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by

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DECOMPOSITION OF THE INITIAL STABILITY PROBLEM FOR A CYLINDRICAL SHELL UNDER NONSYMMETRIC LOADS

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Abstract

Numerical aspects of initial stability analysis of a cylindrical shell of non-constant parameters along the generator and under nonsymmetrical loads are considered. Variational approach based on Sanders's and Donnell's nonlinear equations of thin, elastic shells is applied.

The problem is decomposed to determine : the stability vectors in the axial direction in the first step , and the critical load and the stability vector in the circumferential direction in the second step. The discretization is based on finite Fourier representations and the finite difference method. To find the approximate stability vector in the axial direction an auxiliary problem for axisymmetric loads is solved. The error of the method is defined and the effectiveness of the method is estimated.

The decomposition leads to small and fast algorithms suitable for personal computers. Shells with constant and stepped thicknesses under wind loads are calculated as examples. Tested algorithms shown considerable effectiveness and good accuracy of results.

1.INTRODUCTION.

Non-axisymmetric loading complicates the numerical analysis of the stability of cylindrical shells. Therefore, analyses are mostly based on the initial stability concept (Koiter [12]) and additional attempts are made to reduce solution tasks and improve efficiency of algorithms. This tendency is clearly evident, for example, in literature on the stability analysis of cylindrical shells subjected wind loads.

Many papers concerning the shell stability problem under wind loads employ Fourier representations to the equations. For example Almroth [1] and Maderspach [17] use these representations for discretization of the equations in the axial and circumferential direction. Fourier representations in the circumferential direction are also combined with other methods in the axial direction : with an expansion in Taylor series (Kundurpi et al. [14]) and with a finite differrence method (Sheinman et al. [19]) . The method proposed in [19] is a generalization of a method developed by Bushnell [5] for axisymmetrical problems.

Bushnell in [6,7] approximates a cylindrical shell by a segment of a torus of a very large radius. The stability vector is expanded in a Fourier series along a large circumference of the torus which corresponds to the axial direction of the cylindrical shell. Numerical integration is performed along the small circumference of the torus. This method is an extension of the approach used by Kalnins [10] to solve linear static problems.

In papers of Wang, Billington [21] and Langhaar,Miller [15] Fourier representations are combined with the approximate stability vectors in the axial direction. These vectors are taken from the stability problem for the same shell but axisymmetrically loaded. The stability equations are integrated along a meridian and averaged equations of the problem are used. The above method is based on Timoshenko's equations of shells simplified by an assumption about inextensibility in the circumferential direction, Wang et al. [22].

In the present paper a concept to decompose a problem is treated as a more general approach, which do not have to be combined with simplified variants of the shell theory and averaged equations. This approach is assumed to give a first approximation of the eigenpair after which, upon the error estimation, we decide whether the full eigenvalue problem have to be solved. In cases when full analysis is necessary the determined eigenpair can serve as starting values for eigensolvers.

This approach is intended to allow for efficient tackling much more true to engineering practice cases, when loads and shell parameters vary along a generator of a shell. For such cases two dimensional discretizations [3,4,19] lead to programs which are very large and ineffective. Expansions in Taylor series [14] or trigonometric series [6,17] smooth out variable data and can be also ineffective when discontinuity of data occurs. The semi-analitical method from [15,21] does not allow to take this variability into account at all.

In the present paper the finite difference method is used to descretize the problem in the axial direction thus a variable distribution of parameters along a generator of a shell can be modelled. As the problem is decomposed into small subproblems very complicated forms of deformation may be concerned even in a case when the analyses have to be carried out on a microcomputer.

2. INITIAL STABILITY EQUATIONS

In the present paper loads are treated as quasi-static and deformation independent. Such 'dead ' loads and an elastic shell are a conservative system the stability of which is assured by static criteria based on the potential energy. The stability equations are obtained from the condition

$$\delta P_2 = 0 \qquad (2.1)$$

where P_2 it is a second differential of the potential energy P.

Within the initial stability analysis the following assumptions are made :

(i) the rotational part of the prebuckling deformations is neglected,

(ii) membrane forces for the critical configuration are functions of linear strain measures.

These assumptions linearize the equations of the problem and enable us to transform them into a generalized eigenvalue problem.

Following the assumption (i) components $\overset{*}{\phi}_\alpha$ of the rotation vector characterizing the critical configuration are neglected in P_2 .

Two variants of shell theories are used in this paper : the theory of shallow shells of Donnell [9] and the theory of moderately small rotations of Sanders [18]. The second differential of the potential energy may be found for Donnell's equations in [12] and for Sanders's equations in [20]. For the latter one it is additionally assumed that the normal component ϕ_3 of the rotation vector is a small quantity of the same order as strains $\varepsilon_{\alpha\beta}$ and as the square of tangent component ϕ_α of the rotation vector and consequently terms with (ϕ_3^2) are omitted. The second differentials P_2 for both theories differ only in a form of ϕ_α components of the rotation vector :

$$P_2 = \int_A \frac{h}{2} H^{\alpha\beta\lambda\mu} \left(\varepsilon_{\alpha\beta}\, \varepsilon_{\lambda\mu} + \frac{h^2}{12} \varkappa_{\alpha\beta}\, \varkappa_{\lambda\mu}\right) da + \int_A \overset{*}{N}{}^{\alpha\beta}\, \varepsilon^{NL}_{\alpha\beta}\, da \qquad (2.2)$$

where

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + b_{\alpha\beta} w$$

$$\varkappa_{\alpha\beta} = -\frac{1}{2}(\phi_{\alpha|\beta} + \phi_{\beta|\alpha})$$

$$\overset{*}{N}^{\alpha\beta} = H^{\alpha\beta\lambda\mu} \overset{*}{\varepsilon}_{\lambda\mu}$$

$$\varepsilon^{NL}_{\alpha\beta} = \frac{1}{2}\phi_\alpha \phi_\beta$$

$\phi_\alpha = -w,_\alpha$ for Donnell's theory

$\phi_\alpha = -w,_\alpha + b_{\alpha\beta} u^\beta$ for Sanders's theory.

Tensors $\varepsilon_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ are linear measures of strains and of changes of curvature, $\overset{*}{N}^{\alpha\beta}$ are membrane forces , $H^{\alpha\beta\lambda\mu}$ is the elasticity tensor , h is a thickness of the shell, { u_α , w } are components of displacement vector **u** . Symbols with an asterisk '* ' are related to the primary configuration. A vertical stroke denotes a covariant surface derivative.

The assumption (ii) allows to express the membrane forces for the critical configuration $\overset{*}{N}^{\alpha\beta}$ as

(2.3) $$\overset{*}{N}^{\alpha\beta} = \lambda \overset{\circ}{N}^{\alpha\beta}$$

where $\overset{\circ}{N}^{\alpha\beta}$ are the reference membrane forces taken from a static, linear analysis for unit load.

