For suppose $1u \sim 1v$ and let $f : \mathcal{S}(1u) \to \mathcal{S}(1v)$ be a corresponding map. Then to construct $g : \mathcal{S}(u) \to \mathcal{S}(v)$ we consider two cases for $w \in \mathcal{S}(u)$. If $|w| > |u|$ then $w \in \mathcal{S}(1u)$ so let $g(w) = f(w)$.

Otherwise $|w| = |u|$ and so let $g(w) = v + (w - u)$ where addition and subtraction is done componentwise. It is easy to check that g is well defined and weight preserving. It is also not hard to show that if the answer to question (2) is yes, then the answer to question (3) is yes.

(4) **Find a theorem which, together with the results already proved, explains all the Wilf equivalences in $\mathfrak{S}_5$.** In particular, the results of Section 5.4 show that

$$21345 \sim 21354 \sim 45312 \sim 54312 \quad \text{and} \quad 21453 \sim 21543 \sim 34512 \sim 35412$$

but not why a permutation of the first group is Wilf equivalent to one of the second. The other row of Table 1 which breaks into two groups is

$$31425 \sim 31524 \sim 42513 \sim 52413 \quad \text{and} \quad 32415 \sim 32514 \sim 41523 \sim 51423.$$

(5) **Is it always the case that the number of elements of $\mathfrak{S}_n$ Wilf equivalent to a given permutation is a power of 2?** This is always true in Table 1.

(6) **Is it true that $312 \sim_s 213$?** From our results on strong Wilf equivalence it follows that $12 \sim_s 21$ and $123 \sim_s 132 \sim_s 231 \sim_s 321$. So all the Wilf equivalent elements in $\mathfrak{S}_2$ and $\mathfrak{S}_3$ are actually strongly Wilf equivalent with the possible exception of the pair in the question. Of course, this breaks down in $\mathfrak{S}_4$ as noted in Section 5.5.

(7) **Does Theorem 5.12 remain true if one replaces strong Wilf equivalence with ordinary Wilf equivalence throughout?** If so, a completely different proof will have to be found for that case.

5.8 Tables

The following two tables, Table 5.1 and Table 5.2, were constructed by having a computer calculate, for each composition u, the generating functions $S(u; t, x)$. This was done with the aid of the corresponding automaton from Section 5.2.

5.9 Acknowledgement

Chapter 5, in full, is a partial reprint of the material as it now appears in preprint as "Rationality, irrationality and Wilf equivalence in generalized factor order" by Sergey Kitaev, Jeffrey Liese, Jeffrey Remmel, and Bruce Sagan. The paper is in preparation. The dissertation author was a co-author of this paper.

Table 5.1: Wilf equivalences for permutations of at most 5 elements

12, 21
123, 132, 231, 321
213, 312
1234, 1243, 1342, 1432, 2341, 2431, 3421, 4321
1324, 1423, 3241, 4231
2134, 2143, 3412, 4312
3124, 3214, 4123, 4213
2314, 2413, 3142, 4132
12345, 12354, 12453, 12543, 13452, 13542, 14532, 15432, 23451, 23541, 24531, 25431, 34521, 35421, 45321, 54321
12435, 12534, 14352, 15342, 24351, 25341, 43521, 53421
13245, 13254, 14523, 15423, 32451, 32541, 45231, 54231
21345, 21354, 21453, 21543, 34512, 35412, 45312, 54312
23145, 23154, 45132, 54132
32145, 32154, 45123, 54123
24153, 25143, 34152, 35142
14235, 14325, 15234, 15324, 42351, 43251, 52341, 53241
31425, 31524, 32415, 32514, 41523, 42513, 51423, 52413
24315, 25314, 41352, 51342
24135, 25134, 43152, 53142
34215, 35214, 41253, 51243
34125, 35124, 42153, 52143
41325, 42315, 51324, 52314
41235, 43215, 51234, 53214
42135, 43125, 52134, 53124
13425, 13524, 14253, 15243, 34251, 35241, 42531, 52431
21435, 21534, 43512, 53412
24513, 25413, 31452, 31542
23415, 23514, 41532, 51432
31245, 31254, 45213, 54213

Table 5.2: Wilf equivalences for u with $|u| \leq 3$ and $u_i \leq 3$ for all i

Equivalences	$S(u;t,x)$
1	$\frac{tx}{1-x}$
2	$\frac{tx^2}{(1-x)(1-tx)}$
3	$\frac{tx^3}{(1-x-tx+tx^3)}$
11	$\frac{t^2x^2}{(1-x)^2}$
12,21	$\frac{t^2x^3}{(1-x)^2(1-tx)}$
13,31	$\frac{t^2x^4}{(1-x)^2(1-tx-tx^2)}$
22	$\frac{t^2x^4}{(1-x)(1-x-tx+tx^2-t^2x^3)}$
23,32	$\frac{t^2x^5}{(1-x)(1-x-tx+tx^3-t^2x^4)}$
33	$\frac{t^2x^6}{(1-x)(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
111	$\frac{t^3x^3}{(1-x)^3}$
112,121,211	$\frac{t^3x^4}{(1-x)^3(1-tx)}$
122,221	$\frac{t^3x^5}{(1-x)^2(1-x-tx+tx^2-t^2x^3)}$
212	$\frac{t^3x^5(1+tx^2)}{(1-x)(1-x+t^2x^3)(1-x-tx+tx^2-t^2x^3)}$
113,131,311	$\frac{t^3x^5}{(1-x)^3(1-tx-tx^2)}$
213,312	$\frac{t^3x^6(1+tx^3)}{(1-x)(1-x+t^2x^4)(1-x-tx+tx^3-t^2x^4)}$
123,132,231,321	$\frac{t^3x^6}{(1-x)^2(1-x-tx+tx^3-t^2x^4)}$
222	$\frac{t^3x^6}{(1-x)(1-2x-tx+x^2+2tx^2-tx^3-t^2x^3+t^2x^4-t^3x^5)}$
133,331	$\frac{t^3x^7}{(1-x)^2(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
313	$\frac{t^3x^7(1+tx^3+tx^4)}{(1-x)(1-x+t^2x^4+t^2x^5)(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
223,232,322	$\frac{t^3x^7}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^5-t^3x^6)}$
323	$\frac{t^3x^8(1+tx^3)}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^5-t^3x^6-t^3x^7+t^3x^8-t^4x^9-t^4x^{10})}$
233,332	$\frac{t^3x^8}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^6-t^3x^7)}$
333	$\frac{t^3x^9}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^6-t^3x^7-t^3x^8)}$

6

Counting permutations by consecutive patterns

6.1 Introduction

Suppose we define $\mathrm{nlap}(u,w)$ to be the maximum number of nonoverlapping embeddings of u into w. Notice that the results of Chapter 5 give us a way to compute a generating function involving the statistic $\mathrm{nlap}(u,w)$. Any word w with $\mathrm{nlap}(u,w) = k$ for some $k \geq 1$ can easily be factored into k words embedding u as a suffix and a word avoiding u. Thus we obtain the following.

$$\sum_{w\in\mathbb{P}^*} z^{\mathrm{nlap}(u,w)} x^{\Sigma w} t^{|w|} = \sum_{n\geq 0} z^n S(u)^n A(u) \tag{6.1}$$

So, the study of words in this fashion gives us a relatively easy way to compute the generating function for the statistic nlap. We now turn to a way to try and get a handle on the statistic mentioned before, τ-$\mathrm{nlap}(\sigma)$.

Remmel and Mendes proved Kitaev's Theorem, 3.1, with an alternate approach. We will not present the proof here, but it can be found in [10]. We will first introduce some notation that was adopted by Remmel and Mendes and then will state a corollary that was the impetus to this chapter.

We now define a few auxiliary sets associated with a given permutation $\tau \in S_j$. For a permutation $\sigma \in S_j$, let

$$\mathrm{Mch}_\tau(\sigma) = \{i : \mathrm{red}(\sigma_{i+1} \cdots \sigma_{i+j}) = \tau\} \qquad \text{and}$$
$$I_\tau = \{1 \leq i < j : \text{there exist } \sigma \in S_{j+i} \text{ such that } \mathrm{Mch}_\tau(\sigma) = \{0, i\}\}.$$

One (or one's computer) can find every element in the set I_τ for any $\tau \in S_j$ by finding the set $\mathrm{Mch}_\tau(\sigma)$ for all $\sigma \in S_{j+i}$ for $i = 1, \ldots, j-1$.

Let I_τ^* be the set of all words with letters in the set I_τ. We let ϵ denote the empty word. If $w = w_1 \cdots w_n \in I_\tau^*$ is word with n-letters, we define

$$\ell(w) = n, \qquad \sum w = \sum_{i=1}^{n} w_i, \qquad \text{and} \qquad ||w|| = j + \sum w.$$

In the special case where $w = \epsilon$, we let $\ell(w) = 0$ and $\sum w = 0$. Let

$$A_\tau = \{w \in I_\tau^* : \ell(w) \geq 2 \text{ and } \sum w < j\} \qquad \text{and}$$
$$B_{u,\tau} = \{w_1 \cdots w_n \in I_\tau^* : \sum w_2 \cdots w_n + \sum u < j \leq \sum w_1 \cdots w_n + \sum u\}$$

for each word $u \in I_\tau^*$ with $\sum u < j$.

To help us understand these sets, let us look at the situation for four different permutations τ. These examples should provide insight into the kinds of situations which may arise.

First, consider $\alpha = 2\ 1\ 4\ 3$. We have that $\mathrm{Mch}_\alpha(2\ 1\ 4\ 3\ 6\ 5) = \{0, 2\}$ and $\mathrm{Mch}_\alpha(3\ 2\ 5\ 4\ 1\ 6\ 7) = \{0, 3\}$. Therefore $2, 3 \in I_\alpha$. A quick check can show that $1 \notin I_\alpha$; giving that $I_\alpha = \{2, 3\}$. The set I_α^* is the set of all words in the letters 2 and 3. Therefore $A_\alpha = \varnothing$ since every word w of length ≥ 2 in the letters 2 and 3 has $\sum w \geq 4$. The sets $B_{2,\alpha} = B_{3,\alpha}$ are both equal to $\{2, 3\}$ in this case.

Next, consider $\beta_j = j \cdots 2\ 1$. It may be shown that $I_{\beta_j} = \{1\}$, and therefore $A_{\beta_j} = \{1^k : 2 \leq k < j\}$ (here, 1^k denotes the word $1 \cdots 1$). If $1 \leq i < j$ then $B_{i,\beta_j} = \{1^{j-i}\}$.

When $\gamma = 2\ 1\ 4\ 3\ 6\ 5$, $I_\gamma = \{2, 5\}$, $A_\gamma = \{22\}$, $B_{22,\gamma} = B_{5,\gamma} = \{2, 5\}$ and $B_{2,\gamma} = \{22, 25, 5\}$.

As our last example, let $\delta = 1\ 5\ 2\ 7\ 3\ 8\ 4\ 9\ 6$. Via a computer search, I_δ has been shown to equal $\{2, 4, 6, 8\}$. In this situation,

$$\begin{aligned}
A_\delta &= \{22, 24, 26, 42, 44, 62, 222, 224, 242, 422, 2222\}, \\
B_{2,\delta} &= \{8, 26, 46, 66, 86, 44, 64, 84, 224, 424, 624, 824, \\
&\qquad 82, 62, 242, 442, 642, 842, 422, 622, 822, 2222, 4222, 6222, 8222\}, \\
B_{4,\delta} &= B_{22,\delta} = \{8, 6, 24, 64, 84, 82, 62, 42, 222, 422, 622, 822\}, \\
B_{6,\delta} &= B_{24,\delta} = B_{42,\delta} = B_{222,\delta} = \{4, 6, 8, 22, 42, 62, 82\}, \qquad \text{and} \\
B_{8,\delta} &= B_{26,\delta} = B_{44,\delta} = B_{62,\delta} = B_{224,\delta} = B_{242,\delta} = B_{422,\delta} = \{2, 4, 6, 8\}.
\end{aligned}$$

Form a new alphabet

$$K_\tau = \{\overline{u} : u \in I_\tau\} \cup \{\overline{w} : w \in A_\tau\}.$$

We let $\Psi : K_\tau^* \to I_\tau^*$ be the function such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$. For example, if $\tau = \gamma = 2\,1\,4\,3\,5\,6$ as above, then $\Psi(\overline{5}\,\overline{22}\,\overline{2}\,\overline{2}\,\overline{5}) = 522225$.

Define $\overline{J}_\tau$ in the following manner.

1. $\epsilon \in \overline{J}_\tau$.

2. $\overline{v} \in \overline{J}_\tau$ for all $v \in I_\tau$.

3. If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$ for all $u \in B_{w_1,\tau}$.

4. The only words in $\overline{J}_\tau$ are the result of applying one of the above rules.

Take $J_\tau = \Psi(\overline{J}_\tau)$.

As an example, consider taking $\tau = \gamma = 2\ 1\ 4\ 3\ 5\ 6$. Then

$$\overline{J}_\gamma = \{\epsilon, \overline{2}, \overline{5}, \overline{22}\,\overline{2}, \overline{5}\,\overline{2}, \overline{2}\,\overline{5}, \overline{5}\,\overline{5}, \overline{2}\,\overline{22}\,\overline{2}, \overline{5}\,\overline{22}\,\overline{2}, \overline{2}\,\overline{5}\,\overline{2}, \overline{5}\,\overline{5}\,\overline{222}\,\overline{2}\,\overline{5}, \overline{5}\,\overline{2}\,\overline{5}, \overline{2}\,\overline{5}\,\overline{5}, \overline{5}\,\overline{5}\,\overline{5}, \ldots\}$$

Finally, for $w = w_1 w_2 \ldots w_n$, let

$$\mathcal{P}_w^\tau = \{\sigma \in S_{||w||} : \mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}\}$$

and $\mathcal{P}^{\tau}_{\epsilon} = \{\tau\}$.

We will now make a key observation. Suppose that $\overline{u} = \overline{u}_k \cdots \overline{u}_1 \in \overline{J}_\tau$ and $\Psi(\overline{u}) = w_1 \cdots w_n$. Thus $w = w_1 \cdots w_n \in J_\tau$. Now suppose that $\sigma \in \mathcal{P}^{\tau}_{w}$. Then

$$\mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}.$$

We can scan σ to discover that there are τ-matches at positions $1, 1 + w_1, 1 + w_1 + w_2, \ldots, 1 + w_1 + w_2 + \cdots + w_n$ so that we can recover $w_1, \ldots, w_n$ from σ. In addition, we claim that $\overline{u}$ can also be recovered. Let us now describe an algorithm which does this.

