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INSTITUTE OF ADVANCED STUDIES

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AN INTRODUCTION TO THE STRING THEORY OF HADRONS

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Lectures by

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Reported by

C.J. Burden

Canberra
March, 1979

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The development of these lectures began with lectures given at the University of Tasmania in 1978. Revisions and additions occured when the lectures were given at the University of Melbourne in 1978, and there were further revisions and additions when these lectures were given at the Australian National University. I wish to thank the University of Tasmania and the University of Melbourne for their hospitality. I also with to thank I.G. Enting for helpful discussions, especially about models in statistical mechanics and their relation to lattice gauge theory, and C.J. Burden for preparing these notes.

L.J. Tassie

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1. Introduction

Several aspects of particle physics and of quantum theory have led to the idea that the hadrons - the strongly interacting particles consisting of mesons, π, κ, ψ, and baryons, n, p, Σ, Λ, etc. - are made of string. Perhaps a more proper description is that the hadrons have string-like structure.

According to the theory, the structure of hadrons is modelled as a one-dimensional continuum. There is no direct proof of string structure but rather a number of indications which have pointed in this direction, some of which will be summarized in §2. At present the state of the theory is far from complete and it could require considerable modification before calculations of experimental results such as cross-sections or lifetimes can be made. As such, the idea of a string has perhaps the status of a model rather than a theory.

Introductory accounts of string models have been given by Schwarz [1] and Nambu [2] . More detailed information is available in the reviews [3,4,5] .

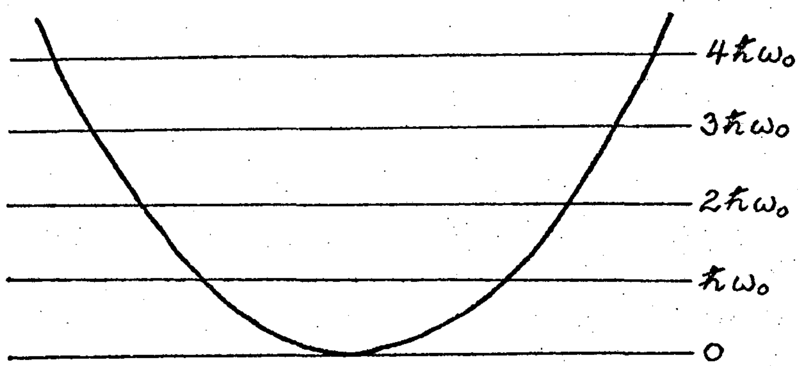
2. Evidence for String Structure of Hadrons

2.1 Statistical Models

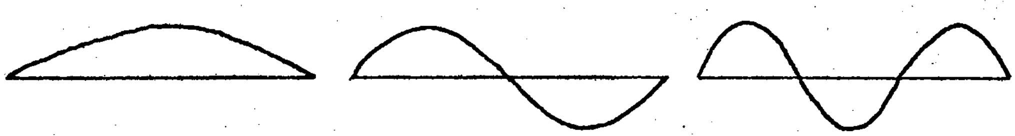
From a study of statistical models, Hagedorn concluded that there may be an infinite number of hadrons whose distribution is an exponentially increasing function of mass. But it is difficult to obtain such a spectrum for hadrons constructed from point particles. However, such a spectrum is obtained for an extended object such as a relativistic string [6] or more generally for an n-dimensional covariant elastic jelly [7] .

A simple non-relativistic example is given here to show the essential difference in the spectrum of a particle structure and an extended object. It should be noted that the Hagedorn type of level density cannot be obtained from this example.

Consider a non-relativistic linear harmonic oscillator. After subtraction of the zero-point energy, we have the energy levels:

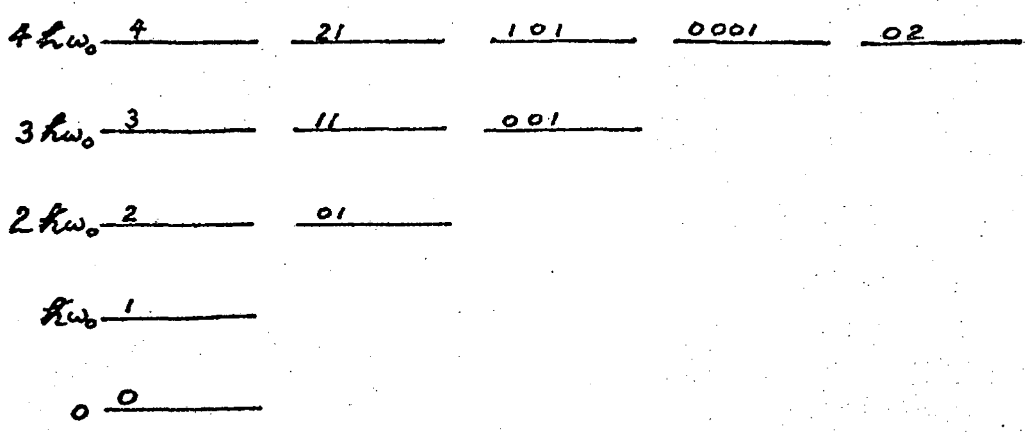


A string can be considered as an infinite set of harmonic oscillators, corresponding to the normal modes of vibration:



	FUNDAMENTAL	1ST HARMONIC	2ND HARMONIC	etc.
	$\omega = \omega_o$	$\omega = 2\omega_o$	$\omega = 3\omega_o$	
ENERGY=	$n_1\hbar\omega_o$	$2n_2\hbar\omega_o$	$3n_3\hbar\omega_o$	

Here n, n_2, n_3 etc. are the occupation numbers of the various modes of vibration. We now tabulate all the possible sets of occupation numbers n_1, n_2, n_3, ... giving rise to total energies of 0, $\hbar\omega_o$, $2\hbar\omega_o$, etc.

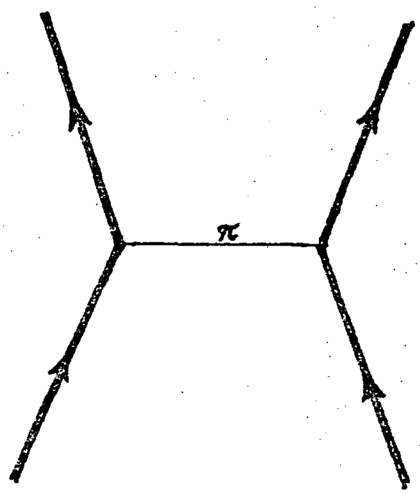


STATES WITH TOTAL ENERGIES OF $0, \hbar\omega_0, 2\hbar\omega_0$, etc.

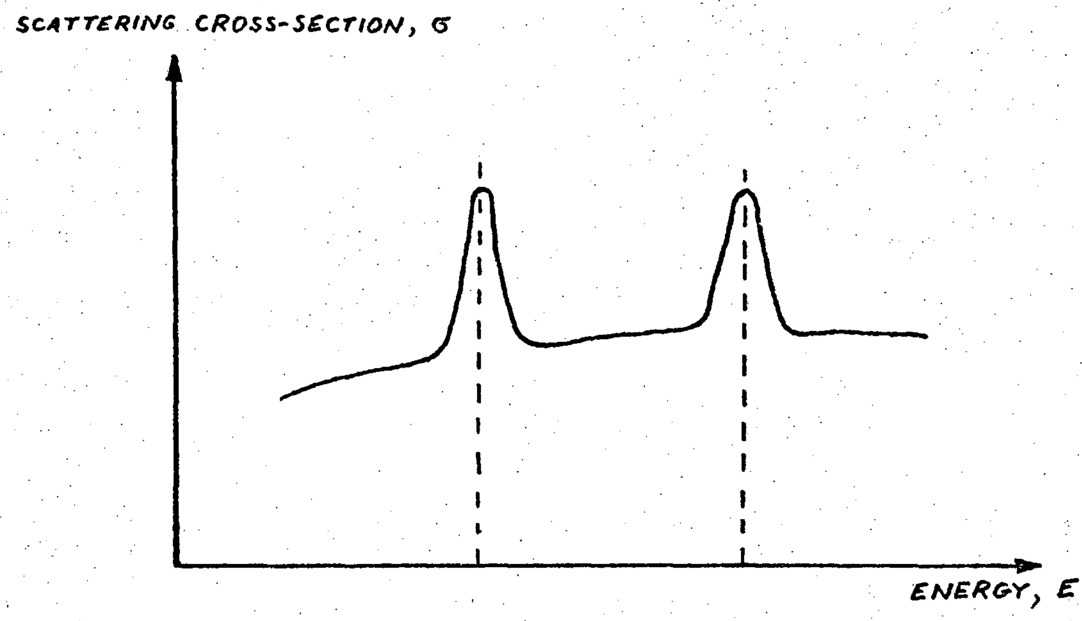
It is obvious that the number of possible states increases rapidly with the total energy.

2.2. Dual Models

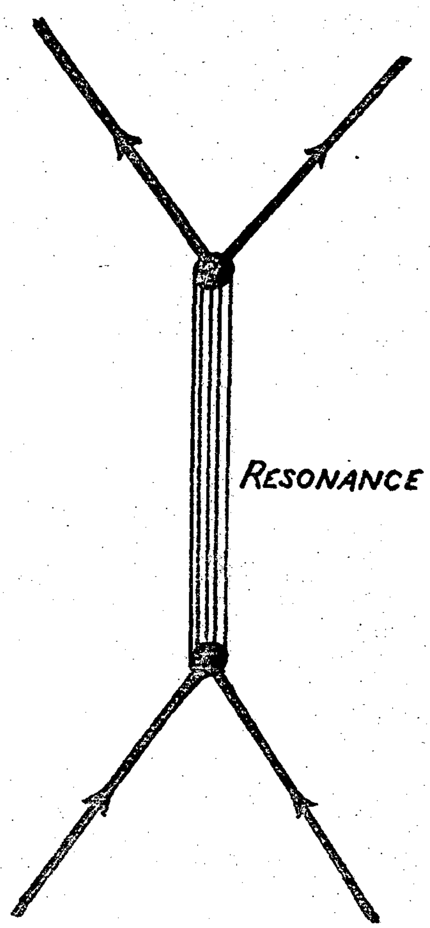
A popular description of hadron interactions is that given in terms of particle exchange. For example, nucleon-nucleon interaction in a nucleus can be described in terms of exchange of pions. These interactions can be visualized diagramatically, e.g.



A second approach to hadron interactions is the idea that particle interactions give rise to intermediate resonances which, for instance, show up as bumps in the energy variation of the scattering cross-section.

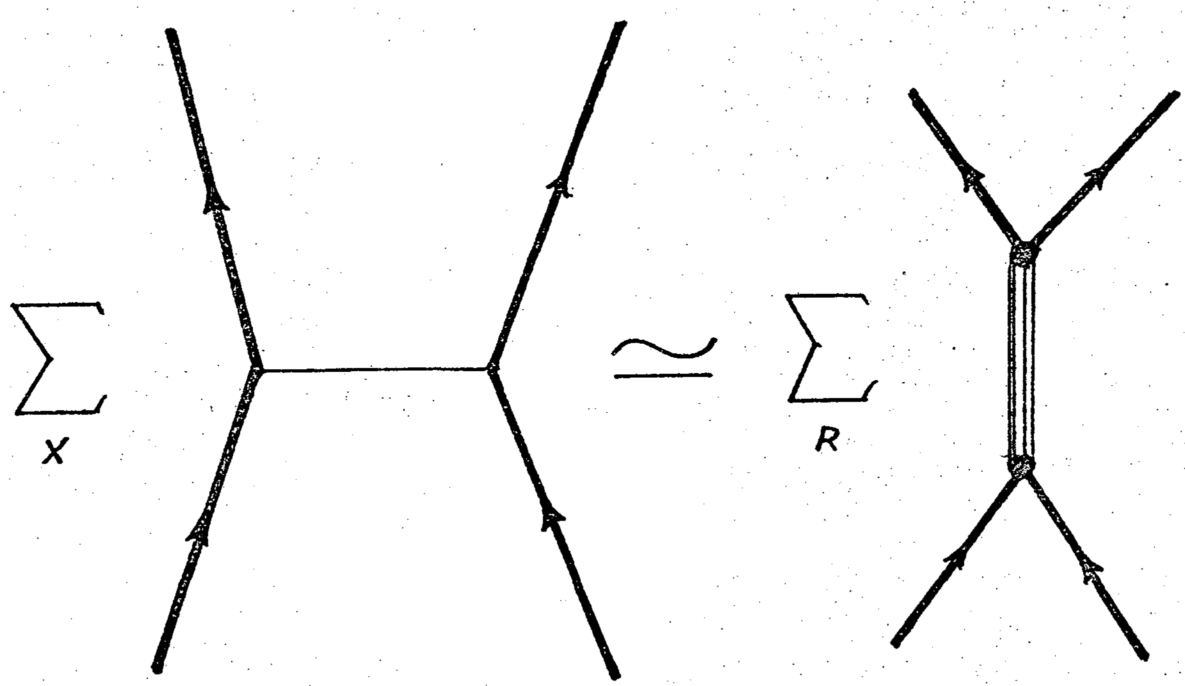


This description of interactions can be shown diagramatically as:



Low energy interactions such as those dealt with in nuclear physics are frequently described in terms of finite sums of resonances.

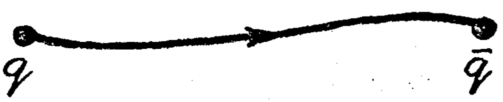
The two approaches are brought together by the principle of duality. Phenomenological duality asserts that the description in terms of exchanged particles should give the low energy behaviour averaged over resonances. Diagramatically:



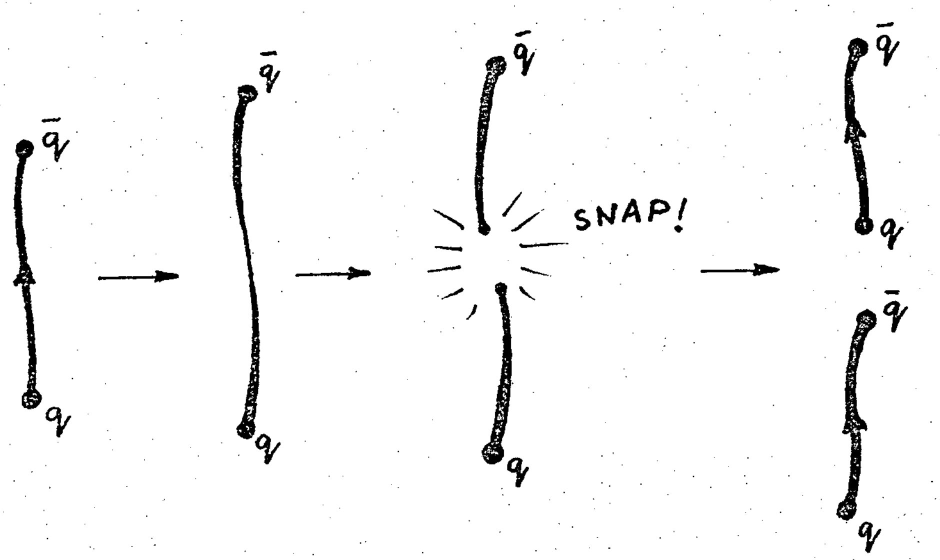
Veneziano has put forward an expression for the ampliutde satisfying the above duality requirement. In this expression, both the sum over particle exchanges and the sum over resonances are infinite and the spectrum of states needed looks like the spectrum of a string.

2.3 Quark Confinement

A rather appealing explanation of quark confinement can be given in terms of strings. Suppose we take a meson to be a finite piece of string and further impose on this string a direction. The two ends now differ and one end of the string can be identified as a quark and the other as an anti-quark.

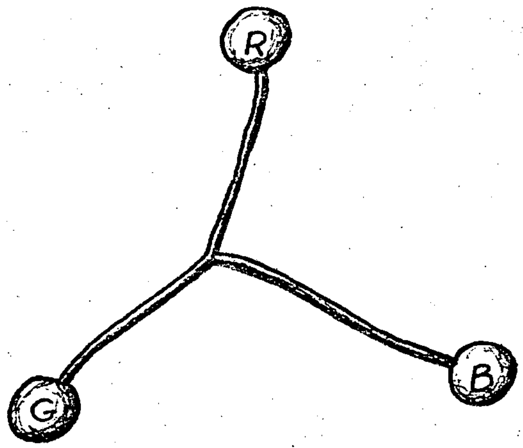


Now in order to isolate a quark we pull on one end of the string. However the string is brittle and as it starts to stretch, it soon breaks, forming a new quark and anti-quark at the broken ends.



So instead we produce a new meson and no free quarks. The brittleness of the string is due to the small mass of, say, the π meson compared with the energies present in high energy collisions which produce vast numbers of mesons.