Taking advantage of these assumptions in the initial stability analysis we seek a value of λ characterizing the configuration of a critical equlibrium . After a discretization of P_2 the shell can be treated as an elastic system of n degress of freedom. From (2.1) we obtain a homogenious system of n stability equations :

(2.4) $$(\mathbf{K_o} + \lambda \mathbf{K_\sigma})\mathbf{q} = 0$$

where

Ko is an elastic stiffness matrix (symmetric and positive definite)

Kσ is an initial stress matrix (symmetric)

q is a vector of generalized coordinates which describe a deformation of a shell.

The solution of above generalized eigenvalue problem is n eigenpairs (λ_k, $\mathbf{q}_k$) where λ_k are eigenvalues and $\mathbf{q}_k$ are corresponding eigenvectors. The critical load λ_{cr} is defined as qthe lowest λ_k while the corresponding vector $\mathbf{q}_k$ is the stability vector.

This kind of analysis may be interpreted [8,11] in two ways : as an uniquness analysis of a solution of a linearized static problem or as a stability analysis of this solution (proved only for discrete systems and shallow shells) . λ_{cr} is correspondingly interpreted as : the point of bifurcation of the static solution or as the point separating stable and unstable parts of the solution (when the Liapunov definition of the stability and the energetic norm in the displacement space are used) . This kind of analysis does not answer a question about the stability of the critical configuration ; in a general case it requires an additional analysis, Koiter [13].

3. METHOD OF DECOMPOSITION

The method of decomposition is intended to split the initial stability problem for nonsymmetric loads into two problems of small sizes : one to establish the stability vectors in the axial direction and one to determine the critical load and the stability vector in the circumferential direction.

Below we describe particular stages of this method.

3.a. Linear static problem

This analysis is to calculate a displacement vector $\overset{\circ}{\mathbf{u}}$ which will be used to calculate reference membrane forces $\overset{\circ}{N}{}^{\alpha\beta}$ (exactly integrals related to these forces) . We adopt the displacement vector $\mathbf{u}$ and the load vector $\mathbf{p}$ in the forms :

$$(3.1) \qquad \mathbf{u}(x,\phi) = \sum_{n=0}^{N} \mathbf{u}_n(x)\ \mathbf{t}_n(\phi)$$

with

$\mathbf{u}_n(x) \equiv \{w_n, u_n, v_n\}$, $\mathbf{t}_n(\phi) = \mathrm{diag}(\cos n\phi, \sin n\phi, \cos n\phi)$

x,ϕ are axial and circumferential coordinates

$(N+1)$ is a number of harmonics in the circumferential direction.

Then the discretized equation of the linear static problem can be decomposed into equations related to particular harmonics :

$$(3.2) \qquad K_{on}\ \mathbf{u}_n = \lambda\ \mathbf{p}_n \qquad n = 0,\ldots,N$$

where λ is a load parameter. Sizes of particular tasks depend only on used discrete model in the axial direction . In the paper the finite difference method on a grid of R points is used. A short description of a variant of this method is given in Appendix.

Reference displacements $\overset{\circ}{\mathbf{u}}_n$, necessary to calculate the reference membrane forces $\overset{\circ}{N}{}^{\alpha\beta}$, are determined in the analysis for $\lambda = 1.0$.

3.b. Auxiliary stability problem.

A purpose of this analysis is to establish the stability vectors in the axial direction. We generate $\overset{\circ}{N}{}^{\alpha\beta}$ using only $\overset{\circ}{\mathbf{u}}_o$ i.e. zeroth harmonic of the reference displacements so the membrane forces are axisymmetrical. . It can be also interpreted as the analysis for the auxiliary axisymmetric loads.

If we take a stability vector in the form :

$$(3.3)\qquad u(x,\phi) = \sum_{j=0}^{M} u_j(x)\ t_j(\phi)$$

then the stability equation (2.4) decomposes into equations related to particular harmonics of representation (3.3) :

$$(3.4)\qquad (K_{oj} + \lambda K_{\sigma j})\ u_n = 0 \qquad j=0,\dots,M$$

A solution of this equation consists of (M+1) pairs { λ_j , u_j }. A size of a particular task depends, as in the previous problem, only on discretization in the axial direction. Here also the finite difference method on a grid of R points was used.

The eigenvectors u_j will serve as approximate stability vectors to solve the problem for nonsymmetric loads.

3.c. Stability problem for nonsymmetric loads.

In this analysis we determine the critical load and the stability vector in the circumferential direction for the nonsymmetrical loads. All harmonics of u_n^o are now used to calculate the reference membrane forces.

The stability vector for this analysis is taken in the form :

$$(3.5)\qquad u(x,\phi) = \sum_{j=0}^{M} Q_j(x)\ q_j\ t_j(\phi)$$

where $q_j = \lfloor \alpha_j , \beta_j , \gamma_j \rfloor^T$

$$Q_j = \mathrm{diag}(W_j , V_j , U_j)$$

$$W_j = \frac{w_j(x)}{\max w_j(x)} , \quad V_j = \frac{v_j(x)}{\max v_j(x)} , \quad U_j = \frac{u_j(x)}{\max u_j(x)}$$

Q_j contains rescaled eigenvectors from the auxiliary analysis ,
q_j is the unknown stability vector for the nonsymmetric problem.

The stability equation is now the following :

$$(3.6)\qquad (K_o + \lambda K_\sigma)\ q = 0$$

with $q = \lfloor q_o ,\dots, q_M \rfloor^T$

Matrix relations to generate K_o and K_σ are given in part 4 of this paper.

A size of this problem is equal 3 (M+1) but there are additional possibilities to reduce this size as not all harmonics distinctly influence the critical eigenpair.

3.d. Control of error.

Because the approximate forms of the stability vectors in the axial direction are used in the nonsymmetric problem it is important to determine an error caused by this approximation.

Let us define the residual vector $\mathbf{r}$:

(3.7) $$\mathbf{r}_M = (\bar{K}_o + \lambda_{cr} \bar{K}_\sigma) \mathbf{u}_{cr}$$

where

$\bar{K}_o$ and $\bar{K}_\sigma$ are matrices of a full problem i.e. are obtained by applying Fourier representations and the finite difference discretization (3.3) directly to equation (2.4).
$\{ \lambda_{cr} , \mathbf{u}_{cr} \}$ is the critical eigenpair from the decomposition analysis.

We transform our problem to the standard form of generalized eigenvalue problem. Then $\mathbf{r}$ can be also expressed as :

(3.8) $$\mathbf{r}_M = (\hat{K}_\sigma - \gamma_{cr} \bar{K}_o) \mathbf{u}_{cr}$$

where

$\hat{K}_\sigma = \bar{K}_\sigma - \rho \bar{K}_o$, $\gamma_{cr} = - 1/\lambda_{cr} - \rho$, ρ is a shift.