Step 1. Set $\overline{u}_1 = \overline{w}_n$. (This must be the case since the only words of length 1 in $\overline{J}_\tau$ are of the form $\overline{v}$ for $v \in I_\tau$.)

Step s+1. Suppose that we have recovered $\overline{u}_1, \ldots, \overline{u}_s$, $\Psi(\overline{u}_s \cdots \overline{u}_1) = w_b \cdots w_n$, and $\Psi(\overline{u}_s) = w_b \cdots w_c$ where $b \leq c$. It must be the case that $w_b + \cdots + w_c = r$ for some $1 \leq r \leq j - 1$. If $w_{b-1} + r \geq j$, then set $\overline{u}_{s+1} = \overline{w}_{b-1}$. Otherwise, let a be the unique integer such that $1 \leq a < b - 1$ such that $w_a + \cdots + w_c \geq j$, but $w_{a+1} + \cdots + w_c < j - 1$ and set $\overline{u}_{s+1} = \overline{w_a \cdots w_{b-1}}$. (In this latter case, it follows from our definitions that $w_a \cdots w_{b-1}$ is the unique cofinal sequence of $w_1 \cdots w_{b-1}$ such that $w_a \cdots w_{b-1} \in B_{w_b \cdots w_c, \tau}$ so that it must be the case that $\overline{u}_{s+1} = \overline{w_a \cdots w_{b-1}}$.)

As an example, consider $w = 222252222225 \in J_\gamma$ where $\gamma = 2\ 1\ 4\ 3\ 6\ 5$. The algorithm would proceed as follows.

Step 1. $\overline{u}_1 = \overline{5}$.

Step 2. $\overline{u}_2 = \overline{2}$ since $2 + 5 \geq 6$.

Step 3. $\overline{u}_3 = \overline{22}$ since $(2 + 2) + 2 \geq 6$.

Step 4. $\overline{u}_4 = \overline{2}$ since $2 + (2 + 2) \geq 6$.

Step 5. $\overline{u}_5 = \overline{22}$ since $(2 + 2) + 2 \geq 6$.

Step 6. $\overline{u}_6 = \overline{5}$ since $5 + (2+2) \geq 6$.

Step 7. $\overline{u}_7 = \overline{2}$ since $2 + 5 \geq 6$.

Step 8. $\overline{u}_8 = \overline{22}$ since $(2+2) + 2 \geq 6$.

Step 9. $\overline{u}_9 = \overline{2}$ since $2 + (2+2) \geq 6$.

Thus $w = \Psi(\overline{2}\ \overline{22}\ \overline{2}\ \overline{5}\ \overline{22}\ \overline{2}\ \overline{22}\ \overline{2}\ \overline{5})$. This given, for each word $w \in J_\tau$, let $\overline{\ell}(w)$ to be the length of the word $\overline{u} \in \overline{J}_\tau$ such that $\Psi(\overline{u}) = w$.

Remmel and Mendes proved the following corollary when they proved Kitaev's formula using their methods. For any permutation τ,

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| = \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}. \tag{6.2}$$

They noted that if the generating function for the number of permutations without any τ-matches is known, then Theorem 3.1 is able to refine all permutations of n by the maximum number of nonoverlapping τ-matches. However, Theorem 3.1 does not give any direct way to find the number of permutations without τ-matches and, in general, it is difficult to count the number of permutations σ in S_n with $\tau\text{-mch}(\sigma) = 0$. However, (6.2) provides an alternative approach to finding the number of permutations σ in S_n with $\tau\text{-mch}(\sigma) = 0$. That is, instead of counting the number of permutations in S_n with $\tau\text{-mch}(\sigma) = 0$, we may try to understand the sum on the right hand side of the statement of the corollary.

As an example of this phenomenon, suppose we wanted to find out more about the maximum number of nonoverlapping τ-matches when $\tau = 1\ 3\ 2$. The set I_τ contains only the integer 2. This greatly simplifies matters since we have that $J_\tau = I_\tau^* = \{2\}^*$. Suppose that $\sigma_1 \ldots \sigma_{2n+3} \in P_{2^n}^\tau$. Since there is τ-match starting at position $2i+1$ for $i = 0, \ldots, n$, it must be the case that $\sigma_{2i+1} < \sigma_{2i+2}, \sigma_{2i+3}$ for $i = 0, \ldots, n$. It follows that $\sigma_1 = 1$ and $\sigma_3 = 2$. It is not difficult to see that σ_2 can be any element of $\{3, \ldots, 2n+3\}$ and that $\text{red}(\sigma_3 \ldots \sigma_{2n+3}) \in P_{2^{n-1}}^\tau$ if $n \geq 1$. It follows that $|P_{2^n}^\tau| = (2n+1)|P_{2^{n-1}}^\tau|$ if $n \geq 1$. Since $P_{2^0}^\tau = 1$, it follows by induction that $|P^{\tau}_{2^n}| = (2n+1)!! = (2n+1)(2n-1)\cdots 3\cdot 1$ for $n \geq 0$. Thus, from 6.2, we have that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}|\{\sigma \in S_n : \mathrm{Mch}_{132}(\sigma) = \emptyset\}| &= \frac{1}{1-t+\sum_{n=0}^{\infty}(-1)^n|P^{132}_{2^n}|\frac{t^{2n+3}}{(2n+3)!}} \\ &= \frac{1}{1-t+\sum_{n=0}^{\infty}(-1)^n(\prod_{i=0}^{n}(2i+1))\frac{t^{2n+3}}{(2n+3)!}} \\ &= \frac{1}{1-\int exp(-t^2/2)dt}.\end{aligned}$$

From Theorem 3.1, the generating function refining the permutations of n by the maximum number of nonoverlapping τ-matches is equal to

$$\left(1-tx+(x-1)\int \exp\left(-t^2/2\right)\,dt\right)^{-1}$$

in the case $\tau = 1\ 3\ 2$.

6.2 Grouping permutations into classes

First we will define σ^c and σ^r. We will then show two lemmas that allow us to group permutations for our analysis.

Definition 6.1. Suppose $\sigma = \sigma_1\sigma_2\ldots\sigma_n$.

$$\sigma^c := (n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$$

For example, if $\sigma = 615324$, $\sigma^c = 162453$ and $(\sigma^c)^c = \sigma$.

Definition 6.2. Suppose $\sigma = \sigma_1\sigma_2\ldots\sigma_n$.

$$\sigma^r := \sigma_n\sigma_{n-1}\ldots\sigma_1$$

For example, if $\sigma = 615324$, $\sigma^r = 423516$ and $(\sigma^r)^r = \sigma$.

Proposition 6.3. *If $\tau, \sigma \in S_n$ and $I_\tau = I_\sigma$ then $J_\tau = J_\sigma$.*

Proof. Given any permutation $\alpha \in S_k$, recall the construction of the alphabet J_α. The only necessary information to construct J_α was the alphabet I_α and the length of α, namely j. Thus any two permutations of the same length which produce the same alphabet I will have the same alphabet J. □

Proposition 6.4. *For any permutation τ and word $w \in J_\tau$,*

$$\sigma \in P_w^\tau \Leftrightarrow \sigma^c \in P_w^{\tau^c}.$$

Proof. Suppose $\sigma \in P_w^\tau$ where $\sigma = \sigma_1\sigma_2\ldots\sigma_{\|w\|}$.

Consider a τ-match at the $(i+1)^{st}$ position, so $red(\sigma_{i+1}\sigma_{i+2}\ldots\sigma_{i+|\tau|}) = \tau$. This means that σ_{i+k} is the $(\tau_k)^{th}$ smallest element of $\sigma_{i+1}\sigma_{i+2}\ldots\sigma_{i+|\tau|}$. This implies that $|\tau|+1-\sigma_{i+k}$ is the $(\tau_k)^{th}$ largest element, or the $(|\tau|+1-\tau_k)^{th}$ smallest element in $(\sigma^c)_{i+1}(\sigma^c)_{i+2}\ldots(\sigma^c)_{i+|\tau|}$. Hence, $(\sigma^c)_{i+k}$ is the $((\tau^c)_k)^{th}$ smallest element in $(\sigma^c)_{i+1}(\sigma^c)_{i+2}\ldots(\sigma^c)_{i+|\tau|}$, showing that $red((\sigma^c)_{i+1}(\sigma^c)_{i+2}\ldots(\sigma^c)_{i+|\tau|}) = \tau^c$. In fact we argue that there could be no other τ^c-matches in σ^c as this would contradict the fact that $\mathrm{Mch}_\tau(\sigma) = \{0, i\}$.

□

Proposition 6.5. *For any permutation τ and word $w \in J_\tau$,*

$$\sigma \in P_w^\tau \Leftrightarrow \sigma^r \in P_{w^r}^{\tau^r}.$$

Proof. The proof is simple and left to the reader.

□

Corollary 6.6. *For any permutation τ,*

$$I_\tau = I_{\tau^c} = I_{\tau^r}.$$

Corollary 6.7. *For any permutation τ and word $w \in J_\tau$,*

$$|P_w^\tau| = |P_w^{\tau^c}| = |P_{w^r}^{\tau^r}|.$$

Corollary 6.8. *For any permutation $\tau \in S_j$ and a given $n \geq j$,*

$$\sum_{w \in J_\tau, \|w\|=n} (-1)^{l(w)} |P_w^\tau| = \sum_{w \in J_{\tau^c}, \|w\|=n} (-1)^{l(w)} |P_w^{\tau^c}| = \sum_{w \in J_{\tau^r}, \|w\|=n} (-1)^{l(w)} |P_w^{\tau^r}| \tag{6.3}$$

Proof. Showing the first two are equivalent is clear. To show that the first and third are equivalent consider the following. Suppose you have a word w in J_τ of length n. Then it is clear that w^r is a word in J_{τ^r}. So there is a one to one correspondence of words in these two alphabets. Therefore, summing over all words in one yields the same value as summing over all words in the other by the previous corollary. □

So, it makes sense to define equivalence classes of permutations closed under complementing and reversing, E_σ. Where $\sigma \in E \Rightarrow \sigma^c \in E$ and $\sigma^r \in E$. Since we are ultimately interested in $\sum_{w \in J_\tau, \|w\|=n} (-1)^{l(w)} |P_w^\tau|$, due to the previous corollaries, we need to only study permutations unique up to equivalence as previously defined. So in S^4, for example, there are 8 equivalence classes of permutations to study.

The representative element of each class is chosen to be the lexicographically least element in the class.

Type 0 - $E_{(1234)} = \{(1234), (4321)\}$
Type 1 - $E_{(1243)} = \{(1243), (4312), (3421), (2134)\}$
Type 2 - $E_{(1342)} = \{(1342), (4213), (2431), (3124)\}$
Type 2 - $E_{(1432)} = \{(1432), (4123), (2341), (3214)\}$
Type 3 - $E_{(1324)} = \{(1324), (4231)\}$
Type 4 - $E_{(1423)} = \{(1423), (4132), (3241), (2314)\}$
Type 5 - $E_{(2143)} = \{(2143), (3412)\}$
Type 6 - $E_{(2413)} = \{(2413), (3142)\}$

The subsequent sections will describe the work attempting to understand

$$\sum_{w \in J_\tau, \|w\|=n} (-1)^{l(w)} |P_w^\tau|$$

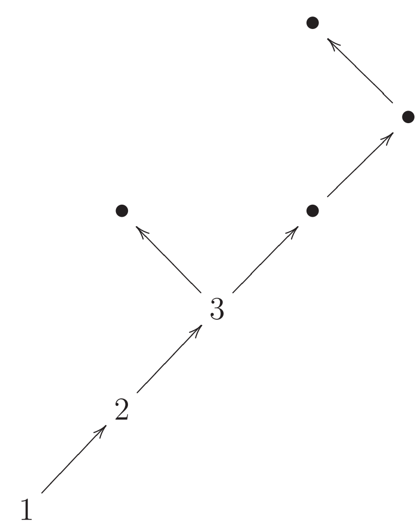
for various permutations of length 4.

6.3 Results on 1243

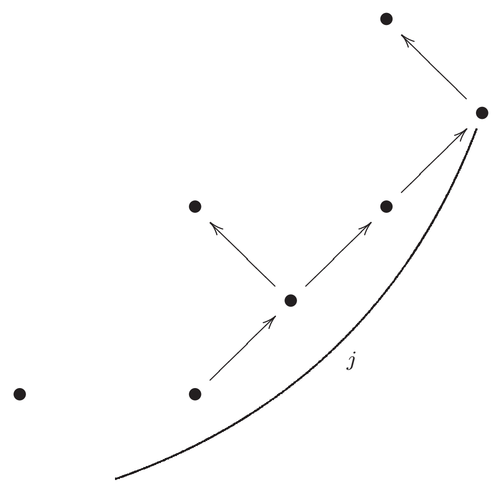
Proposition 6.9. *For* $\tau = (1243)$,

$$|P_{\{3^k\}}| = \prod_{i=0}^{k} (3i+1) \tag{6.4}$$

Proof. The Hasse diagram for a particular permutation in $P_{\{3\}}$ is shown below.



Examining the diagram, it becomes obvious that the bottom left corner must be the smallest element, 1, as there is no element smaller. Similarly, the adjacent element must be 2, and the next 3. After removing these fixed elements, the problem is reduced to applying labels to a Hasse diagram that looks like the following. or a given j, it is clear that there will be a dot and j patterns of τ.



The number of choices for the dot in the diagram is 7, or in general it will be $3j+1$. Continuing recursively up the chain in this fashion yields the result. □

Using this fact, we can derive the following:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}|\{\sigma\in S_n:\mathrm{Mch}_{1243}(\sigma)=\emptyset\}| &= \frac{1}{1-t+\sum_{n=0}^{\infty}(-1)^n|P_{3^n}^{1243}|\frac{t^{3n+4}}{(3n+4)!}}\\ &= \frac{1}{1-t+\sum_{n=0}^{\infty}(-1)^n(\prod_{i=0}^{n}(3i+1))\frac{t^{3n+4}}{(3n+4)!}}\\ &= \frac{1}{1-\sum_{n=0}^{\infty}(-1)^n(\prod_{i=0}^{n-1}(3i+1))\frac{t^{3n+1}}{(3n+1)!}}\end{aligned}$$

From Theorem 3.1, the generating function refining the permutations of n by the maximum number of nonoverlapping τ-matches is equal to

$$\left(1-tx+(x-1)\sum_{n=0}^{\infty}(-1)^n(\prod_{i=0}^{n-1}(3i+1))\frac{t^{3n+1}}{(3n+1)!}\right)^{-1}$$

in the case $\tau=1\ 2\ 4\ 3$. In fact, this idea could be generalized to compute similar generating functions for permutations of the form $\sigma=1\ 2\ 3\ \ldots\ n\ (n-1)$.