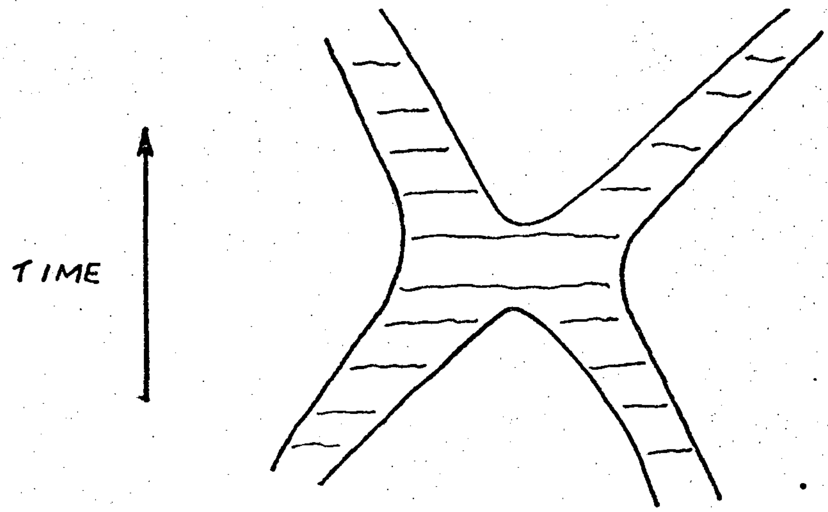
In order tc explain certain inconsistencies in quark models such as violation of the spin-statistics theorem, a new quantum number known as colour has been introduced. The usual "flavours" u(up), d(down) and s(strange) of quarks are further distinguished as coming in three colours, R(red), B(blue) and G(green). Like the original flavour quantum numbers, colour forms on SU(3) symmetry, but unlike the flavours this is an exact symmetry in that it is not broken by any interactions. The corresponding anti-particles are each available in the anti-colours $\overline{R}$,$\overline{B}$ and $\overline{C}$. More recently the flavour spectrum has been extended to an SU(4) symmetry with the introduction of charm, and even more recently to five flavours. Physical baryons are believed to correspond to a colour SU(3) singlet and so are white. According to the string model, a proton, for instance can be drawn as:



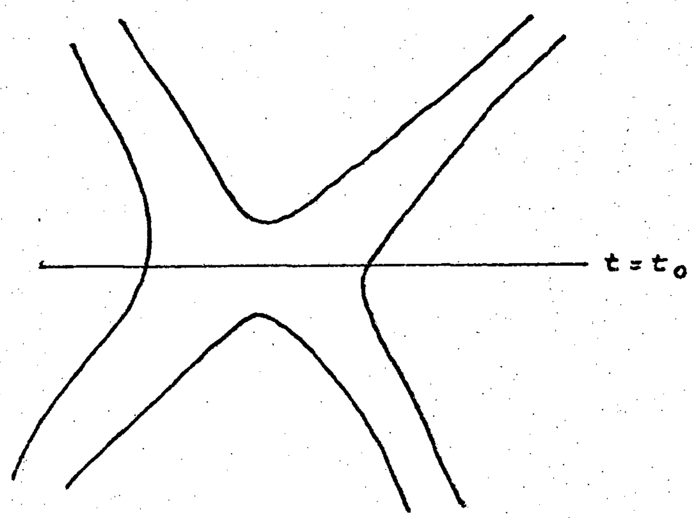
The Feynman diagram for a strong interaction is now a surface traced out in space-time. For instance, for an interaction of the type

meson + meson ⟶ meson + meson

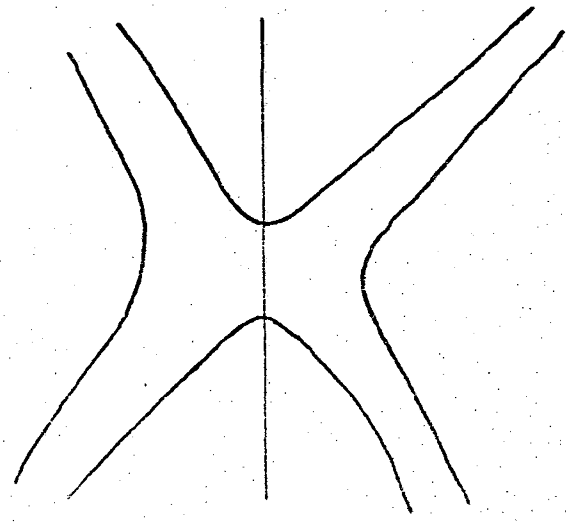
we have



To describe this process we can either parameterise the surface by cutting through the diagram at equal times, ie.



which corresponds to summing over resonances, or parameterise by cutting in this way:

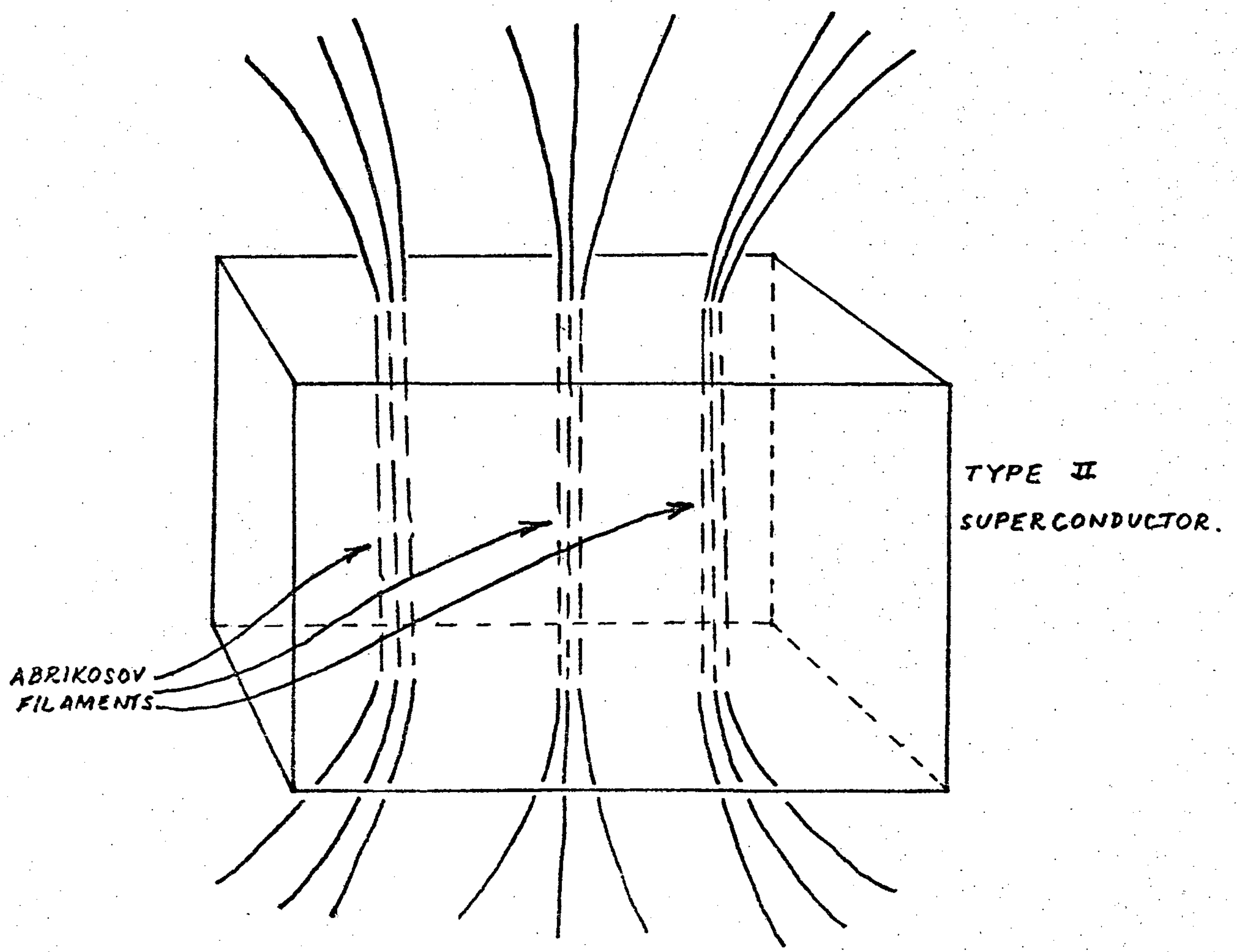


which corresponds to summing over exchanged particles.

2.4 Gauge Field Theories

From gauge theories of vector fields there is the possibility of producing string structures in various ways and so getting quark confinement [8,9,10,11] .

The gauge theory of a vector field that we are most familiar with is electromagnetic theory, which is an Abelian gauge theory. In Abelian gauge theories, strings can arise from the medium squeezing the flux of the field into strings of quantized flux as for instance happens to magnetic flux in a type II superconductor, where the strings are called Abrikosov filaments. A theoretical

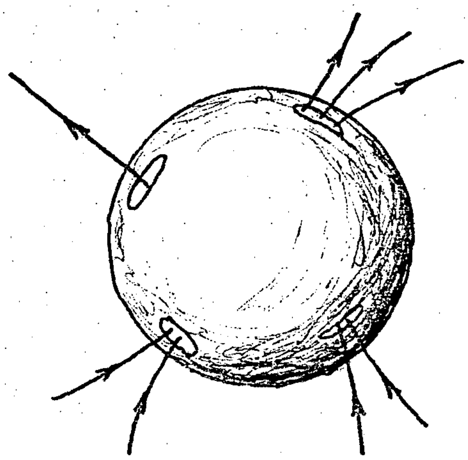


description of this is available using the Ginzburg-Landau hamiltonian.

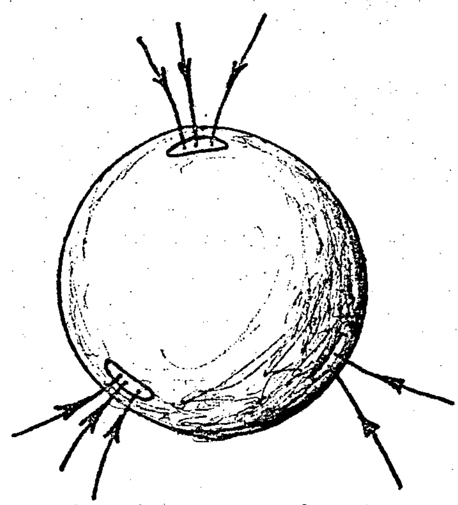
We consider the possibility of an Abelian gauge theory of strong interactions - that is a vector field theory which is mathematically similar to electromagnetic theory. By constructing a hamiltonian for the strong interactions analogous to the Ginzburg-Landau hamiltonian, we will have the vaccuum acting as a type II superconductor for the gauge field, and this will give strings.

As well as Abelian gauge fields, there are more general gauge fields known as non-Abelian gauge fields. The difference between an Abelian gauge field and a non-Abelian gauge field, is described below in physical rather than in mathematical terms.

For an Abelian gauge field, such as electromagnetic theory, Gauss's theorem applies: the nett flux through any surface, not containing any sources, is zero. For a non-Abelian gauge field, Gauss's theorem does not

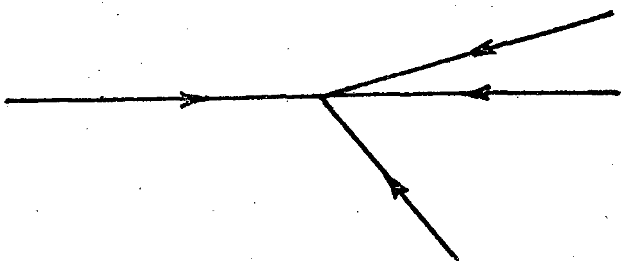


ABELIAN GAUGE FIELD



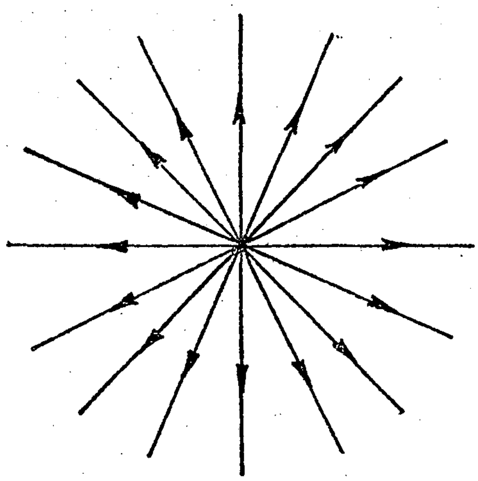
NON-ABELIAN GAUGE FIELD

hold in this simple form. The nett flux through a closed surface is not necessarily zero even in the absence of sources. The flux is not additive. The flux lines can join, for instance as shown



and so we can get tangles of flux lines, which can overcome the tendency of flux lines to repel each other, and because of these tangles the flux lines can form strings of quantized flux.

In the Abelian case, without the effect of a superconducting medium, the flux lines spread out because they repel each other, and the force depends on the number of flux lines per unit area and as we take that area further away we have less flux lines cutting it and so a smaller force and so a smaller potential, ie. $V \propto \frac{1}{r}$.



FORCE DECREASES WITH r

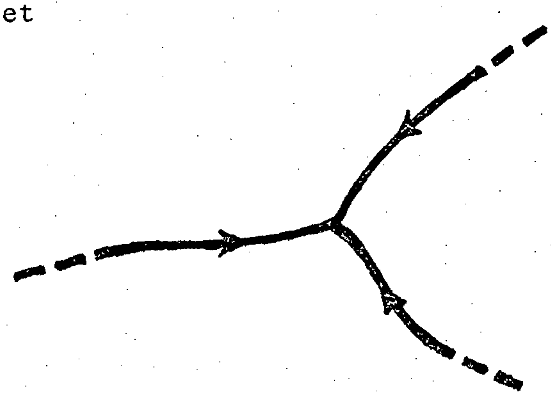
$V \propto \frac{1}{r}$

FORCE INDEPENDENT OF r

$V \propto r$

When the flux lines have come into a string like this, as we take the area along the string, the number of flux lines cutting the area is constant, and so the force is independent of distance - which means the potential is proportional to distance.

For non-Abelian gauge theories, a generalized form of Gauss's theorem holds. As an example, consider SU(3), at present the most popular non-Abelian gauge group for the strong interactions. For this group, the generalization of Gauss's theorem gives that three flux strings can meet



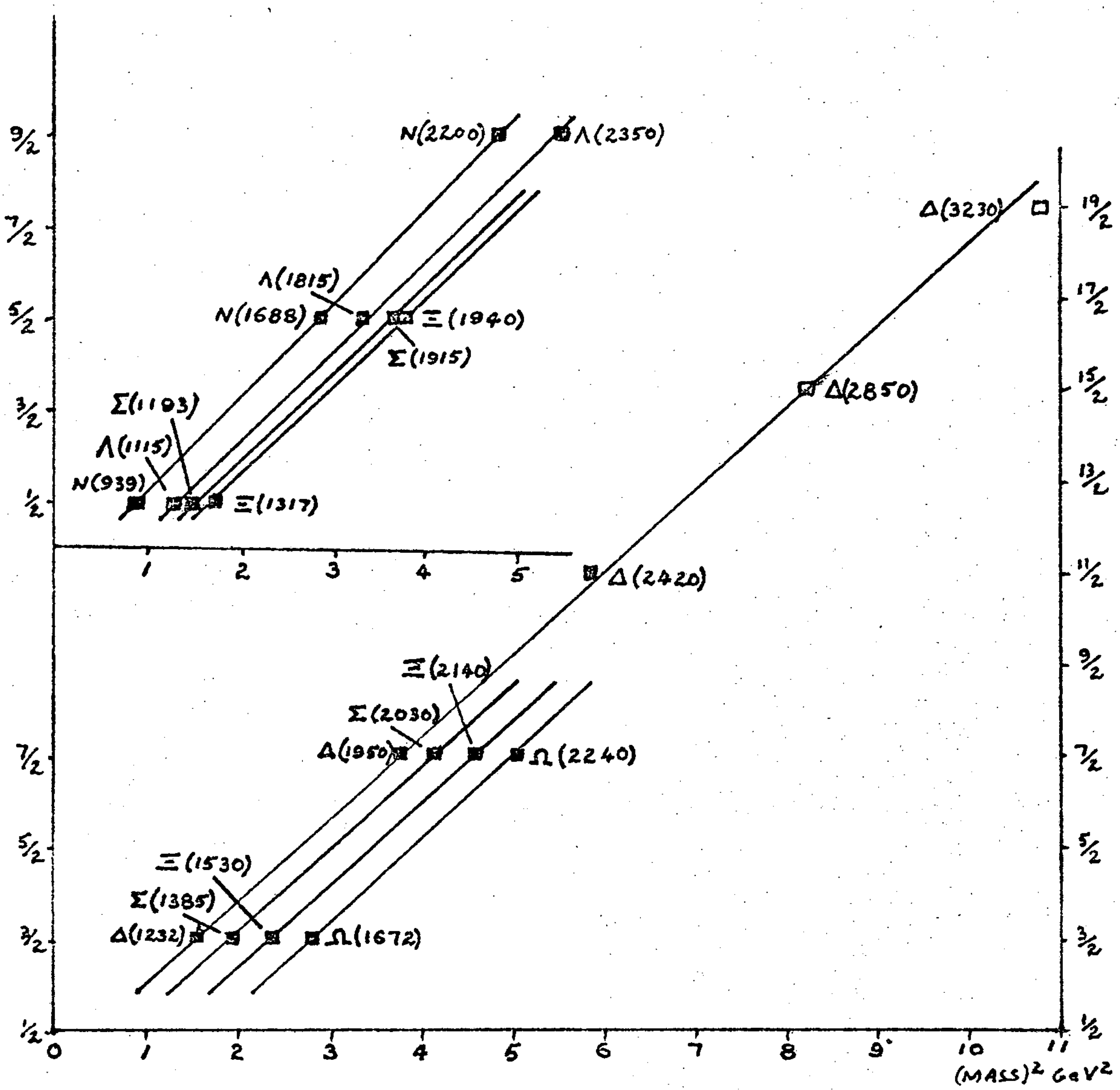
ie. that flux strings are added modulo three. Then identifying the ends of the strings as quarks, we have the pictures of baryons consisting of three quarks and mesons composed of quark and anti-quark described in §2.3. The string traces out a surface of minimal area in space time; at least in the approximation where quark masses are neglected.

