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Pricing and Market Segmentation for Software Upgrades

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Abstract

Upgrades are endemic in the software industry and create the possibility that customers might either postpone purchase or buy early on and never upgrade: When will a customer upgrade? Is it better to upgrade now or to wait for an improved version? When should we release an improved product? How much should we charge for each version? Should we give discounts on upgrades to existing customers? Will today's sales be cannibalized by the anticipated improved version?

We focus on pricing and how it is affected by the degree to which a product is improved between versions. In particular, we are interested how the firm should price different versions of its product and whether it should offer "upgrade discounts" to existing customers. To address these issues, we analyze a two-period model in which a firm sells an initial version of its product in the first period and an improved version the second. In each period, the firm (i.e., the software supplier) selects the selling prices, and customers decide whether to purchase. Customers may purchase the product in either or both periods and, at the firm's discretion, are given a discounted price if they repurchase/upgrade in the second period. We solve this model for subgame perfect equilibrium prices and purchasing decisions and investigate how equilibrium prices, profits, and cash flows are influenced by the degree of product improvement. We also uncover a number of managerial insights; for example, equilibrium pricing induces all of the first period purchasers to upgrade to the improved product in the second period.

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1 Introduction

Upgrades are endemic in the software industry and create the possibility that customers might either postpone purchase or buy early on and never upgrade producers and customers: When will a customer upgrade? Is it better to upgrade now or to wait for an improved version? When should we release an improved product? How much should we charge for each version? Should we give discounts on upgrades to existing customers? Will today's sales be cannibalized by the anticipated improved version?

We focus on pricing and how it is affected by the degree to which a product is improved between versions. In particular, we are interested how the firm should price different versions of its product and whether it should offer "upgrade discounts" to existing customers. To address these issues, we analyze a two-period model in which a firm sells an initial version of its product in the first period and an improved version the second. In each period, the firm (i.e., the software supplier) selects the selling prices, and customers decide whether to purchase. Customers may purchase the product in either or both periods and, at the firm's discretion, are given a discounted price if they repurchase/upgrade in the second period. We solve this model for subgame perfect equilibrium prices and purchasing decisions and investigate how equilibrium prices, profits, and cash flows are influenced by the degree of product improvement. We also uncover a number of managerial insights; for example, equilibrium pricing induces all of the first period purchasers to upgrade to the improved product in the second period.

This article proceeds as follows: Section 2 reviews related literature. Section 3 defines the model in detail. Sections 4 and 5 derive the equilibrium solution and provide a number of structural results. Section 6 illustrates these results for a particular set of parameters/assumptions, and section 7 concludes the discussion. Proofs are relegated to the appendix (section 8).

2 Related Literature

While we know of no other papers that solve this problem in closed form and quantify the profit consequences of upgrades and upgrade pricing, ours is not the first research to address the upgrade pricing issue. One section of Dhebar(1994) touches upon upgrade pricing if only to say that "intertemporal discrimination" does not exist at equilibrium when the product improves sufficiently

rapidly. Kornish(2001) does not consider upgrade pricing but does show that this form of price discrimination is always seen at equilibrium when upgrade pricing is disallowed. Of course, absence of intertemporal discrimination does not imply that rapid product improvement will prove harmful to the firm. Quite the contrary; as we establish later, for rapid product improvement the firm's interests are best served by pricing schemes that avoid this form of discrimination. Bhattacharya et al(2003) considers optimal product sequencing and briefly visits the upgrade pricing issue through an optimization program. A notable element of our work is that we allow upgrades. That is, we assume that purchasers of the initial version of the product can upgrade to the improved version; both Dhebar(1994) and Bhattacharya et al(2003) disallow upgrade purchases.

Fudenberg and Tirole(1998) study the upgrades and trade-ins issue by considering different information structures that the monopolist has about individual customers. Their "semi-anonymouse" case, in which the later period prices are bound by an arbitrage constraint, is the closest to our model. The arbitrage constraint, which we also employ, states that the upgrade price paid by existing cutomers cannot exceed the price charged to new customers. They do not solve this problem but conjecture that there would be a product improvement threshold over which special upgrade pricing would not be optimal. We compute this threshold starting with some assumptions on consumer behavior and also provide additional results relating the optimal pricing strategy to the entire range of product improvement. Ellison and Fudenberg(2000) study the effects of upgrades on social welfare, particularly in the context of network externality. Padmanabhan et al (1997) investigate the positioning decision across periods given network externality effects for a fixed set of homogeneous customers. Again, the current essay assumes heterogeneous consumers but not network externalities.

More generally, our work is related to product development, addressed in both the marketing and operations management literature, and durable goods problems from economics. Within the former, the topics that closely relate to this essay are new product performance, time-to-market, and their connection with sales (and hence revenue/profits). Cohen et al(1996) analyze the relationship between new product performance and time-to-market. They derive a minimal product improvement rate given a target product performance. Product performance is related to demand rate by means of a utility function but the demand rate is assumed and not related to pricing issues. Norton & Bass(1987) model customer adoption of successive multiple generations of products by extending the well-known Bass model with demand rates assumed known (i.e., pricing decisions are not considered). Krishnan et al (1999) incorporate pricing into the Bass model but only for

a single generation of the product. We compute demand rate by relating it to a customer utility function that is a function of product attributes (performance) and price.

Again, the current problem is related to the durable goods monopolist problem in the economics literature. Early work in this area is by Coase(1972) who postulates that a monopolist selling a durable good to rational consumers cannot capture monopoly profits. Consumers will look ahead, anticipate a decreasing price trajectory and hence postpone their purchase till price equals cost. Stokey(1981) models this process over an infinite horizon for a non-depreciating durable good and confirms the results. Bulow(1982) uses a two-period model with a second hand market for a durable good and shows that while a monopolist renter can capture monopoly profits, a monopolist seller cannot do the same. Software renting is a new business model practised by *application service providers* (ASPs) and may provide greater economic benefits as compared to selling, but we pursue that line of research elsewhere.

3   The model

First, a brief description of the two-period model. In the first period, the firm selects a price p_1 at which it offers the initial version of its software. In the second period, the firm offers an improved version at price p_2; however, customers who purchased the initial version are offered a lower price p_u.

Customers may purchase the product in one or both periods. Customers who purchase the initial version enjoy the software for both periods, and, if they choose to upgrade, they use the improved version in the second period. Customers who only purchase in the second period enjoy only one period of use, but it is of the improved version. Of course, some customers may choose not to buy either version. This is thus a "game" between the firm and its potential customer base. We assume complete information and employ the subgame perfect equilibrium concept. The remainder of this section describes the model in detail and establishes some initial results/expressions that are useful later.

Customers, product valuation, purchasing behavior: Customers are heterogenous with respect to the value they ascribe to the software; each is of a particular "type" that is denoted by

a real number θ. We use the term "customer segment" or just "segment" to mean a set of θ-values and refer to a customer whose type is θ as a "θ-customer." A θ-customer is assumed to value the product as follows:

$$
\begin{aligned}
\theta \cdot U_1 \quad &= \quad \text{utility, expressed in monetary units, from using the initial version} \\
&\qquad \text{during the first period} \\
\theta \cdot U_2 \quad &= \quad \text{utility from using the improved version during the second period} \\
\theta \cdot U_{12} \quad &= \quad \text{utility from using the initial version during the second period} \\
\delta \quad &= \quad \text{discount factor for translating cash flows and utilities across time}
\end{aligned}
$$

For example, a customer who purchases the product in the first period and then upgrades in the second period enjoys utility of $\theta \cdot U_1$ in the first period, $\theta \cdot U_2$ in the second, and $\theta \cdot (U_1 + \delta U_2)$ overall (after discounting). If the customer doesn't upgrade, second period utility is instead $\theta \cdot U_{12}$.

The manner in which customer types are spread across the real line is described by a measure with density[1] f, distribution function F, and hazard rate function h_f; thus $F'(\theta) = f(\theta)$ and $h_f(\theta) = \frac{f(\theta)}{1-F(\theta)}$. For convenience, we also use $F^c(\theta) \triangleq 1 - F(\theta)$.

For any customer segment[2] S (i.e., a set of θ-values), the segment's size $|S|$ is

$$|S| \triangleq \int_S f(\theta)\, d\theta.$$

When S is an interval (a, b) this revenue can be written $(F(b) - F(a))$, and $|S|$ is assumed to equal one when S is the entire real line. Under this assumption f satisfies the mathematical definition of a *probability* density function. This is a matter of convenience in that it allows the application of probability theory to these functions. Otherwise, the assumption is just scaling and is without loss of generality. We will also use g and G to denote the pdf and cdf of an arbitrary distribution, and we assume that all distributions herein place strictly positive mass on the positive reals.

