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Examples of algebras of small Gelfand-Kirillov dimension

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Examples of Algebras of Small Gelfand-Kirillov Dimension

A dissertation submitted in partial satisfaction of the
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by

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The dissertation of Alexander A. Young is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2012

DEDICATION

To my sisters, mother, father, and especially my grandparents.

TABLE OF CONTENTS

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Chapters 2, 3, and 4 include reinterpretations of, and borrow heavily from *Nil algebras with restricted growth* (T. H. Lenagan, A. Smoktunowicz, A. A. Young), *On the Kurosh problem for algebras over a general field* (J. P. Bell, A. A. Young), and *Jacobson radical algebras with quadratic growth* (A. Smoktunowicz, A. A. Young), respectively. All three of these papers have been submitted for publication with the dissertation author as a co-author, and may appear in Proc. Edin. Math. Soc., J. Algebra, and Glasgow Math. J., respectively.

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J. P. Bell, A. A. Young, *On the Kurosh problem for algebras over a general field*, to appear in J. Algebra, arXiv:1102.0362v1.

A. Smoktunowicz, A. A. Young *Jacobson radical algebras with quadratic growth*, under consideration in Glasgow Math. J.

ABSTRACT OF THE DISSERTATION

Examples of Algebras of Small Gelfand-Kirillov Dimension

by

Alexander A. Young

Doctor of Philosophy in Mathematics

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Professor Efim Zelmanov, Chair

We construct three examples of affine, associative algebras with relatively low growth. We construct an algebra over an arbitrary countable field that is affine, infinite dimensional, nil, $\mathbb{N}$-graded, and has Gelfand-Kirillov dimension at most 3. We construct an algebra over an arbitrary field that is affine, infinite dimensional, nil, $\mathbb{N}$-graded, and whose growth can be asymptotically bounded above by an arbitrary non-polynomial function. We construct an algebra over an arbitrary, algebraically closed field that is affine, infinite dimensional, $\mathbb{N}$-graded, Jacobson radical, and has quadratic growth.

Chapter 1

Introduction

1.1 Preliminaries

In this paper, we consider associative algebras over a field $\mathbb{K}$. Unless otherwise stated, algebras will not be assumed to be unital.

A *generating space* of an algebra A is a subspace V such that:

$$A = \sum_{i=1}^{\infty} V^i.$$

A set will be said to generate A if its $\mathbb{K}$-span is a generating space.

An algebra is *affine* if it can be generated by a finite dimensional space.

If every affine subalgebra of an algebra is finite dimensional, then we say that algebra is *locally finite*. Trivially, all locally finite affine algebras are finite dimensional.

The notation $\mathbb{K}\langle x_1, ..., x_r\rangle$ will be used to refer to the unital, affine free algebra over indeterminates $\{x_1, ..., x_r\}$. Any affine algebra can be represented as $\mathbb{K}\langle x_1, ..., x_r\rangle/I$ for some generating set $\{x_1, ..., x_r\}$ and some ideal $I \lhd \mathbb{K}\langle x_1, ..., x_r\rangle$.

For any monoid $\mathbb{G}$, an algebra A is $\mathbb{G}$-*graded* if it can be decomposed into subspaces:

$$A = \bigoplus_{i \in \mathbb{G}} A_i,$$

with $A_i A_j \subseteq A_{i \cdot j}$. The elements of A_i are called *homogeneous of degree* i under this grading. This paper will for the most part only concern itself with $\mathbb{N}$-gradings of a certain species: let $A_0 = \mathbb{K} \cap A$ (i.e. either $\mathbb{K}$ or (0)), A_1 be a finite dimensional generating subspace of A, and $A_k = A_1^k$. If $A_i \cap \sum_{j=0}^{i-1} A_j = (0)$ for each i, then $A = \bigoplus_{i=0}^{\infty} A_i$ is an $\mathbb{N}$-grading.

If an affine algebra has such a grading, and $A_0 = \mathbb{K}$, then it is called *connected*. Every connected algebra A has a subalgebra $\bar{A} = \sum_{i=1}^{\infty} A_i$ that may use the same grading, with $\bar{A}_i = A_i$ for all $i > 0$, and $\bar{A}_0 = (0)$. This paper will call such an algebra *almost connected*. Every almost connected algebra can be extended to $\bar{A} \oplus \mathbb{K}$, using multiplication $(a, x) \cdot (b, y) = (ab + xb + ya, xy)$, to make a fully connected algebra isomorphic to A, with $(\bar{A} \oplus \mathbb{K})_i = A_i \oplus (0)$ for each $i > 0$, and $(\bar{A} \oplus \mathbb{K})_0 = (0) \oplus \mathbb{K}$.

If A has a grading $\{A_i\}_{i \in \mathbb{G}}$ and $I \lhd A$ is an ideal, we say the ideal is graded if it is spanned by elements from the A_i subspaces. In other words, $I = \bigoplus_{i \in \mathbb{G}} A_i \cap I$.

In a connected or almost connected algebra, the easiest way of procuring a generating subspace is to use $A_0 + A_1$. If the algebra is almost connected, then $A_0 + A_1 = A_1$, and powers of this generating space form a direct sum.

Every affine free algebra is connected. If a connected algebra is generated by degree-1 elements $x_1, ..., x_n$, the homogeneous elements of degree k can be thought of as the degree k non-commutative homogenous polynomials using these indeterminates. If $A = \mathbb{K}\langle x_1, ..., x_n \rangle$ is free, and $I \lhd A$ is an ideal, then A/I is connected if and only if I is a graded ideal, yielding the grading $(A/I)_k = (A_k + I)/I \cong A_k/(I \cap A_k)$. If we don't count the trivial case when $I = A$, then $I \cap \mathbb{K} = (0)$, $I \lhd \bar{A}$, and $\bar{A}/I$ is $\mathbb{N}$-graded similarly.

An element $a \in A$ of an algebra over $\mathbb{K}$ is *algebraic* if there exists a polynomial $p(x) \in \mathbb{K}[x]$ such that $p(a) = 0$. If every element of A is algebraic, then we say that A is an algebraic algebra.

An element of an algebra is *nilpotent* if it has a zero exponent; i.e. $a \in A$ is nilpotent if there exists some exponent $a^n = 0$. Being nilpotent is a stronger condition that being algebraic. An algebra A comprised of nothing but nilpotent elements is *nil*, likewise a stronger condition than being an algebraic algebra. Nonzero nil algebras can clearly never contain their ground field.

An algebra A satisfies a *polynomial identity* (or, more commonly, is a

"PI algebra") if, for some nonzero non-communtantive polynomial $P(x_1, ..., x_n) \in \mathbb{K}\langle x_1, ..., x_n\rangle$, $P(a_1, ..., a_n) = 0$ for all $a_1, ..., a_n \in A$. All commutative algebras are PI; they satisfy the identity $xy - yx = 0$.

If there exists an n such that for any $a_1, ..., a_n \in A$, $a_1 \cdots a_n = 0$, then A is *nilpotent*, and the minimal such n is its *nilpotence degree*. In other words, A is nilpotent with nilpotence degree $\leq n$ if $A^n = (0)$. This implies being both nil and PI.

Conversely, J. Levitzki [1] and I. Kaplansky [2] proved that all nil PI algebras are locally nilpotent.

Suppose that A is affine and nilpotent, with nilpotence degree n. If V is a finite dimensional space that generates A, then $V^n = (0)$, and $\dim A \leq \sum_{i=1}^{n} \dim V^i < \infty$. Furthermore, since subalgebras of nilpotent algebras are themselves nilpotent, all nilpotent algebras are locally finite.

If A is affine, nil, and commutative, let $V = \mathbb{K}\{v_1, ..., v_k\}$ be a finite k-dimensional space that generates A. If n is such that each $v_i^n = 0$, then $V^{kn} = (0)$, and A is nilpotent.

1.2 The Jacobson radical

A nonzero right module M of an algebra A is *irreducible* if there exist no proper submodules, and $MA \neq (0)$. Irredicuble left A-modules are defined symmetrically. Note that this is a sightly different definition than some sources; some authors specify that all rings are unital, and $m1_A = m$ for all $m \in M$, in which case $MA \neq (0)$ is not needed. If $m \in M$, then mA is a submodule of M, and either $mA = (0)$ or $mA = M$. Further, the set $\{m \in M | mA = (0)\}$ of all "trivially acting" elements is a submodule as well, so it must either be M or (0). The constraint $MA \neq (0)$ eliminates the possibility for all of M being trivially acting, proving that only $0 \in M$ annihilates all of A and for any nonzero $m \in M$, $mA = M$.

(If we ignore the constraint that $MA \neq (0)$, then we leave open the possibility of irreducible right modules where all elements act trivially. In any such module, every $\mathbb{K}$-subspace is a submodule, and irreducible modules are simply one dimensional spaces. This is a very inelegant idea, and one that doesn't work with our following definition of the Jacobson radical, so we discount the possibility.)

The *Jacobson radical* $J(A)$ of an algebra A is the ideal of elements that annihilate all irreducible right modules of A, though it has many equivalent definitions:

- The ideal of elements that annihilate all irreducible left modules of A.
- The intersection of all maximal right ideals of A.[1]
- The intersection of all maximal left ideals of A.[1]
- The (unique) maximal right ideal of elements $a \in A$ such that $\exists b \in A : a + b + ab = 0$, i.e. are *right-quasiregular.*
- The (unique) maximal left ideal of left-quasiregular elements ($a+b+ba = 0$).

Since right units of A are never elements of maximal right ideals, none of them are located in $J(A)$. The same can be said about left units.

An algebra A is itself called Jacobson radical if $A = J(A)$, i.e. no irreducible modules exist. (Remember, if all of A annihilates a module, it doesn't count as irreducible.)

Examples:

- If A is a field or a division ring, then $J(A) = (0)$.
- All maximal ideals of $\mathbb{K}^n$, are of the form $\mathbb{K} \oplus \cdots \oplus \mathbb{K} \oplus (0) \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K}$, and therefore $J(\mathbb{K}^n) = (0)$.
- The Jacobson radical of the algebra of $n \times n$ upper triangular matrices of $\mathbb{K}$ is the ideal of strictly upper trianglar matrices (i.e. with zero diagonals).

[1]We define a maximal right (left) ideal I of an algebra A as a right (left) ideal such that the only right (left) ideals that contain it are A and I, and that $A^2 \not\subseteq I$. The latter stipulation is not included in some sources, but is analogous to our stipulation that if M is an irreducible right (left) module, $MA \neq (0)$.

- All nil algebras are Jacobson radical. If $x^n = 0$ and $y = -x + x^2 - \cdots \pm x^{n-1}$, then $x + y + xy = 0$, and thus all nilpotent elements are quasiregular.

If $I \lhd A$ is a proper ideal of A, every right module M of A/I can be naturally extended to a right module of A by setting $M \cdot I = (0)$. If M is irreducible as a right A/I-module, then for any nonzero $m \in M$, $mA = m(A/I) = M$, and M is an irreducible A-module. Thus, if $x \in J(A)$, then $x + I$ annihilates all irreducible right modules of A/I, and $(J(A) + I)/I \subseteq J(A/I)$.

In the particular case when $I = J(A)$, any irreducible right A-module M can conversely be defined as a right module of $A/J(A)$, as $M \cdot J(A) = (0)$. Annihilating all irreducible right A-modules is equivalent to annihilating all irreducible right $A/J(A)$-modules, and $J(A/J(A)) = (0)$.

1.3 The growth of algebras

Let A be an affine algebra over a field $\mathbb{K}$, and let $V \subseteq A$ be a subspace that generates it:

$$A = \sum_{i=1}^{\infty} V^i.$$

We can define a monotonically increasing *growth function* $f_{A,V}$ using this space:

$$f_{A,V}(n) = \dim \sum_{i=1}^{n} V^i.$$

While the value of this function depends on the choice of V, we can show its long-term ("asymptotic") behavior does not.

For any two functions $f, g : \mathbb{N} \to \mathbb{R}_{>0}$, we say that f is *asymptotically bounded* above by g if there exists $A, B > 0$ such that $Ag(Bx) \geq f(x)$ for all $x \in \mathbb{N}$. We can write this relation as $f \precsim g$. If both $f \precsim g$ and $g \precsim f$, then we say that f and g are *asymptotically equivalent*, or $f \sim g$.

Examples of asymptotic growth relations:

- For any monotonically increasing nonzero f, g, if f is a bounded function, then $f \precsim g$. All monotonically increasing nonzero bounded functions are equivalent.

- If f is a polynomial of degree n, $f \sim x^n$.
- $x^n \precsim x^m$ if and only if $n \leq m$. Polynomials are equivalent if and only if they have the same degree.
- $x^n \precnsim x^n \ln^m x$ for any $m > 0$.
- If $f(x) \precsim x^n$ for some n, we say that f has *polynomial growth.*
- For any $a, b > 1$, $a^x \sim b^x$. We say that any function in this class has *exponential growth.*
- For any n and any $a > 1$, $x^n \precnsim e^{\sqrt{x}} \precnsim a^x$. We thus say that $e^{\sqrt{x}}$ is an example of a function with *intermediate growth.*

Proposition 1.3.1. *If V and V' are both finite dimensional subspaces that generate an affine algebra A, then $f_{A,V} \sim f_{A,V'}$.*

Proof. Since V generates A, let k be such that $V' \subseteq \sum_{i=1}^{k} V^i$.

$$f_{A,V'}(n) = \dim \sum_{i=1}^{n} V'^i \leq \dim \sum_{i=1}^{kn} V^i = f_{A,V}(kn).$$

□

The fastest possible type of asymptotic growth occurs in free algebras (over more than one indeterminate). If $A = \mathbb{K}\langle x_1, ..., x_r\rangle$ and $V = \mathbb{K}\{1, x_1, ..., x_r\}$, then $f_{A,V}(n) = \frac{2^{r(n+1)}-1}{2^r-1}$, which is exponential.