Factorization of $\bar{K}_o$ into $S S^T$ allow us to get a standard eigenvalue problem and define an error :

(3.9) $$\mathbf{r} = (\tilde{K}_\sigma - \gamma_{cr} I) \tilde{\mathbf{u}}_{cr}$$

where

$\mathbf{r} = S^{-1} \mathbf{r}_M$, $\tilde{\mathbf{u}}_{cr} = S^T \mathbf{u}_{cr}$, $\tilde{K}_\sigma = S^{-1} \hat{K}_\sigma S^{-T}$

The accuracy with which γ_{cr} approximates exact γ_j can be estimated from the relation :

(3.10) $$\min_j | \gamma_j - \gamma_{cr} | \le \| \mathbf{r} \|$$

and the accuracy with which $\mathbf{u}_{cr}$ approximates $\mathbf{u}_j$ can be estimated from :

$$(3.11) \qquad \| \mathbf{u}_{cr} - \alpha_j \mathbf{u}_j \| \le \frac{\| \mathbf{r} \|}{s} , \qquad s = \min_{\substack{i \\ i \neq j}} | \gamma_i - \gamma_{cr} | .$$

where

$\{ \gamma_j , \mathbf{u}_j \}$ are exact eigenpairs of the full problem,

$\| \mathbf{r} \| = \mathrm{sqrt} (\Sigma | r_i |^2)$.

Practically, for the error estimation the following formula is used :

$$(3.12) \qquad \varepsilon = \frac{\| \mathbf{r}_M \|}{\| \bar{\mathbf{K}}_o \mathbf{u}_{cr} \|}$$

If the accuracy is not sufficient for our purposes it can be easy improved. In such case we can use approximate λ_{cr} or $\mathbf{u}_{cr}$ as the starting values for the subspace iteration method or for the determinant search method and expect good convergence properties of these algorithms.

3.e. Effectiveness evaluation.

For an effectiveness evaluation the method of decomposition will be compared with the full initial stability problem i.e. when equation (2.4) discretized by representation (3.3) is solved directly. The additional analyses required by the method of decomposition were juxtaposed in Table 3.1. Coefficients which will be used to evaluate the effectiveness of these analyses were also placed in this table.

Table 3.1 . Additional analyses performed in the method of decomposition.

problem	calculating integrals	forming matrices K_0	forming matrices K_σ	solving eigenvalue problem	error estimation
auxiliary	-	-	-	o_{aux}	-
nonsymmetric	o_{int}	r_0	r_σ	o_{non}	-
full	-	-	-	-	o_{err}

1. Calculating integrals [d_1], [d_2], [d_3]

We have to calculate (M+1) integrals [d_1], (M+1) integrals [d_2] and (M+1) (N+1) integrals [d_3], see Part 4 of this paper. (M+1) is a number of harmonics of the stability vector, (N+1) is a number of harmonics of the static problem. As the Newton - Cotes schema of numerical integration requires about R operation for a grid of R points calculations of the integrals require o_{int} operations :

$$o_{int} = R \; (M+1) \; (N+3) \tag{3.13}$$

2. Forming matrices.

We assume that a number of operations neccessary to generate a single element of K_0 and K_σ does not depend on the method of discretization. Consequently , we can compare a number of non-zeroth elements (n.o.e) of basic matrices for both methods. The comparison is done for Donnell's equations of shells and taking into account symmetry of matrices.

$$r_0 = \frac{n.o.e\; K_0}{n.o.e\; \bar{K}_0} \qquad r_0 = \frac{1}{6\; R} \tag{3.14a}$$

$$r_\sigma = \frac{n.o.e\; K_\sigma}{n.o.e\; \bar{K}_\sigma} \qquad r_\sigma = \frac{M+2}{6\; R\; (M+1)} \tag{3.14b}$$

3. Solving of eigenvalue problems.

We assume that the following assumptions hold for both methods :

a. the subspace iteration method is used to solve a generalized eigenvalue problem
b. only one, the lowest eigenvalue is to be determined , p = 1
c. number of iteration vectors used is q = 2 p
d. ten iterations is required to find a solution.

Specifying proper formula from [2] we obtained a total number of operations for the subspace iteration method :

$$o = n \left(m^2 + 86\, m + 144 \right) a \tag{3.15}$$

where n is a number of equations , m is a band-width, a is a number of analyses which have to be carried out. Evaluations for particular methods are placed in Table 3.2 .

Table 3.2 . Number of operations o for compared methods.

problem	m	n	a	o
auxiliary	12	3R	M+1	oaux
nonsymm.	3 (M+1)	3 (M+1)	1	onon
full	12 (M+1)‡	3R (M+1)	1	ofull

$$o_{aux} = 3\,(M+1)\ R\ 1320$$

$$o_{non} = 3\,(M+1)\ \{ 9\,(M+1)^2 + 258\,(M+1) + 144 \}$$

$$o_{full} = 3\,(M+1)\ R\ \{ 144\,(M+1)^2 + 1032\,(M+1) + 144 \}$$

$\ddagger$ Usually, the subspace iteration procedures require K_σ in a form consistent with a form of K_{o_f} (or in$_f$ a lumped form). Therefore, the largest value of K_o^f and K_σ^f half-band widths have to be taken if a standard procedure is to be used.

4. Error estimation.

Calculations of the error defined in (3.12) requires $o_{err} = 2\ n\ m + 4\ n$ operations. For our formulation it gives :

$$o_{err} = 72\ R\ (\ M+1\ )^2 + 12\ R\ (\ M+1\ ) \quad (3.16)$$

This number of operations is the same for both the full and the decomposed problem.

5. Total effectiveness estimation.

Total estimation of effectiveness of the method obtained by adding partial estimations described previously is given in Table 3.3 for selected values of R and M.

Table 3.3 . Effectiveness coefficients for the method of decomposition for different number of points R and harmonics M .

R	M	$\frac{o_1}{o_{full}}$ 100%	$\frac{o_2}{o_{full}}$ 100%
10	10	5.97	6.91
1000	10	4.58	5.51
10	100	0.83	0.99
1000	100	0.09	0.25

where
$o_1 = o_{aux} + o_{non}$, $o_2 = o_1 + o_{int} + o_{err}$, $N = M+1$ was taken to this computations.

The first coefficient is to estimate effectiveness of the eigenvalue analyses only. For $M = 0$ we have $o_{full} = o_{aux}$ and the second analysis (nonsymmetric) is not necessary at all. The difference between the auxiliary and nonsymmetric analysis on the one hand and the full analysis on the other hand increases for increasing values of M and R .
The second coefficient in Table 3.3 compares numbers of all operations for both methods including calculations of integrals and error estimation. Only the number of operations required for generating K_o and K_σ is excluded. We estimate it in a different way using (3.14) . For $M+1 = 20$ we have that $r_\sigma < 0.2 / R$ therefore for $R = 20$ both coefficients r_o and r_σ are below 0.01 , what means that a number of operations to generate these matrices is relatively small.

4. GENERATION OF BASIC MATRICES FOR NONSYMMETRIC PROBLEM.

Matrix relations for generating an elastic, stiffness matrix, K_o, and an initial stress matrix, K_σ, are given for the Donnell's variant of shell equations.

