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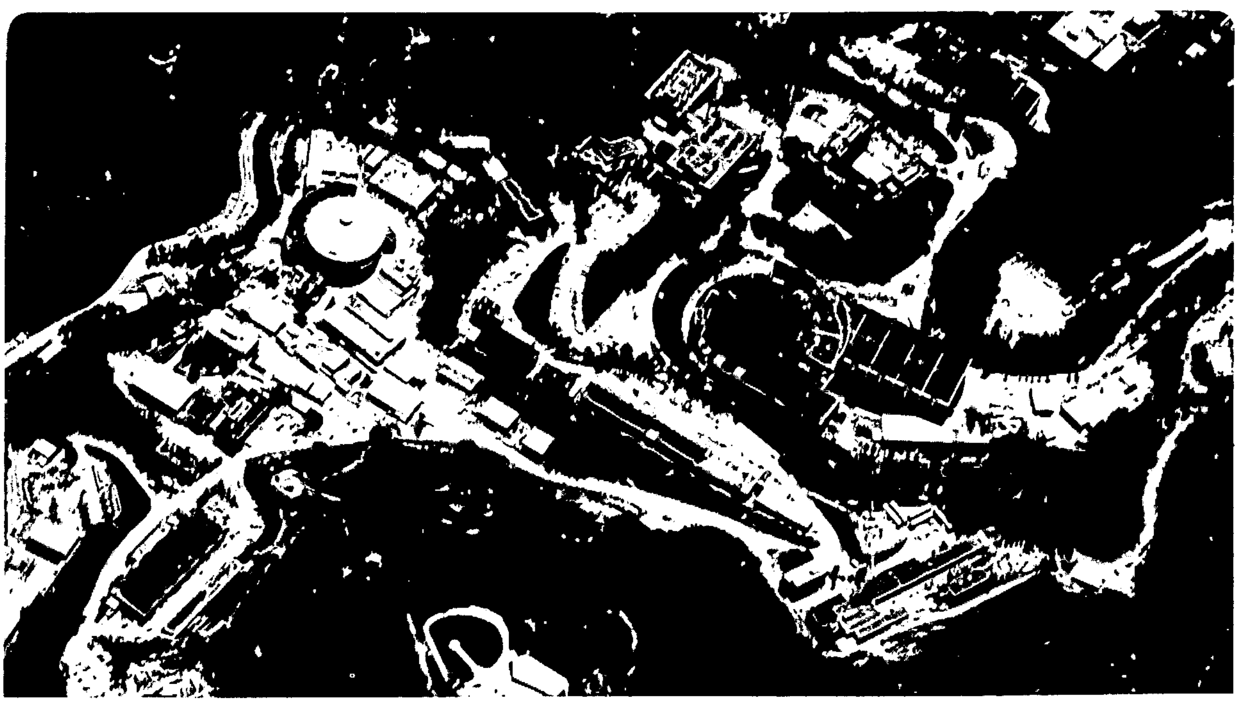

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EXTENSIONS OF 2D GRAVITY[*] [†]

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After reviewing some aspects of gravity in two dimensions, I show that non-trivial embeddings of $sl(2)$ in a semi-simple (super) Lie algebra give rise to a very large class of extensions of 2D gravity. The induced action is constructed as a gauged WZW model and an exact expression for the effective action is given.

1. Introduction

Conformally invariant theories in two dimensions play an important role in the study of string theories, second order phase transitions and integrable systems. The Virasoro algebra and its extensions[1] form the cornerstone of conformal field theories. More recently a lot of attention was devoted to the study of a particular class of conformal field theories: gravity in two dimensions. As I will show in the next section, gravity in two dimensions is a purely quantum mechanical artefact. Contrary to the case of higher dimensions, gravity in two dimensions allows for infinitely many extensions such as higher spin fields, supersymmetry, Yang-Mills symmetries, etc. These extensions are in one to one correspondence with the extensions of the Virasoro algebra.

The interest in 2D gravity arose from its close relation to non-critical string theories. In these theories, the matter sector is a minimal model. A propagating gravity sector contributes to the conformal anomaly in such a way that the conformal anomaly of the combined matter-gravity-ghost sectors vanishes. Non-critical string theories allow for a non-perturbative treatment thus providing a testing ground for techniques and ideas which might be applicable to more "realistic" string theories[2]. More recently 2D gravity gave rise to models which made the study of quantum aspects of black hole evaporation possible[3].

2. Gravity in Two Dimensions

The Einstein-Hilbert action in two dimensions

$$S_{\mathrm{EH}} = \frac{1}{4\pi}\int_\Sigma d^2x\sqrt{g}R^{(2)}, \tag{2.1}$$

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has no dynamical content as it gives the Euler characteristic of the surface Σ: $\chi(\Sigma) = 2-2h$, a topological invariant. Consider a scalar field ϕ coupled to gravity:

$$S_{\rm SF} = \frac{1}{4\pi}\int d^2x\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{2.2}$$

Introducing light-cone coordinates, one parametrizes the metric as $g_{+-}=e^\varphi$, $g_{\pm\pm} = 2e^\varphi\mu_{\pm\pm}$ and Eq. (2.2) gets rewritten as

$$S_{\rm SF} = \frac{1}{2\pi}\int d^2x\frac{1}{\sqrt{1-4\mu_{++}\mu_{--}}}\left(\partial_+\phi\partial_-\phi + 2\mu_{++}T_{--}(\phi) + 2\mu_{--}T_{++}(\phi)\right), \tag{2.3}$$

where the energy-momentum tensor is given by $T_{\pm\pm}(\phi) = -\frac{1}{2}\partial_\pm\phi\partial_\pm\phi$. The modes of the energy-momentum tensor form two commuting copies of the Virasoro algebra with central extension $c=1$. As the action has no explicit dependence on the conformal mode φ anymore, the theory is not only invariant under general coordinate transformations but under local Weyl rescalings of the metric as well. There are as many gauge symmetries as gauge fields so one concludes that here also there is no (except for moduli) gravitational content.