It's worth mentioning for comparison the analoguous aspect in group theory. The growth of a finitely generated group G, with generating subset $S = S^{-1}$, is the function $f_{G,S}(n) = |S^n|$. Again, this function is asymptotically invariant under the choice of S, and can be categorized into exponential, polynomial, and intermediate growth. In 1980, M. Gromov proved [3] that a group has polynomial growth if and only if it has a finite index subgroup that's nilpotent, thereby expanding what was known about polynomial growth groups considerably. (See also [4][5]) Little is known about intermediate groups other than the fact that they exist; see also [6].

Another stratification of polynomial growth algebras is the *Gelfand-Kirillov dimension*. It is defined:

$$\operatorname{GKdim} A = \limsup_{n\to\infty} \log_n f_{A,V}(n) = \limsup_{n\to\infty} \log_n \dim \sum_{i=1}^{n} V^n.$$

If $f \sim g$ then $\limsup_{n\to\infty} \log_n f(n) = \limsup_{n\to\infty} \log_n g(n)$, so Gelfand Kirillov dimension is invariant over the choice of V, and has the same value for algebras with asymptotically equivalent growth. However, the converse is not true in general. For example, if $f_{A,V}(n) \sim n^3$ and $f_{B,W}(n) \sim n^3 \ln n$, then $f_{A,V} \nsim g_{B,W}$, but $\operatorname{GKdim} A = \operatorname{GKdim} B = 3$.

All exponential and intermediate growth algebras have infinite dimensional Gelfand-Kirillov dimension.

If A is not unital, and is generated by V, then the extension $A \oplus \mathbb{K}$ can be generated by $V \oplus \mathbb{K}$. Note that:

$$f_{A\oplus\mathbb{K},V\oplus\mathbb{K}}(n) = \dim \sum_{i=1}^{n} (V \oplus \mathbb{K})^i = \dim \left(\sum_{i=1}^{n} V^i \oplus \mathbb{K} \right) = f_{A,V}(n) + 1.$$

The algebras A and $A \oplus \mathbb{K}$ have asymptotically equivalent growth, and thus the same Gelfand-Kirillov dimension. For the rest of this section, we will assume that A is unital.

If A is connected, then its Gelfand-Kirillov dimension can be calculated by setting $V = A_0 + A_1$:

$$\operatorname{GKdim} A = \limsup_{n\to\infty} \log_n \sum_{i=0}^{n} \dim A_i.$$

If each A_n can be bounded in size polynomially, i.e. each $\dim A_n \leq an^b$ for some $a, b > 0$, then:

$$\operatorname{GKdim} A \leq \limsup_{n\to\infty} \log_n \sum_{i=0}^{n} ai^b \leq \limsup_{n\to\infty} \log_n \frac{a}{b+1}(n+1)^{b+1} = b + 1.$$

If A is not affine, the definition of Gelfand-Kirillov dimension can be extended:

$$\operatorname{GKdim} A = \sup_{B\subseteq A} \{\operatorname{GKdim} B | B \text{ is affine}\},$$

or, equivalently:

$$\operatorname{GKdim} A = \sup_{V \subseteq A, \dim V < \infty} \{\limsup_{n \to \infty} \log_n \dim \sum_{i=1} V^i\}.$$

Basic properties of Gelfand-Kirillov dimension include:

- If $B \subseteq A$ are algebras, then $\operatorname{GKdim} B \leq \operatorname{GKdim} A$.
- For any ideal $I \lhd A$, $\operatorname{GKdim} A/I \leq \operatorname{GKdim} A$.
- $\operatorname{GKdim} A \oplus B = \sup\{\operatorname{GKdim} A, \operatorname{GKdim} B\}$.
- Assuming A and B are unital, $\operatorname{GKdim} A \otimes B = \operatorname{GKdim} A + \operatorname{GKdim} B$.

Some examples:

- Every finite dimensional algebra trivially has Gelfand-Kirillov dimension 0.
- $\operatorname{GKdim} \mathbb{K}[x_1, ..., x_n] = n$.
- The Gelfand-Kirillov dimension of the nth Weyl algebra $\langle x_1, ..., x_n, y_1, ..., y_n | x_i x_j - x_j x_i, y_i y_j - y_j y_i, x_i y_j - y_j x_i - \delta_{i,j} \rangle_{\mathbb{K}}$ is $2n$.
- The Gelfand-Kirillov dimension of any free algebra (over more than one indeterminate) is infinite.
- $\operatorname{GKdim} \mathbb{K}\langle x, y\rangle/(Ay)^n A = n$.

There is, relatively speaking, quite a bit known about affine algebras with low (< 2) Gelfand-Kirillov dimension.

Proposition 1.3.2. *If A is an affine algebra with* $\operatorname{GKdim} A < 1$*, then A is finite dimensional (and thus* $\operatorname{GKdim} A = 0$*).*

Proof. Let V be a finite dimensional $\mathbb{K}$-space that generates A. If A is not finite dimensional, then V^{n+1} is always strictly larger than V^n; if it weren't, then $V^m = V^n$ for all $m > n$ and $\dim A = \dim V^n \leq (\dim V)^n < \infty$. It follows that $\dim V^n \geq n$, and $\log_n \dim V^n \geq 1$. □

Corollary 1.3.1. *If A is an algebra with* $\operatorname{GKdim} A < 1$*, then* $\operatorname{GKdim} A = 0$*, and A is locally finite.*

Proof. Every affine subalgebra $B \subseteq A$ must have $\operatorname{GKdim} B < 1$, and thus be finite dimensional. □

Bergman's Gap Theorem [p. 18][7] proves that no algebras exist with Gelfand-Kirillov dimension in the interval $(1, 2)$. Together these results show 1 to be an "isolated" value of Gelfand-Kirillov dimension, with the nearest other possible values being 2 and 0. On the other hand, there also exists a method [8] to construct an algebra of arbitrary Gelfand-Kirillov dimension ≥ 2.

In the case when A is an affine algebra of Gelfand-Kirillov dimension 1, L. W. Small, J. T. Stafford and R. B. Warfield [9] proved that:

- the Jacobson radical $J(A)$ is nilpotent, and
- the semisimple quotient algebra $A/J(A)$ has a nonzero center Z, $\operatorname{GKdim} Z = 1$, and $A/J(A)$ can be finitely generated as a Z-module.

In particular, A is PI.

1.4 The Kurosh Problem

In 1940, A. G. Kurosh [10] and J. Levitzki [11] (independently) posed what is now known as the Kurosh problem: are all affine algebraic algebras finite dimensional? The answer was provided in 1964 [12] when E. S. Golod and I. R. Shafarevich produced a counterexample.

This problem was an analog of the Burnside problem: if, in a finitely generated group, all elements have finite order, is the group finite? Again the answer is negative, using a group adapted from the Golod-Shafarevich algebra example.

Since then, there has been effort to determine the status of the Kurosh and Burnside problems under certain restrictions [13]. For example: since both the group and algebra counterexamples supplied by E. S. Golod and I. R. Shafarevich have exponential growth, does the Burnside conjecture apply to groups with polynomial growth? As M. Gromov proved [3] that all such groups must have a finite index subgroup that's nilpotent, the conjecture can easily follow.

Eventually, it was asked [9] whether an affine nil algebra that has finite Gelfand-Kirillov dimension could be non-nilpotent, i.e. have infinite dimension. In 2007 [14], T. H. Lenagan and A. Smoktunowicz disproved the conjecture by constructing an example of an affine, infinite dimensional algebra with finite (≤ 20) Gelfand-Kirillov dimension that is nil and almost connected as well.

The dissertation author co-wrote a paper with T. H. Lenagan and A. Smoktunowicz, streamlining this method and lowering the upper bound of Gelfand-Kirillov dimension to ≤ 3. The methods of this paper will be discussed in detail in chapter 2. It works over any ground field, provided the field is countable. In the case of uncountable fields, the method fails, and in fact it is conjectured that no such example exists, or at the very least, it would have to not be almost connected.

In the case of algebras over uncountable fields, a slightly different method must be used. The dissertation author and J. P. Bell put together a paper that constructs a nil, almost connected, infinite dimensional algebra over an arbitrary uncountable field whose growth is asymptotically bounded above by an arbitrary greater-than-polynomial (i.e. exponential or intermediate) function. This paper's method will be discussed in chapter 3.

Another question under consideration concerned affine Jacobson radical algebras: how low can the growth of such an algebra be, assuming it's infinite dimensional? In [9] it was proven that if the Gelfand-Kirillov dimension is 1, the Jacobson radical must be nilpotent, and therefore not equal to the entire algebra. On the other hand, A. Smoktunowicz and L. Bartholdi [15] successfully constructed an example of an affine Jacobson radical algebra with Gelfand-Kirillov dimension 2. Following this, the dissertation author and A. Smoktunowicz wrote a paper constructing an example of one with quadratic growth, establishing once and for all the lowest possible asymptotic growth category for these algebras. This construction will be discussed in chapter 4.

Chapter 2

Nil Algebras with Restricted Growth

The dissertation author has collaborated with T. H. Lenagan, A. Smoktunowicz and J. P. Bell to produce three papers, two of which [16, 17] are to be published and one [18] still currently under review.

The methods used in these papers with be explained in their respective chapters. While some of the notation and theorems won't be exactly the same, the broad approach will be effectively as previously written.

The first paper provides a refinement of a previous Kurosh conjecture counterexample: an affine, nil, almost connected, infinite dimensional algebra with Gelfand-Kirillov dimension ≤ 3.

By Proposition 1.3.2 and Bergman's Gap Theorem, if A is infinite dimensional with $\operatorname{GKdim} A < 2$, then $\operatorname{GKdim} A = 1$. If this is the case, [9] proves that A is PI. Since all algebraic affine PI algebras are finite dimensional, this eliminates the possibility of a Kurosh counterexample with Gelfand-Kirillov dimension < 2.

Let $\mathbb{K}$ be an arbitrary countable field, and let $A = \mathbb{K}\langle x, y\rangle$. A has a natural $\mathbb{N}$-grading (where $\mathbb{N}$ in this instance includes zero) from setting $A_0 = \mathbb{K}$, $A_1 = \mathbb{K}x + \mathbb{K}y$, and $A_n = A_1^n$ for all $n \geq 1$. Let $\bar{A} = \bigoplus_{n=1}^{\infty} A_n \subset A$ be the subalgebra of elements with no constant term. To make an algebra guaranteed to be nil, we will use the countability of $\bar{A}$ to make an enumeration $\{g_1, g_2, ...\} = \bar{A}$, then for each g_i, select an $n_i > 0$, and construct an ideal $I \lhd \bar{A}$ that contains each $g_i^{n_i}$.

The trick will be to make I large enough to make GKdim $\bar{A}/I$ finite and as small as possible, but not so large as to force $\bar{A}/I$ to be finite dimensional.

2.1 The subspaces $\{U_{2^n}\}$

Consider a sequence of proper subspaces $U_{2^n} \subsetneq A_{2^n}$ for each $n \geq 0$, such that $U_{2^n}A_{2^n} + A_{2^n}U_{2^n} \subseteq U_{2^{n+1}}$. The space $\sum_{n=0}^{\infty} U_{2^n}$ can be thought of as "ideal-like" in this manner, despite it clearly not being one. For each $n \geq 1$, let $U'_{2^n} = U_{2^{n-1}}A_{2^{n-1}} + A_{2^{n-1}}U_{2^{n-1}} \subseteq U_{2^n}$.

One useful proposition immediately follows:

Proposition 2.1.1. *For any $n < m$ and any $0 \leq i < 2^{m-n}$,*

$$A_{i2^n}U_{2^n}A_{2^m-(i+1)2^n} \subseteq U'_{2^m}.$$

Proof. This is just simple induction on the value of $m-n$. If $m = n+1$, then the proposition is trivial.