Let us write the condition (2.1), from which we obtain the stability equations, in the following form

$$\{\delta P_2\} = \{\delta U\} + \{\delta U_\sigma\} = \{0\} \tag{4.1}$$

$$\{\delta U\} = \int_0^H \int_0^{2\pi} \left(\{\delta \varepsilon\}^T \{N\} + \{\delta \varkappa\}^T \{M\} \right) r \, d\phi \, dx$$

$$\{\delta U_\sigma\} = \lambda \int_0^H \int_0^{2\pi} \left(\{\delta \varepsilon^{NL}\}^T \{\overset{\circ}{N}\} \right) r \, d\phi \, dx$$

where U is the elastic strain energy and U_σ is the work of internal membrane forces. Other symbols are explained in Appendix.

4.1. Measures of strains and internal forces, variations of strain measures.

Using (3.5) we can write strain measures in the form of series :

$$\{\varepsilon\} = \sum_j [\Phi^2_j] [B_j] \{q_j\} , \qquad \{\varkappa\} = \sum_j [\Phi^2_j] [\bar{\bar{B}}_j] \{q_j\} \tag{4.2}$$

$$\{\varepsilon^{NL}\} = \frac{1}{2} \sum_i \sum_j \alpha_i [B_{ij}] [\Phi^4_i] \{\Phi^3_j\} \alpha_j$$

where particular matrices and vectors depend on :

$[\Phi^2_j]$, $[\Phi^3_j]$, $[\Phi^4_j]$ – trigonometric functions ,

$[B_j]$, $[B_{ij}]$, $[\bar{\bar{B}}_j]$ – rescaled eigenvectors in the axial direction taken from the axiliary stability analysis,

$\{q_j\}$ – generalized coordinates defined in (3.5).

Internal forces and moments are determined using constitutive relations for an elastic, linear material

$$\{N\} = h [C] [\varepsilon], \qquad \{M\} = \frac{h^2}{12} [C] [\varkappa] \tag{4.3}$$

The reference membrane forces $\{\overset{\circ}{N}\}$ are calculated utilizing the displacement vector obtained from the linear static analysis for a unit load.

$$\{\overset{\circ}{N}\}=h\ [C\ ]\ \Sigma\ [\Phi_n^2]\ [\overset{\circ}{B}_n] \tag{4.4}$$

where $[\overset{\circ}{B}_n] = [B_n]\ \{\overset{\circ}{q}_n\}$, $\{\overset{\circ}{q}_n\} = \lfloor \max \overset{\circ}{w}_n , \max \overset{\circ}{v}_n , \max \overset{\circ}{u}_n \rfloor^T$.

Variations of strain measures calculated with respect to vector $\{q_x\}$, $x=0,\ldots,N$ are the following :

(4.5)

$$\{\delta\varepsilon\ \}= [\Phi_x^2]\ [B_x]\ \{\delta q_x\}\ , \qquad \{\delta\varkappa\ \}= [\Phi_x^2]\ [\overset{=}{B}_x]\ \{\delta q_x\}$$

$$\{\delta\varepsilon^{NL}\}=\sum_i\sum_j \alpha_i\ \Delta_{ijx}\ [B_{ij}]\ [\Phi_i^4]\ \ \{\Phi_j^9\ \}\lfloor j\rfloor\ \{\delta q_x\}$$

4.2. Elastic stiffeness matrix Ko

Let us express a variation of the strain energy as :

$$\{\delta U\ \}= \int_0^H \int_0^{2\pi} (\ \{\delta\varepsilon\}^T\ \{N\}\ +\ \{\delta\varkappa\}^T\ \{M\}\ )\ r\ d\Phi\ dx \tag{4.6}$$

Substituting (4.3) and (4.5) after exploiting

$$[C\ ]\ [\Phi_i^2]= [\Phi_i^2]\ [C\ ]\ , \qquad \int_0^{2\pi} [\Phi_x^2]\ [\Phi_i^2]\ d\phi = [\phi_x] \tag{4.7}$$

we obtain

$$\{\delta U\ \}= \{\delta q_x\}^T r \int_0^H (\ h\ [B_x]^T\ [\phi_x]\ [C\ ]\ [B_x]\ + \frac{h^3}{12}\ [\overset{=}{B}_x]^T\ [\phi_x]\ [C\ ]\ [\overset{=}{B}_x]\ )dx\ \{q_x\} \tag{4.8}$$

With the help of identities

(4.9)

$$[B_x]^T\ [\phi_x]\ =\ [\phi_x^3]\ [\tilde{B}_x]\ , \qquad [C\ ]\ [B_x]\ =\ [\overset{\approx}{B}_x\ ]\ [\tilde{C}\ ]$$

$$[\overset{=}{B}_x]^T\ [\phi_x]\ =\ [\phi_x^4]\ [\overset{+}{B}_x]\ , \qquad [C\ ]\ [\overset{=}{B}_x]\ =\ [\overset{++}{B}_x]\ [\overset{\approx}{C}]$$

and defining integrals in the axial direction

$$(4.10) \qquad [d_1] \equiv \int_0^H h\, [\tilde{B}_x]\, [\approx{B}_x]\, dx \, , \qquad [d_2] \equiv \int_0^H \frac{h^3}{12}\, [B_x^+]\, [B_x^{++}]\, dx$$

we obtain

$$(4.11) \qquad \{\delta U\} = \{\delta q_x\}^T [k_{ox}]\, \{q_x\}$$

$$[k_{ox}] \equiv r\, ([\phi_x^3]\, [d_1]\, [\tilde{C}] + [\phi_x^4]\, [d_2]\, [\approx{C}])$$

The strain energy U is a trigonometric polynomial of the second order. Therefore, the integral (4.7b) vanishes for all harmonic numbers i different than the x, to which the variation was calculated. Due to this the elastic stiffeness matrix K_o, obtained by aggregating $[k_{ox}]$, is a one-banded matrix.

4.3. Initial stress matrix Kσ

This matrix is obtained from the variation of the functional of work of membrane forces :

$$(4.12) \qquad \{\delta U_\sigma\} = \lambda \int_0^H \int_0^{2\pi} \{\delta \varepsilon^{NL}\}^T \{\overset{\circ}{N}\}\, r\, d\phi\, dx$$

Substituting (4.4) and (4.5.c) and taking advantage of identities

$$(4.13) \qquad \begin{array}{l} [C]\, [\Phi_n^2] = [\Phi_n^2]\, [C] \\ [C]\, [\overset{\circ}{B}_n] = [\overset{\circ}{\approx{B}}_n]\, [\tilde{C}] \, , \qquad [\overset{\circ}{\approx{B}}_n] = [\approx{B}_n]\, \{\overset{\circ}{q}_n\} \end{array}$$

after a re-arrangement we obtain

$$\{U_\sigma\} = \lambda\, \{\delta q_x\}^T \lfloor j \rfloor^T\, r \sum_i \sum_j \sum_n \alpha_i\, \Delta_{ijx}$$

$$(4.14) \qquad \left(\int_0^{2\pi} \{\Phi_j^3\}^T\, [\Phi_i^4]\, [\Phi_n^2]\, d\phi \right) \left(\int_0^H h\, [B_{ij}]\, [\overset{\circ}{\approx{B}}_n]\, dx \right) [\tilde{C}]$$