From the string tracing out a minimal area in space time it can be shown that the hadrons lie on straight line Regge tranjectories - that for rotational bands

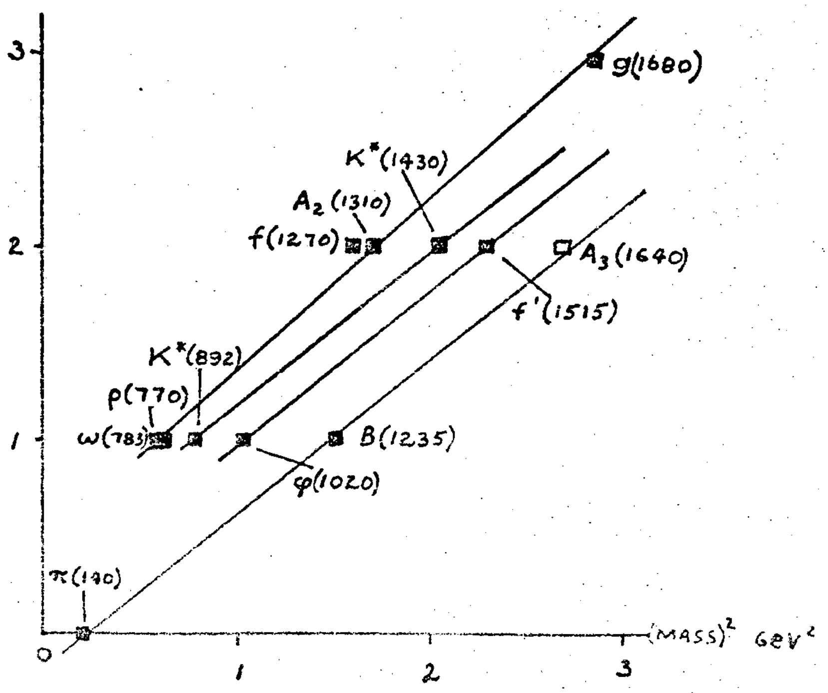
$$M_J^2 = \alpha J + \beta$$

where M_J is the hadron mass and J is the spin. This is in substantial agreement with the Regge trajectories, or more correctly, Chew Frautschi plots for baryons and mesons which are shown on the next page.

CHEW-FRAUTSCHI PLOTS



BARYONS



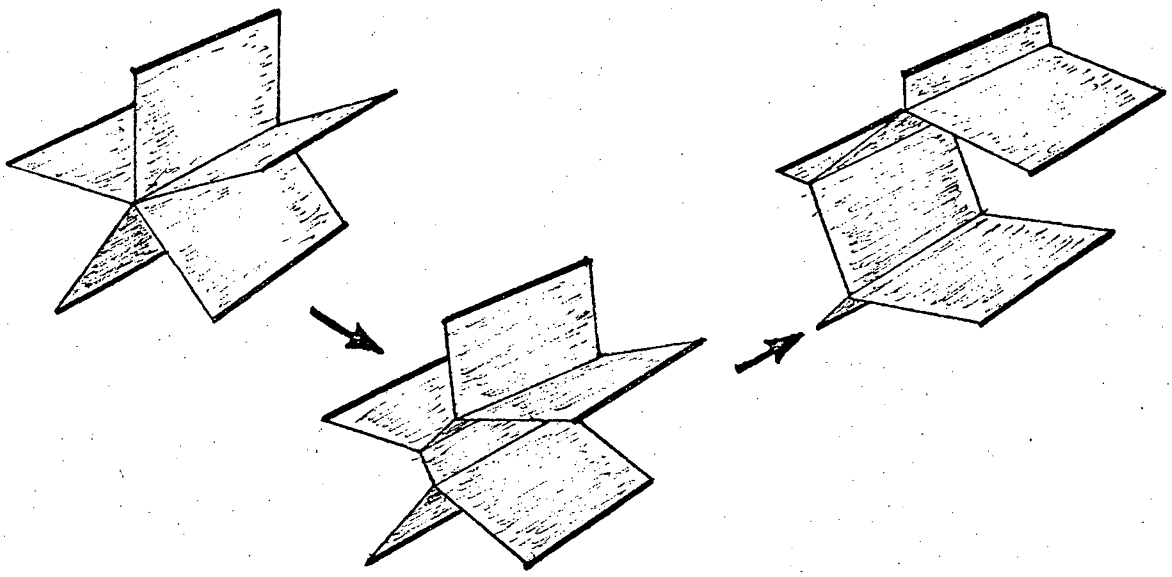
MESONS

3. Minimal Surfaces in Space-Time

The idea that a string should trace out a minimal area in space-time is analogous to the action principle postulated for a single particle, namely that δS should be stationary. For a particle, this leads to uniform motion in a straight line when no forces are present.

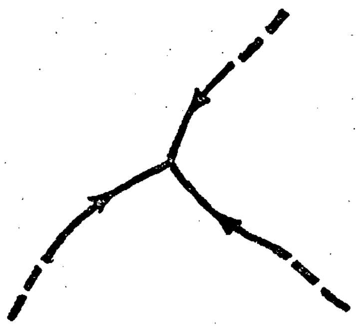
The minimal area problem for a (time-like) area in space-time differs from the corresponding problem in Euclidean space, since in Euclidean space it is necessary to specify the entire boundary. One minimal area of particular interest to us in space-time is that traced out by a rotating straight string where the initial and final postions of the string are specified. The ends of the string move at the speed of light, or alternatively, we ignore those unphysical parts of the solution which correspond to velocities greater than that of light.

To gain some feeling for the problem of determining stationary surfaces in space-time, it is helpful to consider first the corresponding problem in Euclidean space. Work has been done on this topic by Almgren and Taylor [12] and Taylor [13] in connexion with the geometry of soap films. From this work it is known that three surfaces meeting in a line constitute a minimal surface but this is not true for any number of surfaces greater than three. For example, if five infinitely long parallel supporting wires are connected by a soap film so that 5 soap films meet in a line (as shown in the first diagram) this configuration will collapse to one with three soap films meeting at each line as shown in the last diagram.



SOAP BUBBLE WITH FIVE SUPPORTING WIRES COLLAPSING TO MINIMAL SURFACE

The case where the surface is in space-time can be considered by taking the supporting wires of the soap bubble to be aligned with the time-axis. The corresponding result is then that the number of strings meeting at a point can never be greater than three.



MAX. OF 3 STRINGS MEET AT ONE POINT

This is equivalent to saying that the hypothesis that a string should trace out a minimal surface is only consistent with an SU(n) non-abelian gauge theory if n=3.

4. Transverse Vibrational States at about 4GeV.

One requirement of any physical theory should be its ability to make predictions. Tabulated below are the types of states obtained for the

ψ mesons for two different models. In both models the ψ is composed of a charmed quark and a charmed anti-quark but in one case the quark and antiquark are interacting by means of a potential and in the other case they are connected by a piece of string.

$c\ \bar{c}$ potential	c •——→——• $\bar{c}$
Rotational states	Rotational states
Radial excitation	Longitudinal vibrations of string
———	Transverse vibrations of string

Possible states of ψ ($c\bar{c}$) meson if a) linear potential and b) string model assumed

We see that the string model predicts the existence of extra states corresponding to transverse vibrations of the string. Giles and Tye [14] have calculated the energy levels of these states and their results predict the existence of extra resonances at about 4 GeV upwards. So there is the possibility in principle of experimentally testing the string theory by identifying the states of transverse string excitation. However the calculation of Giles and Tye neglects spin-spin and spin-orbit coupling and experimentally there are a lot of states in this region so that this identification has not yet been made.

5. Are Quarks Confined?

As well as asking 'How are quarks confined?', we must also ask 'Are quarks confined?'. La Rue, Fairbank and Hebard [15] have observed charges $\pm \frac{1}{3}$ e on niobium balls. Previously Longo [16] had pointed out the possibility of quarks being unconfined within a heavy nucleus, although this mechanism would not provide an explanation of Fairbank's result.

Consider again the analogy of the gauge theory of quark confinement to superconductivity. In the case of superconductivity, if a sufficiently large magnetic field is applied or the temperature is sufficiently high, the material is no longer superconducting and then the magnetic flux will no longer be squeezed into narrow filaments but spreads out, so that the string structure breaks down. In many of the models of quark confinement, there is the possibility of a similar phase transition leading to the breakup of the string structure and then the quarks are not confined.

The possibilities are:-

A. Quarks are always confined. In this case we must look elsewhere to explain the results of Fairbank's experiment.

B. That there exist both

  i) Confined phase - as in free hadrons

  ii) Unconfined phase - possibly in 1) heavy nuclei
      or 2) neutron stars
      or 3) the first 10^{-4} seconds of the universe
      or 4) who knows?

C. Quarks are not confined but either

  i) We haven't looked hard enough for them

  or ii) their behaviour is very unusual - for instance, they may be the indeterminate mass particles (IMP's) suggested by McCoy and Wu [17] , and so we have failed to see them.

6.1 Classical Mechanics of Strings

We first outline the Lagrangian formalism for a free relativistic particle which can then be used as a guide in setting up the formalism for a string.

The action for a particle is taken to be proportional to the invariant arc length traced out by the particle in space time:

$$S = -mc \int_a^b ds = -mc \int_a^b dt \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = \int_a^b L\, dt$$

where $L = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$ is the Lagrangian.

The principle of least action $\delta S = 0$ leads to the result that the particle traces out a geodesic in space time, ie. it moves with uniform velocity in a straight line.

The momentum is defined by

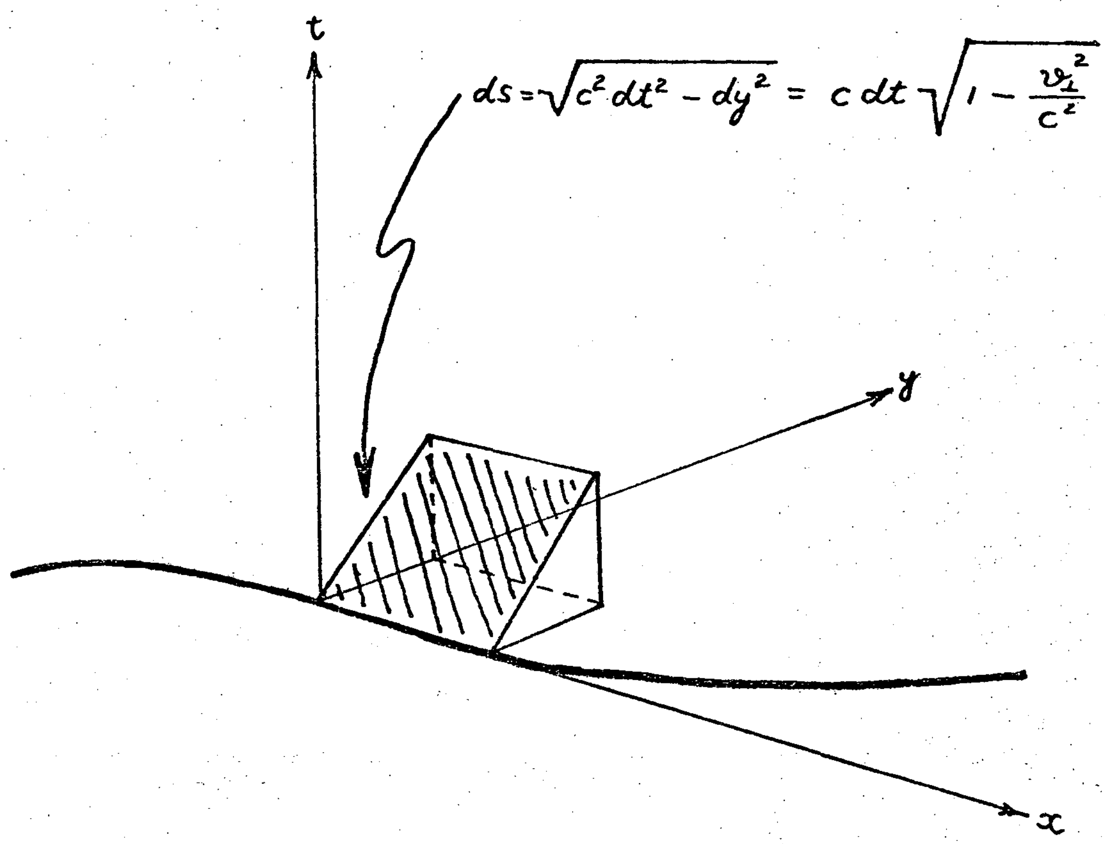
$$\underline{p} = \frac{\partial L}{\partial \dot{\underline{q}}} = \frac{\partial L}{\partial \underline{v}} = \frac{m\underline{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

and the energy is given by

$$E = \underline{p}.\,\dot{\underline{q}} - L = \frac{mc^2}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

Following the analogous procedure for the mechanics of a relativistic string, we take the Lagrangian to be the area traced out in space-time by the string, so that the principle of least action leads to a determination of minimal surfaces.

Considering an infinitesinal element of the string, we can take the x-axis as tangent to the string and y-axis in the direction of the transverse motion of the string.



Then the element of area traced out by the segment dx in time dt is

$$dxds = cdxdt\left(1 - \frac{v_\perp{}^2}{c^2}\right)^{\frac{1}{2}}$$

and the total area swept out by the string is

$$c\int_{t_o}^{t_1} dt \int_{\ell_o}^{\ell_1} dl \left(1 - \frac{v_\perp{}^2}{c^2}\right)^{\frac{1}{2}} .$$

We now define the action as

$$S = \frac{-1}{2\pi\alpha'} \int_{t_o}^{t_1} dt \int_{\ell_o}^{\ell_1} d\ell \left(1 - \frac{v_\perp{}^2}{c^2}\right)^{\frac{1}{2}}$$

and put $L = \frac{-1}{2\pi\alpha'} \int_{\ell_o}^{\ell_1} d\ell \left(1 - \frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}} = \int_{\ell_o}^{\ell_1} d\ell \, \mathcal{L}$

where L is the Lagrangian and the Lagrangian density $\mathcal{L}$.

The constant $\frac{-1}{2\pi\alpha'}$ has historical origins which we won't go into here.

In analogy to particle mechanics we define the transverse momentum at each point of the string by

$$p_\perp = \frac{\partial \mathcal{L}}{\partial \dot{q}_\perp} = \frac{v_\perp / c^2}{2\pi\alpha' \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

The energy density of the string is then given by

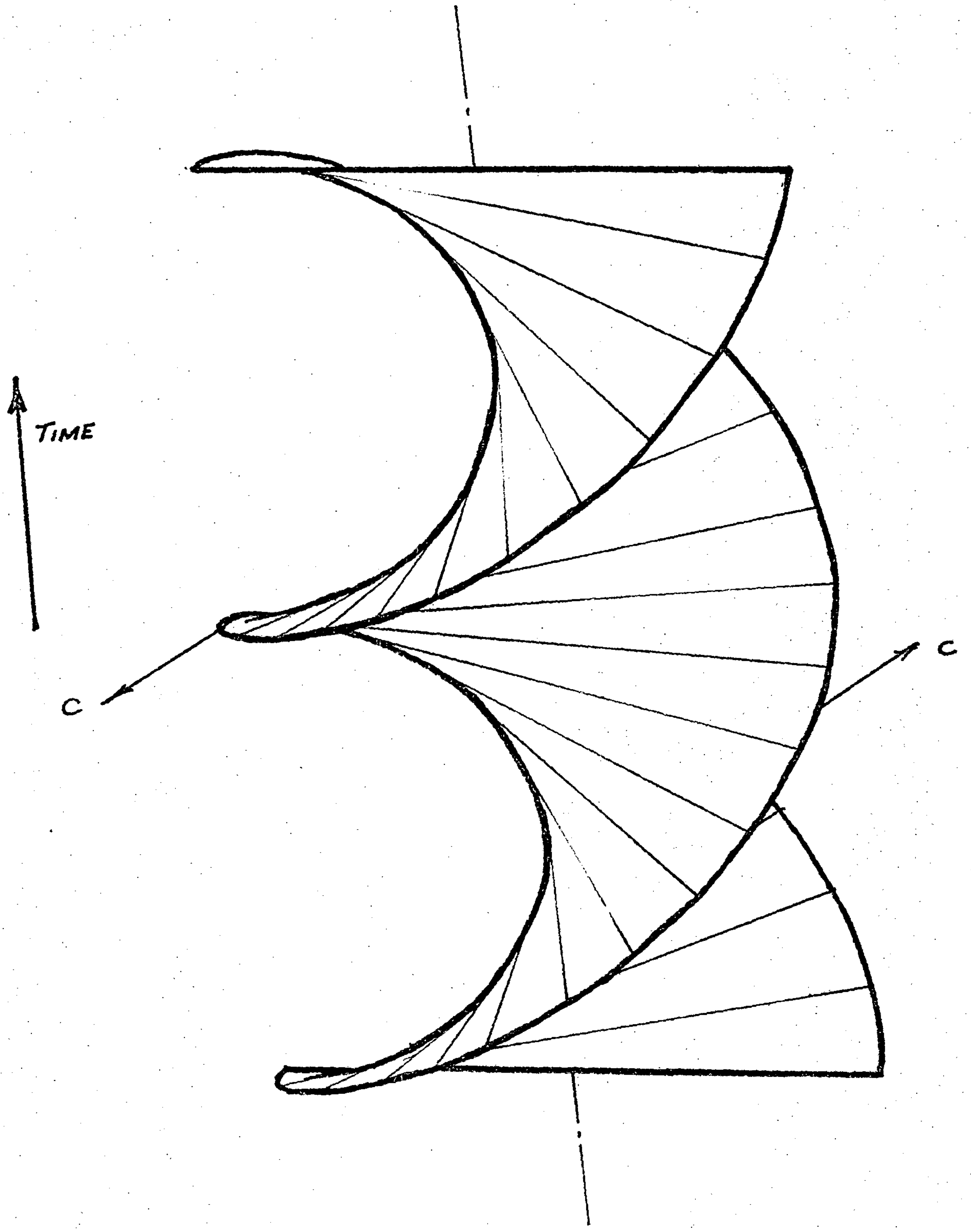
$$\mathcal{E} = p_\perp \dot{q}_\perp - \mathcal{L} = \frac{1}{2\pi\alpha'} \frac{1}{\left(1 - \frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}}}$$

and the total energy by

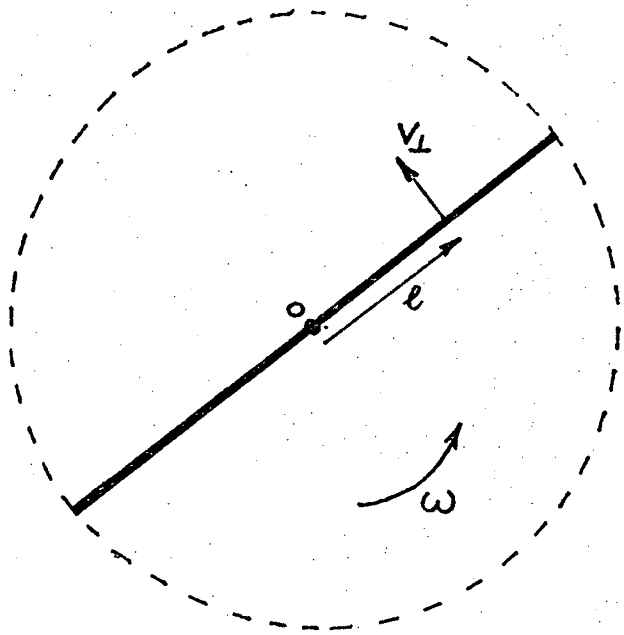
$$E = \int_{\ell_o}^{\ell_1} \mathcal{E} \, d\ell \int_{\ell_o}^{\ell_1} d\ell \, \frac{1}{2\pi\alpha'} \frac{1}{\left(1 - \frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}}} .$$

6.2 Rotating String

We now apply this to the example given in §3 of a string of constant length rotating with constant angular velocity, the ends of the string moving with the velocity c. We take the length of the string as 2a ($-a<\ell<a$) and the angular momentum as ω. It can be checked that this motion satisfies the Euler-Lagrange equation of motion and boundary condition at $\ell = \pm a$ obtained from the action principle by the usual method [18].