If the proposition is true for some n, m, then seek to prove it for $m+1$. For any $0 \leq i < 2^{m-n}$,

$$A_{i2^n}U_{2^n}A_{2^{m+1}-(i+1)2^n} = A_{i2^n}U_{2^n}A_{2^m-(i+1)2^n}A_{2^m} \subseteq U'_{2^m}A_{2^m} \subseteq U_{2^m}A_{2^m} \subseteq U'_{2^{m+1}},$$

and for any $2^{m-n} \leq i < 2^{m-n+1}$,

$$A_{i2^n}U_{2^n}A_{2^{m+1}-(i+1)2^n} = A_{2^m}A_{(i-2^{m-n})2^n}U_{2^n}A_{2^m-(i+1-2^{m-n})2^n} \subseteq$$

$$A_{2^m}U'_{2^m} \subseteq A_{2^m}U_{2^m} \subseteq U'_{2^{m+1}}.$$

□

Using these spaces, we can define a graded ideal $I = \bigoplus_{n=1}^{\infty} I_n$ with each $I_n \subset A_n$. For any $n \in \mathbb{N}$, if $m = \lfloor \log_2 n \rfloor$, i.e. $2^m \leq n < 2^{m+1}$, then we define:

$$I_n = \{r \in A_n | \forall 0 \leq k \leq 2^{m+2} - n, A_k r A_{2^{m+2}-k-n} \subseteq U'_{2^{m+2}}\}$$

To show that I is ideal, it's sufficient to prove that $I_nA_1 + A_1I_n \subseteq I_{n+1}$ for all $n \geq 1$. If $n < 2^{m+1} - 1$, then for any $r \in I_n$ and any $0 \leq k \leq 2^{m+2} - n - 1$,

$$A_k \cdot rA_1 \cdot A_{2^{m+2}-k-n-1} = A_k r A_{2^{m+2}-k-n} \subseteq U'_{2^{m+2}},$$

$$A_k \cdot A_1 r \cdot A_{2^{m+2}-k-n-1} = A_{k+1} r A_{2^{m+2}-k-n-1} \subseteq U'_{2^{m+2}}.$$

Suppose $n = 2^{m+1} - 1$. If $0 \leq k < 2^{m+2} - n$, then:

$$A_k \cdot rA_1 \cdot A_{2^{m+3}-k-n-1} = A_k r A_{2^{m+2}-k-n} A_{2^{m+2}} \subseteq U'_{2^{m+2}} A_{2^{m+2}} \subseteq U'_{2^{m+3}},$$

$$A_k \cdot A_1 r \cdot A_{2^{m+3}-k-n-1} = A_{k+1} r A_{2^{m+2}-k-n-1} A_{2^{m+2}} \subseteq U'_{2^{m+2}} A_{2^{m+2}} \subseteq U'_{2^{m+3}}.$$

If $2^{m+2} - n \leq k < 3 \cdot 2^{m+1} - n$, then:

$$A_k \cdot rA_1 \cdot A_{2^{m+3}-k-n-1} = A_{2^{m+1}} A_{k-2^{m+1}} r A_{3 \cdot 2^{m+1}-k-n} A_{2^{m+1}} \subseteq$$

$$A_{2^{m+1}} U'_{2^{m+2}} A_{2^{m+1}} = A_{2^{m+1}} U_{2^{m+1}} A_{2^{m+2}} + A_{2^{m+2}} U_{2^{m+1}} A_{2^{m+1}} \subseteq$$

$$U_{2^{m+2}} A_{2^{m+2}} + A_{2^{m+2}} U_{2^{m+2}} = U'_{2^{m+3}},$$

$$A_k \cdot A_1 r \cdot A_{2^{m+3}-k-n-1} = A_{2^{m+1}} A_{k-2^{m+1}+1} r A_{3 \cdot 2^{m+1}-k-n-1} A_{2^{m+1}} \subseteq$$

$$A_{2^{m+1}} U'_{2^{m+2}} A_{2^{m+1}} \subseteq U'_{2^{m+3}}.$$

Finally, if $3 \cdot 2^{m+1} - n \leq k \leq 2^{m+3} - n - 1$, then:

$$A_k \cdot rA_1 \cdot A_{2^{m+3}-k-n-1} = A_{2^{m+2}} A_{k-2^{m+2}} r A_{2^{m+3}-k-n} \subseteq A_{2^{m+2}} U'_{2^{m+2}} \subseteq U'_{2^{m+3}},$$

$$A_k \cdot A_1 r \cdot A_{2^{m+3}-k-n-1} = A_{2^{m+2}} A_{k-2^{m+2}+1} r A_{2^{m+3}-k-n-1} \subseteq A_{2^{m+2}} U'_{2^{m+2}} \subseteq U'_{2^{m+3}}.$$

Since I is graded, A/I is connected, and $\bar{A}/I$ is almost connected.

The advantage of using this method to construct I is control over the growth of $\bar{A}/I$. We will see more of how this works later on, but for now, we can note that if $I_{2^n} = A_{2^n}$, then $A_{2^n} \cdot A_{3 \cdot 2^n} = A_{2^{n+2}} \subseteq U'_{2^{n+2}} \subseteq U_{2^{n+2}}$, which is contradicted by $U_{2^{n+2}} \neq A_{2^{n+2}}$. This proves that each $I_{2^n} \neq A_{2^n}$, and $\bar{A}/I$ is infinite dimensional. With our definition of I, we don't have to worry about it being "too big" to work as a counterexample to the Kurosh conjecture.

2.2 The subspaces $\{F_i\}$

As mentioned before, we want an enumeration $\{g_1, g_2, ...\}$ of $\bar{A}$ and a sequence $m_1, m_2, ... \in \mathbb{N}$ such that each $g_i^{m_i} \in I$. In general, the elements g_i are (non-commutative) polynomials over x, y of with many terms, and yield complicated exponents. However, there is a nice property of these exponents than can be used to our advantage.

For any homogeneous subspace $F \subseteq A_N$, we will use $\mathcal{E}(F)$ to represent the graded right A-ideal:

$$\mathcal{E}(F_N) = \sum_{k=1}^{\infty} A_{kN} F A.$$

Proposition 2.2.1. *For any $n \geq 1$, and any $g \in \bar{A}$, if the ideal generated by g is a subspace of $\mathcal{E}(U_{2^n})$, then $g \in I$.*

Proof. Let $g = g_{(1)} + g_{(2)} + \cdots + g_{(d)}$ be the decomposition of g into homogeneous terms, i.e. with each $g_{(i)} \in A_i$. Since I is graded, it is equivalent to prove that each $g_{(i)} \in I$, and since $\mathcal{E}(U_{2^n})$ is graded, we can say that each $Ag_{(i)}A \subseteq \mathcal{E}(U_{2^n})$.

Let q be such that $2^q \leq i < 2^{q+1}$. For any $0 \leq \ell \leq 2^{q+2} - i - \ell$,

$$A_\ell g_{(i)} A_{2^{q+2}-i-\ell} \subseteq A_{2^{q+2}} \cap \mathcal{E}(U_{2^n}).$$

If $q+1 \leq n$, then this intersection is trivial, and $g_{(i)} = 0$. Otherwise, by Proposition 2.1.1,

$$A_{2^{q+2}} \cap \mathcal{E}(U_{2^n}) = \sum_{k=1}^{2^{q-n+2}-1} A_{k2^n} U_{2^n} A_{2^{q+2}-(k+1)2^n} \subseteq U'_{2^{q+2}},$$

and by definition, $g \in I$. □

Lemma 2.2.1. *Let $g \in \bar{A}$ and $d > 0$ be such that $g \in \sum_{i=1}^{d} A_i$. For any $I, J \in \mathbb{N}$ with $0 < I < J - 2d$ and any $m > J$, there exists subspaces $F_{a,b} \subseteq A_{I-J-a-b}$ for each $0 \leq a < d$, $0 \leq b < d$ such that $\dim F_{a,b} \leq I - J - a - b$, and:*

$$g^m \in \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} A_{I+a} F_{a,b} A_b A.$$

Proof. Let $g = g_{(1)} + g_{(2)} + \cdots + g_{(d)}$ be the decomposition of g into homogeneous terms. Let $g^m = g^m_{(m)} + \cdots + g^m_{(dm)}$ be defined similarly.

For any $p, q \in \mathbb{N}$, let S^p_q be the set of all functions $\{1, ..., p\} \to \{1, ..., q\}$. For any $\sigma \in S^p_q$, we define the "sum" of σ to be $\sum_{k=1}^{p} \sigma(k)$. We can write that:

$$g^m = \sum_{\sigma \in S^m_d} g_{(\sigma(1))} \cdots g_{(\sigma(m))} = \sum_{i=m}^{dm} \sum_{\sigma \in S^m_d | \text{sum } \sigma = i} g_{(\sigma(1))} \cdots g_{(\sigma(m))},$$

and that:

$$g^m_{(i)} = \sum_{\sigma \in S^m_d | \text{sum } \sigma = i} g_{(\sigma(1))} \cdots g_{(\sigma(m))}.$$

For any $\sigma \in S^m_d$, we say that for every $1 \le i \le m$, $\sum_{k=1}^{i} \sigma(k)$ is a *splitting point* of σ. The difference between the ith splitting point and the subsequent one is the value of $\sigma(i+1)$, so we know that no adjacent splitting points are more than d apart.

For any a, b, x, y with $0 \le a, b < d$ and $0 < x \le y < m$, we will define the subset $T_{\{a,b,x,y\}} \subseteq S^m_d$ as the set of all functions whose lowest splitting point $\ge I$ is $I + a$, whose highest splitting point $\le J$ is $J - b$, $\sum_{k=1}^{x} \sigma(k) = I + a$, and $\sum_{k=1}^{y} \sigma(k) = J - b$. We can partition S^m_d into disjoint subsets on distinct values of (a, b, x, y).

Working backwards, for any $a, b, a', b', x, y \in \mathbb{N}$ such that:

$$0 \le a, b < d, \quad 0 < a' \le a - d, \quad 0 < b' \le b - d,$$

$$0 < y - x \le J - I - a - b,$$

and any $\sigma_1 \in S^{x-1}_d$, $\sigma_2 \in S^{y-x}_d$, $\sigma_3 \in S^{m-y+1}_d$, with sum $\sigma_1 = I - a'$ and sum $\sigma_2 = J - I - a - b$, there exists a (certainly unique) $\sigma \in T_{\{a,b,x,y\}}$ such that:

$$(\sigma(1), ..., \sigma(m)) =$$

$$(\sigma_1(1), ..., \sigma_1(x-1), a + a', \sigma_2(1), ..., \sigma_2(y-x), b + b', \sigma_3(1), ..., \sigma_3(m-y-1)).$$

Therefore,

$$\sum_{\sigma \in T_{\{a,b,x,y\}}} g_{(\sigma(1))} \cdots g_{(\sigma(m))} =$$

$$\sum_{a'=1}^{a-d}\sum_{b'=1}^{b-d}\sum_{(\sigma_1\in S_d^{x-1}|\text{sum }\sigma_1=I-a')}\sum_{(\sigma_2\in S_d^{y-x}|\text{sum }\sigma_2=J-I-a-b)}\sum_{(\sigma_3\in S_d^{m-y+1})}$$

$$\big(g_{(\sigma_1(1))}\cdots g_{(\sigma_1(x-1))}\big)\cdot g_{(a+a')}\cdot\big(g_{(\sigma_2(1))}\cdots g_{(\sigma_2(y-x))}\big)\cdot g_{(b+b')}\cdot\big(g_{(\sigma_3(1))}\cdots g_{(\sigma_3(m-y+1))}\big)=$$

$$\left(\sum_{a'=1}^{a-d}\sum_{\sigma_1\in S_d^{x-1}|\text{sum }\sigma_1=I-a'} g_{(\sigma_1(1))}\cdots g_{(\sigma_1(x-1))}\cdot g_{(a+a')}\right)\cdot$$

$$\left(\sum_{\sigma_2\in S_d^{y-x}|\text{sum }\sigma_2=J-I-a-b} g_{(\sigma_2(1))}\cdots g_{(\sigma_2(y-x))}\right)\cdot\left(\sum_{b'=1}^{b-d} g_{(b+b')}\right)\cdot$$

$$\left(\sum_{\sigma_3\in S_d^{m-y+1}} g_{(\sigma_3(1))}\cdots g_{(\sigma_3(m-y+1))}\right)\in$$

$$A_{I+a}\cdot\left(\sum_{\sigma_2\in S_d^{y-x}|\text{sum }\sigma_2=J-I-a-b} g_{(\sigma_2(1))}\cdots g_{(\sigma_2(y-x))}\right)\cdot A_bA.$$

Thus, if we set:

$$F_{a,b}=\sum_{c=1}^{J-I-a-b}\mathbb{K}\left(\sum_{\sigma\in S_d^c|\text{sum }\sigma=J-I-a-b} g_{(\sigma(1))}\cdots g_{(\sigma(c))}\right),$$

then $\dim F_{a,b}\leq I-J-a-b$, and:

$$g^m=\sum_{\sigma\in S_d^m} g_{\sigma(1)}\cdots g_{\sigma(m)}=\sum_{a=0}^{d-1}\sum_{b=0}^{d-1}\sum_{x=1}^{I+a}\sum_{y=x}^{J-I-b-a}\sum_{\sigma\in T_{\{a,b,x,y\}}} g_{\sigma(1)}\cdots g_{\sigma(m)}\in$$

$$\sum_{a=0}^{d-1}\sum_{b=0}^{d-1} A_{I+a}F_{a,b}A_bA.$$

□

Theorem 2.2.2. *Let $g\in\bar{A}$ and $d>0$ be such that $g\in\sum_{i=1}^{d}A_i$. For any $n>2d$ and any $m>2n$, there exists a subspace $F\subseteq A_n$ with $\dim F<d^2 2^{2d}n$ such that $g\in\mathcal{E}(F)$.*

Proof. It is sufficient to show that $A_i g^m \in \mathcal{E}(F)$ for each $i \geq 0$. If this can be proven for all $0 \leq i < n$, then it follows for all $i \geq n$ as well; if $i = qn + i'$ with $q \in \mathbb{N}$ and $0 \leq i < n$, then:

$$A_i g^m = A_{qn} A_{i'} g^m \subseteq A_{qn} \cdot \sum_{k=1}^{\lfloor (i'+j)/n \rfloor - 1} A_{kn} F A = \sum_{k=q+1}^{\lfloor (i+j)/n \rfloor - 1} A_{kn} F A.$$

Assume $0 \leq i < n$. Suppose we set $I = n - i$ and $J = I + n$. Using Lemma 2.2.1,

$$A_i g^m \in A_n \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} A_a F_{a,b} A_b A,$$

with each $F_{a,b} \subseteq A_{n-a-b}$ and $\dim F_{a,b} \leq n - a - b$. If we set:

$$F = \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} A_a F_{a,b} A_b \subseteq A_n,$$

then $A_i g^m \in A_n F A \subseteq \mathcal{E}(F)$ and:

$$\dim F < \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} n 2^{a+b} < d^2 2^{2d} n.$$

□

Let $g_1, g_2, g_3, ...$ be an enumeration of $\bar{A}$. Let $\{d_i\}$ each be minimal such that $g_i \in \sum_{k=1}^{d_i} A_k$. Start with $z_1 = \sup\{8, 2d_1 + 2\lceil \log_2 d_1 \rceil + 1\}$. Then, recusively, for each $i > 1$, define z_i as $\sup\{2^{z_{i-1}} + z_{i-1} + 7, 2d_i + 2\lceil \log_2 d_i \rceil + 1\}$. In effect, $\{z_i\}$ will be "sparse" enough to get the growth we need. As we shall see, making it more sparse would keep lowering the growth, though not enough to prove GKdim $\bar{A}/I < 3$.