Quantum mechanically, one of the symmetries becomes anomalous and the metric acquires dynamics. To see this, one passes to the light-cone gauge $\varphi = \mu_{++} = 0$ and the action Eq. (2.3) reads now:

$$S'_{\rm SF} = \frac{1}{2\pi}\int\left(\partial_+\phi\partial_-\phi + 2\mu_{--}T_{++}(\phi)\right). \tag{2.4}$$

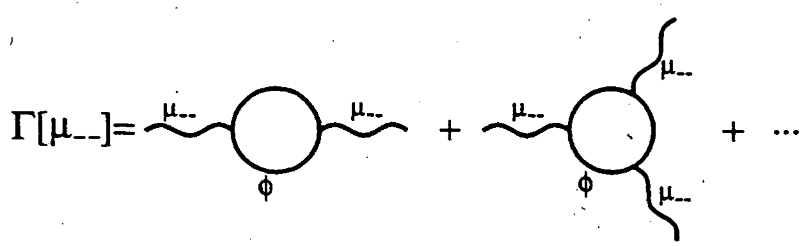
Classically there is still one gauge symmetry left:

$$\begin{aligned}\delta\phi &= \varepsilon_-\partial_+\phi,\\ \delta\mu_{--} &= \partial_-\varepsilon_- + \varepsilon_-\partial_+\mu_{--} - \partial_+\varepsilon_-\mu_{--},\end{aligned} \tag{2.5}$$

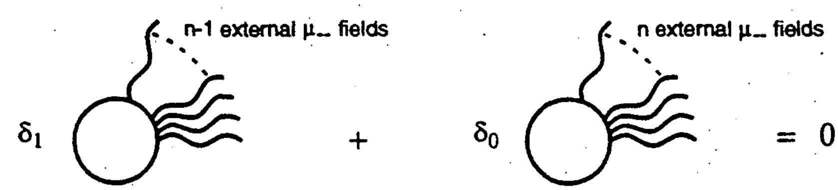
with ε_- an arbitrary infinitesimal parameter. The induced action $\Gamma[\mu_{--}]$, is defined by

$$e^{-\Gamma[\mu_{--}]} = \int[d\phi]e^{-S'_{\rm SF}(\phi,\mu_{--})} = \left\langle e^{-\frac{1}{\pi}\int\mu_{--}T_{++}}\right\rangle. \tag{2.6}$$

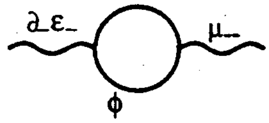
In diagrams $\Gamma[\mu_{--}]$ is given by:



If the symmetry Eq. (2.5) were to persist at quantum level, the induced action would vanish. However one easily shows that



where $\delta_0\mu_{--} = \partial_-\varepsilon_-$ and $\delta_1\mu_{--} = \varepsilon_-\partial_+\mu_{--} - \partial_+\varepsilon_-\mu_{--}$. So the anomaly is given by the δ_0 variation of the two-point diagram:

$\delta\Gamma[\mu_{--}] =$ Anomaly $=$  $= -\frac{c}{12\pi}\int \varepsilon_-\partial_+^3\mu_{--}$

where in our case $c = 1$. Having computed the anomaly, the Ward identity for $\Gamma[\mu_{--}]$ follows:

$$(\partial_- - 2\partial_+\mu_{--} - \mu_{--}\partial_+)\frac{\delta\Gamma[\mu_{--}]}{\delta\mu_{--}} = \frac{c}{12\pi}\partial_+^3\mu_{--}. \tag{2.7}$$

This gives a functional differential equation for $\Gamma[\mu_{--}]$. Methods to solve this equation have been developed[4,5]:

$$\Gamma[\mu_{--}] = \frac{c}{24\pi}\int \partial_+^2\mu_{--}\frac{1}{\partial_-}\frac{1}{1-\mu_{--}\frac{\partial_+}{\partial_-}}\frac{1}{\partial_+}\partial_+^2\mu_{--}. \tag{2.8}$$

Covariantizing this, one obtains the familiar result[6]

$$\Gamma_{\rm cov} = \Gamma[\mu_{--}] + \Gamma[\mu_{++}] + \Delta[\mu_{--},\mu_{++},\varphi] = \frac{c}{96\pi}\int \sqrt{g}R\frac{1}{\Box}R, \tag{2.9}$$

which is manifestly invariant under general coordinate transformations. Adding a cosmological constant to this and coupling it in a diffeomorphic invariant way to a minimal model gives the action for a non-critical string. The resulting model is tuned such that the total central charge of the minimal model and the gravity sector equals 26, precisely cancelling the contribution to the central charge coming from the ghosts.