ROTATING STRING TRACING OUT A MINIMAL SURFACE IN SPACE-TIME



STRING OF CONSTANT LENGTH 2a AND CONSTANT ANGULAR MOMENTUM ω.

Now the angular momentum of the string will be given by

$$J = \int_{\ell_o}^{\ell_1} d\ell \quad \ell p_\perp = \frac{1}{\pi\alpha'} \int_0^a \frac{d\ell \;\; \ell v_\perp / c^2}{\left(1 - \frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}}}$$

and the energy by

$$E = \frac{1}{\pi\alpha'} \int_0^a \frac{d\ell}{\left(1 - \frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}}}$$

Putting $\ell = \frac{v_\perp}{\omega}$ gives

$$E = \frac{1}{\pi\alpha'} \frac{c}{\omega} \int_0^1 \frac{d\left(\frac{v_\perp}{c}\right)}{\left(1 - \frac{v_\perp}{c}^{2}\right)^{\frac{1}{2}}} = \frac{c}{2\alpha'\omega}$$

$$J = \frac{1}{\pi\alpha'} \frac{c}{\omega^2} \int_0^1 \frac{\left(\frac{v_\perp}{c}\right)^2 d\left(\frac{v_\perp}{c}\right)}{\left(1 - \left(\frac{v_\perp}{c}\right)^2\right)^{\frac{1}{2}}} = \frac{c}{4\alpha'\omega^2}$$

from which

$$J = \frac{\alpha'}{c} E^2$$

A quantum mechanical treatment of the same problem leads to the result (with c = 1)

$$J = \alpha' E^2 + \text{constant}.$$

This agrees with the Chew-Frautschi plots for hadrons mentioned in §2.4. From experimental observations, the value of α' is about $1 (\text{GeV})^{2}$.

Replacing ω by $\frac{c}{a}$ in the classical results for E and J gives

$$E = \frac{a}{2\alpha'} \quad \text{and} \quad J = \frac{a^2}{4\alpha' c}$$

so we see that the energy of a hadron is proportional to the radius of the circle traced out by the string in its rest frame. So for hadrons of small mass, and hence also small spin, a is also small but ω is large. For those with large mass and spin, a is large and ω small.

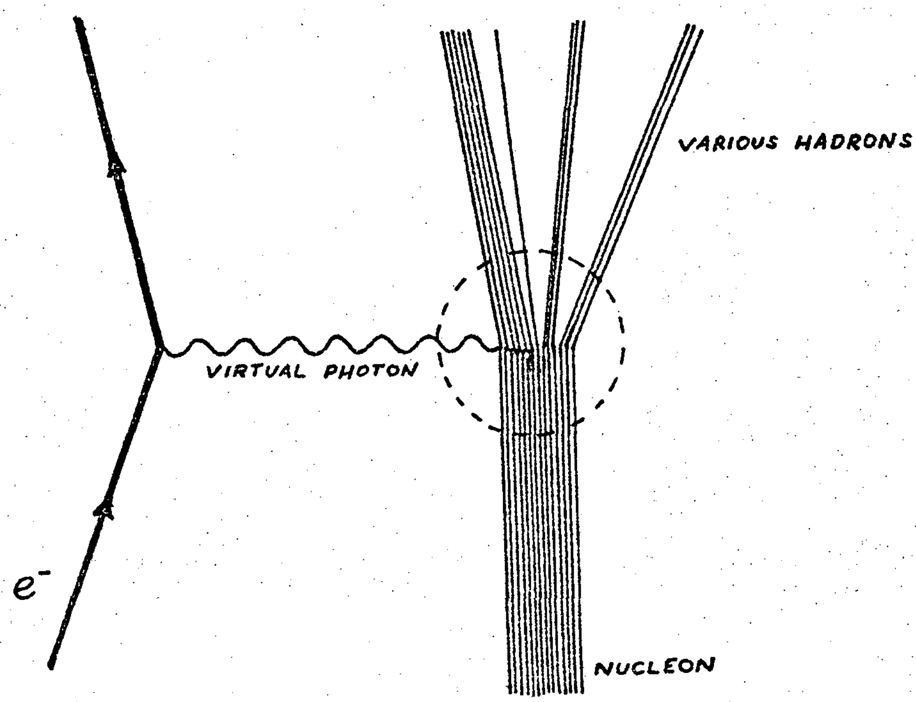
Problems: (Take $a'/c = 1 \hbar \, (\text{GeV})^{-2}$)

1. For a classical rotating string, calculate the length of the string, 2a, when J = 1. (answer: 4×10^{-4} cm)

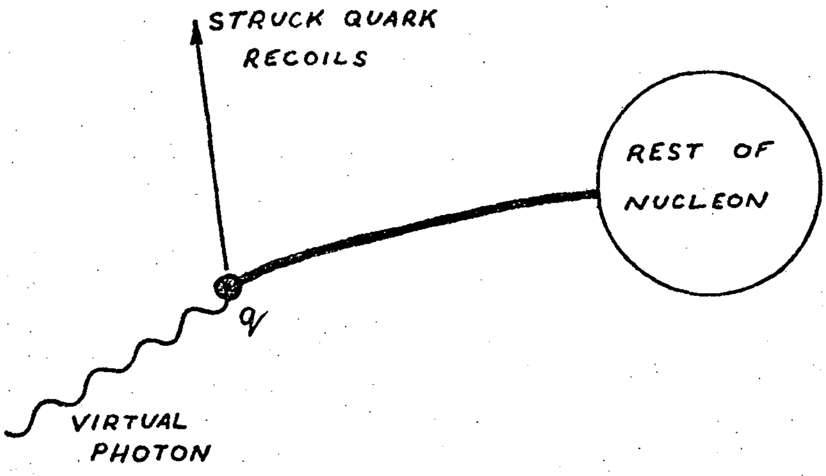
2. A massless particle can be modelled as a non-rotating string of length b moving with velocity c and aligned with its direction of motion. Calculate b for such a particle with an energy of 1 GeV. (answer: 1.2×10^{-13} cm).

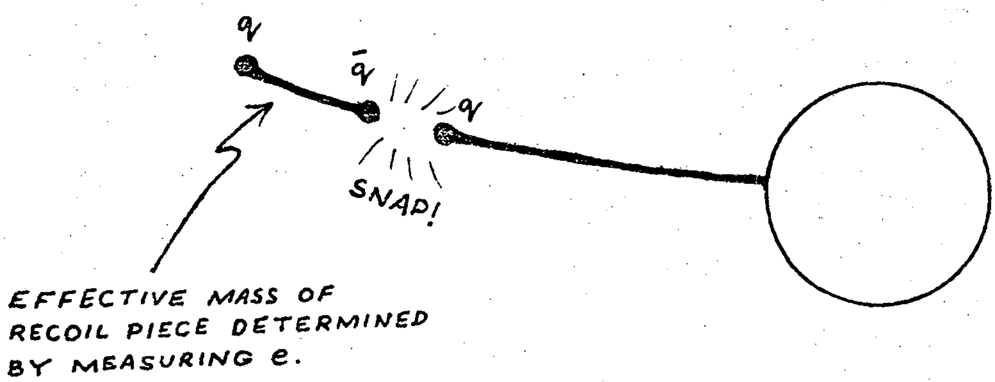
7. Application of Strings to Inelastic Scattering

The inelastic scattering of electrons from nucleons has been described in terms of string theory in the work of Tassie [19] and Braden [20]. The process we are considering is shown diagramatically below.



The emmision of a virtual photon from an electron is well known from quantum electrodynamics and won't be considered further here. To examine the process of absorption of the virtual photon by the nucleon and resultant production of hadrons we assume that one of the quarks from the nucleon is struck by the photon and recoils. As it does so, it draws out the string to which it is attached, breaking the string and so creating a new particle. This first break will determine the effective mass of the recoil piece and can be determined by measuring the inelastically scattered electron.





The recoil piece may subsequently break up into smaller pieces generally moving in the direction of the incident virtual photon. These pieces will form the photon fragmentation region. It is possible that the remaining target piece will break up also, forming the target fragmentation region.

Simple calculations can be made from this model to give relative probabilities for the types of mesons produced in the first recoil piece. Tabulated below are the quantum numbers of electric charge, strangeness and charm for the SU(4) quark flavours:

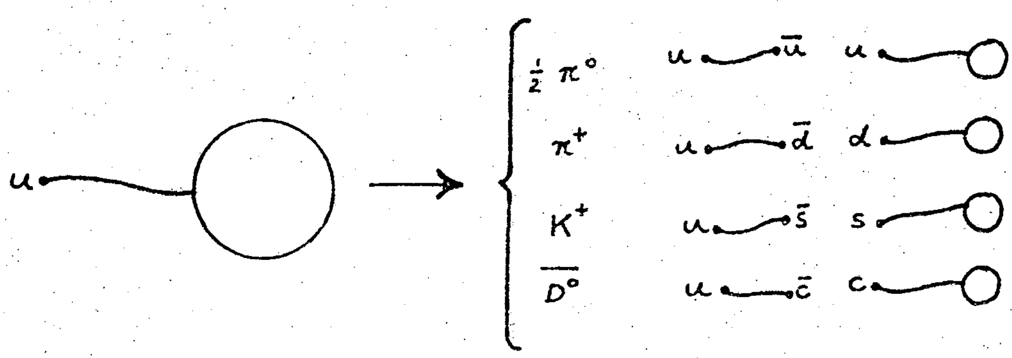
	CHARGE	STRANGENESS	CHARM
u	+ 2/3	0	0
d	- 1/3	0	0
s	- 1/3	-1	0
c	+ 2/3	0	+1

We also list the mesons and their constituent quarks:

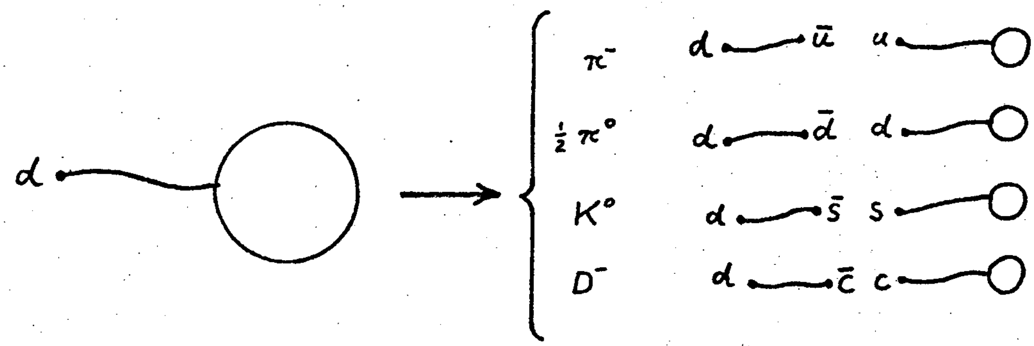
MESONS

π^0	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$	π^-	$d\bar{u}$
π^+	$u\bar{d}$	K^-	$s\bar{u}$
K^+	$u\bar{s}$	D^0	$c\bar{u}$
$\overline{D^0}$	$u\bar{c}$	$\overline{K^0}$	$s\bar{d}$
K^0	$d\bar{s}$	D^+	$c\bar{d}$
$\overline{D^+}=D^-$	$d\bar{c}$	F^-	$c\bar{s}$
$\overline{F^+}=F^-$	$s\bar{c}$		

Nucleons consist only of u and d quarks, so only those mesons containing either u or d quarks can be produced by this model of inelastic scattering. If the quark struck by the virtual photon is a u-quark, the following possibilities arise:



and if the photon strikes a d-quark, we have



Neglecting differences in masses of different types of quarks we assume that each type of quark-antiquark pair can form where the string breaks with an equal probability. However the probability amplitude of producing a particular particle should depend on the type of quark interacting with the virtual photon. We can therefore assume the probability of any of the processes shown in the diagrams above to be proportional to

$$(\text{charge of interacting quark})^2 \times (\text{number of that type of quark in nucleon})$$

For interactions with u-quarks we have $Q^2 = \frac{4}{9}$, with two u-quarks in the proton and one in the neutron, and for d quarks, $Q^2 = \frac{1}{9}$, with two d-quarks in the neutron and one in the proton. This gives the following table of relative probability amplitudes:

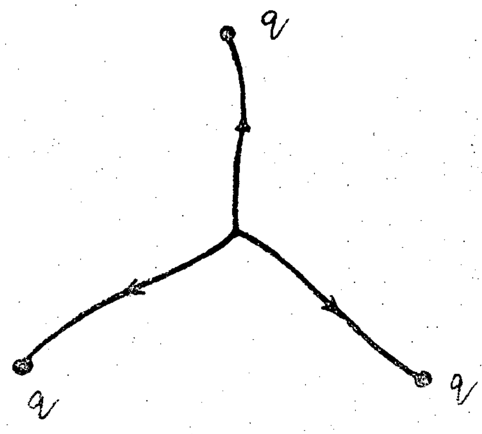
	π^0	π^+	π^-	K^+	K^0	D^0	D^-	Any other mesons
P = uud	$\frac{9}{2}$	8	1	8	1	8	1	0
N = ddu	3	4	2	4	2	4	2	0

The experimental data currently available is still insufficient to make any strong judgements for this model or for similar models or for similar models assuming quark-diquark structure of nucleons (§8). One shortcoming of either model seems to be the predicted ratio of π^+/π^- production off protons. The model used here assumes that the photon fragmentation region consists only of the meson in the initial recoil piece. Such mesons manifest themselves experimentally as having large values of the Feynmann scaling variable x_F, which is the fraction of the longitudinal momentum of the incident photon transferred to the recoiling meson.

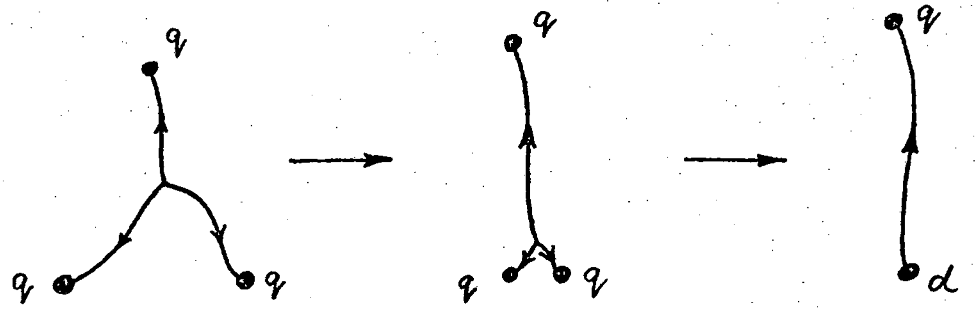
While the predicted ratio of π^+/π^- production is 8, observations indicate that this ratio is never more than about 3. This low value may be due to measurement being taken at low x_F or alternately may indicate the occurence of some other process.

8. Exotics

A model of baryons involving diquarks has been proposed by Lichtenberg and Tassie [21] and Keleman, Lichtenberg and Tassie [22]. For massless quarks, the area traced out in space-time by the configuration so far suggested for baryons, viz.

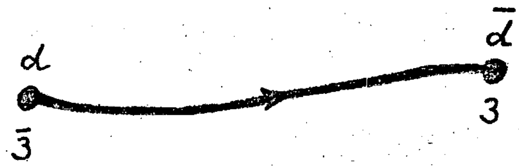


can be reduced if two of the quarks coalesce to give the configuration:

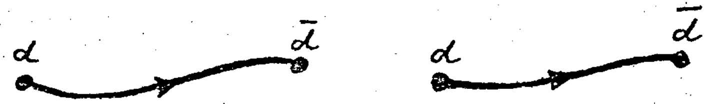


forming a single string bounded by a quark and a diquark. The diquarks belong to the colour SU(3) representation $\bar{3}$ in order that the baryon formed by a quark and a diquark should be a colour singlet.