Apply Theorem 2.2.2 to each g_i, setting $n = 2^{2^{z_i} - z_i}$ and setting $m = 2n + 1$ to find a subspace $F_i \subseteq A_{2^{2^{z_i} - z_i}}$ such that $\dim F_i < d_i^2 2^{2^{z_i} - z_i + 2d_i}$. By Proposition 2.2.1, if $F_i \subseteq U_{2^{2^{z_i} - z_i}}$, and thus $\mathcal{E}(F_i) \subseteq \mathcal{E}(U_{2^{2^{z_i} - z_i}})$, then a power of g_i lies within I.

Let $n_i = 2^{z_i} - z_i$ for each $i > 0$, and let $n_0 = 0$. We need to have $U_{2^{n_i - 1}}$ small enough so that:

$$U'_{2^{n_i}} + F_i \subseteq U_{2^{n_i}} \neq A_{2^{n_i}}.$$

On the other hand, the larger each U_{2^n} is, the larger I is, and the smaller the Gelfand-Kirillov dimension of $\bar{A}/I$ is. So we want each U_{2^n} to be as large as possible, but not so large that it "traps" $U_{2^{n_i}} = A_{2^{n_i}}$.

In practice, the process of defining a particular enumeration of $\bar{A}$ is chaotic, especially when nothing is known about $\mathbb{K}$ besides its countability. We don't much about F_i at all besides $F_i \subseteq A_{2^{n_1}}$ and $\dim F_i < d_i^2 2^{n_i+2d_i} \leq 2^{n_i+z_i-1}$. Our method will work using that knowledge alone. If more were known about each F_i, then perhaps the result could be further refined.

Theorem 2.2.3. *There exists sequences of subspaces $U_{2^n}, V_{2^n} \subseteq A_{2^n}$ with the properties that, for each $n \geq 0$:*

- $U_{2^n} \oplus V_{2^n} = A_{2^n}$,
- $U'_{2^{n+1}} = U_{2^n}A_{2^n} + A_{2^n}U_{2^n} \subseteq U_{2^{n+1}}$
- $V_{2^{n+1}} \subseteq V_{2^n}^2$,
- V_{2^n} *can be generated by monomials (i.e. elements of the semigroup $\langle x, y\rangle$),*
- *if $n = n_i$ for some i, then $F_i \subseteq U_{2^{n_i}}$,*
- *if $n_i - z_i \leq n < n_i$ for some i, then $\dim V_{2^n} = 2^{2^{n-n_i+z_i}}$. Otherwise, $\dim V_{2^n} = 2$. In the latter case, there exists a monomial $m \in V_{2^n}$ such that $mA_{2^n} \subseteq U_{2^{n+1}}$.*

Proof. We are going to build U_{2^n} and V_{2^n} inductively on the value of n.

Start with $U_1 = (0)$ and $V_1 = A_1$. Then, suppose U_{2^m} and V_{2^m} are defined for all $m < n$ and seek to build U_{2^n} and V_{2^n}. Define $U'_{2^n} = U_{2^{n-1}}A_{2^{n-1}} + A_{2^{n-1}}U_{2^{n-1}}$.

Consider three cases:

Case 1: There does not exist an i such that $n_i - z_i < n \leq n_i$.

We need to have $\dim V_{2^n} = 2$. We can say $V_{2^{n-1}} = \mathbb{K}v_1 + \mathbb{K}v_2$, where v_1 and v_2 are monomials. Set $V_{2^n} = v_1 V_{2^{n-1}}$ and set $U_{2^n} = U'_{2^n} + v_2 A_{2^{n-1}}$.

Case 2: There exists some i such that $n_i - z_i < n < n_i$.

In this case, we simply set $U_{2^n} = U'_{2^n}$ and $V_{2^n} = V_{2^{n-1}}^2$. Note that $\dim V_{2^{n-1}} = 2^{2^{n-n_i+z_i-1}}$, even if $n - 1 = n_i - z_i$, so $\dim V_{2^n} = 2^{2^{n-n_i+z_i}}$.

Case 3: $n = n_i$ for some i.

$$\dim U'_{2^n} + F \le \dim A_{2^n} - \dim V^2_{2_{n-1}} + \dim F < \dim A_{2^n} - 2^{2^{z_i}} + 2^{n_i + z_i - 1} < \dim A_{2^n} - 2.$$

Therefore, there exists a two dimensional subspace $V_{2^{n_i}} \subset A_{2^{n_i}}$, generated by monomials, such that $V_{2^{n_i}} \cap (U'_{2^{n_i}} + F_i) = (0)$. We can set $U_{2^{n_i}}$ to be a space containing $U'_{2^{n_i}} + F_i$ such that $U_{2^{n_i}} \oplus V_{2^{n_i}} = A_{2^{n_i}}$. □

Conceptually, each F_i is an obstacle to include. The farther apart we keep the values of $\{z_i\}$, the smaller the difficulty of these obstacles. These obstacles necessitate the rapid surge in the sizes of V_{2^n} in case 2 of Theorem 2.2.3. Our estimate of the growth of $\bar{A}/I$ will depend on the sizes of the spaces V_{2^n}, and this hundle will the "limiting factor" of the strength of this estimate.

2.3 The size of A/I

For any $n \ge 1$, let m be such that $2^m \le n < 2^{m+1}$. Define $R_n \subseteq A_n$ to be the space of all $r \in A_n$ such that $rA_{2^{m+1}-n} \subseteq U_{2^{m+1}}$, and $S_n \subseteq A_n$ to be the space of all $r \in A_n$ such that $A_{2^{m+1}-n} r \subseteq U_{2^{m+1}}$. Additionally, set $S_0 = R_0 = (0)$.

For any m such that $2^m > n$, Proposition 2.1.1 can be used to show that $R_n A_{2^m - n} + A_{2^m - n} S_n \subseteq U_{2^m}$.

Proposition 2.3.1. *Suppose that n has a binary decomposition $2^{p_0} + \cdots + 2^{p_r}$, with $0 \le p_0 < ... < p_r$.*

$$\sum_{i=0}^{r} A_{2^{p_r} + \cdots + 2^{p_{i+1}}} U_{2^{p_i}} A_{2^{p_{i-1}} + \cdots + 2^{p_0}} \subseteq R_n,$$

$$\sum_{i=0} A_{2^{p_0} + \cdots + 2^{p_{i-1}}} U_{2^{p_i}} A_{2^{p_{i+1}} + \cdots + 2^{p_r}} \subseteq S_n.$$

Proof. First examine the first claim. It's equivalent to show that, for any $0 \le i \le r$,

$$A_{2^{p_r} + \cdots + 2^{p_{i+1}}} U_{2^{p_i}} A_{2^{p_r+1} - (2^{p_r} + \cdots + 2^{p_i})} \subseteq U_{2^{p_r+1}}$$

Since 2^{p_i} divides each of the subscripts, the statement follows from 2.1.1. A symmetrical argument can prove the second claim. □

Since $U_{2^m} \oplus V_{2^m} = A_{2^m}$ for each m,

$$\left(\sum_{i=0}^{r} A_{2^{p_r}+\cdots+2^{p_{i+1}}} U_{2^{p_i}} A_{2^{p_{i-1}}+\cdots+2^{p_0}}\right) \oplus (V_{2^{p_r}} \cdots V_{2^{p_0}}) = A_n,$$

$$\left(\sum_{i=0} A_{2^{p_0}+\cdots+2^{p_{i-1}}} U_{2^{p_i}} A_{2^{p_{i+1}}+\cdots+2^{p_r}}\right) \oplus (V_{2^{p_0}} \cdots V_{2^{p_r}}) = A_n.$$

Since each V_{2^i} is generated by monomials, we can choose subspaces $Q_n \subseteq V_{2^{p_r}} \cdots V_{2^{p_0}}$ and $W_n \subseteq V_{2^{p_0}} \cdots V_{2^{p_r}}$, both generated by monomials, such that $R_n \oplus Q_n = S_n \oplus W_n = A_n$. These new spaces will be instrumental in establishing an upper bound of $\dim A_n/I_n$.

Theorem 2.3.1. *For any $n \geq 0$,*

$$\bigcap_{i=0}^{n} S_i A_{n-i} + A_i R_{n-i} \subseteq I_n$$

Proof. Suppose that $r \in \bigcap_{i=0}^{n} S_i A_{n-i} + A_i R_{n-i}$. If $2^m \leq n < 2^{m+1}$ and $0 \leq k \leq 2^{m+1} - n$, then:

$$A_k r A_{2^{m+2}-n-k} \subseteq A_k S_{2^m-k} A_{3\cdot 2^m} + A_{2^m} R_{n-2^m+k} A_{2^{m+2}-n-k} \subseteq$$

$$U_{2^m} A_{3\cdot 2^m} + A_{2^m} U_{2^m} A_{2^{m+1}} \subseteq U_{2^{m+1}} A_{2^{m+1}} \subseteq U'_{2^{m+2}},$$

if $2^{m+1} - n < k \leq 2^{m+1}$, then:

$$A_k r A_{2^{m+2}-n-k} \subseteq A_k S_{2^{m+1}-k} A_{2^{m+1}} + A_{2^{m+1}} R_{n-2^{m+1}+k} A_{2^{m+2}-n-k} \subseteq$$

$$U_{2^{m+1}} A_{2^{m+1}} + A_{2^{m+1}} U_{2^{m+1}} = U'_{2^{m+2}},$$

and if $2^{m+1} < k \leq 2^{m+2} - n$, then:

$$A_k r A_{2^{m+2}-n-k} \subseteq A_k S_{3\cdot 2^m-k} A_m + A_{3\cdot 2^m} R_{k+n-3\cdot 2^m} A_{2^{m+2}-n-k} \subseteq$$

$$A_{2^{m+1}} U_m A_m + A_{3\cdot 2^m} U_m \subseteq A_{2^{m+1}} U_{2^{m+1}} \subseteq U'_{2^{m+2}},$$

proving that $r \in I_n$. ∎

This allows us to put together an upper bound on size of each A_n/I_n:

Corollary 2.3.2. *For any $n \geq 1$,*

$$\dim A_n/I_n \leq \sum_{i=0}^{n} \dim W_i \dim Q_{n-i}.$$

Proof. Since $(S_i A_{n-i} + A_i R_{n-i}) \oplus W_i Q_{n-i} = A_n$,

$$\dim A_n/I_n \leq \dim A_n/\left(\bigcap_{i=0}^{n} S_i A_{n-i} + A_i R_{n-i}\right) \leq \sum_{i=0}^{n} \dim A_n/(S_i A_{n-i} + A_i R_{n-i}) =$$

$$\sum_{i=0}^{n} \dim W_i Q_{n-i} = \sum_{i=0}^{n} \dim W_i \dim Q_{n-i}.$$

□

A few lemmas help us narrow down the sizes of W_n and Q_n.

Lemma 2.3.3. *For any $n \geq 1$, let m be such that $2^m \leq n < 2^{m+1}$.*

$$\dim Q_n \leq \dim V_{2^{m+1}} \dim W_{2^{m+1}-n},$$

$$\dim W_n \leq \dim V_{2^{m+1}} \dim Q_{2^{m+1}-n}.$$

Proof. Examine the first claim first. Let $D = \dim W_{2^{m+1}-n}$, and let $\{w_1, ..., w_D\}$ be a basis of $W_{2^{m+1}-n}$.

We can define a linear transformation $\phi : Q_n \to (A_{2^{m+1}}/U_{2^{m+1}})^D$ by:

$$\phi : x \mapsto (xw_1 + U_{2^{m+1}}, ..., xw_D + U_{2^{m+1}}).$$

If $x \in \ker \phi$, then $xW_{2^{m+1}-n} \subseteq U_{2^{m+1}}$. Recall that, by definition, $xS_{2^{m+1}-n} \subseteq U_{2^{m+1}}$, and since $A_{2^{m+1}-n} = S_{2^{m+1}-n} \oplus W_{2^{m+1}-n}$, $xA_{2^{m+1}-n} \subseteq A_{2^{m+1}}$, and $x \subseteq R_{2^{m+1}-n}$. Since $R_{2^{m+1}-n} \cap Q_{2^{m+1}-n} = (0)$, $\ker \phi = (0)$. The injectivity of ϕ establishes that $\dim Q_n \leq \dim(A_{2^{m+1}}/U_{2^{m+1}})^D = \dim V_{2^{m+1}} \dim W_{2^{m+1}-n}$.