The effective action is obtained from the induced action by integrating over the Beltrami differential μ_{--}:

$$e^{-W[\check{T}_{++}]} = \int [d\mu_{--}]e^{-\Gamma[\mu_{--}] + \frac{1}{\pi}\int \mu_{--}\check{T}_{++}}. \tag{2.10}$$

Introduce the classical effective action $W_{\rm cl}[\check{T}_{++}]$, simply given by the Legendre transform of $\Gamma[\mu_{--}]$:

$$W_{\rm cl}[\check{T}_{++}] = \min_{\{\mu_{--}\}}\left(\Gamma[\mu_{--}] - \frac{1}{\pi}\int \mu_{--}\check{T}_{++}\right). \tag{2.11}$$

The full effective action is equal to the classical one up to multiplicative renormalizations[7]:

$$W[\check{T}_{++}] = Z_c W_{\rm cl}[Z_T\check{T}_{++}], \tag{2.12}$$

where

$$Z_c = \frac{c+\sqrt{(c-1)(c-25)}-37}{2c},$$
$$Z_T = \frac{c}{12+cZ_c}. \quad (2.13)$$

In section four, I will give a general proof for this, not only valid for pure $2D$ gravity but for a very large class of extensions of it as well.

3. Extensions of 2D Gravity

Taking W_3 gravity as an example is most instructive and illustrates all complications of the general case. Consider two free scalar fields, ϕ_1 and ϕ_2. For convenience, I use a matrix notation:

$$\phi = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2-\phi_1 & 0 \\ 0 & 0 & -\phi_2 \end{pmatrix}. \quad (3.1)$$

The action

$$\mathcal{S} = \frac{1}{2\pi}\int tr\partial_+\phi\partial_-\phi, \quad (3.2)$$

is invariant under

$$\delta\phi = \varepsilon_-\partial_+\phi + \lambda_{--}\partial_+\phi\partial_+\phi - \frac{1}{3}\lambda_{--}\, tr\,\{\partial_+\phi\partial_+\phi\} \quad (3.3)$$

provided $\partial_-\varepsilon_- = \partial_-\lambda_{--} = 0$. The Noether currents associated to these symmetries are denoted by T_{++} and W_{+++}: $T_{++} \propto tr\{\partial_+\phi\partial_+\phi\}$ and $W_{+++} \propto tr\{\partial_+\phi\partial_+\phi\partial_+\phi\}$. The modes of the currents T and W satisfy (at quantum level) the W_3 algebra with $c=2$:

$$[L_m, L_n] = \frac{c}{12}m(m^2-1)\delta_{m+n,0} + (m-n)L_{m+n}$$
$$[L_m, W_n] = (2m-n)W_{m+n}$$
$$[W_m, W_n] = \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0} + (m-n)\left\{\frac{1}{15}(m+n+3)(m+n+2)\right.$$
$$\left. -\frac{1}{6}(m+2)(n+2)\right\}L_{m+n} + \beta(m-n)\Lambda_{m+n}, \quad (3.4)$$

where $m,n \in \mathbf{Z}$, $\beta = 16/(22+5c)$ and

$$\Lambda_m = \sum_{n\in\mathbf{Z}} :L_{m-n}L_n: - \frac{3}{10}(m+3)(m+2)L_m. \quad (3.5)$$

The normal ordering prescription is given by $:L_mL_n:=L_mL_n$ if $m \leq -2$ and $:L_mL_n:=L_nL_m$ if $m > -2$. The fact that the commutator of two generators is expressed not only as a linear combination of the generators but contains composites of the generators as well is a generic feature shared by most extensions of the Virasoro algebra.

The symmetry Eq. (3.3) can be gauged by a simple minimal coupling[8]:

$$S_{\mathrm{SF}} = \frac{1}{2\pi}\int tr\partial_+\phi\partial_-\phi + \frac{1}{\pi}\int (\mu_{--}T_{++}(\phi) + \nu_{---}W_{+++}(\phi)). \tag{3.6}$$

The action is invariant under arbitrary transformations Eq. (3.3), provided the gauge fields transform as

$$\begin{aligned}\delta\mu_{--} &= \partial_-\varepsilon_- + \varepsilon_-\partial_+\mu_{--} - \partial_+\varepsilon_-\mu_{--} - 2\left(\lambda_{--}\partial_+\nu_{---} - \nu_{---}\partial_+\lambda_{--}\right)T_{++},\\ \delta\nu_{---} &= \partial_-\lambda_{--} + 2\lambda_{--}\partial_+\mu_{--} - \partial_+\lambda_{--}\mu_{--} + \varepsilon_-\partial_+\nu_{---} - 2\partial_+\varepsilon_-\nu_{---}.\end{aligned} \tag{3.7}$$