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[1] Expositionally, we do not differentiate between a distribution and the functions that define it. For example, f will refer to both the distribution of customer density and the pdf of that distribution.

[2] We will henceforth adopt a convention of using the term "segment" and notation "S" to mean the *interior* of a set of θ-values. This reduces notation and avoids ambiguities related to stong vs. weak inequalities. As a consequence of this assumption, the term "interval" means an open interval.

    This assumption does not affect optimal/equilibrium pricing because it only eliminates sets of zero measure from the firm's revenue calculations.

Customers' equilibrium criteria: With customers valuing the product in the manner stated above, subgame perfection gives the following criteria that defines equilibrium buying behavior by the customers:

Period 2: Customers who purchased in the first period will upgrade if the upgrade price p_u is less than $\theta (U_2 - U_{12})$, the benefit gained over using the version that they already own. Customers who did not buy in period 1 will now buy if the purchase price p_2 is less than θU_2, the utility they would derive from using the improved version over period 2.

Period 1: Customers purchase if

$$\theta (U_1) - p_1 + \delta \max [\theta (U_2 - U_{12}) - p_u, U_{12}] \geq \delta \max [\theta U_2 - p_2, 0]$$

where p_2 and p_u are equilibrium prices in the period 2 subgame. The left hand side of this is the utility net of price that the θ-customer will derive from using the product in the first period $(\theta (U_1) - p_1)$ plus the discounted period 2 utility resulting from an optimal upgrade decision. The right-hand side is the discounted utility earned from declining to purchase in period 1 assuming an optimal second period buying decision then follows.

All-in-all, there are four combinations of decisions available to each customer, so the customers partition themselves into four sets. Notation, descriptions, and brief monikers for these sets are given in the table below.

Notation	Description	Moniker	Utility function
$S_{b,u}$	buys initial version in period 1 – upgrades in period 2	*buy/upgrade*	$u_{b,u} (\theta) \triangleq \theta (U_1 + \delta U_2) - p_1 - \delta p_u$
$S_{b,\emptyset}$	buys initial version, does not upgrade	*buy/decline*	$u_{b,\emptyset} (\theta) \triangleq \theta (U_1 + \delta U_{12}) - p_1$
$S_{\emptyset,b}$	defers purchase until period 2	*decline/buy*	$u_{\emptyset,b} (\theta) \triangleq \theta \delta U_2 - \delta p_2$
$S_{\emptyset,\emptyset}$	never buys	*decline/decline*	$u_{\emptyset,\emptyset} (\theta) \triangleq 0$

Equilibrium criteria for the firm: For the firm, payoff is determined by the prices that it selects and the number of customers attracted by these prices – if S is a segment that pays price

p_i in a period then the payoff from that segment is $p_i\,|S|$. Calculation of the firm's payoffs follows naturally. π_1 is the firm's first period payoff, π_u is the payoff in period 2 from the upgrading segment, and π_2 is the period 2 payoff from new sales:

$$\pi_1 \triangleq p_1\left(|S_{b,u}| + |S_{b,\emptyset}|\right), \quad \pi_u \triangleq p_u\,|S_{b,u}|, \quad \pi_2 \triangleq p_2\left|S_{\emptyset,b}\right|,$$

The firm's overall objective is to maximize its total discounted payoff. Given the timing of decisions in this model, the following criteria define equilibrium pricing by the firm.

Period 2:   Having observed customers' period 1 purchasing decisions, the firm chooses p_u and p_2 to maximize $\pi_u + \pi_2$ subject to $p_u \le p_2$. In computing these values, the firm anticipates equilibrium period-2 purchasing by the customers.

Period 1:   The firm selects a price p_1 that maximizes

$$\pi_1 + \delta \max_{p_u,p_2} \left(\pi_2 + \pi_u\right), \quad s.t. \quad p_u \le p_2.$$

1. Here, the firm assumes equilibrium purchasing by the customers, and the max indicates that the firm anticipates selecting p_u, p_2 to achieve an equilibrium in the period 2 subgame.

Foundational results: Our first proposition is a pair of structural results that greatly simplifies the later analysis. Part (i) partially characterizes customers' optimal buying behavior, and (ii) gives the segmentation structure that results.

Proposition 1 *For any p_1, p_u, p_2:*

(i) Each θ-customer's equilibrium criteria are equivalent to placing the customer in segment $S_{i,j}$ where[3]

$$(i,j) = \arg\max_{(i,j)\in\{(b,u),(b,\emptyset),(\emptyset,b),(\emptyset,\emptyset)\}} \left[u_{b,u}\left(\theta\right),\, u_{b,\emptyset}\left(\theta\right),\, u_{\emptyset,b}\left(\theta\right),\, u_{\emptyset,\emptyset}\left(\theta\right)\right]. \tag{1}$$

(ii) When customers purchase in this manner, the customers are partitioned as:

$$S_{\emptyset,\emptyset} < S_{\emptyset,b} < S_{b,\emptyset} < S_{b,u} \quad \text{if} \quad U_2 < \tfrac{1}{\delta}U_1 + U_{12} \tag{2}$$

$$S_{\emptyset,\emptyset} < S_{b,\emptyset} < S_{\emptyset,b} < S_{b,u} \quad \text{otherwise} \tag{3}$$

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[3] When multiple $u_{i,j}$ functions "tie" for the maximum value, arbitrarily place the θ-customers into one of the $S_{i,j}$ segments associated with those functions.

*The first of these cases occurs for smaller values of U_2, so we refer to these as the **moderate improvement** and **large improvement** cases respectively.*

Part (i) of this proposition gives customers' optimal responses to any pricing strategy employed by the firm across both periods. As we will focus on solving and characterizing equilibrium in this game, we henceforth assume that customers always purchase as specified by the proposition. Part (ii) shows that the "segmentation scheme" that results from optimal purchasing by customers has an appealing structure. Notably, this structure is independent of prices and also of the distribution of customers (i.e., the order of the segments depends only on U_1, U_2, U_{12}, δ). This is, essentially, because the ordering of segments depends only on the slopes of $u_{i,j}$'s constituent functions $u_{b,u}$, $u_{b,\emptyset}$, $u_{\emptyset,b}$, $u_{\emptyset,\emptyset}$; and these slopes depend on the degree of product improvement only.

This allows a very parsimonious representation of the segmentation scheme – all four segments can be specified by stating just the three thresholds between them. α, β, γ will denote these these thresholds as defined by:

$$\text{moderate improvement case:} \quad S_{\emptyset,\emptyset} \leq \alpha \leq S_{\emptyset,b} \leq \beta \leq S_{b,\emptyset} \leq \gamma \leq S_{b,u}$$
$$\text{large improvement case:} \quad S_{\emptyset,\emptyset} \leq \alpha \leq S_{b,\emptyset} \leq \beta \leq S_{\emptyset,b} \leq \gamma \leq S_{b,u}$$

The following corollary to proposition 1 follows directly from these definitions.

Corollary 1 *For a moderate improvement, and assuming optimal purchasing by customers:*

(i) $\alpha < \beta$ *if and only if* $S_{\emptyset,b} \neq \emptyset$, *and* $\gamma > \beta$ *if and only if* $S_{b,\emptyset} \neq \emptyset$.

(ii) $\alpha = \min\left[\beta, \frac{p_2}{U_2}\right]$ *and* $\gamma = \max\left[\beta, \frac{p_u}{U_2 - U_{12}}\right]$.

(iii) $\pi_1 = p_1\left(1 - F\left(\beta\right)\right)$, $\pi_2 = p_2\left(F\left(\beta\right) - F\left(\alpha\right)\right)$,   *and* $\pi_u = p_u\left(1 - F\left(\gamma\right)\right)$

For a large improvement:

(iv) $\alpha < \beta$ *if and only if* $S_{b,\emptyset} \neq \emptyset$, *and* $\gamma > \beta$ *if and only if* $S_{\emptyset,b} \neq \emptyset$.

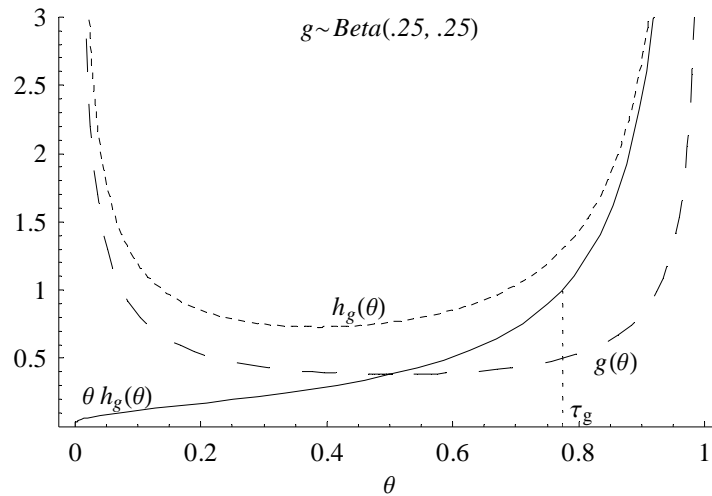Figure 1: Beta distribution properties

(v) If $\alpha < \beta$ then $p_1 = \beta\left(U_1 + \delta U_{12}\right)$. If $\gamma > \beta$ then $p_1 + \delta\left(p_u - p_2\right) = \beta U_1$.