To prove the second claim, use a symmetrical argument: use any basis $(q_1, q_2, ...)$ of $Q_{2^{m+1}-n}$ and define $\phi : W_n \to (A_{2^{m+1}}/U_{2^{m+1}})^{\dim Q_{2^{m+1}-n}}$ through left multiplication:

$$\phi : x \to (q_1 x + U_{2^{m+1}}, q_2 x + U_{2^{m+1}}, ...).$$

We can prove this ϕ to be injective the same way. □

Lemma 2.3.4. *Let $n, i \in \mathbb{N}$ be such that $n < 2^{n_i - z_i}$ and n is divisible by $2^{n_{i-1}}$.*

$$\dim Q_n = 1, \quad \dim W_n \leq 2.$$

Proof. Let $n = 2^{p_0} + \cdots + 2^{p_r}$ be a binary decomposition of n, with $n_{i-1} \leq p_0 < ... < p_r < n_i - z_i$. We know that:

$$Q_n \subseteq V_{2^{p_r}} \cdots V_{2^{p_0}},$$

and each $V_{2^{p_i}}$ generated by two monomials. As Q_n is also generated by monomials, it's sufficient to prove that all monomials of $V_{2^{p_r}} \cdots V_{2^{p_0}}$ except one lie within R_n.

We know that for each 2^{p_i}, there is a monomial $m_i \in V_{2^{p_i}}$ such that $m_i A_{2^{p_i}} \subseteq U_{2^{p_i}}$. Let m_i' be the other monomial that generates $V_{2^{p_i}}$. Using Proposition 2.1.1,

$$V_{2^{p_r}} \cdots V_{2^{p_{i+1}}} \cdot m_i \cdot V_{2^{p_{i-1}}} \cdots V_{2^{p_0}} \cdot A_{2^{p_r+1}-n} \subseteq$$

$$A_{2^{p_r}+\ldots 2^{p_{i+1}}} U_{2^{p_i}} A_{2^{p_r+1}-(2^{p_r}+\cdots+2^{p_i})} \subseteq U_{2^{p_r+1}}.$$

Therefore $V_{2^{p_r}} \cdots V_{2^{p_{i+1}}} \cdot m_i \cdot V_{2^{p_{i-1}}} \cdots V_{2^{p_0}} \subseteq R_n$, and the only monomial of that space that doesn't lie within R_n is $m_r' \cdots m_0'$.

To prove that $\dim W_n \leq 2$, apply Lemma 2.3.3:

$$\dim W_n \leq \dim V_{2^{p_r+1}} \dim Q_{2^{p_r+1}-n} = 2.$$

□

Lemma 2.3.5. *For any $n_1, n_2 \in \mathbb{N}$, if there exists an $m \in \mathbb{N}$ such that $n_1 < 2^m$ and 2^m divides n_2, then:*

$$\dim Q_{n_1+n_2} \leq \dim Q_{n_1} \dim Q_{n_2},$$

$$\dim W_{n_1+n_2} \leq \dim W_{n_1} \dim W_{n_2}.$$

Proof. Let $n_1 = 2^{p_{k+1}} + \cdots + 2^{p_r}$ and $n_2 = 2^{p_0} + \cdots + 2^{p_k}$ be binary decompositions of n_1 and n_2, with $0 \leq p_0 < ... < p_k < m \leq p_{k+1} < ... < p_r$. Without loss of generality, assume $m = p_k + 1$. Recalling the definition of R_n and Proposition 2.1.1,

$$R_{n_1} A_{n_2} \cdot A_{2^{p_r+1}-n_1-n_2} = R_{n_1} A_{2^{p_r+1}-n_1} \subseteq U_{2^{p_r+1}},$$

$$A_{n_1}R_{n_2}A_{2^{p_r+1}-n_1-n_2} \subseteq A_{n_1}U_{2^m}A_{2^{p_r+1}-2^m-n_1} \subseteq U_{2^{p_r+1}},$$

and therefore,

$$R_{n_1}A_{n_2} + A_{n_1}R_{n_2} \subseteq R_{n_1+n_2}.$$

Since $Q_{n+1}Q_{n+2} \oplus R_{n_1}A_{n_2} + A_{n_1}R_{n_2} = A_{n_1+n_2}$,

$$\dim Q_{n_1+n_2} = \dim A_{n_1+n_2} - \dim R_{n_1+n_2} \leq$$

$$\dim A_{n_1+n_2} - \dim(R_{n_1}A_{n_2} + A_{n_1}R_{n_2}) = \dim Q_{n_1} \dim Q_{n_2}.$$

Once again, to prove $\dim W_{n_1+n_2} \leq \dim W_{n_1} \dim W_{n_2}$, use a symmetrical argument. □

Theorem 2.3.6. *For all $k, i \geq 1$, if $k < 2^{n_i - z_i}$, then:*

$$\dim Q_k, \dim W_k \leq n_{i-1} 2^{\frac{1}{2}n_{i-1}+2}$$

and if $k < 2^{n_i - 1}$, then:

$$\dim Q_k, \dim W_k \leq n_i \sqrt{k}$$

Proof. We will prove this inductively on the value of i.

For the base case, seek to prove that for all $k < 2^{n_1 - z_1}$, $\dim Q_k, \dim W_k \leq n_0 2^{\frac{1}{2}n_0+2} = 2^{5/2}$. This follows immediately from Lemma 2.3.4.

We attack the inductive step with three cases:

Case 1: Suppose $k < 2^{n_i-1}$, and assume that:

- for all $j < 2^{n_i - z_i}$, $\dim Q_j, \dim W_j \leq n_{i-1} 2^{\frac{1}{2}n_{i-1}+2}$,
- for all $j < 2^{n_{i-1}-1}$, $\dim Q_j, \dim W_j \leq n_{i-1}\sqrt{j}$.

We want this step to prove that $\dim Q_k, \dim W_k \leq n_i \sqrt{k}$.

If $k < 2^{n_{i-1}-1}$, then the claim follows from $n_{i-1} < n_i$.

If $2^{n_{i-1}-1} \leq k < 2^{n_i - z_i}$, then:

$$\dim Q_k, \dim W_k \leq n_{i-1} 2^{\frac{1}{2}n_{i-1}+2} < 2^{2^{z_{i-1}} + \frac{1}{2}z_{i-1}+2} < 2^{z_i - 1} < n_i < n_i\sqrt{k}.$$

Assume that $2^{n_i - z_i} \leq k < 2^{n_i - 1}$. Let $k = j + 2^{p_0} + \cdots + 2^{p_r}$, with $n_i - z_i \leq p_0 < ... < p_r < n_i - 1$ and $j < 2^{n_i - z_i}$. Using Lemma 2.3.5,

$$\dim Q_k \leq \dim Q_{2^{p_0}+\cdots+2^{p_r}} \dim Q_j \leq \dim V_{2^{p_r}} \cdots \dim V_{2^{p_0}} \dim Q_j \leq$$

$$2^{2^{p_r - n_i + z_i} + \dots 2^{p_0 - n_i + z_i} + \frac{1}{2} n_{i-1} + 2} n_{i-1} < 2^{2^{z_i - n_i} k + \frac{1}{2} n_{i-1} + z_{i-1} + 2},$$

$$\dim W_k \leq \dim V_{2^{p_0}} \cdots \dim V_{2^{p_r}} \dim W_j \leq 2^{2^{z_i - n_i} k + \frac{1}{2} n_{i-1} + z_{i-1} + 2}.$$

If we set:

$$f(k) = \log_2 \frac{2^{2^{z_i - n_i} k + \frac{1}{2} n_{i-1} + z_{i-1} + 2}}{n_i \sqrt{k}} = 2^{z_i - n_i} k - \frac{1}{2} \log_2 k + \frac{1}{2} n_{i-1} + z_{i-1} - \log_2 n_i + 2,$$

then it suffices to show that f is never positive for any $2^{n_i - z_i} \leq k < 2^{n_i - 1}$. Calculating the derivative,

$$f'(k) = 2^{z_i - n_i} - \frac{1}{k \ln 4} \geq 2^{z_i - n_i} \left(1 - \frac{1}{\ln 4} \right) > 0,$$

and thus is it sufficient to prove that $f(2^{n_i - 1}) \leq 0$. Since $z_i \geq 2^{z_{i-1}} + z_{i-1} + 7$,

$$f(2^{n_i - 1}) = 2^{z_i - 1} + \frac{1}{2}(-n_i + n_{i-1} + 5) + z_{i-1} - \log_2 n_i < \frac{1}{2}(z_i + 2^{z_{i-1}} + z_{i-1} + 7) - z_i \leq 0.$$

Case 2: Suppose $k < 2^{n_i}$, and assume that for all $j < 2^{n_i - 1}$, $\dim Q_j$, $\dim W_j \leq n_i \sqrt{k} < n_i 2^{\frac{1}{2}(n_i - 1)}$. We want this step to prove that $\dim Q_k, \dim W_k \leq n_i 2^{\frac{1}{2} n_i + 1}$.

If $k < 2^{n_i - 1}$, the assumption is sufficient. Otherwise, recalling Lemma 2.3.3:

$$\dim Q_k \leq \dim V_{2^{n_i}} \dim W_{2^{n_i} - k} < n_i 2^{\frac{1}{2} n_i + 1},$$

$$\dim W_k \leq \dim V_{2^{n_i}} \dim Q_{2^{n_i} - k} < n_i 2^{\frac{1}{2} n_i + 1}.$$

Case 3: Suppose $k < 2^{n_{i+1} - z_{i+1}}$, and assume that for all $j < 2^{n_i}$, $\dim Q_j$, $\dim W_j \leq n_i 2^{\frac{1}{2} n_i + 1}$.

If $k < 2^{n_i}$, the assumption is sufficient. Otherwise, let $k = j + m$, with $j < 2^{n_i}$ and m divisible by 2^{n_i}. Recalling Lemmas 2.3.4 and 2.3.5,

$$\dim Q_k \leq \dim Q_m \dim Q_j \leq 2^{\frac{1}{2} n_i + 2},$$

$$\dim W_k \leq \dim W_m \dim W_j \leq 2^{\frac{1}{2} n_i + 2}.$$

This completes the induction. □

Corollary 2.3.7. *For all $k \geq 1$,*

$$\dim Q_k, \dim W_k \leq 4 \sqrt{k} \log_2 k$$

Proof. Let $i \geq 1$ be such that $2^{n_{i-1}-1} \leq k < 2^{n_i-1}$. If $2^{n_{i-1}-1} \leq k < 2^{n_i-z_i}$, then Theorem 2.3.6 proves:

$$\dim Q_k, \dim W_k \leq n_{i-1}2^{\frac{1}{2}n_{i-1}+2} < n_{i-1}2^{5/2}\sqrt{k} < 4\sqrt{k}\log_2 k,$$

and if $2^{n_i-z_i} \leq k < 2^{n_i-1}$, then:

$$\dim Q_k, \dim W_k \leq n_i\sqrt{k} < 2(n_i - z_i)\sqrt{k} \leq 2\sqrt{k}\log_2 k.$$

□

Applying Corollary 2.3.2, we can conclude:

$$\dim A_n/I_n \leq \sum_{i=0}^{n} 4\sqrt{i}\log_2 i \cdot 4\sqrt{n-i}\log_2(n-i) <$$

$$\sum_{i=0}^{n} 16n(\log_2 n)^2 = 16n(n+1)(\log_2 n)^2,$$

$$\text{GKdim}\, \bar{A}/I = \limsup_{n\to\infty} \log_n \sum_{i=1}^{n} \dim A_i/I_i \leq \limsup_{n\to\infty} \log_n(16n^2(n+1)(\log_2 n)^2) = 3.$$

The example is complete: $\bar{A}/I$ is nil, infinite dimensional, almost connected, and has a Gelfand-Kirillov dimension ≤ 3.

Chapter 2, includes a reinterpretation of, and borrows heavily from, [16]. This paper has been submitted for publication with the dissertation author as a co-author.

Chapter 3

The Kurosh Problem for Algebras Over a General Field

This chapter reiterates a result from a paper [17] by the dissertation author and J. P. Bell: over an arbitrary uncountable field, for any non-polynomial function f, there exists an algebra that's nil, infinite dimensional, almost connected, and with growth that's asymptotically bounded above by f. Recall that we designate f to be non-polynomial if there exist no $\alpha, C > 0$ such that $f(n) \leq Cn^{\alpha}$ for all n. Combining this result with [16] proves it for the case of general fields.

The method of this paper shares a lot of its reasoning with [16]. Let $\mathbb{K}$ an uncountable field, let $A = \mathbb{K}\langle x, y\rangle$ be the free algebra of two indeterminates over $\mathbb{K}$ with $\mathbb{N}$-grading $A_0 = \mathbb{K}$, $A_i = (\mathbb{K}x + \mathbb{K}y)^i$, and let $\bar{A} = \sum_{i=1}^{\infty} A_i$. The objective will be to find a graded ideal $I = \sum_{i=1}^{\infty} I_i \lhd \bar{A}$ such that $\bar{A}/I$ is nil, and $f_{\bar{A}/I,(A_1+I)/I} \precsim f$. In other words, every $g \in \bar{A}$ has an exponent $g^m \in I$, and there exists some $C, D > 0$ such that for all $n > 0$,

$$Cf(Dn) \geq f_{\bar{A}/I,(A_1+I)/I}(n) = \sum_{i=1}^{n} \dim A_i/I_i.$$

We will be borrowing much of the construction of the subspaces $\{U_{2^n}\}$ from section 2.1. The paths of the two papers begin to diverge at the construction of the subspaces $\{F_n\}$. As no enumeration of $\bar{A}$ exists, we cannot use the same method.