It is not very hard to find the W_3 analogue of the covariant action[9] Eq. (2.3). The main observation is that Eq. (2.3) can be linearized through the introduction of auxiliary fields (termed nested covariant derivatives[9]) F_+ and F_-:

$$S_{\mathrm{SF}} = \frac{1}{\pi}\int d^2x\left(-\frac{1}{2}\partial_+\phi\partial_-\phi - F_+F_- + F_+\partial_-\phi + F_-\partial_+\phi + \hat{\mu}_{++}T_{--}(F_+) + \hat{\mu}_{--}T_{++}(F_-)\right), \tag{3.8}$$

which upon elimination of $F_\pm$ through their equations of motion and identifying $\mu_{\pm\pm} = (1+\hat{\mu}_{++}\hat{\mu}_{--})^{-1}\hat{\mu}_{\pm\pm}$, reduces to Eq. (2.3). The action Eq. (3.8) is gauge invariant with gauge transformations

$$\begin{aligned}\delta\phi &= \varepsilon_-F_+ + \varepsilon_+F_-,\\ \delta F_\pm &= \varepsilon_\mp\partial_\pm F_\pm + \partial_\pm\varepsilon_\mp F_\pm,\\ \delta\hat{\mu}_{\mp\mp} &= \partial_\mp\varepsilon_\mp + \varepsilon_\mp\partial_\pm\hat{\mu}_{\mp\mp} - \partial_\pm\varepsilon_\mp\hat{\mu}_{\mp\mp}.\end{aligned} \tag{3.9}$$

What happened is that because $F_\pm$ only transform under $\varepsilon_\mp$ transformations and not under $\varepsilon_\pm$ transformations, the problem reduced to two copies of the chiral case and minimal coupling ensured gauge invariance. It is clear now that exactly the same procedure can be applied for the case of W_3. The action reads now

$$\begin{aligned}S_{\mathrm{SF}} &= -\frac{1}{\pi}\int\left(\frac{1}{2}tr\partial_+\phi\partial_-\phi + trF_+F_- - trF_+\partial_-\phi - trF_-\partial_+\phi\right)\\ &\quad+\frac{1}{\pi}\int (\mu_{--}T_{++}(F_+) + \nu_{---}W_{+++}(F_+) + \mu_{++}T_{--}(F_-) + \nu_{+++}W_{---}(F_-))\end{aligned} \tag{3.10}$$

However, the auxiliary fields do now appear through cubic order in the action, which prohibits a second order formulation.

The chiral W_3 symmetry Eqs. (3.3, 3.7) is anomalous at the quantum level[4,5]. The induced action in the light-cone gauge is defined similar to Eq. (2.6):

$$e^{-\Gamma[\mu_{--},\nu_{---}]} = \left\langle e^{-\frac{1}{\pi}\int (\mu_{--}T_{++}(\phi) + \nu_{---}W_{+++}(\phi))}\right\rangle. \tag{3.11}$$

A careful treatment of the non-linearities in the W_3 algebra[4] reveals that the Ward identities contain non-local, subleading in $1/c$ terms. This in its turn implies a $1/c$ expansion for the induced action:

$$\Gamma[\mu_{--},\nu_{---}] = \sum_{n\geq 0} c^{1-n}\Gamma^{(n)}[\mu_{--},\nu_{---}]. \tag{3.12}$$

Only $\Gamma^{(0)}[\mu_{--},\nu_{---}]$ has been obtained in a closed form[5]. The effective action is defined by:

$$e^{-W[\check{T}_{++},\check{W}_{+++}]} = \int [d\mu_{--}][d\nu_{---}]\, e^{-\Gamma[\mu_{--},\nu_{---}] + \frac{1}{\pi}\int (\mu_{--}\check{T}_{++} + \nu_{---}\check{W}_{+++})}. \tag{3.13}$$

The classical action is the Legendre transform of the leading term of the induced action Eq. (3.12):

$$W^{(0)}[\check{T}_{++},\check{W}_{+++}] = \min_{\{\mu_{--},\nu_{---}\}}\left(c\Gamma^{(0)}[\mu_{--},\nu_{---}] - \frac{1}{\pi}\int (\mu_{--}\check{T}_{++} + \nu_{---}\check{W}_{+++})\right). \tag{3.14}$$

Just as for pure gravity, the full effective action is, except for a coupling constant and a wavefunction renormalization, equal to the classical action[10,11]:

$$W[\check{T}_{++},\check{W}_{+++}] = Z_c W^{(0)}\left[Z_T\check{T}_{++}, Z_W\check{W}_{+++}\right], \tag{3.15}$$

4. Extended Gravity from Gauged WZW Models

In this section I will present a unifying treatment of extensions of the Virasoro algebra and the corresponding effective gravity theories[12–14]. The principle underlying this approach is quite simple. Consider a matter system, where I denote the matter fields collectively by ϕ, with as action $S[\phi]$ and with a set of n symmetry currents, denoted by $T_i[\phi]$, $i \in \{1,\cdots,n\}$, which forms an extended Virasoro algebra. The induced action in the light-cone gauge is defined by

$$e^{-\Gamma[\mu]} = \int [d\phi] e^{-S[\phi] - \frac{1}{\pi}\int \mu^i T_i[\phi]}, \tag{4.1}$$

where μ_i are sources. Alternatively μ_i can be viewed as chiral gauge fields or generalized Beltrami differentials. The effective action[§] is defined by:

$$e^{-W[\check{T}]} = \int [d\mu] e^{-\Gamma[\mu] + \frac{1}{\pi}\int \mu^i \check{T}_i} = \int [d\phi]\delta(T[\phi] - \check{T})e^{-S[\phi]}. \tag{4.2}$$

In order to evaluate this integral, one has to compute the Jacobian for going from $T[\varphi]$ to φ. Though this is usually impossible, we will be able to do it by realizing the "matter" sector, *i.e.* $S[\phi]$, by a WZW model for which a chiral, solvable

§Strictly speaking this is the generating functional for connected Greens functions of μ, which only upon a Legendre transform becomes the generating functional of 1PI Greens functions which is usually called the effective action.

group is gauged. The possible choices for the gauge group are determined by the inequivalent, non-trivial embeddings of $sl(2)$ in the Lie algebra.