Any meson which is more complex than q——$\bar{q}$, for instance one composed of diquarks, is called an exotic meson. In particular we consider the exotic meson of the form



This can decay by the process of breaking the string into either two exotic mesons or a baryon and antibaryon:



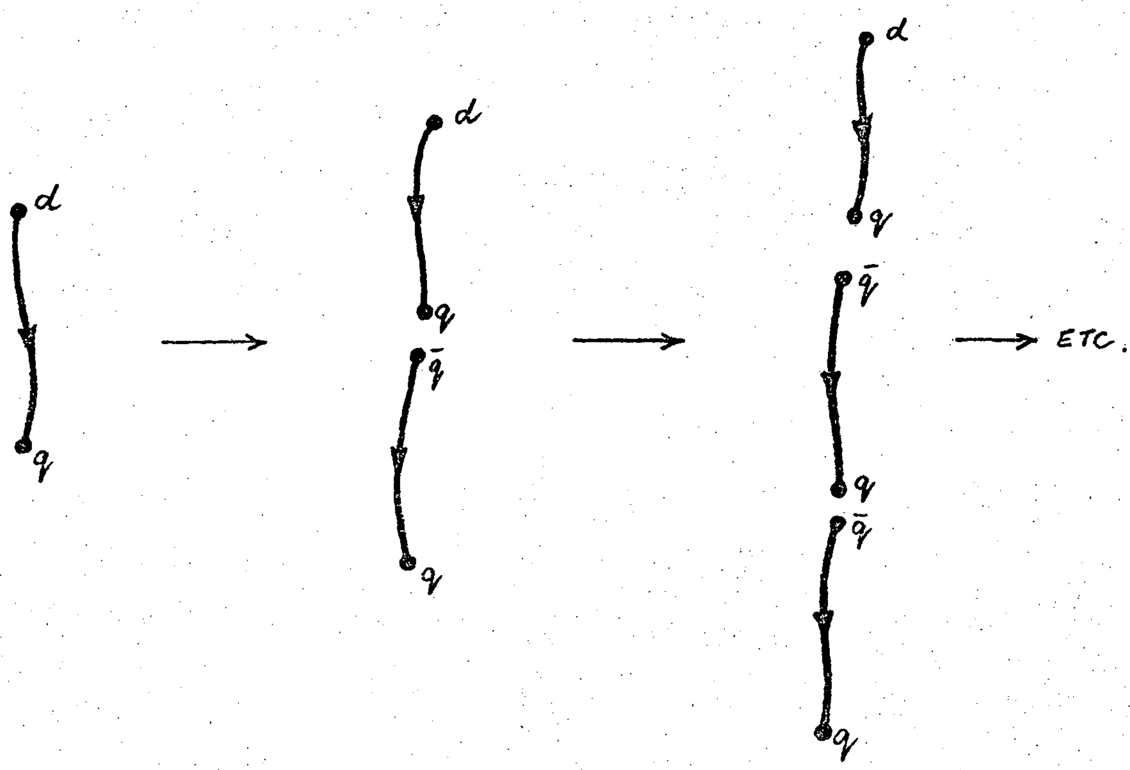
EXOTICS

BARYON – ANTI-BARYON

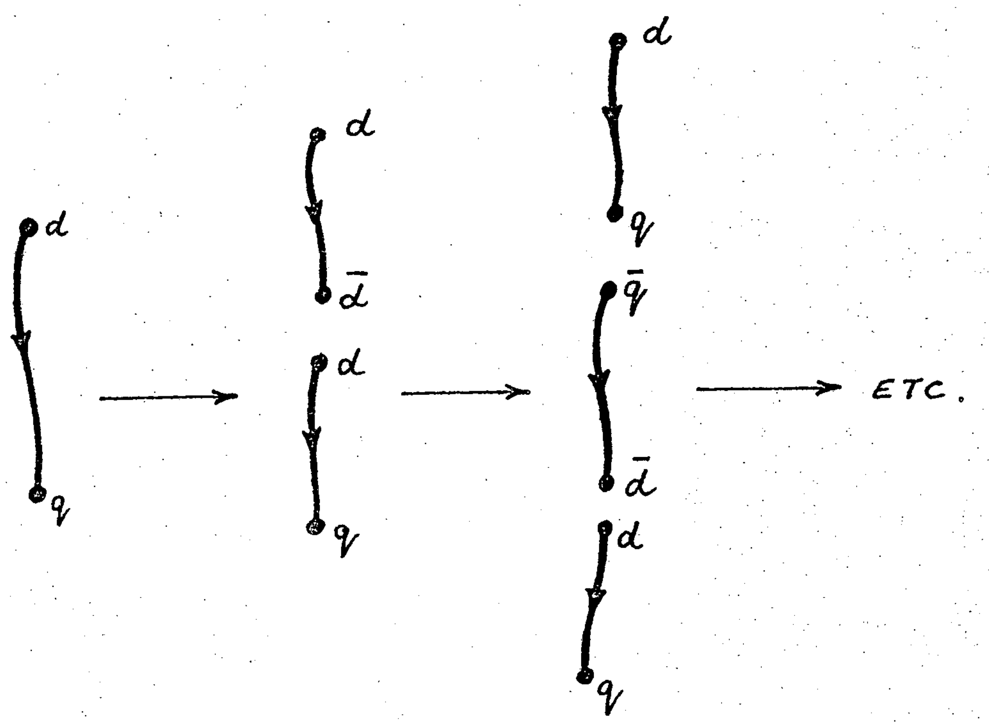
An exotic meson cannot, however, decay by string breaking into only non exotic mesons so that simple string breaking decays of d-$\overline{d}$ exotics must produce baryon-antibaryon pairs.

Consider deep inelastic electron scattering from a baryon constructed from a quark and a diquark.

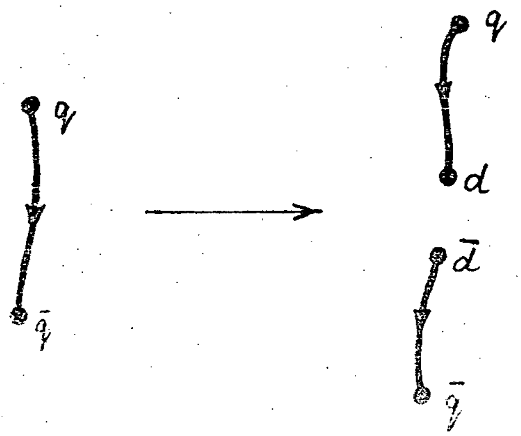
Although it requires less energy to snap a string to create a q$\overline{q}$ pair than to create a $\overline{d}$ d pair, a process such as



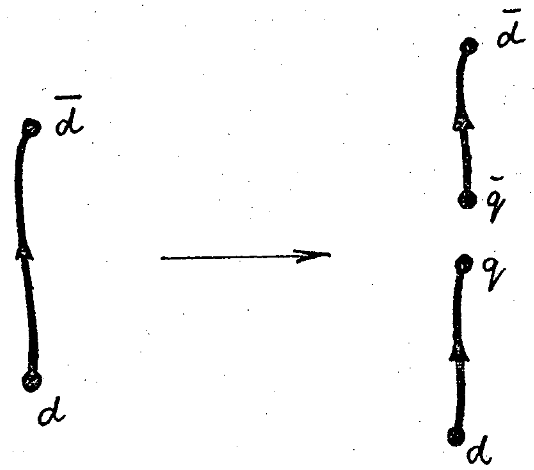
involving only $q\,\bar{q}$ creation will never produce an antibaryon. However antibaryons are produced in deep inelastic scattering and so some $\bar{d}\,d$ string breaking must occur, eg.



In the decay of string pieces antibaryons can arise from the decay of an $q\,\bar{q}$ meson which can decay into either non-exotic mesons or $B\,\bar{B}$ via the process:



or from an exotic meson which decay either into more exotics or into $B\,\overline{B}$ via the process:



Thus we expect that a substantial amount of antibaryons produced in inelastic electron scattering should come from the decay of exotic mesons. It is therefore suggested that the production of baryon-antibaryon resonances be looked for in inelastic electron scattering.

9. Extended Objects in Quantum Field Theory

To examine strings in terms of a field theory it is necessary to develop fields which consist of extended but more or less localized deviations from the ground state which must in themselves be stable, that is, solitons. We shall follow largely the treatment given by 'tHooft [23] , while a more qualitative treatment can be found in [24] .

Such a localized deviation from a vacuum state can be immune to decay into a vacuum state with time evolution if it is topologically stable. To explain what is meant by topological stability we first consider models in one space and one time dimension.

9.1. Solitons in one space and one time dimension

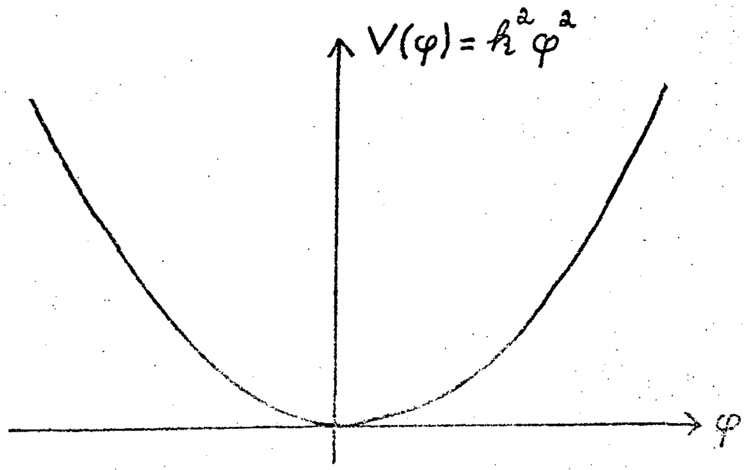
The Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_x \phi)^2 + \tfrac{1}{2}(\partial_t \phi)^2 - V(\phi)$$

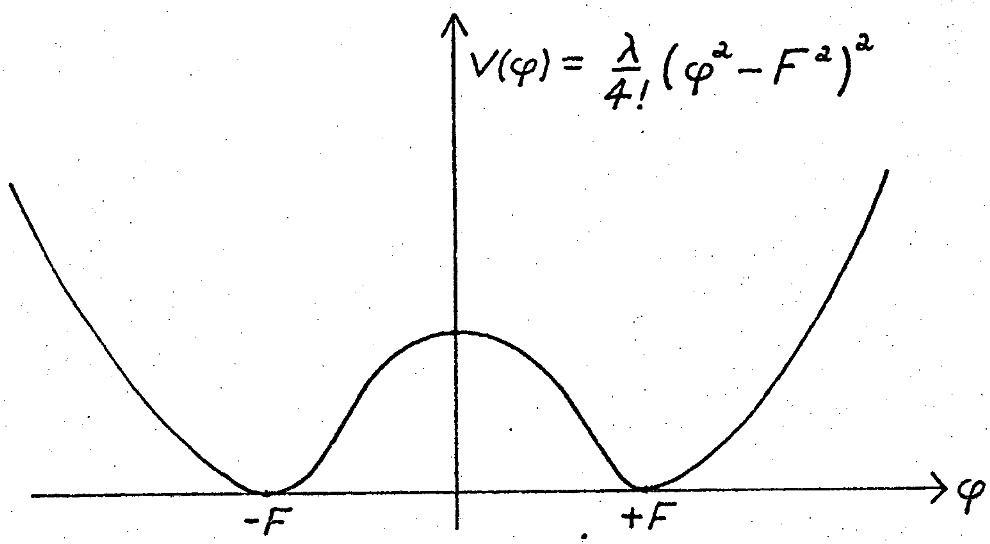
gives rise to a simple scalar field satisfying

$$\partial_t{}^2\phi - \partial_x^2\phi + \frac{d}{d\phi} V(\phi) = 0$$

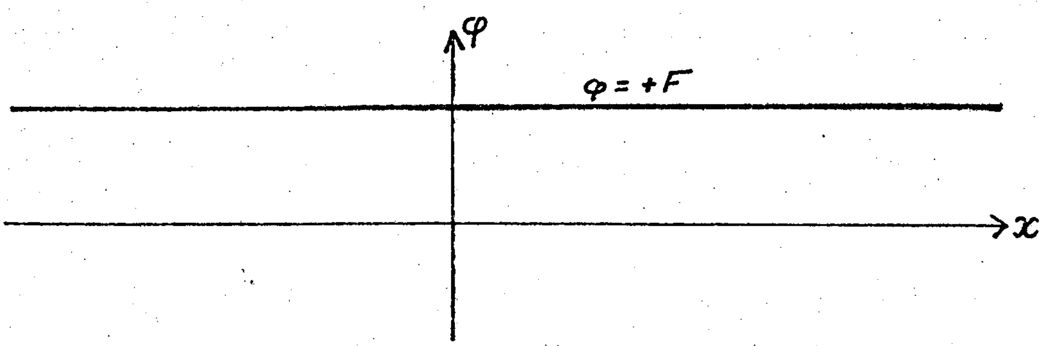
The existence of solitons will depend on the choice of potential $V(\phi)$. If $V(\phi)$ is given by $V(\phi) = k^2\phi^2$

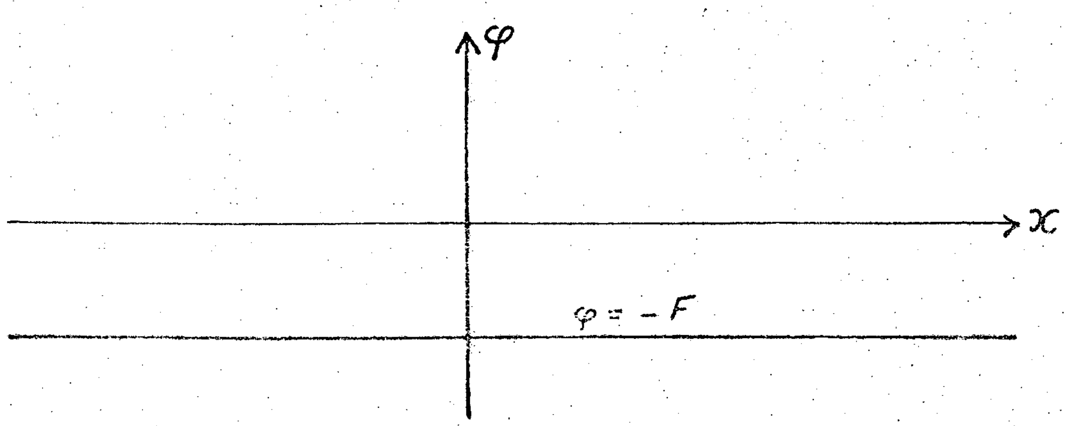


then the vacuum state $\phi \equiv 0$ is clearly a solution. Any solution which initially deviates from the vacuum state in some finite region will eventually decay to the vacuum state as $t \to \infty$ and so no solitons exist for this choice of potential. On the other hand, with the potential $V(\phi) = \lambda(\phi^2-F^2)^2/4!$



there are two degenerate vacuum states, $\phi = \pm F$.





We might also seek solutions satisfying the boundary conditions

$$\phi(+\infty) = +F \quad ; \quad \phi(-\infty) = -F$$

or

$$\phi(+\infty) = -F \quad ; \quad \phi(-\infty) = +F.$$

For static solutions we have

$$\partial_x^2\phi = dV/d\phi$$

so

$$\frac{\partial}{\partial x}\left[\tfrac{1}{2}(\partial_x\phi)^2 - V\right] = 0$$

From the boundary condition $V(\phi(\pm\infty)) = 0$ we have

$$\tfrac{1}{2}(\partial_x\phi)^2 - V(\phi) = \text{const.} = 0$$

so

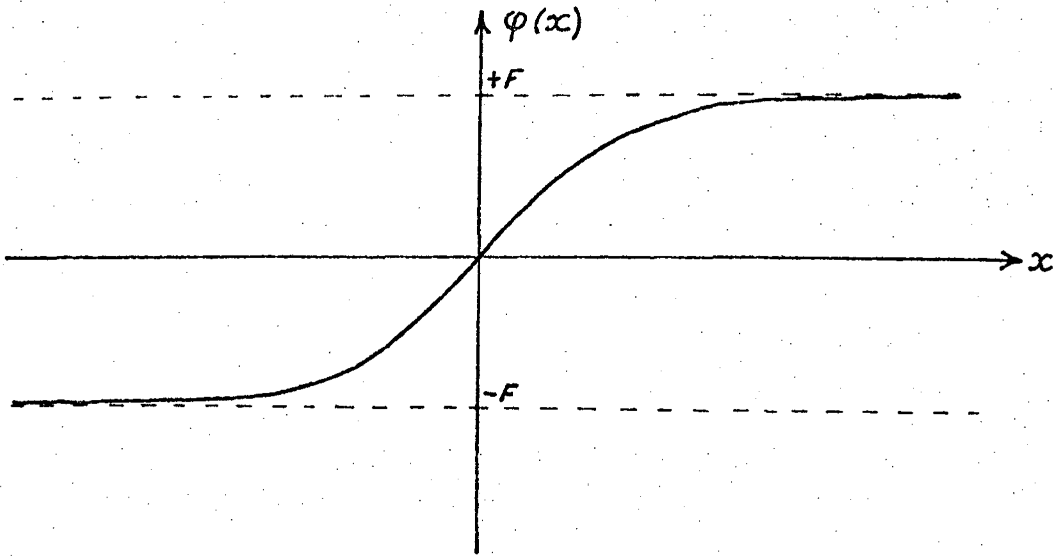
$$\frac{dx}{d\phi} = \frac{1}{(2\,V(\phi))^{\frac{1}{2}}}$$

With $V(\phi) = \lambda(\phi^2 - F^2)^2/4!$ this gives

$$\phi = F \tanh \frac{F\lambda}{2\sqrt{3}}(x+\text{const.})$$

$$= F \tanh \frac{F\lambda}{2\sqrt{3}}\, x$$

by suitable choice of origin.



This solution is topologically stable against decay into either of the vacuum states $\phi = \pm F$ and so represents a soliton at rest. In this case the topological stability is due to the boundary conditions forming a discrete set, so no continuous transformation can convert these boundary conditions to those of either of the vacuum states.

9.2 Solitons in Two Space Dimensions

It is not possible to develop topological solitons in two space dimensions from a scalar field because the boundary condition as $|x| \to \infty$ must differ from a constant. (If the boundary condition were $\phi \to$ const. as $|x| \to \infty$, then there is no topological reason why any solution cannot decay to the vacuum state $\phi \equiv$ const.)