Lemma 3.0.8. *Let $d > 0$. For any $I, J \in \mathbb{N}$ with $0 < I < J - 2d$ and any $m > J$, there exists subspaces $F_{a,b} \subseteq A_{I-J-a-b}$ for each $0 \le a < d$, $0 \le b < d$ such that $\dim F_{a,b} \le (J - I - a - b)^d$, and for any $g \in \sum_{i=1}^{d} A_i$,*

$$g^m \in \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} A_{a+I} F_{a,b} A_b A.$$

Proof. Let $W = \{x, y, 1\}^d \backslash \{1\}$, i.e. the set of all non-trivial monomials of length $\le d$ using letters x, y. We can write $\sum_{i=1}^{d} A_i = \mathbb{K}W$.

Let $T = \{t_w\}_{w \in W}$ be a set of indeterminates, and consider the algebra $A[T]$. Let $g = \sum_{w \in W} t_w w$. We can decompose $g = g_{(1)} + \cdots + g_{(d)}$ with each $g_{(i)} = \sum_{|w|=i} t_w w$.

Using this value of g, we can copy almost all the work done in the proof of Lemma 2.2.1, and end up with:

$$F'_{a,b} = \sum_{c=1}^{J-I-a-b} \mathbb{K} \left(\sum_{\sigma \in S_d^c | \text{sum } \sigma = J-I-a-b} g_{(\sigma(1))} \cdots g_{(\sigma(c))} \right) \subseteq A_{J-I-a-b}[T],$$

$$g^m \in \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} A_{a+I} F'_{a,b} A_b A.$$

Let $F'_{a,b,c}$ be the element:

$$\sum_{\sigma \in S_d^c | \text{sum } \sigma = J-I-a-b} g_{(\sigma(1))} \cdots g_{(\sigma(c))} \in A_{J-I-a-b} \cdot T^c,$$

so that $F'_{a,b} = \sum_{c=1}^{J-I-a-b} \mathbb{K} F'_{a,b,c}$.

Let $E(c, m)$ be the set of all sequences $\{i_w\}_{w \in W}$ of non-negative integers such that:

$$\sum_{w \in W} i_w = c, \quad \sum_{w \in W} |w| i_w = m.$$

This way,

$$F'_{a,b,c} \in A_{J-I-a-b} \cdot \left\{ \prod_{w \in W} t_w^{i_w} \middle| \{i_w\} \in E(c, J - I - a - b) \right\}.$$

Note that there are at most $(m + 1)^{d-1}$ elements of $E(c, m)$.

For any $h \in \sum_{i=1}^{d} A_i$, there exists a homomorphism $\phi_h : A[T] \to A$ that maps $g \mapsto h$ by mapping each t_w to the w-coefficient of h, an element of $\mathbb{K}$. We can compute that:

$$\dim \sum_{h \in \mathbb{K}W} \mathbb{K}\phi_h(F'_{a,b,c}) \leq |E(c, J-I-a-b)| \leq (J-I-a-b)^{d-1}.$$

If we set $F_{a,b} = \sum_{h \in \mathbb{K}W} \phi_h(F'_{a,b})$, then $F_{a,b} \subseteq A_{J-I-a-b}$,

$$h^m \in \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} A_{a+I} F_{a,b} A_b A,$$

and:

$$\dim F_{a,b} = \dim \sum_{h \in \mathbb{K}W} \phi_h(F'_{a,b}) \leq$$

$$\sum_{c=1}^{J-I-a-b} \dim \sum_{h \in \mathbb{K}W} \mathbb{K}\phi_h(F'_{a,b,c}) \leq (J-I-a-b)^d.$$

□

Theorem 3.0.9. *For any $d > 0$ any $n > 2d$, and any $m > 2n$, there exists a subspace $F \subseteq A_n$ with $\dim F < d^2(4n)^d$ such that for any $g \in \sum_{i=1}^{d} A_i$, $Ag^m A \in \mathcal{E}(F)$.*

Proof. As Lemma 3.0.8 copies from Lemma 2.2.1, this theorem uses the same steps as 2.2.2, which gives us:

$$F = \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} A_a F_{a,b} A_b \subseteq A_n,$$

and $\dim F < d^2 2^{2d} n^d$. □

Recall our non-polynomial function f. For each $\alpha \in \mathbb{N}$, $n^\alpha \precsim f(n)$, and there exists a $C > 0$ such that $n^\alpha < f(Cn)$ for all $n \in \mathbb{N}$, and if $n \geq C^\alpha$, then $n^{\alpha-1} < f(n)$. Thus, for each $\alpha > 0$, we can choose a B_α such that for all $n \geq B_\alpha$, $n^\alpha < f(n)$.

Define the sequence $\{z_i\}_{i=1}^{\infty}$ recursively, by setting $z_1 = 5$ and each $z_i = \sup\{z_{i-1}+2, \log_2(i(\log_2 B_{9i+20}+5))\}$. Given this sequence, define $\{n_i\}_{i=0}^{\infty}$ by setting

$n_i = \lfloor i^{-1}2^{z_i} \rfloor - 4$ for each $i > 0$ and setting $n_0 = 0$. Finally, use Theorem 3.0.9 to select $F_i \subseteq A_{2^{n_i}}$ as a subspace with $\dim F_i < i^2 2^{i(n_i+2)}$ such that for each $g \in \sum_{k=1}^{i} A_k$, $Ag^{2^{n_i+1}+1}A \in \mathcal{E}(F_i)$.

With this established, we're going to build the spaces $\{U_{2^i}, V_{2^i}\}$ in much the same way as in Theorem 2.2.3.

Theorem 3.0.10. *There exists sequences of subspaces $U_{2^n}, V_{2^n} \subseteq A_{2^n}$ with the properties that, for each $n \geq 0$:*

- $U_{2^n} \oplus V_{2^n} = A_{2^n}$,
- $U'_{2^{n+1}} = U_{2^n}A_{2^n} + A_{2^n}U_{2^n} \subseteq U_{2^{n+1}}$
- $V_{2^{n+1}} \subseteq V_{2^n}^2$,
- *V_{2^n} can be generated by monomials (i.e. elements of the semigroup $\langle x, y\rangle$),*
- *if $n = n_i$ for some i, then $F_i \subseteq U_{2^{n_i}}$,*
- *if $n_i - z_i \leq n < n_i$ for some i, then $\dim V_{2^n} = 2^{2^{n-n_i+z_i}}$. Otherwise, $\dim V_{2^n} = 2$. In the latter case, there exists a monomial $m \in V_{2^n}$ such that $mA_{2^n} \subseteq U_{2^{n+1}}$.*

Proof. This can be done with the exact same proof as 2.2.3. The only relevant difference is the size of F and the spacing between each n_i; it's needed to show that $n_i - z_i - n_{i-1} \geq 0$ and $\dim U'_{2^{n_i}} + F_i < \dim A_{2^{n_i}} - 2$ for each n_i.

For the first inequality,

$$n_i - z_i - n_{i-1} = \lfloor i^{-1}2^{z_i} \rfloor - z_i - \lfloor (i-1)^{-1}2^{z_{i-1}} \rfloor \geq$$

$$i^{-1}(2^{z_i} - 2^{z_{i-1}+1}) - z_i - 1 \geq i^{-1}(3 \cdot 2^{z_i-2} - \frac{1}{2}(z_i - 4)(z_i+1)) > 0.$$

For the second,

$$\dim U'_{2^{n_i}} + F_i = \dim A_{2^{n_i}} - \dim V_{2^{n_i}} + \dim F_i < \dim A_{2^{n_i}} - 2^{2^{z_i}} + i^2 2^{i(n_i+2)} \leq$$

$$\dim A_{2^{n_i}} - 2^{2^{z_i}} + i^2 2^{i(i^{-1}2^{z_i} - i^{-1} - 1)} = \dim A_{2^{n_i}} + 2^{2^{z_i}}\left(i^2 2^{-i-1} - 1\right) < \dim A^{2^{n_i}} - 2.$$

□

The construction of $I \lhd \bar{A}$ will be the same as in section 2.1, using this version of $\{U_{2^n}\}$. As before, we can show that each $I_{2^n} \neq A_{2^n}$, and $\bar{A}/I$ is infinite dimensional. Finally, Theorem 3.0.9 combined with Proposition 2.2.1 shows that if $g \in \sum_{k=1}^{i} A_k$, then $g^{2^{n_i+1}+1} \in I$.

The one remaining piece of this section is to prove that the growth of A/I is asymptotically bounded above by f.

Recall the definitions of R_n and S_n from section 2.3, and the existence of Q_n and W_n such that $R_n \oplus Q_n = S_n \oplus W_n = A_n$ and, if $n = 2^{p_0} + \cdots + 2^{p_r}$ is a binary decomposition of n with $0 \leq p_0 < ... < p_r$,

$$Q_n \subseteq V_{2^{p_r}} \cdots V_{2^{p_0}},$$

$$W_n \subseteq V_{2^{p_0}} \cdots V_{2^{p_r}}.$$

Lemmas 2.3.4 and 2.3.5 and Corollary 2.3.2 apply as well.

Lemma 3.0.11. *For any $n, i \geq 1$, if $n < 2^{n_i - z_i}$, then:*

$$\dim Q_n, \dim W_n \leq 2^{2^{z_{i-1}+2}}.$$

Proof. We can decompose $n = b_i + a_{i-1} + b_{i-1} + \cdots + a_1 + b_1$, with each $b_k < 2^{n_k - z_k}$, $2^{n_{k-1}} | b_k$, and each $a_k < 2^{n_k}$, $2^{n_k - z_k} | a_k$. Using Lemma 2.3.4, each $\dim Q_{b_k}, \dim W_{b_k} \leq 2$, and using Lemma 2.3.5,

$$\dim Q_n \leq \prod_{k=1}^{i-1} \dim Q_{a_k} \cdot \prod_{k=1}^{i} \dim Q_{b_k} < 2^i \cdot \prod_{k=1}^{i-1} \dim V_{2^{n_k - z_k}} \cdots \dim V_{2^{n_k - 1}} \leq$$

$$2^i \cdot \prod_{k=1}^{i-1} 2^{2^0 + 2^{z_k}} = 2^i \cdot \prod_{k=1}^{i-1} 2^{2^{z_k+1}-1} = 2^{\sum_{k=1}^{i-1} 2^{z_k+1}} < 2^{2^{z_{i-1}+2}},$$

$$\dim W_n \leq \prod_{k=1}^{i-1} \dim W_{a_k} \cdot \prod_{k=1}^{i} \dim W_{b_k} < 2^{2^{z_{i-1}+2}}.$$

∎

Theorem 3.0.12.

$$f_{\bar{A}/I, (A_1+I)/I} \precsim f.$$

Proof. Consider two cases: when $n < 2^{n_2 - z_2}$, and when $n \geq 2^{n_2 - z_2}$.

In the former case, the size of $f_{\bar{A}/I,(A_1+I)/I}(n)$ is bounded, and it's clear that there exists some $C \geq 1$ such that $f_{\bar{A}/I,(A_1+I)/I}(n) \leq f(Cn)$ for all such values of n.

In the latter case, it's sufficient to prove that $f_{\bar{A}/I,(A_1+I)/I}(n) < f(n)$.

Let $i \geq 3$ be such that $2^{n_{i-1}-z_{i-1}} \leq n < 2^{n_i - z_i}$. We can show:

$$n \geq 2^{n_{i-1}-z_{i-1}} \geq 2^{n_{i-2}} \geq 2^{(i-2)^{-1}2^{z_{i-2}-5}} \geq B_{9i+2},$$

and therefore $f(n) \geq n^{9i+2}$.

From Corollary 2.3.2 and Lemma 3.0.11,

$$\dim A_n/I_n \leq \sum_{k=0}^{n} \dim W_k \dim Q_{n-k} \leq \sum_{k=0}^{n} \left(2^{2^{z_{i-1}+2}}\right)^2 = (n+1)2^{2^{z_{i-1}+3}},$$

$$\sum_{k=1}^{n} \dim A_k/I_k \leq \sum_{k=1}^{n} (k+1)2^{2^{z_{i-1}+3}} < n^2 2^{2^{z_{i-1}+3}} < n^2 2^{9\cdot 2^{z_{i-1}}-9z_{i-1}-45(i-1)} \leq$$

$$n^2 2^{9i(n_{i-1}-z_{i-1})} \leq n^{2+9i} \leq f(n).$$

□

Chapter 3 includes a reinterpretation of, and borrows heavily from, [17]. This paper has been submitted for publication with the dissertation author as a co-author.

Chapter 4

Jacobson radical algebras with quadratic growth

In this chapter, we will discuss a paper [18] by the dissertation author and A. Smoktunowicz that producesan almost connected Jacobson radical algebra over an arbitrary countable and algebraically closed field that has precisely quadratic growth.