Consider a (super) Lie algebra $\bar{g}$. Call the affine extension of $\bar{g}$ with level κ: $\hat{g}$. The affine algebra is realized by a Wess-Zumino-Witten theory with action $\kappa S^-[g]$. Given a nontrivial embedding of $sl(2)$ in $\bar{g}$, the adjoint representation of $\bar{g}$ branches into irreducible representations of $sl(2)$ which allows us to write the generators of $\bar{g}$ as $t_{(jm,\alpha_j)}$ where $j \in \frac{1}{2}\mathbb{N}$ labels the irreducible representation of $sl(2)$, m runs from $-j$ to j and α_j counts the multiplicity of the irreducible representation j in the branching. The $sl(2)$ generators $e_\pm$ and e_0 where $e_\pm \equiv t_{(1\pm1,0)}/\sqrt{2}$ and $e_0 \equiv t_{(10,0)}$, satisfy the standard commutation relations: $[e_0, e_\pm] = \pm 2e_\pm$ and $[e_+, e_-] = e_0$. The action of the $sl(2)$ algebra on the other generators is given by

$$\begin{aligned}
[e_0, t_{(jm,\alpha_j)}] &= 2m\, t_{(jm,\alpha_j)}, \\
[e_\pm, t_{(jm,\alpha_j)}] &= (-)^{j+m-\frac{1}{2}\pm\frac{1}{2}} \sqrt{(j\mp m)(j\pm m+1)}\, t_{(jm\pm1,\alpha_j)}.
\end{aligned} \tag{4.3}$$

The $sl(2)$ embedding introduces a natural grading on $\bar{g}$ given by the eigenvalue of e_0. I use the projection operators Π to project Lie algebra valued fields onto certain subsets of the $sl(2)$ grading, *e.g.* $\Pi_+\bar{g} = \{t_{(jm,\alpha_j)}|m > 0; \forall j, \alpha_j\}$, $\Pi_{\geq m}\bar{g} = \{t_{(jn,\alpha_j)}|n \geq m; \forall j, \alpha_j\}$, $\Pi_m\bar{g} = \{t_{(jm,\alpha_j)}|\forall j, \alpha_j\}$. All other conventions are as in previous papers[12].

The action S_1

$$S_1 = \kappa S^-[g] + \frac{1}{\pi x}\int str\, A_-\left(J_+ - \frac{\kappa}{2}e_- - \frac{\kappa}{2}[\tau, e_-]\right) + \frac{\kappa}{4\pi x}\int str[\tau, e_-]\partial_-\tau, \tag{4.4}$$

with the affine currents $J_+ = \frac{\kappa}{2}\partial_+ g g^{-1}$, the gauge fields $A_- \in \Pi_+\bar{g}$ and the "auxiliary" fields $\tau \in \Pi_{+1/2}\bar{g}$, is invariant under the gauge transformations

$$g \to hg \qquad A_- \to \partial_- h h^{-1} + hA_-h^{-1} \qquad \tau \to \tau + \Pi_{+\frac{1}{2}}\eta, \tag{4.5}$$

where $h = \exp\eta$, $\eta \in \Pi_+\bar{g}$.

The gauge fields A_- (Lagrange multipliers) impose the constraint $\Pi_- J_+ = \frac{\kappa}{2}e_- + \frac{\kappa}{2}[\tau, e_-]$. Calling the constrained current J_+^c, one performs the gauge transformation which brings J_+^c in the form $T + \frac{\kappa}{2}e_-$ where $T \in \ker ad\, e_+$, and obtain in this way the fields T which are gauge invariant modulo the constraints, *i.e.* modulo the equations of motion of the gauge fields A_-. They are of the form $T \propto \Pi_{\ker ad\, e_+} J_+ + \cdots$ These currents are coupled to sources and the action is modified to

$$S_2 = S_1 + \frac{1}{4\pi xy}\int str \mu T, \tag{4.6}$$

with the sources $\mu \in \ker ad\, e_-$. As the fields T are only gauge invariant modulo terms proportional to the equations of motion of the gauge fields, the resulting non-invariance terms in δS_2 are cancelled by modifying the transformation rules for the gauge fields. These modifications are proportional to the μ-fields and do not depend on the gauge fields themselves. Because the gauge fields occur linearly in Eq. (4.6), gauge invariance is restored.

In the next I will argue that the fields T generate an extended Virasoro algebra. A strong hint for this is the observation[15,16] that constraining a chiral

WZW current as $J_+ = T + \frac{\kappa}{2}$ where $T \in \ker ad\, e_+$, reduces the WZW Ward identities to the classical Ward identities of some extended Virasoro algebra with currents $T = T^{(j,\alpha_j)} t_{(jj;\alpha_j)}$ and $T^{(j,\alpha_j)}$ has conformal dimension $j+1$.