Unfortunately, this is the only case that we have been able to produce this type of result. However, for $\sigma=1243$ and 1423 we have been able to produce an algorithm to compute $|P_w^\sigma|$ for any word $w \in J_\tau$. When the number of elements in the alphabet J_τ is more than one, things become much more complicated. The next few sections contain partial results on our research into other permutations.

It also turns out that when dealing with "short" permutations it is often the case that $I_\tau^* = J_\tau$. In fact for all permutations of length 4 this is the case, thus J_τ is never difficult for us to compute.

6.4 Results on 1423

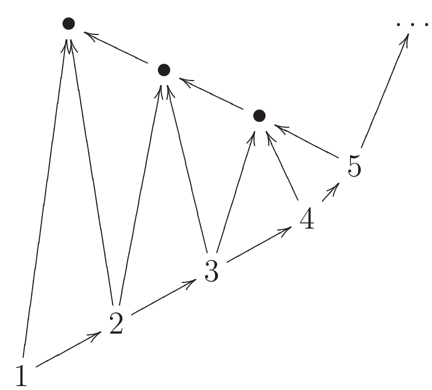
Here are two propositions which will yield an algorithm to find $|P_w|$ for any word w. We should mention that $I_{1423} = \{2, 3\}$. We first note a trivial case without proof.

Proposition 6.10. *If $\tau = (1423)$ and $w = 2^k$, then $|P_w| = 1$.*

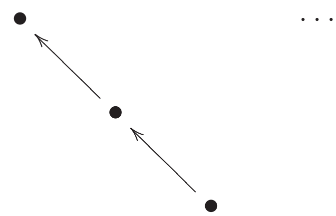
Proposition 6.11. *If $\tau = (1423)$ and $w = 2^k 3\overline{w}$, then*

$$|P_w| = \binom{\|w\| - (k+3)}{k+1} |P_{\overline{w}}| \tag{6.5}$$

Proof. The Hasse diagram for a particular permutation in $P_{2^2\overline{w}}$ is shown below.



Examining the diagram, it becomes obvious that the bottom left corner must be the smallest element, 1, as there is no element smaller. Similarly, the adjacent element must be 2, and so on to 5. After removing 1,2,3, and 4, the problem is reduced to applying labels to a Hasse diagram that looks like the following. For a given k, it is clear that there will be $k+1$ connected dots followed by the diagram for $P_{\overline{w}}$, designated by "...".



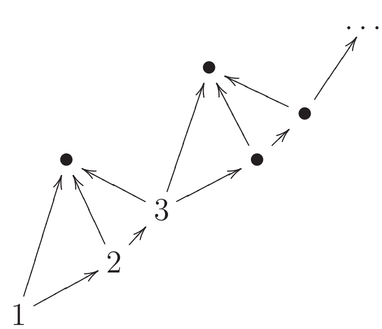
Since these dots have been broken off of the rest of the Hasse diagram, we need only choose these $k+1$ numbers and put them in decreasing order and then label the rest of the diagram with the remaining numbers. In general, there are $\|w\| - (k+3)$ numbers to choose from. Hence $|P_w| = \binom{\|w\|-(k+3)}{k+1}|P_{\overline{w}}|$.

□

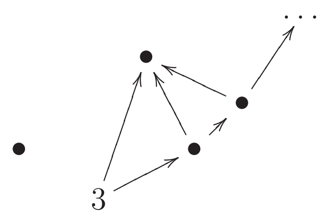
Proposition 6.12. *If* $\tau = (1423)$ *and* $w = 3^k\overline{w}$, *then*

$$|P_w| = |P_{\overline{w}}|(\prod_{i=1}^{k}(\|w\| - 3i)) \tag{6.6}$$

Proof. The Hasse diagram for a particular permutation in $P_{3\overline{w}}$ is shown below.



Examining the diagram, it becomes obvious that the bottom left corner must be the smallest element, 1, as there is no element smaller. Similarly, the adjacent element must be 2, and the next 3. After removing these fixed elements, the problem is reduced to applying labels to a Hasse diagram that looks like the following. For a given j, it is clear that there will be a separated dot followed by the diagram for $P_{2^{j-1}}$ and then the diagram for $P_{\overline{w}}$, designated by "…".



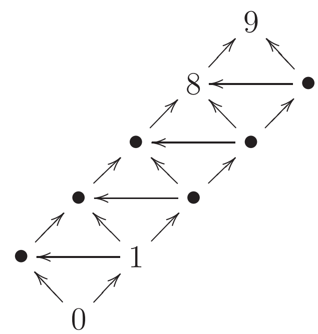
Since this dot has been broken off of the rest of the Hasse diagram, we need only choose a value for the dot and then label the rest of the diagram with the remaining numbers. There are $\|w\| - 3$ numbers to choose from. Continuing down the chain in this fashion yields the result, on the i^{th} step there will be $\|w\| - 3i$ numbers to choose from. Hence, $|P_w| = (\prod_{i=1}^{k} (\|w\| - 3i))|P_{\overline{w}}|$.

□

6.5 Results on 1324

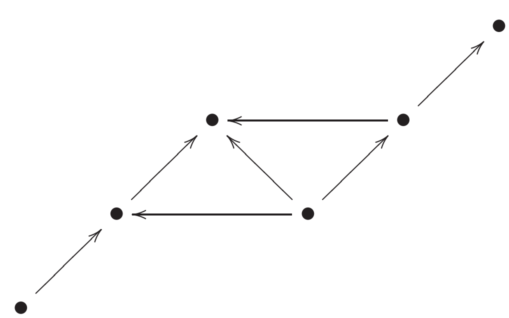
Proposition 6.13. *For* $\tau = (1324)$*,* $|P_{\{2^k\}}| = C_{k+1}$ *(where* C_k *is the* k^{th} *Catalan number.)*

Proof. The Hasse diagram for a particular permutation in $P_{\{2^3\}}$ is shown below. It is clear that for a permutation in $P_{\{2^j\}}$, there will be $j+1$ "boxes" and we need to apply labels $\{0, 1, \ldots, 2j+3\}$ to the vertices so that the ordering on the particular Hasse diagram is satisfied.

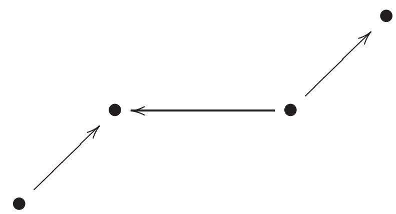


Examining the diagram, it becomes obvious that the top right corner must be the largest element, as there is no element larger. Similarly, the bottom left corner must be the smallest element, as there is no element smaller. By a similar argument, the adjacent elements to the previous must be the next largest and smallest respectively. In general, $\{0, 1, 2j+2, 2j+3\}$ are forced to be placed in these locations.

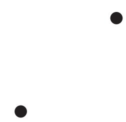
So the problem is reduced to applying labels $\{0, 1, \ldots, 2j-1\}$ to a Hasse diagram that looks like the following. For a given j, it is clear that there will be $j-2$ slanted "boxes". In our previous example, where $j=3$, there is one slanted box.



Define $C_{n+1} = |P_{\{2^n\}}|$. Also, Define $c_{n,n-k}$ = number of placements of $\{0, 1, \ldots, 2n-1\}$ on a Hasse diagram as shown above where the bottom left element is chosen to be k and there are $n-2$ slanted "boxes". We interpret the expression 0 slanted boxes to be the following,



and the expression -1 slanted boxes to be



so that description makes sense when $n = 1, 2$.

In this fashion, it is easy to see that

$$C_{j+1} = c_{j,0} + c_{j,1} + \ldots + c_{j,j} \tag{6.7}$$

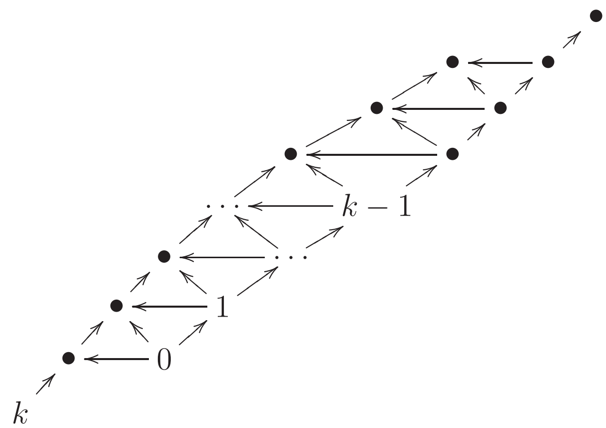
because the only choices for the bottom left element are $(0, 1, \ldots, j)$. This is because there are $j - 1$ elements forced to be larger in the diagram, so the largest it could be is $2j - 1 - (j - 1) = j$, and there are 0 elements forced to be smaller, so the smallest it could be is 0.

It will suffice to show the following recursion.

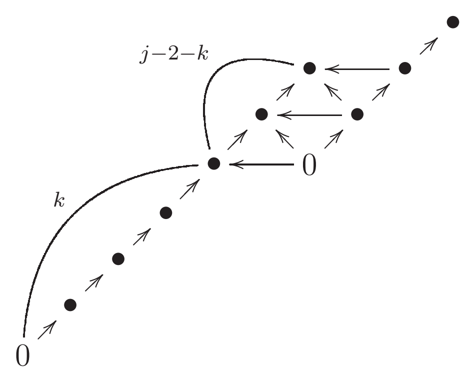
$$c_{n,k} = c_{n,k-1} + c_{n-1,k} \tag{6.8}$$

This recursion, coupled with the fact that $c_{n,0} = 1$ and $c_{n,n+1} = 0$, shows that the terms $c_{n,k}$ is the n, k element of the Catalan triangle and thus C_k is the k^{th} Catalan number.

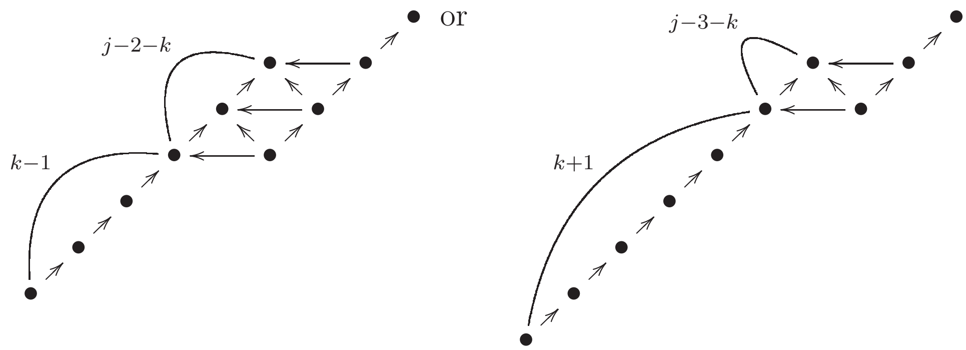
Consider $c_{j,j-k}$. We can clearly see the diagonals of these diagrams form two increasing sequences, the choice of k for the bottom left element forces the placement of $0, 1, \ldots, k - 1$ onto the minimal elements of the lower increasing sequence as depicted below.



After removing what is fixed, we are looking at a Hasse diagram that looks like the following.



In this diagram, we could place 0 at either the bottom left element or at the minimal element in the lower increasing sequence. This is pictured graphically below after having removed the fixed element 0.



It is clear to see that after the placement of 0 in the lower left position, the rest of the placement is the same as $c_{j-1,j-1-(k-1)} = c_{j-1,j-k}$. Also, after the placement of 0 in the minimal element of the lower increasing sequence, the rest of the placement is the same as $c_{j,j-(k+1)}$ so

$$c_{j,j-k} = c_{j-1,j-k} + c_{j,j-k-1} \tag{6.9}$$

and thus the recursion is satisfied.

□

Proposition 6.14. *For* $\tau = (1324)$, *if* $\omega = \{3^{\beta_0}2^{\alpha_1}3^{\beta_1}2^{\alpha_2}3^{\beta_2}\dots 2^{\alpha_n}3^{\beta_n}\}$ *where* $\beta_1, \beta_2, \dots, \beta_{n-1} \neq 0$ *and* $\alpha_1, \alpha_2, \dots, \alpha_n \neq 0$ *then*

$$|P_{\{\omega\}}| = \prod_{i=1}^{n} |P_{\{2^{\alpha_i}\}}| = \prod_{i=1}^{n} C_{\alpha_i+1} \tag{6.10}$$

Proof. For any $\omega, ||\omega|| = n$, consider $\sigma \in P_{\{\omega\}}$. It is clear to see that the largest element of σ must be in the last position.

Now consider $\gamma \in P_{\{\omega 3\}}$. Now since $\tau = 1324$ the last three elements of γ are forced to be $n+1, n+2$, and $n+3$ respectively. So after thinking of these elements as being fixed, it is clear that $|P_{\{\omega 3\}}| = |P_{\{\omega\}}|$.

In effect, $|P_{\{\omega\}}|$ is invariant to adding 3 to any word ω.

So thinking of building words by adding sequences of twos and threes. Once a sequence of j threes has been added to ω, the last $3j$ elements of a particular

$\sigma \in P_{\{\omega\}}$ are fixed, meaning $|P_{\{\omega\}}| = |P_{\{\omega 3^j\}}|$. Now consider adding a sequence of k twos to the word. Notice that adding onto σ is the same as starting a new permutation in $P_{\{2^k\}}$ but with shifted indices. So, we can say that

$$|P_{\{\omega 3^j 2^k\}}| = |P_{\{\omega\}}| \cdot |P_{\{2^k\}}| \tag{6.11}$$

Continuing inductively yields the result.

□

6.6 Results on 2143

First we define $S_{1,\{\}} := 3$ and $S_{1,\{1\}} := 3$. Then we recursively define S as follows.

$$S_{n,\{k_1,\ldots,k_m\}} = \left(\sum_{i=1}^{m} S_{n,\{k_1,\ldots,k_m\}\setminus k_i}\right) + (2n+m+1)S_{n-1,\{k_1-1,\ldots,k_m-1\}}\chi(k_1 \neq 1) \tag{6.12}$$

We then define $T_{1,\{\}} := 1$ and $T_{1,\{1\}} := 1$. Then we recursively define T as follows.