Defining integrals

(4.15)
$$\lfloor \phi_{jin} \rfloor \equiv \int_{0}^{2\pi} \{\Phi_j^3\}^T \, [\Phi_i^4] \, [\Phi_n^2] \, d\phi$$

$$[d_3] \equiv \int_{0}^{H} [B_{ij}] \, [\overset{\circ}{\tilde{\tilde{B}}}_n] \, dx$$

finally we have

(4.16)
$$\{\delta U_\sigma\} = \lambda \quad \{\delta q_x\}^T \sum_i [k_{\sigma i}] \, \{q_i\}$$

$$[k_{\sigma i}] \equiv r \quad \lfloor j \rfloor^T \quad \sum_{jn}\sum \Delta_{ijx} \, \lfloor \phi_{jin} \rfloor \, [d_3] \, [\tilde{C}] \, \lfloor j \rfloor$$

The subintegral of $\lfloor \phi_{jin} \rfloor$ is a trigonometric polynomial of the third order . Therefore the initial stress matrix K_σ , aggregated from sub-matrices $[k_{\sigma i}]$, is a multi- banded matrix, if we only keep a numeration of unknowns which is optimal for the linear analysis or for the auxiliary stability problem. Though it is not possible to decompose the stability equation for this analysis the size of the task is considerably diminished when the stability vectors from the auxiliary stability problem are used.

5.NUMERICAL EXAMPLES.

In this section two numerical examples are presented to assess the performance of the method. They are calculated with the program INSTAB written in FORTRAN and run on the IBM PC/AT microcomputer.

Example 1. Shell of a constant thickness

A shell open at top, with a free upper and a clamped lower edge, was analysed, Fig.1 . The shell has the following geometrical and material characteristics :

$r=40$ [in] , $H=120$ [in] , $h_1=h_2=h_3=h_4=h_5=0.1064$ [in]

$E=3\ 10^7$ [psi] , $\nu=0.3$

A load distribution in the circumferential direction is shown in Fig.2. Loads were assumed in the form :

$$C(\phi)=\sum_{n=0}^{6} c_n \cos n\phi \tag{5.1}$$

with

$c_0=-0.220$, $c_1=-0.338$, $c_2=-0.533$, $c_3=-0.471$

$c_4=-0.166$, $c_5=+0.066$, $c_6=+0.055$

Distribution of loads in the axial direction is constant.

At first, for all harmonics of the load , linear static analysis was done to determine the reference membrane forces. A program applying Fourier representations and the finite difference method for a displacement vector and based on the Sanders-Koiter's variant of shell equations was used.

Later, the auxiliary stability analysis for axisymmetrical load was performed. The finite difference model consisted of 11 uniformly distributed points . Analyses were executed for each particular harmonic (n=1,...,9) of the stability vector. The critical load $\lambda_{cr}=2.108$ was obtained with harmonic n=6. The normalized stability vector in the axial direction is shown in Fig.3 (broken lines). The normalized displacement vector for the axisymmetrical load is also shown in this figure (continious lines).

Finally the stability analysis for the nonsymmetrical loads was done . For comparison we carried out this analysis for Donnell's and Sanders's shell equations. In Table 5.1 critical loads for both variants are compared with results obtained by other authors. The experimental critical load (see [14]) is equal to 3.045. The obtained results are very close to results of

[14]. They are better than results of the finite element analysis (three dimensional degenerated elements S16) from paper [3].

Table 5.1. Critical loads λ_{cr} for a shell of constant thickness

LOADS	Axisymmetric		Wind			
References		pp	[14]	[3]	pp	pp
Equations	D	S	D	R	D	S
Solution method	FR 2dim.	FR+ FD	FR+ TP 5o	FE S16	FR+ VAA	FR+ VAA
λ_{cr}	2.03	2.108	3.857	4.69	3.860	3.927
$\lambda_{cr}/\lambda_{exper}$	0.667	0.692	1.267	1.54	1.268	1.289

pp- present paper
D- Donnell's S- Sanders's R- Reissner's
FR- Fourier Representations , FD- Finite Differences ,
FE- Finite Element , TP 5o- Taylor Polynomial of 5th order ,
VAA- Vectors from Auxiliary Analysis

There is no significant difference between results we obtained for Donnell's and for Sanders's equations. The normalized normal component W_n of the loss of stability vector for Donnell's equations is shown in Fig.4 (broken lines). The form of this vector obtained from Sanders's equations was almost identical. The normalized displacement vector for the static problem is marked by a continious line in this figure.

Example 2. Shell of a stepped thickness.

Due to using the finite difference method in the auxiliary problem the method can tackle shells with nonconstant parameters in the axial direction. As an example a shell with stepped thickness was calculated. Thicknesses of strips decrease from the bottom to upper edge, see Fig.1 :

h_1=0.1596 [in], h_2=0.1330 [in], h_3=0.1064 [in]
h_4=0.0798 [in], h_5=0.0532 [in] .

Width of strips is d=24 [in] . The other parameters are the same as in Example 1.

The auxiliary analysis for axisymmetric load is based on Sanders's shell equations while the analysis for nonsymmetric loads is based on Donnell's equations. The finite difference model consists of 21 points and the placement of points preserved a constant thickness within each range of integration. The obtained critical loads were λ_{cr}=0.879 for symmetrical loads and λ_{cr}=0.920 for nonsymmetrical loads. Normalized components W_n of the displacement vector (continious lines) and the stability vector (broken lines) are shown in Fig.5 and Fig.6.

In simplified engineering analyses a shell with constant thickness is often substituted for a shell with stepped thickness ; provided they both have equal average thicknesses. As the shells from Example 1 and Example 2 have the same average thicknesses we compare in Table 5.2 critical loads for them. It is apparent that the critical values for the stepped shell are considerably lower and ,thus,the commonly used substitution is not appropriate.

Table 5.2. Critical loads λ_{cr} for shell with constant and stepped thickness.

Shell	Loads Axisymmetric	Wind
Constant	2.108	3.860
Stepped	0.879	0.920

Let us compare displacement vectors (continuous lines) and stability vectors (broken lines) in Figures 3 and 5 and later in Figures 4 and 6 for both shells. Normalized displacements and stability vectors in the axial direction (Fig.3 and 5) have different distributions which correspond however to different thicknesses of the shells. The differences in normalized displacements of upper edges of both shells (continious lines) , see Fig.4 and 6, are localized in the front area of a shell, next to the symmetry plane of loads. The stability vector (broken lines) for the stepped shell is characterized by a greater number of small waves also concentrated in the front zone. This phenomenon can be explained by greater r/h parameter for upper strips of the stepped shell then for the shell of constant thickness. This parameter also effects that the differences between critical loads for symmetrical and nonsymmetrical analyses are smaller for the stepped shell then for a shell of constant thickness .