The functional $\Gamma[\mu]$

$$\exp -\Gamma[\mu] = \int [\delta g g^{-1}][d\tau][dA_-] \left(\mathrm{Vol}\,(\Pi_+\bar{g})\right)^{-1} \exp - \left(\mathcal{S}_2 - \frac{1}{4\pi x y} \int str \mu \hat{T}\right), \tag{4.7}$$

is, if T forms an extended Virasoro algebra, the induced action in the light-cone gauge of the corresponding extended gravity theory. The price paid for modifying the transformation rule of the gaugefields A_+ is that the gauge algebra only closes on-shell. Such a system calls for the Batalin-Vilkovisky formalism[17] to gauge fix it. Introducing ghostfields $c \in \Pi_+\bar{g}$ and anti-fields $J_+^* \in \bar{g}$, $A_-^* \in \Pi_-\bar{g}$, $\tau^* \in \Pi_{-1/2}\bar{g}$ and $c^* \in \Pi_-\bar{g}$, the solution to the BV master equation is given by:

$$\begin{aligned}\mathcal{S}_{\rm BV} &= \mathcal{S}_2 - \frac{1}{2\pi x}\int str c^* c c + \frac{1}{2\pi x}\int str J_+^* \left(\frac{\kappa}{2}\partial_+ c + [c, J_+]\right) + \frac{1}{2\pi x}\int str \tau^* c \\ &+ \frac{1}{2\pi x}\int str A_-^* \left(\partial_- c + [c, A_-] + \mu\text{-dependent terms}\right).\end{aligned} \tag{4.8}$$

The μ-dependent terms proportional to A_-^* absorb all complications arising from the non-invariance of T.

The gauge choice $A_- = 0$ is made by performing a canonical transformation which changes A_-^* into a field, the antighost $b \in \Pi_-\bar{g}$, and A_- into an antifield b^*. The gauge-fixed action reads:

$$\mathcal{S}_{\rm gf} = \kappa S^-[g] + \frac{\kappa}{4\pi x}\int str[\tau, e_-]\partial_-\tau + \frac{1}{2\pi x}\int str\, b\partial_- c + \frac{1}{4\pi x y}\int str \mu \hat{T}, \tag{4.9}$$

and the nilpotent BRST charge is:

$$Q = \frac{1}{4\pi i x}\oint str\left\{c\left(J_+ - \frac{\kappa}{2}e_- - \frac{\kappa}{2}[\tau, e_-] + \frac{1}{2}J_+^{\rm gh}\right)\right\}, \tag{4.10}$$

where $J_+^{\rm gh} = \frac{1}{2}\{b, c\}$.

The only unknown in the action is the current $\hat{T}$. This reflects the fact that I did not specify the explicit form of the μ dependent terms in Eq. (4.8). In order to guarantee BRST invariance of the action the currents $\hat{T}$ themselves have to be BRST invariant. This determines them up to BRST exact pieces.

Following initial studies of this system[18-20], the BRST cohomology of Q was solved in its full generality[14] using spectral sequence techniques[21]. I will omit most details here and just summarize the results. The fields, Φ, in the theory are assigned a double grading $[\Phi] = (k, l)$, $k, l \in \frac{1}{2}\mathbf{Z}$, with $k + l \in \mathbf{Z}$ the ghostnumber: $[J_+] = (m, -m)$ for $J_+ \in \Pi_m\bar{g}$, $m \in \frac{1}{2}\mathbf{Z}$, $[b] = (-m, m-1)$ for $b \in \Pi_{-m}\bar{g}$, $m > 0$, $[c] = (m, -m+1)$ for $c \in \Pi_m\bar{g}$, $m > 0$ and $[\tau] = (0, 0)$. The operator product expansions (OPE) are compatible with the grading. The BRST operator Q is decomposed as $Q = Q_0 + Q_1 + Q_2$ where

$$Q_0 = -\frac{\kappa}{8\pi i x}\oint strce_- \qquad Q_1 = -\frac{\kappa}{8\pi i x}\oint strc[\tau, e_-], \tag{4.11}$$

such that $[Q_0]=(1,0)$, $[Q_1]=(\frac{1}{2},\frac{1}{2})$ and $[Q_2]=(0,1)$. One computes that $Q_0^2=Q_2^2=\{Q_0,Q_1\}=\{Q_1,Q_2\}=Q_1^2+\{Q_0,Q_2\}=0$, but $Q_1^2=-\{Q_0,Q_2\}=\frac{\kappa}{32\pi i x}\oint str\{c[\Pi_{1/2}c,e_-]\}$. A first fact is that, because of the existence of the subcomplex with trivial cohomology and generated by $\{b,\Pi_-\hat{J}_z-\frac{\kappa}{2}[\tau,e_-]\}$, the full cohomology is isomorphic to the cohomology computed on the reduced complex generated by $\{\Pi_{\geq 0}\hat{J}_+,\tau,c\}$. I denoted the total currents by $\hat{J}_+=J_++J_+^{gh}$. Note that the OPEs also close on this subcomplex.