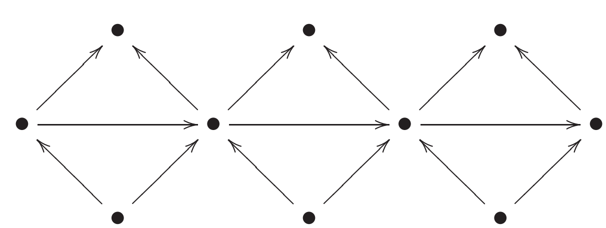
$$T_{n,\{k_1,\ldots,k_m\}} = \left(\sum_{i=1}^{m} T_{n,\{k_1,\ldots,k_m\}\setminus k_i}\right) + S_{n-1,\{k_1-1,\ldots,k_m-1\}}\chi(k_1 \neq 1) \tag{6.13}$$

Proposition 6.15. *For $\tau = (2143)$,*

$$|P_{\{3^k\}}| = T_{k+1,\{1,2,\ldots,k+1\}} \tag{6.14}$$

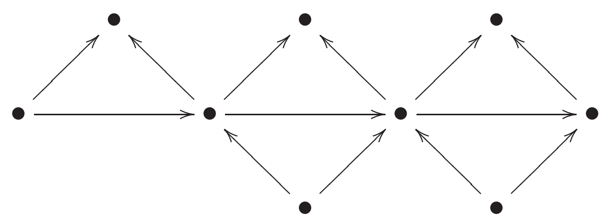
Proof. The Hasse diagram for a particular permutation in $P_{\{3^2\}}$ is shown below. We will define $\hat{T}_{3,\{1,2,3\}}$ to be the number of placements on this diagram. We interpret the first 3 to mean there are 3 diamonds and the set $\{1,2,3\}$ to mean that they are full diamonds (this will become clear soon).

$\hat{T}_{3,\{1,2,3\}} \equiv$

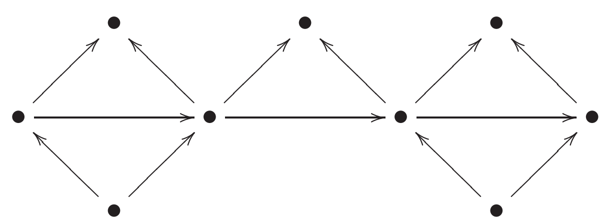


Examining the diagram, it becomes obvious that one of the three bottom elements must be the smallest element, 1, as there is no element smaller than these. So we have three choices to place the smallest element. So the number of placements on this Hasse diagram is the sum of the number of placements on the following 3 Hasse diagrams. In general, $\hat{T}_{n,\{k_1,k_2,...,k_m\}}$ = number of placements on a Hasse diagram that has n diamonds where the k_1^{th}, k_2^{th}, ..., and k_m^{th} diamonds are full diamonds and the others are diamonds missing the bottom element. In this fashion,

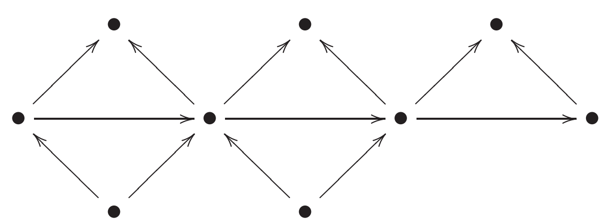
$\hat{T}_{3,\{2,3\}} \equiv$



$\hat{T}_{3,\{1,3\}} \equiv$



$\hat{T}_{3,\{1,2\}} \equiv$

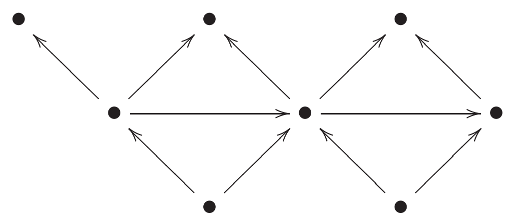


So clearly, as long as $k_1 = 1$,

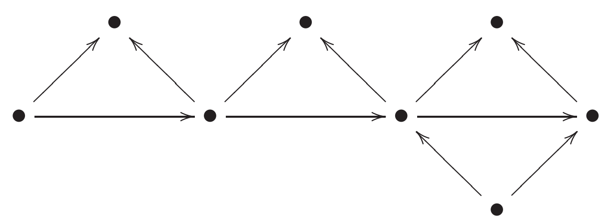
$$\hat{T}_{n,\{k_1,...,k_m\}} = (\sum_{i=1}^{m} \hat{T}_{n,\{k_1,...,k_m\}\backslash k_i}) \tag{6.15}$$

Lets examine what happens when $k_1 \neq 1$. Lets consider the first diagram of the previous. Again, upon examination, it is clear that the far left element, or one of the bottom elements must be the smallest element, 1, as there is no element smaller than these. So we have three choices to place the smallest element. So the number of placements on this Hasse diagram is the sum of the number of placements on the following 3 Hasse diagrams. In general, $\hat{S}_{n,\{k_1,k_2,\ldots,k_m\}}$ = number of placements on a Hasse diagram that has n diamonds with a "handle" on the left where the k_1^{th}, k_2^{th}, ..., and k_m^{th} diamonds are full diamonds and the others are diamonds missing the bottom element. In this fashion,

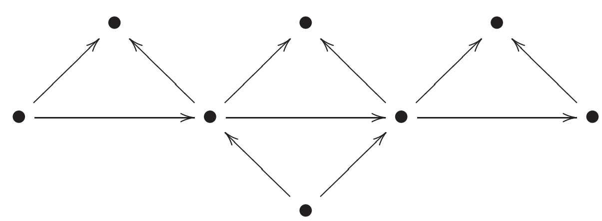
$\hat{S}_{2,\{1,2\}} \equiv$



$\hat{T}_{3,\{3\}} \equiv$



$\hat{T}_{3,\{2\}} \equiv$

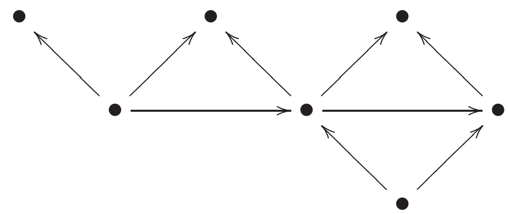


So you can see that when $k_1 \neq 1$ and you choose the left element as the smallest, you add a "handle" and shift the other diamonds to the left one while decreasing the total number of diamonds by 1. This shows that

$$\hat{T}_{n,\{k_1,\ldots,k_m\}} = (\sum_{i=1}^{m} \hat{T}_{n,\{k_1,\ldots,k_m\}\setminus k_i}) + \hat{S}_{n-1,\{k_1-1,\ldots,k_m-1\}}\chi(k_1 \neq 1) \qquad (6.16)$$

Now consider, $\hat{S}_{n,\{k_1,\ldots,k_m\}}$, the number of placements on a Hasse diagram that looks like the following ($\hat{S}_{2,\{2\}}$ in our example.)

$\hat{S}_{2,\{2\}} \equiv$

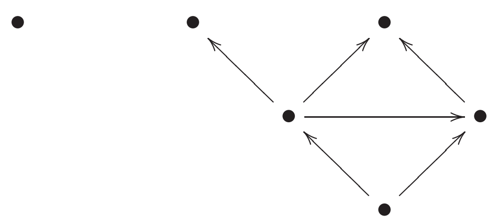


Similar to $\hat{T}$, when $k_1 = 1$, it is clear that

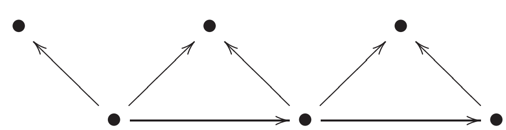
$$\hat{S}_{n,\{k_1,\ldots,k_m\}} = (\sum_{i=1}^{m} \hat{S}_{n,\{k_1,\ldots,k_m\}\backslash k_i}) \tag{6.17}$$

Lets examine what happens when $k_1 \neq 1$, as in our example. Again, upon examination, it is clear that the element under the "handle", or one of the bottom elements must be the smallest element, 1, as there is no element smaller than these. So we have two choices to place the smallest element. So the number of placements on this Hasse diagram is the sum of the number of placements on the following 2 Hasse diagrams.

$6\hat{S}_{1,\{1\}} \equiv$



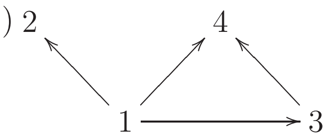
$\hat{S}_{2,\{\}} \equiv$

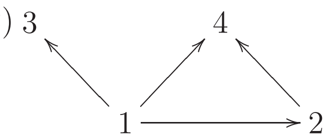


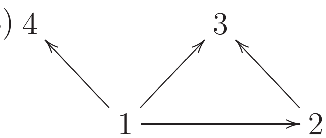
So you can see that when $k_1 \neq 1$ and you choose the element under the "handle" as the smallest, you break off a dot and shift the other diamonds to the left one while decreasing the total number of diamonds by 1. There are 6 choices or in general, $(2n + m + 1)$. This shows that

$$\hat{S}_{n,\{k_1,\ldots,k_m\}} = (\sum_{i=1}^{m} \hat{S}_{n,\{k_1,\ldots,k_m\}\backslash k_i}) + (2n+m+1)\hat{S}_{n-1,\{k_1-1,\ldots,k_m-1\}}\chi(k_1 \neq 1) \quad (6.18)$$

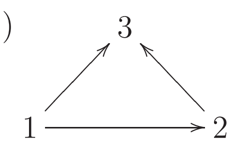
And $\hat{S}_{1,\{\}} = \hat{S}_{1,\{1\}} = 3$ since the only placements are

1) 

2) 

3) 

And $\hat{T}_{1,\{\}} = \hat{T}_{1,\{1\}} = 1$ since the only placement is

1) 

This shows that $\hat{T} = T$ and $\hat{S} = S$, proving the result. □

7

Enumerating descents of a fixed value

We will now alter the descent statistic in another way. For example, suppose we only wanted to count descents that drop by a fixed value, k. If we let

$$\sigma = 3\ 8\ 6\ 2\ 4\ 7\ 5\ 1,$$

we can see that $Des(\sigma) = \{2, 3, 4\}$ and thus $des(\sigma) = 3$. We can see that descent in the second position is from 8 to 6, and thus we would consider this a drop by 2, or a 2-drop. In general, if $\sigma_i - \sigma_{i+1} = k > 0$, then we call the descent at position i a k-drop.

We can now try to enumerate permutations by the number of k-drops. To be more rigorous, suppose we define, for $k > 0$,

$$\begin{aligned} Des_k(\sigma) &= \{i | \sigma_i - \sigma_{i+1} = k\} \\ des_k(\sigma) &= |Des_k(\sigma)| \\ P_{n,k}(x) &= \sum_{\sigma \in S_n} x^{des_k(\sigma)} = \sum_{s=0}^{n-k} P_{n,k,s} x^s. \end{aligned}$$

We will later extend this definition so that $P_{n,k}(x)$ is defined in the case when $k = 0$.

7.1 An approach using inclusion/exclusion

We can employ the same techniques we used in Chapter 3 to obtain an expression for $P_{n,k,s}$.

Suppose that we fix k and define the following. Let $\Omega_n = \{(i+k,i)|1 \leq i \leq n-k\}$. Given $\sigma \in S_n$, we define $Pr_{\Omega,n}(\sigma)$, the property set of σ relative to Ω, to be the set of all $(i+k,i) \in \Omega_n$ where i follows $i+k$ in σ. Then we define the following.

1. For each $T \subseteq \Omega_n$, let $E_{=T,\Omega,n} = \{\sigma \in S_n : Pr_{\Omega,n}(\sigma) = T\}$ and $\beta_{T,\Omega,n} = |E_{=T,\Omega,n}|$.

2. For each $T \subseteq \Omega_n$, let $E_{\supseteq T,\Omega,n} = \{\sigma \in S_n : Pr_{\Omega,n}(\sigma) \supseteq T\}$ and $\alpha_{T,\Omega,n} = |E_{\supseteq T,\Omega,n}|$.

3. For each $r \geq 0$, let $\beta_{r,\Omega,n} = \sum_{S\subseteq\Omega_n,|S|=r} \beta_{S,\Omega,n}$ and $\alpha_{r,\Omega,n} = \sum_{S\subseteq\Omega_n,|S|=r} \alpha_{S,\Omega,n}$.

It is an easy consequence of the inclusion-exclusion principle that

$$\sum_{t\geq 0} \beta_{t,\Omega,n} x^t = \sum_{t\geq 0} \alpha_{t,\Omega,n}(x-1)^t. \tag{7.1}$$

It is also easy to see from our definitions that

$$\sum_{t\geq 0} \beta_{t,\Omega,n} x^t = P_{n,k}(x). \tag{7.2}$$

Thus we get an expression for $P_{n,k}(x)$ by calculating the RHS of (7.1).

Next we observe that it is easy to compute $\alpha_{T,\Omega,n}$. $\alpha_{T,\Omega,n} = (n-|T|)!$. To show this, we need to construct $E_{\supseteq T,\Omega,n}$ which consists of all permutations $\sigma \in S_n$ such that each pattern in T occurs consecutively in σ. We do this by first constructing the maximal blocks of elements of $\{1,\ldots,n\}$ where xy occurs consecutively in a block if and only if $xy \in T$. For example, if $n=7$, $k=2$ and $T = \{(5,3),(3,1),(6,4)\}$, then the maximal blocks constructed from T are 531, 64, 2 and 7. Then it is easy to see that any permutation of the maximal blocks constructed from T corresponds to a permutation $\sigma \in E_{\supseteq T,\Omega,n}$. It is easy to see that the number of maximal blocks of $\{1,\ldots,n\}$ constructed from T is $n-|T|$. Thus $\alpha_{T,\Omega,n} = |E_{\supseteq T,\Omega,n}| = (n-|T|)!$.

Thus to compute $\alpha_{t,\Omega,n}$, we need only count the number of subsets of size t in Ω_n. We can think of this problems as counting the number of rook placements of size t in a certain board associated with Ω_n. That is, given Ω_n, let $B_{\Omega,n}$ be the set of all (x,y) such that $xy \in \Omega_n$. For example, if $k=2$ so that $\Omega_7 = \{(3,1),(4,2),(5,3),(6,4),(7,5)\}$, then $B_{\Omega,7}$ consists of the shaded cells on the board pictured in Figure 7.1.

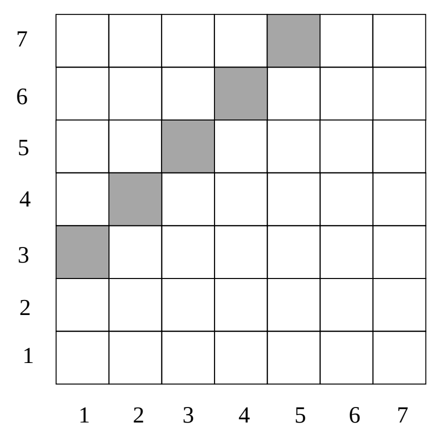


Figure 7.1: The board $B_{\Omega,7}$.