6. FINAL REMARKS.

The method of decomposition presented has the following features :

a) it does not depend on shell equations and can be used with any variant of equations of thin, elastic shells of the Kirchhoff-Love type without modifications,

b) it allows one to take into account variable distribution of parameters of a shell along the generator and any, classical boundary conditions - provided they are axisymmetrical.

c) a series of small tasks is solved instead of one large task. Maximal size of a task is 3 * max (number of finite difference points or harmonics of Fourier representations). Therefore the method can be used on microcomputers with short times of execution even for large problems,

For many cases the critical eigenpair from the decomposition analysis has satisfactory accuracy. In these cases the method is much more effective than a standard method. However often the eigenvalue analysis for the full stability equations will be necessary. Previously obtained results can then be used as starting values for the eigenvalue analysis and improve the convergence properties of the eigenproblem solver. This is why the effectiveness of the method of decomposition can be expected to be better than the direct solution of the full eigenvalue problem.

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8.APPENDIX.

8.A.Strain measures.

$$\{\varepsilon\}=\lfloor \varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{12} \rfloor^T \quad , \qquad \{\varkappa\}=\lfloor \varkappa_{11},\ \varkappa_{22},\ \varkappa_{12}\rfloor^T .$$

$$\{\varepsilon^{NL}\}= \lfloor \varepsilon^{NL}_{11},\ \varepsilon^{NL}_{22},\ \varepsilon^{NL}_{12} \rfloor^T$$

$$[B_i]=\begin{bmatrix} b_1, & 0 & ,b_2 \\ 0 & , b_3 & , 0 \\ 0 & , b_4 & , b_5 \end{bmatrix} \qquad [\tilde{B}_i]=\begin{bmatrix} b_1, & 0, & 0, & b_2, & 0 \\ 0, & b_3, & 0, & 0, & 0 \\ 0, & 0, & b_4, & 0, & b_5 \end{bmatrix}^T$$

$$[\tilde{\tilde{B}}_i]=\begin{bmatrix} b_1, & b_2, & b_3, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & b_1, & b_2, & b_3, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & b_4, & b_5 \end{bmatrix}$$

$$[B^+_i]= \text{diag}\ (b_6\ ,\ b_7\ ,\ b_8) \qquad [B_{ij}]= \text{diag}\ (b_9\ ,\ b_{10},\ b_{11})$$

$$[B^=_i]=\begin{bmatrix} b_6, & 0, & 0 \\ b_7, & 0, & 0 \\ b_8, & 0, & 0 \end{bmatrix} \qquad [B^{++}_i]=\begin{bmatrix} b_6, & b_7, & 0 \\ b_7, & b_6, & 0 \\ 0, & 0, & b_8 \end{bmatrix}$$

$$b_1= \frac{1}{r}\, W_i \ ,\quad b_2= \frac{i}{r}\, U_i \ ,\quad b_3= \dot{V}_i \ ,\quad b_4= -\frac{i}{2r}\, V_i \ ,\quad b_5= \frac{1}{2}\, \dot{U}_i$$

$$b_6= \frac{i^2}{r^2}\, W_i \ ,\quad b_7= -\ddot{W}_i \ ,\quad b_8= \frac{i}{r}\, \dot{W}_i$$

$$b_9= \frac{ij}{r^2}\, W_i\, W_j \ ,\quad b_{10}= \dot{W}_i \dot{W}_j \ ,\quad b_{11}= -\frac{j}{r}\, \dot{W}_i\, W_j$$

Dots above letters mean differantiation with respect coordinate x.

8.B.Variations of strains

Variations are calculated with respect to $\{q_x\}=\lfloor \alpha_x\ ,\beta_x,\ \gamma_x \rfloor^T$ corresponding to harmonic x. For example

$$\{\delta\varepsilon\}_{3\times 1} = \delta_{\alpha x}\{\varepsilon\}_{3\times 1} + \delta_{\beta x}\{\varepsilon\}_{3\times 1} + \delta_{\gamma x}\{\varepsilon\}_{3\times 1}$$

Analogically are defined $[\delta\varkappa]$ and $[\delta\varepsilon^{NL}]$.

$$\{\delta q_x\} = \lfloor \delta\alpha_x , \delta\beta_x , \delta\gamma_x \rfloor^T \quad , \quad \lfloor j \rfloor = (1 , 0 , 0)^T$$

$\Delta_{ijx} = \frac{1}{2} (\delta_{jx} + \delta_{ix})$ where δ_{jx}, δ_{ix} are Kronecker's deltas.

8.C. Internal forces and moments.

$$\{N\} = \lfloor N^{11}, N^{22}, 2N^{12} \rfloor^T \quad , \quad \{M\} = \lfloor M^{11}, M^{22}, 2M^{12} \rfloor^T$$

$$[C] = \begin{bmatrix} c_1 , & c_2 , & 0 \\ c_2 , & c_1 , & 0 \\ 0 , & 0 , & c_3 \end{bmatrix} \qquad [\tilde{C}] = \begin{bmatrix} c_1, & 0 , & 0 , & c_2, & 0 , & 0 , & 0 , & 0 \\ 0 , & 0 , & c_1, & 0 , & 0 , & c_1, & c_3, & 0 \\ 0 , & c_1, & 0 , & 0 , & c_2, & 0 , & 0 , & c_3 \end{bmatrix}$$

$$[\approx C] = \begin{bmatrix} c_1 , & 0 , & 0 \\ c_2 , & 0 , & 0 \\ c_3 , & 0 , & 0 \end{bmatrix} \qquad c_1 = E/(1-\nu^2) \ , \quad c_2 = \nu \ c_1 \ , \quad c_3 = 2 \ (1-\nu) \ c_1$$

where h is thickness of shell, E is Young's moduli, ν is Poisson's ratio.

8.D. Matrices of trigonometric functions and integrals of trigonometric polynomials.

$$[\Phi_i^2] = \mathrm{diag}\ (\cos i\phi \ , \ \cos i\phi \ , \ \sin i\phi)$$

$$[\Phi_i^3] = \lfloor \sin i\phi \ , \ \cos i\phi \ , \ \cos i\phi \rfloor^T$$

$$[\Phi_i^4] = \mathrm{diag}\ (\sin i\phi \ , \ \cos i\phi \ , \ \sin i\phi)$$

For calculations of integrals of trigonometric polynomials the trigonometric identities for polynomials of 3rd degree and the orthogonality conditions for trigonometric functions were applied.

$$[\phi_n] = \mathrm{diag}\ (c_n, \ c_n, \ s_n)$$

$$[\phi^3_n]=\begin{bmatrix} c_n, & 0, & 0, & 0, & 0 \\ 0, & c_n, & s_n, & 0, & 0 \\ 0, & 0, & 0, & c_n, & s_n \end{bmatrix} \qquad [\phi^4_n]=\begin{bmatrix} c_n, & 0, & 0 \\ c_n, & 0, & 0 \\ s_n, & 0, & 0 \end{bmatrix}$$

$$\lfloor \phi_{jin} \rfloor = \frac{1}{2} \lfloor \delta_{ln}-\delta_{kn}, \ \delta_{ln}+\delta_{kn}, \ \delta_{kn}-\delta_{ln} \rfloor \ [\phi_n]$$

with $l=i+j$, $k=|i-j|$, δ_{ln}, δ_{kn} are Kronecker's deltas.

$$c_n = \int_0^{2\pi} \cos^2 n\phi \, d\phi = \begin{cases} n=0 \Rightarrow c_n = 2\pi \\ n\neq 0 \Rightarrow c_n = \pi \end{cases}$$

$$s_n = \int_0^{2\pi} \sin^2 n\phi \, d\phi = \begin{cases} n=0 \Rightarrow s_n = 0 \\ n\neq 0 \Rightarrow s_n = \pi \end{cases}$$

8.E. Finite difference operators.