(vi) $\pi_1 = p_1\left(1 - F\left(\gamma\right) + F\left(\beta\right) - F\left(\alpha\right)\right)$, $\pi_2 = p_2\left(F\left(\gamma\right) - F\left(\beta\right)\right)$,   and $\pi_u = p_u\left(1 - F\left(\gamma\right)\right)$.

The next section, which solves for equilibrium, uses one additional assumption and a lemma. We assume that f is in the following class of functions: C is the class of continuous probability distributions g with positive mass on $\Re > 0$ and for which $\theta \cdot h_g\left(\theta\right)$ is increasing in θ. Fortunately, this class is very large. It includes the class of increasing hazard rate distributions that has been widely studied in the equipment reliability literature and that includes uniform, normal, and exponential distributions. It also includes a very large number of other distributions; for example, as illustrated by figure 1, the $Beta\left(.25, .25\right)$ distribution is in class C – as indicated by the upward sloping $\theta \cdot h_g\left(\theta\right)$ curve – even though neither its density $g\left(\theta\right)$ nor hazard rate $h_g\left(\theta\right)$ are monotonic. As shown in the lemma below, this class also includes truncations of all other distributions in the class. Broadly, the function $\theta \cdot h_g\left(\theta\right)$ is sometimes called a "generalized failure (or hazard) rate" of a distribution, and, under modest regularity conditions, it is increasing in θ if and only if g is log-concave (Lariviere 2004).

Next, for g in this class, define τ_g as the solution to[4]

$$\tau_g \cdot h_g\left(\tau_g\right) = 1. \tag{4}$$

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[4]This definition presumes a continuous hazard rate but is easily generalized without loss of generality.
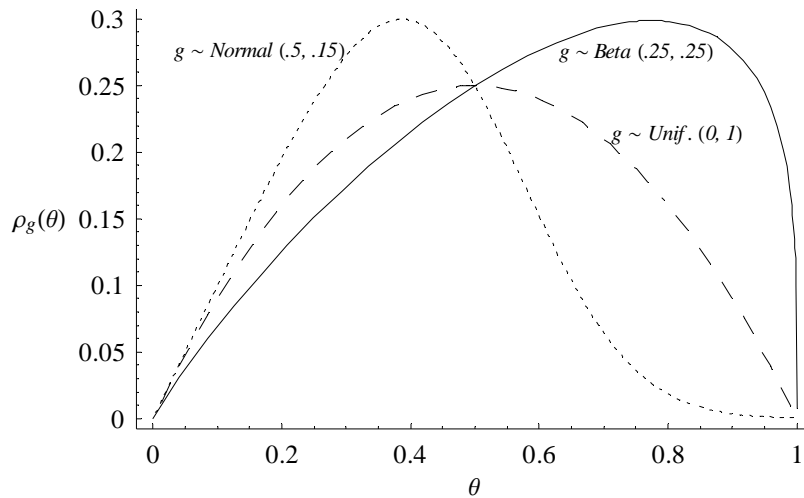
Figure 2: ρ_g examples

For any distribution g, let the pdf g_b (and cdf G_b) denote the distribution that results from truncating g above at b. That is, $g_b(\theta) = \frac{g(\theta)}{G(\theta)}$ for $\theta < b$. Thus, f_b and F_b refer to truncations of the customer density distribution.

One final definition – for distribution $g \in C$ (equivalently $G \in C$), define the function ρ_g by

$$\rho_g(\theta) \triangleq \theta G^c(\theta). \tag{5}$$

Figure 2 illustrates this function for several distributions in class C.

To interpret the function ρ_g, consider the following "single-period-single-product" version of the model. The firm offers a product to θ-customers who each derive utility θU from using the product (there is no second period and no second version). The θ-values are distributed as per distribution function G, and the firm's problem is to find the price p the maximizes revenue. If customers behave optimally, each customer will purchase the item if $\theta U - p > 0$, the firm will sell $G^c\left(\frac{p}{U}\right)$ units and receive revenue of $pG^c\left(\frac{p}{U}\right)$. Equivalently, the firm could set price by specifying the θ of the "marginal customer" thereby establishing price $p = \theta U$ and revenue $\theta U G^c(\theta)$. Comparing this with (5), we can think of ρ_g as the firm's equilibrium revenue in this model after normalizing by the value of the product.

The lemma below gives several foundational results about class C, customer distributions, and the τ_g function. Part (ii) of the lemma extends the above analogy – the firm maximizes revenue when it prices such that $\theta = \tau_g$ for the marginal customer.

Lemma 1 *Let g (equivalently G) be a distribution in class C.*

(i) Class C is closed to truncation – so $g_b \in C$.

(ii) $\rho_g(\theta)$ is strictly quasiconcave in θ and is maximized when $\theta = \tau_g$.

(iii) $\rho_g(\theta)$ is concave at all $\theta \le \tau_g$.

(iv) If g is a uniform, beta, normal, lognormal, logistic, or exponential distribution, then $\rho_g(\theta)$ is concave-convex. It has a concave region at lower values of θ followed by a (possibly empty) convex region.

Corollary: *(ii) through (iv) are inherited by the truncated distribution g_b. That is, on the domain $\theta \le b$: the function $\rho_{g_b}(\theta)$ equals $\theta \left(1 - \frac{G(\theta)}{G_b(\theta)}\right)$; it is strictly quasiconcave in θ, maximized when $\theta = \tau_{g_b}$, and concave for $\theta \le \tau_{g_b}$; furthermore, ρ_g is concave at τ_{g_b} for every b.*

The "τ" values will be important when deriving optimal/equilibrium prices. In this regard, the fact that C is closed to truncation – part (i) of the lemma and the corollary – are important when finding the optimal p_2. This importance follows from the fact that the set of customers in period 2 who will potentially pay p_2 is a subset of the original customer set. Nonetheless, (i) implies that the distribution of these customers, which is a truncation of the original distribution of customer types, remains in class C.

Part (ii) of the lemma gives the two important properties of class C. First is quasiconcavity of the normalized payoff function ρ_g when g is in this class; this guarantees that first order optimality conditions are sufficient for optimizing ρ_g. Second is that τ_g, easily found via (4), solves this optimization. Additionally notable is that τ_g, the optimizer of ρ_g, depends only on the distribution of customers – it is independent of the value of the product. (iii) and (iv) show that ρ_g has a couple of other appealing structural properties for $g \in C$. Generally, it is concave up to at least the optimizer τ_g (part (iii)), and, for several commonly used distributions, this concave region may only be followed by a single convex region. Several related properties are also useful to note:

1. Via an application of the implicit function theorem to equation (4), the derivative of τ_{g_b} with

respect to the truncation point b can be written as

$$\frac{d}{db}\tau_{g_b} = \frac{g(b)}{2f(\tau_{g_b}) + \tau_{g_b} f'(\tau_{g_b})}.$$

2. Furthermore, $0 < \frac{d}{db}\tau_{g_b} < 1$, and one implication of this is that $\tau_{g_b} < \tau_g$ for $g \in class \quad C$.

3. The inclusion of distribution functions for which closed form expressions do not exist disallows solving for the τ variables in closed form. We will consequently give equilibrium results and solutions in less convenient, albeit concise, implicit forms.

With these results in place we turn to the actual derivation and characterization of equilibria. The next section considers cases of moderate improvements to the product and section 5 addresses cases of large improvements.

4 Equilibrium for moderate improvements $\left(U_2 < \frac{1}{\delta}U_1 + U_{12}\right)$

We now derive conditions that define an equilibrium for cases of moderate improvements to the product. Using the typical backward recursion approach, we establish period 2 results first and then move to period 1.