Similar to what we did before, it is easy to see that number of subsets of size t in Ω_n equals $r_t(B_{\Omega,n})$ and thus, $\alpha_{t,\Omega,n} = (n-t)!r_t(B_{\Omega,n})$.

However, counting rook placements on this board is simple as no two cells are in the same row or column. Thus, to place t rooks on $B_{\Omega,n}$, you just need to choose the t cells to place the rooks. There are $n-k$ cells on the board $B_{\Omega,n}$, and thus, we have that $r_t(B_{\Omega,n}) = \binom{n-k}{t}$ meaning that

$$\alpha_{t,\Omega,n} = (n-t)!\binom{n-k}{t}$$

It follows that

$$\begin{aligned} P_{n,k}(x) &= \sum_{t\geq 0}\beta_{t,\Omega,n}x^t = \sum_{t\geq 0}\alpha_{t,\Omega,n}(x-1)^t \\ &= \sum_{t\geq 0}(n-t)!\binom{n-k}{t}\sum_{s=0}^{t}(-1)^{t-s}\binom{t}{s}x^s \\ &= \sum_{s\geq 0}x^s\sum_{t=s}^{n-k}(n-t)!(-1)^{t-s}\binom{t}{s}\binom{n-k}{t}. \end{aligned} \tag{7.3}$$

Thus we have the following formula for $P_{n,k,s}$.

Theorem 7.1. *For $n \geq 1$, $0 \leq k \leq n$ and $s \geq 0$,*

$$P_{n,k,s} = \sum_{t=s}^{n-k}(-1)^{t-s}(n-t)!\binom{t}{s}\binom{n-k}{t}. \tag{7.4}$$

We also have the following trivial fact.

Theorem 7.2. *For $n \geq 1$, $n < k$ and $s \geq 0$,*

$$P_{n,k,s} = \begin{cases} n!, & s = 0 \\ 0, & s > 0 \end{cases} \tag{7.5}$$

Following directly from Theorem 7.1, we have the following corollary.

Corollary 7.3. *For $n \geq k$,*

$$P_{n,k,n-k} = k!.$$

Proof. We have the following formula for $P_{n,k,n-k}$ from Theorem 7.1,

$$P_{n,k,n-k} = \sum_{t=n-k}^{n-k}(n-t)!(-1)^{t-(n-k)}\binom{t}{n-k}\binom{n-k}{t}.$$

So, there is only the $t = n-k$ summand and thus we have that

$$P_{n,k,n-k} = (n-(n-k))!(-1)^{(n-k)-(n-k)}\binom{n-k}{n-k}\binom{n-k}{n-k} = k!.$$

However, we can also provide a combinatorial proof of this fact. Suppose we wish to construct a permutation in S_n having the maximum number of k-drops. If we break up n into sets of equivalence classes mod k, each element of any given equivalence class should be involved in a k-drop except for the smallest element. This means that the elements of a given equivalence class must appear in descending order in the permutation. This is true for each of the k equivalence classes. Thus we can think of just gluing together the elements of each equivalence class in descending order into blocks. We can now permute the order of these k blocks and thus we have that $P_{n,k,n-k} = k!$. □

7.2 A connection to excedences

The excedence set of a permutation is defined as follows.

$$Exc(\sigma) = \{i | i < \sigma_i\}$$

$$exc(\sigma) = |Exc(\sigma)|.$$

It is a well known fact that $des(\sigma)$ and $exc(\sigma)$ are equidistributed over S_n. In other words,

$$\sum_{\sigma \in S_n} x^{des(\sigma)} = \sum_{\sigma \in S_n} x^{exc(\sigma)}.$$

In fact there is a very nice bijection, Θ, that verifies this fact. We now present this bijection by demonstrating how it acts on an example.

Suppose

$$\sigma = 935164287.$$

Note that $des(\sigma) = 4$. We scan from left to right until we reach the smallest number, namely 1 and then cut the permutation at this location. We then continue to the right and recursively cut after the smallest remaining number until we reach the end of the permutation. Performing these cuts on σ yields

$$9351/642/87/.$$

Now, construct a permutation by making cycles out of the elements in the blocks in reverse order. This yields

$$(1,5,3,9)(2,4,6)(7,8) = \Theta(\sigma),$$

the image of our bijection. Notice also that $exc(\Theta(\sigma)) = 4$, as expected.

More importantly notice that each descent $\sigma_i > \sigma_{i+1}$ corresponds to an excedence $\Theta(\sigma_{i+1}) = \sigma_i$.

This fact means that if we define, for $k \geq 0$,

$$\begin{aligned} Exc_k(\sigma) &= \{i|\sigma(i) - i = k\} \\ exc_k(\sigma) &= |Exc_k(\sigma)|. \end{aligned}$$

We have the following,

$$\sum_{\sigma \in S_n} x^{des_k(\sigma)} = \sum_{\sigma \in S_n} x^{exc_k(\sigma)}. \tag{7.6}$$

In particular, when $k = 0$, the LHS of (7.6) does not make sense with our definitions, but the RHS does. A zero excedence at i means that $i = \sigma(i)$, or that i is a fixed point of σ. So we can extend our definition of $P_{n,k}(x)$ in the case where $k = 0$. In particular, $P_{n,0,s}$ counts the number of permutations in S_n that have exactly s fixed points.

Recall that the set $\mathbb{D}_n$, is the set of derangements of $[n]$. A derangement of $[n]$ is a permutation of $[n]$ that has no fixed points. Formally,

$$\mathbb{D}_n = \{\sigma \in S_n | \sigma(i) \neq i \text{ for all } i\}.$$

We define $D_n = |\mathbb{D}_n|$.

Thus, we have a natural connection with derangements with the following equation.

$$P_{n,0,0} = D_n \tag{7.7}$$

Examine Table 7.1, which illustrates the values of the polynomials, $P_{n,k}$ for $1 \leq n \leq 9$ and $1 \leq k \leq 3$.

Studying these values, certain things become apparent, some of the most interesting of these observations are proved in the following section.

Table 7.1: Values of $P_{n,k}(x)$ for select n and k

$P_{1,0}(x) = x$

$P_{2,0}(x) = 1 + x^2$

$P_{3,0}(x) = 2 + 3x + x^3$

$P_{4,0}(x) = 9 + 8x + 6x^2 + x^4$

$P_{5,0}(x) = 44 + 45x + 20x^2 + 10x^3 + x^5$

$P_{6,0}(x) = 265 + 264x + 135x^2 + 40x^3 + 15x^4 + x^6$

$P_{7,0}(x) = 1854 + 1855x + 924x^2 + 315x^3 + 70x^4 + 21x^5 + x^7$

$P_{8,0}(x) = 14833 + 14832x + 7420x^2 + 2464x^3 + 630x^4 + 112x^5 + 28x^6 + x^8$

$P_{9,0}(x) = 133496 + 133497x + 66744x^2 + 22260x^3 + 5544x^4 + 1134x^5 + 168x^6 + 36x^7 + x^9$

$P_{1,1}(x) = 1$

$P_{2,1}(x) = 1 + x$

$P_{3,1}(x) = 3 + 2x + x^2$

$P_{4,1}(x) = 11 + 9x + 3x^2 + x^3$

$P_{5,1}(x) = 53 + 44x + 18x^2 + 4x^3 + x^4$

$P_{6,1}(x) = 309 + 265x + 110x^2 + 30x^3 + 5x^4 + x^5$

$P_{7,1}(x) = 2119 + 1854x + 795x^2 + 220x^3 + 45x^4 + 6x^5 + x^6$

$P_{8,1}(x) = 16687 + 14833x + 6489x^2 + 1855x^3 + 385x^4 + 63x^5 + 7x^6 + x^7$

$P_{9,1}(x) = 148329 + 133496x + 59332x^2 + 17304x^3 + 3710x^4 + 616x^5 + 84x^6 + 8x^7 + x^8$

$P_{1,2}(x) = 1$

$P_{2,2}(x) = 2$

$P_{3,2}(x) = 4 + 2x$

$P_{4,2}(x) = 14 + 8x + 2x^2$

$P_{5,2}(x) = 64 + 42x + 12x^2 + x^3$

$P_{6,2}(x) = 362 + 256x + 84x^2 + 16x^3 + 2x^4$

$P_{7,2}(x) = 2428 + 1810x + 640x^2 + 140x^3 + 20x^4 + 2x^5$

$P_{8,2}(x) = 18806 + 14568x + 5430x^2 + 1280x^3 + 210x^4 + 24x^5 + 2x^6$

$P_{9,2}(x) = 165016 + 131642x + 50988x^2 + 12670x^3 + 2240x^4 + 294x^5 + 28x^6 + 2x^7$

$P_{1,3}(x) = 1$

$P_{2,3}(x) = 2$

$P_{3,3}(x) = 6$

$P_{4,3}(x) = 18 + 6x$

$P_{5,3}(x) = 78 + 36x + 6x^2$

$P_{6,3}(x) = 426 + 234x + 54x^2 + 6x^3$

$P_{7,3}(x) = 2790 + 1704x + 468x^2 + 72x^3 + 6x^4$

$P_{8,3}(x) = 21234 + 13950x + 4260x^2 + 780x^3 + 90x^4 + 6x^5$

$P_{9,3}(x) = 183822 + 127404x + 41850x^2 + 8520x^3 + 1170x^4 + 108x^5 + 6x^6$

7.3 Facts about $P_{n,k,s}$

We can easily construct a recursion for $P_{n,k,s}$ using the techniques from Chapter 3. In fact we arrive at the following.

Theorem 7.4. *For $n \geq 2$, $0 \leq k < n$, and $s \geq 1$,*

$$P_{n,k,s} = (n-s-1)P_{n-1,k,s} + (s+1)P_{n-1,k,s+1} + P_{n-1,k,s-1}. \tag{7.8}$$

Proof. We have been able to verify the theorem algebraically using our formula given in 7.1. In fact we prove a generalization of this which can be found in Section 7.4. A combinatorial proof is much more simple and is presented here.

Imagine inserting n into a permutation $\tau \in S_{n-1}$ to obtain $\sigma \in S_n$ such that σ has s k-drops. Inserting n will either increase by 1, decrease by 1, or not change the number of k-drops. Thus we need to only consider the cases where τ has $s-1$, s, or $s+1$ k-drops. If τ has $s-1$ k-drops, then n must create a k-drop and must be placed in front of $n-k$. Thus, there is a contribution of $P_{n-1,k,s-1}$ to $P_{n,k,s}$. If τ has $s+1$ k-drops, then n must destroy a k-drop and must be placed between one of the $s+1$ existing k-drops. Thus, there is a contribution of $(s+1)P_{n-1,k,s+1}$ to $P_{n,k,s}$. If τ has s k-drops, then n must not create or destroy a k-drop and must not be placed between one of the s existing k-drops or in front of $n-k$. Thus, there is a contribution of $(n-s-1)P_{n-1,k,s}$ to $P_{n,k,s}$. □

Theorem 7.5. *For $n \geq 1$, $0 \leq k < n$, and $s \geq 0$,*

$$P_{n,k,s} = \binom{n-k}{s} P_{n-s,k,0}. \tag{7.9}$$

Proof. It is easy to verify this theorem algebraically using the formula given in 7.1. In fact a more general version of this proof appears in Section 7.4. So instead, we will present both a combinatorial proof and a bijective proof of this fact. We first will present the combinatorial proof. We will first introduce a rook board that is associated with $P_{n,k,s}$. Since $P_{n,k,s}$ counts the number of permutations of S_n having s k-excedences. It is easy to see that if we look at the graph of any permutation satisfying this property with i on the x-axis and $\sigma(i)$ on the y-axis, it must intersect the diagonal connecting $(1, 1+k)$ and $(n-k, n)$ exactly s times. Thus, we define this diagonal to be $B(P_{n,k,s})$. See Figure 7.2 for an example of $B(P_{7,3,2})$.

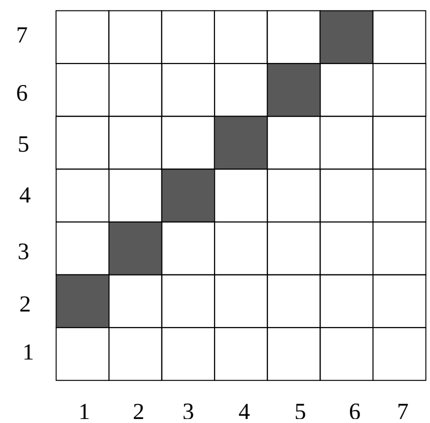


Figure 7.2: The board associated with $P_{7,3,2}$.

The LHS counts the number of placements of n non-attacking rooks on the $n \times n$ grid that hit $B(P_{n,k,s})$ exactly s times. For the RHS, we can think of the binomial coefficient as choosing the cells on the board that are hit by a placement of rooks. Once we remove the columns and rows that contain these s cells, we are left with placing non-attacking rooks on an $(n-s) \times (n-s)$ grid that never hit $B(P_{n-s,k,0})$. See Figure 7.3 for an example.

Now we will present a bijective proof of the theorem. To do so, we will show how the bijection acts on a particular example and will make comments on how it works in general. The reader should be convinced that the bijection holds in the general case. Suppose that $n = 16$, $k = 4$ and $s = 4$. We interpret the $\binom{n-k}{s}$ term as choosing the s elements from $\{k+1, k+2, \ldots, n\}$ to be the first element of each of the s k-drops. In this example, let us suppose that we choose $15, 11, 8,$ and 7. We now interpret the $P_{n-s,k,0}$ as choosing a permutation in S_{n-s} having no k-drops. For our example, suppose we choose the permutation,

$$\sigma = 12\ 1\ 6\ 10\ 2\ 9\ 3\ 4\ 11\ 5\ 7\ 8.$$

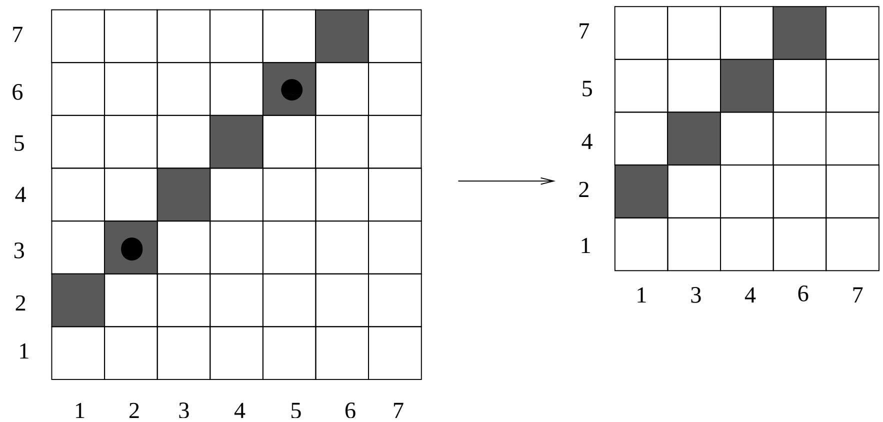


Figure 7.3: Showing how a placement on $B(P_{7,3,2})$ reduces to a placement on $B(P_{5,3,0})$.