In the finite difference model two kinds of points {A,B,C} and {1,2,3,4} are distinguished, Fig.7. Areas of integration are assigned to points {A,B,C} and the finite difference operators are defined in points {A,B,C}. These operators are expressed by means of values in points {1,2,3,4}. The following operators were used :

$$f(B) = \frac{1}{2} \left(f(2)+f(3) \right) \qquad f,_x(B) = \frac{1}{2\Delta} \left(f(3)-f(2) \right)$$

$$f,_{xx}(B) = \frac{1}{2\Delta^2} \left(f(1)-f(2)-f(3)+f(4) \right)$$

where f is a component of the displacement vector.

The finite difference operators were generated using a condition of minimum of the error of expansion of f in the Taylor series in a vicinity of B , Liszka,Orkisz [16]. The obtained operators ensure a smooth solution of an eigenvalue problem for shells with constant thickness , similarly as operators of Bushnell [7].

8.F. Integrals along a generator.

Integrals $[d_1]$, $[d_2]$, $[d_3]$ were calculated with the Newton-Cotes's schema of a numerical integration. This schema is based on a linear interpolation of a subintegral function for uniformly distributed computational points.

$$\int_0^H Z\,dx = \frac{H}{L-1}\left[\,0.5\,(Z^1+Z^L) + \sum_{l=2}^{L-1} Z^l\,\right]$$

where r=1,...,R and Z^l are values of Z in points l.

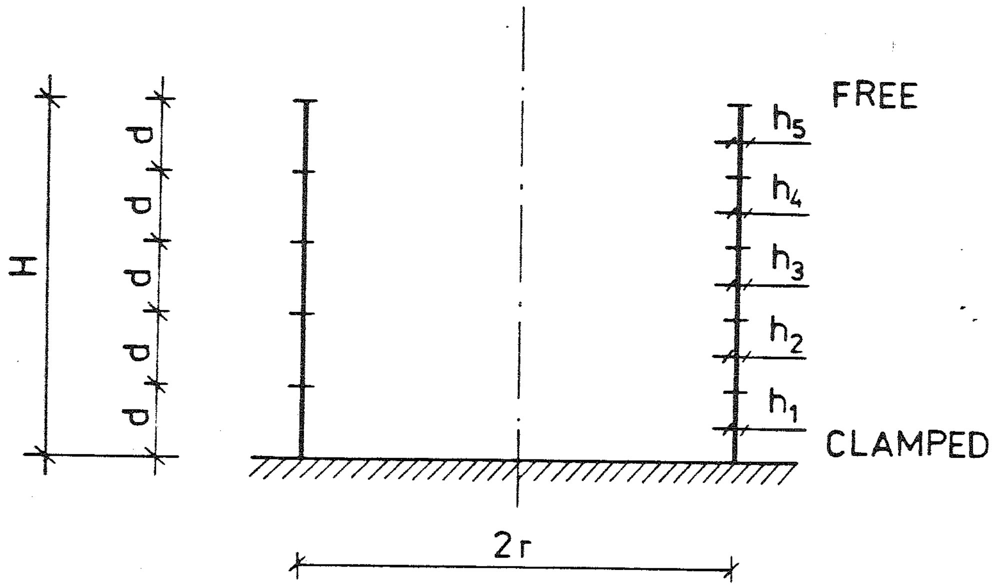

Fig.1. Geometry of a shell.

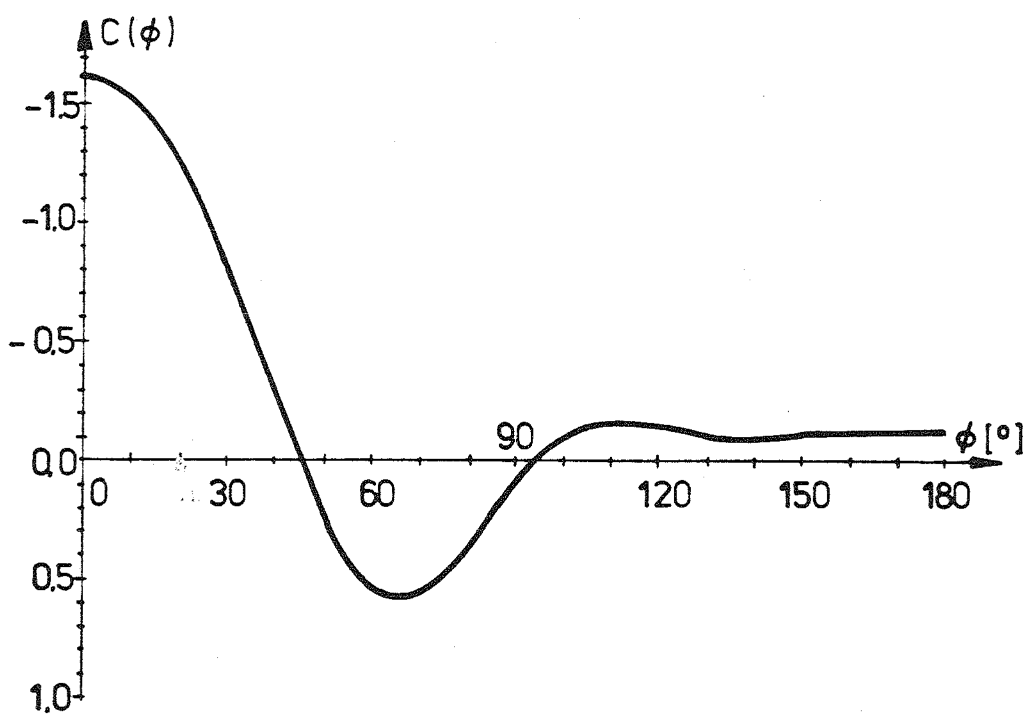

Fig.2. Wind pressure distribution.