4.1 Equilibrium in the period 2 subgame

The firm's problem in this subgame is to find p_2 and p_u that maximize $\pi_u + \pi_2$. By the first corollary, equilibrium is achieved when the firm selects p_u and p_2 to solve

$$\max_{p_u,p_2} [p_u (1 - F(\gamma)) + p_2 (F(\beta) - F(\alpha))] \quad s.t. \quad p_u \le p_2. \tag{6}$$

Since β separates the period 1 buyers from non-buyers, it is known in period 2 when the firm chooses p_u and p_2. When solving for equilibrium in the period 2 subgame, we take period 1 decisions as given, so β in (6) is treated as a constant. In contrast, α and γ are determined by customers' reactions to p_u and p_u and so are endogenous to period 2. Next, define p_2^* and p_u^* by

$$p_2^* \triangleq \arg\max_{p_2} [\pi_2] = \arg\max_{p_2} [p_2 (F(\beta) - F(\alpha))] \tag{7}$$

$$p_u^* \triangleq \arg\max_{p_u} [\pi_u] = \arg\max_{p_u} [p_u (1 - F(\gamma))]. \tag{8}$$

(p_2^* and p_u^* can be thought of as functions of β, but we suppress this dependence to minimize notation.) Given β, α is independent of p_u, and γ is independent of p_2. Thus p_2^* and p_u^* jointly optimize (6) whenever its pricing constraint is non-binding (i.e., whenever $p_u^* \geq p_2^*$).

We can thus distinguish the *upgrade discounting case* from the *shared price case* as follows. When $p_u^* < p_2^*$, we have the upgrade discounting case, and it is immediate that p_u^* and p_2^* are the equilibrium prices. The shared price case results when this inequality is reversed. In the shared price case, the pricing constraint is binding at equilibrium, so the firm selects a *shared price p_s* that is offered to all customers. We now consider these cases separately.

The upgrade discounting case: A first observation is that it would be suboptimal to set p_2 so high that α equals β because that would drive π_2 to 0. It would also be suboptimal to set p_u any lower than the price at which $\gamma = \beta$; doing so would unnecessarily lower the upgrade price without an attendant increase in the number of upgrading customers. Thus,

$$p_2^* = \alpha \cdot U_2 \quad \text{and} \quad p_u^* = \gamma \left(U_2 - U_{12} \right). \tag{9}$$

$$\pi_u = \gamma \left(U_2 - U_{12} \right) \left(1 - F\left(\gamma \right) \right) = \left(U_2 - U_{12} \right) \rho_f \left(\gamma \right), \quad \text{and} \tag{10}$$

$$\pi_2 = \alpha \cdot U_2 \left(F\left(\beta \right) - F\left(\alpha \right) \right) = U_2 F\left(\beta \right) \alpha \left(1 - \frac{F\left(\alpha \right)}{F\left(\beta \right)} \right) = U_2 F\left(\beta \right) \rho_{f_\beta} \left(\alpha \right). \tag{11}$$

Here, (9) is implied by the above observations together with part (i) of proposition 1's corollary. Substituting these prices and then the definition of ρ into part (iii) of that corollary then gives (10) and (11).

By lemma 1(ii), maximizing $\rho_f \left(\gamma \right)$ is equivalent to pricing such that $\gamma = \tau_f$. Inspecting (10) and noting that $\left(U_2 - U_{12} \right)$ is a positive constant, this is also equivalent to maximizing π_u. Modifying this slightly to guarantee feasiblity gives $\gamma = \max \left(\beta, \tau_f \right)$ at equilibrium. Similarly, maximizing π_2 is equivalent to pricing such that $\alpha = \tau_{f_\beta}$.[5] Applying (9) then solves for the equilibrium prices[6]

$$p_2^* = \tau_{f_\beta} U_2 \quad \text{and} \quad p_u^* = \left(U_2 - U_{12} \right) \max \left(\beta, \tau_f \right) = \left(U_2 - U_{12} \right) \beta. \tag{12}$$

This also gives a condition that determines whether this case is active – if

$$\tau_{f_\beta} > \beta \left(1 - \frac{U_{12}}{U_2} \right) \quad \text{then} \quad p_u^* < p_2^*$$

---

[5] Also important is that the class C is closed to truncation (lemma 1(i)); this ensures that $f_\beta \in C$.

[6] The last equality in (12) comes from the fact that equilibrium pricing in period 1 will always induce $\beta > \tau_f$. This is established a later.

and upgrades are discounted. Otherwise, the shared price case is active.

The shared-price case: In this case the firm selects a common or shared price p_s that is offered to all customers regardless of whether they purchased the first version of the product. The breakpoints $\alpha \leq \beta \leq \gamma$ are defined as before except that p_s substitutes for both p_u and p_2, so

$$\alpha = \min\left[\beta, \frac{p_s}{U_2}\right] \quad \text{and} \quad \gamma = \max\left[\beta, \frac{p_s}{U_2 - U_{12}}\right]. \tag{13}$$

With a shared price, the firm's period 2 pricing problem is

$$\max_{p_s}\left[\pi_u + \pi_2\right] = \max_{p_s}\left[p_s\left(1 - F\left(\gamma\right) + F\left(\beta\right) - F\left(\alpha\right)\right)\right]. \tag{14}$$

Also, we assume that $p_2^* < p_u^*$ (otherwise, the upgrade discounting case is active), and we restrict attention to cases of $\beta > \tau_f$, a condition which must hold at equilibrium (this is verified later). The next lemma characterizes and solves the period 2 equilibrium for this case.

Lemma 2 *At equilibrium in the shared price case:*

(i) $p_2^* < p_s < p_u^*$

(ii) $S_{b,\emptyset}$ *is empty;* $S_{\emptyset,b}$ *is not empty.*

(iii) $p_s = \beta\left(U_2 - U_{12}\right).$

(iv) $\alpha = \beta\left(1 - \frac{U_{12}}{U_2}\right).$

Given that $p_2^* < p_u^*$ defines the shared price case, it is intuitive that the equilibrium p_s is between p_2^* and p_u^* – part (i) of the lemma formalizes this. (ii) establishes that all eligible customers will upgrade to the improved version. (iii) gives the equilibrium price, and (iv) establishes that the previous case, in which the firm differentiates its pricing based on whether the customers are upgraders or new purchasers, yield a larger number of buyers across both periods. The shared price case results in fewer total customers than in the discounted upgrades case. [7]

Synthesizing the two cases: The two cases considered here are very different with regard to the analysis used – for example, the proof of the previous lemma is very different than the analysis of the upgrade discounting case – yet the price expressions are very similar. For example, upgrading customers are charged the same price in both cases. The next proposition synthesizes the results of this section; it follows directly from earlier results, so it is given without additional proof.

---

[7] (this is because: (1) the number of total customers is $F^c\left(\alpha\right)$ in both cases, (2) α equals τ_{f_β} in the discounted upgrades case, and (3) α in the shared-price case is $> \tau_{f_\beta}$).

Proposition 2 *For a moderate product improvement, at equilibrium in the period two subgame:*

(*i*)   *All eligible customers upgrade. That is, $S_{b,\emptyset} = \emptyset$; equivalently, $\gamma = \beta$*

(*ii*) $\alpha = \max \left[\beta \left(1 - \frac{U_{12}}{U_2} \right), \tau_{f_\beta} \right]$. *This is less than β, and $S_{\emptyset,b}$ is thus non-empty – i.e., period 2 always sees some first-time buyers.*

(*iii*) $p_u = \beta (U_2 - U_{12})$ *and* $p_2 = U_2 \alpha$.

(*iv*) *There exists a threshold β below which upgrade discounting is used and above which a shared price is used.*

(*v*) $\pi_u + \pi_2$ *is increasing in β.*

4.2   Equilibrium in the complete game (for moderate improvements):

Having fully solved the second period subgame, we now move to period 1. The firm's problem in this initial period, is to find p_1 to maximize total discounted profits assuming equilibrium behavior in the period 2 subgame. First is to establish the following relationship between the equilibrium β and the (previously defined) τ_f.