Notice that $des_4(\sigma) = 0$. We will describe how to transform these choices into a permutation, $\tau \in S_n$ such that $des_k(\tau) = s$.

We begin by relabeling the elements of σ to obtain $L(\sigma)$. We begin with 1 and we label each element equivalent to $1 \mod k$ from smallest to largest, then we move on to 2 and label each element equivalent to $2 \mod k$ from smallest to largest. Continue in this fashion until σ has been entirely relabeled. So, in our example

$$L(12\ 1\ 6\ 10\ 2\ 9\ 3\ 4\ 11\ 5\ 7\ 8) = 12\ 1\ 5\ 6\ 4\ 3\ 7\ 10\ 9\ 2\ 8\ 11.$$

We will now claim that $0 \le des_1(L(\sigma)) \le k-1$. The reasoning is as follows. Since σ had no k-drops, when we applied the relabeling L, there could be no 1-drops within each equivalence class. Therefore, the only possible way $L(\sigma)$ could contain a 1-drop from $i+1$ to i would be if the $i+1$ label was applied to an element from a different equivalence class $\mod k$ than the element for which the i label was applied. In particular, we can list the $k-1$ possible 1-drops of $L(\sigma)$ in lexicographic order as the sequence, $S(\sigma) = S_1^\sigma, \ldots, S_{k-1}^\sigma$. We then define the set, $T(\sigma)$ as follows.

$$T(\sigma) = \{i | S_i^\sigma \text{ appears in } L(\sigma)\}.$$

In our particular example, we labeled the elements equivalent to $1 \mod 4$ with the labels, $\{1, 2, 3\}$, the elements equivalent to $2 \mod 4$ with the labels, $\{4, 5, 6\}$, the elements equivalent to $3 \mod 4$ with the labels, $\{7, 8, 9\}$, and the elements equivalent to $4 \mod 4$ with the labels, $\{10, 11, 12\}$. Thus, in our example, the 1-drops of $L(\sigma)$ are contained in the ordered set, $S(\sigma) = \{(4, 3), (7, 6), (10, 9)\}$. Also in our example, $T(\sigma) = \{1, 3\}$ as we notice that 4 3 and 10 9 both appear in $L(\sigma)$.

We will now define a set $S(\tau)$. Using the choices for the first element of each of the s k-drops from step one, we think of gluing together each of these elements with its successor into a block. We then apply the same relabeling procedure L, and define $S(\tau) = \{i + 1, i\}$ where the $i + 1$ label was applied to an element from a different equivalence class $\mod k$ than the element for which the i label was applied. Again, we will order the elements of $S(\tau)$ in lexicographic order.

In our particular example, the blocks are the following,

$$\underline{15\ 11\ 7\ 3}, \underline{8\ 4}, \underline{1}, \underline{2}, \underline{5}, \underline{6}, \underline{9}, \underline{10}, \underline{12}, \underline{13}, \underline{14}, \underline{16}.$$

If we label the relabel the blocks by L, we obtain,

$$L(\underline{15\ 11\ 7\ 3}, \underline{8\ 4}, \underline{1}, \underline{2}, \underline{5}, \underline{6}, \underline{9}, \underline{10}, \underline{12}, \underline{13}, \underline{14}, \underline{16}) = 9, 10, 1, 5, 2, 6, 3, 7, 11, 4, 8, 12.$$

We labeled the elements equivalent to $1 \mod 4$ with the labels, $\{1, 2, 3, 4\}$, the elements equivalent to $2 \mod 4$ with the labels, $\{5, 6, 7, 8\}$, the elements equivalent to $3 \mod 4$ with the label, $\{9\}$, and the elements equivalent to $4 \mod 4$ with the labels, $\{10, 11, 12\}$. Thus, $S(\tau) = \{(5, 4), (9, 8), (10, 9)\}$.

We now think of gluing every occurrence or $i + 1, i$ in $L(\sigma)$ to form blocks and relabel these blocks preserving the relative order among the elements of $L(\sigma)$ to yield $R(L(\sigma))$.

In our example, since 4 3 and 10 9 both appear in $L(\sigma)$, the blocks of $L(\sigma)$ are

$$\underline{12}\ \underline{1}\ \underline{5}\ \underline{6}\ \underline{4\ 3}\ \underline{7}\ \underline{10\ 9}\ \underline{2}\ \underline{8}\ \underline{11}.$$

And applying the relabeling gives

$$R(\underline{12}\ \underline{1}\ \underline{5}\ \underline{6}\ \underline{4\ 3}\ \underline{7}\ \underline{10\ 9}\ \underline{2}\ \underline{8}\ \underline{11}) = 10\ 1\ 4\ 5\ 3\ 6\ 8\ 2\ 7\ 9.$$

Starting with the smallest $i \in T(\sigma)$, we will think of creating an occurrence of S_i^τ by finding the second element of S_i^τ, call this element j, in $R(L(\sigma))$ and expanding it into a block $j+1$, j and just shifting every other element that is larger than j up by 1. We then recursively apply this technique using the next smallest $i \in T(\sigma)$ until we've done this expanding process for each $i \in T(\sigma)$.

In our particular example, $T(\sigma) = \{1, 3\}$, so we first must create an occurrence of 5 4 by finding 4 in $R(L(\sigma))$ and expanding it into a block 5 4 and applying the appropriate shifting, this yields

$$11\ 1\ 5\ 4\ 6\ 3\ 7\ 9\ 2\ 8\ 10.$$

Secondly, we must create an occurrence of 10 9 by finding 9 in the above and expanding it into a block 10 9 and applying the appropriate shifting, this yields

$$12\ 1\ 5\ 4\ 6\ 3\ 7\ 10\ 9\ 2\ 8\ 11.$$

We now obtain τ by replacing i with the block that was labeled with an i when we were obtaining $S(\tau)$.

Thus, the above becomes

$$\underline{16}\ \underline{1}\ \underline{2}\ \underline{13}\ \underline{6}\ \underline{9}\ \underline{10}\ \underline{8\ 4}\ \underline{15\ 11\ 7\ 3}\ \underline{5}\ \underline{14}\ \underline{12}.$$

This yields the image of the bijection, τ, by just removing the blocks and thinking of the above as a permutation. In our example,

$$\tau = 16\ 1\ 2\ 13\ 6\ 9\ 10\ 8\ 4\ 15\ 11\ 7\ 3\ 5\ 14\ 12.$$

Notice that $des_4(\tau) = 4$ as expected.

This process is also reversible and therefore the theorem holds. □

Theorem 7.6. *For $n \geq 2$ and $0 \leq k < n$,*

$$P_{n,k,0} = (n-1)P_{n-1,k,0} + (n-1-k)P_{n-2,k,0}. \tag{7.10}$$

Proof. We first note that

$$P_{n,k,0} = (n-1)P_{n-1,k,0} + P_{n-1,k,1}.$$

It is simple to verify this fact combinatorially. For any permutation in S_n having no k-drops, n must be either between a k-drop or not. If it is, then when it is removed you are a left with a permutation in S_{n-1} having no k-drops. Note that there are $n-1$ positions where this can occur. If n is not between a k-drop, then when it is removed you are a left with a permutation in S_{n-1} having one k-drop. Note that there is only 1 position where this can happen, namely between the existing k-drop. Thus, the claim is verified. We can then use Theorem 7.5 to obtain that $P_{n-1,k,1} = \binom{n-k-1}{1} P_{n-2,k,0} = (n-1-k)P_{n-2,k,0}$. Putting these two facts together gives the result. □

Theorem 7.6 is a nice generalization of derangements. If we let $k = 0$ in the theorem we obtain a known recursion for derangements, namely,

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}. \tag{7.11}$$

It is surprising to note that the recursion involving derangements is almost trivial to prove combinatorially, but the generalization provided in the theorem is certainly not. Note that we could extend the bijection provided in the proof of Theorem 7.5 to prove the theorem bijectively as well.

We also observe the following.

Theorem 7.7. *For $n \geq 2$, and $0 < k < n$,*

$$P_{n,k,0} = P_{n,k-1,0} + P_{n-1,k-1,0}. \tag{7.12}$$

Proof. Again we will prove this theorem both combinatorially and bijectively. We have also been able to verify this theorem algebraically using our formula given in 7.1. In fact we prove a generalization of this which can be found in Section 7.5.

Again, the LHS counts the number of placements of n non-attacking rooks on the $n \times n$ grid that never hit $B(P_{n,k,0})$. Imagine taking the first row of the $n \times n$

grid and moving it to the top. Now, if there is no rook in the $(n-k+1)^{\text{st}}$ cell of this row, then we are left with a placement of n non-attacking rooks on the $n \times n$ grid that never hit $B(P_{n,k-1,0})$.

If there is a rook in this cell then we can think of removing the row and column containing this cell and we are left with a placement of n non-attacking rooks on the $(n-1)\times(n-1)$ grid that never hit $B(P_{n-1,k-1,0})$. See Figure 7.4 for an example.

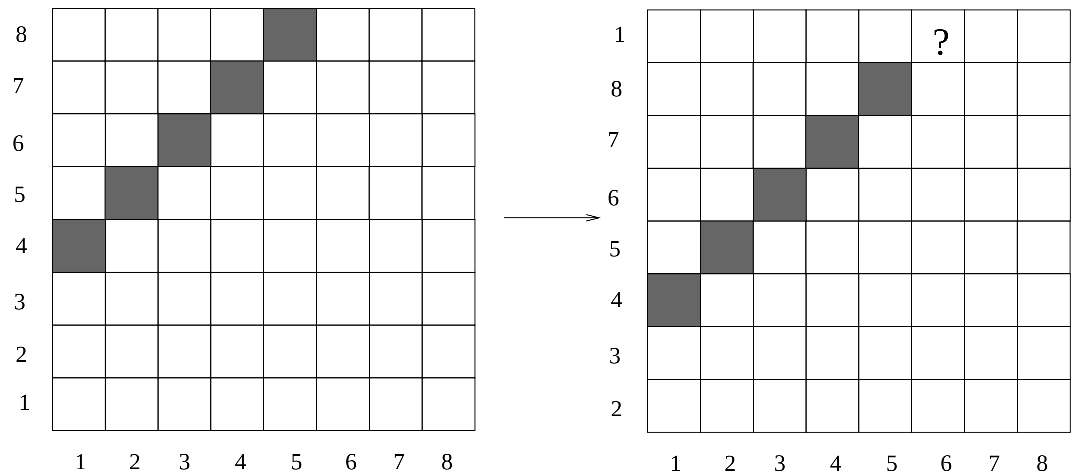


Figure 7.4: The $(n-k+1)^{\text{st}}$ cell of the top row determines which case we are in.

Now we will present a bijective proof. The idea used in the bijection above will be applied here as well. We will provide a bijection which maps a permutation $\sigma \in S_n$ such that $des_k(\sigma) = 0$ to a permutation $\tau \in S_n \cup S_{n-1}$ satisfying $des_{k-1}(\tau) = 0$.

First, we apply the relabeling procedure, L to σ as described in the previous proof. Suppose $n = 9$ and $k = 3$, and $\sigma = 2\ 7\ 1\ 5\ 6\ 4\ 9\ 3\ 8$. Then,

$$L(\sigma) = 4\ 3\ 1\ 5\ 8\ 2\ 9\ 7\ 6.$$

As before, $0 \leq des_1(L(\sigma)) \leq k-1$. We will also list the $k-1$ possible 1-drops of $L(\sigma)$ in lexicographic order as the sequence, $S(\sigma) = S_1^\sigma, \ldots, S_{k-1}^\sigma$. T is also defined in the same way, i.e.,

$$T(\sigma) = \{i | S_i^\sigma \text{ appears in } L(\sigma)\}.$$

In our particular example, $S(\sigma) = \{43, 76\}$ and $T(\sigma) = \{1, 2\}$ as both 43 and 76 appear in $L(\sigma)$.

Just as before, we now think of gluing every occurrence or $i+1, i$ in $L(\sigma)$ to form blocks and relabel these blocks preserving the relative order among the elements of $L(\sigma)$ to yield $R(L(\sigma))$.

In our example, since 4 3 and 7 6 both appear in $L(\sigma)$, the blocks of $L(\sigma)$ are

$$\underline{4\ 3}\ \underline{1}\ \underline{5}\ \underline{8}\ \underline{2}\ \underline{9}\ \underline{7\ 6}.$$

Applying the relabeling gives

$$R(\underline{4\ 3}\ \underline{1}\ \underline{5}\ \underline{8}\ \underline{2}\ \underline{9}\ \underline{7\ 6}) = 3\ 1\ 4\ 6\ 2\ 7\ 5.$$

We will now attempt to build back a permutation with no $k-1$ drops of length m where $m = n$ if $k-1 \notin T(\sigma)$ or $m = n-1$ if $k-1 \in T(\sigma)$. We will now define a set $S(\tau)$ similar to before. We then apply the same relabeling procedure L to the integers from 1 to m, but using equivalence classes mod $k-1$ and define $S(\tau) = \{i+1, i\}$ where the $i+1$ label was applied to an element from a different equivalence class mod k than the element for which the i label was applied. Again, we will order the elements of $S(\tau)$ in lexicographic order. Since we applied the labeling using equivalence classes mod $k-1$, we will know that $|S(\tau)| = k-2$.

In our example, we will be apply the relabeling procedure to the integers from 1 to 8 using equivalence classes mod 2. Thus, the only element of $S(\tau)$ is 54.

Starting with the smallest $i \in T(\sigma)$, provided that $i \neq k-1$, we will think of creating an occurrence of S_i^τ by finding the second element of S_i^τ, call this element j, in $R(L(\sigma))$ and expanding it into a block $j+1$, j and just shifting every other element that is larger than j up by 1. We then recursively apply this technique using the second smallest $i \in T(\sigma)$. If $k-1 \in T(\sigma)$, we do nothing.