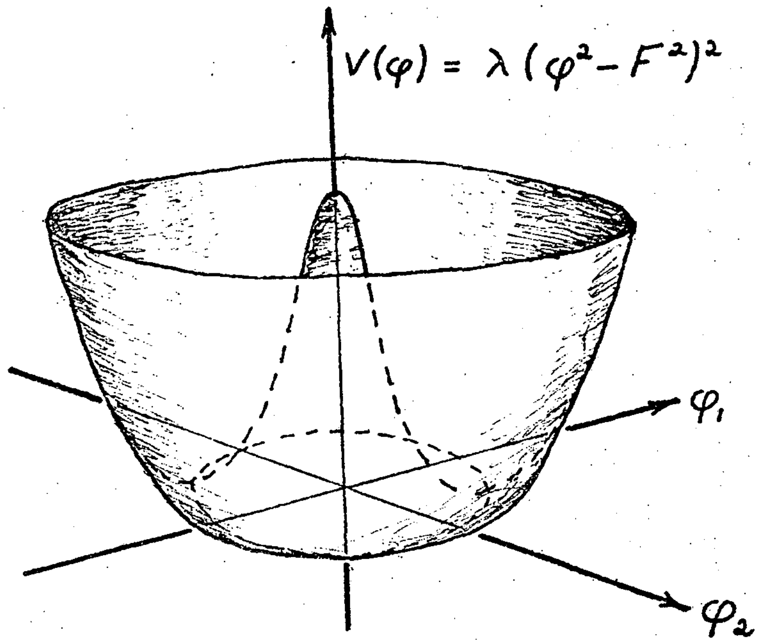
Suppose then that we consider a two component field

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

with the "Mexican hat" potential:

$$V(\phi) = \lambda(\phi^2 - F^2)^2$$

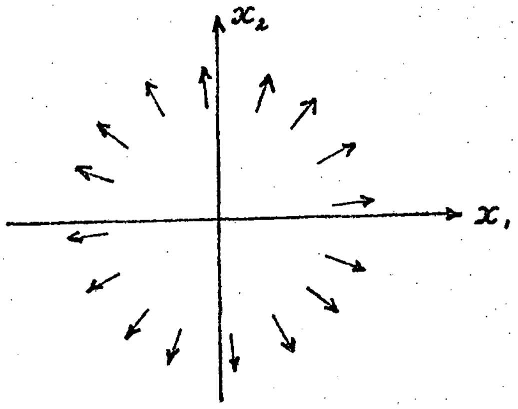
where $\phi^2 = \phi_1^2 + \phi_2^2$



For this potential we have an infinitely degenerate set of vacuum states given by $\phi^2 = F^2$.

The boundary condition

$$\underline{\phi}(x) \to \underline{x}/|\underline{x}| \quad \text{as} \quad |\underline{x}| \to \infty$$



could be expected to yield topologically stable solutions. However the term $\frac{1}{2}(\partial_{\underline{x}}\phi)^2 \; d^2\underline{x}$ in the Lagrangian asymptotes to $\frac{1}{2}(F/|x|)^2 \; d^2\underline{x}$ as $|x| \to \infty$, and upon integrating, this is logarithmically divergent. By the same argument it can be shown that the corresponding n-component field and boundary conditions in n space dimensions also lead to divergent integrals within the Lagrangian.

It is possible to overcome this problem with the introduction of gauge fields. We modify ∂_i with the gauge transformation

$$\partial_i \rightarrow D_i = \partial_i + gA_i$$

where A_i is a gauge field.

9.3. Gauge Fields

Before preceeding further we consider the effect of gauge transformations on a complex scalar field. The Lagrangian

$$\mathcal{L}(\phi,\phi^*) = (\partial_\mu\phi)(\partial^\mu\phi^*) - \kappa^2\phi\phi^*$$

gives the Klein-Gordon equation

$$(\Box + \kappa^2)\phi = 0 \qquad \text{where} \qquad \Box = \partial^2/\partial t^2 - \nabla^2$$

Clearly the phase of ϕ is unimportant, that is, the Klein-Gordon equation is invariant with respect to transformation of the form

$$\phi \rightarrow e^{i\theta}\phi .$$

In this case, the gauge group is the Abelian group U(1). Generalizations can be made to include non-Abelian groups but we will not go into the mathematics of these here.

If a gauge transformation is used in which the phases are position dependent in space time, ie.

$$\phi(x) \rightarrow e^{i\theta(x)}\phi(x)$$

the Lagrangian and field equations are no longer invariant since

$$\partial_\mu\phi \rightarrow e^{i\theta(x)}\,\partial_\mu\phi + ie^{i\theta(x)}\,\partial_\mu\theta(x)$$

that is, the derivative of the field no longer transforms like the field itself. To restore the invariance we replace ∂_μ by a covariant derivative D_μ such that

$$D_\mu\,\phi(x) \rightarrow e^{i\theta(x)}\,D_\mu\,\phi(x) \qquad (1)$$

This can be done by introducing the gauge field $A_\mu(x)$ which is defined to transform as

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{g}\,\partial_\mu\theta(x)$$

with g a constant. D_μ is then defined by

$$D_\mu \equiv \partial_\mu + igA_\mu$$

so that $D_\mu\phi$ now transforms according to (1).

We can still add a gauge invariant scalar to the Lagrangian:

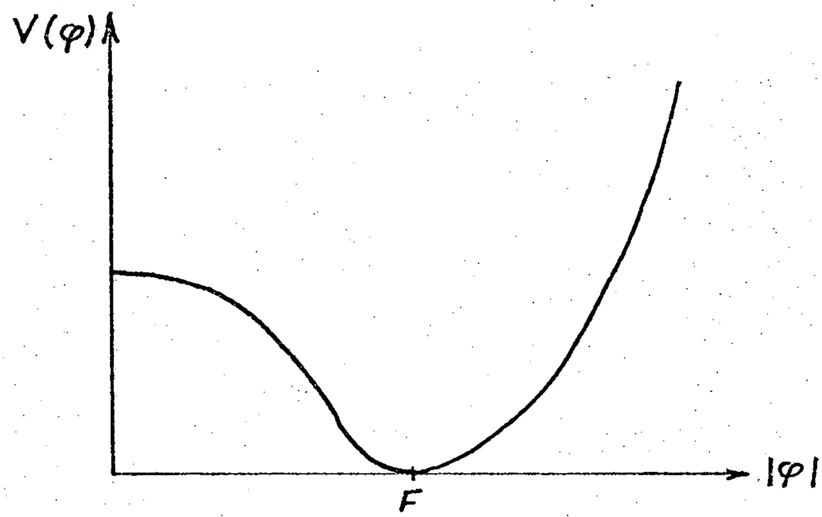
$$\begin{aligned}\mathcal{L} &= \mathcal{L}_1 - \frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \\ &= -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - igA_\mu)\phi^*\,(\partial^\mu + igA^\mu)\phi - \kappa^2\phi\phi^*\end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, giving the Lagrangian for an electromagnetic field with a scalar field as source. The electromagnetic field is an Abelian gauge field corresponding to a particle of zero mass, ie. a photon. By use of the Higgs mechanism we can introduce a mass for the vector particle by altering the Lagrangian to give degenerate vacuum states.

The Higgs Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \left[(\partial_\mu - igA_\mu)\phi^*\right](\partial^\mu + igA^\mu)\phi - \tfrac{1}{2}\lambda(\phi^*\phi - F^2)^2 \qquad (2)$$

Whereas previously the vacuum was described by $\phi=0$, we now have the degenerate Higgs vacuum $\phi = \exp(i\alpha F)$.



An alternative choice of gauge which makes some of the physics clearer is to put

$$\phi(x) = (F + \psi)e^{i\alpha(x)}$$

where $\alpha(x)$ is chosen so as to make ψ real. It is also necessary to regauge A, so we write

$$A_\mu = B_\mu - \frac{1}{g}\partial_\mu \alpha(x)$$

Then the Higgs Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \left[(\partial_\mu - ig B_\mu)\psi\right](\partial^\mu + igB^\mu)\psi$$

$$+ 2g^2FB_\mu B^\mu\psi + \tfrac{1}{2}\lambda\psi^4 - 2\lambda F\psi^3 - 2\lambda F^2\psi^2 + g^2F^2B_\mu B^\mu$$

From the term $+g^2F^2B_\mu B^\mu$ we identify the mass of the vector field as

$$m_V = \sqrt{2}\ gF$$

and from the term $-\ 2\lambda F^2\psi^2$, the Higgs scalar field has a mass of

$$m_H = (2\lambda)^{\frac{1}{2}}F$$

9.4. String-like solutions to the Higgs Lagrangian

In this section we shall follow the work of Nielsen and Olesen [25] and Olesen [26].

Returning to the original Higgs Lagrangian (2), we note that it is similar to the Lagrangian of the Ginzburg-Landau theory of type II superconductivity and so there are vortex solutions. That is, it should be possible to derive string-like solutions.

We are seeking a solution which decays to the vacuum at infinity, so for two spatial dimensions a suitable boundary condition is

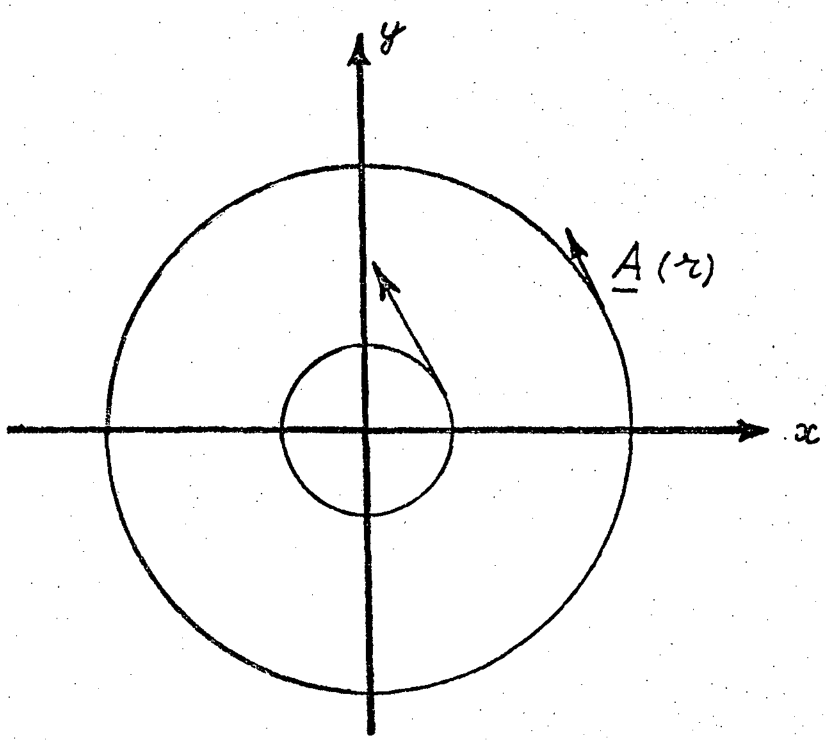
$$\phi \to F\ e^{in\theta} \quad \text{as } |x| \to \infty$$

where n is an integer in order that ϕ should be single-valued. We shall consider the simplest case, n = 1, and assume cylindrical symmetry and z-independence of the solution. Using polar co-ordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

we put

$$\phi = \chi(r)\ \exp i\theta$$
$$\underline{A} = \underline{r} \times \underline{e}_z a(r), \quad A_o = 0$$



The action is given by

$$\int \mathcal{L}\, d^2\underline{x} = \int d^2\underline{x}\left[-\frac{1}{4}(\text{curl } A)^2 - (\partial_i - igA_i)\varphi^*(\partial_i + igA_i)\varphi - \frac{1}{2}\lambda(|\varphi|^2 - F^2)^2\right]$$

$$= \int d^2\underline{x} \left\{ -\frac{1}{4}\left(2a(r) + r\frac{da}{dr}\right)^2 - \left[\left(r^{-1}\partial_\Theta - igra(r)\right)e^{-i\Theta}\chi(r)\right.\right.$$

$$\left.\left.(r^{-1}\partial_\Theta + igra(r))\, e^{i\Theta}\chi(r)\right] - \left(\frac{d\chi}{dr}\right)^2 - \frac{1}{2}\lambda\,(\chi^2 - F^2)^2 \right\}$$

$$= 2\pi \int_0^\infty r\, dr \left[-\frac{1}{4} r^2 (da/dr)^2 - (d\chi/dr)^2\right.$$

$$\left. - (r^{-1} + gra(r))^2\, \chi^2 - \frac{1}{2}\lambda\,(\chi^2 - F^2)^2 - (a^2 + ra(da/dr))\right] .$$

Integrating the last two terms:

$$\int_0^\infty r\, dr(a^2 + ra(da/dr)) = \int_0^\infty dr(ra^2 + r^2 a\, (da/dr))$$

$$= \frac{1}{2}\int_0^\infty dr\, \frac{d}{dr}\,(r^2 a^2)$$

$$= 0 \quad , \quad \text{assuming } |A| \to 0 \text{ at infinity.}$$

So the action is given by

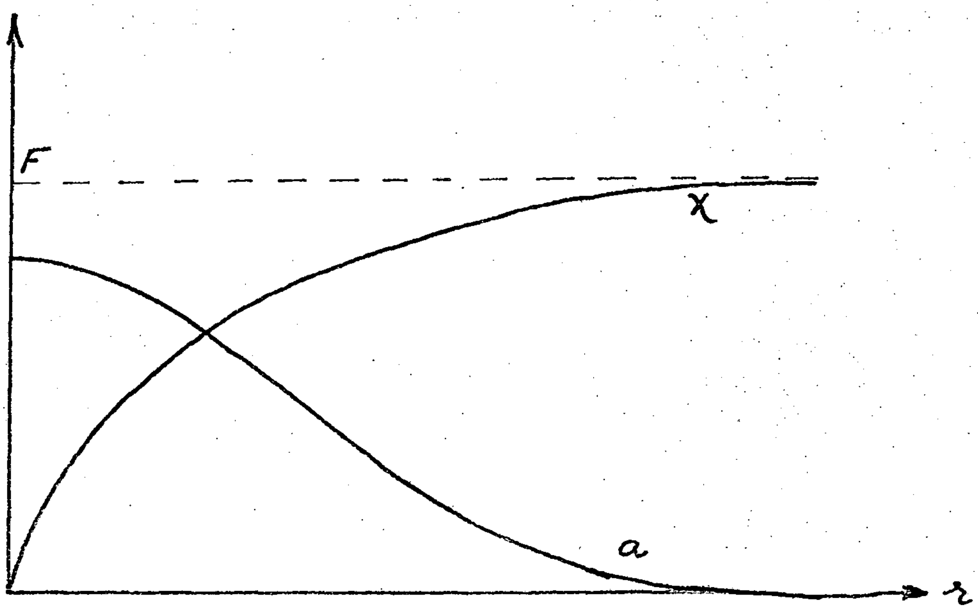
$$\int \mathcal{L}\, d^2\underline{x} = 2\pi \int r\, dr \left[-\frac{1}{4} r^2 (da/dr)^2 - (d\chi/dr)^2 - (r^{-1} + gra(r))^2 \chi^2 - \frac{1}{2}\lambda(\chi^2 - F^2)^2 \right]$$

For the last two terms to converge at $r \to \infty$ we must have

$$\left.\begin{array}{l} \chi \to F \\ a \to -(gr^2)^{-1} \end{array}\right\} \text{as } r \to \infty \; .$$

Also we must have

$$\left.\begin{array}{l} \chi \to 0 \\ a \;\; \text{finite} \end{array}\right\} \text{as } r \to 0 \; .$$



From the conditions on a as $r \to \infty$ the flux Φ in the x-direction is

$$\Phi = \int F_{\mu\nu}\, d\sigma^{\mu\nu} = \oint A_\mu(x) dx^\mu$$

$$= \oint \underline{A}.d\underline{l} = 2\pi r.ra = -2\pi/g \; .$$

In fact, one can see that the flux is quantized if we consider the possible cylindrically symmetric boundary conditions at $|r| \to \infty$, viz. $\varphi \to F \exp(in\theta)$. This gives

$$\Phi = 2\pi n/g \quad .$$

The Higgs Lagrangian in the form (2) yields the field equation:-

$$(\partial_\mu + igA_\mu)^2\varphi = \lambda F^2\varphi - \lambda \varphi^2\varphi^* \tag{3}$$

$$\partial^\nu F_{\mu\nu} = j_\mu = ig(\varphi^*\partial_\mu\varphi - \varphi\,\partial_\mu\varphi^*) - 2g^2A_\mu|\varphi|^2 \quad . \tag{4}$$

To deal with the cylindrically symmetric static case we rewrite these equations in cylindrical co-ordinates. For $\varphi = \chi(r)\exp(i\theta)$ this gives

$$-r^{-1}\frac{d}{dr}\left(r\frac{d\chi}{dr}\right) + \left[(r^{-1} - g|A|)^2 - \lambda F^2 + \lambda\chi^2\right]\chi = 0 \tag{5}$$

$$-\frac{d}{dr}\left(r^{-1}\frac{d}{dr}\,r|A|\right) + 2\,\chi^2(|A|\,g^2 - gr^{-1}) = 0 \tag{6}$$

Equation (6) is the $\hat{\underline{\theta}}$ component of (4). To obtain the first term of (6) we note that the $\hat{\theta}$ component of $\partial^\nu F_{\mu\nu}$ is just

$$(\text{curl}\ \underline{H})_{\hat{\theta}} = \partial H/\partial r = \frac{d}{dr}\left(r^{-1}\frac{d}{dr}\,r|A|\right) \quad .$$

Although (5) and (6) have not so far been solved analytically it is possible to find the asymptotic behaviour of the solutions as $r \to \infty$. For $|A|$ we have

$$|A| \sim (gr)^{-1} - C\,r^{-\frac{1}{2}}\,e^{\sqrt{2}\,gFr} \quad \text{as} \quad r \to \infty$$

where C is a constant of integration. The vector field $\underline{H} = \nabla \times \underline{A}$ is then given by

$$H = \frac{1}{r}\ \frac{d}{dr}\ (r|A|) \sim \sqrt{2}\ gFC\ r^{-\frac{1}{2}}\ e^{-\sqrt{2}\ gFr} \quad , \quad r \to \infty \ .$$

The asymptotic behaviour of χ is given by

$$\chi \sim F + C'\ r^{-\frac{1}{2}}\ e^{-(2\lambda)^{\frac{1}{2}}Fr} \quad , \quad r \to \infty \ .$$