Proposition 3 *For moderate improvements, at equilibrium $\beta > \tau_f$.*

Combining this with earlier results gives us that $\alpha < \tau_f < \beta = \gamma$ under equilibrium pricing. The fact that eligible customers always upgrade (because $\beta = \gamma$) has some obvious managerial implications, and it also has an important analytical implication – the *buy/decline* segment is empty at equilibrium, and this implies that a β-customer is indifferent between *buy/upgrade* and *decline/buy*. Setting $u_{b,u}(\beta)$ equal to $u_{\emptyset,b}(\beta)$ and simplifying gives

$$p_1 = \beta (U_1 + \delta U_{12} - \delta U_2) + \delta \alpha U_2. \tag{15}$$

Equation (15) together with proposition 2(*iii*) provides a reformulation of the firm's first period strategy. Rather than selecting the optimal price p_1, the firm can equivalently select the threshold

β that maximizes

$$\pi_1 + \delta\left(\pi_2 + \pi_u\right) = U_1\beta F^c\left(\beta\right) + \delta U_2\alpha F^c\left(\alpha\right) \quad s.t.\ \beta \geq \tau_f \tag{16}$$

$$= U_1\rho_f\left(\beta\right) + \delta U_2\rho_f\left(\alpha\right)$$

$$\text{with } \alpha = \max\left[\beta\left(1 - \frac{U_{12}}{U_2}\right), \tau_{f_\beta}\right].$$

Then, with β and α known, the equilibrium p_1 is given by (15), and the equilibrium p_u and p_2 are given by proposition $2(iii)$. Upgrading customers are given a discount when α equals τ_{f_β}, and a shared price is used otherwise. The first derivative of ρ_f is $\rho_f'\left(\theta\right) = -\theta f\left(\theta\right) + F^c\left(\theta\right)$ – applying this, critical β-values occur when

$$U_1\left(-\beta f\left(\beta\right) + F^c\left(\beta\right)\right) + \delta U_2\frac{d\alpha}{d\beta}\left(-\alpha f\left(\alpha\right) + F^c\left(\alpha\right)\right) = 0. \tag{17}$$

$$\text{with} \quad \frac{d\alpha}{d\beta} = \begin{cases} f\left(\alpha\right)/\left(2f\left(\alpha\right) + \alpha f'\left(\alpha\right)\right) & \text{if} \quad \alpha = \tau_{f_\beta} \\ 1 - U_{12}/U_2 & \text{if} \quad \alpha < \tau_{f_\beta} \end{cases}$$

This is the standard (necessary) first order condition for optimality, and ρ's characteristics, established in lemma 1, can be used to show that this always has a solution that identifies a (local) maximum of (16). While the analysis herein does not eliminate the possibility of multiple solutions, we have been unable to find an example with multiple critical points let alone multiple local maximums. However, should one encounter a particular distribution and parameters for which multiple local maximums exist, it is a simple matter to compute their values numerically and select the one delivering the highest profit to the firm. Next is to consider the case of a large product improvement.

5 Equilibrium for large improvements $\left(U_2 > \frac{1}{\delta}U_1 + U_{12}\right)$

As shown in proposition 1 and its corollary, the ordering of the *decline/buy* and *buy/decline* segments is reversed from the previous section and the thresholds α, β, γ are redefined. Specifically, we now have

$$S_{\emptyset,\emptyset} \leq \alpha \leq S_{b,\emptyset} \leq \beta \leq S_{\emptyset,b} \leq \gamma \leq S_{b,u}. \tag{18}$$

It is not immediately obvious, but the analysis of this case is simpler than for moderate improvements. The solution is also simpler; it is given in the next proposition.

Proposition 4 *With large improvements, at equilibrium:*

(*i*) *All eligible customers upgrade.*

(*ii*) *Nobody purchases afresh in period 2.*

(*iii*) $\alpha = \beta = \gamma = \tau_f$.

(*iv*) *Equilibrium prices are:* $\quad p_1 = \tau_f \left(U_1 + \delta U_{12} \right)$

$$p_2 = \tau_f U_2 \text{ or any higher price}$$

$$p_u = \tau_f \left(U_2 - U_{12} \right).$$

(*v*) *Total discounted profits increase with* U_2.

As in the previous section, it remains true that all period 1 buyers upgrade to the improved version (part (*i*) of the proposition). What does change (part (*ii*)) is that period 2 sees no new buyers; all period 2 revenue comes from the upgrading customers. Intuition might suggest that this is suboptimal because it results in $\pi_2 = 0$. However, analysis (not presented here) has shown that setting p_2 low enough to attract some new customers results in some of the *buy/upgrade* customers defecting to the *decline/buy*, and the revenue generated by the new sales is overshadowed by the revenue lost by these defections.

Another change from the moderate improvement cases is part (*iii*). As discussed earlier, τ_f can be interpreted as the purchasing threshold in a single-product/single-period analog of our model – i.e., all/only customers with $\theta > \tau_f$ purchase at equilibrium in the simpler model. For moderate improvements, the equilibrium number of period 1 buyers is smaller and the number of total buyers larger than in the single-period/single-product model. With a large improvement in the product, (*iii*) tells us that exactly the same number buy in period 1 as in the simpler model, and, because nobody buys afresh in period 2, this also equals the number of total buyers.

To extend this analogy, for large improvements, the equilibrium price structure is "myopic" in the sense that it exactly duplicates the following problem. A firm offers two different products, each for a single period, to two different sets of consumers (with the distribution f applying to each set independently of the other). The first product has quality $U_1 (1 + \delta)$ and the second has quality $(U_2 - U_{12})$, and the model is otherwise as described herein. The equilibrium prices for these two

problems are respectively p_1 and p_u as given in the previous proposition and the quantities sold are also the the same $(1 - F(\tau_f))$. In other words, the extreme case is myopic in that it can be solved by solving the two periods as unrelated problems.

As a corollary to this proposition, the firm does use upgrade discounting, but it is moot in the sense that no customers actually pay the non-discounted price p_2. Interestingly, this is nonetheless important – equilibrium requires a p_2 that is sufficiently high that none of the customers prefer to delay purchase beyond the first period.

6   Extended Example: Customer density $\sim Uniform(0,1)$

In this section, we assume that customers' θs are uniformly distributed over the $(0,1)$ interval, and we: (1) provide an extended example of our results, (2) demonstrates how equilibrium profits and cash flows are affected by the degree of improvement in the product, (3) co

nsider costs associated with developing the improved product, and (4) extend the model by treating U_2 as a strategic variable. For the $Uniform(0,1)$ distribution

$$
\begin{aligned}
f(\theta) &= 1 & F(\theta) &= \theta \\
\theta \cdot h_f(\theta) &= \tfrac{\theta}{1-\theta} & & \\
\tau_f &= \tfrac{1}{2} & \tau_{f_b} &= \tfrac{b}{2}
\end{aligned}
$$

Moderate improvement cases are solved by proposition 2 and expression (17). Large improvment cases are solved by $\tau_f = \frac{1}{2}$ (for the Uniform(0,1) distribution) together with proposition 4. The next two tables summarize these results.

Moderate Improvements		
	updgrade discounting	shared price
occurs when	$U_2 < 2U_{12}$	$2U_{12} \leq U_2 < 1 + \frac{U_1}{\delta U_{12}}$
p_1	$\frac{(2U_1+2\delta U_{12}-\delta U_2)(2U_1+\delta U_2)}{8U_1+2\delta U_2}$	$\frac{U_1^2 U_2 + \delta U_1 U_2 (U_2 - U_{12})}{2U_1 U_2 + 2\delta (U_2 - U_{12})^2}$
p_u	$(U_2 - U_{12})\left(1 - \frac{2U_1}{4U_1+\delta U_2}\right)$	$\frac{U_2(U_2-U_{12})(U_1+\delta(U_2-U_{12}))}{2U_1 U_2 + 2\delta(U_2-U_{12})^2}$
p_2	$U_2\left(\frac{1}{2} - \frac{U_1}{4U_1+\delta U_2}\right)$	same as p_u
α	$\frac{1}{2} - \frac{U_1}{4U_1+\delta U_2}$	$\frac{1}{2} - \frac{U_1 U_{12}}{2U_1 U_2 + 2\delta(U_2-U_{12})^2}$
β	$1 - \frac{2U_1}{4U_1+\delta U_2}$	$\frac{U_2(U_1+\delta(U_2-U_{12}))}{2U_1 U_2 + 2\delta(U_2-U_{12})^2}$
upgrade discounts used?	Yes	No
period 1 sales	$1 - \beta$	
period 2 upgrades	$1 - \beta$	
period 2 new sales	$\beta - \alpha$	

Large Improvements	
occurs when	$\left(1 + \frac{U_1}{\delta U_{12}} \leq U_2\right)$
$p_1 = \frac{1}{2}(U_1 + \delta U_{12})$	$\alpha = \beta = \gamma = \frac{1}{2}$
$p_u = \frac{1}{2}(U_2 - U_{12})$	period 1 sales = upgrades = $\frac{1}{2}$
$p_2 = \frac{U_2}{2}$ (or any larger value)	period 2 new sales = 0

6.1 Financial Implications

We now illustrate the financial implications of product improvement. We maintain the assumption that customer types are uniformly distributed over $(0, 1)$, and we assume that U_1 and U_{12} are equal. As a basis for comparison, the graphs below include cases in which upgrade discounting is disallowed or infeasible.[8]

Of primary interest is how the firm's total discounted revenue changes with the degree of improvement of the product. This is illustrated in figure 3[9]; the solid line illustrates the equilibrium payoff earned by the firm as U_2 varies. The dashed line is for comparison, it gives the equilibrium payoffs for cases in which upgrade discounting is not used.

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[8] These "updgrade discounting" cases were solved using analysis similar to that already presented.