In our particular example, $T(\sigma) = \{1, 2\}$, so we first must create an occurrence of 5 4 by finding 4 in $R(L(\sigma))$ and expanding it into a block 5 4 and applying the appropriate shifting, this yields

$$3\ 1\ 5\ 4\ 7\ 2\ 8\ 6.$$

The next smallest element of $T(\sigma)$ is 2 which is $k-1$ in our example and thus we do nothing with this element. It is precisely at this step that determines whether the image of this bijection is of length n or $n-1$.

We now obtain τ by replacing i with the integer that was labeled with an i when we were obtaining $S(\tau)$.

This yields the image of the bijection,

$$\tau = 5\ 1\ 2\ 7\ 6\ 3\ 8\ 4.$$

Notice that $des_2(\tau) = 0$ as expected.

This process is also reversible and therefore the theorem holds. □

Another interesting observation is,

Theorem 7.8. *For $n \geq 2$ and $0 < k < n$,*

$$P_{n,k,0} = kP_{n-1,k-1,0} + (n-k)P_{n-1,k,0}. \tag{7.13}$$

Proof. We will first provide a simple algebraic proof of this fact. By Theorem 7.1, the RHS becomes

$$k\left(\sum_{t=0}^{n-k}(-1)^t(n-1-t)!\binom{n-k}{t}\right)+(n-k)\left(\sum_{t=0}^{n-k-1}(-1)^t(n-1-t)!\binom{n-k-1}{t}\right).$$

Removing the $n-k$ term from the first sum yields

$$k\left(\sum_{t=0}^{n-k-1}(-1)^t(n-1-t)!\frac{(n-k)!}{t!(n-k-t)!}\right)+(n-k)\left(\sum_{t=0}^{n-k-1}(-1)^t(n-1-t)!\frac{(n-k-1)!}{t!(n-k-t-1)!}\right)+(-1)^{n-k}k!.$$

Combining these sums gives

$$\begin{aligned}
&\sum_{t=0}^{n-k-1}(-1)^t(n-t)!\frac{(n-k)!}{t!(n-k-t)!}\left(\frac{k}{n-t}+\frac{n-k-t}{n-t}\right)+(-1)^{n-k}k! \\
=\ &\sum_{t=0}^{n-k}(-1)^t(n-t)!\frac{(n-k)!}{t!(n-k-t)!} \\
=\ &P_{n,k,0}.
\end{aligned}$$

We can also prove this theorem combinatorially.

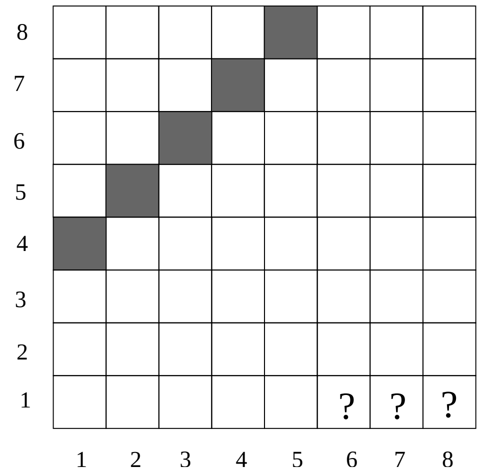


Figure 7.5: The location of the rook in the first row determines which case we are in.

Again, the LHS counts the number of placements of n non-attacking rooks on the $n \times n$ grid that never hit $B(P_{n,k,0})$. Consider the first row of the grid. If there is a rook in one of the last k cells of this row, we can imagine removing the row and column containing this cell and we are left with a placement of $n-1$ non-attacking rooks on the $(n-1) \times (n-1)$ grid that never hit $B(P_{n-1,k-1,0})$.

If there is a rook in one of the first $n-k$ cells of this row, then we can think of removing the row and column containing this cell and we are left with a placement of $n-1$ non-attacking rooks on the $(n-1) \times (n-1)$ grid that never hit $B(P_{n-1,k,0})$. See Figure 7.5 for an example.

□

Theorem 7.8 gives us the following interesting result.

Theorem 7.9. *For $n \geq 1$ and $0 \leq k < n$,*

$$P_{n,k,0} = k! \sum_{r=0}^{k} \binom{k}{r} \binom{n-k}{k-r} P_{n-k,k-r,0}. \tag{7.14}$$

Proof. We will provide a simple combinatorial proof of this fact. Remember that the LHS counts the number of placements of n non-attacking rooks on the $n \times n$ grid that never hit $B(P_{n,k,0})$. Consider the lightly shaded cells in the lower right

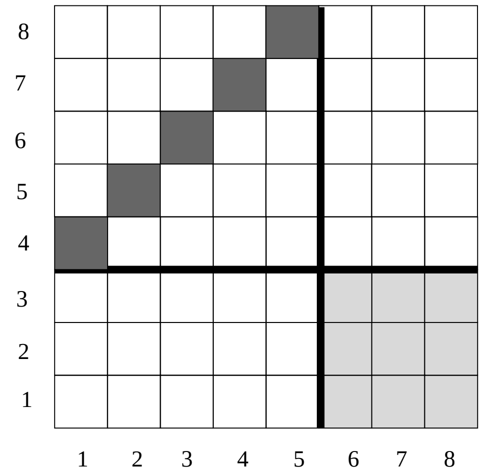


Figure 7.6: The board, $B(P_{8,3,0})$, with some lightly shaded cells.

hand corner of the board. There can be anywhere from 0 to k rooks placed in this square area. Suppose that we choose to place r rooks in this area. We first will choose the r rows which will contain the rooks. Since there must be a rook in each column, there must be $k-r$ rooks in the rectangular region above the lightly shaded cells so we will also choose the $k-r$ rows which will contain these rooks. Then we place these k rooks in some order, contributing $k!\binom{k}{r}\binom{n-k}{k-r}$ to $P_{n,k,0}$.

Now, if we think to completely remove each row and column that contains a rook, each rook placed in a lightly shaded cell will reduce to a placement with the value of k reduced by one. Also, each rook placed will reduce to a placement with the value of n reduced by one. Thus, to complete the placement, we are left with placing $n-k$ non-attacking rooks on the $(n-k)\times(n-k)$ grid that never hit $B(P_{n-k,k-r,0})$. Thus the total number of placements having r rooks in the lightly shaded cells is given by $k!\binom{k}{r}\binom{n-k}{k-r}P_{n-k,k-r,0}$. Summing over all possible values of r yields the result.

□

Theorem 7.7 gives us the following interesting result.

Theorem 7.10. *For $n \geq 1$ and $0 \leq k < n$,*

$$P_{n,k,0} = \sum_{r=0}^{k} \binom{k}{r} D_{n-k+r}. \tag{7.15}$$

Proof. We can prove this theorem using induction on k. When $k = 0$ and $n \geq 0$, it is clear that $P_{n,0,0} = D_n$. Now assume that $P_{n,i,0} = \sum_{r=0}^{i} \binom{i}{r} D_{n-i+r}$, for all $n \geq 0$ and all $i < k$. Our goal is now to show that $P_{n,k,0} = \sum_{r=0}^{k} \binom{k}{r} D_{n-k+r}$. Firstly, using 7.7 we have that

$$P_{n,k,0} = P_{n,k-1,0} + P_{n-1,k-1,0}.$$

We then apply the induction hypothesis to the RHS of the above to get

$$\begin{aligned} & \sum_{r=0}^{k-1} \binom{k-1}{r} D_{n-(k-1)+r} + \sum_{r=0}^{k-1} \binom{k-1}{r} D_{(n-1)-(k-1)+r} \\ = & \sum_{r=0}^{k-1} \binom{k-1}{r} D_{n+1-k+r} + \sum_{r=0}^{k-1} \binom{k-1}{r} D_{n-k+r}. \end{aligned}$$

Then, peeling off the $k-1$ term of the first sum and the 0 term of the second sum gives

$$\begin{aligned} & \binom{k-1}{k-1} D_n + \sum_{r=0}^{k-2} \binom{k-1}{r} D_{n+1-k+r} + \binom{k-1}{0} D_{n-k} + \sum_{r=1}^{k-1} \binom{k-1}{r} D_{n-k+r} \\ = & \binom{k}{k} D_n + \sum_{r=1}^{k-1} \left(\binom{k-1}{r-1} + \binom{k-1}{r} \right) D_{n-k+r} + \binom{k}{0} D_{n-k} \\ = & \binom{k}{k} D_n + \sum_{r=1}^{k-1} \binom{k}{r} D_{n-k+r} + \binom{k}{0} D_{n-k} \\ = & \sum_{r=0}^{k} \binom{k}{r} D_{n-k+r}, \end{aligned}$$

which was to be shown. □

The interesting thing to note here is that the task of computing an arbitrary $P_{n,k,s}$, can be reduced to computing $P_{n-s,k,0}$ using Theorem 7.5. Then, the task of computing $P_{n-s,k,0}$ can be reduced to computing $D_{n-s-k+r}$ for specific values of r using Theorem 7.7. It is surprising to find that computing an arbitrary $P_{n,k,s}$ can be reduced to counting the number of derangements of n. In fact we can provide the expression and it is the following direct corollary of the previous theorems.

Corollary 7.11. *For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$,*

$$P_{n,k,s} = \binom{n-k}{s} \sum_{r=0}^{k} \binom{k}{r} D_{n-s-k+r}. \tag{7.16}$$

7.4 Two q-Analogues of $P_{n,k,s}$

We will first begin with the definitions of two q-analogues of the formula given in Theorem 7.1. For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$, they are

$$P_{n,k,s}(q) := \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{s}{2}} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q$$

and

$$\overline{P}_{n,k,s}(q) := \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{s+1}{2} + \binom{t}{2} + t(k-s)} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q.$$

If $k \geq n$, then we simply define

$$P_{n,k,s}(q) = \overline{P}_{n,k,s}(q) := \begin{cases} 0, & \text{if } s > 0 \\ [n]_q!, & \text{if } s = 0 \end{cases}.$$

We will now go on to show that we can obtain very nice q-analogues of the recursions and formulas obtained from the previous section. It is also simple to verify that

$$P_{n,k,s}(q) = q^{\binom{n}{2}} \overline{P}_{n,k,s}(q^{-1}). \tag{7.17}$$

This means that we can begin by proving formulas involving $P_{n,k,s}(q)$ and can use (7.17) to prove similar formulas involving $\overline{P}_{n,k,s}(q)$. We employ this strategy and will present the corresponding formulas for $\overline{P}_{n,k,s}(q)$ in Table 7.2. We will present the q-analogues in a different order than the previous section where $q = 1$.

This is because the proofs in this section build on each other in a required order, whereas in the previous section the order was more intuitive from a combinatorial standpoint.

Theorem 7.12. *For $n \geq 3$ and $0 \leq k < n$,*

$$P_{n,k,0}(q) = q\,[n-1]_q\, P_{n-1,k,0}(q) + [n-1-k]_q\, P_{n-2,k,0}(q). \tag{7.18}$$

Proof. By definition of P, we have that the RHS of the theorem, $q\,[n-1]_q\, P_{n-1,k,0}(q) + [n-1-k]_q\, P_{n-2,k,0}(q)$, equals

$$q\,[n-1]_q \sum_{t=0}^{n-k-1} (-1)^t\,[n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q +$$
$$[n-1-k]_q \sum_{t=0}^{n-k-2} (-1)^t\,[n-t-2]_q! \begin{bmatrix} n-k-2 \\ t \end{bmatrix}_q.$$

The 0 term and the 1 term from the first sum contribute

$$\begin{aligned} & q\,[n-1]_q\,([n-1]_q!) - q\,[n-1]_q\,[n-2]_q! \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}_q \\ = \;& ([n]_q - [1]_q)([n-1]_q!) - q\,[n-1]_q\,[n-2]_q! \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}_q \\ = \;& [n]_q! - ([n-1]_q!)(1 + q \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}_q) \\ = \;& [n]_q! - ([n-1]_q!) \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q. \end{aligned}$$

The LHS of the theorem is $P_{n,k,0}(q)$, which by definition is

$$\begin{aligned} & \sum_{t=0}^{n-k} (-1)^t\,[n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\ = \;& [n]_q! - ([n-1]_q!) \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q + \sum_{t=2}^{n-k} (-1)^t\,[n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\ = \;& [n]_q! - ([n-1]_q!) \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q + \sum_{t=0}^{n-k-2} (-1)^t\,[n-2-t]_q! \begin{bmatrix} n-k \\ t+2 \end{bmatrix}_q. \end{aligned}$$

Notice that the first two terms here match the contribution from the 0 term and 1 term from the sum mentioned above. Thus, to verify the theorem, it remains to show that

$$\begin{aligned} & q\left[n-1\right]_q \sum_{t=2}^{n-k-1} (-1)^t \left[n-t-1\right]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q + \\ & \left[n-1-k\right]_q \sum_{t=0}^{n-k-2} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k-2 \\ t \end{bmatrix}_q \\ = & \sum_{t=0}^{n-k-2} (-1)^t \left[n-2-t\right]_q! \begin{bmatrix} n-k \\ t+2 \end{bmatrix}_q . \end{aligned}$$

The LHS of the above becomes

$$\begin{aligned} & q\left[n-1\right]_q \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-3\right]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q + \\ & \sum_{t=0}^{n-k-2} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q \left[t+1\right]_q \\ = & \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-3\right]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q (q\left[t+1\right]_q + q^{t+2}\left[n-2-t\right]_q) + \\ & \sum_{t=0}^{n-k-2} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q (1+q\left[t\right]_q). \end{aligned}$$

Writing this as four separate sums gives

$$\begin{aligned} & \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-3\right]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q q\left[t+1\right]_q + \\ & \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q q^{t+2} + \\ & \sum_{t=0}^{n-k-2} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q + \\ & \sum_{t=1}^{n-k-2} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q q\left[t\right]_q . \end{aligned}$$

Notice that the first sum and the fourth sum are off by a factor of -1 and thus they cancel. Combining the remaining two sums gives

$$\begin{aligned}
& \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-2\right]_q! q^{t+2} \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q + \\
& (-1)^{n-2-k} \left[k\right]_q! + \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q \\
= \; & (-1)^{n-k} \left[k\right]_q! + \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-2\right]_q! \left(q^{t+2} \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q \right) \\
= \; & (-1)^{n-k} \left[k\right]_q! + \sum_{t=0}^{n-k-3} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k \\ t+2 \end{bmatrix}_q \\
= \; & \sum_{t=0}^{n-k-2} (-1)^t \left[n-t-2\right]_q! \begin{bmatrix} n-k \\ t+2 \end{bmatrix}_q .
\end{aligned}$$