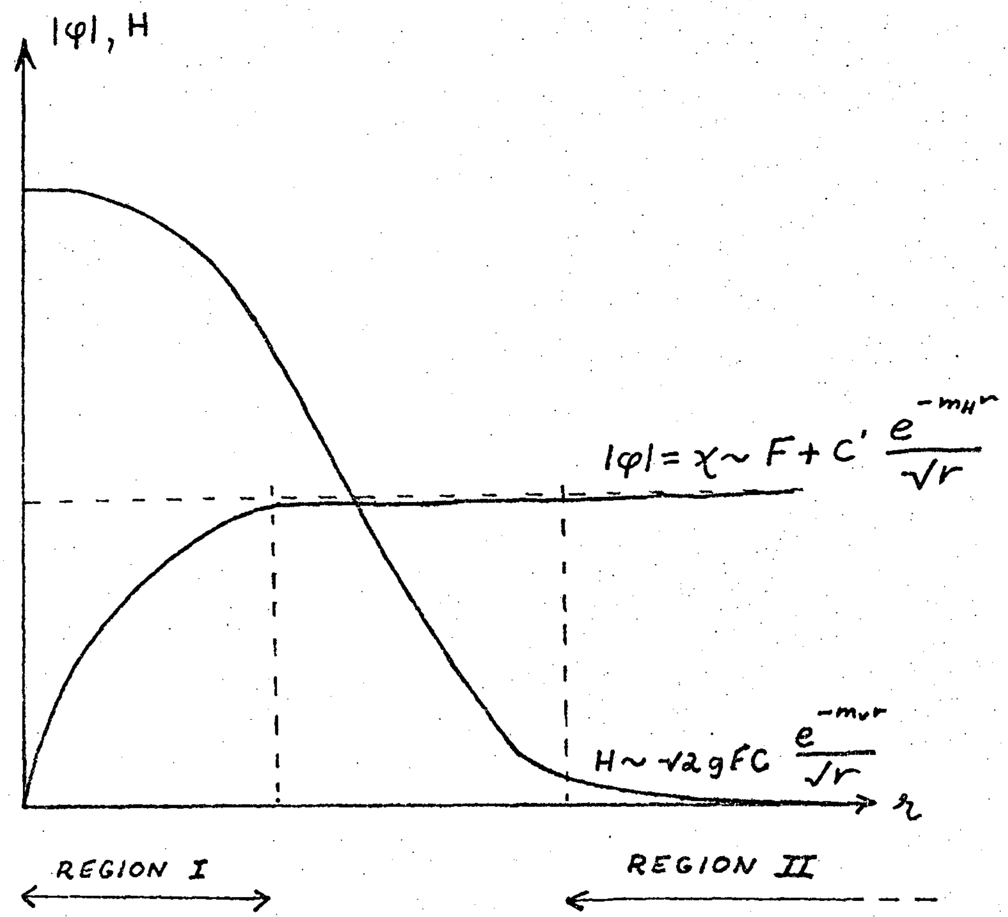
Recalling the definitions of m_V and m_H, the masses of the vector and scalar Higg's fields from §9.3, we see that

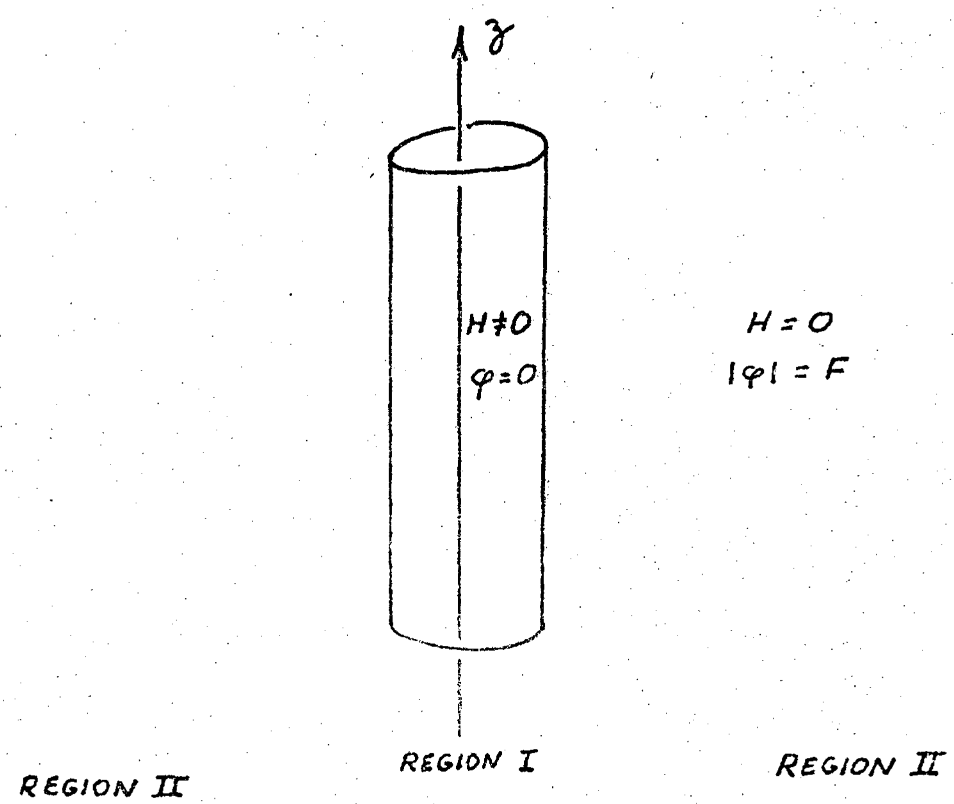
$$H \sim \sqrt{2}\ g\ FC\ r^{-\frac{1}{2}}\ e^{-m_V r} \quad , \quad r \to \infty$$

$$\chi \sim F + C'\ r^{-\frac{1}{2}}\ e^{-m_H r} \quad , \quad r \to \infty \ .$$

Plots of H and χ as functions of r are given on the next page.

Because of the term $-\frac{1}{2}\lambda F^2|\varphi|^2$ in the Lagrangian, the Higgs mechanism forces φ from 0 to F. However, we also have the term $g^2 A_\mu A^\mu |\varphi|^2$, so if in some region of space A^2 is sufficiently large, this term wins and there is no longer a Higgs mechanism, so the ground state is $\varphi=0$. These phenomena lead to the string-like structure of the vortex solution, where in region I we have $H \neq 0$, $\varphi = 0$ and in region II, $H = 0$, $|\varphi| = F$. The distances over which H and φ differ noticably from their limiting values are characterised by the inverse Compton wavelengths m_V^{-1} and m_H^{-1} of the vector and scalar masses respectively.


|φ|, H
|φ| = χ ~ F + c' e^{-m_H r}/√r
H ~ -√2 gFC e^{-m_V r}/√r
r
REGION I
REGION II


z
H≠0
φ=0
H = 0
|φ| = F
REGION II
REGION I
REGION II

9.5 Strong Coupling Limit

We now identify the vortex solution of §9.4 with the dual string discussed in §6.

Recall that the rest energy per unit length for a dual string is given by

$$\mathcal{E}_{string} = \frac{1}{2\pi\alpha'}$$

For the vortex solution, we have for the magnetic energy

$$\mathcal{E}_{vortex} = \frac{1}{2}\int H^2 \, 2\pi r \, dr \quad .$$

This can be evaluated approximately by assuming H to take its average value for $0 \leqslant r \leqslant R = m_v^{-1}$, so that

$$\mathcal{E}_{vortex} = \frac{1}{2} H^2 \pi R^2 = \frac{1}{2} \frac{\Phi^2}{\pi R^2} = \frac{1}{2} \cdot \frac{4\pi^2}{g^2} \cdot \frac{1}{\pi R^2}$$

$$= \frac{2\pi}{g^2} m_v^2 = 2\pi F^2 \quad .$$

There is also a contribution to the energy per unit length of the string due to the core, where $|\varphi|$ deviates from F, and this contribution is of the same order of magnitude as the magnetic energy [25].

So $\alpha' \sim F^{-2}$, that is, F should remain finite. For the vortex to constitute a string of negligible thickness we must have $m_v = \sqrt{2}\, gF$ and $m_H = (2\lambda)^{\frac{1}{2}} F \rightarrow \infty$. This implies that the coupling constants λ and g must both tend to infinity. We refer to this limit as the strong coupling limit.

In this limit the field $F_{\mu\nu}$ is non-vanishing only in a small region and the field φ is nearly equal to the vacuum everywhere apart from that region in space where $F_{\mu\nu} \neq 0$ provided we take m_V and m_H approximately equal. Because of this behaviour, the terms $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $-\frac{1}{2}\lambda(\varphi^2-F^2)^2$ in the Lagrangian are non-vanishing only within the line vortex where they are of order m_V^2 and m_H^2 respectively and so act like smeared out δ - functions. The remaining term in the Lagrangian $[(\partial_\mu+igA_\mu)\varphi]^2$ remains finite everywhere, and because of the choice of gauge vanishes outside the vortex in the strong coupling limit.

The action for the line vortex is then given by

$$S_{vortex} = \int d^4x \mathcal{L}_{vortex}$$

$$\simeq \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\lambda(\varphi^2-F^2)^2 \right]$$

$$\simeq \int dt\, d\ell \left[-\frac{1}{4}\overline{F_{\mu\nu}F^{\mu\nu}}\,\sigma_V - \frac{1}{2}\lambda\overline{(\sigma^2-F^2)^2}\,\sigma_H \right]$$

where ℓ is the distance along the string and σ_V and σ_H the areas over which the Higgs vector and scalar fields do not vanish.

The cross-sectional area of a moving string is related to the cross-sectional area in the rest frame by

$$\sigma = \sigma_0(1 - v_\perp^2)^{\frac{1}{2}} ,$$

so finally we recover the action for a dual string:

$$S = \text{const.} \int dt\, d\ell\, (1-v_\perp^2)^{\frac{1}{2}} .$$

10 Lattice Gauge Theory

Wilson [27] has treated gauge theory on a discrete lattice in four dimensional Euclidean space-time. Other treatments of lattice gauge theory have been given by Balian et al. [28] and Kadanoff [29]. By defining field values at each site in a lattice it is possible to quantize the field by means of Feynman path integrals.

We take the field to be $\varphi(n)$, where n labels the lattice sites. Assuming a model in which only fields of neighbouring sites interact, the action can be taken to be, for instance,

$$S = \sum_{\substack{n,m \text{ nearest} \\ \text{neighbours}}} \varphi(n)^*\varphi(m) + \varkappa^2 \sum_n \varphi(n)^*\varphi(n) \quad .$$

By analogy with the treatment of the abelian gauge field in §9 we now demand the action to be invariant under the gauge transformation

$$\varphi(n) \rightarrow e^{i\alpha(n)}\varphi(n) \quad .$$

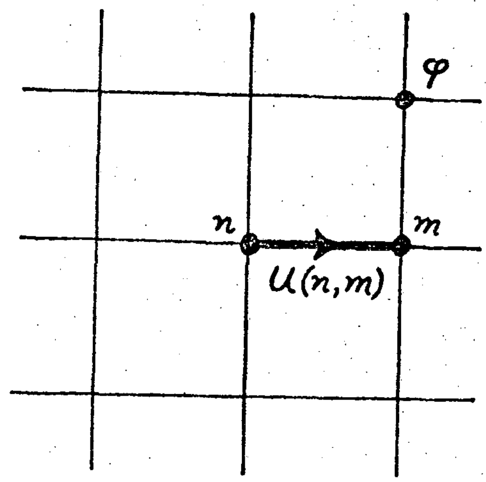
Invariance of S with respect to this transformation can be guaranteed by the introduction of a gauge field U(n,m) transforming according to

$$U(n,m) \rightarrow e^{i\alpha(n)}\, U(n,m)e^{-i\alpha(m)} \quad .$$

The action

$$S = \sum_{\substack{n,m \text{ nearest} \\ \text{neighbours}}} \varphi(n)^*U(n,m)\varphi(m) + \varkappa^2 \sum_n \varphi(n)^*\varphi(n)$$

is then gauge invariant. We can loosely refer to the U(n,m) as a piece of string joining the lattice sites n and m.



corresponding to the addition of the gauge invariant term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ in the continuum case, terms of the form

$$U(n_1,n_2)U(n_2,n_3)\ \ldots\ U(n_k,n_1)$$

can be added to the Lagrangian without destroying the gauge invariance.

10.1 Quantization by Feynman Path Integrals

In order to quantize the field we turn to Feynman path integrals. If the action

$$S = \int d^4x\, \mathcal{L}$$

includes a source term J then the vacuum-to-vacuum field is written in terms of an integral over all possible fields of the amplitude exp(iS(φ,J)). That is

$$\langle 0|0\rangle_J = A\int \mathcal{D}\varphi\ e^{i\int d^4x\,\mathcal{L}(\varphi,\partial_\mu\varphi)} .$$

The convenience of a lattice becomes apparent when we try to give a meaning to the operator $\int \mathcal{D}\varphi$. By writing the field $\varphi(n)$ at each lattice site n in the lattice as discussed before we define

$$\int \mathcal{D}\varphi \quad \int \prod_{\text{sites}} d\varphi(n) \quad .$$

If the field φ is further restricted to admit only discrete values σ, say, then we can write the integral operator as a statistical sum over configurations, viz :

$$\int \mathcal{D}\varphi = \prod_{\text{sites}} tr_{\sigma(n)}$$

$$\overset{\text{def.}}{=} \quad Tr_{\sigma(n)}$$

For instance, with an Ising model we have $\sigma(n) = \pm 1$ and so

$$\int \mathcal{D}\varphi = \int d\varphi(1) d\varphi(2) \ldots d\varphi(N)$$

$$= \sum_{\sigma(1)=\pm 1} \sum_{\sigma(2)=\pm 1} \cdots \sum_{\sigma(N)=\pm 1}$$

$$= tr_{\sigma(1)} \, tr_{\sigma(2)} \ldots tr_{\sigma(N)}$$

where we have defined

$$tr_{\sigma} f(\sigma) = f(1) + f(-1) \quad .$$

If a cubic lattice is chosen with an imaginary time co-ordinate, so that

$$x^{\mu} = (x^0, \underline{x}) = (i\, a\, n^0, a\underline{n})$$

where a is the lattice spacing, the amplitude $\exp(\frac{i}{\hbar} S(\varphi))$ becomes

$$\exp\left(\frac{i}{\hbar}\int\mathcal{L}(\varphi,\varphi_{,\mu})d^4x\right) = \exp\left(-\frac{a^4}{\hbar}\sum_n \mathcal{L}(n,m)\right) \quad .$$

Here n and m are four-vectors with integer components labelling neighbouring sites. The replacement

$$\varphi_{,\mu} \rightarrow a^{-1}(\varphi_{n+\hat{\mu}} - \varphi_n) \quad , \quad |\hat{\mu}| = a$$

has been made in the Lagrangian so that interaction only occurs between neighbouring sites. The vacuum-to-vacuum amplitude is then

$$\langle 0|0\rangle = A \, \mathrm{Tr}_\sigma \exp\left(-\frac{a^4}{h}\sum_{\substack{\text{nearest}\\ \text{neighbours}}} \mathcal{L}(m,n)\right) \quad .$$

The resemblence with the partition function

$$Z = \mathrm{Tr}_\sigma \exp(-H/kT)$$

from statistical mechanics is obvious, and the methods employed in statistical mechanics can be carried over to here. We need therefore only concern ourselves with calculating functions of the form

$$Z = \sum_{\{\varphi\}} \exp\left\{-H(\varphi(n),\varphi(m))\right\} \quad \left(\begin{matrix}n,m \\ \text{nearest neighbours}\end{matrix}\right)$$

where the sum is taken over all possible configurations $\{\varphi(n)\}$.

10.2 Gauge Ising Models

The treatment of gauge Ising models was first given by Balian et al. [30] with further work concerning the lattice independence being done by Enting [31].

As an example consider a cubic lattice on which the field $\varphi(n)$ at each site may take values ±1. The Hamiltonian is given by

$$H = \sum_n \varphi_n \varphi_{n+\hat{\mu}}$$

We introduce the gauge transformation

$$\varphi_n \to A_n \varphi_n \quad , \quad A_n = \pm 1 \quad .$$

To make H gauge invariant we introduce an Ising gauge field $\sigma(n,m)$:

$$H = \sum_{\substack{\text{nearest} \\ \text{neighbours}}} \varphi(n)\sigma(n,m)\varphi(m)$$

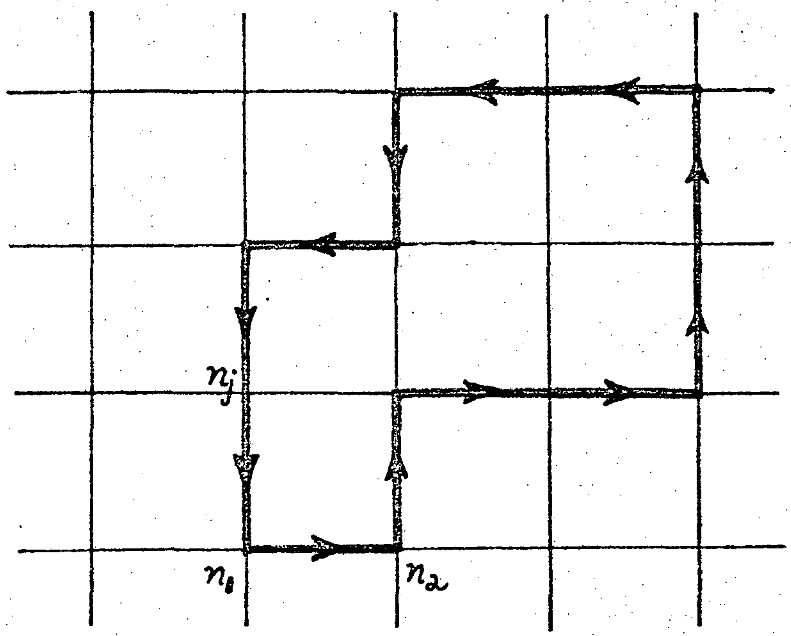
where $\sigma(n,m)$ takes values ±1 and transforms according to

$$\sigma(n,m) \to A_n \, \sigma(n,m) A_m \quad .$$

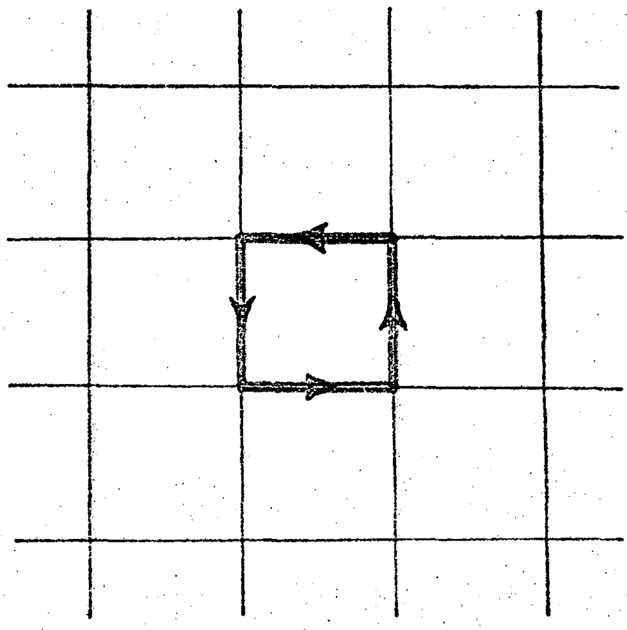
We can add to H a gauge invariant term for the free gauge field analogous to the term $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ introduced in the continuum case in §9. For example, the term

$$\sigma(n_1,n_2) \, \sigma(n_2,n_3) \, \dots \, \sigma(n_j,n_1)$$

corresponding to a closed loop such as



is gauge invariant. Any such term can be written in terms of terms corresponding to elementary plaquettes such as



For the 1+1 dimensional case, consider a lattice whose sites are labelled (n,m); $1 \leq n \leq N$; $1 \leq m \leq M$.