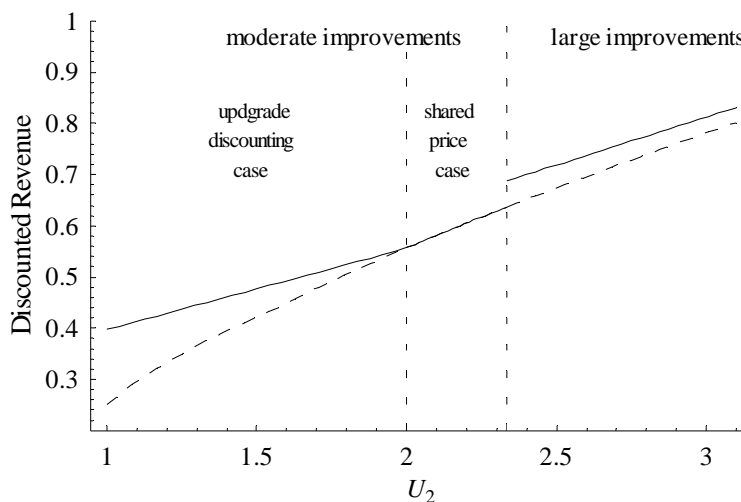[9] U_1 and δ are respectively 1.0 and 0.75 in all figures.

Figure 3: Equilibrium payoff

Two observations are immediate. First, revenue unambiguously increases with U_2. Second, for most U_2 values, disallowing the discounting of upgrades reduces the firm's revenue. Of course, the rate that payoff increases with U_2 depends on the pricing structure at equilibrium, and this differs depending on whether the upgrade is moderate or large and whether upgrade discounting is used. Also recall the sequence of the *buy/decline* and *decline/buy* segments changes at the transition to large improvements; this causes the continuity in the figure.

Next we consider product improvement costs and ask, "What is the maximum acceptable cost to improve the product from U_1 to U_2?" To answer this, we will first write $\pi^{eq}(U_1, U_2)$ to denote the firm's equilibrium payoff from offering the "product pair" (U_1, U_2) and define $\underline{\pi}^{eq} \triangleq \lim_{U_2 \to U_1} \pi_1^{eq}(U_1, U_2)$. Now, $\pi_1^{eq}(U_1, U_2) - \underline{\pi}^{eq}$ is the maximum development cost that the firm currently offering quality U_1 would willingly pay to improve the product to quality level U_2. Analogously defined, let π^{sh} and $\underline{\pi}^{sh}$ be the equilibrium payoff and its limit as $U_2 \to U_1$ when upgrade pricing is disallowed (i.e., the "shared pricing" is enforced for all U_2); so, without upgrade pricing, the maximum development cost the firm would be willing to bear is $\pi^{sh}(U_1, U_2) - \underline{\pi}^{sh}$.

These values are plotted in figure 4 (dashed line is with upgrade pricing disallowed). Interestingly, (as indicated by the fact that the dashed line is above the solid line) a firm that cannot offer upgrade discounts would be more willing to pay product improvement costs.

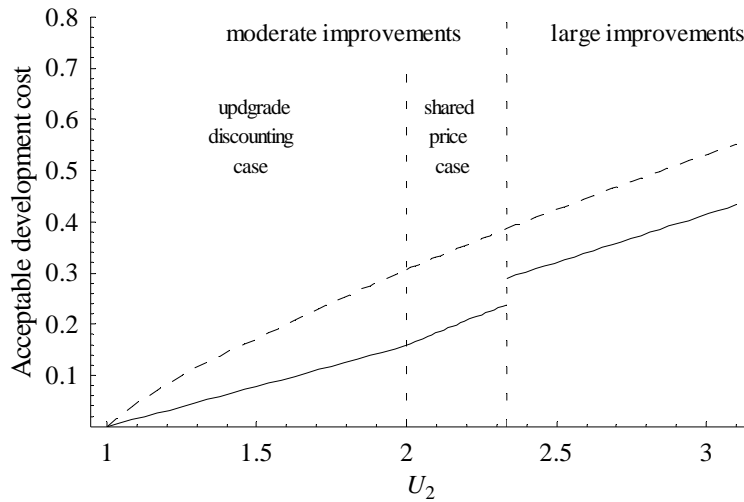Cash flow is another ubiquitous concern to managers; the individual periods' cash flows are shown

Figure 4: Acceptable development cost

in figure 5. An important observation is that, relative to the baseline of $U_2 = U_1$, offering the improved product together with upgrade pricing requires additional cash in the first period for but provides positive cash flow for large improvements. This is of course in addition to any cash requirements of developing the improved product.

Figure 5 tells another interesting story. We have thus far assumed that U_2 represents the product that is actually released by the firm in the second period. But, suppose instead the actual product released by the firm is modelled as a strategic decision that is made at the beginning of period 2; the firm's period 2 decision how has three dimensions: p_u and p_2 as before, and now U_2 which is from a range $(U_1, \overline{U_2})$. That is, the firm's choice U_2 is constrained below by its first period offering and above by $\overline{U_2}$ which represents the most advanced version of the product that the firm is able to product.[10] When selecting which U_2 to actually offer, the firm is trying to maximize its second period revenues. This implies, as can be seen in figure 5, that there is a range of U_2 values in the large improvement region that the firm will never select. Specifically, the subgame perfect equilibrium will never have the firm offering a product with U_2 in the interval $\left(U_1 \left(1 + \frac{1}{\delta}\right), \mathcal{U}_2\right)$ with $\mathcal{U}_2$ given by the expression

$$\mathcal{U}_2 = \frac{\left(1 - \delta^2\right) \left[\delta \left(U_1 - 1\right) - 1\right] \left[\delta \left(U_1 - 1\right) - U_1 - 1\right] \left[U_1 + 1 + \delta \left(U_1 - 1\right)^2\right]}{\delta \left[1 + U_1 + \delta \left(U_1 - 2\right) + \left(U_1 - 1\right)^2 \delta^2\right]^2}.$$

Generally, and disregarding any development costs, the firm's equilibrium decision is to offer the

---

[10] As a technical point, we now assume that $\overline{U_2}$ is common knowledge in period 1. The actual U_2 is selected by the firm at the beginning of period 2, so it is common knowledge at the customers' period 2 purchasing epoch.
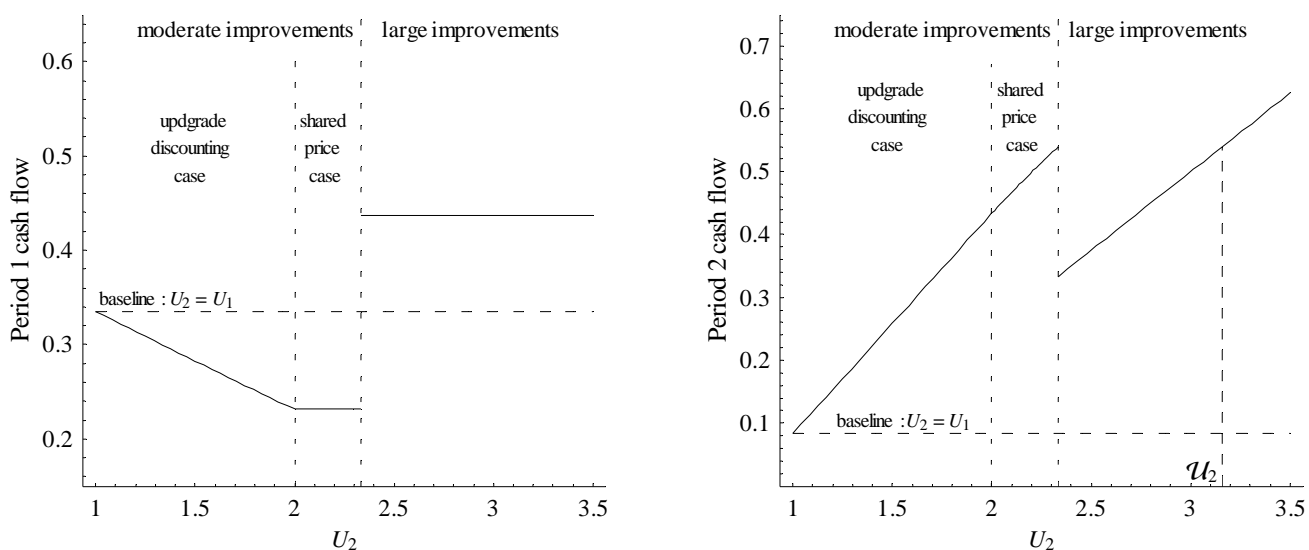
Figure 5: Cash flows by period

largest possible U_2 unless it falls within this interval; in that case, the firm will offer $U_2 = U_1 \left(1 + \frac{1}{\delta}\right)$. In any event, the equilibrium prices will be as given previously albeit using the equilibrium U_2 in the second period. This discussion does not hold if upgrade pricing is disallowed. In that case, period two revenues strictly increase with U_2 and the firm's equilibrium decision is to release the largest possible U_2.