As mentioned above, if an algebra has growth that is strictly less than quadratic, then it has Gelfand-Kirillov dimension either 1 or 0. In the former case, it can be proven that the algebra is not Jacobson radical (see [9]), and in the latter, it is finite dimensional. Therefore, it's sufficient to take a countable, algebraically closed field $\mathbb{K}$, a pair of indeterminates x, y, and an ideal $I \lhd A = \mathbb{K}\langle x, y\rangle$ such that:

- $I = \bigoplus_{n=1}^{\infty} I_n$, where each $I_n \subseteq A_n = \mathbb{K}\{x, y\}^n$,
- $I_n \neq A_n$ for an infinite number of values of n,
- $\sum_{k=1}^{n} \dim A_k/I_k \precsim n^2$,
- For every $g \in A$, there exists an h such that $g + h + gh \in I$.

Once again, we will stay close to the method in chapter 2.

4.1 The subspaces $\{U_{2^n}\}$

Suppose we have a strictly increasing sequence of natural numbers $\{N_i\}_{i=0}^{\infty}$ with $N_0 = 1$ and a sequence of homogeneous subspaces $\{F_i\}_{i=0}^{\infty}$ with each $F_i \subseteq A_{2^{N_i}}$ and $F_0 = (0)$.

In this section, we ask the question: does there exist, for every $i \geq 0$, a subspace $U_{2^i} \subset A_{2^i}$ and two elements $v_{i,1}, v_{i,2} \in \{x, y\}^{2^i}$ such that, for each $i \geq 0$:

- $U_{2^i} \oplus \mathbb{K}v_{i,1} \oplus \mathbb{K}v_{i,2} = A_{2^i}$,
- There exists a $v \in \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$ such that $U_{2^{i+1}} = A_{2^i}U_{2^i} + U_{2^i}A_{2^i} + vA_{2^i}$,
- $F_i \subseteq U_{2^{N_i}}$.

We shall attack the question with induction. For the base case, set $U_1 = (0)$, $v_{0,1} = x$, $v_{0,2} = y$.

For the inductive step, assume the existence of $U_{2^{N_i}}, v_{N_i,1}, v_{N_i,2}$ for some $i \geq 0$, and find possible $U_{2^k}, v_{k,1}, v_{k,2}$ for all $N_i < k \leq N_{i+1}$.

Let $W \cong \mathbb{K}^{2(N_{i+1}-N_i)}$ be a subspace with indices $\{x_{k,1}, x_{k,2}\}_{k=N_i}^{N_{i+1}-1}$, let W_k be the subspace of all elements where $(x_{k,1}, x_{k,2}) = (0, 0)$, and let $\overline{W} = W \setminus \bigcup_{k=N_i}^{N_{i+1}-1} W_k$.

Given some vector $\vec{w} \in \overline{W}$, define $U_{2^k}(\vec{w}), v_{k,1}(\vec{w}), v_{k,2}(\vec{w})$ recursively for each $N_i \leq k \leq N_{i+1}$, as follows: first, set $U_{2^{N_i}}(\vec{w}) = U_{2^{N_i}}$, $v_{N_i,1}(\vec{w}) = v_{N_i,1}$, $v_{N_i,2}(\vec{w}) = v_{N_i,2}$.

Then, assuming $U_{2^k}(\vec{w}), v_{k,1}(\vec{w}), v_{k,2}(\vec{w})$ are defined for some $N_i \leq k < N_{i+1}$:

$$U_{2^{k+1}}(\vec{w}) = A_{2^k}U_{2^k}(\vec{w}) + U_{2^k}(\vec{w})A_{2^k} + (x_{k,2}(\vec{w})v_{k,1}(\vec{w}) - x_{k,1}(\vec{w})v_{k,2}(\vec{w}))A_{2^k}.$$

If $x_{k,1}(\vec{w}) \neq 0$, set:

$$v_{k+1,1}(\vec{w}) = x_{k,1}(\vec{w})^{-1}v_{k,1}^2(\vec{w}),$$

$$v_{k+1,2}(\vec{w}) = x_{k,1}(\vec{w})^{-1}v_{k,1}(\vec{w})v_{k,2}(\vec{w}),$$

and if $x_{k,1}(\vec{w}) = 0$, then $x_{k,2}(\vec{w}) \neq 0$, so set:

$$v_{k+1,1}(\vec{w}) = x_{k,2}(\vec{w})^{-1}v_{k,2}(\vec{w})v_{k,1}(\vec{w}),$$

$$v_{k+1,2}(\vec{w}) = x_{k,2}(\vec{w})^{-1} v_{k,2}^2(\vec{w}).$$

The only task remaining in this section is to determine a sufficient condition of a $\vec{w} \in \overline{W}$ such that $F_{i+1} \subseteq U_{2^{N_{i+1}}}(\vec{w})$.

Lemma 4.1.1. *For any $N_i \leq k < N_{i+1}$, $a, b \in \{1, 2\}$, $\vec{w} \in \overline{W}$,*

$$v_{k,a}(\vec{w}) v_{k,b}(\vec{w}) \in x_{k,a}(\vec{w}) v_{k+1,b}(\vec{w}) + U_{2^{k+1}}(\vec{w}).$$

Proof. If $x_{k,1}(\vec{w}) \neq 0$, and $a = 1$, $v_{k,a}(\vec{w}) v_{k,b}(\vec{w}) = x_{k,a}(\vec{w}) v_{k+1,b}(\vec{w})$.

If $x_{k,1}(\vec{w}) \neq 0$, and $a = 2$,

$$v_{k,a}(\vec{w}) v_{k,b}(\vec{w}) =$$

$$x_{k,a}(\vec{w}) v_{k+1,b}(\vec{w}) + x_{k,1}(\vec{w})^{-1} (x_{k,2}(\vec{w}) v_{k,1}(\vec{w}) - x_{k,1}(\vec{w}) v_{k,2}(\vec{w})) v_{k,b}(\vec{w}).$$

If $x_{k,1}(\vec{w}) = 0$ and $a = 1$,

$$v_{k,a}(\vec{w}) v_{k,b}(\vec{w}) = x_{k,2}(\vec{w})^{-1} (x_{k,2}(\vec{w}) v_{k,1}(\vec{w}) - x_{k,1}(\vec{w}) v_{k,2}(\vec{w})) v_{k,b}(\vec{w}).$$

And if $x_{k,1}(\vec{w}) = 0$ and $a = 2$, $v_{k,a}(\vec{w}) v_{k,b}(\vec{w}) = x_{k,2}(\vec{w}) v_{k+1,b}(\vec{w})$. □

Let $P = \mathbb{K}[x_{k,1}, x_{k,2}]_{k=N_i}^{N_{i+1}-1}$, i.e. the (commutative) algebra of polynomial functions $W \to \mathbb{K}$. Let $Q = \prod_{k=N_i}^{N_{i+1}-1} (\mathbb{K} x_{k,1} + \mathbb{K} x_{k,2})^{2^{N_{i+1}-k-1}}$ be a homogenous subspace of P.

Theorem 4.1.2. *For any sequence $\{s_k\}_{k=1}^{2^{N_{i+1}-N_i}}$ of $\{1, 2\}^{2^{N_{i+1}-N_i}}$, there exists some $p_s \in Q$ such that for any $\vec{w} \in \overline{W}$,*

$$\prod_{k=1}^{2^{N_{i+1}-N_i}} v_{N_i, s_k} \in p_s(\vec{w}) v_{N_{i+1}, s_{2^{N_{i+1}-N_i}}}(\vec{w}) + U_{2^{N_{i+1}}}(\vec{w}).$$

Proof. We will use induction to show that, for any $0 \leq h \leq N_{i+1} - N_i$ and any sequence $\{s_k\}_{k=1}^{2^h}$ of $\{1, 2\}^{2^h}$,

$$\prod_{k=1}^{2^h} v_{N_i, s_k} \in \left(\prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j-1}} x_{N_i+j, s_{2^j(2k-1)}}(\vec{w}) \right) v_{N_i+h, s_{2^h}}(\vec{w}) + U_{2^{N_i+h}}(\vec{w}),$$

with the end result of the theorem proven when $h = N_{i+1} - N_i$.

The base case is simply $v_{N_i,s_1} \in v_{N_i,s_1}(\vec{w}) + U_{2^{N_i}}(\vec{w})$.

For the inductive step, let $\{s_k\}_{k=1}^{2^{h+1}}$ be a sequence of $\{1,2\}^{2^{h+1}}$, and assume the inductive statement is true for $\{s_k\}_{k=1}^{2^h}$ and $\{s_k\}_{k=2^h+1}^{2^{h+1}}$. Lemma 4.1.1 shows that:

$$v_{N_i+h,s_{2^h}}(\vec{w})v_{N_i+h,s_{2^{h+1}}}(\vec{w}) \in x_{N_i+h,s_{2^h}}(\vec{w})v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{2^{N_i+h+1}}(\vec{w}).$$

Therefore,

$$\prod_{k=1}^{2^{h+1}} v_{N_i,s_k} \in \left(\left(\prod_{j=0}^{h-1}\prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w})\right) v_{N_i+h,s_{2^h}}(\vec{w}) + U_{N_i+h}(\vec{w})\right)\cdot$$
$$\left(\left(\prod_{j=0}^{h-1}\prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)+2^h}}(\vec{w})\right) v_{N_i+h,s_{2^{h+1}}}(\vec{w}) + U_{2^{N_i+h}}(\vec{w})\right) \subseteq$$
$$\left(\prod_{j=0}^{h-1}\prod_{k=1}^{2^{h-j}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w})\right) x_{N_i+h,s_{2^h}}(\vec{w})v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{2^{N_i+h+1}}(\vec{w}) =$$
$$\left(\prod_{j=0}^{h}\prod_{k=1}^{2^{h-j}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w})\right) v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{2^{N_i+h+1}}(\vec{w}).$$

□

Corollary 4.1.3. *For any $f \in A_{2^{N_{i+1}}}$, there exists $p, q \in Q$ such that $\forall \vec{w} \in \overline{W}$, $f \in p(\vec{w})v_{N_{i+1},1}(\vec{w}) + q(\vec{w})v_{N_{i+1},2}(\vec{w}) + U_{2^{N_{i+1}}}(\vec{w})$.*

Proof. First, note that:

$$A_{2^{N_{i+1}}} = (U_{2^{N_i}} + \mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2^{N_{i+1}-N_i}} =$$
$$(\mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2^{N_{i+1}-N_i}} + \sum_{k=1}^{2^{N_{i+1}-N_i}} A_{(k-1)2^{N_i}} U_{2^{N_i}} A_{2^{N_{i+1}}-k2^{N_i}},$$

and for each $f \in A_{2^{N_{i+1}}}$, there exists a $f' \in (\mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2^{N_{i+1}-N_i}}$ such that, for any $\vec{w} \in \overline{W}$, $f \in f' + U_{2^{N_{i+1}}}(\vec{w})$.

Since f' can be written as a linear combination of the elements of the form $\prod_{k=1}^{2^{N_{i+1}}} v_{N_i,s_k}$, it's sufficient to prove the corollary over these elements, which is done in Theorem 4.1.2. □

Let $d = \dim F_{i+1}$, let $\{f_k\}_{k=1}^d$ be elements that generate F_{i+1}, and let $\{p_k, q_k\} \subseteq Q$ be such that $\forall \vec{w} \in \overline{W}, f_k \in p_k(\vec{w})v_{N_{i+1},1}(\vec{w}) + q_k(\vec{w})v_{N_{i+1},2}(\vec{w}) + U_{2^{N_{i+1}}}(\vec{w})$, as detailed in Corollary 4.1.3. If there exists a $\vec{w} \in \overline{W}$ such that each $p_k(\vec{w}) = q_k(\vec{w}) = 0$, then $F_{i+1} \subseteq U_{2^{N_{i+1}}}(\vec{w})$.

Let $G = \sum_{k=1}^d \mathbb{K}p_k + \mathbb{K}q_k \subseteq Q$ be the vector space generated by $\{p_k, q_k\}$. Our remaining goal is to show $\exists \vec{w} \in \overline{W} : G(\vec{w}) = (0)$.

Let R be the algebra over $\mathbb{K}$ generated by Q, i.e. $R = \bigoplus_{k=1}^\infty Q^k$.

Lemma 4.1.4. *If G, P are defined as above, then:*

$$R \cap GP \subseteq G + GR.$$

Proof. Let $M = \bigcup_{n=1}^\infty \{x_{N_i,1}, x_{N_i,2}, ..., x_{N_{i+1}-1,1}, x_{N_{i+1}-1,2}\}^n$, i.e. the set of all non-trivial monomials of P (without coefficient). Let M_Q be the monomials that generate Q, let $M_R = \bigcup_{j=1}^\infty M_Q^j$ be the monomials that generate R, and let $M'_R = M \backslash M_R$. P can be decomposed: $P = \mathbb{K} \oplus R \oplus \mathbb{K}M'_R$.

Note that for any $m \in M_Q$ and any $m' \in M'_R$, $mm' \in M'_R$. As R is generated by monomials, $R \cap QM'_R = (0)$.

Let $g \in G$, and let $p \in P$ have the decomposition $p = k + r + s$, with $k \in \mathbb{K}$, $r \in R$ and $s \in \mathbb{K}M'_R$. Suppose that $gp \in R$. Since $gk + gr \in R$, $gs \in R \cap QM'_R = (0)$. Therefore, $gp \in \mathbb{K}g + gR$, and $R \cap GP \subseteq G + GR$. □

Theorem 4.1.5. *If $\{\vec{w} \in W : G(\vec{w}) = (0)\} \subseteq W \backslash \overline{W} = \bigcup_{k=N_i}^{N_{i+1}-1} W_k$, then:*

$$d \geq \frac{1}{2}(N_{i+1} - N_i + 1).$$

Proof. Given an ideal I of P, we define $Z(I) = \{\vec{w} \in W : I(\vec{w}) = (0)\}$. This is an affine subvariety of W. It's our goal to show that if $Z(GP) \subseteq \bigcup_{k=N_i}^{N_{i+1}-1} W_k$, then $d \geq \frac{1}{2}(N_{i+1} - N_i + 1)$.