We can take H to be

$$H = -J \sum_{1 \le m < M} \sum_{1 \le n < N} \sigma(n,m;n+1,m)\sigma(n+1),m;n+1,m+1)$$

$$\sigma(n+1,m+1;n,m+1)\sigma(n,n+1;n,m) \quad .$$

Following Enting we define new variables t by

$$t(n,m;n,m+1) = \begin{cases} \sigma(n,m;n+1,m)\sigma(n+1,m;n+1,m+1) \\ \qquad \sigma(n+1,m+1;n,m+1)\sigma(n,n+1;n,m) & \text{if } n \ne N \\ \\ \sigma(1,m;\ 1,m+1) & \text{if } n \ne N \end{cases} ,$$

$$t(n,m;n+1,m) = \sigma(n,m;n+1,m) \quad .$$

Then,

$$H = -J \sum_{1 \le m < M} \sum_{1 \le n < N} t(n,m;\ n,\ m+1) \quad .$$

Since the mapping from the σ variables to the t variables is one-to-one we have

$$Z_{NM} = \sum_{\{\sigma\}} \exp(-H/kT) = \sum_{\{t\}} \exp(-H/kT)$$

$$= \sum_{\{t\}} \prod_{\substack{1 \le m < M \\ 1 \le n < N}} \exp\{(J/kT)t(n,m;n,m+1)\}$$

$$= 2^{|E|} \cosh^{|F|}(K)$$

where

$$|F| = M(N-1) + N(M-1)$$

is the number of edges in the lattice,

$$|F| = (N-1)(M-1)$$

is the number of internal faces and $K = J/kT$.

In the limit of an infinite lattice, the free energy per site is given by

$$f = -\lim_{N,M\to\infty} (NM)^{-1} \ln Z_{NM} = -\ln 4 - \ln \cosh K$$

which does not show any singular behaviour for positive K and so no phase transition occurs.

Enting [31] has extended this result to any two dimensional lattice, i.e. a gauge invariant Ising model of the type described above on an arbitrary planar graph will not have a phase transition.

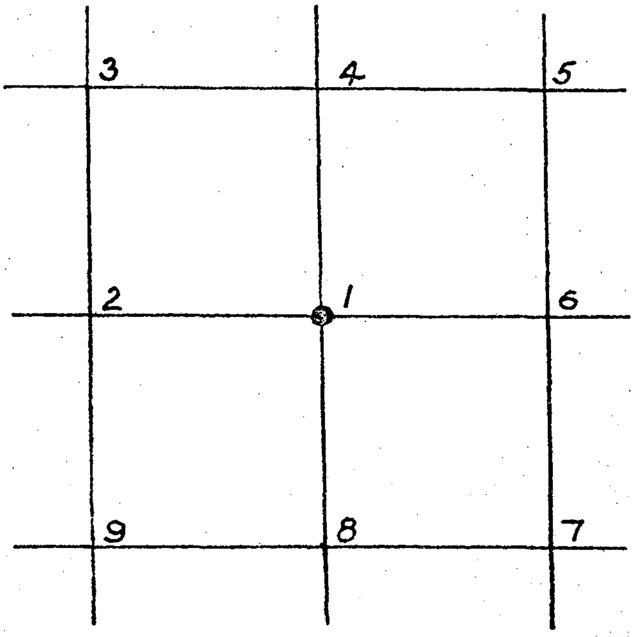
10.3 Another Soluble Model

Illustrated below is another soluble model on a 1+1 dimensional lattice which, like the previous model, has little to do with the real world. It does, however, illustrate some of the techniques currently in use. Attempts at solutions of physical models so far have generally been unsuccessful.

Once again a square lattice is considered and the field at each point is given by $\sigma_i = \pm 1$. The action is taken to be

$$S = \sum_{\text{lattice sites}} \{ \beta_1 \sigma_4 \sigma_6 \sigma_8 \sigma_2 + \beta_2 (\sigma_2 \sigma_6 + \sigma_4 \sigma_8 + 2\sigma_2 \sigma_3 \sigma_4 \sigma_1) \}$$

where the convention for numbering of neighbouring sites is as illustrated;



β_1 and β_2 are constants and the term in chain brackets refers to site 1.

We take the gauge transformation to be

$$\sigma_i \rightarrow e^{i\alpha_i} \sigma_i \quad , \quad \alpha_i = 0 \text{ or } \pi$$

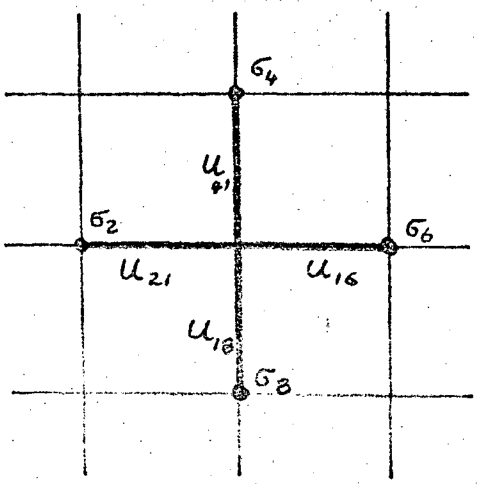
and introduce a gauge field $U_{ij} = \pm 1$ transforming according to

$$U_{ij} \rightarrow e^{-i\alpha_i} U_{ij} e^{i\alpha_i} \quad .$$

To make S gauge invariant, the following replacements are made:

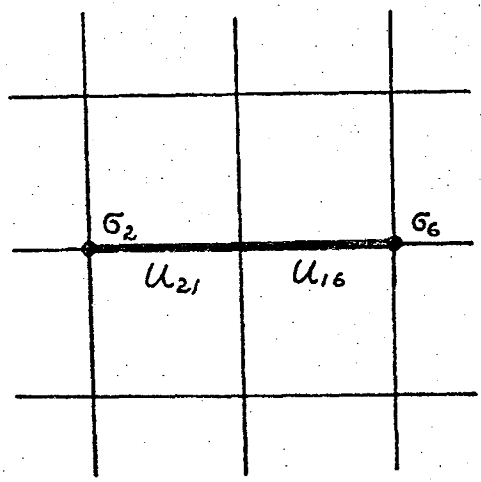
the term $\sigma_4\sigma_6\sigma_8\sigma_2$ is

replaced by $\sigma_4 U_{41} U_{16} \sigma_6 \sigma_8 U_{81} U_{12} \sigma_2$



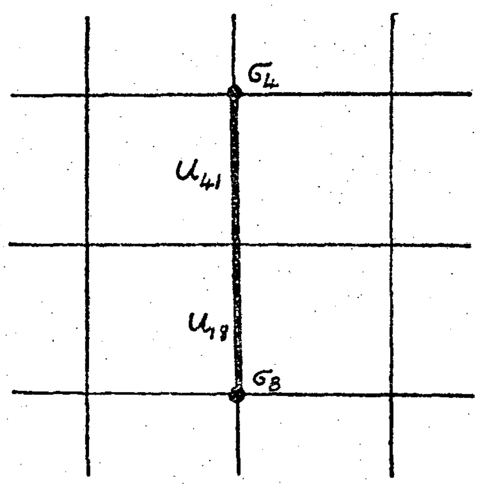
the term $\sigma_2\sigma_6$ is replaced by

$\sigma_2 U_{21} U_{16} \sigma_6$



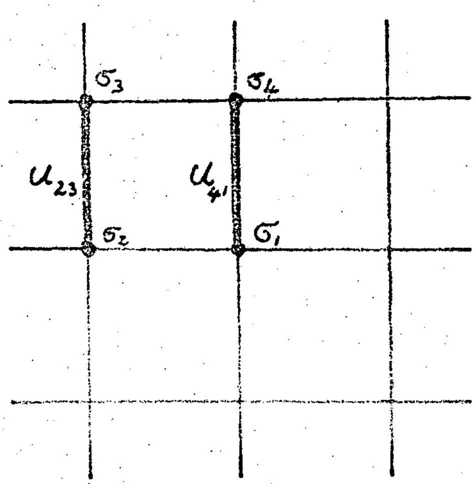
the term $\sigma_4\sigma_8$ is replaced by

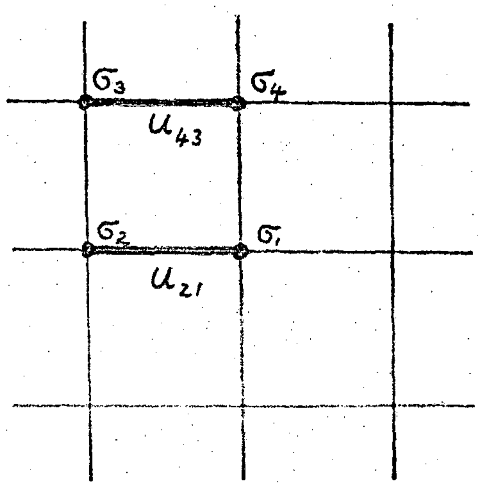
$\sigma_4 U_{41} U_{18} \sigma_8$



and the term $2\sigma_2\sigma_3\sigma_4\sigma_1$ is replaced by

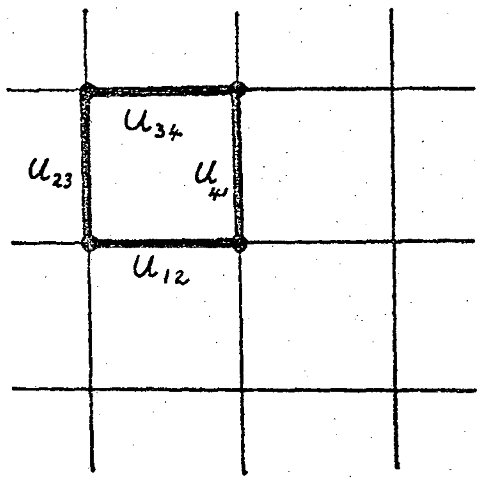
$$(\sigma_2 U_{23} \sigma_3)\ (\sigma_4 U_{41} \sigma_1) + (\sigma_2 U_{21} \sigma_1)\ (\sigma_4 U_{43} \sigma_3)$$





Also, by analogy with the term $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ introduced in the continuum case, we add the gauge invariant term

$$\beta_3 U_{12} U_{23} U_{34} U_{41}.$$



The full action is then given by

$$S = \sum_{\substack{\text{lattice}\\\text{sites}}} \beta_1 \sigma_4 U_{41} U_{16} \sigma_6 \sigma_8 U_{81} U_{12} \sigma_2$$

$$+ \quad \beta \left[\sigma_2 U_{21} U_{16} \sigma_6 + \sigma_4 U_{41} U_{18} \sigma_8\right.$$

$$+ \quad \left.(\sigma_2 U_{23} \sigma_3)(\sigma_4 U_{41} \sigma_1) + (\sigma_2 U_{21} \sigma_1)(\sigma_4 U_{43} \sigma_3)\right]$$

$$+ \quad \beta_3 U_{12} U_{23} U_{41}$$

This problem is exactly soluble for the case $\beta_3 = \beta_1$. Since $\sigma_1^2 = 1$, we can force the term σ_1^2 into appropriate places in the action to give

$$S = \sum_{\substack{\text{lattice}\\ \text{sites}}} \Big\{ \beta_1(\sigma_4U_{41}\sigma_1)(\sigma_1U_{16}\sigma_6)(\sigma_8U_{81}\sigma_1)(\sigma_1U_{12}\sigma_2)$$

$$+ \quad \beta_2\big[(\sigma_2U_{21}\sigma_1)(\sigma_1U_{16}\sigma_6)+(\sigma_4U_{41}\sigma_1)(\sigma_1U_{18}\sigma_8)$$

$$+ \quad (\sigma_2U_{23}\sigma_3)(\sigma_4U_{41}\sigma_1)+(\sigma_2U_{21}\sigma_1)(\sigma_4U_{23}\sigma_3)\big]$$

$$+ \quad \beta_3U_{12}U_{23}U_{34}U_{41} \Big\} .$$

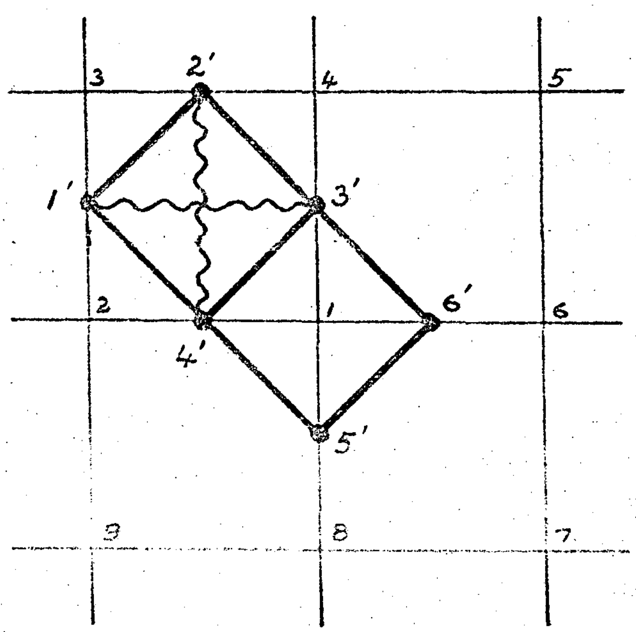
Introducing the new variables $V_{ij} = \sigma_iU_{ij}\sigma_j$, the action becomes

$$S = \sum_{\substack{\text{lattice}\\ \text{sites}}} \Big\{ \beta_1V_{41}V_{16}V_{81}V_{12} + \beta_2\big[V_{21}V_{16}+V_{41}V_{18} + V_{23}V_{41} + V_{21}V_{43}\big]$$

$$+ \quad \beta_3V_{12}V_{23}V_{34}V_{41} \Big\} .$$

The introduction of the $V_{ij's}$ is a valuable trick, but is only effective when the end terms cancel. For more general models the method will fail.

We now number the lattice edges according to the following scheme, so creating a new lattice with the primed figures as vertices:



Relabeling V_{ij} as σ_i', S becomes

$$S = \sum_{\substack{\text{old}\\ \text{lattice}}} \beta_1\sigma_3'\sigma_6'\sigma_5'\sigma_4' + \beta_2\left[\sigma_4'\sigma_6'+\sigma_3'\sigma_5' + \sigma_1'\sigma_3' + \sigma_4'\sigma_2'\right]$$

$$+ \quad \beta_3\sigma_4'\sigma_1'\sigma_2'\sigma_3'$$

$$= \sum_{\substack{\text{old}\\ \text{lattice}}} \beta_1\sigma_1'\sigma_2'\sigma_3'\sigma_4' + \beta_2(\sigma_1'\sigma_3' + \sigma_4'\sigma_2')$$

$$+ \quad \beta_1\sigma_3'\sigma_6'\sigma_5'\sigma_4' + \beta_2(\sigma_4'\sigma_6' + \sigma_3'\sigma_5')$$

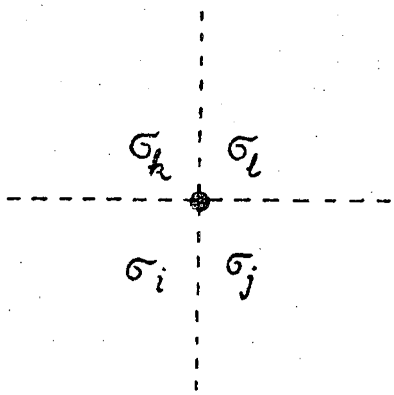
$$= \sum_{\substack{\text{new}\\ \text{lattice}}} \beta_1\sigma_1\sigma_2\sigma_3\sigma_4 + \beta_2(\sigma_1\sigma_3 + \sigma_4\sigma_2)$$

where β_1 has been put equal to β_3.

The last form for S is mathematically equivalent to the Hamiltonian of Baxter's eight vertex model [32] given by

$$E = - \sum \left[J\, \sigma_j\sigma_k + J'\sigma_i\sigma_\ell + J_4\, \sigma_i\sigma_j\sigma_k\sigma_\ell\right]$$

for the case J=J'. In this model the spins $\sigma_i = \pm 1$ are associated with the spaces around each vertex site.



This model can be solved exactly [32].

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