7   Conclusions

Upgrades are endemic to any industry with rapid technological innovation. But firms would benefit from an understanding of market reaction to their technological choices. This is particularly true for software upgrades that may vary only slightly from older versions. Intuition suggests that customer look ahead is a bigger problem when new versions do not differ much from the old ones. Indeed, this implies that firms should keep different versions far apart in order to minimize this problem. However, modest upgrades provide firms with the opportunity to price discriminate in the future by offering old customers special upgrade prices and this optimal pricing strategy mitigates customer balking. While our results seem to provide an explanation for the adoption of such strategies by software producers, they also seem to indicate that in reality firms might be pricing upgrades higher than what consumers perceive to be optimal.

As the degree of product improvement increases, it becomes important that software producers are (usually) unable to charge upgrading customers more than new purchasers. This induces the firm to offer a common second period price to all customers. When the improvement reaches the "large" range (as per the definitions herein), the equilibrium behavior for both firms and consumers is myopic in nature, and the problem decouples into two single period problems. Even though the different levels of product improvement result in different prices and in different segmentation patterns, it remains true that all eligible customers choose to upgrade to the improved product.

These results are quite robust to different distributions of customer types, but, as always, our results are restricted by the reality of our assumptions. As topics for future research, we believe that the maximum new mileage will be achieved by generalizing the information structure available to the customer in terms of product improvement, future prices, and perhaps additional generations of product improvement. This research will also explore interesting questions about the mechanisms that firms and customers will use to convey and acquire such information.

8   Appendix

Proof of proposition 1 (p. 6): (i) By construction, the four $u_{i,j}$ functions are the utility values to a customer from each of the four combinations: *buy/upgrade, buy/decline, decline/buy, and decline/decline.*

Suppose a θ-customer is placed into $S_{b,u}$ meaning $u_{b,u}$ came out of the maximum in (1). The proposition states that this customer's optimal response to the firm's prices is to buy the product in period 1 and then upgrade in period 2. This is verified as follows:

(1) Assuming that a θ-customer bought in period 1, we require that the optimal period 2 decision is to upgrade. $\theta \cdot (U_2 - U_{12})$ is the marginal value the customer ascribes to the upgrade and p_u is the price to be paid. Upgrading is thus optimal if $\theta \cdot (U_2 - U_{12}) - p_u > 0$. This inequality does indeed hold after applying simple algebra to fact that $u_{b,u} > u_{b,\emptyset}$ (which comes from the fact that $u_{b,u}$ is the maximizer of (1)).

(2) Moving to period 1, we require that the customer's optimal decision is to buy the initial version. The previous item establishes that the customer would follow up a purchase in this period with an upgrade in period 2, so we know that a buy decision now would translate into an overall strategy of *buy/upgrade*. This strategy is preferable to *decline/buy* because $u_{b,u} > u_{\emptyset,b}$ and is preferable to *decline/decline* because $u_{b,u} > u_{\emptyset,\emptyset}$.

Placing the customer into the *buy/upgrade* segment thus fulfills the customer's optimality conditions in both of the periods. Verification of the other cases is nearly identical and is omitted.

(ii) The objective function in (1) is the max of linear increasing functions, so it is continuous, piecewise linear, increasing, and convex in θ. This then implies that the sets $S_{b,u}$, $S_{b,\emptyset}$, $S_{\emptyset,b}$, $S_{\emptyset,\emptyset}$ are ordered by the slope, but not the intercept, of $u_{b,u}$, $u_{b,\emptyset}$, $u_{\emptyset,b}$, $u_{\emptyset,\emptyset}$. For instance, $S_{b,\emptyset} < S_{\emptyset,b}$ if and only if $\frac{du_{b,\emptyset}}{d\theta} < \frac{du_{\emptyset,b}}{d\theta}$. The actual orderings given in (2) and (3) follow immediately after extracting the slopes of the four associated functions.**QED**

Proof of lemma 1(p. 10):   In this proof, $G_{(a,b)}$ is a truncation of a distribution $G \in class\ C$; a is the lower truncation point, and b is the upper truncation point. $g_{(a,b)}$ is the pdf of this distribution. Thus, part (i)

(i) The claim here is that C is closed to truncation; that is, $G_{(a,b)} \in class\ C$. We first show that $\theta h_{g_{(a,b)}}(\theta)$ is increasing in θ.

$$\theta h_{g_{(a,b)}}(\theta) = \frac{\theta g_{(a,b)}(\theta)}{1 - G_{(a,b)}(\theta)} \tag{19a}$$

$$= \frac{\theta g(\theta) \frac{1}{G(b) - G(a)}}{1 - G(\theta) \frac{1}{G(b) - G(a)}} = \frac{\theta g(\theta)}{G(b) - G(a) - G(\theta)} \tag{19b}$$

$$= \theta \frac{g(\theta)}{1 - G(\theta)} \cdot \frac{1 - G(\theta)}{G(b) - G(a) - G(\theta)} = \theta \cdot h_g(\theta) \cdot \frac{1 - G(\theta)}{G(b) - G(a) - G(\theta)} \tag{19c}$$

[Here, $(19a)$ is expanding the hazard rate $h_{g_{(a,b)}}(\theta)$, $(19b)$ is from the definition of $g_{(a,b)}$, and $(19c)$is from the definition of h_g and algebraic manipulation.] $\frac{1 - G(\theta)}{G(b) - G(a) - G(\theta)}$ and $\theta \cdot h_g(\theta)$ are both increasing in θ, the latter because $g \in Class\ C$, so expression $(19c)$ is increasing in θ. All that remains is to show continuity of $\theta \cdot h_{g_{(a,b)}}$, and this is inherited from h_g; thus, $g_{(a,b)}$ is in class C.

(ii) We first show that $\rho_g(\theta)$ has a unique critical point defined by

$$\theta \cdot h_g(\theta) = 1. \tag{20}$$

By definition, $G(\theta) \triangleq \int_{-\infty}^{\theta} g(x)\,dx$. Substituting this into $\rho_g(\theta)$ and differentiating via Leibnitz's rule gives

$$\frac{d}{d\theta}\rho_g(\theta) = -u\,\theta\,g(\theta) + u(1 - G(\theta)).$$

Setting this equal to zero, solving, and then applying the definition of h_g verifies that $\frac{d}{d\theta}\rho_g(\theta) = 0$ iff (20) holds, so (20) defines critical points of $\rho_u(\theta)$.

Next observe that

$$\rho_u(0) = 0 = \lim_{\theta \to \infty} \rho_u(\theta). \tag{21}$$

This guarantees that at least one critical point exists (by Rolle's theorem). Uniqueness then follows from the fact that $\theta \cdot h_g(\theta)$ is increasing in θ (because g is in class C) implying (20) cannot have multiple solutions. Next observe that (21)together with the fact that $\rho_u(\theta) \geq 0$ for all $\theta > 0$ implies that the critical point must be a maximum rather than a minimum. The claim of quasiconcavity then follows directly – a continuous function whose only critical point is a maximum is quasiconcave.

(iii) Straightforward differentiation and algebraic manipulation of $\theta h_g(\theta)$ shows that it is increasing in θ if and only if

$$\theta g'(\theta) > -g(\theta)(1 - \theta h_g(\theta)).$$

Twice differentiating ρ_g and applying this inequality

$$\frac{\partial^2 \rho_g}{\partial \theta^2} = (-1)\left(2g(\theta) + \theta g'(\theta)\right) < -g(\theta)\left(1 - \theta h_g(\theta)\right). \tag{22}$$

By inspection of this $\theta h_g(\theta) < 1$ implies concave ρ_g. Since $\theta h_g(\theta) = 1$ defines τ_g and $\theta h_g(\theta)$ is increasing in θ (by g's membership in class C), the function ρ is concave for all $\theta < \tau_g$.

(iv) From part (iii), ρ_g is concave for $\theta < \tau_g$. This concave region may be followed by a convex region, but a second concave region must be disallowed. A sufficient condition to rule out a second concave region is to show that ρ_g cannot have more than one inflexion point on the region $\theta > \tau_g$. From (22), an inflexion point is defined by

$$\theta \frac{g'(\theta)}{g(\theta)} = -2. \tag{23}$$

· If g has a uniform distribution then $g'(\theta) = 0$ and ρ_g has no inflexion points.

· If $g \sim \text{Exponential}(\lambda)$ then $\theta \frac{g'(\theta)}{g(\theta)} = -\theta \lambda$ and ρ_g has exactly one inflexion point.