Computing a spectral sequence, one can show[14] that the cohomology is generated by $\hat{T}\equiv\sum_{j,\alpha_j}\hat{T}^{(j,\alpha_j)}t_{(jj;\alpha_j)}\in\ker ad\,e_+$ and $\hat{T}^{(j,\alpha_j)}$ has the form

$$\hat{T}^{(j,\alpha_j)}=\sum_{r=0}^{2j}\hat{T}_r^{(j,\alpha_j)}, \tag{4.12}$$

where $\hat{T}_r^{(j,\alpha_j)}$ has grading $(j-\frac{r}{2},-j+\frac{r}{2})$. The leading term is of the form

$$\hat{T}_0^{(j,\alpha_j)}=C^j\left\{\hat{J}_+^{(jj;\alpha_j)}+\frac{\kappa}{4}\sum_{\alpha_0}\delta_{j,0}\delta_{\alpha_j,\alpha_0}[\tau,[e_-,\tau]]^{(00;\alpha_0)}\right\}, \tag{4.13}$$

where the normalization constant C will be fixed later on and the remaining terms are recursively determined by a generalized tic-tac-toe construction:

$$Q_0\hat{T}_r^{(j,\alpha_j)}=-Q_1\hat{T}_{r-1}^{(j,\alpha_j)}-Q_2\hat{T}_{r-2}^{(j,\alpha_j)}. \tag{4.14}$$

The OPEs of $T^{(j,\alpha_j)}$ close modulo BRST exact terms. However, because $T^{(j,\alpha_j)}$ has ghostnumber 0, a BRST exact term must be derived from a ghostnumber -1 field. As the cohomology was computed on a reduced complex which has no fields of negative ghostnumber, one concludes that the operator algebra of $T^{(j,\alpha_j)}$ closes!

One can also show that the map $T^{(j,\alpha_j)}\to T_{2j}^{(j,\alpha_j)}$ is an algebra isomorphism[14,20]. This is the so-called quantum Miura transformation.

The only thing which remains to be shown is that the algebra is an extension of the Virasoro algebra, *i.e.* it does contain the Virasoro algebra. One can show that the energy-momentum tensor¶:

$$\begin{aligned}\hat{T}^{\rm EM} &= \frac{\kappa}{x(\kappa+\tilde{h})}\left(str\left\{\hat{J}_ze_-\right\}+str\left\{[\tau,e_-]\hat{J}_z\right\}+\frac{1}{\kappa}str\left\{\Pi_0(\hat{J}_z)\Pi_0(\hat{J}_z)\right\}+\frac{\kappa+\tilde{h}}{\kappa}str\left\{e_-\partial\hat{J}_z'\right\}\right.\\ &\left.+\frac{1}{\kappa}str\left\{\left[\Pi_0(t^A),\left[\Pi_0(t_A),\partial\hat{J}_z'\right]\right]e_-\right\}-\frac{\kappa+\tilde{h}}{4}str\left\{[\tau,e_-]\partial\tau\right\}\right),\end{aligned} \tag{4.15}$$

satisfies the Virasoro algebra with

$$c=\frac{1}{2}c_{\rm crit}-\frac{(d_B-d_F)\tilde{h}}{\kappa+\tilde{h}}-6y(\kappa+\tilde{h}), \tag{4.16}$$

where $c_{\rm crit}$ is the critical central extension of the algebra under consideration, $c_{\rm crit}=\sum_{j,\alpha_j}(-)^{(\alpha_j)}(12j^2+12j+2)$, y is the index of the $sl(2)$ embedding, *i.e.* the ratio of the

¶The first term in Eq. (4.15) is $\hat{T}_0^{(1,0)}$, the second term $\hat{T}_1^{(1,0)}$ and the remainder forms $\hat{T}_2^{(1,0)}$. Requiring that this forms the Virasoro algebra in the standard normalization fixes C: $C=\frac{4y\kappa}{\sqrt{2(\kappa+\tilde{h})}}$.

length squared of the longest root of $\bar{g}$ with the length squared of the $sl(2)$ root, d_B, d_F respectively, is the number of bosonic, fermionic respectively, generators of $\bar{g}$ and $\tilde{h}$ is the dual Coxeter number of $\bar{g}$. Adding a BRST exact term to Eq. (4.15), one gets the energy-momentum tensor in the familiar KPZ form:

$$\begin{aligned} \tilde{T}^{\mathrm{IMP}} &\equiv \frac{1}{x(\kappa+\tilde{h})} str J_+ J_+ - \frac{1}{8xy} str e_0 \partial_+ J_+ - \frac{\kappa}{4x} tr\left([\tau, e_-]\partial_+\tau\right) \\ &+ \frac{1}{4x} strb[e_0, \partial_+ c] - \frac{1}{2x} strb\partial_+ c + \frac{1}{4x} str\partial_+ b[e_0, c]. \end{aligned} \qquad (4.17)$$

A simple example is provided by the embeddings of $sl(2)$ in $sl(3)$. There are two inequivalent embeddings of $sl(2)$ in $sl(3)$. For the first, the adjoint of $sl(2)$ branches according to $\underline{8} \rightarrow [3] \oplus [5]$ which gives rise to the W_3 algebra containing a dimension 2 current, the energy-momentum tensor, and a conformal dimension 3 current. The other embedding is characterized by $\underline{8} \rightarrow [1] \oplus [2] \oplus [2] \oplus [3]$ and corresponds to the $W_3^{(2)}$ algebra containing the energy-momentum tensor, two *bosonic* dimension 3/2 currents and a $U(1)$ current.