Since Q annihilates each W_k, it must annihilate $Z(GP)$ as well. Hilbert's Nullstellensatz states that since $\mathbb{K}$ is algebraically closed, for each $q \in Q$, there must be an exponent $q^\pi \in GP$.

Using Lemma 4.1.4, $q^\pi \in R \cap GP \subseteq G + GR$, and so the quotient algebra $R/(G + GR)$ is nil. Since $G^2 \subseteq GR$, R/GR is nil as well. All affine commutative nil algebras are finite dimensional, so GKdim $R/GR = 0$.

In Lemma 3.2 of [19], L. Bartholdi and A. Smoktunowicz prove that, if R is a affine commutative graded algebra, and $I = AgA \lhd R$ is an ideal generated by a simple homogeneous element g, then $\operatorname{GKdim} R/I \geq \operatorname{GKdim} R - 1$. Extrapolating this property, if I is generated by d homogeneous elements, then $\operatorname{GKdim} R/I \geq \operatorname{GKdim} R - d$. In our case, GR is generated by $2d$ homogeneous elements, and so $\operatorname{GKdim} R/GR \geq \operatorname{GKdim} R - 2d = 0$, and $d \geq \frac{1}{2} \operatorname{GKdim} R$.

Remember that for any $j \geq 0$, $Q^j = \prod_{k=N_i}^{N_{i-1}-1} (\mathbb{K}x_{k,1} + \mathbb{K}x_{k,2})^{j2^{N_{i+1}-k-1}}$, and:

$$\dim Q^j = \prod_{k=N_i}^{N_{i+1}-1} (j2^{N_{i+1}-k-1} + 1) \geq 2^{\frac{1}{2}(N_{i+1}-N_i-1)(N_{i+1}-N_i)} j^{N_{i+1}-N_i}.$$

Therefore $d \geq \frac{1}{2} \operatorname{GKdim} R \geq \frac{1}{2}(N_{i+1} - N_i + 1)$. □

We can thus conclude that as long as $\dim F_{i+1} < \frac{1}{2}(N_{i+1} - N_i + 1)$, there is a $\vec{w} \in \overline{W}$ such that $G(\vec{w}) = 0$, and we have an appropriate space $U_{2^k} = U_{2^k}(\vec{w})$ and monomials $v_{k,1} = v_{k,1}(\vec{w})$, $v_{k,2} = v_{k,2}(\vec{w})$ for each $k \leq N_{i+1}$. If this holds for all $i \geq 0$, the induction can proceed.

4.2 The size of A_n/I_n

We define $R_n, S_n \subseteq A_n$ the same way as in chapter 2: if m is such that $2^m \leq n < 2^{m+1}$, then $R_n = \{r \in A_n : rA_{2^{m+1}-n} \subseteq U_{2^{m+1}}\}$ and $S_n = \{r \in A_n : A_{2^{m+1}-n} r \subseteq U_{2^{m+1}}\}$.

For each $i \in \mathbb{N}$, let $v_i \in \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$ be such that $U_{2^{i+1}} = A_{2^i}U_{2^i} + U_{2^i}A_{2^i} + v_iA_{2^i}$, let $U'_{2^i} = U_{2^i} + \mathbb{K}v_i$. If $v_{i,1} \notin U'_{2^i}$, then set $V'_{2^i} = \mathbb{K}v_{i,1}$, otherwise, set $V'_{2^i} = \mathbb{K}v_{i,2}$. This way,

$$U_{2^i} \subseteq U'_{2^i}, \quad V'_{2^i} \subseteq V_{2^i}, \quad U'_{2^i} \oplus V'_{2^i} = A_{2^i},$$

$$U_{2^{i+1}} = A_{2^i}U_{2^i} + U'_{2^i}A_{2^i}.$$

For any $n \in \mathbb{N}$, and let $n = 2^{p_0} + \cdots + 2^{p_r}$ be the usual binary decomposition, with $0 \leq p_0 < ... < p_r$. Define $Q_n = V'_{2^{p_r}} \cdots V'_{2^{p_0}}$. Note that $\dim Q_n = 1$.

Lemma 4.2.1. *For every $n \in \mathbb{N}$, $Q_n \oplus R_n = A_n$.*

Proof. Use the same binary decomposition. Consider the space:

$$R = \sum_{i=0}^{r} A_{2^{p_r}+\dots+2^{p_{i+1}}} U'_{2^{p_i}} A_{2^{p_{i-1}}+\dots+2^{p_0}}$$

Since each $U'_{2^{p_i}} \oplus V'_{2^{p_i}} = A_{2^{p_i}}$, $R \oplus Q_n = A_n$. It's sufficient to prove that $R \subseteq R_n$; since $\dim Q_n = 1$, $R \subseteq R_n$ implies either $R = R_n$ or $R_n = A_n$, and the latter is contradicted by the definition of R_n and the fact that $U_{2^{p_r+1}} \neq A_{2^{p_r+1}}$.

For each $0 \leq i \leq r$, let $n_i = n - (2^{p_{i-1}} + \cdots 2^{p_0}) < 2^{p_r+1}$. Since $n_i < 2^{p_r+1}$ and $2^{p_i}|n_i$, $2^{p_r+1} - n_i \geq 2^{p_i}$. Since Proposition 2.1.1 still applies,

$$R_n A_{2^{p_r+1}-n} = \sum_{i=0}^{r} A_{2^{p_r}+\dots+2^{p_{i+1}}} U'_{2^{p_i}} A_{2^{p_r+1}-n_i} =$$

$$\sum_{i=0}^{r} A_{2^{p_r}+\dots+2^{p_{i+1}}} \left(U'_{2^{p_i}} A_{2^{p_i}}\right) A_{2^{p_r+1}-n_i} = \sum_{i=0}^{r} A_{2^{p_r}+\dots+2^{p_{i+1}}} U_{2^{p_i+1}} A_{2^{p_r+1}-n_i} \subseteq U_{2^{p_r+1}},$$

and $R \subseteq R_n$. □

Copying our work in section 2.3, there also exists a subspace $W_n \subseteq V_{2^{p_0}} \cdots V_{2^{p_r}}$ such that $W_n \oplus Q_n$. Lemma 2.3.3 still applies, with:

$$\dim W_n \leq \dim(\mathbb{K}v_{p_r+1,1} + \mathbb{K}v_{p_r+1,2}) \dim Q_{2^{p_r+1}-n} = 2.$$

Corollary 2.3.2 still applies as well:

$$\dim A_n/I_n \leq \sum_{i=0}^{n} \dim W_n \dim Q_{n-i} \leq 2n + 2,$$

$$f_{\bar{A}/I,(A_1+I)/I}(n) = \sum_{i=1}^{n} 2i + 2 = n^2 + 3n.$$

Therefore, $\bar{A}/I$ has quadratic growth.

4.3 The subspaces $\{F_i\}$

Let $g \in \bar{A}$, and let d be minimal such that $g \in \sum_{i=1}^{d} A_i$. Let $g = g_{(1)} + \cdots g_{(d)}$ be the homogeneous decomposition of g, with each $g_{(i)} \in A_i$. For each $n \geq 0$, define the element $s_n \in A_n$ recursively:

- $s_0 = 1$,
- $s_n = -\sum_{i=1}^{\min\{n,d\}} g_{(i)} s_{n-i}$.

One can inductively show that:

$$s_n = \sum_{k=0}^{n} \sum_{(1 \leq i_1,\ldots,i_k \leq d, i_1+\cdots+i_k=n)} (-1)^k g_{(i_1)} \cdots g_{(i_k)},$$

and by symmetry,

$$s_n = -\sum_{i=1}^{\min\{n,d\}} s_{n-i} g_{(i)}.$$

Lemma 4.3.1. *For any a, b, k with $0 \leq a \leq b - 2n \leq k - 2n$,*

$$s_k \in \sum_{i,j=0}^{d-1} A_{a+i} s_{b-a-j-i} A_{k-b+j}.$$

Proof. First, we wish prove the claim:

$$s_k \in \sum_{i=0}^{d-1} A_{a+i} s_{k-a-i}.$$

Use induction on the value of a. The base case, $a = 0$, is trivial from the definition of s_k. For the inductive step,

$$s_k \in \sum_{i=0}^{d-1} A_{a+i} s_{k-a-i} = A_a s_{k-a} + \sum_{i=1}^{d-1} A_{a+i} s_{k-a-i} =$$

$$-\sum_{i=1}^{d} A_a g_{(i)} s_{k-a-i} + \sum_{i=1}^{d-1} A_{a+i} s_{k-a-i} \subseteq \sum_{i=0}^{d-1} A_{(a+1)+i} s_{k-(a+1)-i}.$$

Through symmetry, and the fact that $s_k = \sum_{i=1}^{d} s_{k-i} g_{(i)}$, we can also prove, for each $0 \leq i \leq d-1$:

$$s_{k-a-i} \in \sum_{j=0}^{d-1} s_{b-a-j-i} A_{k-b+j}.$$

Combining these,

$$s_k \in \sum_{i=0}^{d-1} A_{a+i} s_{k-a-i} \subseteq \sum_{i,j=0}^{d-1} A_{a+i} s_{b-a-j-i} A_{k-b+j}.$$

□

For each $N \geq 2d$, define the space $F_N(g) \subseteq A_N$ as:

$$F_N(g) = \sum_{i,j=0}^{d-1} A_i s_{N-i-j} A_j.$$

Lemma 4.3.2. *For any $k \geq 2N$,*

$$A s_k A \in \mathcal{E}(F_N(g)).$$

Proof. As $A_N \mathcal{E}(F_N(g)) A \subseteq \mathcal{E}(F_N(g))$, it's sufficient to prove that $A_m s_k \in \mathcal{E}(F_N(g))$ for any $0 \leq m < N$.

Using Lemma 4.3.1,

$$A_m s_k \subseteq A_m \sum_{i,j=0}^{d-1} A_{N-m+i} s_{N-i-j} A_{k-2N+j} \subseteq A_N F_N(g) \subseteq \mathcal{E}(F_N(g)).$$

□

Theorem 4.3.3. *For any $N \geq 2d$, there exists an $h \in \bar{A}$ such that $g + h + gh \in \mathcal{E}(F_N(g))$.*

Proof. Let $h = \sum_{i=1}^{2N+d} s_i$.

$$g + h = g + \sum_{i=1}^{2N+d} s_i = g - \sum_{i=1}^{2N+d} \sum_{j=1}^{\min\{i,d\}} g_{(j)} s_{i-j} = g - \sum_{j=1}^{d} \sum_{i=j}^{2N+d} g_{(j)} s_{i-j} =$$

$$- \sum_{j=1}^{d} \sum_{i=j+1}^{2N+d} g_{(j)} s_{i-j} = - \sum_{j=1}^{d} \sum_{i=1}^{2N+d-j} g_{(j)} s_i =$$

$$- \left(\sum_{j=1}^{d} g_{(j)} \right) \left(\sum_{i=1}^{2N+d} s_i \right) + \sum_{j=1}^{d} \sum_{i=2N+d-j+1}^{2N+d} g_{(j)} s_i \in -gh + \sum_{i=1}^{d} A s_{2N+i}.$$

Finally, lemma 4.3.2 proves that $\sum_{i=1}^{d} A s_{2N+i} \subseteq \mathcal{E}(F_N(g))$. □

If we can get $F_{2^n}(g) \subseteq U_{2^n}$ for some n, then Theorem 4.3.3 and Proposition 2.2.1 combined prove that there exists an $h \in \bar{A}$ such that $g + h + gh \in I$. In other words, $g + I$ is right-quasiregular in $\bar{A}/I$.

As $\mathbb{K}$ is countable, we can construct an enumeration $\bar{A} = \{g_1, g_2, ...\}$. Let each d_i be minimal such that $g_i \in \sum_{j=1}^{d_i} A_j$. Define the series $\{N_i\}_{i=0}^{\infty}$ recursively, with $N_0 = 0$, and for each $i > 0$, $N_i = N_{i-1} + 2^{2d_i+1} - 1$.

Set $F_i = F_{2^{N_i}}(g_i)$. Section 4.1 establishes that we can have each $F_i \subseteq U_{2^{N_i}}$, so long as $\dim F_i < \frac{1}{2}(N_i - N_{i-1} + 1)$, which is indeed the case:

$$\dim F_i \leq \sum_{j,k=0}^{d_i-1} \dim A_j \dim A_k = \sum_{j,k=0}^{d_i-1} 2^{j+k} < 2^{2d_i} = \frac{1}{2}(N_i - N_{i-1} + 1).$$

We have thus proven the existence of a set $\{U_{2^n}\}$, following the specifications of section 4.1, that results in an ideal $I \lhd \bar{A}$ such that every element of $\bar{A}/I$ is right-quasiregular, from which it follows that $\bar{A}/I$ is Jacobson radical.

Chapter 4 includes a reinterpretation of, and borrows heavily from, [18]. This paper has been submitted for publication with the dissertation author as a co-author.

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