This is what needed to be shown. □

Theorem 7.13. *For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$,*

$$P_{n,k,s}(q) = q^{\binom{s}{2}} \begin{bmatrix} n-k \\ s \end{bmatrix}_q P_{n-s,k,0}(q). \tag{7.19}$$

Proof. By definition, $P_{n,k,s}(q)$ equals

$$\begin{aligned}
& \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{s}{2}} \left[n-t\right]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\
= \; & q^{\binom{s}{2}} \sum_{t=s}^{n-k} (-1)^{t-s} \left[n-t\right]_q! \frac{\left[t\right]_q!}{\left[s\right]_q! \left[t-s\right]_q!} \frac{\left[n-k\right]_q!}{\left[t\right]_q! \left[n-k-t\right]_q!} \\
= \; & q^{\binom{s}{2}} \begin{bmatrix} n-k \\ s \end{bmatrix}_q \sum_{t=s}^{n-k} (-1)^{t-s} \left[n-t\right]_q! \begin{bmatrix} n-k-s \\ t-s \end{bmatrix}_q \\
= \; & q^{\binom{s}{2}} \begin{bmatrix} n-k \\ s \end{bmatrix}_q \sum_{t=0}^{n-s-k} (-1)^{t} \left[n-s-t\right]_q! \begin{bmatrix} n-k-s \\ t \end{bmatrix}_q \\
= \; & q^{\binom{s}{2}} \begin{bmatrix} n-k \\ s \end{bmatrix}_q P_{n-s,k,0}(q).
\end{aligned}$$

□

Theorem 7.14. *For $n \geq 2$, $0 \leq k < n$, and $s \geq 1$,*

$$P_{n,k,s}(q) = q^{s+1}\,[n-s-1]_q\,P_{n-1,k,s}(q) + [s+1]_q\,P_{n-1,k,s+1}(q) + q^{s-1}P_{n-1,k,s-1}(q). \tag{7.20}$$

Proof. We can use Theorem 7.13 to show that the LHS of the theorem is

$$q^{\binom{s}{2}} \begin{bmatrix} n-k \\ s \end{bmatrix}_q P_{n-s,k,0}(q).$$

To prove the theorem, we will now show that the RHS of the theorem divided by $q^{\binom{s}{2}}$ is equal to $\begin{bmatrix} n-k \\ s \end{bmatrix}_q P_{n-s,k,0}(q)$ Applying Theorem 7.13 to the RHS of the theorem and dividing by $q^{\binom{s}{2}}$ yields

$$\begin{aligned}
& q^{s+1}\,[n-s-1]_q \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q P_{n-1-s,k,0}(q) + \\
& q^{s+1}\,[s+1]_q \begin{bmatrix} n-1-k \\ s+1 \end{bmatrix}_q P_{n-s-2,k,0}(q) + \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q P_{n-s,k,0}(q) \\
= \; & q^{s} \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q \Big(q\,[n-s-1]_q\,P_{n-1-s,k,0}(q) + [n-s-1-k]_q\,P_{n-s-2,k,0}(q) \Big) \\
& + \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q P_{n-s,k,0}(q).
\end{aligned}$$

We can now apply Theorem 7.12 to the expression in parenthesis to obtain

$$\begin{aligned}
& q^{s} \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q P_{n-s,k,0}(q) + \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q P_{n-s,k,0}(q) \\
= \; & \left(q^{s} \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q + \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q \right) P_{n-s,k,0} \\
= \; & \begin{bmatrix} n-k \\ s \end{bmatrix}_q P_{n-s,k,0},
\end{aligned}$$

which verifies the theorem. □

Theorem 7.15. *For $n \geq 2$ and $0 < k < n$,*

$$P_{n,k,0}(q) = [k]_q\,P_{n-1,k-1,0}(q) + q^{k}\,[n-k]_q\,P_{n-1,k,0}(q). \tag{7.21}$$

Proof. It is easy to verify that $\frac{[k]_q+q^k[n-k+t]_q}{[n-t]_q} = 1$. We will use this fact in the second step of the proof. First we have that $P_{n,k,0}(q)$ equals

$$\begin{aligned}
& \sum_{t=0}^{n-k}(-1)^t [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\
= & \sum_{t=0}^{n-k}(-1)^t [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \left(\frac{[k]_q}{[n-t]_q} + \frac{q^k [n-k+t]_q}{[n-t]_q} \right) \\
= & [k]_q \sum_{t=0}^{n-k}(-1)^t [n-t-1]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\
& + q^k [n-k]_q \sum_{t=0}^{n-k-1} (-1)^t [n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q \\
= & [k]_q P_{n-1,k-1,0}(q) + q^k [n-k]_q P_{n-1,k,0}(q)
\end{aligned}$$

□

Theorem 7.15 gives us the following interesting result.

Theorem 7.16. *For $n \geq 1$ and $0 < k < n$,*

$$P_{n,k,0}(q) = [k]_q! \sum_{r=0}^{k} q^{(k-r)^2} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} n-k \\ k-r \end{bmatrix}_q P_{n-k,k-r,0}(q). \tag{7.22}$$

Proof. To prove this fact, we just need to apply the recursion from Theorem 7.15 to $P_{n,k,0}(q)$ a total of k times. Consider how we would get a term of the form $P_{n-k,k-r,0}(q)$ for some r satisfying $0 \leq r \leq k$. We can track terms that come out of the recursion by whether it came out of the first term of the recursion or the second. In this situation, we needed to use the first term of the recursion a total of r times second term of the recursion a total of $k-r$ times. Thus, we will obtain a factor of $[k]_q \cdot [k-1]_q \ldots [k-r+1]_q$ from the first terms and a factor of $[n-k]_q \cdot [n-k-1]_q \ldots [n-2k-r+1]_q$ from the second. Multiplying these together gives a factor of

$$[k]_q! \begin{bmatrix} n-k \\ k-r \end{bmatrix}_q.$$

We interpret the $\begin{bmatrix} k \\ r \end{bmatrix}_q$ as choosing the r times we use the first term of the recursion from the total of k steps and record this as a rearrangement of r 1's and $k-r$ 0's and we count q raised to the power of the number of coinversions in the rearrangement. Notice that each coinversion means that we used the second term of the recursion before the first at some other step. Thus, the factor of q that comes along when using the second term will be one higher. The smallest possible factor of q that you could obtain would be by using the first term in the recursion r times consecutively and then using the second term in the recursion $k-r$ times consecutively and this would yield $q^{(k-r)^2}$. Thus, we have accounted for every term in the formula.

□

Theorem 7.17. *For $n \geq 2$,*

$$P_{n,0,0}(q) = [n]_q \, P_{n-1,0,0}(q) + (-1)^n. \tag{7.23}$$

Proof.

$$\begin{aligned} P_{n,0,0}(q) &= \sum_{t=0}^{n} (-1)^t \, [n-t]_q! \begin{bmatrix} n \\ t \end{bmatrix}_q \\ &= [n]_q! \sum_{t=0}^{n} \frac{(-1)^t}{[t]_q!} \\ &= [n]_q \left([n-1]_q! \sum_{t=0}^{n-1} \frac{(-1)^t}{[t]_q!} + [n-1]_q! \frac{(-1)^n}{[n]_q!} \right) \\ &= [n]_q \, P_{n-1,0,0}(q) + (-1)^n \end{aligned}$$

□

Notice that Theorem 7.17 shows that $P_{n,0,0}(q)$ satisfies a known recursion for a q-analogue of derangements. Using (7.17), we can get q-analogues of $\overline{P}_{n,k,s}(q)$ which are presented in Table 7.2.

Notice that corresponding q-analogue to equation (7.23) shows that $\overline{P}_{n,0,0}(q)$ satisfies a known recursion for a different q-analogue of derangements. Also notice that with $P_{n,k,s}(q)$ and $\overline{P}_{n,k,s}(q)$, we were able to find q-analogues of the recursions

Table 7.2: Corresponding q-analogues for $\overline{P}_{n,k,s}(q)$

Equation	Corresponding q-analogue for $\overline{P}$
(7.18)	$\overline{P}_{n,k,0}(q) = [n-1]_q \overline{P}_{n-1,k,0}(q) + q^{n+k-1}[n-1-k]_q \overline{P}_{n-2,k,0}(q)$
(7.19)	$\overline{P}_{n,k,s}(q) = q^{ks}\left[{n-k \atop s}\right]_q \overline{P}_{n-s,k,0}(q)$
(7.20)	$\overline{P}_{n,k,s}(q) = [n-s-1]_q \overline{P}_{n-1,k,s}(q) + q^{n-s-1}[s+1]_q \overline{P}_{n-1,k,s+1}(q) + q^{n-s}\overline{P}_{n-1,k,s-1}(q)$
(7.21)	$\overline{P}_{n,k,0}(q) = q^{n-k}[k]_q \overline{P}_{n-1,k-1,0}(q) + [n-k]_q \overline{P}_{n-1,k,0}(q)$
(7.22)	$\overline{P}_{n,k,0}(q) = [k]_q! \sum_{r=0}^{k} q^{r(n+r-2k)}\left[{k \atop r}\right]_q \left[{n-k \atop k-r}\right]_q \overline{P}_{n-k,k-r,0}(q)$
(7.23)	$\overline{P}_{n,0,0}(q) = [n]_q \overline{P}_{n-1,0,0}(q) + (-1)^n q^{\binom{n}{2}}$

found in Theorem 7.5 and Theorem 7.8, but not in Theorem 7.7. Strangely, if we introduce a new q-analogue, $\hat{P}_{n,k,s}(q)$, we find just the opposite.

7.5 A third q-Analogue of $P_{n,k,s}$

For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$, we define

$$\hat{P}_{n,k,s}(q) := \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q.$$

Again, if $n \geq k$ we define

$$\hat{P}_{n,k,s}(q) := \begin{cases} 0, & \text{if } s > 0 \\ [n]_q!, & \text{if } s = 0. \end{cases}$$

Here are some recursions satisfied by $\hat{P}_{n,k,s}(q)$.

Theorem 7.18. *For $n \geq 2$ and $0 < k < n$,*

$$\hat{P}_{n,k,0}(q) = \hat{P}_{n,k-1,0}(q) + q^{n-k}\hat{P}_{n-1,k-1,0}(q). \tag{7.24}$$

Proof. By the definition of $\hat{P}_{n,k,s}(q)$, $\hat{P}_{n,k-1,0}(q)$ is

$$\sum_{t=0}^{n+1-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n+1-k \\ t \end{bmatrix}_q.$$

Now, using the fact that $\begin{bmatrix} n+1-k \\ t \end{bmatrix}_q = \begin{bmatrix} n-k \\ t \end{bmatrix}_q + q^{n+1-k-t}\begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q$, the above becomes

$$\begin{aligned}
&\sum_{t=0}^{n+1-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \left(\begin{bmatrix} n-k \\ t \end{bmatrix}_q + q^{n+1-k-t} \begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q \right) \\
= &\sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + \sum_{t=1}^{n+1-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! q^{n+1-k-t} \begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q \\
= &\sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + \sum_{t=1}^{n+1-k} (-1)^t q^{\binom{t-1}{2}} q^{n-k} [n-t]_q! \begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q \\
= &\sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + \sum_{t=0}^{n-k} (-1)^{(t+1)} q^{\binom{t}{2}} q^{n-k} [n-t-1]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q.
\end{aligned}$$

Notice now though, that the first term is the definition of $\hat{P}_{n,k,0}(q)$, so it remains to show that

$$\sum_{t=0}^{n-k} (-1)^{(t+1)} q^{\binom{t}{2}} q^{n-k} [n-t-1]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + q^{n-k}\hat{P}_{n-1,k-1,0}(q) = 0.$$

Which is certainly true, as $q^{n-k}\hat{P}_{n-1,k-1,0}(q)$ is

$$\sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} q^{n-k} [n-t-1]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q.$$

The i-th summand in each of the sums is equal up to a minus sign and so they cancel off in pairs, which was to be shown. □

Theorem 7.19. *For $n \geq 1$ and $0 \leq k < n$,*

$$\hat{P}_{n,k,0}(q) = \sum_{r=0}^{k} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q). \tag{7.25}$$

Proof. We can prove this theorem using induction on k. When $k = 0$ and $n \geq 0$, it is clearly true. Now assume that $\hat{P}_{n,i,0}(q) = \sum_{r=0}^{i} q^{r(n-i)} \begin{bmatrix} i \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q)$, for all $n \geq 0$ and all $i < k$. Our goal is now to show that $\hat{P}_{n,k,0}(q) = \sum_{r=0}^{k} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q)$. Firstly, using Theorem 7.18 we have that

$$\hat{P}_{n,k,0}(q) = \hat{P}_{n,k-1,0}(q) + q^{n-k}\hat{P}_{n-1,k-1,0}(q).$$

We then apply the induction hypothesis to the RHS of the above to get

$$\begin{aligned} & \sum_{r=0}^{k-1} q^{r(n-k+1)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q) + q^{n-k} \sum_{r=0}^{k-1} q^{r(n-k)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{P}_{n-1-r,0,0}(q) \\ = \; & \sum_{r=0}^{k-1} q^{r(n-k+1)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q) + \sum_{r=0}^{k-1} q^{(r+1)(n-k)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{P}_{n-1-r,0,0}(q). \end{aligned}$$

Then, peeling off the 0 term of the first sum and the $k-1$ term of the second sum gives

$$\begin{aligned} & \begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q \hat{P}_{n,0,0}(q) + \sum_{r=1}^{k-1} q^{r(n-k+1)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q) \\ & + q^{k(n-k)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q \hat{P}_{n-k,0,0}(q) + \sum_{r=0}^{k-2} q^{(r+1)(n-k)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{P}_{n-1-r,0,0}(q) \\ = \; & q^{k(n-k)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q \hat{P}_{n-k,0,0}(q) + \begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q \hat{P}_{n,0,0}(q) \\ & + \sum_{r=1}^{k-1} q^{r(n-k)} \left(q^r \begin{bmatrix} k-1 \\ r \end{bmatrix}_q + \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q \right) \hat{P}_{n-r,0,0}(q) \\ = \; & q^{k(n-k)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q \hat{P}_{n-k,0,0}(q) + \sum_{r=1}^{k-1} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q) + \begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q \hat{P}_{n,0,0}(q) \\ = \; & \sum_{r=0}^{k} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{P}_{n-r,0,0}(q), \end{aligned}$$

which was to be shown. Note, that when going from the second line to the third, we used the fact that $q^r \begin{bmatrix} k-1 \\ r \end{bmatrix}_q + \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q = \begin{bmatrix} k \\ r \end{bmatrix}_q$, which is easy to verify. $\square$

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