The effective action in the light-cone gauge, $W[\tilde{T}]$, of the corresponding gravity theory[12] is defined by

$$\exp -W[\tilde{T}] = \int [d\mu] exp - \left(\Gamma[\mu] - \frac{1}{4\pi xy}\int str \mu \tilde{T}\right). \qquad (4.18)$$

where $\Gamma[\mu]$ was given in Eq. (4.7). In order to compute the effective action, one first feeds the information gained in the previous analysis back into the solution of the BV master equation and one then chooses a different gauge: $\tau = \Pi_+[e_+, J_+] = 0$. To achieve this, one makes a canonical transformation in Eq. (4.8) which interchanges fields and anti-fields for $\{\tau, \tau^*\}$ and $\{\Pi_+[e_+, J_+], \Pi_-[e_-, J_+^*]\}$. One finds

$$W[\tilde{T}] = \kappa_c S_-[g], \qquad (4.19)$$

where $\kappa_c = \kappa + 2\tilde{h}$ and I used $[\delta g g^{-1}] = [dJ_+] \exp\left(-2\tilde{h} S^-[g]\right)$. From Eq. (4.16) one gets the level as a function of the central charge:

$$12y\kappa_c = 12y\tilde{h} - \left(c - \frac{1}{2}c_{\mathrm{crit}}\right) - \sqrt{\left(c - \frac{1}{2}c_{\mathrm{crit}}\right)^2 - 24(d_B - d_F)\tilde{h}y}. \qquad (4.20)$$

Eq. (4.20) provides an all-order expression for the coupling constant renormalization. The WZW model in Eq. (4.19) is constrained by

$$\partial_+ g g^{-1} + \frac{1}{4xy} str\left\{\Pi_{\mathrm{NA}}\left(\partial_+ g g^{-1}\right)\right)\Pi_{\mathrm{NA}}\left(\partial_+ g g^{-1}\right)\} e_+ = e_- + \frac{1}{\kappa+\tilde{h}} \sum_{j,\alpha_j} \frac{1}{2^{\frac{3}{2}j-1} y^j} \tilde{T}^{(j\alpha_j)} t_{(jj,\alpha_j)} \qquad (4.21)$$

where $\Pi_{\mathrm{NA}}\bar{g}$ is the projection on the centralizer of $sl(2)$ in $\bar{g}$. I also used[12,22] $J_+ = \frac{\alpha_\kappa}{2}\partial_+ g g^{-1}$ with $\alpha_\kappa = \kappa + \tilde{h}$.

5. Conclusions

I showed that one can associate an extended Virasoro algebra with every non-trivial embedding of $sl(2)$ in a semi-simple (super) Lie algebra. The algebra is realized as a WZW model where a chiral, solvable group is gauged—the gauge group being determined by the $sl(2)$ embedding. Algebras that can be obtained in this way can be called "simple" extensions of the Virasoro algebra. All other extended Virasoro algebras, can be called "non-simple" extended Virasoro algebras and I conjecture that they can be obtained from "simple" ones by canonical manipulations such as adding free fermions or $U(1)$ currents[23], orbifolding[24], further gauging of affine subalgebras, etc. Relating the representation theory of the extended Virasoro algebra with the representation theory of the underlying WZW model remains a very interesting, open problem.

From Eq. (4.20), one finds that no renormalization of the coupling constant beyond one loop occurs if and only if either $d_B = d_F$ or $\tilde{h} = 0$ (or both). One gets $d_B = d_F$ for $su(m \pm 1|m)$, $osp(m|m)$ and $osp(m+1|m)$ and $\tilde{h} = 0$, for $su(m|m)$, $osp(m+2|m)$ and $D(2,1,\alpha)$. This hints towards the existence of a generalized non-renormalization theorem whose precise nature remains to be elucidated.

The close relation between this method of constructing extended Virasoro algebras and extended 2D gravity theories, provides a good starting point for the study of non-critical strings. Given an embedding of $sl(2)$ in $\bar{g}$, one considers the corresponding (p,q) minimal model as the matter sector of the string theory. Its central charge c_M is given by Eq. (4.16), where $\kappa_M + \tilde{h} = p/q$. In order to cancel the conformal anomaly, one needs to supplement the matter sector by a gauge sector whose central charge c_L is again given by eq, (4.16) but now $\kappa_L + \tilde{h} = -p/q$. The corresponding W string is now determined by currents $T_{\rm tot} = T_M + T_L$ and a BRST charge $Q = \frac{1}{2\pi i} \oint strc(T_{\rm tot} + \frac{1}{2} T_{ghost})$, where the ghost system contributes $-c_{\rm crit}$ to the central charge. In order to explicitly perform this program, a covariant formulation is needed so that one is not restricted to the light-cone gauge but that the conformal gauge, which is more convenient, can be used as well. This can again be achieved using WZW like techniques[25].

A most challenging problem is the understanding of the geometry behind the extensions of $d = 2$ gravity. We saw that the Virasoro algebra appeared as the algebra of residual symmetry after gauge fixing a theory invariant under general coordinate transformations in $d = 2$. A similar statement for extensions of the Virasoro algebra remains to be found. Finally, a most exciting application of the methods developed in this paper would be the study of reductions of continuum algebras[26] which will lead to integrable theories in $d > 2$! Work in these directions is in progress.

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