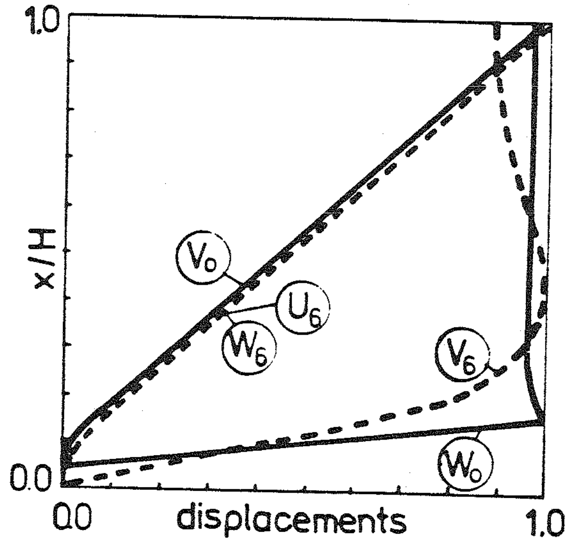

Fig.3. Normalized generator displacements for n=0 (continuous lines) and normalized stability vectors for n=6 (broken lines). Shell of a constant thickness.

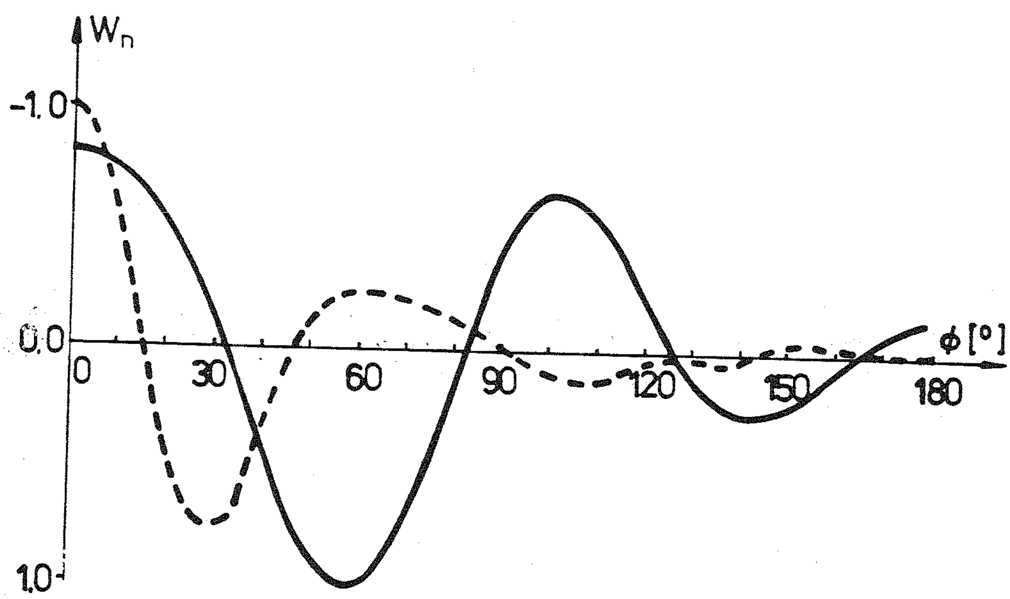

Fig.4. Normalized displacement (continuous line) and normalized stability vectors for the upper edge of the shell. Shell of a constant thickness.

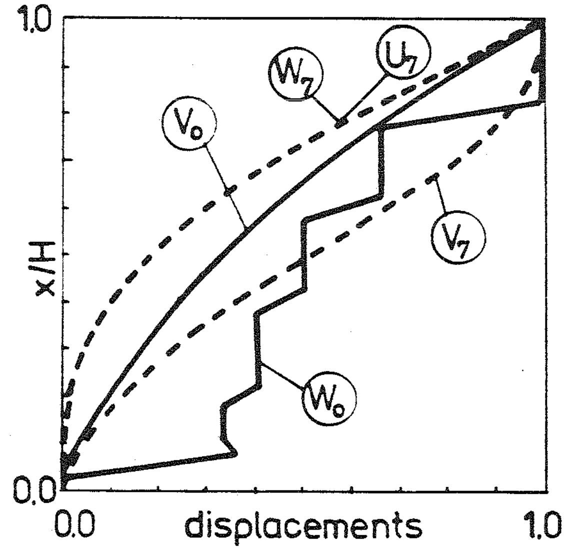

Fig.5. Normalized generator displacements for n=0 (continuous lines) and normalized stability vectors for n=7 (broken lines). Shell of a stepped thickness.

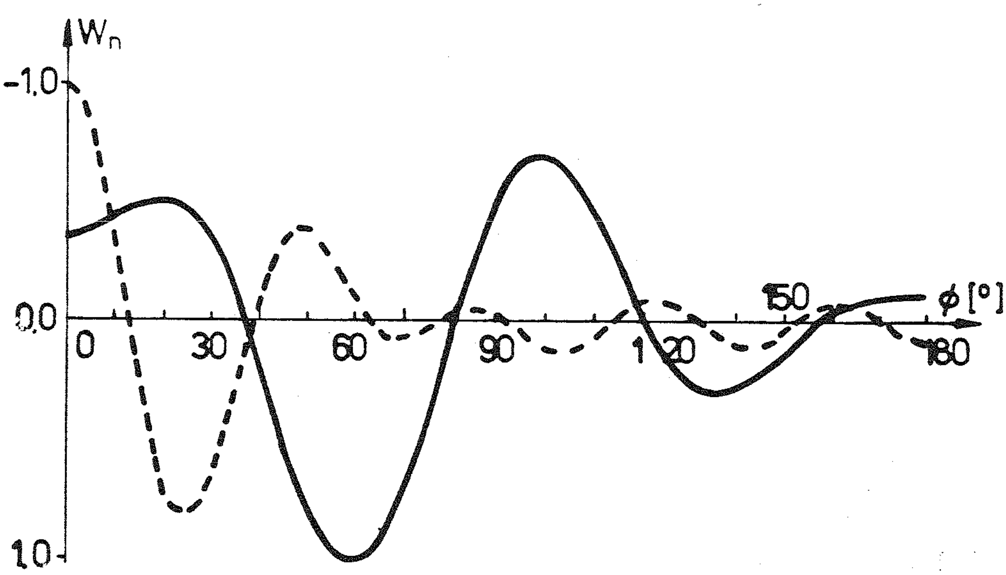

Fig.6. Normalized displacement (continuous line) and normalized stability vectors for the upper edge of the shell. Shell of a stepped thickness.

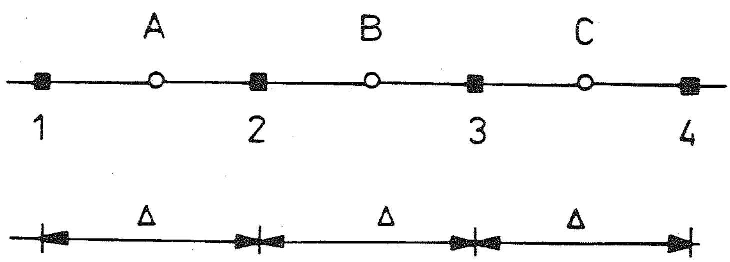

Fig.7. Numeration of points and assignment of areas of integration.