· If $g \sim \text{Normal}\left(\mu, \sigma^2\right)$ then $\theta \frac{g'(\theta)}{g(\theta)} = \frac{\theta(\mu - \theta)}{\sigma^2}$ and is monotonically decreasing in θ for $\theta > \frac{\mu}{2}$. It can be shown that $\tau_g > \mu$ for normally distributed g, so this monotonicity implies at most 1 solution to (23) and at most 1 inflexion point.

· If $g \sim \text{Beta}$ with shape parameters s_1 and s_2, then $\frac{d}{d\theta}\left(\theta \frac{g'(\theta)}{g(\theta)}\right) = \frac{1-b}{(1-\theta)^2}$. This is monotone in θ for every β implying at most one inflexion point.

· If $g \sim \text{LogNormal}$ then $\frac{d}{d\theta}\left(\theta \frac{g'(\theta)}{g(\theta)}\right) = \frac{-1}{\theta} < 0$. If $g \sim \text{Logistic}(\mu, s)$ then $\frac{d}{d\theta}\left(\theta \frac{g'(\theta)}{g(\theta)}\right) = -\frac{\theta + \sinh(\theta)}{1 + \cosh(\theta)} < 0$. For both of these, monotonicity again implies at most one inflexion point. **QED**

Proof of lemma 2 (p. 13): (i) follows directly from the fact π_u and π_2 are quasiconcave with optimums occurring p_u^* and p_2^* respectively and because $p_2^* < p_u^*$ (by the definition of the shared price case).

(ii) We first show by contradiction that $\gamma = \beta$, so assume that $\gamma > \beta$. Then,

$$
\begin{aligned}
\gamma \;&=\; \max\left[\beta, \tfrac{p_s}{U_2 - U_{12}}\right] && \text{(expression (13))}\\
&=\; \tfrac{p_s}{U_2 - U_{12}} && \text{(because } \gamma > \beta\text{)}\\
&<\; \tfrac{p_u^*}{U_2 - U_{12}} && \text{(because } p_s < p_u^* \text{ in the shared case)}\\
\gamma \;&<\; \beta && \text{(because } p_u^* = \beta\,(U_2 - U_{12}) \text{ by (12))}
\end{aligned}
$$

This contradicts the original assumption, so it must be that $\gamma \le \beta$. The definition of γ and β precludes $\gamma < \beta$ however, and thus $\gamma = \beta$ at equilibrium. Equivalently, $S_{b,\emptyset}$ is empty.

Next is to show that $S_{\emptyset,b}$ is not empty by showing that $\alpha < \beta$ at equilibrium. The fact that $p_s > p_2^*$ (from part (i)) implies that $\alpha < \tau_{f_\beta}$ and the result then follows because $\tau_{f_\beta} < \beta$.

(iii) follows immediately from $\gamma = \tfrac{p_s}{U_2 - U_{12}}$ and $\gamma = \beta$ as established in part (ii).

$$
\begin{aligned}
(iv)\quad \alpha \;&=\; \min\left[\beta, \tfrac{p_s}{U_2}\right] && \text{(from (13))}\\
&=\; \min\left[\beta, \tfrac{\beta}{U_2}\,(U_2 - U_{12})\right] && \text{(because } p_s = \beta\,(U_2 - U_{12}) \text{ from } (iii)\text{)}\\
&=\; \beta\left(1 - \tfrac{U_{12}}{U_2}\right) && \text{(after simplifying).}\textbf{QED}
\end{aligned}
$$

Proof of proposition 3 (p. 14): Equilibrium payoffs $\pi_u + \pi_2$ in the period 2 subgame strictly increase with β. Thus, equilibrium pricing in period 1 must place β at a point at which π_1 is decreasing in β. Otherwise, the aggregate payoffs $\pi_1 + \delta\,(\pi_2 + \pi_u)$ will be decreasing in β and (thus) increasing in p_1, and this implies suboptimality. Therefore β must be $> \tau_f$ at equilibrium because this is the region over which π_1 is decreasing in β.**QED**

Proof of proposition 4 (p. 16): Since the α, β, γ define boundaries between segments, they are points of indifference between the segments that they abut and this gives the following three conditions that must hold at equilibrium:

$$
0 = \alpha\,(U_1 + \delta U_{12}) - p_1, \quad \beta\,(U_1 + \delta U_{12}) - p_1 = \beta \delta U_2 - p_2 \tag{24}
$$
$$
\text{and} \quad \gamma \delta U_2 - p_2 = \gamma\,(U_1 + \delta U_2) - p_1 - \delta p_u
$$

The second period equilibrium criteria give two more conditions:

$$
\beta U_2 = p_2^{eq} \quad \text{and} \quad \gamma\,(U_2 - U_{21}) = p_u^{eq}. \tag{25}
$$

If the first of these did not hold, either: customers in $S_{\emptyset,b}$ would defect in period 2 from *decline/buy* to *decline/decline* or the firm could raise p_2 without losing any second period demand. If the second constraint did not hold, either customers in $S_{b,u}$ would defect from *buy/upgrade* to *buy/decline* or the firm could raise p_u without losing any upgrade demand. Any of these possibilities violates the equilibrium criteria of either the customers, if defection occurs, or of the firm (raising price without losing demand would increase profits thereby violating the presumption that p_u and p_2 are equilibrium prices in subgame 3).

The equations of (24) and (25) simultaneously solve to

$$\alpha = \beta = \gamma = \frac{p_1}{U_1 + \delta U_{12}}, \ p_u = p_1 \frac{U_2 - U_1}{U_1 + \delta U_{12}}, \ p_2 = p_1 \frac{U_2}{U_1 + \delta U_{12}}; \tag{26}$$

and the fact that $\alpha = \beta = \gamma$ implies parts (i) and (ii) of the proposition.

Part (iii): For the large-upgrade case

$$\pi = (p_1 + \delta p_u)(1 - F(\gamma)) + \delta p_2 (F(\gamma) - F(\beta)) + \delta p_u (F(\beta) - F(\alpha)).$$

Substituting from (26), this simplifies to

$$\pi = (U_1 + \delta U_2)\rho_f(\beta) \tag{27}$$

which, by lemma ($1ii$), is maximized when $\beta = \tau_f$; together with $\alpha = \beta = \gamma$, this verifies part (iii) of the proposition.

Part (iv): Substituting $\beta = \tau_f$ into (26) and solving gives

$$p_1 = (U_1 + \delta U_{12})\tau_f, \ p_u = (U_1 - U_{12})\tau_f, \ \text{and} \ p_2 = U_2 \tau_f.$$

Part (v): From $\beta = \tau_f$ together with (27), the firm's payoff π at equilibrium is

$$\pi = (U_1 + \delta U_2)\rho_f(\tau_f).$$

This is strictly increasing in U_2 after noting that $\rho_f(\tau_f)$ is independent of U_2. **QED**

9   References

Bulow, Jeremy. 1982. Durable-Goods Monopolists. *Journal of Political Economy.* 90(21) 314-332

Bhattacharya, Shantanu, V. Krishnan, Vijay Mahajan. 2003. Operationalizing technology improvements in product development decision-making. *European Journal of Operational Research.* *149. 102-130*

Coase, Ronald. 1972. Durability and Monopoly. *Journal of Law and Economics.* 15(4) 143-49

Cohen, Morris, Jehoshua Eliashberg, Teck-Hua Ho. 1996. New product development: The performance and time-to-market trade-off. *Management Science. 42(2) 173-186*

Dhebar, Anirudh. 1994. Durable-goods monopolists, rational consumers, and improving products. *Marketing Science. 13(1) 100-120*

Ellison, Glenn, Drew Fudenberg. 2000. The neo-Luddite's lament: excessive upgrades in the software industry. *RAND Journal of Economics. 31(2) 253-272*

Fudenberg, Drew, Jean Tirole. 1998. Upgrades, tradeins, and buybacks. *RAND Journal of Economics. 29(2) 235-258*

Krishnan, Trichy, Frank Bass, Dipak Jain. 1999. Optimal pricing strategy for new products. *Management Science. 1999. 45(12) 1650-1663*

Kornish, Laura. 2001. Pricing for a durable goods monopolist under rapid sequential innovation. *Management Science. 47(11) 1552-1561*

Lariviere, Martin A. 2004. A note on Probability Distributions with Increasing Generalized Failure Rates. Forthcoming in *Operations Research.*

Norton, John, Frank Bass. 1987. A diffusion theory model of adoption and substitution for successive generations of high-technology products. *Management Science. 33(9) 1069-1086*

Padmanabhan, V., Surendra Rajiv, Kannan Srinivasan. 1997. New products, upgrades, and new

releases: A rationale for sequential product introduction. *Journal of Marketing Research.* *34(4)* *456-472*

Stokey, Nancy. 1981. Rational expectations and durable goods pricing. *Bell Journal of Economics.* 12 112-128.